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Inverse Problems for Generalized Subdiffusion Equations

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

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List of Publications

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Author’s Contributions to the Publications

- **I** In Publication I, I was formulating and proving mathematical statements, performing numerical computations, writing a draft of the paper.
- **II** In Publication II, I was formulating and proving mathematical statements, writing a draft of the paper.
- **III** In Publication III, I was formulating and proving mathematical statements, writing a manuscript, acting as a corresponding author.

Introduction

This work mainly focuses on inverse source problems for subdiffusion equations.

Let us firstly describe what the subdiffusion process is and what is the difference between the usual diffusion and subdiffusion processes:

- Usual diffusion is described by Random Walk model (RW): the elementary steps taken on a microlevel are independent from each other and happen with the SAME time pace.
- Subdiffusion is described by Continuous Time Random Walk model (CTRW) and stands for the models with the VARIABLE waiting time.

A common way to describe CTRW is the time-fractional diffusion equation. The derivation of this equation and the generalized subdiffusion equation will be described in the next Chapter. In order to write it down we need the definitions of Riemann–Liouville and Caputo fractional derivatives of the order $\beta \in (0,1)$ [70]:

$$({}^R D_a^\beta v)(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} v(\tau)d\tau, \quad ({}^C D_a^\beta v)(t) = \int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} v'(\tau)d\tau.$$

The Riemann-Liouville fractional integral of the order $\beta \in (0,1)$ is defined as

$$(I_a^\beta v)(t) = \int_a^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} v(\tau)d\tau.$$

Then the time-fractional diffusion equation in self-similar medium is:

$$u_t = {}^R D_0^{1-\beta} Lu + Q \tag{1.1}$$

with $0<\beta<1$, $L=\varkappa\Delta$, where Δ is the Laplace operator, $\varkappa$ is a positive constant and Q is a source term. The time-factional diffusion equation (1.1) is used to describe subdiffusion (slow diffusion) processes [5, 13, 18, 67]. These are diffusion in fractal and porous media such as propagation of underground pollution, dynamics of protein in cells, heat flow in media with memory. Fractional diffusion equation (1.1) is also used in describing Hamiltonian chaos, transport in dielectrics and semiconductors, application of optical tweezers, etc. [12, 13, 18, 73, 87].

In order to incorporate linear reaction in the model we replace $L=\varkappa\Delta$ by

$$L = \varkappa\Delta + r(x)I,$$

where I is the unity operator. The addend $r(x)I$ is also referred to as a potential term [98].

By applying the operator $I_0^{1-\beta}$ to (1.1) we obtain the equivalent equation in the Caputo form

$${}^C D_0^\beta u = Lu + F \qquad (F = I_0^{1-\beta} Q). \tag{1.2}$$

Unlike (1.1), this equation is not overdifferentiated from the mathematical point of view, thus mathematicians generally prefer to work with this model.

In case the medium is not self-similar, the power function has to be replaced by some other kernel under the derivative. We utilize the generalized fractional derivatives in Riemann–Liouville ${}^R D_a^{\{k\}}$ and Caputo sense ${}^C D_a^{\{k\}}$:

$$({}^R D_a^{\{k\}} v)(t) = \frac{d}{dt}\int_a^t k(t-\tau)v(\tau)d\tau,\ ({}^C D_a^{\{k\}} v)(t) = \int_a^t k(t-\tau)v'(\tau)d\tau,$$

$t>a$, k is a locally integrable function.

The generalized subdiffusion equation is [13, 18, 82]

$$u_t = {}^R D_0^{\{M\}} Lu + Q \tag{1.3}$$

where M is an arbitrary locally integrable kernel. In case there exists a kernel k such that $k * M = 1$ then (1.3) can be transformed to the Caputo form

$${}^C D_0^{\{k\}} u = Lu + F \qquad (F = k * Q) \tag{1.4}$$

where $*$ is the time convolution:

$$v_1 * v_2(t) = \int_0^t v_1(t-\tau) v_2(\tau) d\tau.$$

The generalized subdiffusion (the medium is not self-similar) equation (1.3) describes the cases, when the medium is not self-similar. These include multiterm and distributed diffusion models [12, 48, 66, 82, 87], tempered subdiffusion [10, 19, 82, 84, 95], some models with bounded kernels [27].

Let us explain what an inverse problem is. In the classical theory of PDEs developed by Laplace and Hadamard the goal is to solve the direct problem. This means to reconstruct the process, given the nature law (the PDE itself), the measurements (the boundary data) and its characteristics (the coefficients of the equation) .

In practice, however, the coefficients and the source term of the equation are often unknown. Thus, in order to apply the model created before, additional measurements that allow to reconstruct the unknown coefficient are required. This type of problem is called **an inverse problem** [28, 30, 32, 58, 76].

For example, the inverse problem in case of the diffusion equation can be to determine space-dependent components of source terms and space-dependent coefficients by means of final overdetermination data

$$u(T,x) = \psi(x), \quad x \in \Omega, \tag{1.5}$$

where $T > 0$ and Ω is the space domain where a process is going on. Inverse source problems for diffusion equations have important applications in location of groundwater and atmospheric pollution sources [43, 89] . The problem to determine u where final overdetermination condition of type (1.5) replaces the initial condition is called a backward problem. This type of problem has many applications, including the reconstruction of geothermal history of Earth [30].

Theoretical solvability of problems in PDEs addresses the issue of the well-posedness. In sense of Hadamard the problem is **well-posed** if [30]:

1. the solution is unique;
2. the solution exists;
3. the solution is stable, i.e., it continuously depends on data.

Let the operator A map a metric space X to a metric space Y. Then the definition of well-posedness is equivalent to:
The equation $Ax = y$, $x \in X$, $y \in Y$ represents a **well-posed** problem in sense of Hadamard if the operator A has a continuous inverse from Y to X.

If one of the conditions 1-3 is violated the problem is called ill-posed. There are many examples of inverse problems that are ill-posed [41]. That is because the forward operator

A is usually smoothing and, therefore, an inverse of A from Y to X is not continuous. However, a proper redefinition of spaces may lead to a well-posed problem. Thus, one of the goals of theoretical analysis of the ill-posed problem is to choose a setting of function spaces where A^{-1} is continuous.

Ill-posed problems are classified as mildly, moderately and severely ill-posed. The definition of degree of ill-posedness depends on the formulation of the problem. For example, in case of linear operators the degree of ill-posedness of the operator is defined via its singular values. For the moderately ill-posed problems the degree of ill-posedness can be defined as the highest order of the derivative that must be included in the stability estimate of the solution. If derivatives of all orders are involved in the reconstruction then the problem is said to be severely ill-posed.

It is worth to point out that in practice one may not be able to compute the derivatives necessary for the stability estimate, which makes the problem ill-posed problem in its essence, not a well-posed. Knowing the degree of ill-posedness is important, since it helps to choose an appropriate regularization technique to reconstruct the solution in practice.

Let us give general remarks on problems with final overdetermination.

Theoretical issues of a problem to reconstruct a space-dependent factor $f(x)$ of the source term

$$F(t,x) = f(x)g(t,x) \tag{1.6}$$

and a problem to identify a space-dependent reaction (potential) coefficient $r(x)$ in parabolic equation by means of the final data (1.5) were studied in [29]. There the problem for f was reduced to a fixed-point equation with a compact operator and uniqueness was proved by means of maximum principles. Existence and continuous dependence of f on data ψ follow from the uniqueness.

This approach was further developed in [6, 34, 59, 60]. More precisely, in [34, 59], inverse problems for parabolic equations including a lower-order integral term were studied.

If the known factor of the source term g depends only on t then the source function F has separated variables and the problem to recover f from final measurements ψ can be handled by means of the Fourier method: the original inverse problem is reduced to a family of inverse problems for Fourier coefficients of f that are explicitly solved [39].

The uniqueness of solution of a backward in time problem for parabolic equation equation was shown in [91].

Inverse problems for the fractional equation have often been studied by the same methods as in non-fractional case. For example, positivity principle, the Fourier method, the method of Laplace transform have been successfully extended to the fractional case.

Existence, uniqueness and continuous dependence on data of solution of a problem to determine the factor $f(x)$ of a source function included in the **time fractional diffusion equation** (1.2) from final data (1.5) in the particular case $g = g(t)$ were established by means of the Fourier method [69, 78, 93, 94].

The method enables to handle problems with non-classical boundary conditions and different non-local space operators, too (see [1, 46, 47]). The paper [86] treated a more general case when a space operator L contains coefficients depending both on x and t but g is still independent of x. Uniqueness of reconstruction of f from final data was proved by means of monotonicity arguments.

If $g = g(t)$ and the unknown $f(x)$ is a priori smoother than an intial state then the final data ψ contain enough information to recover simultaneously f and the order of derivative β in (1.2) [37].

An inverse problem to reconstruct factor $f(x)$ of a source function of (1.2) from final data in the general case $g = g(t,x)$ was considered in [79, 90]. The existence, uniqueness and continuous dependence of a solution on data were proved except for a finite set of values of $\varkappa$. This study uses analyticity arguments and is a generalization of an analogous result obtained for usual parabolic problem [15].

An equation (1.2) with the semilinear term $F = F(t,x,u) = f(x)g(t,x,u)$ was considered in [36]. Uniqueness of the reconstruction of f from final data was proved by means of a positivity principle provided in the same paper. That falls into category of maximum principle results [34, 54, 61].

Estimates for $r(x)$ in a subdomain of Ω in terms of the final data $\psi(x)$, $x \in \Omega$, were deduced by means of Carleman estimates in [96]. However, this method assumes essential restrictions on the equation (only the one-dimensional equation (1.2) in case of half derivative was considered). Similar results for a problem to determine a diffusion coefficient depending on spatial variables were obtained in [75].

There is a number of papers that are concerned with inverse problems for (1.2) that use *overdetermination conditions that are different from* (1.5).

The reconstruction of the source factor $f(x)$ in case $g = g(t)$ from the weighted integral overdetermination of the form $\int_0^T \eta_0(t)u(t,x)dt = \psi(x)$ was considered in [55]. Existence and uniqueness of a distributional solution were proved.

Several works have been concerned with inverse problems with local, boundary or integrated overdetermination along time. For example, in [31, 56, 57, 78, 85] existence and uniqueness of reconstruction of time-dependent factors of sources and boundary conditions were proved.

The paper [44] was concerned with the reconstruction of the source function F that depends on time and part of spatial variables from boundary measurements over the time. Estimates for the solution in terms of the data were deduced.

The inverse problem to determine the factor $f(x)$ in the usual parabolic equation or in the fractional diffusion equation (1.2) from final data is moderately ill-posed. But unlike the backward in time problem for the parabolic equation, that is a classical example of a severely ill-posed problem [30], such a problem for the fractional equation (1.2) is moderately ill-posed [39, 78]. This difference in the regularity of these problems is caused by a difference in behavior of Fourier coefficients of the state function u for large eigenvalues. They have an exponential decay in the usual parabolic case but a power-type decay in the fractional case.

The asymptotics of Fourier coefficients of u was used to prove moderate ill-posedness of a problem to identify the coefficient $r(x)$ in (1.2) from final data (1.5) under the assumption that T is sufficiently large [98].

Inverse problems for **the generalized time fractional diffusion equations** (1.3), (1.4) have found less attention in the literature.

A couple of papers was concerned with inverse problems for (1.4) in case the kernel k is a sum of power functions, i.e. the equation involves a sum of Caputo derivatives of different orders. In such a case the corresponding ODE in Fourier domain can be handled by means of multinomal Mittag-Leffler functions [53]. More precisely, in [38] uniqueness of determination of the source factor $f(x)$ from measurements in a subdomain of $(0,T) \times \Omega$ in the case $g = g(t)$ was proved. The paper [53] dealt with a reconstruction of orders of derivatives of such an equation.

Another group of papers deals with identification of kernels M and k in (1.3) and (1.4) by means of measurements along the time axis. Reconstruction of weight functions of distributed Caputo derivatives in (1.4) was considered in [77] and determination of a kernel

M of a perturbed Riemann-Liouville derivative was studied in [33]. Most general result in this direction was obtained in [35] where the existence, uniqueness and continuous dependence on data for a problem to identify M satisfying certain monotonicity and convexity conditions were proved.

Inverse problems to determine x-dependent source terms and coefficients in equation (1.3) from final data as well as backward problems for (1.3) **had not been studied before**. Such problems will be one investigation object of the thesis.

In another problem under investigation in this thesis the aim is to reconstruct the unknown source term f which depends on both space and time variables. In order to do that we formulate an inverse problem in a different way. Instead of a pointwise final-time overdetermination condition (1.5) we consider an overdetemination condition on a final time subinterval:

$$u|_{(t_0,T)\times\Omega} = \varphi, \tag{1.7}$$

where $t_0 \in (0,T)$.

The inverse source problem for the generalized fractional diffusion equation (1.3) with the overdetermination condition (1.7) **has not yet been considered** even in the usual fractional case (1.2).

Before solving this inverse problem, however, we first consider a different problem that serves as a good starting point for the further applications. This is the problem to recover a history of a function u at $0 < t < T$ by means of measurements of $u(t)$ and its generalized fractional derivative in a left neighborhood of T:
given $\varphi, g : (t_0,T) \to \mathbb{R}$, find $u : (0,T) \to \mathbb{R}$ such that

$$u|_{(t_0,T)} = \varphi \quad \text{and} \quad D_0^{\{k\}} u|_{(t_0,T)} = g, \tag{1.8}$$

where $D_0^{\{k\}}$ is either Riemann-Liouville or Caputo generalized fractional derivative.

Such a problem makes sense only in case of fractional or generalized fractional derivative due to the unlocal nature of it. In case of a usual derivative it would have been impossible to reconstruct the function backward in time based on the measurements of derivative on a final time subinterval. The problem is new and in this situation the techniques working successfully in a usual parabolic case cannot be simply extended to the fractional case.

The **objectives** of the thesis include theoretical study that focuses on establishing the conditions of uniqueness, existence and stability for the problems related to integrated versions of the generalized subdiffusion equation (1.3):

- inverse problems to reconstruct the space-dependent part of the source term $f(x)$ from the data (1.5);
- a backward in time problem;
- a problem to identify the unknown reaction term $r(x)$ from the data (1.5);

the objectives also include

- theoretical analysis of a problem to recover a history of function u given its value and the value of its generalized fractional derivative on a final-time subinterval;
- theoretical study of a problem to reconstruct an unknown source $f(x,t)$ in (1.3) from the overdetermination data on a stripe (t_0,T).

The scientific **novelty** of the thesis is justified by the following results

- inverse problems with final overdetermination for generalized subdiffusion equations have been studied first time;
- these problems describe much wider range of processes than problems posed for the usual fractional diffusion equation (1.1);
- the problem of recovery of a history of function u given its value and the value of its generalized fractional derivative on a final-time subinterval has been addressed for the first time;
- inverse source problems with observations on final time subintervals have been investigated for the first time.

Let us provide a **content overview** and short summary of **methodology**.

In the next Chapter we discuss the motivation to consider such a type of problems and address their history. Next we describe basic mathematical concepts, including the setting of functional spaces, the Sonine kernels and Mittag-Leffler functions. We provide the examples of kernels M and k used in the thesis. We also show different approaches to derive the model from a physical perspective and discuss the processes that such a model elaborates.

In the Chapter 2 we consider two inverse problems for a generalized diffusion equation (1.3) that use final observation data. We prove our results under certain monotonicity and convexity assumptions on M and k. The first problem is to identify a space-dependent factor f of a source term $g(t,x)f(x)$ and the second one is to reconstruct a coefficient $r(x)$ of a linear reaction term. We prove the uniqueness of the solution to the inverse source problem by applying a modified version of the positivity principle from [36]. Next we prove the existence and stability of the solution to the inverse source problem by means of the Fredholm alternative. The uniqueness of reconstruction of the reaction term follows from the results for the inverse source problem. Finally, we prove local existence and stability of the solution to the problem of reconstruction a reaction coefficient by means of the contraction argument.

In Chapter 3 firstly an inverse problem for an equation (1.3) with $M = \frac{t^{\beta-1}}{\Gamma(\beta)} + m * \frac{t^{\beta-1}}{\Gamma(\beta)}$ is considered, that is equivalent to a fractional diffusion equation (1.2) with an additional perturbation term $m * Lu$. The objective here is to reconstruct a space-dependent component $f(x)$ of the source term $f(x)g(t)$, given the final overdetermination condition (1.5). Since the variables in the source term are separated, the Fourier method is used to reduce the original inverse problem to the family of inverse problems for the Fourier coefficients of f. Then a family of fractional ODEs for the Fourier coefficients of u is solved using Mittag-Leffler functions. By composing this back into series the closed solution formula for f is obtained. Next we use norms with the exponential weights to obtain the solution estimates in the setting of L_p spaces and based on that formulate uniqueness, existence and stability theorems.

Next in Chapter 3 given a final overdetermination condition we solve the same equation backward in time and again we derive the closed solution formula by means of Fourier method.

In Chapter 4 we consider the inverse problem of a backward reconstruction of a history of u from (1.8). By means of the Laplace transform we prove the uniqueness for a general class of kernels k and reduce backward continuation problem to an integral equation that is further used to derive the solution formulas. Then the solution formulas are derived in some particular cases of k based on the expansion with the Legendre polynomials.

Further we apply the results obtained for backward continuation problem to an inverse problem of reconstruction of a history of a source in a general PDE from the measurements in a left neighborhood of final time T. Straightforwardly from the results for backward continuation problem we obtain the uniqueness of solution for source reconstruction problem. Finally we deduce explicit solution formulas for some particular cases.

Basic mathematical concepts

1.1 Functional spaces and integral transforms

The symbol $\mathscr{L}(X,Y)$ denotes the set of all bounded linear operators from Banach space X to another Banach space Y. If $X=Y$ we use an abbreviated notation $\mathscr{L}(X)$.

Let X be a Banach space and $G\subseteq\mathbb{R}^n$. The space $C(G;X)$ denotes the space of all continuous functions $w:G\to X$. If G is compact then $C(G;X)$ is a Banach space with a norm:

$$\|w\|_{C(G;X)}=\sup_{y\in G}\|w(y)\|_X.$$

We denote by $C^m(G,X)$ the the space of all functions $w:G\to X$ which admit continuous derivatives of order $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\geq 0$, $i=1,\ldots,n$ and $|\alpha|=\alpha_1+\ldots+\alpha_n\leq m$. If G is compact then it is a Banach space with a norm:

$$\|w\|_{C^m(G,X)}=\sum_{|\alpha|\leq m}\left\|\frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\ldots\partial y_n^{\alpha_n}}w\right\|_{C(G;X)}.$$

We also define

$$C^\infty(G;X)=\bigcap_{m\geq 0}C^m(G;X).$$

For open G we denote by $L_p(G;X)$, $p\in[1,+\infty)$ the Lebesgue spaces. The space $L_p(G;X)$ contains the equivalence classes of all Bochner-measurable functions $w:G\to X$, such that $\|w(y)\|_X^p$ is integrable. This is a Banach space with a norm

$$\|w\|_{L_p(G;X)}=\left(\int_G\|w(y)\|_X^p dy\right)^{\frac{1}{p}}.$$

In case if G is unbounded

$$L_{p,loc}(G;X)=\left\{w: w|_{G'}\in L_p(G';X)\quad\forall G'\subset G,\ \text{such that } G' \text{ is bounded}\right\}.$$

The space $L_\infty(G;X)$ contains the equivalence classes of all Bochner-measurable functions $w:G\to X$, such that $\|w(y)\|_X$ is essentially bounded. This is a Banach space with a norm:

$$\|w\|_{L_\infty(G;X)}=\operatorname{ess\,sup}_{y\in G}\|w(y)\|_X.$$

We denote by $W_p^m(G;X)$, $p\in[1,+\infty), n\in\mathbb{N}$ the Sobolev space, i.e. the space of all functions $w:G\to X$ having distributional derivatives of order $\alpha=(\alpha_1,\ldots,\alpha_n)$ in $L_p(G;X)$ for $\alpha_i\geq 0$, $i=1,\ldots,n$ and $|\alpha|\leq m$. The norm in this Banach space is:

$$\|w\|_{W_p^m(G;X)}=\left(\sum_{|\alpha|\leq m}\left\|\frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\ldots\partial y_n^{\alpha_n}}w\right\|^p_{L_p(G;X)}\right)^{\frac{1}{p}}.$$

For the exponentially bounded $f\in L_{1,loc}(\mathbb{R}_+;X)$ (i.e. $\int_0^\infty e^{-\omega t}\|f(t)\|_X dt<\infty$ for some $\omega\in\mathbb{R}$) we will denote the Laplace transform [74]

$$\widehat{f}(s)=(\mathscr{L}_{t\to s}f)(s)=\int_0^\infty e^{-st}f(t)dt,\quad \text{Re}s>\omega.$$

Fourier transform for the function $f \in L_1(\mathbb{R};X)$ is defined as [74]:

$$\mathscr{F} f(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

Fourier transform of the distribution f is defined by formula:

$$< \mathscr{F} f, \varphi >=< f, \mathscr{F}\varphi >, \quad \forall \varphi \in C^\infty(\mathbb{R}), \text{ such that } |x^k \varphi(x)| \underset{|x|\to\infty}{\to} 0 \quad \forall k > 0.$$

Then for $f \in L_p(\mathbb{R};X)$ we have that $< f, \varphi >= \int_{\mathbb{R}^n} f(x)\varphi(x)dx$ is absolutely convergent and Fourier transform is defined in a distributional sense

$$< \mathscr{F} f, \varphi >= \int_{\mathbb{R}^n} f(\xi)\mathscr{F}\varphi(\xi)d\xi, \quad \forall \varphi \in C^\infty(\mathbb{R}), \text{ such that } |x^k \varphi(x)| \underset{|x|\to\infty}{\to} 0 \quad \forall k > 0.$$

We introduce the spaces

$$H^s_p((0,T);X) = \{w|_{(0,T)} : w \in H^\beta_p(\mathbb{R};X)\}, \ p \in (1,\infty), s > 0,$$

where

$$H^s_p(\mathbb{R};X) = \{w \in L_p(\mathbb{R};X) : \mathscr{F}^{-1}|\xi|^s \mathscr{F} w \in L_p(\mathbb{R};X)\}.$$

Moreover, we define

$${}_0H^s_p((0,T);X) = \{w|_{(0,T)} : w \in H^s_p(\mathbb{R};X), \operatorname{supp} w \subseteq [0,\infty)\}, \ p \in (1,\infty), s > 0,$$

where the support of w, i.e. $\operatorname{supp} w$ is the complement in $\mathbb{R}$ of the largest open set on which $w = 0$ almost everywhere.

By default we drop the symbol of value space for $X = \mathbb{R}$ or $X = \mathbb{C}$, but we show it if necessary.

A useful sentence is the *Young's theorem for convolutions* which states that for $m \in L_q(0,T)$ and $w \in L_p((0,T);X)$ with $p,q \in [1,\infty]$, the convolution $m * w$ belongs to the space $m * w \in L_s((0,T);X)$ where $1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ and the inequality

$$\|m * w\|_{L_s((0,T);X)} \leqslant \|m\|_{L_q(0,T)} \|w\|_{L_p((0,T);X)} \tag{1.9}$$

is valid.[1]

1.2 Hölder spaces

Let us denote

$$C_0([0,T];X) = \{u \in C([0,T];X) : u(0) = 0\}.$$

Next for $0 < \alpha < 1$ we introduce the abstract Hölder spaces with corresponding norms

$$C^\alpha_0([0,T];X) = \left\{u \in C_0([0,T];X) : \|u\|_{C^\alpha_0([0,T];X)} := \sup_{0<t_1<t_2<T} \frac{\|u(t_2)-u(t_1)\|_X}{(t_2-t_1)^\alpha} < \infty\right\},$$

$$\begin{aligned} &C^\alpha([0,T];X) = C^\alpha_0([0,T];X) + X \\ &= \{u : u(t) = u_1(t) + u_2, t \in [0,T], u_1 \in C^\alpha_0([0,T];X), u_2 \in X\}, \\ &\|u\|_{C^\alpha([0,T];X)} = \|u - u(0)\|_{C^\alpha_0([0,T];X)} + \|u(0)\|_X, \end{aligned}$$

$$\begin{aligned} &C^{1+\alpha}_0([0,T];X) = \{u : u, u' \in C^\alpha_0([0,T];X)\}, \\ &\|u\|_{C^{1+\alpha}_0([0,T];X)} = \|u\|_{C^\alpha_0([0,T];X)} + \|u'\|_{C^\alpha_0([0,T];X)}. \end{aligned}$$

[1]Here $\frac{1}{s} = 0 \Leftrightarrow s = +\infty$. The same relation works for p and q in place of s.

1.3 Sonine kernels and completely monotonic functions

The function $M \in L_{1,loc}(0,\infty)$ is called **Sonine kernel** if the equation

$$M * k(t) = 1, \quad t > 0, \tag{1.10}$$

has a solution $k \in L_{1,loc}(0,\infty)$ [80]. The solution k if it exists is *unique* (Theorem 5.2, p.158 in [22]) and is referred to as **associate** to M. Since the convolution is commutative, k is also a Sonine kernel and M is its associate.

The Sonine kernel is unbounded at $t = 0$, since $k * M(t) \not\to 0$ as $t \to 0^+$.

The Laplace transform can be useful to derive the associate kernel to a given Sonine kernel. For this purpose we present the analogue of relation (1.10) in the Laplace domain

$$\widehat{M}(s)\widehat{k}(s) = \frac{1}{s}. \tag{1.11}$$

The kernel of a usual fractional derivative is a Sonine kernel. Indeed, in this case $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \in L_{1,loc}(0,\infty)$ and its associate is $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} \in L_{1,loc}(0,\infty)$. Let us check it by computing the corresponding Laplace transforms: $\widehat{M}(s) = \frac{1}{s^\beta}$ and $\widehat{k}(s) = \frac{1}{s^{1-\beta}}$, thus the relation (1.11) holds. Further examples of Sonine kernels are presented in the Section 1.7.

Let M be Sonine kernel and k its associate, then we have

$${}^R D_0^{\{k\}}(M * v) = \frac{d}{dt} k * M * v = \frac{d}{dt} 1 * v = v, \quad \forall v \in L_1((0,T);X). \tag{1.12}$$

Therefore, the operator $M*$ is a one-to-one mapping from $L_1((0,T);X)$ to

$$M * L_1((0,T);X) = \{M * v \,:\, v \in L_1((0,T);X)\}$$

and ${}^R D_0^{\{k\}}$ is the inverse of $M*$. The reversed relation to (1.12) is

$$M * \left({}^R D_0^{\{k\}} v\right) = v, \quad \forall v \in M * L_1((0,T);X).$$

This justifies the transformation of (1.3) to (1.4) by applying the operator $k *$.

We provide the Lemma that follows from Theorems 1 and 2 in [21]:

Lemma 1.1. *Let $z \in L_{1,loc}(0,\infty) \bigcap C^1(0,\infty)$, $z \geq 0$, $z' \leq 0$, $\lim_{t\to 0^+} z(t) = \infty$. Then z is Sonine kernel.*

Next we present the definition of completely monotonic functions. The function $z \in C^\infty(0,\infty)$ is called completely monotonic if

$$(-1)^i z^{(i)}(t) \geq 0, t > 0, i = 0,1,2...$$

We denote by $\mathscr{CM}$ a subclass of completely monotonic functions:

$$\mathscr{CM} = \{z \in L_{1,loc}(0,\infty) \bigcap C^\infty(0,\infty) \,:\, \lim_{t\to 0^+} z(t) = \infty, (-1)^i z^{(i)} > 0, i = 0,1,\ldots\}.$$

According to Lemma 1.1 and [21], Theorem 3:

Lemma 1.2. *The class $\mathscr{CM}$ consists of Sonine kernels. Moreover, $M \in \mathscr{CM}$ if and only if its associate kernel $k \in \mathscr{CM}$.*

1.4 C- and Hölder spaces related to the Sonine kernels

Let M be a Sonine kernel and k its associate. Based on the relation (1.12), we introduce the functional space

$$C_0^{\{k\}}([0,T];X) := M * C([0,T];X) = \{M * v \,:\, v \in C([0,T];X)\}.$$

It is a Banach space with the norm

$$\|u\|_{C_0^{\{k\}}([0,T];X)} = \|{}^R D_0^{\{k\}} u\|_{C([0,T];X)}.$$

Since $M* \in \mathscr{L}(C([0,T];X), C_0([0,T];X))$, the continuous embedding holds

$$C_0^{\{k\}}([0,T];X) \hookrightarrow C_0([0,T];X).$$

We also define

$$\begin{aligned}
&C^{\{k\}}([0,T];X) := C_0^{\{k\}}([0,T];X) + X, && (1.13)\\
&\|u\|_{C^{\{k\}}([0,T];X)} = \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X, &&\\
&C_0^{\{k\},\alpha}([0,T];X) = M * C_0^{\alpha}([0,T];X), && (1.14)\\
&\|u\|_{C_0^{\{k\},\alpha}([0,T];X)} = \|{}^R D_0^{\{k\}} u\|_{C_0^{\alpha}([0,T];X)}, &&\\
&C^{\{k\},\alpha}([0,T];X) = M * C^{\alpha}([0,T];X) + X, && (1.15)\\
&\|u\|_{C^{\{k\},\alpha}([0,T];X)} = \|{}^R D_0^{\{k\}}(u - u(0))\|_{C^{\alpha}([0,T];X)} + \|u(0)\|_X. &&
\end{aligned}$$

Let us establish some connections between the space $C^{\{k\}}$, $C^{\{k\},\alpha}$ and the usual C, C^1- and Hölder spaces. For $C^{\{k\}}([0,T];X)$ the continuous embeddings

$$C^1([0,T];X) \hookrightarrow C^{\{k\}}([0,T];X) \hookrightarrow C([0,T];X) \qquad (1.16)$$

are valid. The right embedding follows from $M* \in \mathscr{L}(C([0,T];X))$. To prove the left embedding, we choose some $u \in C^1([0,T];X)$. Then

$$\begin{aligned}
\|u\|_{C^{\{k\}}([0,T];X)} &= \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X = \|{}^R D_0^{\{k\}}(u - u(0))\|_{C([0,T];X)}\\
&+ \|u(0)\|_X = \|k * u'\|_{C([0,T];X)} + \|u(0)\|_X
\end{aligned}$$

and since $k* \in \mathscr{L}(C([0,T];X))$, the left relation in (1.16) follows.

Analogous relations for the space $C_0^{\{k\},\alpha}([0,T];X)$ are

$$C_0^{1+\alpha}([0,T];X) \hookrightarrow C_0^{\{k\},\alpha}([0,T];X) \hookrightarrow C_0^{\alpha}([0,T];X). \qquad (1.17)$$

The right embedding in (1.17) is a consequence of the fact that $M* \in \mathscr{L}(C_0^{\alpha}([0,T];X))$ (see Lemma 4.2 in [35]) and the left embedding in (1.17) can be proved similarly to the left embedding in (1.16).

The embeddings (1.16) and (1.17) are strict. Let us show it for the right embedding in (1.17). For arbitrary $v \in C_0^{\{k\},\alpha}([0,T];X) = M * C_0^{\alpha}([0,T];X)$ we have

$$\|v(t)\| \le \int_0^t M(\tau)d\tau\, O(t^{\alpha}) = o(t^{\alpha}) \quad \text{as} \quad t \to 0^+.$$

Thus, $t^{\alpha}x \in C_0^{\alpha}([0,T];X) \setminus C_0^{\{k\},\alpha}([0,T];X)$, $x \in X$, $x \neq 0$. The strictness of other mentioned embeddings can be shown in a similar manner.

Under additional assumptions on M it is possible to show that the operator $M*$ increases the order of Hölder continuity of a function.

Lemma 1.3. *Let* $M(t) = ct^{\beta-1}$, $c > 0, 0 < \beta < \alpha < 1$. *Then* $M * C_0^{\alpha-\beta}([0.T];X) = C_0^{\alpha}([0,T];X)$.

Lemma 1.4. *If* $|M(t)| \leq C_1 t^{\beta-1}$, $|M'(t)| \leq C_2 t^{\beta-2}$, $t \in (0,T)$ *for some* $C_1, C_2 \in \mathbb{R}_+$, $0 < \beta \leq \alpha < 1$ *then* $M* \in \mathscr{L}(C_0^{\alpha-\beta}([0,T];X), C_0^{\alpha}([0,T];X))$.

The proof of Lemma 1.3 can be found in [36] and the proof of Lemma 1.4 is in the Appendix of Publication II.
Under conditions of Lemma 1.4, $C_0^{\{k\},\alpha-\beta}([0,T];X) \hookrightarrow C_0^{\alpha}([0,T];X)$. Under conditions of Lemma 1.3 we have that $k(t) = \frac{\Gamma(\beta)}{c\Gamma(1-\beta)} t^{-\beta}$ and we obtain an equality $C_0^{\{k\},\alpha-\beta}([0,T];X) = C_0^{\alpha}([0,T];X)$.

1.5 Mittag-Leffler functions and their main properties

An important tool in the analysis of fractional differential equations is the family of Mittag-Leffler functions

$$E_\alpha(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \quad E_{\alpha,\gamma}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n+\gamma)}, \quad z \in \mathbb{C}. \tag{1.18}$$

The function $E_{\alpha,\gamma}$ is entire in case $\alpha > 0$, $\gamma > 0$ [20]. The formulas (1.18) immediately imply $E_{\alpha,1} = E_\alpha$ and

$$E_\alpha(0) = 1\,, \quad E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}\,, \quad E'_\alpha = \frac{1}{\alpha} E_{\alpha,\alpha}. \tag{1.19}$$

Let us point out some useful properties of $E_\beta(-z)$ and $E_{\beta,\beta}(-z)$ in case $\beta \in (0,1)$. The restrictions of functions $E_\beta(-z)$ and $E_{\beta,\beta}(-z)$ to the interval $(0,\infty)$ are completely monotonic and satisfy the asymptotic relations (see [20])

$$zE_\beta(-z) = \frac{1}{\Gamma(1-\beta)} + O(z^{-1}) \quad \text{as} \quad z \to \infty, \tag{1.20}$$

$$z^2 E_{\beta,\beta}(-z) = -\frac{1}{\Gamma(-\beta)} + O(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{1.21}$$

Physical background

1.6 Derivation of subdiffusion equation

CTRW was first introduced in [68] to describe the carriage of the charge in the amorphous semiconductors. Later on CTRW has become a popular framework to describe anomalous and Brownian diffusion in complex systems. It describes the diffusion in porous media, including gels and geological formations. The experiments have justified the choice of waiting time pdf proportional to $t^{-1-\alpha}$ in anomalous transport applications. The parameter α is constant if the medium is self-similar.

This particular choice of the waiting time pdf leads us to Riemann–Liouville and Caputo fractional derivatives. In the CTRW model the pdf $\Psi(x,t)$ is often decoupled as $\Psi(x,t) = \psi(t)\varphi(x)$, where $\varphi(x)$ is the jump length and $\psi(t)$ is the waiting time pdf. Different choices of the pdf yield different types of CTRW.

We would like to present three different ways to derive the subdiffusion equation that can be found in the literature:

1. Firstly, we consider the approach that was initially presented by Scher and Lax [83], it is also considered in [18]. This is a general approach that allows to obtain the CTRW model, when it is not possible to derive it from the RW analogue straightforwardly.

We begin by deriving the equation that determines the pdf $p(x,t|x_0,0)$ that the walker is situated at x at time t via the probability density $\eta(x,t|x_0,0)$ the walker arrives at point x at time t, given that he started at point x_0 and time $t_0 = 0$. Let us firstly write the conditional $n-$step probability density $\eta_n(x,t|x_0,0)$ that the walker arrives at point x at time t within n steps:

$$\eta_n(x,t|x_0,0) = \sum_{x'} \int_0^t \Psi(x-x',t-t')\eta_{n-1}(x',t'|x_0,0)dt'.$$

Then the conditional probability density of arriving at x at time t irrespective of number of steps is

$$\eta(x,t|x_0,0) = \sum_{n=0}^{\infty} \eta_n(x,t|x_0,0) = \delta(x-x_0)\delta(t) + \sum_{x'} \int_0^t \Psi(x-x',t-t')$$
$$\times \sum_{n=1}^{\infty} \eta_n(x',t'|x_0,0)dt' = \delta(x-x_0)\delta(t) + \sum_{x'} \int_0^t \Psi(x-x',t-t')\eta(x',t'|x_0,0)dt', \quad (1.22)$$

where $\delta(x-x_0)\delta(t)$ is the initial condition, that is the conditional probability of being at point x at time t if the particle did not perform any step.

Let us denote by $\Phi(t) = 1 - \int_0^t \psi(t')dt'$ that is a probability of a particle not taking a step during the period $[0,t]$. Thus, the probability density $p(x,t|x_0,0)$ is:

$$p(x,t|x_0,0) = \int_0^t \eta(x,t'|x_0,0)\Phi(t-t')dt'.$$

Thus, after convolving (1.22) with Φ we obtain the Generalized Master Equation (GME) for the probability $p(x,t|x_0,0)$:

$$p(x,t|x_0,0) = \delta(x-x_0)\Phi(t) + \int_0^t \sum_{x'} p(x',t'|x_0,0)\Psi(x-x',t-t')dt'. \quad (1.23)$$

Next after applying the Fourier and Laplace transforms to the GME, some algebra to the obtained equation and inverting the transforms we obtain that

$$\frac{\partial p(x,t|x_0,0)}{\partial t} = \frac{\partial}{\partial t} \int_0^t M(t-t') \left[-p(x,t'|x_0,0) + \sum_{x'} \varphi(x-x')p(x',t'|x_0,0) \right] dt'.$$

Time-dependent kernel in the equation points out the non-Markovian nature of the process. Here the kernel M is such that $\widehat{M}(s) = \frac{\widehat{\psi}(s)}{1-\widehat{\psi}(s)}$.

We switch to the continuous description of the evolution of $p(x,t)$, since it allows to solve the problems with different types of boundaries and sources. The transition is done by letting the jump rate to infinity and the spacing of the underlying lattice to zero. With the Gaussian jump-length pdf $\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$ [18], p. 46 we obtain

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial t} \int_0^t M(t-t')\Delta p(x,t')dt',$$

or equivalently

$$p_t = {}^R D_0^{\{M\}} \Delta p.$$

Fixation of a power-law waiting-time pdf $\psi(t) \propto t^{-1-\alpha}$ with $0 < \alpha < 1$ this leads us to the time-fractional diffusion equation [9, 67]:

$$\frac{\partial p(x,t)}{\partial t} = K_\alpha D_0^{1-\alpha} \Delta p(x,t), \text{ where } K_\alpha \text{ is a constant coefficient.}$$

Usual RW model can be obtained as the limit case of such a CTRW, then the waiting-time pdf becomes Poisson and the jump-length pdf remains Gaussian.

It is possible to reformulate (1.23) in terms of particle concentration C instead of the probabilities p by using the formula $p(x,t|x_0,0) = \frac{C(x,t|x_0,0)}{C(x_0,0|x_0,0)}$ and multiplying (1.23) by $C(x_0,0|x_0,0)$:

$$C(x,t|x_0,0) = C(x_0,0|x_0,0)\Phi(t) + \int_0^t \sum_{x'} C(x',t'|x_0,0)\Psi(x-x',t-t')dt'.$$

2. Next approach to derive a subdiffusion equation works well to describe the particle flow in some chemical reactions [13, 18]. For this purpose we consider the mass balance at the lattice site i:

$$\frac{\partial C_i(t)}{\partial t} = j_i^+(t) - j_i^-(t) + f_i(t)$$

where C_i is the number of particles at the site i, j_i^+ is the gain flux , j_i^- is the loss flux and f_i is the source term, that provides the number of particles that enter the site at time $t > 0$. Particles depart equally to the left and right. Therefore,

$$j_i^+(t) = \frac{1}{2} j_{i-1}^-(t) + \frac{1}{2} j_{i+1}^-(t)$$

and we get

$$\frac{\partial C_i(t)}{\partial t} = \frac{1}{2} j_{i-1}^-(t) + \frac{1}{2} j_{i+1}^-(t) - j_i^-(t) + f_i(t). \tag{1.24}$$

Particles located at the site at initial time or arriving there at later times "wait" before leaving. This is be expressed by

$$j_i^-(t) = \psi_i(t)C_i(0) + \int_0^t \psi(t-t')(j_i^+(t') + f_i(t'))dt',$$

where ψ is the waiting time density (here $j_i^+(t') + f_i(t')$ is the total gain from the flux and the source at time t'). Therefore

$$j_i^-(t) = \psi_i(t)C_i(0) + \int_0^t \psi(t-t') \left(\frac{\partial C_i(t')}{\partial t'} + j_i^-(t') \right) dt'.$$

This is a Volterra equation of 2nd kind for j_i^-. Solving it we have

$$j_i^-(t) = \frac{\partial}{\partial t}\int_0^t M(t-t')C_i(t')dt'$$

where the Laplace transforms of ψ and M are related by the formula $\widehat{M}(s) = \frac{\widehat{\psi}(s)}{1-\widehat{\psi}(s)}$. Plugging this into (1.24) we have

$$\frac{\partial C_i(t)}{\partial t} = \frac{\partial}{\partial t}\int_0^t M(t-t')[\frac{1}{2}C_{i-1}(t') + \frac{1}{2}C_{i+1}(t') - C_{i-1}(t')]dt' + f_i(t).$$

Taking the continuous limit we obtain

$$\frac{\partial C(x,t)}{\partial t} = \varkappa\, {}^R D_0^{\{M\}} \Delta C(x,t) + f(x,t) \tag{1.25}$$

where $\varkappa > 0$ is some constant.

In order to obtain the reaction-subdiffusion equation the elliptic operator is complemented by the reaction term under the fractional derivative, in other words we replace Lu by $Lu+R$, where R is the reaction term [18, 26, 50, 52].

For example, L takes the form $Lu = \varkappa\Delta u + ru$ in case of a linear reaction, $R = ru$, where r is a reaction rate independent of u [26].

3. Finally, we consider a convenient way to derive a subdiffusion equation for a heat flow presented by Povstenko [72], p.300. Firstly, time-nonlocal constitutive equation for the heat flux q is considered:

$$q(x,t) = -k\frac{\partial}{\partial t}\int_0^t a(t-\tau)\nabla T(x,\tau)d\tau, \tag{1.26}$$

where T is the temperature, k is a thermal conductivity of a solid and a is a thermal diffusivity coefficient. After combining it with the conservation law

$$c\frac{\partial T}{\partial t} + \operatorname{div} q = Q, \tag{1.27}$$

here Q is source function and c is a constant, we obtain heat conduction equation

$$c\frac{\partial T}{\partial t}(x,t) = k\,{}^R D_0^{\{a\}} \Delta T(x,t) + Q(x,t).$$

In case of "long-tale" power time-nonlocal kernel in a constitutive equation it transforms into

$$q(x,t) = -\frac{k}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}\nabla T(x,\tau)d\tau$$

or in other words

$$q(x,t) = -k\,{}^R D_0^{1-\alpha}\nabla T(x,t), \quad 0 < \alpha < 1,$$

that in combination with (1.27) yields time-fractional heat conduction equation

$$c\frac{\partial T}{\partial t} = k\,{}^R D_0^{1-\alpha}\Delta T + Q.$$

The choice of power law waiting time pdfs in CTRW or the power-type kernel in a subdiffusion equation is not the only reasonable possibility. In fact, the other choice of the memory kernel can work better to describe certain subdiffusion processes. Therefore, in this work we prefer to use a generalized fractional derivative that opens new opportunities for the applications of CTRW model.

1.7 Examples of kernels M and k

In this thesis we are solving problems with a generalized fractional derivative. This concept has been used in [35, 48, 63].

We use the Sonine kernels as the kernels for the generalized fractional derivative since they allow to switch between the equations of Riemann-Liouville (1.3) and Caputo (1.4) type. We separate the description of the kernels M and k, because play a different role in the models (1.3) and (1.4) respectively.

We provide the examples of kernels with their Laplace transforms that are used further.

(M1) $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ is the basic case. It has the Laplace transform $\widehat{M}(s) = \frac{1}{s^\beta}$.

(k1) $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$ with the Laplace transform $\widehat{k}(s) = \frac{1}{s^{1-\beta}}$.

It was shown in Section 1.3 that (M1) and (k1) are Sonine kernels and they are associate to each other, i.e. $M * k = 1$. They are used in the celebrated time fractional diffusion equation (1.1) or (1.2) [13, 45, 54, 57, 69], where ${}^R D_0^{\{k\}}$ and ${}^C D_0^{\{k\}}$ become Riemann-Liouville and Caputo fractional derivatives of order β.

Often a memory is not of power-type. A direct generalization of (M1) and (k1) leads to **multiterm and distributed order fractional derivatives** [49, 66, 87]. The equations (1.3) and (1.4) with multiterm derivatives contain the following kernels:

(M2) $M(t) = \sum_{j=1}^N q_j \frac{t^{\beta_j-1}}{\Gamma(\beta_j)}$, $0 < \beta_j < \beta_{j+1} < 1$, $q_j > 0$ with $\widehat{M}(s) = \sum_{j=1}^N q_j \frac{1}{s^{\beta_j}}$, corresponding to the retarded diffusion [14, 66, 87];

(k2) $k(t) = \sum_{j=1}^N q_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $0 < \beta_{j+1} < \beta_j < 1$, $q_j > 0$ with $\widehat{k}(s) = \sum_{j=1}^N q_j \frac{1}{s^{1-\beta_j}}$, corresponding to the accelerated diffusion [14, 66, 87].

The equations (1.3), (1.4) with the distributed order fractional derivative contain respectively the kernels (M3) and (k3) described below.

(M3) $M(t) = \int_0^1 q(\beta) \frac{t^{\beta-1}}{\Gamma(\beta)} d\beta$ where $q \in L_1(0,1)$, $q \geq 0$ is nonvanishing (cf. [12, 66, 87]). Then $\widehat{M}(s) = \int_0^1 q(\beta) \frac{1}{s^\beta} d\beta$.This type of kernel stands for the distributed order fractional derivative that is used in physical literature for modeling diffusion with a logarithmic growth of the mean square displacement [48].

(k3) $k(t) = \int_0^1 q(\beta) \frac{t^{-\beta}}{\Gamma(1-\beta)} d\beta$, where $q \in L_1(0,1), q \geq 0$ is nonvanishing. Then $\widehat{k}(s) = \int_0^1 q(\beta) \frac{1}{s^{1-\beta}} d\beta$. A proper choice of q in (k3) allows modelling ultraslow diffusion [66].

The kernels (M2), (k2), (M3), (k3) are Sonine kernels since they satisfy the conditions of Lemma 1.1. We would like to point out, however, that the kernels (M2) and (k2), (M3) and (k3) are not associate to each other. Thus, in case of multiterm and distributed order derivatives the equations (1.3) and (1.4) represent different models.

Actually, the cases (M2) and (M3), (k2) and (k3) can be unified to a form of Lebesgue-Stiltjes integral as $M(t) = \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)} d\mu(\beta)$, $k(t) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)} d\mu(\beta)$.

Tempered fractional derivatives are used to describe slow transition of anomalous diffusion to a normal one. There are three models of this type in the literature that differ in their mathematical derivations.

The kernels corresponding to the tempered fractional diffusion and their associate kernels are described below.

(M4) $M(t)=\frac{1}{\Gamma(\beta)}e^{-\lambda t}t^{\beta-1}+\frac{\lambda}{\Gamma(\beta)}\int_0^t e^{-\lambda\tau}\tau^{\beta-1}d\tau$, $\lambda>0, 0<\beta<1$, [19, 84, 95]. Then $\widehat{M}(s)=\frac{(s+\lambda)^{1-\beta}}{s}$.
The relation (1.11) implies $\widehat{k}(s)=\frac{1}{(s+\lambda)^{1-\beta}}$. By taking the inverse Laplace transform we obtain

(k4) $k(t)=\frac{1}{\Gamma(1-\beta)}e^{-\lambda t}t^{-\beta}$ which is associate to (M4).

(M5) $M(t)=\frac{1}{\Gamma(\beta)}e^{-\lambda t}t^{\beta-1}, 0<\beta<1, \lambda>0$, [82] with the Laplace transform $\widehat{M}(s)=\frac{1}{(s+\lambda)^{\beta}}$.
Again, from the relation (1.11) we calculate $\widehat{k}(s)=\frac{(s+\lambda)^{\beta}}{s}$ and after taking the inverse Laplace transform obtain the associate kernel to (M5)

(k5) $k(t)=\frac{1}{\Gamma(1-\beta)}e^{-\lambda t}t^{-\beta}+\frac{\lambda}{\Gamma(1-\beta)}\int_0^t e^{-\lambda\tau}\tau^{-\beta}d\tau.$

(M6) $M(t)=e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}(\lambda^{\beta}t^{\beta})$, $0<\beta<1$, $\lambda>0$, [19, 95]. According to [95] $\widehat{M}(s)=\frac{1}{(s+\lambda)^{\beta}-\lambda^{\beta}}$ and we get $\widehat{k}(s)=\frac{(s+\lambda)^{\beta}-\lambda^{\beta}}{s}$. Then the inverse Laplace transform implies the associate of (M6)

(k6) $k(t)=\frac{1}{\Gamma(1-\beta)}e^{-\lambda t}t^{-\beta}+\frac{\lambda}{\Gamma(1-\beta)}\int_0^t e^{-\lambda\tau}\tau^{-\beta}d\tau-\lambda^{\beta}.$

The models with the kernels (M4), (k4) and (M5), (k5) look similar, but we describe them separately, since they represent different physical models.

Models with generalized fractional derivatives that contain bounded kernels highlight memory effects better [4]. In this thesis we consider the following bounded kernels:

(k7) $k(t)=\frac{1}{1-\beta}e^{-\frac{\beta}{1-\beta}t}$, $0<\beta<1$ is the kernel of Caputo-Fabrizio derivative [4, 11] and has a Laplace transform $\widehat{k}(s)=\frac{1}{(1-\beta)s+\beta}$;

(k8) $k(t)=\frac{1}{1-\beta}E_{\beta}\left(-\frac{\beta t^{\beta}}{1-\beta}\right), 0<\beta<1$ is a kernel of Atangana-Baleanu fractional derivative [3, 23]. It follows from [20] that $\widehat{k}(s)=\frac{s^{\beta-1}}{(1-\beta)s^{\beta}+\beta}$.

Since the kernels (k7), (k8) are bounded, they are not Sonine kernels.

2 Inverse problems for a generalized time fractional diffusion equation in C- and Hölder spaces

This Chapter contains results of Publication II with some modifications and additions. Throughout the Chapter we work in the complex-valued scalar functional spaces by default, we additionally specify if the space is real-valued.

2.1 Formulation of direct and inverse problems

Let us consider a subdiffusion process that is supplemented by linear reaction and is going on in an open bounded domain $\Omega \in \mathbb{R}^n$ with the boundary $\partial\Omega$. We will denote a state function satisfying nonhomogeneous boundary conditions by U. The lowercase letter u will stand for the translated state function that satisfies the corresponding homogeneous boundary conditions.

The process is governed by the generalized subdiffusion equation

$$U_t(t,x) = {}^R D_0^{\{M\}} LU(t,x) + Q(t,x), \quad x \in \Omega, t \in (0,T), \tag{2.1}$$

where Q is a source term, $L = \varkappa\Delta + rI$ and $r = r(x)$ is a reaction coefficient.

For the sake of mathematical generality, we replace $L = \varkappa\Delta + rI$ by the more general operator $L = L(x)$ defined by

$$L(x) = L_1(x) + r(x)I, \quad L_1(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}, \tag{2.2}$$

where a_{ij}, a_j are given coefficients.

We assume that the kernel M is Sonine and its associate is k. Thus, by applying the operator $k*$ to (2.1) we obtain the equivalent equation in Caputo form:

$${}^C D_0^{\{k\}} U(t,x) = LU(t,x) + H(t,x), \quad x \in \Omega, t \in (0,T), \tag{2.3}$$

where $H = k * Q$. Let us transform the Caputo derivative ${}^C D_0^{\{k\}} U(t,x)$ as follows:

$$\begin{aligned} &{}^C D_0^{\{k\}} U(t,x) = \int_0^t k(t-\tau) U_\tau(\tau,x) dx = \frac{\partial}{\partial t}\int_0^t k(t-\tau)(U(\tau,x) - U(0,x)) d\tau \\ &= {}^R D_0^{\{k\}} (U(t,x) - U(0,x)). \end{aligned}$$

Since the term ${}^R D_0^{\{k\}} (U(t,x) - U(0,x))$ does not contain the first order derivative of U, for the sake of generality we use it instead of ${}^C D_0^{\{k\}} U(t,x)$.

Now we formulate a direct problem for the function U:

$$\begin{aligned} &{}^R D_0^{\{k\}} (U - \Phi)(t,x) = LU(t,x) + H(t,x), \quad x \in \Omega, t \in (0,T), \\ &U(0,x) = \Phi(x), \quad x \in \Omega, \\ &\mathscr{B}(U - b)(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T). \end{aligned} \tag{2.4}$$

Here Φ and b are given functions and $\mathscr{B}$ is a boundary operator such that

$$\mathscr{B}v(x) = v(x) \quad \text{or} \quad \mathscr{B}v(x) = \omega(x) \cdot \nabla v(x),$$

where ω is a vector function such that $\omega(x) \cdot \vartheta(x) > 0$, $\vartheta(x) = (\vartheta_1(x), ..., \vartheta_n(x))$ denoting the outer normal of $\partial\Omega$ at $x \in \Omega$.

Let us proceed to inverse problems. To this end we introduce the condition

$$U(T,x) = \Psi(x), \quad x \in \Omega, \tag{2.5}$$

with a given observation function Ψ.

Firstly, we formulate of an inverse source problem. Let

$$H(t,x) = g(t,x)f(x) + h_0(t,x) \tag{2.6}$$

where the components gf and h_0 may correspond to different sources or sinks. The factor f is unknown and to be reconstructed by means of the data (2.5).

IP1nh. Determine a pair of functions (f,U) that satisfies (2.4), (2.5) and (2.6).

Next we aim to identify the reaction coefficient $r(x)$.

IP2nh. Find a pair (r,U) that satisfies (2.4) and (2.5).

We can handle the case of zero initial condition $\Phi = 0$ in IP2nh.

Methods to be used in this Chapter require homogeneous boundary conditions. Therefore, let us perform the change of variable $u = U - b$ in the formulated problems. The direct the problem (2.4) is transformed to the form

$$\begin{aligned}
&{}^{R}D_0^{\{k\}}(u-\varphi)(t,x) = Lu(t,x) + F(t,x), \quad x \in \Omega, t \in (0,T),\\
&u(0,x) = \varphi(x), \quad x \in \Omega,\\
&\mathscr{B}u(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T),
\end{aligned} \tag{2.7}$$

where

$$\varphi(x) = \Phi(x) - b(0,x), \tag{2.8}$$

$$F(t,x) = Lb(t,x) - {}^{R}D_0^{\{k\}}(b - b(0,\cdot))(t,x) + H(t,x). \tag{2.9}$$

The overdetermination condition is changed in the following way:

$$u(T,x) = \psi(x), \quad x \in \Omega, \tag{2.10}$$

where

$$\psi(x) = \Psi(x) - b(T,x). \tag{2.11}$$

Plugging (2.6) into (2.9) we obtain

$$F(t,x) = g(t,x)f(x) + h(t,x), \tag{2.12}$$

where $h(t,x) = h_0(t,x) + Lb(t,x) - {}^{R}D_0^{\{k\}}(b - b(0,\cdot))(t,x)$.

The reformulated first inverse problem is

IP1. Find the pair of functions (f,u) that satisfies (2.7), (2.10) and (2.12).

Let us reformulate the second inverse problem, too. From the relations (2.4), (2.5) with $\Phi = 0$ by means of the change of variable $u = U - b$, we obtain the following problem for the pair (r,u):

$$\begin{aligned}
&{}^{R}D_0^{\{k\}}u(t,x) = L_1u(t,x) + r(x)(u+b)(t,x) + F_1(t,x) \quad x \in \Omega, t \in (0,T),\\
&u(0,x) = 0, \quad x \in \Omega, \qquad \mathscr{B}u(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T),\\
&u(T,x) = \psi(x), \quad x \in \Omega,
\end{aligned} \tag{2.13}$$

where we assume that $b(0,x)=0$, $x\in\Omega$, the function ψ is expressed by $\psi(x)=\Psi(x)-b(T,x)$ and $F_1(t,x)=H(t,x)+L_1b(t,x)-{}^RD_0^{\{k\}}b(t,x)$.

Thus, the reformulated second inverse problem is

IP2. Find the pair of functions (r,u) that satisfies (2.13).

If b is sufficiently regular then the problems IP1nh and IP2nh are equivalent to IP1 and IP2 respectively. The problem IP1 has a solution in some Banach space $\mathscr{F}\times\mathscr{U}$ if and only if IP1nh has a solution in $\mathscr{F}\times(\mathscr{U}+b)$. Similarly, IP2 has a solution in a space $\mathscr{R}\times\mathscr{U}$ if and only if IP2nh has a solution in $\mathscr{R}\times(\mathscr{U}+b)$. Thus, we will focus on the problems IP1 and IP2 in this chapter.

2.2 Basic assumptions

In this section we collect basic conditions on the domain, operator L and kernels k and M that will be assumed throughout the Chapter.

We assume that $\partial\Omega$ is uniformly of class C^2. Moreover, we assume that $a_{ij},a_j,r\in C(\overline{\Omega};\mathbb{R})$ and the principal part of L is uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2 \quad \forall\xi\in\mathbb{R}^n, \quad x\in\Omega, \text{ for some the } c>0.$$

In addition, we assume that the vector function $\omega\in(C^1(\partial\Omega;\mathbb{R}))^n$.

Concerning the function k, we assume that

1. k is Sonine kernel with associate M such that

$$\begin{gathered} M\in C^1((0,\infty);\mathbb{R}), \quad \lim_{t\to0^+}M(t)=\infty, \quad M>0, \quad M'\le0, \\ -M' \text{ is nonincreasing and convex;} \end{gathered} \tag{2.14}$$

2. k has the following properties:

$$\begin{gathered} k\in C((0,\infty);\mathbb{R}), \ \lim_{t\to0^+}k(t)=\infty, \ k>0, \ k \text{ is nonincreasing and} \\ \exists t_k>0 : k(t) \text{ is strictly decreasing in } (0,t_k). \end{gathered} \tag{2.15}$$

The assumptions (2.14) ensure the existence of a sufficiently regular solution of the direct problem (see Lemma 2.2 and its proof) and the assumptions (2.15) are needed for the application of a positivity principle to this solution.

We mention that it is possible to reduce all assumptions concerning the pair (k,M) to the assumptions on M simply. The assumption $M\in L_{1,loc}((0,T);\mathbb{R})$ and (2.14) imply that M is Sonine kernel by Lemma 1.1. All properties (2.15) follow from conditions that are a bit more restrictive than (2.14). It is shown in the following lemma. Proof is in Appendix of Publication II.

Lemma 2.1. *Let $M\in L_{1,loc}((0,\infty);\mathbb{R})$ satisfy* (2.14) *and $M'<0$, $\log M$ - convex, $\log(-M')$ - convex. Then the solution of $M*k=1$ satisfies* (2.15).

The assumptions (2.14) and (2.15) imposed on M and k hold for weakly singular completely monotonic kernels from $\mathscr{CM}$ introduced in Section 1.3.

The Lemma 1.2 implies that $M\in\mathscr{CM}$ if and only if $k\in\mathscr{CM}$.

The kernels M and k described in Section 1.7 are of the class $\mathscr{CM}$, except for (k7) and (k8). Therefore, they satisfy conditions (2.14) and (2.15). By computing the derivatives it is easy to check that for the kernels (M1), (k1), (M2), (k2), (M3), (k3). Let us consider the other kernels from this Section.

In case (M4) we see that $M(t) > 0$ and $M'(t) = \frac{1}{\Gamma(\beta)}(\beta-1)t^{\beta-2}e^{-\lambda t} < 0$. By continuing the differentiation we obtain $(-1)^i M^{(i)}(t) > 0, \quad i = 0,1,\dots$.

In case (M5), similarly, we obtain $M(t) > 0, M'(t) = \frac{1}{\Gamma(\beta)}(-\lambda t^{\beta-1} + (\beta-1)t^{\beta-2})e^{-\lambda t} < 0$ and $(-1)^i M^{(i)}(t) > 0, \quad i = 0,1,\dots$.

In case (M6) we investigate the corresponding associate kernel $k(t)$ given by the formula (k6). The derivative of k is $k'(t) = -\beta \frac{e^{-\lambda t}t^{-\beta-1}}{\Gamma(1-\beta)}$. Immediately, $(-1)^i k^{(i)} > 0$, $i = 1,2,\dots$. To show that $k > 0$ let us compute the limit:

$$\lim_{t\to\infty} k(t) = \lambda \lim_{t\to\infty} \int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)} d\tau - \lambda^\beta = \lambda^\beta \lim_{t\to\infty} \int_0^{\lambda t} \frac{e^{-\sigma}\sigma^{-\beta}}{\Gamma(1-\beta)} d\sigma - \lambda^\beta$$
$$= \lambda^\beta \int_0^\infty \frac{e^{-\sigma}\sigma^{-\beta}}{\Gamma(1-\beta)} d\sigma - \lambda^\beta = \lambda^\beta \frac{\Gamma(1-\beta)}{\Gamma(1-\beta)} - \lambda^\beta = 0.$$

Since k is strictly decreasing, we obtain $k > 0$. Thus, $k \in \mathscr{CM}$ and $M \in \mathscr{CM}$.

2.3 Abstract Cauchy problem

Let $A : \mathscr{D}(A) \to X$ be a linear densely defined operator in a complex Banach space X. We say that A belongs to the class $\mathscr{S}(\eta,\theta)$ for $\eta \in \mathbb{R}$, $\theta \in (0,\pi)$ if

$$\rho(A) \supset \Sigma(\eta,\theta) = \{\lambda \in \mathbb{C} : \lambda \neq \eta, arg|\lambda-\eta| < \theta\} \quad \text{and}$$
$$\|(\mu - A)^{-1}\|_{\mathscr{L}(X)} \leq \tfrac{C}{|\mu-\eta|} \quad \forall \mu \in \Sigma(\eta,\theta) \quad \text{for some constant } C > 0.$$

An operator $A \in \mathscr{S}(\eta,\theta)$ is closed. This implies that $X_A := \mathscr{D}(A)$ is a Banach space with the graph norm
$$\|w\|_{X_A} = \|w\|_X + \|Aw\|_X.$$

Now let us consider the Cauchy problem

$$^R D_0^{\{k\}}(u-\varphi)(t) = Au(t) + F(t), \quad t \in [0,T], \quad u(0) = \varphi, \tag{2.16}$$

with given $F : [0,T] \to X$ and $\varphi \in X$.

Lemma 2.2. *Let $A \in \mathscr{S}(\eta, \frac{\pi}{2})$ for some $\eta \in \mathbb{R}$. Then the following statements are valid.*
(i) (uniqueness) *Let $u \in C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$ solve (2.16) and $\varphi = 0$, $F = 0$. Then $u = 0$.*
(ii) *Let $F \in C_0^\alpha([0,T];X)$, $0 < \alpha < 1$ and $\varphi = 0$. Then (2.16) has a solution u in the space $C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^\alpha([0,T];X_A)$. This solution satisfies the estimate*

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^\alpha([0,T];X_A)} \leq C_3 \|F\|_{C_0^\alpha([0,T];X)}. \tag{2.17}$$

(iii) *Let $F \in C^\alpha([0,T];X)$, $0 < \alpha < 1$ and $\varphi \in X_A$. Then (2.16) has a solution u in the space $C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$. This solution satisfies the estimate*

$$\|u\|_{C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)} \leq C_4 (\|F\|_{C^\alpha([0,T];X)} + \|\varphi\|_{X_A}). \tag{2.18}$$

The constants C_3 and C_4 depend on M, A and α.

Proof. The change of variable $v = {}^R D_0^{\{k\}}(u-\varphi) \;\Leftrightarrow\; u = M*v+\varphi$ reduces (2.16) of the integral equation

$$v(t) = A(M*v)(t)+F(t)+A\varphi, \quad t\in[0,T]. \tag{2.19}$$

Provided $F\in C([0,T];X)$, $\varphi\in X_A$, the function $u\in C^{\{k\}}([0,T];X)\bigcap C([0,T];X_A)$ solves (2.16) if and only if

$$v\in V := \{v\in C([0,T];X) \,:\, M*v\in C_0([0,T];X_A)\} \text{ solves (2.19).}$$

In the particular case $F\in C_0^\alpha([0,T];X)$, $\varphi=0$ similar one-to-one correspondence holds for $u\in C_0^{\{k\},\alpha}([0,T];X)\bigcap C_0^\alpha([0,T];X_A)$ and

$$v\in V^\alpha := \{v\in C_0^\alpha([0,T];X) \,:\, M*v\in C_0^\alpha([0,T];X_A)\}.$$

Since M satisfies the conditions (2.14) and $A\in\mathscr{S}(\eta,\frac{\pi}{2})$, we can apply results of Ch. 3 of [74] to (2.19). Namely, it follows from [74] that there exists a family of operators $S:[0,\infty)\to\mathscr{L}(X)$ (called resolvent of (2.19)) so that a solution $v\in V$ (if it exists) is represented by the formula

$$v = \frac{d}{dt}S*(F+A\varphi)$$

(see Theorem 3.2, Corollary 1.1 and Proposition 1.2 in [74]).

(i) Since there exists a solution u to (2.16), the equation (2.19) has a solution $v\in V$. Due to the assumptions $F=0$, $\varphi=0$ the representation formula implies $v=0$. Thus, $u=0$.

(ii) Theorem 3.3 (i) [74] implies that for $F\in C_0^\alpha([0,T];X)$ there exists a solution $v\in V^\alpha$ of (2.19). This proves the existence of the solution $u\in C_0^{\{k\},\alpha}([0,T];X)\bigcap C_0^\alpha([0,T];X_A)$ of (2.16). The estimate (2.17) follows from the bounded inverse theorem.

(iii) It is sufficient to prove this assertion in case $F(t)\equiv\xi\in X$, because the problem with given pair of data (F,φ) can be splitted into two problems with the data $(F-F(0),0)$ and $(F(0),\varphi)$, respectively. For the first problem, the assertion (ii) applies. Having proved (iii) for the second one, u is expressed as the sum of solutions of these two problems and satisfies (iii), too.

Thus, let us assume that $F(t)\equiv\xi\in X$. Due to Proposition 1.2 (ii) [74], (2.19) has the solution $v=\frac{d}{dt}S*\hat{F}=S(\xi+A\varphi)\in V$. This implies the existence assertion of (iii). Due to the strong continuity of $S(t)$ [74], $\|S(t)\|_{\mathscr{L}(X)}\le C_5$, $t\in[0,T]$, where C_5 is a constant. Thus,

$$\|v\|_{C([0,T],X)} \le C_5\,(\|\xi\|_X+\|A\varphi\|_X).$$

Then we continue as follows

$$\begin{aligned}&\|u\|_{C^{\{k\}}([0,T];X)\bigcap C([0,T];X_A)} = \|u\|_{C^{\{k\}}([0,T];X)}+\|u\|_{C([0,T];X_A)}\\ &\le \|u\|_{C^{\{k\}}([0,T];X)}+\|u\|_{C([0,T];X)}+\|Au\|_{C([0,T];X)}.\end{aligned}$$

Using the embedding $C^{\{k\}}([0,T];X)\hookrightarrow C([0,T];X)$ and the definition of the norm in $C^{\{k\}}([0,T];X)$ we have

$$\begin{aligned}&\|u\|_{C^{\{k\}}([0,T];X)\bigcap C([0,T];X_A)} \le C\|u\|_{C^{\{k\}}([0,T];X)}+\|Au\|_{C([0,T];X)}\\ &\le C\left(\|{}^R D_0^{\{k\}}(u-\varphi)\|_{C([0,T];X)}+\|\varphi\|_X\right)+\|Au\|_{C([0,T];X)}\end{aligned}$$

with some constant C. Since $v = {}^R D_0^{\{k\}}(u-\varphi)$, due to the equation for u (there $F = \xi$), it holds $Au = {}^R D_0^{\{k\}}(u-\varphi) - \xi = v - \xi$. Therefore,

$$\|u\|_{C^{\{k\}}([0,T];X)\bigcap C([0,T];X_A)} \le C\left(\|v\|_{C([0,T];X)} + \|\varphi\|_X\right) + \|v-\xi\|_{C([0,T];X)}.$$

Finally, using the triangle inequality and substituting the estimate of v we deduce

$$\|u\|_{C^{\{k\}}([0,T];X)\bigcap C([0,T];X_A)} \le C_6(\|\xi\|_X + \|\varphi\|_{X_A})$$

with another constant C_6. This implies (2.18). □

2.4 Statements on direct problem

In order to apply Lemma 2.2 to the direct problem (2.7), we introduce appropriate Banach spaces of x-dependent functions and define realizations of the operator L in these spaces so that they belong to $\mathscr{S}(\eta, \frac{\pi}{2})$:

1. $X_p = L_p(\Omega)$, $1 < p < \infty$,
   $A_p : X_{A_p} \to X_p$ with $X_{A_p} = \{z \in W_p^2(\Omega) : \mathscr{B}z|_{\partial\Omega} = 0\}$ and
   $A_p z = Lz, \ z \in X_{A_p}$.
2. $X_0 = \begin{cases} \{z \in C(\overline{\Omega}) : z|_{\partial\Omega} = 0\} \text{ in case } \mathscr{B} = I, \\ C(\overline{\Omega}) \text{ in case } \mathscr{B} = \omega \cdot \nabla, \end{cases}$
   $A_0 : X_{A_0} \to X_0$ with $X_{A_0} = \{z \in \bigcap\limits_{1<p<\infty} W_p^2(\Omega) : \mathscr{B}z|_{\partial\Omega} = 0, Lz \in X_0\}$ and
   $A_0 z = Lz, \ z \in X_{A_0}$.

Corollary 2.1. *Operators A_p, $p \in \{0\}\bigcup(1,\infty)$ and are from $\mathscr{S}(\eta, \frac{\pi}{2})$. Thus, Lemma 2.2 holds in cases $X = X_p$, $A = A_p$, $p \in \{0\}\bigcup(1,\infty)$ and applies to problem (2.7).*

Proof. The fact that $A_p \in \mathscr{S}(\eta, \frac{\pi}{2})$ in different cases of p and $\mathscr{B}$ follows from Theorems 3.1.2, 3.1.3 and Corollaries 3.1.21(ii) and 3.1.24 (ii) in [64]. □

Next let us focus on the real case. Let us define the spaces

$$X_{p,\mathbb{R}} = \{\mathrm{Re} z : z \in X_p\}, \quad p \in \{0\}\bigcup(1,\infty),$$
$$X_{A_p,\mathbb{R}} = \{\mathrm{Re} z : z \in X_{A_p}\}, \quad p \in \{0\}\bigcup(1,\infty).$$

The spaces $X_{p,\mathbb{R}}$ and $X_{A_p,\mathbb{R}}$ constitute real Banach spaces with the norms

$$\|y\|_{X_{p,\mathbb{R}}} = \|y + 0i\|_{X_p}, \ y \in X_{p,\mathbb{R}}, \quad \|y\|_{X_{A_p,\mathbb{R}}} = \|y + 0i\|_{X_{A_p}}, \ y \in X_{A_p,\mathbb{R}}. \tag{2.20}$$

The spaces $X_{p,\mathbb{R}}$ and $X_{A_p,\mathbb{R}}$ can also be identified as

$$X_{p,\mathbb{R}} = L_p(\Omega;\mathbb{R}), \ X_{A_p,\mathbb{R}} = \{y \in W_p^2(\Omega;\mathbb{R}) : \mathscr{B}y|_{\partial\Omega} = 0\}, \ 1 < p < \infty,$$
$$X_{0,\mathbb{R}} = \begin{cases} \{y \in C(\overline{\Omega};\mathbb{R}) : y|_{\partial\Omega} = 0\} \text{ in case } \mathscr{B} = I, \\ C(\overline{\Omega};\mathbb{R}) \text{ in case } \mathscr{B} = \omega \cdot \nabla, \end{cases}$$
$$X_{A_0,\mathbb{R}} = \{y \in \bigcap_{1<p<\infty} W_p^2(\Omega;\mathbb{R}) : \mathscr{B}y|_{\partial\Omega} = 0, Ly \in X_{0,\mathbb{R}}\}.$$

Lemma 2.3. *Let $p \in \{0\}\bigcup(1,\infty)$. Then the following statements are valid.*
(i) (uniqueness) *Let $u \in C^{\{k\}}([0,T];X_{p,\mathbb{R}})\bigcap C([0,T];X_{A_p,\mathbb{R}})$ solve (2.7) and $\varphi = 0$, $F = 0$.*

Then $u = 0$.
(ii) *Let* $F \in C_0^{\alpha}([0,T];X_{p,\mathbb{R}})$ *for some* $0 < \alpha < 1$ *and* $\varphi = 0$. *Then* (2.7) *has a solution* u *in the space* $C_0^{\{k\},\alpha}([0,T];X_{p,\mathbb{R}}) \bigcap C_0^{\alpha}([0,T];X_{A_p,\mathbb{R}})$. *This solution satisfies the estimate*

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X_{p,\mathbb{R}}) \bigcap C_0^{\alpha}([0,T];X_{A_p,\mathbb{R}})} \leq C_7 \|F\|_{C_0^{\alpha}([0,T];X_{p,\mathbb{R}})}. \tag{2.21}$$

(iii) *Let* $F \in C^{\alpha}([0,T];X_{p,\mathbb{R}})$ *for some* $0 < \alpha < 1$ *and* $\varphi \in X_{A_p,\mathbb{R}}$. *Then* (2.7) *has a solution* u *in the space* $C^{\{k\}}([0,T];X_{p,\mathbb{R}}) \bigcap C([0,T];X_{A_p,\mathbb{R}})$. *This solution satisfies the estimate*

$$\|u\|_{C^{\{k\}}([0,T];X_{p,\mathbb{R}}) \bigcap C([0,T];X_{A_p,\mathbb{R}})} \leq C_{22}(\|F\|_{C^{\alpha}([0,T];X_{p,\mathbb{R}})} + \|\varphi\|_{X_{A_p,\mathbb{R}}}). \tag{2.22}$$

The constants C_7 *and* C_{22} *depend on* M, L, p, α.

Proof. Firstly, we prove (i). Since $u + 0i \in C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ solves (2.7) with vanishing data, the assertion follows from Lemma 2.2 (i).

Secondly, we prove (iii). Lemma 2.2 (iii) implies that (2.7) with the data $\widetilde{F} = F + 0i$, $\widetilde{\varphi} = \varphi + 0i$ has a solution $\widetilde{u} \in C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$. On the other hand, since the coefficients of L and the kernel k are real, the complex problem for $\widetilde{u}$ consists of two independent real subproblems for $\mathrm{Re}\,\widetilde{u}$ and $\mathrm{Im}\,\widetilde{u}$. These problems have the data $\mathrm{Re}\,\widetilde{F} = F$, $\mathrm{Re}\,\widetilde{\varphi} = \varphi$ and $\mathrm{Im}\,\widetilde{F} = 0$, $\mathrm{Im}\,\widetilde{\varphi} = 0$, respectively. The solution of the first subproblem $u = \mathrm{Re}\,\widetilde{u} \in C^{\{k\}}([0,T];X_{p,\mathbb{R}}) \bigcap C([0,T];X_{A_p,\mathbb{R}})$ is the desired one. The assertion (i) applied to second subproblem for $\mathrm{Im}\,\widetilde{u}$ implies $\mathrm{Im}\,\widetilde{u} = 0$. Therefore, $\widetilde{u} = u + 0i$ and the estimate (2.22) follows from (2.20) and (2.18) applied to $\widetilde{u}$.

The assertion (ii) can be proved in an similar manner. □

Now we prove a positivity principle that can be applied to the direct problem with either homogeneous or nonhomogeneous boundary conditions. Therefore, we use the notation that differs from the notation used in formulation of (2.7).

Lemma 2.4. *Let* $K \in L_1((0,T);\mathbb{R}) \bigcap C^1((0,T);\mathbb{R})$, $\lim_{t\to 0^+} K(t) = \infty$, $K > 0$, K *be nonincreasing and* $\exists t_K > 0 : K$ *is strictly decreasing in* $(0,t_K)$. *Moreover, let* $\mathrm{F} \in C([0,T];C(\overline{\Omega};\mathbb{R}))$. *Assume that the function* u *solves the problem*

$$\begin{aligned} &{}^{R}D_0^{\{K\}}(\mathrm{u} - \phi)(t,x) = L\mathrm{u}(t,x) + \mathrm{F}(t,x),\ t \in (0,T),\ x \in \Omega, \\ &\mathrm{u}(0,x) = \phi,\ x \in \Omega \end{aligned}$$

and satisfies the smoothness conditions $\mathrm{u} \in C([0,T];C(\overline{\Omega};\mathbb{R}))$, $\mathrm{u} \in C((0,T];W_p^2(\Omega;\mathbb{R}))$ *for some* $p > n$, $L_1\mathrm{u} \in C((0,T];C(\overline{\Omega};\mathbb{R}))$, ${}^{R}D_0^{\{K\}}(\mathrm{u} - \phi) \in C((0,T];C(\overline{\Omega};\mathbb{R}))$. *Finally, let*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_0^{\varepsilon} K(\tau) d\tau \sup_{0\leq s\leq \varepsilon} |\mathrm{u}(t-s,x) - \mathrm{u}(t,x)| = 0 \quad \forall t \in (0,T],\, x \in \overline{\Omega}. \tag{2.23}$$

If $\phi \geq 0$, $\mathrm{F} \geq 0$ *and* $\mathscr{B}\mathrm{u}|_{\partial\Omega} \geq 0$ *then the following assertions are valid.*

(i) $\mathrm{u} \geq 0$;

(ii) *if* $\mathrm{u}(t_0,x_0) = 0$ *at some point* $(t_0,x_0) \in (0,T] \times \Omega_N$, *where*

$$\Omega_N = \begin{cases} \Omega & \textit{in case } \mathscr{B} = I \\ \overline{\Omega} & \textit{in case } \mathscr{B} = \omega \cdot \nabla, \end{cases} \tag{2.24}$$

then $\mathrm{u}(t,x_0) = 0$ *for any* $t \in [0,t_0]$.

This lemma is a slight modification of a positivity principle that was proved in [36] for a semilinear equation in case of more smooth solution $\mathrm{u} \in C((0,T];C^2(\overline{\Omega};\mathbb{R}))$ and strictly decreasing in $(0,T)$ kernel K.

To prove Lemma 2.4, we need the following auxiliary result. It is proved in Appendix of the Publication II.

Lemma 2.5. *Let $w \in W_p^2(\Omega;\mathbb{R})$ for some $p > n$, $L_1 w \in C(\overline{\Omega};\mathbb{R})$ and $x^* = \operatorname{argmin}_{x\in\overline{\Omega}} w(x)$. In case $x^* \in \partial\Omega$ we assume additionally that $(\omega \cdot \nabla w)(x^*) \geq 0$. Then $L_1 w(x^*) \geq 0$.*

Proof of Lemma 2.4. Without a restriction of generality we assume that $r \leq 0$. Otherwise it is possible to define $\widetilde{\mathrm{u}} = e^{-\sigma t}\mathrm{u}$ as in [36] and to consider the corresponding problem for $\widetilde{\mathrm{u}}$. Such a problem also satisfies the assumptions of Lemma 2.4 and has the coefficient $\widetilde{r} = r - \sigma \int_0^T e^{-\sigma s} K(s)ds$ in place of r. Since $\lim_{t\to 0^+} K(t) = \infty$ and K is positive and nonincreasing, we have that for $\sigma > \frac{1}{T}$

$$
\begin{aligned}
\sigma \int_0^T e^{-\sigma s} K(s)ds &\geq \sigma \int_0^{\frac{1}{\sigma}} e^{-\sigma s} K(s)ds \geq K(\frac{1}{\sigma})\sigma \int_0^{\frac{1}{\sigma}} e^{-\sigma s} ds \\
&= K(\frac{1}{\sigma})(1-e^{-1}) \to \infty, \ \sigma \to \infty.
\end{aligned}
$$

Therefore, for sufficiently large σ, $\widetilde{r} \leq 0$.

Thus, under the assumption $r \leq 0$ let us suppose that (i) does not hold. Then there exists $(t_1,x_1) \in (0,T] \times \overline{\Omega}$ such that

$$
\mathrm{u}(t_1,x_1) < 0 \quad \text{and} \quad (t_1,x_1) = \operatorname*{argmin}_{x\in\overline{\Omega},\, t\in[0,T]} \mathrm{u}(t,x).
$$

It was shown in [36] (formula (37)) that the assumptions ${}^R D_0^{\{K\}}(\mathrm{u}-\phi) \in C((0,T];C(\overline{\Omega};\mathbb{R}))$, (2.23), $K > 0$ and K – nonincreasing together with the relations $\mathrm{u}(t,x_1) \geq \mathrm{u}(t_1,x_1)$ and $\mathrm{u}(t_1,x_1) < 0$ imply

$$
{}^R D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1) < 0. \tag{2.25}
$$

On the other hand, let us consider the function $w = \mathrm{u}(t_1,\cdot)$ and its minimum point $x^* = x_1$. It satisfies the regularity conditions of Lemma 2.5. In case of $\mathscr{B} = I$ the minimum point x_1 must lie inside the domain Ω, since $\mathscr{B}\mathrm{u}|_{\Omega} \geq 0$ and $\mathrm{u}(t_1,x_1) < 0$. In case $\mathscr{B} = \omega \cdot \nabla$ the condition $\omega \cdot \nabla \mathrm{u}(t_1,x_1) \geq 0$ is satisfied if $x_1 \in \partial\Omega$. Thus, by Lemma 2.5 we obtain

$$
L_1 \mathrm{u}(t_1,x_1) \geq 0.
$$

Also $r(x_1)\mathrm{u}(t_1,x_1) \geq 0$ and $\mathrm{F} \geq 0$. Thus, the left-hand side of the equation

$$
{}^R D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1) = [L\mathrm{u} + \mathrm{F}](t_1,x_1)
$$

is negative, but the right-hand side is nonnegative. We have reached a contradiction. The assertion (i) is valid.

Let us prove (ii). Let $\mathrm{u}(t_0,x_0) = 0$ at $(t_0,x_0) \in (0,T] \times \Omega_N$. Define

$$
\hat{t}_0 = \inf\big\{t \,:\, t \leq t_0,\, \mathrm{u}(\tau,x_0) = 0 \text{ for } \tau \in [t,t_0]\big\}.
$$

If (ii) is not valid, then $\hat{t}_0 > 0$ and $\mathrm{u}(t,x_0) \geq \delta$, $t \in (t_2,t_3)$ for some $\delta > 0$ and $(t_2,t_3) \subset (0,\hat{t}_0)$ such that $\hat{t}_0 - t_2 < t_K$. Then, similarly to the proof in [36] p.138, from the assumptions

${}^RD_0^{\{K\}}(\mathrm{u}-\phi) \in C((0,T];C(\overline{\Omega};\mathbb{R}))$, (2.23), $K>0$, K – nonincreasing and relations $\mathrm{u}\geq 0$, $\mathrm{u}(t,x_0)\geq\delta>0$, $t\in(t_2,t_3)$, we derive

$$ {}^RD_0^{\{K\}}(\mathrm{u}-\phi)(\hat{t}_0,x_0) \leq \delta(K(\hat{t}_0-t_2)-K(\hat{t}_0-t_3)). \tag{2.26} $$

Since $0<\hat{t}_0-t_3<\hat{t}_0-t_2<t_K$ and K is strictly decreasing in $(0,t_K)$, inequality (2.26) implies

$$ {}^RD_0^{\{K\}}(\mathrm{u}-\phi)(\hat{t}_0,x_0)<0. $$

On the other hand, from $\mathrm{u}(\hat{t}_0,x_0)=0$ and $\mathrm{u}(t,x)\geq 0$, $(t,x)\in(0,T]\times\Omega$, we conclude that

$$ (\hat{t}_0,x_0)=\operatorname*{argmin}_{x\in\overline{\Omega}}\mathrm{u}(\hat{t}_0,x). $$

By Lemma 2.5, $L_1\mathrm{u}(\hat{t}_0,x_0)\geq 0$. Moreover, $(r\mathrm{u})(\hat{t}_0,x_0)=0$ and $\mathrm{F}\geq 0$. Left-hand side of the equation

$$ {}^RD_0^{\{K\}}(\mathrm{u}-\phi)(\hat{t}_0,x_0)=[L\mathrm{u}+\mathrm{F}](\hat{t}_0,x_0) $$

is negative, but right-hand side is nonnegative. Again, we have reached the contradiction. Thus, (ii) holds. □

To help the reader some details of the proof, such as the derivation of the problem for $\widetilde{u}$ and inequalities (2.25), (2.26) are presented in the Appendix of this Chapter.

At this point we present sufficient conditions on the input data of the direct problem (2.7) that together with the basic assumptions on the kernels k and M (2.14), (2.15) imply the assumptions of Lemma 2.4.

Corollary 2.2. *Let $F\geq 0$, $\varphi=0$ and one of the assumptions* (a1) – (a3) *hold:*
(a1) $F\in C^{\{M\},\alpha}([0,T];X_{0,\mathbb{R}})$ *for some* $0<\alpha<1$ *and* $F(0,\cdot)=0$;
(a2) $F\in C_0^{\alpha}([0,T];X_{0,\mathbb{R}})$ *and* $M(t)\geq ct^{\gamma-1}$, $t\in(0,T)$ *for some* $c\in\mathbb{R}_+$, $0<\gamma<\alpha<1$;
(a3) $F\in C_0^{\alpha-\beta}([0,T];X_{0,\mathbb{R}})$ *and* $c_1t^{\gamma-1}\leq M(t)\leq c_2t^{\beta-1}$, $|M'(t)|\leq c_3t^{\beta-2}$, $t\in(0,T)$, *for some* $c_1,c_2,c_3\in\mathbb{R}_+$, $0<\beta\leq\gamma<\alpha<1$.
Then the solution u of the problem (2.7) *is a real function and satisfies the assertions of Lemma* 2.4, *namely:*
(i) $u\geq 0$;
(ii) *if $u(t_0,x_0)=0$ at some point $(t_0,x_0)\in(0,T]\times\Omega_N$, where Ω_N is given by* (2.24), *then $u(t,x_0)=0$ for any $t\in[0,t_0]$.*

Proof. Lemma 2.3 implies that the solution of (2.7) exists in the space $C^{\{k\}}([0,T];X_{0,\mathbb{R}})\bigcap C([0,T];X_{A_0,\mathbb{R}})$. The smoothness conditions of Lemma 2.4 yield from the embeddings $X_{A_0;\mathbb{R}}\hookrightarrow W_p^2(\Omega;\mathbb{R})$ for $p\in(1,\infty)$ and $W_p^2(\Omega;\mathbb{R})\hookrightarrow C^1(\overline{\Omega};\mathbb{R})$ for $p\in(n,\infty)$.

It remains to show that (2.23) holds.

The case (a1). The relations $F\in C^{\{M\},\alpha}([0,T];X_{0,\mathbb{R}})$, $F(0,\cdot)=0$ mean that $F=k*F_2$, where $F_2\in C^{\alpha}([0,T];X_{0,\mathbb{R}})$. Let consider the problem

$$ \begin{aligned} &{}^RD_0^{\{k\}}u_2(t,x)=Lu_2(t,x)+F_2(t,x), \quad x\in\Omega,\, t\in(0,T),\\ &u_2(0,x)=0, \quad x\in\Omega,\\ &\mathscr{B}u_2(t,x)=0, \quad x\in\partial\Omega,\, t\in(0,T). \end{aligned} \tag{2.27} $$

By Lemma 2.3 (iii) the problem (2.27) has a solution $u_2\in C_0^{\{k\}}([0,T];X_{0,\mathbb{R}})$. Next we denote $u=k*u_2$. After convolving (2.27) with k it is easy to see that the function u solves (2.7) with $F = k * F_2$. Therefore,

$$u \in k * C_0^{\{k\}}([0,T];X_{0,\mathbb{R}}) = k * M * C([0,T];X_{0,\mathbb{R}}) = 1 * C([0,T];X_{0,\mathbb{R}}) \subset C^1([0,T];X_{0,\mathbb{R}}).$$

Hence, for $t \in (0,T]$, $x \in \overline{\Omega}$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon k(\tau) d\tau \sup_{0 \le s \le \varepsilon} |u(t-s,x) - u(t,x)| = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon k(\tau) d\tau \cdot O(\varepsilon) = 0.$$

The case (a2). By Lemma 2.3 (ii), $u \in C_0^{\{k\},\alpha}([0,T];X_{0,\mathbb{R}})$ and by (1.17), $u \in C_0^{\alpha}([0,T];X_{0,\mathbb{R}})$. Then the relation (2.23) follows from the estimate

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon k(\tau) d\tau \sup_{0 \le s \le \varepsilon} |u(t-s,x) - u(t,x)| = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon k(\tau) d\tau \cdot O(\varepsilon^\alpha) \\
&\le \lim_{\varepsilon \to 0^+} \frac{O(\varepsilon^\alpha)}{\varepsilon M(\varepsilon)} \int_0^\varepsilon M(\varepsilon - \tau) k(\tau) d\tau = \lim_{\varepsilon \to 0^+} O(\varepsilon^{\alpha - \gamma}) = 0 \quad \forall t \in (0,T],\ x \in \overline{\Omega}.
\end{aligned}$$

The case (a3). According to Lemma 2.3 (ii), $F \in C_0^{\alpha-\beta}([0,T];X_{0,\mathbb{R}})$ implies that

$$u \in C_0^{\{k\},\alpha-\beta}([0,T];X_{0,\mathbb{R}}) = M * C_0^{\alpha-\beta}([0,T];X_{0,\mathbb{R}}).$$

Lemma 1.4 yields $u \in C_0^{\alpha}([0,T];X_{0,\mathbb{R}})$. This enables us finish the proof as in case (a2). $\square$

2.5 Inverse source problem

We will study IP1 in context of Hölder spaces with respect to t. For the sake of generality, we will assume different orders of spaces related to g and h: for g we use α_1 and for h we use α_2. Firstly, we prove uniqueness theorem and then continue with existence and stability.

Theorem 2.1. *Let one of the following assumptions be valid:*
(A1) $g \in C_0^{1+\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$ *for some* $0 < \alpha_1 < 1$;
(A2) $g \in C_0^{\{k\},\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$ *and* $M(t) \ge ct^{\gamma-1}, t \in (0,T)$ *for some* $c \in \mathbb{R}_+$, $0 < \gamma < \alpha_1 < 1$;
(A3) $g \in C_0^{\{k\},\alpha_1-\beta}([0,T];C(\overline{\Omega};\mathbb{R}))$ *and* $c_1 t^{\gamma-1} \le M(t) \le c_2 t^{\beta-1}$, $|M'(t)| \le c_3 t^{\beta-2}$, $t \in (0,T)$, *for some* $c_1, c_2, c_3 \in \mathbb{R}_+$, $0 < \beta \le \gamma < \alpha_1 < 1$.
Additionally, we assume that $g \ge 0$, $g_1 := {}^R D_0^{\{k\}} g - r_{max} g \ge 0$ *where* $r_{max} := \max_{x \in \overline{\Omega}} r(x)$ *and*

$$\text{a.e. } x \in \Omega \quad \exists t_x \in (0,T] : g(t_x,x) > 0. \tag{2.28}$$

In case $\mathscr{B} = I$ *we also assume that for any* $x \in \partial\Omega$, *either* $g(T,x) > 0$ *or* $g(\cdot,x) = 0$.

Finally, let $(f,u) \in C(\overline{\Omega}) \times \left(C_0^{\{k\}}([0,T];C(\overline{\Omega})) \bigcap C_0([0,T];W_p^2(\Omega)) \right)$ *for some* $p > 1$ *solve* IP1 *for* $\varphi = 0$, $\psi = 0$, $h = 0$. *Then* $(f,u) = (0,0)$.

Proof. It is sufficient to prove the assertion of the Theorem in the particular case when f and u are real functions. That is because the the problem for the complex (f,u) can be split into two independent subproblems for $(\mathrm{Re} f, \mathrm{Re} u)$ and $(\mathrm{Im} f, \mathrm{Im} u)$.

We start the proof by showing that in case $\mathscr{B} = I$, for any $x \in \partial\Omega$ such that $g(T,x) > 0$, the equality $f(x) = 0$ is valid. To show this, we consider the equality

$${}^R D_0^{\{k\}} u(T,x) = f(x) g(T,x), \quad x \in \overline{\Omega}$$

that follows from equation (2.7) in view of $\psi = 0$. If $x \in \partial\Omega$ and $\mathscr{B} = I$ then the left-hand side of this equality equals zero. Thus, $f(x)g(T,x) = 0$ and provided $g(T,x) > 0$ we obtain $f(x) = 0$.

Let us introduce the functions $f^+ = \frac{|f|+f}{2}$ and $f^- = \frac{|f|-f}{2}$. Due to the definition, $f^\pm \in C(\overline{\Omega};\mathbb{R})$ and $f^\pm \geq 0$. Moreover,

$$\text{in case } \mathscr{B} = I, \text{ for any } x \in \partial\Omega \text{ such that } g(T,x) > 0, \text{ it holds } f^\pm(x) = 0. \tag{2.29}$$

Firstly, we consider the problems

$$\begin{aligned} &{}^R D_0^{\{k\}} u^\pm(t,x) = L u^\pm(t,x) + g(t,x) f^\pm(x), \quad x \in \Omega, t \in (0,T), \\ &u^\pm(0,x) = 0, \quad x \in \Omega, \qquad \mathscr{B} u^\pm(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T). \end{aligned} \tag{2.30}$$

By assumptions of the theorem and (2.29), $g(t,\cdot)f^\pm \in X_{0,\mathbb{R}}$, $t \in [0,T]$. Therefore, in cases (A1) and (A2) due to (1.17) we have $gf^\pm \in C_0^{\{M\},\alpha_1}([0,T];X_{0,\mathbb{R}})$ and $gf^\pm \in C_0^{\alpha_1}([0,T];X_{0,\mathbb{R}})$, respectively. Similarly, in case (A3) due to (1.17) and Lemma 1.4 we obtain $gf^\pm \in C_0^{\alpha_1}([0,T];X_{0,\mathbb{R}})$. Moreover, $gf^\pm \geq 0$. The assumptions of Corollary 2.2 are satisfied for the functions $F = gf^\pm$. Hence, the solutions $u^\pm$ of (2.30) satisfy the assertions (i) and (ii) of Corollary 2.2.

Secondly, let us consider the problems

$$\begin{aligned} &{}^R D_0^{\{k\}} v^\pm(t,x) = L v^\pm(t,x) + g_1(t,x) f^\pm(x), \quad x \in \Omega, t \in (0,T), \\ &v^\pm(0,x) = 0, \quad x \in \Omega, \qquad \mathscr{B} v^\pm(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T). \end{aligned} \tag{2.31}$$

In case (A1) we have $g' \in C_0^{\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$. Thus,

$$g_1 = {}^R D_0^{\{k\}} g - r_{max} g = k * g' - r_{max} g \in C_0^{\{M\},\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R})).$$

From $g(t,\cdot)f^\pm \in X_{0,\mathbb{R}}$, $t \in [0,T]$ we immediately get $g_1(t,\cdot)f^\pm \in X_{0,\mathbb{R}}$, $t \in [0,T]$. Therefore, $g_1 f^\pm \in C_0^{\{M\},\alpha_1}([0,T];X_{0,\mathbb{R}})$.

Using similar reasoning, we deduce $g_1 f^\pm \in C_0^{\alpha_1}([0,T];X_{0,\mathbb{R}})$ in case (A2) and $g_1 f^\pm \in C_0^{\alpha_1-\beta}([0,T];X_{0,\mathbb{R}})$ in case (A3). Moreover, $g_1 f^\pm \geq 0$. Again, the assumptions of Corollary 2.2 are satisfied for $F = g_1 f^\pm$. The solutions $v^\pm$ of (2.31) satisfy the assertions (i) and (ii) of Corollary 2.2.

The problem for $M * v^\pm$ is equivalent to the problem for $u^\pm - r_{\max} M * u^\pm$. Thus,

$$v^\pm = {}^R D_0^{\{k\}} u^\pm - r_{\max} u^\pm. \tag{2.32}$$

Moreover, since $f = f^+ - f^-$, we have $u = u^+ - u^-$. Thus, $\psi = u(T,\cdot) = 0$ implies that $u^+(T,\cdot) = u^-(T,\cdot)$. Let us denote

$$x^* = \underset{x\in\overline{\Omega}}{\operatorname{argmax}}\, u^+(T,x) = \underset{x\in\overline{\Omega}}{\operatorname{argmax}}\, u^-(T,x).$$

By definition, either $f^+(x^*) = 0$ or $f^-(x^*) = 0$. Let us assume that $f^+(x^*) = 0$ (the situation when $f^-(x^*) = 0$ can be considered in a similar manner).

Let us suppose that either $x^* \in \Omega$ or $\mathscr{B} = \omega \cdot \nabla$ (the case $x^* \in \partial\Omega$ and $\mathscr{B} = I$ will be considered later separately). Then we can apply Lemma 2.5 to the function $w = -u^+(T,\cdot)$. We get $L_1 u^+(T,x^*) \leq 0$. Thus, from (2.30), (2.32) and $u^+ \geq 0$, $r \leq r_{max}$ it follows:

$$v^+(T,x^*) = L_1 u^+(T,x^*) + (r(x^*) - r_{max}) u^+(T,x^*) \leq 0. \tag{2.33}$$

Due to Corollary 2.2 (i),

$$v^+(t,x) \geq 0,\ (t,x) \in (0,T) \times \Omega. \tag{2.34}$$

Hence, (2.33) and (2.34) imply $v^+(T,x^*) = 0$. Thus, by Corollary 2.2 (ii),

$$v^+(t,x^*) = 0, t \in [0,T].$$

By formula (2.32) the latter inequality implies ${}^R D_0^{\{k\}} u^+(t,x^*) - r_{max} u^+(t,x^*) = 0, t \in [0,T]$. Applying $M*$ to to this equality, we obtain the following homogeneous Volterra equation of the second kind:

$$u^+(t,x^*) - r_{max} M * u^+(t,x^*) = 0, \quad t \in [0,T].$$

It has only the trivial solution $u^+(t,x^*) = 0, t \in [0,T]$. Hence, $u^+(T,x^*) = 0$.

Since x^* is a maximum point of $u^+(T,x)$ and $u^+(T,x) \geq 0$, we also get

$$u^+(T,x) = 0,\ \ x \in \Omega. \tag{2.35}$$

Now we consider the case $x^* \in \partial\Omega$, $\mathscr{B} = I$, too. Then by $\mathscr{B}u^+|_{\partial\Omega} = 0$, immediately $u^+(T,x^*) = 0$ and again we have (2.35).

Since $u = u^+ - u^-$ and $\psi = u(T,\cdot) = 0$ holds, from (2.35) we get

$$u^\pm(T,x) = 0, x \in \Omega.$$

Corollary 2.2 (ii) implies $u^\pm(t,x) = 0$, $(t,x) \in [0,T] \times \Omega$. Therefore,

$$u(t,x) = 0,\ (t,x) \in [0,T] \times \Omega.$$

From the differential equation for u we obtain $f(x)g(t,x) = 0$, $(t,x) \in [0,T] \times \Omega$. Finally, (2.28) yields $f = 0$. □

Next we deduce simple sufficient conditions for g and k that imply the assumption ${}^R D_0^{\{k\}} g - r_{max} g \geq 0$ in Theorem 2.1. For that reason we need the following Lemma.

Lemma 2.6. *Let $w \in C^{\{k\}}([0,T];\mathbb{R})$ be nonnegative and nondecreasing. Then*

$${}^R D_0^{\{k\}} w \geq k(T) w.$$

Proof. The assertion follows from the estimate

$$\begin{aligned}
{}^R D_0^{\{k\}} w(t) &= \lim_{\delta\to 0^+} \frac{1}{\delta}\Big[\int_0^{t+\delta} k(\tau)w(t+\delta-\tau)d\tau - \int_0^t k(\tau)w(t-\tau)d\tau\Big] \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta}\Big[\int_t^{t+\delta} k(\tau)w(t+\delta-\tau)d\tau + \int_0^t k(\tau)(w(t+\delta-\tau) - w(t-\tau))d\tau\Big] \\
&\geq \lim_{\delta\to 0^+} k(T+\delta)\frac{1}{\delta}\Big[\int_t^{t+\delta} w(t+\delta-\tau)d\tau + \int_0^t (w(t+\delta-\tau) - w(t-\tau))d\tau\Big] \\
&= k(T)\frac{d}{dt}\int_0^t w(t-\tau)d\tau = k(T)w(t), \quad 0 < t < T.
\end{aligned}$$

□

Due to that Lemma 2.6, ${}^R D_0^{\{k\}} g - r_{max} g \geq 0$ holds provided along with other assumptions on g in Theorem 2.1 the following conditions are satisfied:

g is nondecreasing in case $r_{max} \leq 0$;

g is nondecreasing and $k(T) \geq r_{max}$ in case $r_{max} > 0$.

Theorem 2.2. *Let g, M satisfy the assumptions of Theorem 2.1 and the inequality*

$$g(T,x) \geq g_0 > 0, x \in \overline{\Omega},$$

hold. If $\varphi, \psi \in X_{A_p}$ and $h \in C^{\alpha_2}([0,T];X_p)$, where $p \in \{0\} \bigcup (1,\infty)$, $0 < \alpha_2 < 1$, then IP1 *has a unique solution $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ and the following estimate holds:*

$$\|f\|_{X_p} + \|u\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})} \leq C_9 \left(\|\varphi\|_{X_{A_p}} + \|\psi\|_{X_{A_p}} + \|h\|_{C^{\alpha_2}([0,T];X_p)} \right). \quad (2.36)$$

If additionally $\varphi = h(0,\cdot) = 0$, then $u \in C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^{\alpha}([0,T];X_{A_p})$ where

$$\alpha = \begin{cases} \alpha_2 & \text{in case (A1)} \\ \min\{\alpha_1, \alpha_2\} & \text{in cases (A2), (A3)} \end{cases}$$

and the estimate

$$\|f\|_{X_p} + \|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \leq C_{10} \left(\|\psi\|_{X_{A_p}} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)} \right) \quad (2.37)$$

is valid. The constants C_9 and C_{10} depend on the parameters M, L, g, p, α_2.

Proof. Firstly, we are going to replace the overdetermination condition (2.10) by a fixed-point equation with respect to f.

Suppose that $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ solves IP1. Then, since (2.10) holds, the equation (2.7) at $t = T$ with $F = fg + h$ yields

$$f(x) = \frac{\left({}^R D_0^{\{k\}} (u - \varphi) - \eta u \right)(T,x) - (A_p - \eta)\psi(x) - h(T,x)}{g(T,x)}, \quad (2.38)$$

where η is chosen so that $0 \in \rho(A_p - \eta I)$.

Let us split u into the sum of two functions: $u = u_1 + u_2$, such that

$${}^R D_0^{\{k\}} u_1 = A_p u_1 + fg, \quad u_1(0,\cdot) = 0, \quad (2.39)$$

$${}^R D_0^{\{k\}} (u_2 - \varphi) = A_p u_2 + h, \quad u_2(0,\cdot) = \varphi. \quad (2.40)$$

In the context of IP1, u_2 is a known function. According to Lemma 2.2, the solution u_2 to (2.40) belongs to $C^{\{k\}}([0,T];X_p)$.

Next we consider the functions v_1 and v_2. The function v_2 is given by the relation

$$v_2 := {}^R D_0^{\{k\}} (u_2 - \varphi) - \eta u_2 \quad (2.41)$$

and since $u_2 \in C^{\{k\}}([0,T];X_p)$ we obtain $v_2 \in C([0,T];X_p)$. The function v_1 is defined as the solution of the problem:

$${}^R D_0^{\{k\}} v_1 = A_p v_1 + f({}^R D_0^{\{k\}} g - \eta g), \quad v_1(0,\cdot) = 0. \quad (2.42)$$

Due to the assumptions (A1) - (A3) and (1.17), it holds ${}^R D_0^{\{k\}} g \in C_0^{\hat{\alpha}}([0,T];C(\overline{\Omega}))$ where

$$\hat{\alpha} = \begin{cases} \alpha_1 & \text{in cases (A1), (A2)} \\ \alpha_1 - \beta & \text{in case (A3).} \end{cases} \quad (2.43)$$

Thus, $f({}^R D_0^{\{k\}} g - \eta g) \in C_0^{\hat{\alpha}}([0,T];X_p)$. According to Lemma 2.2 (ii) the problem (2.42) has a solution v_1 in $C_0^{\{k\},\hat{\alpha}}([0,T];X_p) \bigcap C_0^{\hat{\alpha}}([0,T];X_{A_p})$. It is easy to check that

$$v_1 = {}^R D_0^{\{k\}} u_1 - \eta u_1.$$

The notations introduced allow us to rewrite (2.38) in the form

$$f = \mathscr{F} f + \mathscr{G}, \tag{2.44}$$

$$\text{where } \mathscr{G}(x) = \frac{v_2(T,x) - (A_p - \eta)\psi(x) - h(T,x)}{g(T,x)}, \quad x \in \Omega, \tag{2.45}$$

$$(\mathscr{F} f)(x) = \frac{1}{g(T,x)} v_1[f](T,x) \tag{2.46}$$

and $v_1[\cdot]$ stands for the operator that assigns to f the solution v_1 of (2.42). Thus, (2.7), (2.10), (2.12) imply (2.44).

On the other hand, taking into account all the substitutions performed, we can move back from (2.44) to (2.38). Together with (2.7) at $t = T$ and (2.12) it implies $(A_p - \eta)u(T,x) = (A_p - \eta)\psi(x)$. Since $(A_p - \eta)$ is injective, it yields (2.10).

Consequently, IP1 is equivalent to the problem of finding the pair of functions (f,u) that solves (2.7), (2.12), (2.44) in the space $X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$.

We point out that (2.44) is an independent equation for the first component f of the solution of IP1. Let us analyse properties of the operator $\mathscr{F}$ involved in this equation. By Lemma 2.2, $v_1[\cdot] \in \mathscr{L}(X_p, C_0^{\hat{\alpha}}([0,T];X_{A_p}))$. Thus, $v_1[\cdot](T,\cdot) \in \mathscr{L}(X_p, X_{A_p})$.

Furthermore, the compact embedding holds $X_{A_p} \hookrightarrow\hookrightarrow X_p$. In case $p \in (1,\infty)$ it is a direct consequence of $W_p^2(\Omega) \hookrightarrow\hookrightarrow L_p(\Omega)$. In case $p = 0$ it follows from the continuous embedding of X_{A_0} in $C^1_{\mathscr{B}}(\overline{\Omega}) := X_0 \bigcap C^1(\overline{\Omega})$ (see Theorems 3.1.19, 3.1.22 in [64]) and $C^1_{\mathscr{B}}(\overline{\Omega}) \hookrightarrow\hookrightarrow X_0$.

Therefore, $v_1[\cdot](T,\cdot) : X_p \to X_p$ is compact. Since $\frac{1}{g(T,\cdot)} \in C(\overline{\Omega})$ due to the assumptions of this theorem, $\mathscr{F} : X_p \to X_p$ is also compact.

Next, let us show that $1 \notin \sigma(\mathscr{F})$.

Firstly, let us consider the case $p = 0$. Suppose that $1 \in \sigma(\mathscr{F})$. Then the equation $f = \mathscr{F} f$ has a solution $f \in X_0$, $f \neq 0$. This means that the problem (2.7), (2.12), (2.44) with homogeneous data $\varphi = 0$, $\psi = 0$, $h = 0$ has the nontrivial solution (f,u_1) in the space $X_0 \times C_0^{\{k\}}([0,T];X_0) \bigcap C_0([0,T];X_{A_0})$. But due to the Theorem 2.1, IP1 with a homogeneous data has only the trivial solution in such a space. We came to a contradiction. Consequently, $1 \notin \sigma(\mathscr{F})$.

Secondly, let us consider the case $p \in (1,\infty)$. We again suppose that $1 \in \sigma(\mathscr{F})$, hence the equation $f = \mathscr{F} f$ has a nontrivial solution $f \in X_p$. The idea is to show that this solution actually belongs to X_0. We can then apply the arguments from the previous case to show that $1 \in \sigma(\mathscr{F})$ leads to a contradiction.

If $p > \frac{n}{2}$, then $v_1[f](T,\cdot) \in X_{A_p} \hookrightarrow X_0$. Thus, $f = \mathscr{F} f = \frac{1}{g(T,x)} v_1[f](T,\cdot) \in X_0$.

If $p \leq \frac{n}{2}$, then according to embedding theorems, $X_{A_p} \hookrightarrow X_{p_1} = L_{p_1}(\Omega)$, where $p_1 = \frac{np}{n-2p} > p$. Therefore, $v_1[f](T,\cdot) \in X_{p_1}$ and $f = \mathscr{F} f = \frac{1}{g(T,x)} v_1[f](T,\cdot) \in X_{p_1}$. After a finite number of iterations we obtain $f \in X_{p_i}$, where $p_i = \frac{np}{n-2ip} > \frac{n}{2}$ (works for $i > \frac{n}{2p} - 1$). Next iteration gives $f \in X_0$.

We have shown that the first case of Fredholm alternative is satisfied for the equation (2.44). Consequently, the solution to (2.44) exists and is unique for any $\mathscr{G} \in X_p$ and $(I - \mathscr{F})^{-1} \in \mathscr{L}(X_p)$.

Since $F = fg + h$ is Hölder-continuous with values in X_p, Lemma 2.2 implies that the problem (2.7), (2.12) has unique solution $u \in C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})$. This completes the proof of the existence and uniqueness assertion of the theorem.

In the rest of the proof, $\widehat{C}$ stands for a generic constant depending on the parameters M, L, g, p, α_2.

Let us deduce the stability estimate (2.36). We obtain

$$\|f\|_{X_p} \le \|(I-\mathscr{F})^{-1}\|_{\mathscr{L}(X_p)}\|\mathscr{G}\|_{X_p} \le \widehat{C}\Big(\|h(T,\cdot)\|_{X_p} + |\eta|\|\psi\|_{X_p} + \|\psi\|_{X_{A_p}} + \\ + \|v_2\|_{C([0,T];X_p)}\Big).$$

Since v_2 is given by the relation (2.41) and the function u_2 solves (2.40) we estimate it by means of Lemma 2.2.

$$\|v_2\|_{C([0,T];X_p)} = \|{}^R D_0^{\{k\}}(u_2-\varphi) - \eta u_2\|_{C([0,T];X_p)} \le \|{}^R D_0^{\{k\}}(u_2-\varphi)\|_{C([0,T];X_p)} \\ + \eta\|u_2\|_{C([0,T];X_p)} \le \|A_p u_2\|_{C([0,T];X_p)} + \|h\|_{C([0,T];X_p)} + \eta\|u_2\|_{C([0,T];X_p)} \\ \le \widehat{C}\left(\|u_2\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})} + \|h\|_{C([0,T];X_p)}\right) \le \widehat{C}(\|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_A}).$$

Therefore,

$$\|f\|_{X_p} \le \widehat{C}(\|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_A} + \|\psi\|_{X_{A_p}}). \tag{2.47}$$

Further, we note that $g \in C_0^{\gamma}([0,T];C(\overline{\Omega}))$ for any $\gamma \in (0,1)$ in case (A1) and for $\gamma = \alpha_1$ in cases (A2), (A3). By applying Lemma 2.2 to the problems (2.39) and (2.40) we obtain

$$\|u\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})} = \|u_1+u_2\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})} \\ \le \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^{\gamma}([0,T];C(\overline{\Omega}))} + \|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_{A_p}}).$$

Together with the estimate of f (2.47) it implies (2.36).

In case $\varphi = h(0,\cdot) = 0$, the solution of (2.7), (2.12) belongs to the space $C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})$ and can be estimated as

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \le \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^{\gamma}([0,T];C(\overline{\Omega}))} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)}).$$

This with (2.47) implies (2.37). $\square$

We point out that in case $p = 0$ and $\mathscr{B} = I$, the assumptions of Theorem 2.2 allow to recover $f \in X_0 = \{f \in C(\overline{\Omega}) : f|_{\partial\Omega} = 0\}$ only. In order to fix that in the following theorem we provide some additional conditions that are sufficient to restore $f \in C(\overline{\Omega})$ in case $\mathscr{B} = I$. The idea is as follows. We treat the problem in the Lebesgue space X_p, $p > \frac{n}{2}$ and show that in case of sufficient regularity of the data the unknown f whose existence follows from the previous theorem belongs to $C(\overline{\Omega}) \subset X_p$.

Theorem 2.3. *Let g, M satisfy the assumptions of Theorem 2.2. If φ, ψ, $L\varphi \in X_{A_p}$ for some $p > \frac{n}{2}$, $L\psi \in C(\overline{\Omega})$, $h \in C^{\{k\},\alpha_2}([0,T];X_p)\bigcap C([0,T];C(\overline{\Omega}))$, where $0 < \alpha_2 < 1$ and $h(0,\cdot) \in X_{A_p}$ then IP1 has a unique solution $(f,u) \in C(\overline{\Omega}) \times C^{\{k\}}([0,T];X_{A_p})$. Moreover, $Lu \in C([0,T];C(\overline{\Omega}))$ and the estimate*

$$\|f\|_{C(\overline{\Omega})} + \|u\|_{C^{\{k\}}([0,T];X_{A_p})} + \|Lu\|_{C([0,T];C(\overline{\Omega}))} \le C_{11}\Big(\|\varphi\|_{X_p} + \|L\varphi\|_{X_{A_p}} \\ + \|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)\bigcap C([0,T];C(\overline{\Omega}))} + \|h(0,\cdot)\|_{X_{A_p}}\Big) \tag{2.48}$$

holds. If additionally $\varphi = h(0,\cdot) = {}^R D_0^{\{k\}} h(0,\cdot) = 0$ *then* $u \in C_0^{\{k\},\alpha'}([0,T];X_{A_p})$ *and the estimate*

$$\begin{aligned} &\|f\|_{C(\overline{\Omega})} + \|u\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} + \|Lu\|_{C_0([0,T];C(\overline{\Omega}))} \\ &\le C_{12}\left(\|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)\bigcap C_0([0,T];C(\overline{\Omega}))}\right) \end{aligned} \tag{2.49}$$

is valid where $\alpha' = \min\{\hat{\alpha};\alpha_2\}$ *and* $\hat{\alpha}$ *is given by* (2.43). *The constants* C_{11} *and* C_{12} *depend on* M, L, g, p, α_2.

Proof. Throughout the proof, $\widehat{C}$ denotes a generic constant depending on M, L, g, p, α_2 and RHS stands for the expression in brackets at the right-hand side of (2.48).

By Theorem 2.2, IP1 has a unique solution $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})$. Let us consider the problem

$$ {}^R D_0^{\{k\}}(w_2 - w_2(0,\cdot)) = A_p w_2 + {}^R D_0^{\{k\}}(h - h(0,\cdot)),\ w_2(0,\cdot) = L\varphi + h(0,\cdot). \tag{2.50}$$

Under the assumptions of this theorem, Lemma 2.2 implies that (2.50) has a unique solution $w_2 \in C^{\{k\}}([0,T];X_p)\cap C([0,T];X_{A_p})$. Moreover, due (2.18) and (1.15)

$$\|w_2\|_{C([0,T];X_{A_p})} \le \widehat{C}(\|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)} + \|h(0,\cdot)\|_{X_{A_p}} + \|L\varphi\|_{X_{A_p}}).$$

It is easy to check that $w_2 = {}^R D_0^{\{k\}}(u_2 - \varphi)$ and $u_2 = M * w_2 + \varphi$ where u_2 solves (2.40). Since w_2 is in $C([0,T];X_{A_p})$ we have $u_2 \in C^{\{k\}}([0,T];X_{A_p})$ and

$$\|u_2\|_{C^{\{k\}}([0,T];X_{A_p})} \le \widehat{C}\Big(\|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)} + \|h(0,\cdot)\|_{X_{A_p}} + \|L\varphi\|_{X_{A_p}}\Big) + \|\varphi\|_{X_{A_p}}. \tag{2.51}$$

Let us consider the function $\mathscr{G}$ given by (2.45). (Recall that there $v_2 = w_2 - \eta u_2$.) Due the proved properties of w_2 and u_2 and the assumptions of the theorem and the embedding

$$X_{A_p} \hookrightarrow C(\overline{\Omega})$$

it holds $\mathscr{G} \in C(\overline{\Omega})$ and $\|\mathscr{G}\|_{C(\overline{\Omega})} \le \widehat{C}\mathrm{RHS}$.

Now, let us provide an estimate for $\|f\|_{C(\overline{\Omega})}$ using the formulas (2.44) and (2.46). Since $\frac{1}{g(T,\cdot)} \in C(\overline{\Omega})$ and $v_1[\cdot](T,\cdot) \in \mathscr{L}(X_p, X_{A_p})$, we have

$$\begin{aligned} \|f\|_{C(\overline{\Omega})} &\le \|\mathscr{F} f\|_{C(\overline{\Omega})} + \|\mathscr{G}\|_{C(\overline{\Omega})} \le \widehat{C}\|v_1[f](T,\cdot)\|_{C(\overline{\Omega})} + \|\mathscr{G}\|_{C(\overline{\Omega})} \\ &\le \widehat{C}\|v_1[f](T,\cdot)\|_{X_{A_p}} + \|\mathscr{G}\|_{C(\overline{\Omega})} \le \widehat{C}\|f\|_{X_p} + \|\mathscr{G}\|_{C(\overline{\Omega})}. \end{aligned}$$

Since $(I - \mathscr{F})$ is invertible in X_p, the estimate holds

$$\|f\|_{X_p} \le \|(I-\mathscr{F})^{-1}\|_{\mathscr{L}(X_p)}\|\mathscr{G}\|_{X_p} \le \widehat{C}\|\mathscr{G}\|_{C(\overline{\Omega})}.$$

Thus, we obtain $\|f\|_{C(\overline{\Omega})} \le \widehat{C}\|\mathscr{G}\|_{C(\overline{\Omega})}$ and therefore

$$\|f\|_{C(\overline{\Omega})} \le \widehat{C}\mathrm{RHS}. \tag{2.52}$$

Finally, let us derive an estimate for u and finish the proof of the first part of the theorem. We have $u = u_1 + u_2$, where $u_1 = M * w_1$, $w_1 = {}^R D_0^{\{k\}} u_1$ and w_1 solves the problem

$$ {}^R D_0^{\{k\}} w_1 = A_p w_1 + f\, {}^R D_0^{\{k\}} g, \quad w_1(0,\cdot) = 0. \tag{2.53}$$

Since $f\,{}^R D_0^{\{k\}} g \in C_0^{\alpha'}([0,T];X_p)$, Lemma 2.2 implies $w_1 \in C_0^{\alpha'}([0,T];X_{A_p})$ and

$$\|u_1\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} = \|w_1\|_{C_0^{\alpha'}([0,T];X_{A_p})} \le \widehat{C}\|f\|_{C(\overline{\Omega})}\|{}^R D_0^{\{k\}} g\|_{C_0^{\alpha'}([0,T];X_p)}.$$

Using here (2.52) we have

$$\|u_1\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} \le \widehat{C}\mathrm{RHS}. \tag{2.54}$$

From (2.51) and (2.54) we obtain for $u = u_1 + u_2$ the estimate

$$\|u\|_{C^{\{k\}}([0,T];X_{A_p})} \le \widehat{C}\mathrm{RHS}. \tag{2.55}$$

It remains to estimate Lu in the space $C([0,T];C(\overline{\Omega}))$. Using (2.55) we deduce

$$\|{}^R D_0^{\{k\}}(u-\varphi)\|_{C([0,T];C(\overline{\Omega}))} \le \widehat{C}\|{}^R D_0^{\{k\}}(u-\varphi)\|_{C([0,T];X_{A_p})} \le \widehat{C}\mathrm{RHS}.$$

From the expression $Lu = {}^R D_0^{\{k\}}(u-\varphi) - fg - h$ due to the proved estimates for ${}^R D_0^{\{k\}}(u-\varphi)$ and f we obtain

$$\|Lu\|_{C([0,T];C(\overline{\Omega}))} \le \widehat{C}\mathrm{RHS}. \tag{2.56}$$

Summing up, (2.52), (2.55) and (2.56) imply (2.48).

Now let us focus on the second part of this theorem that is concerned with the particular case $\varphi = h(0,\cdot) = {}^R D_0^{\{k\}} h(0,\cdot) = 0$. Then RHS reduces to the expression in brackets at the right-hand side of (2.49). Lemma 2.2 implies that the function w_2 which solves (2.50) belongs the space $C_0^{\alpha'}([0,T];X_{A_p})$, the function $u_2 = M * w_2$ belongs to $C_0^{\{k\},\alpha'}([0,T];X_{A_p})$ and $\|u_2\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} \le \widehat{C}\|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)}$. This relation by $u = u_1 + u_2$ and the estimates (2.52), (2.54) and (2.56) implies (2.49). □

Provided the assumptions of Theorem 2.3 hold and $\mathscr{B} = I$, an explicit expression of the unknown function f at the boundary can be derived. Namely, setting $t = T$ and $x \in \partial\Omega$ in (2.7) and taking the relations $F = fg + h$ and $u(T,\cdot) = \psi$ into account we obtain

$$f(x) = -\frac{1}{g(T,x)}[L\psi(x) + h(T,x)], \quad x \in \partial\Omega.$$

Remark 2.1. In case the data h, φ and ψ are real-valued functions, the solution (f,u) of IP1 is also real. That holds because the coefficients of the operator L and the functions k and g are real. The complex IP1 consists of two independent real subproblems for $(\mathrm{Re}\, f, \mathrm{Re}\, u)$ and $(\mathrm{Im}\, f, \mathrm{Im}\, u)$. These problems depend on the data $\mathrm{Re}\, h, \mathrm{Re}\,\varphi, \mathrm{Re}\,\psi$ and $\mathrm{Im}\, h, \mathrm{Im}\,\varphi, \mathrm{Im}\,\psi$, respectively. If $\mathrm{Im}\, h = 0, \mathrm{Im}\,\varphi = 0, \mathrm{Im}\,\psi = 0$ then by applying Theorem 2.1 to the subproblem for $(\mathrm{Im}\, f, \mathrm{Im}\, u)$ we obtain $(\mathrm{Im}\, f, \mathrm{Im}\, u) = (0,0)$ that means that the pair (f,u) is a real.

At the end of this section we consider a problem with perturbations. Firstly, let us replace the kernel M in the equation (2.1) by the sum $M + M * m$, where M satisfies the conditions listed in Section 2.2 and m is a small perturbation factor. Then the initial boundary value problem for $u = U - b$ reads

$$\begin{aligned} &{}^R D_0^{\{k\}}(u-\varphi)(t,x) = Lu(t,x) + m * Lu(t,x) + F(x,t), \quad x \in \Omega, t \in (0,T), \\ &u(0,x) = \varphi(x), \ x \in \Omega, \quad \mathscr{B}u(t,x) = 0, \ x \in \partial\Omega, t \in (0,T). \end{aligned} \tag{2.57}$$

Secondly, let us replace (2.12) by the formula

$$F(t,x) = (g(t,x)+\gamma(t,x))f(x)+h(t,x), \tag{2.58}$$

where g satisfies the assumptions of Theorem 2.2 and γ is a small perturbation term.

In the modified IP1 (we call it **IP1a**), one has to find a pair (f,u) that satisfies (2.57), (2.58) and (2.10).

Theorem 2.4. (i) *Let the assumptions of Theorem* 2.2 *be satisfied and* $\varphi = 0$, $h(0,\cdot) = 0$. *We also assume that* $m \in L_1(0,T)$, $\gamma \in C_0^{\alpha}([0,T];C(\overline{\Omega}))$. *Then there exist constants* ε_1, C_{13} *depending on* M, L, g, p, α_2 *such that if*

$$\max\{\|m\|_{L_1(0,T)}; \|\gamma\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))}\} < \varepsilon_1$$

then IP1a *has a unique solution* $(f,u) \in X_p \times C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^{\alpha}([0,T];X_{A_p})$ *and the following estimate is valid:*

$$\|f\|_{X_p} + \|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \le C_{13}\left(\|\psi\|_{X_{A_p}} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)}\right). \tag{2.59}$$

(ii) *Let the assumptions of Theorem* 2.3 *hold and* $\varphi = 0$, $h(0,\cdot) = 0$, ${}^{R}D_0^{\{k\}}h(0,\cdot) = 0$. *We also assume that* $m \in L_1(0,T)$, $\gamma \in C_0^{\{k\},\alpha'}([0,T];X_p) \bigcap C_0([0,T];C(\overline{\Omega}))$. *Then there exist constants* ε_2, C_{14} *depending on* M, L, g, p, α_2 *such that if*

$$\max\{\|m\|_{L_1(0,T)}; \|\gamma\|_{C_0^{\{k\},\alpha'}([0,T];X_p)\bigcap C_0([0,T];C(\overline{\Omega}))}\} < \varepsilon_2$$

then IP1a *has a unique solution in the space* $\{(f,u) \in C(\overline{\Omega}) \times C_0^{\{k\},\alpha'}([0,T];X_{A_p}) : Lu \in C_0([0,T];C(\overline{\Omega}))\}$ *and the following estimate holds:*

$$\begin{aligned}&\|f\|_{C(\overline{\Omega})} + \|u\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} + \|Lu\|_{C_0([0,T];C(\overline{\Omega}))}\\ &\le C_{14}\left(\|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)\bigcap C_0([0,T];C(\overline{\Omega}))}\right).\end{aligned} \tag{2.60}$$

Proof. Let $\mathscr{A}$ be the solution operator of IP1. This means that the solution of IP1 can be represented as $(f,u) = \mathscr{A}(h,\varphi,\psi)$.
(i) According to the definition of $\mathscr{A}$, IP1a is equivalent to the operator equation

$$(f,u) = \hat{\mathscr{A}}(f,u) + \mathscr{A}(h,0,\psi) \tag{2.61}$$

where $\hat{\mathscr{A}}$ is a linear operator defined by $\hat{\mathscr{A}}(f,u) = \mathscr{A}(m*Lu+\gamma f,0,0)$. By Theorem 2.2, $\mathscr{A}(h,0,\psi) \in X_p \times C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^{\alpha}([0,T];X_{A_p})$ and

$$\|\mathscr{A}(h,0,\psi)\|_{X_p\times C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \le C_{10}\left(\|\psi\|_{X_{A_p}} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)}\right). \tag{2.62}$$

On the other hand, Lemma 4.2 in [35] implies

$$\|m*v\|_{C_0^{\alpha}([0,T];X_p)} \le 2\|m\|_{L_1(0,T)}\|v\|_{C_0^{\alpha}([0,T];X_p)} \quad \forall v \in C_0^{\alpha}([0,T];X_p).$$

Thus, by means of (2.37) we deduce the estimate

$$\|\hat{\mathscr{A}}(f,u)\|_{X_p\times C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \le C_{10}\|m*Lu+\gamma f\|_{C_0^{\alpha}([0,T];X_p)}$$
$$\le C_{10}\Big(2\|m\|_{L_1(0,T)}\|Lu\|_{C_0^{\alpha}([0,T];X_p)}+\|\gamma\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))}\|f\|_{X_p}\Big)$$
$$\le 2C_{10}\max\{\|m\|_{L_1(0,T)};\|\gamma\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))}\}\|(f,u)\|_{X_p\times C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})}.$$

In case $\max\{\|m\|_{L_1(0,T)},\|\gamma\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))}\}<\varepsilon_1=\frac{1}{2C_{10}}$, the operator $\hat{\mathscr{A}}$ is a contraction in the space $X_p\times C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})$. This proves the existence and uniqueness assertions of (i). From (2.61) by means of $\|\hat{\mathscr{A}}\|<1$ and (2.62) we deduce (2.59) with the constant $C_{13}=\frac{C_{10}}{1-\|\hat{\mathscr{A}}\|}$.
(ii) The proof of this assertion repeats the proof of (i) with appropriate changes of spaces and norms.

□

2.6 Inverse coefficient problem

In this section we apply results on IP1 to study IP2. In this connection there is a need to impose conditions similar to (A1) - (A3) on the factor $u+b$ of the unknown coefficient r. Those conditions depend on the upper bound r_{max} of r. This means that we are faced with a situation where assumptions of theorem depend on the unknown. Therefore we introduce the following sets of r that have their upper bounds less than some given number ρ and use ρ instead of $r_{\max}$ in the mentioned assumptions:

$$\mathscr{K}_\rho=\{r\in C(\overline{\Omega};\mathbb{R}) : r(x)\le\rho,\ x\in\overline{\Omega}\}\qquad \text{where } \rho\in\mathbb{R}.$$

In the next theorem we prove global uniqueness of the solution.

Theorem 2.5. *Let $\rho\in\mathbb{R}$, the data of IP2 be real and IP2 have 2 solutions (r,u), (r_1,u_1), such that*

$$r\in C(\overline{\Omega};\mathbb{R}),\quad r_1\in\mathscr{K}_\rho,\quad u,u_1\in C_0^{\{k\}}([0,T];L_1(\Omega;\mathbb{R}))\bigcap C_0([0,T];W_1^2(\Omega;\mathbb{R})),$$
$$u_1-u\in C_0^{\{k\}}([0,T];C(\overline{\Omega};\mathbb{R}))\bigcap C_0([0,T];W_p^2(\Omega;\mathbb{R}))$$

for some $p>1$ and the function; $U=u+b$ (and M) satisfy one of the following assumptions:
(A4) *$U\in C_0^{1+\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$ for some $0<\alpha_1<1$;*
(A5) *$U\in C_0^{\{k\},\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$ and $M(t)\ge ct^{\gamma-1}, t\in(0,T)$ for some $c\in\mathbb{R}_+$, $0<\gamma<\alpha_1<1$;*
(A6) *$U\in C_0^{\{k\},\alpha_1-\beta}([0,T];C(\overline{\Omega};\mathbb{R}))$ and $c_1t^{\gamma-1}\le M(t)\le c_2t^{\beta-1}$, $|M'(t)|\le c_3t^{\beta-2}$, $t\in(0,T)$, for some $c_1,c_2,c_3\in\mathbb{R}_+$, $0<\beta\le\gamma<\alpha_1<1$.*
Additionally, we assume that

$$U\ge 0,\quad {}^{R}D_0^{\{k\}}U-\rho U\ge 0, \tag{2.63}$$

$$\text{a.e. } x\in\Omega\quad \exists t_x\in(0,T] : U(t_x,x)>0. \tag{2.64}$$

In case $\mathscr{B}=I$ we also assume that for any $x\in\partial\Omega$, either $U(T,x)>0$ or $U(\cdot,x)=0$. Then $(r_1,u_1)=(r,u)$.

Proof. Let us denote the difference $(\hat{r},\hat{u}) = (r_1 - r, u_1 - u)$. Then

$$(\hat{r},\hat{u}) \in C(\overline{\Omega};\mathbb{R}) \times \left(C_0^{\{k\}}([0,T];C(\overline{\Omega};\mathbb{R})) \bigcap C_0([0,T];W_p^2(\Omega;\mathbb{R})) \right)$$

and solves the problem

$$\begin{aligned} &{}^R D_0^{\{k\}} \hat{u}(t,x) = (L_1 + r_1)\hat{u}(t,x) + U(t,x)\hat{r}(x), \quad x \in \Omega, t \in (0,T), \\ &\hat{u}(0,x) = 0, \quad x \in \Omega, \qquad \mathscr{B}\hat{u}(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T), \\ &\hat{u}(T,x) = 0, \quad x \in \Omega. \end{aligned} \tag{2.65}$$

The inequalities (2.63) imply that ${}^R D_0^{\{k\}} U - r_{max} U \geq 0$ where $r_{max} := \max_{x\in\overline{\Omega}} r_1(x) \leq \rho$. Consequently, the assumptions of Theorem 2.1 are satisfied for the problem (2.65) and we obtain $\hat{r} = 0$, $\hat{u} = 0$. □

Let us formulate a problem that contains approximate data:

$$\begin{aligned} &{}^R D_0^{\{k\}} (\widetilde{u} - \widetilde{\varphi})(t,x) = L_1 \widetilde{u}(t,x) + \widetilde{r}(x)(\widetilde{u} + \widetilde{b})(t,x) + \widetilde{F}_1(t,x), \;\; x \in \Omega, t \in (0,T), \\ &\widetilde{u}(0,x) = 0, \quad x \in \Omega, \qquad \mathscr{B}\widetilde{u}(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T), \\ &\widetilde{u}(T,x) = \widetilde{\psi}, \quad x \in \Omega. \end{aligned} \tag{2.66}$$

We are going to prove an existence and approximation theorem for this problem in case its data vector $\widetilde{D} = (\widetilde{b}, \widetilde{F}_1, \widetilde{\psi})$ is close to the data vector $D = (b, F_1, \psi)$ of the exact problem IP2. In general we will work with complex (2.66).

Theorem 2.6. *Assume that* $\rho \in \mathbb{R}$, IP2 *has real data and a real solution*

$$(r,u) \in \mathscr{K}_\rho \times C_0^{\{k\}}([0,T];L_1(\Omega;\mathbb{R})) \bigcap C_0([0,T];W_1^2(\Omega;\mathbb{R}))$$

such that $U = u + b$ *(and* M*) satisfy one of the assumptions* (A4) - (A6), *the inequalities* (2.63) *and* $U(T,x) \geq U_0 > 0, x \in \overline{\Omega}$. *Then the following statements are valid.*

(i) *Let* $p \in \{0\} \bigcup (\frac{n}{2}, \infty)$, $\alpha_2 \in (0,1)$. *There exist constants* $\delta_1 > 0$ *and* $K_1 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$\widetilde{D} - D \in \mathscr{D}_1 = C_0^{\alpha_2}([0,T];C_{(p)}(\overline{\Omega})) \times C_0^{\alpha_2}([0,T];X_p) \times X_{A_p}$$

and $\|\widetilde{D} - D\|_{\mathscr{D}_1} \leq \delta_1$ *where* $C_{(p)}(\overline{\Omega}) = \begin{cases} C(\overline{\Omega}) & \text{in case } p \in (\frac{n}{2}, \infty) \\ X_0 & \text{in case } p = 0 \end{cases}$ *then problem* (2.66) *has a unique solution in the set*

$$\Big\{ (\widetilde{r},\widetilde{u}) : (\widetilde{r} - r, \widetilde{u} - u) \in \mathscr{X}_1 := X_p \times \left(C_0^{\{k\},\alpha}([0,T];X_p) \bigcap C_0^{\alpha}([0,T];X_{A_p}) \right), \\ \|(\widetilde{r} - r, \widetilde{u} - u)\|_{\mathscr{X}_1} \leq K_1 \|\widetilde{D} - D\|_{\mathscr{D}_1} \Big\}$$

where $\alpha = \begin{cases} \alpha_2 & \text{in case (A4)} \\ \min\{\alpha_1, \alpha_2\} & \text{in cases (A5), (A6).} \end{cases}$

(ii) *Let* $p \in (\frac{n}{2}, \infty)$, $\alpha_2 \in (0,1)$. *There exist constants* $\delta_2 > 0$ *and* $K_2 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$\widetilde{D} - D \in \mathscr{D}_2 = \left(C_0^{\{k\},\alpha_2}([0,T];X_p) \bigcap C_0^{\alpha_2}([0,T];C(\overline{\Omega})) \right)^2 \times Y_p$$

and $\|\widetilde{D}-D\|_{\mathscr{D}_2} \le \delta_2$ *where* $Y_p = \{\psi : \psi \in X_{A_p}, L\psi \in C(\overline{\Omega})\}$ *then the problem* (2.66) *has a unique solution in the set*

$$\left\{(\widetilde{r},\widetilde{u}) : (\widetilde{r}-r,\widetilde{u}-u) \in \mathscr{X}_2 := C(\overline{\Omega}) \times \mathscr{U}_{p,\alpha'},\ \|(\widetilde{r}-r,\widetilde{u}-u)\|_{\mathscr{X}_2} \le K_2\|\widetilde{D}-D\|_{\mathscr{D}_2}\right\}$$

where $\mathscr{U}_{p,\alpha'} = \{v \in C_0^{\{k\},\alpha'}([0,T];X_{A_p}) : Lv \in C_0([0,T];C(\overline{\Omega}))\}$, $\alpha' = \min\{\hat{\alpha};\alpha_2\}$ *and* $\hat{\alpha} = \begin{cases} \alpha_1 & \textit{in cases } \text{(A4), (A5)} \\ \alpha_1 - \beta & \textit{in case } \text{(A6)}. \end{cases}$

We mention that in this theorem, the operator A_p and the space X_{A_p} defined on the basis of $L = L_1 + rI$ depend on the component r of the solution of the exact problem IP2.

Proof. Let us denote the difference $(\hat{r},\hat{u}) = (\widetilde{r}-r, \widetilde{u}-u)$. Then the problem for the pair $(\hat{r},\hat{u})$ reads

$$\begin{aligned} &{}^R D_0^{\{k\}} \hat{u} = (L_1+r)\hat{u} + \hat{r}(u+b) + \left[\hat{r}\hat{u} + \widetilde{F}_1 - F_1 + (\hat{r}+r)(\widetilde{b}-b)\right], \\ &\hat{u}(0,\cdot) = 0, \quad \mathscr{B}\hat{u}|_{\partial\Omega} = 0, \quad \hat{u}(T,\cdot) = \widetilde{\psi} - \psi. \end{aligned} \tag{2.67}$$

This problem can be treated as IP1 with $f = \hat{r}$, $g = u+b$, $h = \hat{r}\hat{u} + \widetilde{F}_1 - F_1 + (\hat{r}+r)(\widetilde{b}-b)$. Therefore, applying the solution operator of IP1 $\mathscr{A}$ to (2.67), it is reduced to the operator equation

$$\begin{aligned} &(\hat{r},\hat{u}) = \mathscr{F}_2(\hat{r},\hat{u}), \\ &\text{where } \mathscr{F}_2(\hat{r},\hat{u}) = \mathscr{A}(\hat{r}\hat{u} + \widetilde{F}_1 - F_1 + (\hat{r}+r)(\widetilde{b}-b), 0, \widetilde{\psi}-\psi). \end{aligned} \tag{2.68}$$

We are going to show that $\mathscr{F}_2$ is a contraction in a ball $\|(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon$ with a suitable chosen $\varepsilon > 0$. Firstly, we have to prove that this ball remains invariant with respect to the operator $\mathscr{F}_2$. Let $\|(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon$. According to (2.37),

$$\|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le C_{10}\left(\|\widetilde{\psi}-\psi\|_{X_{A_p}} + \|\hat{r}\hat{u} + \widetilde{F}_1 - F_1 + (\hat{r}+r)(\widetilde{b}-b)\|_{C_0^{\alpha_2}([0,T];X_p)}\right).$$

Let c_p be an embedding constant such that $\|w\|_{C(\overline{\Omega})} \le c_p\|w\|_{X_{A_p}}$. Then

$$\|\hat{r}\hat{u}\|_{C_0^{\alpha_2}([0,T];X_p)} \le \|\hat{r}\|_{X_p}\|\hat{u}\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))} \le \|\hat{r}\|_{X_p} c_p \|\hat{u}\|_{C_0^{\alpha}([0,T];X_{A_p})} \le c_p\varepsilon^2.$$

Therefore,

$$\begin{aligned} \|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} &\le C_{10}\Big(\|\widetilde{\psi}-\psi\|_{X_{A_p}} + c_p\varepsilon^2 + \|\widetilde{F}_1 - F_1\|_{C_0^{\alpha_2}([0,T];X_p)} \\ &+ (\varepsilon + R_1)\|\widetilde{b}-b\|_{C_0^{\alpha_2}([0,T];C_{(p)}(\overline{\Omega}))}\Big) \le C_{10}\left(c_p\varepsilon^2 + (\varepsilon+1+R_1)\|\widetilde{D}-D\|_{\mathscr{D}_1}\right), \end{aligned}$$

where $R_1 = \|r\|_{X_p}$ in case $p \in (\frac{n}{2},\infty)$ and $R_1 = \|r\|_{C(\overline{\Omega})}$ in case $p = 0$. Now let us take $\varepsilon = K_1\|\widetilde{D}-D\|_{\mathscr{D}_1}$ with a constant K_1. Then

$$\|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le C_{10}\left((c_pK_1^2 + K_1)\|\widetilde{D}-D\|_{\mathscr{D}_1} + 1 + R_1\right)\|\widetilde{D}-D\|_{\mathscr{D}_1}.$$

In case $\|\widetilde{D}-D\|_{\mathscr{D}_1} \le \delta_1$ we have

$$\|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le C_{10}\left((c_pK_1^2 + K_1)\delta_1 + 1 + R_1\right)\|\widetilde{D}-D\|_{\mathscr{D}_1}.$$

Let us define the constants as follows: $K_1 = C_{10}(2+R_1)$, $\delta_1 = \frac{1}{c_p K_1^2 + K_1}$. Then

$$\|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le K_1 \|\widetilde{D} - D\|_{\mathscr{D}_1}.$$

Consequently, for $\|(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon$ we have $\|\mathscr{F}_2(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon$.

Secondly, inside the set $\|(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon = K_1\|\widetilde{D} - D\|_{\mathscr{D}_1}$ let us consider the difference of $\mathscr{F}_2$ at $(\hat{r}_1,\hat{u}_2)$ and $(\hat{r}_2,\hat{u}_2)$. Assuming $\|\widetilde{D} - D\|_{\mathscr{D}_1} \le \delta_1$, we deduce the estimate

$$\begin{aligned}
&\|\mathscr{F}_2(\hat{r}_1,\hat{u}_1) - \mathscr{F}_2(\hat{r}_2,\hat{u}_2)\|_{\mathscr{X}_1} \le C_{10}\|(\hat{r}_1-\hat{r}_2)\hat{u}_1 + \hat{r}_2(\hat{u}_1-\hat{u}_2) + (\hat{r}_1-\hat{r}_2)\\
&\times(\widetilde{b}-b)\|_{C_0^{\alpha_2}([0,T];X_p)} \le C_{10}\Big(c_p\varepsilon\|\hat{r}_1-\hat{r}_2\|_{X_p} + c_p\varepsilon\|\hat{u}_1-\hat{u}_2\|_{C_0^{\alpha}([0,T];X_{A_p})} + \delta_1\|\hat{r}_1-\hat{r}_2\|_{X_p}\Big)\\
&\le C_{10}(c_pK_1\delta_1+\delta_1)\|(\hat{r}_1-\hat{r}_2,\hat{u}_1-\hat{u}_2)\|_{\mathscr{X}_1} = \frac{1}{(2+R_1)}\|(\hat{r}_1-\hat{r}_2,\hat{u}_1-\hat{u}_2)\|_{\mathscr{X}_1}.
\end{aligned}$$

It shows that the operator $\mathscr{F}_2$ is a contraction in the ball $\|(\hat{r},\hat{u})\|_{\mathscr{X}_1} \le \varepsilon$. According to the Banach fixed point theorem there exists a unique solution to the equation (2.68) in that ball. This proves the assertion (i).

(ii) The proof of (ii) repeats the proof of (i) with appropriate changes of spaces and norms. For $\mathscr{A}$, the estimate (2.49) is used instead of (2.37). □

Remark 2.2. If the data of the approximate problem (2.66) are real, then the solution $(\widetilde{r},\widetilde{u})$ to the problem (2.66) is also real. This is due to the fact that the operator $\mathscr{A}$ (and therefore $\mathscr{F}_2$) maps the real functions into real functions and the subspace of real functions in $\mathscr{X}_1$ constitutes a real Banach space with the norm of $\mathscr{X}_1$. That enables one to follow the proof of the Theorem 2.6 and obtain corresponding results for the real functions.

Remark 2.3. Physically, a particular case of the exact solution $(0,u)$ corresponds to the reaction-free case. Then (2.66) governs a slow reaction process and Theorem 2.6 implies the identifiability of a small reaction coefficient from final data.

Remark 2.4. Let us construct sufficient conditions on the data that imply the inequalities (2.63) and $U(T,x) \ge U_0 > 0$, $x \in \overline{\Omega}$ in Theorems 2.5, 2.6. For this purpose we consider the problem (2.4) for U and set there $\Phi = H(0,\cdot) = 0$. Let us suppose that U is sufficiently smooth.

Constructing a corresponding problem for ${}^RD_0^{\{k\}}U - r_{max}U$ and assuming ${}^RD_0^{\{k\}}H - r_{max}H \ge 0$, $({}^RD_0^{\{k\}}\mathscr{B}b - r_{max}\mathscr{B}b)|_{\partial\Omega} \ge 0$, Lemma 2.4 (i) implies the inequality ${}^RD_0^{\{k\}}U - r_{max}U \ge 0$.

Next, we consider the inequalities $U \ge 0$ and $U(T,x) \ge U_0 > 0$, $x \in \overline{\Omega}$. Let us assume that

$$\begin{aligned}
&\exists\mu \in C([0,T];\mathbb{R}),\ \mu \ge 0,\ \mu \ne 0,\ \mu - \text{nondecreasing}:\\
&H(t,x) \ge \mu(t),\ x \in \overline{\Omega}, t \in [0,T],\quad \mathscr{B}b(t,x) \ge \mu(t),\ x \in \partial\Omega, t \in [0,T].
\end{aligned}$$

Define $V = U - \delta 1 * \mu$ with $\delta > 0$. The function V solves the problem

$${}^RD_0^{\{k\}}V = LV + H_1, \quad V(0,\cdot) = 0, \quad \mathscr{B}(V - (b - \delta 1 * \mu))|_{\partial\Omega} = 0,$$

where $H_1 = H + \delta(r1 * \mu - {}^RD_0^{\{k\}}1 * \mu)$.

Since ${}^{R}D_0^{\{k\}}1*\mu = k*\mu$, we get that for sufficiently small δ,

$$H_1(t,x) \geq \mu(t)[1-\delta(\max_{x\in\overline{\Omega}}|r(x)|t + \|k\|_{L_1(0,T)})] \geq 0, \quad t\in[0,T],\ x\in\Omega.$$

Let us also show that $\mathscr{B}V|_{\partial\Omega} \geq 0$ for sufficiently small δ. We obtain

$$\mathscr{B}V|_{\partial\Omega} = \mathscr{B}(b-\delta 1*\mu)|_{\partial\Omega} \geq \mu - \delta\mathscr{B}1*\mu.$$

- If $\mathscr{B}=I$ we have $\mathscr{B}V|_{\partial\Omega} \geq \mu(t) - \delta\int_0^t \mu(\tau)d\tau \geq \mu(t)(1-\delta t) \geq 0$.
- If $\mathscr{B}=\omega\cdot\nabla$ we have $\mathscr{B}V|_{\partial\Omega} \geq \mu(t) - 0 \geq 0$.

Thus, Lemma 2.4 (i) yields $V\geq 0$. Consequently, we obtain the desired inequalities $U = V+\delta 1*\mu \geq 0$ and

$$U(T,x) = V(T,x) + \delta\int_0^T \mu(\tau)d\tau \geq \delta\int_0^T \mu(\tau)d\tau = U_0 > 0, x\in\overline{\Omega}.$$

At the end of this Section, we would like to point out that we have applied results on IP1 to analyze IP2. In a similar manner, results on IP1 can be applied to study inverse problems to determine other coefficients of L, too.

We mention that the restriction $g(0,\cdot)=0$ for the function g in IP1 as well as the related zero initial condition $U(0,\cdot)=0$ in IP2 result from strong smoothness assumption (2.23) of a positivity principle. This principle is one of bases of our theory.

Solutions of IP1 and IP2 depend continuously on derivatives of the data of finite order. This means that these problems are moderately ill-posed. In case approximate data are given with errors, regularization procedures can be effectively applied.

2.7 Inverse problem with an integral overdetermination condition

The results for problems with final overdetermination can be applied to study problems with integral overdetermination condition of the following form:

$$\int_0^T u(t,x)dt = \psi(x). \tag{2.69}$$

For example let us consider the inverse problem to determine the pair of functions (f,u) that satisfies (2.7), (2.12), (2.69).

Theorem 2.7. *Let $g\in C^{\alpha_0}([0,T];C(\overline{\Omega};\mathbb{R}))$ for some $0<\alpha_0<1$ and let one of the following assumptions be valid:*
$(\widehat{\mathbf{A}}\mathbf{1})$ $g\in C_0^{\alpha_0}([0,T];C(\overline{\Omega};\mathbb{R}))$;
$(\widehat{\mathbf{A}}\mathbf{2})$ $k*g \in C_0^{\alpha_1}([0,T];C(\overline{\Omega};\mathbb{R}))$ *and* $M(t)\geq ct^{\gamma-1}, t\in(0,T)$ *for some* $c\in\mathbb{R}_+$, $0<\gamma<\alpha_1<1$;
$(\widehat{\mathbf{A}}\mathbf{3})$ $k*g\in C_0^{\alpha_1-\beta}([0,T];C(\overline{\Omega};\mathbb{R}))$ *and* $c_1t^{\gamma-1}\leq M(t)\leq c_2t^{\beta-1}$, $|M'(t)|\leq c_3t^{\beta-2}$, $t\in(0,T)$, *for some* $c_1,c_2,c_3\in\mathbb{R}_+$, $0<\beta\leq\gamma<\alpha_1<1$.
Additionally, we assume that for $\widehat{g}(t,x) := \int_0^t g(\tau,x)d\tau, (t,x)\in[0,T]\times\overline{\Omega}$ *it holds*

$$\widehat{g}\geq 0;\quad {}^{R}D_0^{\{k\}}\widehat{g} - r_{max}\widehat{g} \geq 0,\quad r_{max} := \max_{x\in\overline{\Omega}} r(x);$$
$$\widehat{g}(T,x) \geq g_0 > 0, x\in\overline{\Omega}.$$

If $\varphi, \psi \in X_{A_p}$ and $h \in C^{\alpha_2}([0,T];X_p)$, where $p \in \{0\} \bigcup (1,\infty)$, $0 < \alpha_2 < 1$, then the inverse problem (2.7), (2.12), (2.69) *has a unique solution*

$$(f,u) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$$

and the following estimate holds:

$$\|f\|_{X_p} + \|u\|_{C^{\{k\}}([0,T];X_p) \cap C([0,T];X_{A_p})} \leq C_{15} \left(\|\varphi\|_{X_{A_p}} + \|\psi\|_{X_{A_p}} + \|h\|_{C^{\alpha_2}([0,T];X_p)} \right). \quad (2.70)$$

The constant C_{15} depends on the parameters M, L, g, p, α_2.

Proof. Firstly, let us denote $\widehat{h}(t,x) = 1 * h(t,x) + tL\varphi(x)$ and consider the IP1 for $(f, \widehat{u})$:

$$\begin{aligned} &{}^R D_0^{\{k\}} \widehat{u}(t,x) = L\widehat{u}(t,x) + \widehat{g}(t,x) f(x) + \widehat{h}(t,x), \quad x \in \Omega, t \in (0,T), \\ &\widehat{u}(0,x) = 0, \quad x \in \Omega, \\ &\mathscr{B}\widehat{u}(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T), \end{aligned} \quad (2.71)$$

with a final condition

$$\widehat{u}(T,x) = \psi(x) - T\varphi(x). \quad (2.72)$$

By applying Theorem 2.2 we obtain the existence of a unique solution $(f, \widehat{u}) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ to the problem (2.71), (2.72) and the estimation

$$\begin{aligned} \|f\|_{X_p} &\leq C_9 \left(\|\psi - T\varphi\|_{X_{A_p}} + \|\widehat{h}\|_{C^{\alpha_2}([0,T];X_p)} \right) \\ &\leq C_{16} \left(\|\varphi\|_{X_{A_p}} + \|\psi\|_{X_{A_p}} + \|h\|_{C^{\alpha_2}([0,T];X_p)} \right), \end{aligned} \quad (2.73)$$

where C_{16} is a constant.

Next we consider the direct problem (2.7) such that F is given by (2.12) and f is a first component of a solution to IP1 (2.71), (2.72). By Lemma 2.2 (iii) there is a solution to (2.7) $u \in C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ that satisfies the estimate:

$$\|u\|_{C^{\{k\}}([0,T];X_p) \cap C([0,T];X_{A_p})} \leq C_4(\|gf\|_{C^{\alpha_0}([0,T];X_p)} + \|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_{A_p}}). \quad (2.74)$$

Finally, we integrate the equation and boundary condition (2.7) with respect to time from 0 to t. Then we perform the substitution

$$\widehat{u}(x,t) = \int_0^t (u(x,\tau) - \varphi(x)) d\tau, \quad (x,t) \in \Omega \times [0,T] \quad (2.75)$$

in the obtained problem. These transformations result in problem (2.71). The substitution (2.75) in the condition (2.72) results into (2.69). Therefore, (f,u) solves the inverse problem for (2.7), (2.12), (2.69) and the estimates (2.73) and (2.74) imply that (2.70) is valid. □

2.8 Appendix: details of proof of Lemma 2.4

To help the reader, we present the treatment of a problem for $\widetilde{\mathrm{u}} = e^{-\sigma t}\mathrm{u}$ and derivation of the inequalities (2.25), (2.26) in Lemma 2.4. This repeats the material of the paper [36] with some little modifications.

Substituting $\mathrm{u} = e^{\sigma t}\widetilde{\mathrm{u}}$ to the problem for u and performing some transformations we obtain

$$
\begin{aligned}
&{}^{R}D_0^{\{\widetilde{K}\}}(\widetilde{\mathrm{u}}-\phi)(t,x) = L_1\widetilde{\mathrm{u}}(t,x) + \widetilde{r}(x)\widetilde{\mathrm{u}}(t,x) + \widetilde{\mathrm{F}}(t,x),\ t\in(0,T),\ x\in\Omega,\\
&\widetilde{\mathrm{u}}(0,x) = \phi,\ x\in\Omega,
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde{K}(t) = e^{-\sigma t}K(t) - \sigma\int_t^T e^{-\sigma s}K(s)ds, \quad \widetilde{r}(x) = r(x) - \sigma\int_0^T e^{-\sigma s}K(s)ds,\\
&\widetilde{\mathrm{F}}(t,x) = e^{-\sigma t}\mathrm{F}(t,x) + \phi(x)\sigma\int_t^T e^{-\sigma s}K(s)ds.
\end{aligned}
$$

It is easy to see that $\widetilde{\mathrm{u}}$ and $\widetilde{\mathrm{F}}$ satisfy the assumptions of Lemma 2.4. Let us verify that $\widetilde{K}$ meets the conditions of that lemma, as well. Integrating by parts we obtain

$$
\widetilde{K}(t) = e^{-\sigma T}K(T) - \int_t^T e^{-\sigma s}K'(s)ds.
$$

The monotonicity properties of K imply $K'\leq 0$, $K'(t)<0$ a.e. $t\in(0,t_K)$. Therefore, $\widetilde{K}>0$ and for any $t_1,t_2\in(0,T)$, $t_1<t_2$, it holds $\widetilde{K}(t_2)-\widetilde{K}(t_1) = \int_{t_1}^{t_2} e^{-\sigma s}K'(s)ds \leq 0$. Moreover, in the particular case $t_1,t_2\in(0,t_K)$, we have $\widetilde{K}(t_2)-\widetilde{K}(t_1)<0$. Thus, $\widetilde{K}(t)$ is nonincreasing in $(0,T)$ and strictly decreasing in $(0,t_K)$.

Next, we prove (2.25). Let us represent the derivative ${}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1)$ by means of the limit:

$$
{}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1) = \frac{d}{dt}\int_0^t K(t-\tau)[\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau\Big|_{t=t_1} = \lim_{\varepsilon\to0+}\frac{1}{\varepsilon}I_\varepsilon,
$$

where

$$
\begin{aligned}
I_\varepsilon &= \int_0^{t_1} K(t_1-\tau)[\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau - \int_0^{t_1-\varepsilon} K(t_1-\varepsilon-\tau)[\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau\\
&= \int_0^{t_1-\varepsilon}[K(t_1-\tau)-K(t_1-\varepsilon-\tau)][\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau\\
&+ \int_{t_1-\varepsilon}^{t_1} K(t_1-\tau)[\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau.
\end{aligned}
$$

Since $\mathrm{u}(\tau,x_1)\geq\mathrm{u}(t_1,x_1)$, $\tau\in(0,t_1-\varepsilon)$, and K is noninceasing, we have

$$
[K(t_1-\tau)-K(t_1-\varepsilon-\tau)]\mathrm{u}(\tau,x_1) \leq [K(t_1-\tau)-K(t_1-\varepsilon-\tau)]\mathrm{u}(t_1,x_1), \quad \tau\in(0,t_1-\varepsilon).
$$

Therefore, we can estimate as follows:

$$
\begin{aligned}
I_\varepsilon &\leq \int_0^{t_1-\varepsilon}[K(t_1-\tau)-K(t_1-\varepsilon-\tau)]d\tau\,[\mathrm{u}(t_1,x_1)-\phi(x_1)]\\
&+ \int_{t_1-\varepsilon}^{t_1} K(t_1-\tau)[\mathrm{u}(\tau,x_1)-\phi(x_1)]d\tau\\
&= \int_{t_1-\varepsilon}^{t_1} K(\tau)d\tau\,[\mathrm{u}(t_1,x_1)-\phi(x_1)] + \int_0^\varepsilon K(\tau)[\mathrm{u}(t_1-\tau,x_1)-\mathrm{u}(t_1,x_1)]d\tau.
\end{aligned}
$$

Thus

$$ {}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1) \le \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int_{t_1-\varepsilon}^{t_1} K(\tau)d\tau\,[\mathrm{u}(t_1,x_1)-\phi(x_1)] $$
$$ +\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int_0^{\varepsilon} K(\tau)[\mathrm{u}(t_1-\tau,x_1)-\mathrm{u}(t_1,x_1)]d\tau. $$

By (2.23), the second addend at the right-hand side equals zero and due to the continuity of K, it holds $\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int_{t_1-\varepsilon}^{t_1} K(\tau)d\tau = K(t_1)$. Therefore,

$$ {}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(t_1,x_1) \le K(t_1)[\mathrm{u}(t_1,x_1)-\phi(x_1)]. $$

Since $K>0$ and $\mathrm{u}(t_1,x_1)<0\le\phi(x_1)$, we obtain (2.25).

Finally, we prove (2.26). We have

$$ {}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(\hat{t}_0,x_0) = \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}J_\varepsilon, $$

where

$$ J_\varepsilon = \int_0^{\hat{t}_0} K(\hat{t}_0-\tau)[\mathrm{u}(\tau,x_0)-\phi(x_0)]d\tau - \int_0^{\hat{t}_0-\varepsilon} K(\hat{t}_0-\varepsilon-\tau)[\mathrm{u}(\tau,x_0)-\phi(x_0)]d\tau $$
$$ = \int_0^{\hat{t}_0-\varepsilon}[K(\hat{t}_0-\tau)-K(\hat{t}_0-\varepsilon-\tau)]\mathrm{u}(\tau,x_0)d\tau $$
$$ + \int_{\hat{t}_0-\varepsilon}^{\hat{t}_0} K(\hat{t}_0-\tau)\mathrm{u}(\tau,x_0)d\tau - \int_{\hat{t}_0-\varepsilon}^{\hat{t}_0} K(\tau)d\tau\,\phi(x_0). $$

Let $\varepsilon<\min\{\hat{t}_0-t_3;t_2\}$. Since

$$ \mathrm{u}(\tau,x_0)\ge z(\tau):=\begin{cases} 0, & \tau\in(0,\hat{t}_0-\varepsilon)\setminus(t_2,t_3) \\ \delta, & \tau\in(t_2,t_3)\end{cases}, \quad \tau\in(0,\hat{t}_0-\varepsilon), $$

and K is nonincreasing we have for $\tau\in(0,\hat{t}_0-\varepsilon)$

$$ [K(\hat{t}_0-\tau)-K(\hat{t}_0-\varepsilon-\tau)]\mathrm{u}(\tau,x_0) \le [K(\hat{t}_0-\tau)-K(\hat{t}_0-\varepsilon-\tau)]z(\tau). $$

Moreover, $K>0$ and $\phi(x_0)\ge 0$. Consequently,

$$ J_\varepsilon \le \delta\int_{t_2}^{t_3}[K(\hat{t}_0-\tau)-K(\hat{t}_0-\varepsilon-\tau)]d\tau + \int_{\hat{t}_0-\varepsilon}^{\hat{t}_0} K(\hat{t}_0-\tau)\mathrm{u}(\tau,x_0)d\tau $$
$$ = \delta\left[\int_{\hat{t}_0-t_3}^{\hat{t}_0-t_2} K(\tau)d\tau - \int_{\hat{t}_0-t_3-\varepsilon}^{\hat{t}_0-t_2-\varepsilon} K(\tau)d\tau\right] + \int_0^{\varepsilon} K(\tau)\mathrm{u}(\hat{t}_0-\tau,x_0)d\tau. $$

Due to (2.23) and $\mathrm{u}(\hat{t}_0,x_0)=0$ we have $\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\int_0^{\varepsilon} K(\tau)\mathrm{u}(\hat{t}_0-\tau,x_0)d\tau=0$. Therefore

$$ {}^{R}D_0^{\{K\}}(\mathrm{u}-\phi)(\hat{t}_0,x_0) \le \lim_{\varepsilon\to 0+}\frac{\delta}{\varepsilon}\left[\int_{\hat{t}_0-t_3}^{\hat{t}_0-t_2} K(\tau)d\tau - \int_{\hat{t}_0-t_3-\varepsilon}^{\hat{t}_0-t_2-\varepsilon} K(\tau)d\tau\right] $$
$$ = \delta\frac{d}{ds}\int_{\hat{t}_0-t_3+s}^{\hat{t}_0-t_2+s} K(\tau)d\tau\Big|_{s=0} = \delta[K(\hat{t}_0-t_2)-K(\hat{t}_0-t_3)]. $$

We obtain (2.26).

3 Inverse problems for a perturbed time fractional diffusion equation in Lebesgue spaces

This Chapter comprises results of Publication I with some modifications. As in the previous Chapter, we consider complex-valued scalar spaces by default, if the space is real-valued then we additionally specify it.

3.1 Perturbed time fractional diffusion equation

Let us consider the generalized subdiffusion equation (1.3) with the operator

$$L = L(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + r(x)I.$$

We assume that (see [33])

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + m * \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad 0 < t < T, \tag{3.1}$$

where $m \in L_1((0,T);\mathbb{R})$ and $0 < \beta < 1$.

Plugging (3.1) into (1.3) we arrive at the equation

$$u_t = {}^{R}D_0^{1-\beta}(Lu + m * Lu) + Q. \tag{3.2}$$

The kernel M is Sonine and the equation (3.2) can be reduced to the form (1.4), but in this Chapter we use a different method to integrate (3.2).

Applying the operator of fractional integration $I_0^{1-\beta} = \frac{t^{-\beta}}{\Gamma(1-\beta)} *$ of the order $1-\beta$ to (3.2), we obtain the **perturbed time-fractional diffusion equation**

$${}^{C}D_0^{\beta} u = Lu + m * Lu + F, \tag{3.3}$$

where $F = I_0^{1-\beta} Q$.

The equation (3.3) can also be obtained by means as an extension of the parabolic integro-differential equation $u_t = Lu + m * Lu + F$ that describes hereditary heat processes [2, 42] to the fractional case and is referred to as the fractional diffusion equation with memory [51].

Let us consider some examples of m based on the kernels from Section 1.7.

Firstly, we consider the case of kernel (M2). Without loss of generality we take $q_1 = 1$ and redenote the principal term $\beta = \beta_1$. Based on the formula

$$\frac{t^a}{\Gamma(a+1)} * \frac{t^b}{\Gamma(b+1)} = \frac{t^{a+b+1}}{\Gamma(a+b+2)}, \quad a, b > -1 \tag{3.4}$$

that can be verified by means of Laplace transform, we deduce that in case (M2) the relation (3.1) is valid with the function

$$m(t) = \sum_{j=2}^{N} q_j \frac{t^{\beta_j - \beta - 1}}{\Gamma(\beta_j - \beta)}. \tag{3.5}$$

Next we consider the case (k2). We would like to represent the associate to k kernel M in the form (3.1). Again we assume that $q_1 = 1$ and redenote $\beta = \beta_1$. That allows us to represent k in the form

$$k = \frac{t^{-\beta}}{\Gamma(1-\beta)} + l * \frac{t^{-\beta}}{\Gamma(1-\beta)}$$

such that:

$$l(t) = \sum_{j=2}^{N} q_j \frac{t^{\beta-\beta_j-1}}{\Gamma(\beta-\beta_j)}, \quad 0 < t < T.$$

We consider the Volterra equation of the second kind with respect to m

$$l + l * m + m = 0. \tag{3.6}$$

It has a unique solution in $L_1(0,T);\mathbb{R})$ [22]. Since $\frac{t^{-\beta}}{\Gamma(1-\beta)} * \frac{t^{\beta-1}}{\Gamma(\beta)} = 1$ the equation (3.6) is equivalent to

$$\left(\frac{t^{-\beta}}{\Gamma(1-\beta)} + l * \frac{t^{-\beta}}{\Gamma(1-\beta)}\right) * \left(\frac{t^{\beta-1}}{\Gamma(\beta)} + m * \frac{t^{\beta-1}}{\Gamma(\beta)}\right) = 1.$$

This implies the relation $k * M = 1$ for M of the form (3.1).

Further, if M satisfies the conditions $M - \frac{t^{\beta-1}}{\Gamma(\beta)} \in W_1^1(0,T)$, $\lim\limits_{t\to 0^+} (M(t) - \frac{t^{\beta-1}}{\Gamma(\beta)}) = 0$ then the formula (3.1) is valid with the function

$$m(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} * \left(M(t) - \frac{t^{\beta-1}}{\Gamma(\beta)}\right)'.$$

This observation is useful in cases (M4) and (M5). We obtain

$$m(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} * \left[\left(e^{-\lambda t} - 1\right)\frac{t^{\beta-2}}{\Gamma(\beta-1)}\right] \quad \text{in case (M4)}, \tag{3.7}$$

$$m(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} * \left[\left(e^{-\lambda t} - 1\right)\frac{t^{\beta-2}}{\Gamma(\beta-1)} - \lambda e^{-\lambda t}\frac{t^{\beta-1}}{\Gamma(\beta)}\right] \quad \text{in case (M5)}. \tag{3.8}$$

To handle the kernel (M6) we use the definition of $E_{\beta,\beta}$ (1.18) and express M as

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{j=1}^{J} \frac{\lambda^{\beta j} t^{\beta j+\beta-1}}{\Gamma(\beta j+\beta)} + w(t),$$
$$w(t) = \sum_{j=J+1}^{\infty} \frac{\lambda^{\beta j} t^{\beta j+\beta-1}}{\Gamma(\beta j+\beta)} + t^{\beta-1}(e^{-\lambda t} - 1)E_{\beta,\beta}(\lambda^{\beta} t^{\beta}),$$

where J is chosen so that $\beta J + \beta - 1 \le 0$ and $\beta(J+1) + \beta - 1 > 0$. Then $w \in W_1^1(0,T)$, $\lim\limits_{t\to 0^+} w(t) = 0$. The kernel m in (3.1) is a sum of two kernels $m = m_1 + m_2$ where

$$\sum_{j=1}^{J} \frac{\lambda^{\beta j} t^{\beta j+\beta-1}}{\Gamma(\beta j+\beta)} = \frac{t^{\beta-1}}{\Gamma(\beta)} * m_1(t), \quad w(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} * m_2(t).$$

The addend $m_1(t)$ can be expressed as in case (M2). We obtain

$$m_1(t) = \sum_{j=1}^{J} \frac{\lambda^{\beta j} t^{\beta j-1}}{\Gamma(\beta j)} \quad \text{and} \quad m_2(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} * w'(t).$$

The cases (M3) and (k3) are not the particular cases of (3.1), so they are not covered by this Chapter.

But $M(t)$ in the following special form of the Lebesgue-Stiltjes integral

$$M(t) = \int_0^1 \frac{t^{\beta-1}}{\Gamma(\beta)} d\mu(\beta) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \int_\beta^1 a(s) \frac{t^{s-1}}{\Gamma(s)} ds$$

is still representable as (3.1) where $m(t) = \int_\beta^1 a(s) \frac{t^{s-\beta-1}}{\Gamma(s-\beta)} ds$.

3.2 Formulation of direct and inverse problems

For the sake of generality, let us transform the Caputo derivative ${}^C D_0^\beta u$ contained in (3.3) to the form ${}^R D_0^\beta (u - u(0,x))$ that does not contain the first order derivative of u. We obtain the following equation: ${}^R D_0^\beta (u - u(0,x)) = Lu + m * Lu + F$.

Now we are going to formulate problems to be treated in the present Chapter. Let $\Omega \in \mathbb{R}^n$ be an n-dimensional open bounded domain. Firstly, we formulate a *direct problem* for the function u:

$$
\begin{aligned}
&{}^R D_0^\beta (u - \varphi)(t,x) = Lu(t,x) + (m * Lu)(t,x) + F(t,x), \quad x \in \Omega, t \in (0,T), && (3.9)\\
&u(0,x) = \varphi(x), \quad x \in \Omega, && (3.10)\\
&\mathscr{B}u(t,x) = 0, \quad x \in \partial\Omega, t \in (0,T). && (3.11)
\end{aligned}
$$

Here F and φ are given functions and $\mathscr{B}$ is a boundary operator:

$$\mathscr{B}v(x) = v(x) \quad \text{or} \quad \mathscr{B}v(x) = \vartheta_L(x) \cdot \nabla v(x) + \theta v(x),$$

where ϑ_L is the conormal vector. i.e.

$$\vartheta_L(x) = (\vartheta_{L,1}(x), \ldots, \vartheta_{L,n}(x)), \quad \vartheta_{L,i}(x) = \sum_{j=1}^n a_{ij}(x) \vartheta_j(x)$$

and $\vartheta(x) = (\vartheta_1(x), \ldots, \vartheta_n(x))$ is the outer normal of $\partial\Omega$ at $x \in \partial\Omega$.

A problem with non-homogeneous boundary conditions can be transformed to the problem (3.9) - (3.11) by means of a standard change of variables as in Section 2.1.

Next we formulate two *inverse problems* that use the final overdetermination condition

$$u(T,x) = \psi(x), \quad x \in \Omega, \tag{3.12}$$

with a given observation function ψ.

Firstly, we pose an inverse source problem. Let F have the form

$$F(t,x) = g(t) f(x) + h(t,x), \tag{3.13}$$

where g and h are given functions. The aim is to reconstruct the factor f.

IP1. Find a pair of functions (f,u) that satisfies (3.9) - (3.13).

Secondly, we formulate a bacward in time problem.

IP2. Find a function u that satisfies (3.9), (3.11) and (3.12).

3.3 Preliminaries and basic assumptions

Let us introduce basic conditions on the data that will be assumed throughout the Chapter.

Regarding the kernel m we assume that $m \in L_1((0,T);\mathbb{R})$.

The assumptions on the domain Ω and the operator L are as follows:

$$\partial\Omega \text{ is of class } C^2, \quad a_{ij} \in C^1(\overline{\Omega};\mathbb{R}), \quad r \in C(\overline{\Omega};\mathbb{R}),$$
$$a_{ij} = a_{ji}, \ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \ \forall \xi \in \mathbb{R}^n, x \in \Omega \text{ for some } c > 0$$
$$r \leq 0, \theta \geq 0 \quad \text{and either } \exists r_0 > 0 \,:\, -r(x) \geq r_0 \ \forall x \in \Omega \text{ or } \theta > 0.$$

Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ stand for the norm and the inner product in the space $L_2(\Omega)$, respectively.

We present some important features of the operator L that follow from the listed basic assumptions. The operator $-L$ with the domain $\mathscr{D}(-L) = \mathscr{D}(L) = \{z \in W_2^2(\Omega) : \mathscr{B}z = 0 \text{ in } \partial\Omega\}$ is a bijection from $\mathscr{D}(-L)$ to $L_2(\Omega)$ and its inverse is compact, self-adjoint and positive definite (Theorems 3.8 and 3.10 in [16]). Therefore, its eigenvalues and orthonormed in $L_2(\Omega)$ eigenfunctions $\{\lambda_k, v_k\}_{k\in\mathbb{N}}$ are such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots, \lambda_k \to \infty$, and v_k, $k \in \mathbb{N}$, form a basis in the space $L_2(\Omega)$. Moreover,

$$\|z\|_{\mathscr{D}(-L)} = [\sum_{k=1}^{+\infty} \lambda_k^2 |\langle z, v_k\rangle|^2]^{\frac{1}{2}}$$

is a an equivalent norm in the space $\mathscr{D}(-L)$.

We introduce fractional powers of $-L$ and related domains. The operator $(-L)^\zeta$, $\zeta \geq 0$, can be defined by the relation $(-L)^\zeta z = \sum_{k=1}^{+\infty} \lambda_k^\zeta \langle z, v_k\rangle v_k$ and has the domain

$$\mathscr{D}((-L)^\zeta) = \left\{ z \in L_2(\Omega) \,:\, \|z\|_{\mathscr{D}((-L)^\zeta)} := \left[\sum_{k=1}^{+\infty} \lambda_k^{2\zeta} |\langle z, v_k\rangle|^2\right]^{\frac{1}{2}} < \infty \right\}$$

in the space $L_2(\Omega)$ [78]. Evidently, $\mathscr{D}((-L)^0) = L_2(\Omega)$. Moreover, we have the continuous embedding

$$\mathscr{D}((-L)^{\zeta_1}) \hookrightarrow \mathscr{D}((-L)^{\zeta_2}), \quad \zeta_1 > \zeta_2.$$

We formulate a lemma that follows from Corollary 2.8.1 and discussions in p.29 of [97].

Lemma 3.1. *Let X be a complex Hilbert space. Let $\beta \in (0,1)$, $p \in (1,\infty)$. The operator of fractional integration of the order β, i.e. $I_0^\beta = \frac{t^{\beta-1}}{\Gamma(\beta)} *$, is a bijection from $L_p((0,T);X)$ onto ${}_0H_p^\beta((0,T);X)$, the inverse of I_0^β is the Riemann-Liouville fractional derivative ${}^R D_0^\beta = \frac{d}{dt} I_0^{1-\beta}$ and*

$$\|w\|_{{}_0H_p^\beta((0,T);X)} = \|{}^R D_0^\beta w\|_{L_p((0,T);X)}$$

is a norm in the space ${}_0H_p^\beta((0,T);X)$. Moreover, in case $p \in (\frac{1}{\beta},\infty)$ it holds $H_p^\beta((0,T);X) \hookrightarrow C([0,T];X)$ and $w(0) = 0$ for $w \in {}_0H_p^\beta((0,T);X)$.

In treatment of convolutional terms we will apply **norms with exponential weights**. Let us define these norms in the spaces of scalar functions $L_p(0,T)$, $p \in [1,\infty]$:

$$\|w\|_{p;\sigma} = \|e^{-\sigma t} w\|_{L_p(0,T)}, \quad \text{where } \sigma \geqslant 0.$$

If $\sigma = 0$ then $\|\cdot\|_{p;\sigma}$ becomes the usual norm in $L_p(0,T)$ and we denote it by $\|\cdot\|_p$. The following equivalence relations are valid:

$$e^{-\sigma T}\|w\|_p \leqslant \|w\|_{p;\sigma} \leqslant \|w\|_p. \tag{3.14}$$

Note that the weight can be easily brought into the convolution, i.e.

$$e^{-\sigma t} m * w = (e^{-\sigma t} m) * (e^{-\sigma t} w)$$

and the Young's inequality (1.9) extended to the weighted norms:

$$\|m * w\|_{s;\sigma} \leqslant \|m\|_{q;\sigma}\|w\|_{p;\sigma}, \quad 1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{q}. \tag{3.15}$$

Finally, in case $p < \infty$, $\|w\|_{p;\sigma} \to 0$ as $\sigma \to \infty$.

Next we provide some extra properties of Mittag-Leffler functions required for this Chapter.

It follows from the complete monotonicity of $E_\beta(-z)$ and $E_\beta(0) = 1$ that

$$0 < E_\beta(-z) \leqslant 1, \quad z \geqslant 0. \tag{3.16}$$

Since (1.20) and (3.16) hold, there exist $C_{17}, C_{18} > 0$ such that

$$\frac{C_{17}}{1+z} \leqslant E_\beta(-z) \leqslant \frac{C_{18}}{1+z} \leqslant \frac{C_{18}}{z} \quad \text{for} \quad z \geqslant 0. \tag{3.17}$$

In addition to the Mittag-Leffler functions, we introduce the **α-exponential function** [8]:

$$e_\alpha^{\lambda t} = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha), \quad \alpha > 0. \tag{3.18}$$

The relations (1.19) and (3.18) yield the following useful formula:

$$\int_0^t \lambda e_\beta^{-\lambda\tau} d\tau = 1 - E_\beta(-\lambda t^\beta). \tag{3.19}$$

Moreover, the formula (3.19) in view of the relations (3.16) implies

$$\|\lambda_k e_\beta^{-\lambda_k t}\|_1 \leqslant 1. \tag{3.20}$$

Let us prove a technical lemma. It will be applied in proofs of Theorem 3.2 (ii), Theorem 3.3 and Lemma 3.4.

Lemma 3.2. *There exists a constant $C_{19} > 0$ such that*

$$(\lambda e_\beta^{-\lambda t} *)^i E_\beta(-\lambda t^\beta) \leqslant \frac{C_{18} C_{19}^i}{\lambda t^\beta}, \quad t > 0,\ \lambda > 0,\ i \in \mathbb{N}. \tag{3.21}$$

Proof. The convolution formula of Mittag-Leffler functions (see [25], (11.12)) implies

$$\lambda e_\beta^{-\lambda t} * E_\beta(-\lambda t^\beta) = \lambda t^\beta E^2_{\beta,\beta+1}(-\lambda t^\beta) = \frac{\lambda t^\beta}{\beta} E_{\beta,\beta}(-\lambda t^\beta). \tag{3.22}$$

Here $E^\gamma_{\alpha,\beta}$ is the three-parametric Mittag-Leffler function and we used the formula $E^2_{\beta,\beta+1}(z) = \frac{1}{\beta} E_{\beta,\beta}(z)$ ([25], (11.4)), too. The asymptotic relations (1.20), (1.21) and $\Gamma(1-\beta) = (-\beta)\Gamma(-\beta)$ yield

$$z^2 E_{\beta,\beta}(-z) = \beta z E_\beta(-z) + O(z^{-1}) \quad \text{as} \quad z \in \mathbb{R}, z \to \infty,$$

where $\beta z E_\beta(-z)$ is the dominating term at the right hand side. Thus, there exists $z_0 > 0$ such that $\frac{z}{\beta} E_{\beta,\beta}(-z) \leqslant 2E_\beta(-z)$ for $z > z_0$. On the other hand, since $zE_{\beta,\beta}(-z) \in C[0,z_0]$ and $E_\beta(-z)$ is positive and decreasing, we obtain $\frac{z}{\beta} E_{\beta,\beta}(-z) \leqslant C_{20} E_\beta(-z)$ for $0 \leqslant z \leqslant z_0$ where $C_{20} = \frac{\max\limits_{0\leqslant y\leqslant z_0} yE_{\beta,\beta}(-y)}{\beta E_\beta(-z_0)}$. Therefore, $\frac{z}{\beta} E_{\beta,\beta}(-z) \leqslant C_{19} E_\beta(-z)$ for any $z \geqslant 0$, where $C_{19} = \max\{2; C_{20}\}$ and from (3.22) we have

$$\lambda e_\beta^{-\lambda t} * E_\beta(-\lambda t^\beta) \leqslant C_{19} E_\beta(-\lambda t^\beta). \tag{3.23}$$

So we continue the iterations and obtain

$$(\lambda e_\beta^{-\lambda t} *)^i E_\beta(-\lambda t^\beta) \leqslant C_{19}^i E_\beta(-\lambda t^\beta).$$

Finally, estimating $E_\beta(-\lambda t^\beta)$ by means of (3.17), we reach (3.21). □

3.4 Direct problem

In the sequel we will search for the solution u of (3.9)-(3.11) from the following space:

$$\begin{aligned} &\mathscr{U}_{s,\beta} = \{u \in L_s((0,T);\mathscr{D}(-L)) \cap C([0,T];L_2(\Omega)) \,:\, u - u(0,\cdot) \in {}_0H_s^\beta((0,T);L_2(\Omega))\}, \\ &\|u\|_{\mathscr{U}_{s,\beta}} = \|u\|_{L_s((0,T);\mathscr{D}(-L))} + \|u\|_{C([0,T];L_2(\Omega))} + \|u - u(0,\cdot)\|_{{}_0H_s^\beta((0,T);L_2(\Omega))}, \end{aligned}$$

where $s \in (1,\infty)$.

For $s_1 > s_2$ the embedding holds $\mathscr{U}_{s_1,\beta} \hookrightarrow \mathscr{U}_{s_2,\beta}$.

Let us introduce a notation for the Fourier coefficients of data functions involved in the direct problem:

$$u_k(t) = \langle u(t,\cdot), v_k\rangle, \;\; F_k(t) = \langle F(t,\cdot), v_k\rangle, \;\; \varphi_k = \langle \varphi, v_k\rangle, \qquad k \in \mathbb{N}.$$

Proposition 3.1. *Let $F \in L_p((0,T);L_2(\Omega))$ with some $p \in (1,\infty)$ and $\varphi \in L_2(\Omega)$. Then the following assertions are valid.*

(i) *If $u \in \mathscr{U}_{s,\beta}$ with some $s \in (1,\infty)$ is a solution of the direct problem* (3.9)-(3.11)*, then the Fourier coefficients u_k, $k \in \mathbb{N}$, belong to*

$$\widetilde{\mathscr{U}_{s,\beta}} = \{w \in C[0,T] \,:\, w - w(0) \in {}_0H_s^\beta(0,T)\}$$

and are solutions of the following sequence of problems for $k \in \mathbb{N}$:

$$\begin{aligned} &{}^R D_0^\beta (u_k - \varphi_k)(t) + \lambda_k u_k(t) + \lambda_k (m * u_k)(t) = F_k(t), \;\; t \in (0,T), && (3.24)\\ &u_k(0) = \varphi_k. && (3.25) \end{aligned}$$

(ii) *If* (3.24), (3.25) *have solutions $u_k \in \widetilde{\mathscr{U}_{s,\beta}}$, $k \in \mathbb{N}$, with some $s \in (1,\infty)$ such that $u = \sum\limits_{k=1}^{+\infty} u_k v_k \in \mathscr{U}_{s,\beta}$, then u is a solution of the direct problem* (3.9)-(3.11).

Proof. (i) Let $u \in \mathscr{U}_{s,\beta}$ with some $s \in (1,\infty)$ solve (3.9)-(3.11). Since $u - u(0,\cdot) = u - \varphi \in {}_0H_s^\beta((0,T);L_2(\Omega))$, by Lemma 3.1 there exists $\widetilde{u} \in L_s((0,T);L_2(\Omega))$ such that $u - \varphi = I_0^\beta \widetilde{u}$ and $\widetilde{u} = {}^R D_0^\beta (u - \varphi)$.

Let us denote $\widetilde{u}_k(t) = \langle \widetilde{u}(t,\cdot), v_k \rangle$. Due to $\widetilde{u} \in L_s((0,T);L_2(\Omega))$, we have $\widetilde{u}_k \in L_s(0,T)$. On the other hand, $u_k - \varphi_k = \langle u - \varphi, v_k \rangle = \langle I_0^\beta \widetilde{u}, v_k \rangle = I_0^\beta \langle \widetilde{u}, v_k \rangle = I_0^\beta \widetilde{u}_k$. This relation with Lemma 3.1 implies

$$u_k - \varphi_k \in {}_0H_s^\beta(0,T) \text{ and } {}^R D_0^\beta (u_k - \varphi_k) = \widetilde{u}_k.$$

Further, from $u \in \mathscr{U}_{s,\beta} \subset C([0,T];L_2(\Omega))$ we immediately have $u_k \in C[0,T]$. Moreover, taking the inner product of the initial condition $u(0,\cdot) = \varphi$ with v_k, we deduce (3.25). The relation $u_k - \varphi_k \in {}_0H_s^\beta(0,T)$ with (3.25) and $u_k \in C[0,T]$ proves that $u_k \in \widetilde{\mathscr{U}_{s,\beta}}$. The deduced equalities $\widetilde{u} = {}^R D_0^\beta (u - \varphi)$ and ${}^R D_0^\beta (u_k - \varphi_k) = \widetilde{u}_k$ imply

$$\langle {}^R D_0^\beta (u - \varphi), v_k \rangle = {}^R D_0^\beta (u_k - \varphi_k).$$

Moreover, $\langle -Lu, v_k \rangle = \langle u, -Lv_k \rangle = \lambda_k \langle u, v_k \rangle = \lambda_k u_k$. Consequently, taking the inner product of the equation ${}^R D_0^\beta (u - \varphi) - Lu - m * Lu = F$ with v_k, we obtain the equation (3.24).
(ii) Let the assumptions of (ii) hold for u_k. Denote

$$R = {}^R D_0^\beta (u - \varphi - \rho) - Lu - m * Lu - F, \qquad \rho = u(0,\cdot) - \varphi.$$

Then $u \in \mathscr{U}_{s,\beta}$ solves the problem

$$\begin{aligned} &{}^R D_0^\beta (u - \widetilde{\varphi}) - Lu - m * Lu = \widetilde{F}, \\ &u(0,\cdot) = \widetilde{\varphi}, \end{aligned}$$

where $\widetilde{F} = F + R$ and $\widetilde{\varphi} = \varphi + \rho$.

Applying the proved statement (i) to this problem, we see that u_k, $k \in \mathbb{N}$, solve the problems

$$\begin{aligned} &{}^R D_0^\beta (u_k - \widetilde{\varphi}_k) + \lambda_k u_k + \lambda_k (m * u_k) = \widetilde{F}_k, \\ &u_k(0) = \widetilde{\varphi}_k, \end{aligned}$$

where $\widetilde{F}_k = F_k + \langle R, v_k \rangle$ and $\widetilde{\varphi}_k = \varphi_k + \langle \rho, v_k \rangle$. Comparing these problems with (3.24), (3.25), we see that $\langle R, v_k \rangle = 0$, $\langle \rho, v_k \rangle = 0$, $k \in \mathbb{N}$. This implies $R = 0$, $\rho = 0$. Consequently, u is a solution of (3.9)-(3.11). □

Theorem 3.1. *Let $k \in \mathbb{N}$. Then the following statements hold.*

(i) (uniqueness) *If $F_k = 0$, $\varphi_k = 0$ and $u_k \in \widetilde{\mathscr{U}_{s,\beta}}$ with some $s \in (1,\infty)$ solves* (3.24), (3.25) *then $u_k = 0$.*

(ii) *If $F_k \in L_p(0,T)$ with some $p \in (\frac{1}{\beta},\infty)$ then the problem* (3.24), (3.25) *has a solution u_k in the space $\widetilde{\mathscr{U}_{p,\beta}}$. This solution is represented by the uniformly in $[0,T]$ converging series*

$$u_k(t) = \varphi_k \sum_{i=0}^{+\infty} (M_k *)^i E_\beta(-\lambda_k t^\beta) + \sum_{i=0}^{+\infty} (M_k *)^i \int_0^t e_\beta^{-\lambda_k (t-\tau)} F_k(\tau) d\tau, \tag{3.26}$$

where $$M_k(t) = -\lambda_k \int_0^t e_\beta^{-\lambda_k (t-\tau)} m(\tau) d\tau. \tag{3.27}$$

Proof. (i) Let $F_k = 0$, $\varphi_k = 0$ and $u_k \in \widetilde{\mathscr{U}_{s,\beta}}$ with some $s \in (1,\infty)$ solve (3.24), (3.25). Since $u_k(0) = \varphi_k = 0$, we have $u_k \in {}_0H_s^{\beta}(0,T)$.

Denoting $y_k = {}^R D_0^{\beta} u_k$, we obtain $u_k = I_0^{\beta} y_k$ and $y_k \in L_s(0,T)$, by Lemma 3.1. Moreover, from the equation for u_k we deduce the homogeneous Volterra equation of the second kind

$$y_k + K_k * y_k = 0, \quad \text{with} \quad K_k = \lambda_k \frac{t^{\beta-1}}{\Gamma(\beta)} + \lambda_k m * \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t \in (0,T).$$

Such an equation has only the trivial solution. Consequently, $y_k = 0$ and $u_k = 0$.

(ii) Assume $F_k \in L_p(0,T)$ for some $p \in (\frac{1}{\beta},\infty)$. Let us consider the Volterra equation of the second kind

$$y_k + K_k * y_k = R_k, \quad \text{where} \quad R_k = F_k - \lambda_k \varphi_k - \lambda_k m * \varphi_k \in L_p(0,T).$$

This equation has a solution $y_k \in L_p(0,T)$ ([22], Sect. 2.3).

Let us define $u_k = I_0^{\beta} y_k + \varphi_k$. By Lemma 3.1, $u_k - \varphi_k \in {}_0H_p^{\beta}(0,T)$ and $y_k = {}^R D_0^{\beta}(u_k - \varphi_k)$. From the equation of y_k we deduce the equation (3.24) for u_k. Since $p \in (\frac{1}{\beta},\infty)$ we obtain $u_k - \varphi_k \in C[0,T]$ and $u_k(0) - \varphi_k = 0$. This implies (3.25) and $u_k \in \widetilde{\mathscr{U}_{p,\beta}}$. The existence assertion of (ii) is proved.

Finally, let us deduce the formula (3.26) with (3.27). To this end, we need a solution formula of the fractional differential equation ${}^R D_0^{\beta} w + \lambda w = z$. It can be found e.g. in [81], Example 42.2. Provided $z \in L_p(0,T)$, the solution $w \in {}_0H_p^{\beta}(0,T)$ of this equation is $w = e_{\beta}^{-\lambda t} * z$. After rewriting (3.24) in the form of the equation

$${}^R D_0^{\beta} w_k + \lambda_k w_k = z_k, \quad \text{where} \quad w_k = u_k - \varphi_k \text{ and } z_k = F_k - \lambda_k \varphi_k - \lambda_k m * u_k$$

and applying the mentioned solution formula to it we obtain $u_k = e_{\beta}^{-\lambda t} * z_k + \varphi_k$. Using (3.19), (3.27) we transform the latter relation to the Volterra equation

$$u_k(t) = Q_k(t) + M_k * u_k(t), \quad t \in (0,T), \qquad (3.28)$$
$$\text{with } Q_k = \varphi_k E_{\beta}(-\lambda_k t^{\beta}) + e_{\beta}^{-\lambda_k t} * F_k \in C[0,T].$$

Next let us show that $M_k* \in \mathscr{L}(C[0,T])$. Since $m \in L_1(0,T)$, it holds that $M_k \in L_1(0,T)$, hence for any $w \in C[0,T]$ we have $M_k * w \in C[0,T]$. Due to (3.15) and (3.20) we obtain

$$\|M_k\|_{1;\sigma} \leqslant \|m\|_{1;\sigma} \|\lambda_k e_{\beta}^{-\lambda_k t}\|_{1;\sigma} \leqslant \|m\|_{1;\sigma} \|\lambda_k e_{\beta}^{-\lambda_k t}\|_1 \leqslant \|m\|_{1;\sigma}. \qquad (3.29)$$

For any $w \in C[0,T]$, we have $\|M_k * w\|_{\infty;\sigma} \leqslant \|M_k\|_{1;\sigma} \|w\|_{\infty;\sigma} \leqslant \|m\|_{1;\sigma} \|w\|_{\infty,\sigma}$. Consequently, $M_k* \in \mathscr{L}(C[0,T])$. Moreover, there exists sufficiently large σ such that

$$\|M_k *\|_{\mathscr{L}(C[0,T])} \leqslant \|m\|_{1;\sigma} < 1.$$

Thus, applying the theorem about the continuously inverse operator (see [92], p.140), we express the solution of (3.28) by means of the uniformly convergent Neumann series (3.26). ☐

Theorem 3.2. (i) (uniqueness) *If $F = 0$, $\varphi = 0$ and $u \in \mathscr{U}_{s,\beta}$ with some $s \in (1,\infty)$ solves the direct problem* (3.9)-(3.11) *then $u = 0$.*

(ii) *If $\varphi \in L_2(\Omega)$ and $F = 0$ then the direct problem* (3.9)-(3.11) *has a solution u that belongs to $\mathscr{U}_{s,\beta}$ for any $s \in (1, \frac{1}{\beta})$. This solution has the form*

$$u(t,x) = \sum_{k=1}^{+\infty} \varphi_k \sum_{i=0}^{+\infty} (M_k *)^i E_\beta(-\lambda_k t^\beta) v_k(x) \tag{3.30}$$

and satisfies the estimate

$$\|u\|_{\mathscr{U}_{s,\beta}} \le C_{21}\|\varphi\|, \quad \text{where } C_{21} \text{ is a constant.} \tag{3.31}$$

(iii) *If $\varphi = 0$ and $F \in L_p((0,T); L_2(\Omega))$ with some $p \in (\frac{1}{\beta}, \infty)$ then direct problem has a solution $u \in \mathscr{U}_{p,\beta}$. The solution has the form*

$$u(t,x) = \sum_{k=1}^{+\infty} \sum_{i=0}^{+\infty} (M_k *)^i \int_0^t e_\beta^{-\lambda_k (t-\tau)} F_k(\tau) d\tau \, v_k(x) \tag{3.32}$$

and satisfies the estimate

$$\|u\|_{\mathscr{U}_{p,\beta}} \le C_{22}\|F\|_{L_p((0,T);L_2(\Omega))} \quad \text{where } C_{22} \text{ is a constant.} \tag{3.33}$$

Proof. (i) is an immediate consequence of Proposition 3.1 (i) and Theorem 3.1 (i).

(ii) Let us consider the sequence of problems (3.24), (3.25) with $F_k = 0$. By Theorem 3.1 (ii), they have solutions $u_k \in \widetilde{\mathscr{U}}_{p,\beta}$ for any $p \in (\frac{1}{\beta}, \infty)$. We construct the solution to (3.9)-(3.11) in form of series $u = \sum\limits_{k=1}^{+\infty} u_k v_k$ and show that it satisfies the assertions of Theorem 2(ii).

We start by showing $u \in C([0,T]; L_2(\Omega))$. Since $u_k \in C[0,T]$ and $v_k \in L_2(\Omega)$, it follows that $u_k v_k \in C([0,T]; L_2(\Omega))$. Now let us show that the series $u = \sum\limits_{k=1}^{+\infty} u_k v_k$ is uniformly convergent in $[0,T]$ and therefore defines a continuous function. From (3.26) by means of Young's inequality (3.15), (3.16) and (3.29) we obtain

$$e^{-\sigma T}|u_k(t)| \leqslant |\varphi_k| \Big(\sum_{i=0}^{+\infty} \|M_k\|_{1;\sigma}^i\Big) \|E_\beta(-\lambda_k t^\beta)\|_{\infty;\sigma} \leqslant |\varphi_k| \Big(\sum_{i=0}^{+\infty} \|m\|_{1;\sigma}^i\Big) \leqslant \frac{|\varphi_k|}{1 - \|m\|_{1;\sigma}}$$

provided σ is sufficiently large to guarantee $\|m\|_{1;\sigma} < 1$. In view of $\varphi \in L_2(\Omega)$, for any $\varepsilon > 0$ there exists $K_\varepsilon \in \mathbb{N}$ such that $\sum_{k=K_\varepsilon}^{+\infty} \varphi_k^2 < \frac{(1-\|m\|_{1;\sigma})^2}{e^{2\sigma T}} \varepsilon$. Thus,

$$\Big\| \sum_{k=K_\varepsilon}^{+\infty} u_k(t) v_k \Big\|^2 = \sum_{k=K_\varepsilon}^{+\infty} |u_k(t)|^2 \leqslant \frac{e^{2\sigma T}}{(1 - \|m\|_{1;\sigma})^2} \sum_{k=K_\varepsilon}^{+\infty} \varphi_k^2 < \varepsilon \quad \forall t \in [0,T]. \tag{3.34}$$

Therefore this series is uniformly convergent and $u \in C([0,T]; L_2(\Omega))$. Similarly to (3.34) we derive

$$\|u(t,\cdot)\|_{L_2(\Omega)}^2 = \Big\| \sum_{k=1}^{+\infty} u_k(t) v_k \Big\|^2 = \sum_{k=1}^{+\infty} |u_k(t)|^2 \leqslant \frac{e^{2\sigma T}}{(1 - \|m\|_{1;\sigma})^2} \sum_{k=1}^{+\infty} \varphi_k^2 \tag{3.35}$$
$$= \frac{e^{2\sigma T}}{(1 - \|m\|_{1;\sigma})^2} \|\varphi\|^2, \quad \forall t \in [0,T].$$

Secondly, we prove that $u \in L_s((0,T);\mathscr{D}(-L))$. To this end, we investigate

$$\|u(t,\cdot)\|_{\mathscr{D}(-L)} = \Big\{ \sum_{k=1}^{+\infty} \lambda_k^2 \varphi_k^2 \Big[\sum_{i=0}^{+\infty} (M_k*)^i E_\beta(-\lambda_k t^\beta) \Big]^2 \Big\}^{\frac{1}{2}}. \tag{3.36}$$

For each term of the inner series in view of (3.27), we get

$$|(M_k*)^i E_\beta(-\lambda_k t^\beta)| \leqslant (|m|*)^i (\lambda_k e_\beta^{-\lambda_k t} *)^i E_\beta(-\lambda_k t^\beta).$$

Hence Lemma 3.2 implies $|(M_k*)^i E_\beta(-\lambda_k t^\beta)| \leqslant (|m|*)^i C_{18} C_{19}^i [\lambda_k t^\beta]^{-1}$. Now we use this inequality in (3.36). We reach the following estimate:

$$\|u(t,\cdot)\|_{\mathscr{D}(-L)} \leqslant \Big\{ \sum_{k=1}^{+\infty} \varphi_k^2 \Big[\sum_{i=0}^{+\infty} (|m|*)^i \frac{C_{18} C_{19}^i}{t^\beta} \Big]^2 \Big\}^{\frac{1}{2}} = \Big[\sum_{i=0}^{+\infty} (|m|*)^i \frac{C_{18} C_{19}^i}{t^\beta} \Big] \|\varphi\|.$$

Let us choose σ such that $C_{19}\|m\|_{1;\sigma} < 1$. Since $\frac{1}{t^\beta} \in L_s(0,T)$ for $s \in (1, \frac{1}{\beta})$, due to (3.15) we obtain the estimate

$$\|u\|_{L_s((0,T);\mathscr{D}(-L))} \leqslant C_{18} e^{\sigma T} \Big[\sum_{i=0}^{+\infty} C_{19}^i \|m\|_{1;\sigma}^i \Big] \|t^{-\beta}\|_{s;\sigma} \|\varphi\| < \infty. \tag{3.37}$$

This proves $u \in L_s((0,T);\mathscr{D}(-L))$.

Next we show that $u - u(0,\cdot) \in {}_0H_s^\beta((0,T);L_2(\Omega))$ and u satisfies the direct problem. Due to $u \in C([0,T];L_2(\Omega))$ and $u_k(0) = \varphi_k$, $k \in \mathbb{N}$, we have the initial condition $u(0,\cdot) = \varphi$. In previous part of the proof of (ii) we showed that $Lu \in L_s((0,T);L_2(\Omega))$. By Young's theorem, $m * Lu \in L_s((0,T);L_2(\Omega))$. Let $y = I_0^\beta(Lu + m * Lu)$. Lemma 3.1 implies $y \in {}_0H_s^\beta((0,T);L_2(\Omega))$. Let us compute the Fourier coefficients of y:

$$y_k = \langle I_0^\beta(Lu + m * Lu), v_k \rangle = (I_0^\beta + I_0^\beta m*)\langle Lu, v_k \rangle = I_0^\beta(-\lambda_k u_k - m * \lambda_k u_k), \quad k \in \mathbb{N}.$$

On the other hand, applying the operator I_0^β to the equation (3.24) (there $F_k = 0$), we obtain $u_k - \varphi_k = I_0^\beta(-\lambda_k u_k - m * \lambda_k u_k)$, $k \in \mathbb{N}$. Therefore, $y_k = u_k - \varphi_k$, $k \in \mathbb{N}$, and $y = u - \varphi = u - u(0,\cdot)$. We get that $u - u(0,\cdot) \in {}_0H_s^\beta((0,T);L_2(\Omega))$. Substituting y by $u - \varphi$ in $y = I_0^\beta(Lu + m * Lu)$ and applying the operator ${}^R D_0^\beta$ we obtain the equation (3.9) with $F = 0$:

$${}^R D_0^\beta(u - u(0,\cdot)) = Lu + m * Lu. \tag{3.38}$$

Finally, we prove (3.31). The equation (3.38) implies the estimate

$$\|u - u(0,\cdot)\|_{{}_0H_s^\beta((0,T);L_2(\Omega))} = \|Lu + m * Lu\|_{L_s((0,T);L_2(\Omega))} \leq (1 + \|m\|_1) \tag{3.39}$$
$$\times \|Lu\|_{L_s((0,T);L_2(\Omega))} = (1 + \|m\|_1)\|u\|_{L_s((0,T);\mathscr{D}(-L))}.$$

Then the estimates (3.35), (3.37), (3.39) imply (3.31). The solution formula (3.30) follows from (3.26).

(iii) Applying the operator I_0^β to the equation in (3.9), it is transformed to the following evolutionary integral equation in the space $L_2(\Omega)$:

$$u(t) - (a * Lu)(t) = (a * m * Lu)(t) + (a * F)(t), \quad t \in (0,T), \tag{3.40}$$

where $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. By Lemma 3.1 the equation (3.40) is equivalent to the direct problem (3.9)-(3.11) with zero initial condition in the space $\mathscr{U}_{p,\beta}$.

The assertion (iii) partially follows from Theorem 8.7 of [74] applied to the equation (3.40). We verify the validity of the assumptions of Theorem 8.7 with the corresponding reasonings:

1. $(-L) \in \mathscr{BIP}$ [2], because $(-L)$ is a normal and sectorial operator, and $\theta_{(-L)} = 0$, because $(-L)$ has positive real spectrum (cf. [74] Sect. 8.7, comment c) (i));
2. a is 1-regular and θ_a-sectorial with $\theta_a = \pi/2$, because a is completely monotonic (it follows from Proposition 3.3 of [74]);
3. $\theta_a + \theta_{(-L)} < \pi$, because $\theta_a = \frac{\pi}{2}$ and $\theta_{(-L)} = 0$;
4. $\lim\limits_{\mu\to\infty} |\hat{a}(\mu)|\mu^\beta < \infty$, because $\hat{a}(\mu) = \frac{1}{\mu^\beta}$.

Thus, Theorem 8.7 (a) of [74] implies that (3.40) has a solution u in the space $L_p((0,T);\mathscr{D}(-L)) \bigcap {}_0H_p^\beta((0,T);L_2(\Omega))$ and

$$\|u\|_{L_p((0,T);\mathscr{D}(-L))} + \|u\|_{{}_0H_p^\beta((0,T);L_2(\Omega))} \leq C\|F\|_{L_p((0,T);L_2(\Omega))}.$$

Since $p > \frac{1}{\beta}$, Lemma 3.1 implies that ${}_0H_p^\beta((0,T);L_2(\Omega)) \subset C([0,T];L_2(\Omega))$ and

$$\|u(t,\cdot)\|_{C([0,T];L_2(\Omega))} \leq C\|u\|_{{}_0H_p^\beta((0,T);L_2(\Omega))}.$$

This proves that $u \in \mathscr{U}_{p,\beta}$ and the estimate (3.33). Finally, the solution formula (3.32) follows from the Proposition 3.1 (i) and Theorem 3.1 (ii). □

Theorem 8.7 of [74] implies the existence of a solution of (3.9)-(3.11) in case $\varphi \neq 0$, too, but under the stronger assumption $\varphi \in \mathscr{D}(-L)$. The assertion (ii) of Theorem 3.2 in the particular case $m = 0$ follows from Theorem 2.1 of [78].

3.5 Inverse source problem

Let us introduce the notation for Fourier coefficients of functions involved in IP1:

$$f_k = \langle f, v_k\rangle,\ h_k(t) = \langle h(t,\cdot), v_k\rangle,\ \psi_k = \langle \psi, v_k\rangle,\ k \in \mathbb{N}.$$

Proposition 3.2. *Assume that $g \in L_p(0,T)$, $h \in L_p((0,T);L_2(\Omega))$ with some $p > \frac{1}{\beta}$ and $\varphi, \psi \in L_2(\Omega)$.*

If $(f,u) \in L_2(\Omega) \times \mathscr{U}_{s,\beta}$ for some $s > 1$ is a solution of IP1, then f_k, $k \in \mathbb{N}$, are solutions of the sequence of linear equations

$$A_k f_k = \psi_k - B_k, \quad A_k = \sum_{i=0}^{+\infty} \big((M_k *)^i e_\beta^{-\lambda_k t} * g\big)(T), \tag{3.41}$$

$$B_k = \varphi_k \sum_{i=0}^{+\infty} \big((M_k *)^i E_\beta(-\lambda_k t^\beta)\big)(T) + \sum_{i=0}^{+\infty} \big((M_k *)^i e_\beta^{-\lambda_k t} * h_k\big)(T).$$

Conversely, let f_k, $k \in \mathbb{N}$, be the solutions of the equations (3.41) and $\sum_{k=1}^{+\infty} f_k^2 < \infty$. Then $f = \sum_{k=1}^{+\infty} f_k v_k \in L_2(\Omega)$, direct problem (3.9)-(3.11) with $F = fg + h$ has a solution $u \in \mathscr{U}_{s,\beta}$, $s \in (1, \frac{1}{\beta})$ and the pair (f,u) solves IP1.

[2] $\mathscr{BIP}$ is the space of operators with bounded imaginary powers. Here we refer to the definition of sectorial operator provided in [74].

Proof. Let $(f,u) \in L_2(\Omega) \times \mathscr{U}_{s,\beta}$ solve IP1. Using Proposition 3.1 (i) and Theorem 3.1, we deduce the formula (3.26) with $F_k = gf_k + h_k$. Setting there $t = T$ and replacing $u_k(T)$ by ψ_k, we obtain (3.41).

Conversely, let f_k, $k \in \mathbb{N}$ be the solutions of (3.41) such that $\sum_{k=1}^{+\infty} f_k^2 < \infty$. Then by Theorem 3.2, the problem (3.9) with $F = gf + h$ and $f = \sum_{k=1}^{+\infty} f_k v_k$ has a solution $u \in \mathscr{U}_{s,\beta}$ for any $s \in (1, \frac{1}{\beta})$. Again, by Proposition 3.1 (i) and Theorem 3.1 we reach (3.26). Comparing it with (3.41), we see that $u_k(T) = \psi_k$, $k \in \mathbb{N}$. This implies (3.12). Thus, (f,u) solves IP1. □

Now we prove a basic lower estimate of A_k in (3.41). We do it separately for the different cases of m.

Lemma 3.3. *Assume that there exist $T_1 \in (0,T)$, $g_0 > 0$ such that one of the following conditions is valid:*

(A1) $m \leqslant 0$, $g \in L_p((0,T);\mathbb{R})$ *with some* $p > \frac{1}{\beta}$, $g \geqslant 0$ *and* $g(t) \geqslant g_0$ *a.e.* $t \in (T_1,T)$*;*

(A2) $g \in L_\infty((0,T);\mathbb{R})$, $g \geqslant 0$, $g(t) \geqslant g_0$ *a.e.* $t \in (T_1,T)$, *and* $\|m\|_1 < \frac{g_0 C_{23}}{g_0 C_{23} + \|g\|_\infty}$, *where* $C_{23} = 1 - E_\beta(-\lambda_1 (T-T_1)^\beta)$*;*

(A3) $m \geqslant 0$, $g \in W_1^1((0,T);\mathbb{R})$, $g - m*g \geqslant 0$, $g' \geqslant 0$ *and* $(g - m*g)(t) \geqslant g_0$ *for* $t \in (T_1,T)$.

Then $A_k \geqslant \frac{C_{24}}{\lambda_k}$, $k \in \mathbb{N}$, *where* $C_{24} > 0$ *is a constant independent of* k.

Proof. Firstly, we consider the case (A1). Note that $m \leqslant 0$ implies $M_k \geqslant 0$. Thus, due to (3.19), the properties of g and the monotonicity of $E_\beta(-z)$, we obtain that each term of the series for A_k in (3.41) is nonnegative. Therefore, we estimate A_k from below by the first term of the series:

$$A_k \geqslant (e_\beta^{-\lambda_k t} * g)(T) \geqslant g_0 \int_0^{T-T_1} e_\beta^{-\lambda_k t} dt = \frac{(1 - E_\beta(-\lambda_k (T-T_1)^\beta)) g_0}{\lambda_k} \geqslant \frac{C_{24}}{\lambda_k}, \quad (3.42)$$

where $C_{24} = [1 - E_\beta(-\lambda_1 (T-T_1)^\beta)] g_0$.

In case (A2), by means of (3.15) and (3.42) we deduce

$$A_k \geqslant (e_\beta^{-\lambda_k t} * g)(T) - \Big| \sum_{i=1}^{+\infty} \big((M_k *)^i e_\beta^{-\lambda_k t} * g \big)(T) \Big| \geqslant \frac{(1 - E_\beta(-\lambda_1 (T-T_1)^\beta)) g_0}{\lambda_k}$$
$$- \sum_{i=1}^{+\infty} \|M_k\|_1^i \, \|e_\beta^{-\lambda_k t}\|_1 \, \|g\|_\infty.$$

Using (3.20) and (3.29) we obtain $A_k \geqslant \frac{C_{24}}{\lambda_k}$, where

$$C_{24} = [(1 - E_\beta(-\lambda_1 (T-T_1)^\beta))] g_0 - \frac{\|m\|_1}{1 - \|m\|_1} \|g\|_\infty > 0.$$

Finally, we treat the case (A3). We point out that A_k can be represented as

$$A_k = \sum_{i=0}^{+\infty} (M_k *)^{2i} e_\beta^{-\lambda_k t} * (g - M_k * g)(T)$$
$$= \sum_{i=0}^{+\infty} (M_k *)^{2i} e_\beta^{-\lambda_k t} * \big(g - \lambda_k e_\beta^{-\lambda_k t} * g * m \big)(T). \quad (3.43)$$

Let us estimate the term inside the braces in (3.43) $\left(g-\lambda_k e_\beta^{-\lambda_k t}*g*m\right)$. By means of the integration by parts we have

$$(\frac{d}{dt}E_\beta(-\lambda_k t^\beta))*g = E_\beta(-\lambda_k t^\beta)g(0) - g + E_\beta(-\lambda_k t^\beta)*g'.$$

Since $-\lambda_k e_\beta^{-\lambda_k t} = \frac{d}{dt}E_\beta(-\lambda_k t^\beta)$ (see (3.19)) it holds that

$$g-\lambda_k e_\beta^{-\lambda_k t}*g*m = g+\Big((\frac{d}{dt}E_\beta(-\lambda_k t^\beta))*g\Big)*m = g(0)E_\beta(-\lambda_k t^\beta)*m$$
$$+g'*E_\beta(-\lambda_k t^\beta)*m+g-m*g.$$

Therefore, since $g(0)=(g-g*m)(0)\geq 0$ and in view of the assumptions (A3), we have

$$g-\lambda_k e_\beta^{-\lambda_k t}*g*m \geqslant g-m*g \geqslant 0.$$

Since $M_k*M_k = (-m)*\lambda_k e_\beta^{-\lambda_k t}*(-m)*\lambda_k e_\beta^{-\lambda_k t}\geqslant 0$, each term of series (3.43) is nonnegative. Therefore, we estimate A_k from below by the first addend from (3.43):

$$A_k \geqslant \left(e_\beta^{-\lambda_k t}*(g-m*g)\right)(T).$$

Then similarly to (3.42), $A_k \geqslant \frac{C_{24}}{\lambda_k}$, where $C_{24} = [1-E_\beta(-\lambda_1(T-T_1)^\beta)]g_0$. □

From Proposition 3.2 and Lemma 3.3 we easily deduce the uniqueness assertion for IP1.

Corollary 3.1. *Let the assumptions of Lemma* 3.3 *be satisfied,* $\varphi=0$, $h=0$ *and* $\psi=0$. *If* $(f,u)\in L_2(\Omega)\times\mathscr{U}_{s,\beta}$ *for some* $s>1$ *is a solution of the* IP1, *then* $f=0$, $u=0$.

Proof. If $(f,u)\in L_2(\Omega)\times\mathscr{U}_{s,\beta}$ is a solution of the inverse problem, then by Proposition 3.2, the formulas (3.41) are valid and it yields from the assumptions of the corollary that $\psi_k=B_k=0$, $k\in\mathbb{N}$. On the other hand, Lemma 3.3 implies $A_k>0$, $k\in\mathbb{N}$. Therefore, the solution of (3.41) is $f_k=0$, $k\in\mathbb{N}$. This implies that $f=0$ and therefore $F=fg+h=0$. Thus by Theorem 3.2 (i) $u=0$. □

The functions m corresponding to the kernels (M2) and (M4) are the examples of $m\geqslant 0$, this follows directly from their representation (3.5) and (3.7).

The examples of $m\leqslant 0$ include the kernels (k2) with 2 terms and (M5).

Let us take a closer look at (k2) with 2 terms. For $k=\frac{t^{-\beta}}{\Gamma(1-\beta)}+q_2\frac{t^{-\beta_2}}{\Gamma(1-\beta_2)}$, $\quad\beta_2<\beta$ the function m satisfies the equation:

$$\left(\frac{t^{-\beta}}{\Gamma(1-\beta)}+q_2\frac{t^{-\beta_2}}{\Gamma(1-\beta_2)}\right)*\left(\frac{t^{\beta-1}}{\Gamma(\beta)}+m*\frac{t^{\beta-1}}{\Gamma(\beta)}\right)=1.$$

Let us apply the Laplace transform to this relation:

$$\left(\frac{1}{s^{1-\beta}}+\frac{q_2}{s^{1-\beta_2}}\right)\left(\frac{1}{s^\beta}+\hat{m}\frac{1}{s^\beta}\right)=\frac{1}{s}.$$

Therefore

$$\hat{m}=\frac{-q_2}{s^{\beta-\beta_2}+q_2}$$

and according to [20] p.312

$$m(t) = -q_2 t^{\beta-\beta_2-1} E_{\beta-\beta_2,\beta-\beta_2}(-q_2 t^{\beta-\beta_2}) < 0.$$

Now let us check the case (M5). For this purpose let us substitute the expansion $e^{-\lambda t} = \sum_{i=0}^{+\infty} \frac{(-\lambda t)^i}{i!}$ into (3.8)

$$\begin{aligned} m(t) &= \frac{t^{-\beta}}{\Gamma(1-\beta)} * \left[-\lambda(\beta-1) \sum_{i=0}^{+\infty} \frac{(-\lambda)^i t^{i+\beta-1}}{(i+1)!\Gamma(\beta)} - \lambda \sum_{i=0}^{+\infty} \frac{(-\lambda)^i t^{i+\beta-1}}{i!\Gamma(\beta)} \right] \\ &= -\lambda \frac{t^{-\beta}}{\Gamma(1-\beta)} * \sum_{i=0}^{+\infty} \frac{(-\lambda)^i t^{i+\beta-1}(i+\beta)}{(i+1)!\Gamma(\beta)} \end{aligned}$$

and then apply the formula (3.4) for the convolution. We obtain that

$$m(t) = -\lambda\beta \sum_{i=0}^{+\infty} \frac{(-\lambda t)^i \Gamma(2)\Gamma(i+1+\beta)}{\Gamma(i+2)\Gamma(1+\beta) i!} = -\lambda\beta\, {}_1F_1(1+\beta, 2, -\lambda t) < 0,$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

The coupled conditions for m and g in (A3) cover all positive integrable m. This means that for any $m \in L_1((0,T);\mathbb{R})$, $m \geqslant 0$, it is possible to find a function g so that (A3) is valid. Let us construct such a g. Choose an arbitrary $z \in W_1^1((0,T);\mathbb{R})$ so that $z \geqslant 0$, $z' \geqslant 0$ and $z(t) \geqslant z_0 > 0$, $t \in (T_1, T)$ and define g as a solution of the Volterra equation of the second kind $g - m * g = z$. Then $g' - m * g' - g(0)m = z'$, hence $g' = \sum_{i=0}^{+\infty} (m*)^i (z' + g(0)m) \geqslant 0$. So, the conditions (A3) are satisfied.

Theorem 3.3. *Let the assumptions of Lemma* 3.3 *be satisfied,* $\psi \in \mathscr{D}(-L)$, $h \in L_\rho((0,T);L_2(\Omega))$, *with some* $\rho \in (\frac{1}{\beta}, \infty]$ *and* $\sum_{k=0}^{+\infty} \lambda_k^{2\omega} \|h_k\|_\rho^2 < \infty$ *where* ω *is some number satisfying inequality*

$$\omega \begin{cases} > \frac{1}{\beta\rho} & \text{in case } \rho \in (\frac{1}{\beta}, \infty), \\ \geqslant 0 & \text{in case } \rho = \infty. \end{cases}$$

Moreover, let one of the following conditions be valid:

(A4) $\varphi \in \mathscr{D}(-L)$;

(A5) $\varphi \in \mathscr{D}((-L)^\zeta)$ *for some* $\zeta \in [0,1)$ *and* $m \in L_{s'}((0,T);\mathbb{R})$ *for some* $s' > \frac{1}{1-\beta(1-\zeta)}$;

(A6) $\varphi \in L_2(\Omega)$ *and* $\exists c_m \geqslant 0, \gamma_m < 1 : |m(t)| \leq \frac{c_m}{\Gamma(1-\gamma_m)} t^{-\gamma_m}$ *a.e.* $t \in (0,T)$.

Then IP1 *has a unique solution* $(f,u) \in L_2(\Omega) \times \mathscr{U}_{s,\beta}$ *for any* $s \in (1, \frac{1}{\beta})$. *The components of this solution satisfy the estimates*

$$\|f\| \leqslant C_{25} \Big\{ \|\psi\|_{\mathscr{D}(-L)} + \|\varphi\|_{\mathscr{D}((-L)^\Theta)} + \Big[\sum_{k=0}^{+\infty} \lambda_k^{2\omega} \|h_k\|_\rho^2 \Big]^{\frac{1}{2}} \Big\} \tag{3.44}$$

and

$$\|u\|_{\mathscr{U}_{s,\beta}} \leqslant C_{26} \Big\{ \|\psi\|_{\mathscr{D}(-L)} + \|\varphi\|_{\mathscr{D}((-L)^\Theta)} + \Big[\sum_{k=0}^{+\infty} \lambda_k^{2\omega} \|h_k\|_\rho^2 \Big]^{\frac{1}{2}} + \|h\|_{L_\rho((0,T);L_2(\Omega))} \Big\}, \tag{3.45}$$

where C_{25}, C_{26} *are the constants and the exponent* Θ *equals* 1, ζ *and* 0 *in cases* (A4), (A5) *and* (A6), *respectively.*

Proof. Let us consider the formula of B_k in (3.41). Since $h \in L_\rho((0,T);L_2(\Omega))$ the coefficients $h_k \in L_\rho(0,T)$. Firstly, we estimate the term containing h_k by means of (3.14), (3.15) and (3.29):

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i e_\beta^{-\lambda_k t} * h_k\big)(T)\Big| \leqslant e^{\sigma T}\sum_{i=0}^{+\infty}\|M_k\|_{1;\sigma}^i \|e_\beta^{-\lambda_k t} * h_k\|_{\infty;\sigma}$$
$$\leqslant e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i \|e_\beta^{-\lambda_k t} * h_k\|_\infty. \tag{3.46}$$

In case $\rho = \infty$ we have $\omega \geq 0$ and by means of (3.15), (3.20) we obtain

$$\|e_\beta^{-\lambda_k t} * h_k\|_\infty \leqslant \|e_\beta^{-\lambda_k t}\|_1 \|h_k\|_\infty \leqslant \lambda_k^{-1}\|h_k\|_\infty \leqslant \frac{1}{\lambda_1^\omega}\lambda_k^{\omega-1}\|h_k\|_\infty.$$

Next let $\rho \in (\frac{1}{\beta},\infty)$. Without restriction of generality we assume that $\omega \leqslant 1$. We note that the boundedness of $E_{\beta,\beta}(-z)$ for $z \geqslant 0$ and the asymptotical relation (1.21) imply the inequality $E_{\beta,\beta}(-z) \leqslant \frac{C_{27}}{z^{1-\omega}}$ for $z \geqslant 0$ with some constant C_{27}. Thus

$$e_\beta^{-\lambda_k t} = t^{\beta-1}E_{\beta,\beta}(-\lambda_k t^\beta) \leqslant C_{27}\lambda_k^{\omega-1}t^{\beta\omega-1}.$$

Due to the assumed inequality $\omega > \frac{1}{\beta\rho}$ it holds $t^{\beta\omega-1} \in L_{\rho'}(0,T)$, where $\frac{1}{\rho}+\frac{1}{\rho'}=1$. Thus, by Hölder inequality we obtain

$$\|e_\beta^{-\lambda_k t} * h_k\|_\infty \leqslant C_{27}\lambda_k^{\omega-1}\|t^{\beta\omega-1}\|_{\rho'}\|h_k\|_\rho = C_{28}\lambda_k^{\omega-1}\|h_k\|_\rho.$$

Let us continue the estimation of (3.46). For any $\rho \in (\frac{1}{\beta},\infty]$ we have

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i e_\beta^{-\lambda_k t} * h_k\big)(T)\Big| \leqslant C_{29}\, e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i \lambda_k^{\omega-1}\|h_k\|_\rho = C_{30}\lambda_k^{\omega-1}\|h_k\|_\rho \tag{3.47}$$

with $C_{29} = \max\{\lambda_1^{-\omega};C_{28}\}$, $C_{30} = \frac{e^{\sigma T}C_{29}}{1-\|m\|_{1;\sigma}}$, provided σ is large enough to guarantee $\|m\|_{1;\sigma} < 1$.

Secondly, we estimate the factor of φ_k in (3.41). In the general case when $m \in L_1(0,T)$ (it is so in the case (A4)), we have due to (3.16) and (3.29) that

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big| \leqslant e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i \|E_\beta(-\lambda_k t^\beta)\|_\infty \leqslant C_{31}, \tag{3.48}$$

where $C_{31} = \frac{e^{\sigma T}}{1-\|m\|_{1;\sigma}}$.

In case (A5), by means of (3.20) and (3.29) we obtain the estimate

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big| = \Big|E_\beta(-\lambda_k T^\beta) - \sum_{i=0}^{+\infty}\big((M_k*)^i \lambda_k e_\beta^{-\lambda_k t}$$
$$*m*E_\beta(-\lambda_k t^\beta)\big)(T)\Big| \leqslant E_\beta(-\lambda_k T^\beta) + e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i \|m*E_\beta(-\lambda_k t^\beta)\|_\infty. \tag{3.49}$$

Since by formula (3.17) $E_\beta(-\lambda_k t^\beta) \leqslant \frac{C_{18}}{1+\lambda_k t^\beta} \leqslant \frac{C_{18}}{(\lambda_k t^\beta)^{1-\zeta}}$, $\zeta \in [0,1)$, we estimate

$$\|m*E_\beta(-\lambda_k t^\beta)\|_\infty \leqslant C_{18}\lambda_k^{\zeta-1}\|m\|_{s'}\|t^{-\beta(1-\zeta)}\|_{s''}$$

where $\frac{1}{s'}+\frac{1}{s''}=1$. In this point we have $\|t^{-\beta(1-\zeta)}\|_{s''}<\infty$ because of the assumption $s'>\frac{1}{1-\beta(1-\zeta)}$. Thus, from (3.49) we obtain

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big|\leqslant C_{32}\lambda_k^{\zeta-1} \tag{3.50}$$

with $C_{32}=\frac{C_{18}}{T^{\beta(1-\zeta)}}+\frac{e^{\sigma T}C_{18}}{(1-\|m\|_{1;\sigma})}\|m\|_{s'}\|t^{-\beta(1-\zeta)}\|_{s''}$.

Finally, if the assumptions (A6) hold for m, we deduce

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big|\leqslant \sum_{i=0}^{+\infty}\Big(\big(\frac{c_m t^{-\gamma_m}}{\Gamma(1-\gamma_m)}*\big)^i(\lambda_k e_\beta^{-\lambda_k t}*)^i E_\beta(-\lambda_k t^\beta)\Big)(T).$$

Using Lemma 3.2 and the formula (3.4) repeatedly, we continue the estimation:

$$\begin{aligned}\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big| &\leqslant \sum_{i=0}^{+\infty}\Big(\frac{c_m^i t^{i(1-\gamma_m)-1}}{\Gamma(i(1-\gamma_m))}*\frac{C_{18}C_{19}^i}{\lambda_k t^\beta}\Big)(T)\\ &=\frac{1}{\lambda_k}C_{18}\Gamma(1-\beta)\sum_{i=0}^{+\infty}\frac{(C_{19}c_m)^i T^{i(1-\gamma_m)-\beta}}{\Gamma(i(1-\gamma_m)+1-\beta)}=C_{33}\lambda_k^{-1},\end{aligned} \tag{3.51}$$

where $C_{33}=C_{18}T^{-\beta}\Gamma(1-\beta)E_{1-\gamma_m,1-\beta}(C_{19}c_m T^{1-\gamma_m})$. Summing up, (3.48), (3.50) and (3.51) imply

$$\Big|\sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\Big|\leqslant C_{34}\lambda_k^{\Theta-1} \tag{3.52}$$

with $C_{34}=\max\{C_{31};C_{32};C_{33}\}$ for all cases (A4) - (A6).

Now we are able estimate the quantity f_k in (3.41). Lemma 3.3 and the relations (3.47), (3.52) yield $|f_k|\le C_{25}\{\lambda_k|\psi_k|+\lambda_k^\Theta|\varphi_k|+\lambda_k^\omega\|h_k\|_\rho\}$, where $C_{25}=\frac{1}{C_{24}}\max\{1;C_{30};C_{34}\}$. Assumptions of the theorem yield $\sum_{k=1}^{+\infty}f_k^2<\infty$. Therefore, existence assertion of the Theorem follows from the Proposition 3.2.

Plugging the deduced estimate for $|f_k|$ into the relation $\|f\|=\Big[\sum_{k=1}^{+\infty}|f_k|^2\Big]^{1/2}$ and using the triangle inequality in l_2-space, we obtain (3.44).

By estimates (3.31) and (3.33) from Theorem 3.2 with $F=fg+h$ we obtain the estimate for u:

$$\|u\|_{\mathscr{U}_{s,\beta}}\leqslant C_{35}\Big(\|\varphi\|+\|g\|_{L_{p_1}(0,T)}\|f\|+\|h\|_{L_\rho((0,T);L_2(\Omega))}\Big)$$

for any $s\in\Big(1,\frac{1}{\beta}\Big)$, where $C_{35}=\max\{C_{21},C_{22}\}$ and $p_1=p$ in case (A1), $p_1\in(\frac{1}{\beta},\infty)$ in cases (A2), (A3). After inserting the estimate for f (3.44) into this inequality we obtain (3.45). □

The assumption (A6) is satisfied by kernels m corresponding to the cases (k2), (M2), (M4), (M5), (M6).

3.6 Backward in time problem

Proposition 3.3. *Assume that $F\in L_p((0,T);L_2(\Omega))$ with some $p>\frac{1}{\beta}$ and $\psi\in L_2(\Omega)$. If $u\in\mathscr{U}_{s,\beta}$ for some $s>1$ is a solution of* IP2*, then $\varphi_k=u_k(0)$, $k\in\mathbb{N}$, are solutions of the* *sequence of linear equations*

$$\hat{A}_k\varphi_k = \psi_k - \hat{B}_k, \quad \hat{A}_k = \sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T), \tag{3.53}$$

$$\hat{B}_k = \sum_{i=0}^{+\infty}\big((M_k*)^i e_\beta^{-\lambda_k t} * F_k\big)(T),$$

where $\psi_k = \langle\psi, v_k\rangle$ *as in the case of* IP1.

Conversely, let $\varphi_k, k\in\mathbb{N}$*, be solutions of the equations* (3.53) *and* $\sum_{k=1}^{+\infty}\varphi_k^2 < \infty$*. Then the direct problem* (3.9)-(3.11) *with* $\varphi = \sum_{k=1}^{+\infty}\varphi_k v_k \in L_2(\Omega)$ *has a solution* $u\in\mathscr{U}_{s,\beta}$ *for any* $s\in(1,\frac{1}{\beta})$*. The function* u *is also a solution to* IP2.

The proof is similar to the proof of Proposition 3.2.

Next we derive a basic lower estimate for $\hat{A}_k$ in different cases of m. Unlike IP1, we have no results in case of general positive m. Lack of an additional degree of freedom (as the function g in IP1) makes the study of the case $m \geqslant 0$ very complicated.

Lemma 3.4. *Let one of the following conditions hold:*

(A7) $m \leqslant 0$;

(A8) $\|m\|_1 < 1$, $m\in L_{s'}((0,T);\mathbb{R})$ *for some* $s' > \frac{1}{1-\beta}$ *and*

$$\frac{\|m\|_{s'}}{1-\|m\|_1} < \frac{C_{17}(1-\beta s'')^{1/s''}}{C_{18}(1/\lambda_1 + T^\beta)T^{1/s''-\beta}},$$

where C_{17} *and* C_{18} *are the constants from* (3.17) *and* $\frac{1}{s'}+\frac{1}{s''}=1$;

(A9) $|m(t)| \leqslant \frac{c_m}{\Gamma(1-\gamma_m)}t^{-\gamma_m}$ *a.e.* $t\in(0,T)$ *with some* $\gamma_m < 1$ *and a sufficiently small* $c_m > 0$*, such that*

$$c_m E_{1-\gamma_m, 2-\gamma_m-\beta}(C_{19}T^{1-\gamma_m}c_m) < \frac{C_{17}}{C_{18}C_{19}\Gamma(1-\beta)T^{1-\gamma_m-\beta}(1/\lambda_1+T^\beta)}, \tag{3.54}$$

where C_{19} *is the constant from* (3.21).

Then $\hat{A}_k \geqslant \frac{C_{36}}{\lambda_k}$, $k\in\mathbb{N}$*, where* $C_{36} > 0$ *is a constant independent of* k.

Since $E_{1-\gamma_m,2-\gamma_m-\beta}$ in the left hand side of (3.54) is locally bounded as an entire function, the inequality (3.54) is satisfied for sufficiently small c_m.

Proof. In case (A7), we have $M_k \geq 0$ and by applying (3.17) we estimate:

$$\hat{A}_k = \sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T) \geqslant E_\beta(-\lambda_k T^\beta) \geqslant \frac{C_{17}}{1+\lambda_k T^\beta} \geqslant \frac{C_{36}}{\lambda_k},$$

where we take $C_{36} = \frac{C_{17}}{1/\lambda_1 + T^\beta}$.

Secondly, let us consider the case (A8). We have the relation

$$\hat{A}_k = \sum_{i=0}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T) \geqslant E_\beta(-\lambda_k T^\beta) - \left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right|$$

$$\geqslant \frac{C_{36}}{\lambda_k} - \left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right|,$$

where we treat the series similarly to (3.49):

$$\left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right| = \left|\sum_{i=0}^{+\infty}\big((M_k*)^i \lambda_k e_\beta^{-\lambda_k t} * m * E_\beta(-\lambda_k t^\beta)\big)(T)\right|$$
$$\leqslant \sum_{i=0}^{+\infty}\|m\|_1^i \|m*E_\beta(-\lambda_k t^\beta)\|_\infty \leqslant \frac{C_{18}\|m*t^{-\beta}\|_\infty}{\lambda_k(1-\|m\|_1)} \leqslant \frac{C_{18}\|t^{-\beta}\|_{s''}}{\lambda_k(1-\|m\|_1)}\|m\|_{s'}.$$

Then

$$\hat{A}_k \geqslant \frac{C_{36}}{\lambda_k}, \quad \text{where } C_{36} = \frac{C_{17}}{1/\lambda_1 + T^\beta} - \frac{C_{18}\|m\|_{s'}}{1-\|m\|_1}\left(\frac{T^{1-\beta s''}}{1-\beta s''}\right)^{1/s''}.$$

Finally, the case (A9) can be treated similarly to (A8) in the sense that we start from the estimate

$$\hat{A}_k \geqslant \frac{C_{17}}{\lambda_k(1/\lambda_1 + T^\beta)} - \left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right|,$$

and estimate the series from above. As in (3.51) by means of Lemma 3.2, we obtain

$$\left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right| \leqslant \frac{1}{\lambda_k}C_{18}\Gamma(1-\beta)\sum_{i=1}^{+\infty}\frac{(C_{19}c_m)^i T^{i(1-\gamma_m)-\beta}}{\Gamma(i(1-\gamma_m)+1-\beta)}.$$

The series starts with $i=1$, thus we can extract the factor c_m and reach the estimate

$$\left|\sum_{i=1}^{+\infty}\big((M_k*)^i E_\beta(-\lambda_k t^\beta)\big)(T)\right| \leqslant \frac{1}{\lambda_k}C_{18}C_{19}c_m\Gamma(1-\beta)T^{1-\gamma_m-\beta}$$
$$\times\sum_{i=0}^{+\infty}\frac{(C_{19}c_m)^i T^{i(1-\gamma_m)}}{\Gamma(i(1-\gamma_m)+2-\beta-\gamma_m)} = \frac{C_{37}}{\lambda_k}c_m E_{1-\gamma_m,2-\gamma_m-\beta}(C_{19}c_m T^{1-\gamma_m}),$$

where $C_{37} = C_{18}C_{19}\Gamma(1-\beta)T^{1-\gamma_m-\beta}$. We obtain the relation

$$\hat{A}_k \geqslant \frac{C_{36}}{\lambda_k}, \quad \text{where} \quad C_{36} = \frac{C_{17}}{1/\lambda_1 + T^\beta} - C_{37}c_m E_{1-\gamma_m,2-\gamma_m-\beta}(C_{19}c_m T^{1-\gamma_m}).$$

□

Corollary 3.2. *Let the assumptions of Lemma* 3.4 *be satisfied,* $F=0$ *and* $\psi=0$*. If* $u\in\mathscr{U}_{s,\beta}$ *for some* $s>1$ *is a solution of* IP2*, then* $u=0$.

The proof is similar to the proof of the previous corollary.

Theorem 3.4. *Let the assumptions of Lemma* 3.4 *be satisfied,* $\psi\in\mathscr{D}(-L)$ *and* $F\in L_\rho((0,T);L_2(\Omega))$*, with some* $\rho\in(\frac{1}{\beta},\infty]$ *and* $\sum_{k=0}^{+\infty}\lambda_k^{2\omega}\|F_k\|_\rho^2<\infty$ *where* ω *is some number satisfying the inequality*

$$\omega\begin{cases} > \frac{1}{\beta\rho} & \text{in case } \rho\in(\frac{1}{\beta},\infty), \\ \geqslant 0 & \text{in case } \rho=\infty. \end{cases}$$

Then IP2 *has a unique solution* $u\in\mathscr{U}_{s,\beta}$ *for any* $s\in(1,\frac{1}{\beta})$*. This solution satisfies the estimates*

$$\|u(0,\cdot)\| \leqslant C_{38}\Big\{\|\psi\|_{\mathscr{D}(-L)} + \Big[\sum_{k=0}^{+\infty}\lambda_k^{2\omega}\|F_k\|_\rho^2\Big]^{\frac{1}{2}}\Big\},$$

and

$$\|u\|_{\mathscr{U}_{s,\beta}} \leqslant C_{39}\Big\{\|\psi\|_{\mathscr{D}(-L)} + \Big[\sum_{k=0}^{+\infty} \lambda_k^{2\omega}\|F_k\|_\rho^2\Big]^{\frac{1}{2}} + \|F\|_{L_\rho((0,T);L_2(\Omega))}\Big\}, \tag{3.55}$$

where C_{38}, C_{39} are constants.

Proof. Let us estimate $\hat{B}_k$ from above. As in (3.47), we deduce the relation

$$|\hat{B}_k| = \Big|\sum_{i=0}^{+\infty}\big((M_k*)^i e_\beta^{-\lambda_k t} * F_k\big)(T)\Big| \leqslant C_{30}\lambda_k^{\omega-1}\|F_k\|_\rho.$$

This estimate together with Lemma 3.4 and (3.53) yields $|\varphi_k| \leqslant C_{38}\{\lambda_k|\psi_k| + \lambda_k^\omega\|F_k\|_\rho\}$. Now the assertions of the theorem follow by means of arguments similar to the proof of Theorem 3.3. □

4 Inverse problem for a generalized fractional derivative and reconstruction of time- and space-dependent sources

In this Chapter we assume that the overdetermination condition is given not only at the final moment of time T, but in its neighbourhood. Main results of this Chapter have appeared in the Publication III. All scalar functional spaces are real by default in this Chapter.

4.1 Formulation of problems

We are solving problems with higher order generalized fractional derivatives in Riemann-Liouville ${}^R D_a^{\{k\},n}$ and Caputo sense ${}^C D_a^{\{k\},n}$:

$$({}^R D_a^{\{k\},n} v)(t) = \frac{d^n}{dt^n} \int_a^t k(t-\tau) v(\tau) d\tau, \; ({}^C D_a^{\{k\},n} v)(t) = \int_a^t k(t-\tau) v^{(n)}(\tau) d\tau,$$
$$t > a, n \in \{0\} \cup \mathbb{N}, k \in L_{1,loc}(0,\infty).$$

We utilize $D_a^{\{k\},n}$ as a unified notation that stands either for ${}^R D_a^{\{k\},n}$ or ${}^C D_a^{\{k\},n}$.

In case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$ we have that ${}^R D_a^{\{k\},n}$ and ${}^C D_a^{\{k\},n}$ are the Riemann-Liouville and Caputo fractional derivatives of the order $n+\beta-1$, i.e.

$$({}^R D_a^{\{k\},n} v)(t) = ({}^R D_a^{n+\beta-1} v)(t) = \frac{d^n}{dt^n} \int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} v(\tau) d\tau, \tag{4.1}$$

$$({}^C D_a^{\{k\},n} v)(t) = ({}^C D_a^{n+\beta-1} v)(t) = \int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} v^{(n)}(\tau) d\tau. \tag{4.2}$$

Our basic inverse problem consists in reconstruction of a function u at $0 < t < T$ by means of measurements of $u(t)$ and its generalized fractional derivative in a left neighborhood of T.

Let $0 < t_0 < T$. **IP1**. Given $\varphi, g : (t_0, T) \to \mathbb{R}$, find $u : (0,T) \to \mathbb{R}$ such that

$$u|_{(t_0,T)} = \varphi \quad \text{and} \quad D_0^{\{k\},n} u|_{(t_0,T)} = g\,. \tag{4.3}$$

An example of IP1 is the reconstruction of physical quantities in constitutive relations involving fractional derivatives. In the subdiffusive model of heat flow discussed in the Section 1.6 the flux is proportional to a time fractional derivative of antigradient of the temperature (see (1.26)). In this context IP1 means the reconstruction of the history of temperature by means of measurement of temperature and flux in a left neighborhood of a time value T. Similar meaning for IP1 can be given in the Scott-Blair's model of viscoelasticity. Then the stress is proportional to a time fractional derivative of the strain [65].

We use the results obtained for IP1 in order to investigate an inverse problem of reconstruction of a history of a source in a general PDE that includes as particular cases fractional diffusion and wave equations from the measurements in a left neighborhood of final time T. That is formulated as follows:

IP2. Given $\varphi, \Phi : \Omega \times (t_0, T) \to \mathbb{R}$, find $u, F : \Omega \times (0,T) \to \mathbb{R}$, such that

$$(D_0^{\{k\},n} Bu)(x,t) + D^l u(x,t) - Au(x,t) = F(x,t), \quad x \in \Omega,\, t \in (0,T), \tag{4.4}$$

is fulfilled and

$$u|_{\Omega \times (t_0,T)} = \varphi, \quad F|_{\Omega \times (t_0,T)} = \Phi. \tag{4.5}$$

Here $\Omega \subseteq \mathbb{R}^N$ with some $N \in \mathbb{N}$, $D^l = \sum_{j=1}^{l} q_j \frac{\partial^j}{\partial t^j}$ with some $l \in \mathbb{N}$, $q_j \in \mathbb{R}$, and A and B are operators that act on functions depending on x. Throughout the Chapter we assume that A and B with their domains $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are such that $A : \mathscr{D}(A) \subseteq C(\Omega) \to C(\Omega)$, $B : \mathscr{D}(B) \subseteq C(\Omega) \to C(\Omega)$ and B is injective.

The equation (4.4) generalizes different subdiffusion equations in Riemann-Liouville or Caputo form, these are

$$\frac{\partial}{\partial t} u - {}^R D_0^{\{M\},1} Lu = Q \quad \text{and} \quad {}^C D_0^{\{k\},1} u - Lu = F,$$

where L is an elliptic operator. In case of Riemann-Liouville subdiffusion equation $B = -L$ and in order to guarantee the injectivity of B, proper boundary conditions must be specified in the domain $\mathscr{D}(B)$.

The equation (4.4) also includes the fractional wave equation [24, 62, 99]

$${}^C D_0^{\beta} u + \lambda(-\Delta)^{\alpha} u = F, \quad \beta \in (1,2), \quad \alpha \in [0.5,1], \quad \lambda > 0$$

and the attenuated wave equation [17, 88]

$$\frac{\partial^2}{\partial t^2} u + \mu\, {}^R D_0^{\beta} u - \lambda \Delta u = F, \quad \beta \in (0,1) \cup (1,2).$$

We point out that the operators A and B in (4.4) are not necessarily linear.

In case if $\Phi = 0$, IP2 means a reconstruction of a source that was active in the past using a measurement of the state of u in a left neighbourhood of T. Such an inverse problem may occur in ground water pollution, seismology, etc.

Now we reduce IP2 to IP1. Let (u,F) solve IP2. Then the equation (4.4) restricted to $\Omega \times (t_0,T)$ has the form $(D_0^{\{k\},n} Bu)(x,t) + D^l \varphi(x,t) - A\varphi(x,t) = \Phi(x,t)$. Therefore, Bu is a solution of the following family of IP1:

$$Bu|_{\Omega \times (t_0,T)} = B\varphi \quad \text{and} \quad D_0^{\{k\},n} Bu|_{\Omega \times (t_0,T)} = g, \tag{4.6}$$

where

$$g(x,t) = \Phi(x,t) + A\varphi(x,t) - D^l \varphi(x,t), \quad x \in \Omega,\ t \in (t_0,T). \tag{4.7}$$

The solution of IP2 is expressed by means of Bu explicitly:

$$u = B^{-1} Bu, \quad F = D_0^{\{k\},n} Bu + D^l u - Au.$$

4.2 Dual problem for IP1

Let us consider the case $n = 1$. We assume that k is a Sonine kernel and M is its associate, i.e. $M * k = 1$. Then firstly for $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$ and $u \in W_1^1(0,T)$ we have

$${}^R D_0^{\{M\},1}\, {}^R D_0^{\{k\},1} u(t) = \frac{d}{dt} M * \frac{d}{dt} k * u = \frac{d^2}{dt^2} M * k * u = u'(t), \quad t \in (0,T).$$

Secondly, in case $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$ for $u \in W_\infty^1(0,T)$, $k * u' \in W_1^1(0,T)$ we have

$${}^C D_0^{\{M\},1}\, {}^C D_0^{\{k\},1} u(t) = M * \frac{d}{dt} k * u' = \frac{d}{dt} M * k * u' = u'(t), \quad t \in (0,T).$$

Then, based on the relations (4.3), we write the problem for $v(t) = D_0^{\{k\},1} u(t)$

$$v|_{(t_0,T)} = g, \quad D_0^{\{M\},1} v|_{(t_0,T)} = \varphi', \tag{4.8}$$

which we call dual to the IP1 (4.3). After solving the problem (4.8) we can compute u that satisfies (4.3). In case $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$ we compute u by the formula

$$u(t) = M * v(t), \quad t \in (0,T). \tag{4.9}$$

In case $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$ we obtain $u' = \frac{d}{dt} M * v$ and therefore

$$u(t) = \lim_{\tau \to t_0^+} \varphi(\tau) + M * v(t) - M * v(t_0), \quad t \in (0,T). \tag{4.10}$$

4.3 Uniqueness results

Lemma 4.1. *Let k be real analytic in $(0,\infty)$ and $v \in L_1(0,t_0)$. Then $w(t) = \int_0^{t_0} k(t-\tau)v(\tau)d\tau$ is real analytic in (t_0,∞).*

Proof. The function k can be extended as a complex analytic function $k_{\mathbb{C}}$ in an open domain $D \subset \mathbb{C}$ containing the positive part of the real axis. Let us define

$$w_{\mathbb{C}}(z) = \int_0^{t_0} k_{\mathbb{C}}(z-\tau)v(\tau)d\tau \quad \text{for} \quad z \in D_{t_0} = \{z : z = \xi + t_0,\, \xi \in D\}.$$

Using the analyticity of $k_{\mathbb{C}}$, it is possible to show that functions u and v involved in the formula $w_{\mathbb{C}}(t+is) = u(t,s) + iv(t,s)$, are continuously differentiable and satisfy Cauchy-Riemann equations in $\{(t,s) : t+is \in D_{t_0}\}$. This implies that $w_{\mathbb{C}}$ is complex analytic in D_{t_0}. Its restriction to the subset $\{z = t + i0 \,:\, t \in (t_0,\infty)\}$ is the real function w, therefore w is real analytic in (t_0,∞). □

We prove a uniqueness theorem for IP1.

Theorem 4.1. *Assume that k satisfies the following conditions:*

$$\exists \mu \in \mathbb{R} : \int_0^\infty e^{-\mu t}|k(t)|dt < \infty, \tag{4.11}$$

k is real analytic in $(0,\infty)$, (4.12)

$\widehat{k}(s)$ cannot be meromorphically extended to the whole complex plane $\mathbb{C}$. (4.13)

Then the following assertions hold.

(i) *If $u \in L_1(0,T)$, $k * u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^R D_0^{\{k\},n} u|_{(t_0,T)} = 0$ then $u = 0$.*

(ii) *If $u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^C D_0^{\{k\},n} u|_{(t_0,T)} = 0$ then $u = 0$.*

Proof. (i) Let us extend $u(t)$ by zero for $t > T$ and define the function $f : (0,\infty) \to \mathbb{R}$:

$$f = {}^R D_0^{\{k\},n} u.$$

Since $u(t) = 0$, $t > t_0$, it holds that

$$f(t) = \frac{d^n}{dt^n} \int_0^{t_0} k(t-\tau)u(\tau)d\tau = \int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau, \quad t > t_0.$$

The function k is real analytic, therefore $k^{(n)}$ is also real analytic. Hence, Lemma 4.1 implies that f is real analytic in (t_0,∞). Since $f(t)=0, t\in(t_0,T)$, and f is real analytic we obtain that $f(t)=0, t>t_0$.

Due to (4.11) the $\widehat{k}(s)$ exists and is holomorphic for $\mathrm{Re}s>\mu$. Moreover, in view the properties of f, the $\widehat{f}(s)$ also exists and is expressed by the formula

$$\widehat{f}(s)=s^n\widehat{k}(s)\widehat{u}(s)-p_0s^{n-1}-...-p_{n-1},\quad \mathrm{Re}s>\mu,\quad p_j=\frac{d^j}{dt^j}(k*u)(t)\Big|_{t=0}. \tag{4.14}$$

Since the values $f(t)$ and $u(t)$ vanish for $t>t_0$, $\widehat{f}$ and $\widehat{u}$ are entire functions. Thus, the functions $s^n\widehat{u}(s)$ and $\widehat{f}(s)+p_0s^{n-1}+...+p_{n-1}$ are also entire. Assume that $\widehat{u}$ does not vanish on $\mathbb{C}$. The Identity theorem and the fact that $\widehat{u}$ is entire imply that the set of zeros of $\widehat{u}$ does not contain accumulation points. Then it follows from (4.14) that

$$\widehat{k}(s)=\frac{\widehat{f}(s)+p_0s^{n-1}+...+p_{n-1}}{s^n\widehat{u}(s)}\quad \text{for any } s \text{ such that } \mathrm{Re}s>\mu \text{ and } s^n\widehat{u}(s)\neq 0.$$

Therefore, the extension of $\widehat{k}$ is meromorphic on $\mathbb{C}$. This contradicts to the assumption (4.13) of the theorem. Thus, the assumption $\widehat{u}\not\equiv 0$ is invalid, which implies $u=0$ in $L_1(0,T)$.

(ii) At this part of the proof let us use the notation $v:=u^{(n)}$. Then

$$v|_{(t_0,T)}={}^RD_0^{\{k\},0}v|_{(t_0,T)}=0$$

and v, $k*v\in L_1(0,T)$. Therefore, by the assertion (i) of this theorem $v=0$. Consequently, $u^{(n)}=0$ and $u|_{(t_0,T)}=0$ imply that $u=0$ in $W_1^n(0,T)$. □

Let us check if the kernels from the Section 1.7 satisfy the conditions of the Theorem 4.1.

All of the kernels satisfy (4.11) and (4.12). Moreover, it is evident that the kernels (k1), (M1), (k2), (M2), (k4), (M4), (k5), (M5), (k6), (M6), (k8) satisfy (4.13), because Laplace transforms of these functions have branch points. To show that (k3) also satisfies (4.13) we compute the limit under assumption that $q\geq 0$ and $q\neq 0$

$$\lim_{\substack{\mathrm{Arg}\,s\to\pm\pi\\ |s|=1}}\mathrm{Im}\widehat{k}(s)=\int_0^1 q(\beta)\sin((\beta-1)(\pm\pi))d\beta \ \substack{<\\>}\ 0.$$

This shows that $\widehat{k}(s)$ has a jump at $s=-1$, hence (4.13) holds. Similar result is valid for (M3).

Summing up, the solution of IP1 for a derivative containing a kernel (k1) - (k6), (M1) - (M6) or (k8) is unique.

The kernel of Caputo-Fabrizio fractional derivative (k7) does not satisfy (4.13), because it has the meromorphic in $\mathbb{C}$ Laplace transform. IP1 with this kernel has infinitely many solutions. In case $D_0^{\{k\},n}={}^RD_0^{\{k\},n}$, the solution to homogeneous IP1 is any function u that satisfies the condition

$$\int_0^{t_0}e^{\frac{\beta}{1-\beta}\tau}u(\tau)d\tau=0,\quad u|_{(t_0,T)}=0.$$

Similarly, in case $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$, the solution to homogeneous IP1 is u such that

$$\int_0^{t_0} e^{\frac{\beta}{1-\beta}\tau} u^{(n)}(\tau)d\tau = 0, \quad u|_{(t_0,T)} = 0.$$

Now we proceed to IP2. We introduce the set of functions $\mathscr{U}$ related to the operators A, B and D^l:

$$\mathscr{U} = \{u : \Omega \times (0,T) \to \mathbb{R} : u(\cdot,t) \in \mathscr{D}(A) \cap \mathscr{D}(B)\ \forall t \in (0,T),$$
$$u, Au, Bu \in C(\Omega \times (0,T)) \text{ and } q_j \frac{\partial^j}{\partial t^j} u \in C(\Omega \times (0,T)),\ j = 1,\dots,l\}.$$

From Theorem 4.1 we can immediately deduce a uniqueness statement for IP2.

Corollary 4.1. *Let k satisfy* (4.11) - (4.13). *Then the following assertions hold.*

(i) *If* $(u_j, F_j) \in \{u \in \mathscr{U} : (k * Bu)(x,\cdot) \in W_1^n(0,T)\, \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, *solve* (4.4) *with* $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$ *and* $(u_1,F_1)|_{\Omega\times(t_0,T)} = (u_2,F_2)|_{\Omega\times(t_0,T)}$ *then* $(u_1,F_1) = (u_2,F_2)$.

(ii) *If* $(u_j, F_j) \in \{u \in \mathscr{U} : Bu(x,\cdot) \in W_1^n(0,T)\, \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, *solve* (4.4) *with* $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$ *and* $(u_1,F_1)|_{\Omega\times(t_0,T)} = (u_2,F_2)|_{\Omega\times(t_0,T)}$ *then* $(u_1,F_1) = (u_2,F_2)$.

Proof. Proof is technically the same in cases (i) and (ii). The condition $u_1|_{\Omega\times(t_0,T)} = u_2|_{\Omega\times(t_0,T)}$ implies

$$(Bu_1 - Bu_2)|_{\Omega\times(t_0,T)} = 0. \tag{4.15}$$

After subtracting the equations (4.4) corresponding to (u_1,F_1) and (u_2,F_2) we obtain the equation

$$D_0^{\{k\},n}(Bu_1 - Bu_2)(x,t) + D^l(u_1-u_2)(x,t) - (Au_1 - Au_2)(x,t) = (F_1 - F_2)(x,t),$$
$$x \in \Omega,\ t \in (0,T). \tag{4.16}$$

Since $(u_1,F_1)|_{\Omega\times(t_0,T)} = (u_2,F_2)|_{\Omega\times(t_0,T)}$ we get from (4.16)

$$D_0^{\{k\},n}(Bu_1 - Bu_2)|_{\Omega\times(t_0,T)} = 0. \tag{4.17}$$

Then by applying Theorem 4.1 to the problem (4.15), (4.17) we obtain that $Bu_1 = Bu_2$. Consequently, since the operator B is injective it holds $u_1 = u_2$. Finally, the equation (4.16) implies $F_1 = F_2$. □

4.4 Reduction to integral equations

In this subsection we reduce IP1 to integral equations. Let us assume that k satisfies (4.12).

Firstly, we consider the case $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$. Assume that $u \in L_1(0,T)$ solves IP1 and $k * u \in W_1^n(0,T)$. Then for $t \in (t_0,T)$

$$\int_0^t k(t-\tau)u(\tau)d\tau = \int_0^{t_0} k(t-\tau)u(\tau)d\tau + \int_{t_0}^t k(t-\tau)\varphi(\tau)d\tau, \tag{4.18}$$

where the left hand side belongs to $W_1^n(t_0,T)$ and the first addend in the right-hand side belongs to $C^\infty(t_0,T]$. Thus, for any $\delta\in(t_0,T)$ the data φ necessarily satisfies

$$\int_{t_0}^{t} k(t-\tau)\varphi(\tau)d\tau \in W_1^n(t_0+\delta,T).$$

Applying $\frac{d^n}{dt^n}$ to (4.18) we obtain:

$${}^R D_0^{\{k\},n} u(t) = \int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau + {}^R D_{t_0}^{\{k\},n}\varphi(t), \quad t\in(t_0,T).$$

Using the second condition in (4.3) and rearranging the terms we obtain the following integral equation of the first kind for $u|_{(0,t_0)}$:

$$\int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau = f(t),\ t\in(t_0,T), \quad \text{where} \quad f = g - {}^R D_{t_0}^{\{k\},n}\varphi. \tag{4.19}$$

Secondly, let us consider the case $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$, $n\geq 1$. In a similar manner we conclude that if $u\in W_1^n(0,T)$ solves IP1 then $u^{(n)}|_{(0,t_0)}$ is a solution of the integral equation

$$\int_0^{t_0} k(t-\tau)u^{(n)}(\tau)d\tau = f(t),\ t\in(t_0,T), \quad \text{where} \quad f = g - {}^C D_{t_0}^{\{k\},n}\varphi. \tag{4.20}$$

Since $\lim\limits_{\tau\to t_0^-} u^{(j)}(\tau) = \lim\limits_{\tau\to t_0^+}\varphi^{(j)}(\tau)$, $j=0,\dots,n-1$, the function $u|_{(0,t_0)}$ is obtained from $u^{(n)}|_{(0,t_0)}$ by the integration:

$$u(t) = \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{(n-1)!}u^{(n)}(\tau)d\tau + \sum_{j=0}^{n-1}\lim_{\tau\to t_0^+}\varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!}, \quad t\in(0,t_0).$$

Due to Lemma 4.1, the integral operators involved in (4.19) and (4.20) map $L_1(0,t_0)$ into the space of functions that are real analytic in $t>t_0$. This means that IP1 is severely ill-posed and necessarily, f is real analytic in (t_0,T). In the next section we will derive solution formulas for IP1 that contain the quantities

$$f^{(m)}(t_1), \quad m\in\{0\}\cup\mathbb{N},$$

where t_1 is an arbitrary point in (t_0,T).

4.5 Solution formula to an integral equation with a power-type kernel

Theorem 4.2. *Let $\alpha\in\mathbb{R}\setminus\mathbb{Z}$, $t_1>t_0>0$ and $f\in C^\infty(t_0,\infty)$. Let us introduce the following family of sums that depend on a variable $t\in(0,t_0)$ and parameters α, f, t_1, t_0:*

$$V_N(\alpha,f,t_1,t_0)(t) = (t_1-t)^{-\alpha-2}\sum_{n=0}^{N} A_n P_n\left(\frac{2t_1(t_1-t_0)}{t_0(t_1-t)} - \frac{2t_1-t_0}{t_0}\right).$$

Here $N\in\{0\}\cup\mathbb{N}\cup\{\infty\}$, P_n are normalized in $L_2(-1,1)$ Legendre polynomials

$$P_n(t) = \sum_{l=0}^{\lfloor\frac{n}{2}\rfloor} c_{n,l}t^{n-2l}, \text{ where } c_{n,l} = \sqrt{\frac{2n+1}{2}}\frac{1}{2^n}(-1)^l\binom{n}{l}\binom{2n-2l}{n},$$

and

$$A_n = A_n(\alpha, f, t_1, t_0) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m} \left(\frac{t_0 - 2t_1}{t_0}\right)^{n-2l-m}$$
$$\times \left(\frac{2t_1(t_1-t_0)}{t_0}\right)^{m+1} \Gamma(\alpha - m + 1) f^{(m)}(t_1).$$

Assume that the given function f is such that the equation

$$\int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau = f(t), \quad t > t_0 \tag{4.21}$$

has a solution $v \in L_2(0,t_0)$.

Then the series $V_\infty(\alpha, f, t_1, t_0)(t)$ converges almost everywhere in $(0,t_0)$ and

$$v(t) = V_\infty(\alpha, f, t_1, t_0)(t), \quad a.e.\ t \in (0,t_0). \tag{4.22}$$

Moreover, $V_N(\alpha, f, t_1, t_0) \to v$ in $L_2(0,t_0)$ as $N \to \infty$. If in addition, $v \in BV[0,t_0]$ [3]*, then $V_\infty(\alpha, f, t_1, t_0)(t)$ converges pointwise in $(0,t_0)$ and the estimate is valid:*

$$|v(t) - V_N(\alpha, f, t_1, t_0)(t)| \leq \frac{c(t)}{N}, \quad t \in (0,t_0),$$

where $c(t)$ is a positive constant depending on t.

Proof. After taking the n-th derivative of (4.21) we obtain for $t_1 > t_0$

$$\frac{1}{\Gamma(\alpha - n + 1)} \int_0^{t_0} (t_1 - \tau)^{\alpha - n} v(\tau) d\tau = f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N}. \tag{4.23}$$

The substitution $s = \frac{1}{t_1 - \tau}$ under the integral takes (4.23) to the form

$$\int_{\frac{1}{t_1}}^{\frac{1}{t_1 - t_0}} s^n w(s) ds = \Gamma(\alpha - n + 1) f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N} \tag{4.24}$$

where $w(s) = s^{-\alpha-2} v\left(t_1 - \frac{1}{s}\right)$.

We would like to expand our function into series by means of orthonormal Legendre polynomials $P_n \in L_2(-1,1)$. Thus, we apply a linear substitution that takes us from $[\frac{1}{t_1}, \frac{1}{t_1 - t_0}]$ to the interval $[-1,1]$:

$$\widetilde{s} = as + b, \quad \text{where} \quad a = \frac{2t_1(t_1 - t_0)}{t_0}, \quad b = -\frac{2t_1 - t_0}{t_0}.$$

By applying this substitution to (4.24) we obtain

$$\int_{-1}^{1} \frac{1}{a^{n+1}} (\widetilde{s} - b)^n \widetilde{w}(\widetilde{s}) d\widetilde{s} = \Gamma(\alpha - n + 1) f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N} \tag{4.25}$$

where $\widetilde{w}(\widetilde{s}) = w(s)$.

Since the performed changes of variables under the integrals are diffeomorphic, $v \in L_2(0,t_0)$ implies $w \in L_2(\frac{1}{t_1}, \frac{1}{t_1 - t_0})$ and $\widetilde{w} \in L_2(-1,1)$ (cf. [40] Section 16.4). Similarly, $v \in BV[0,t_0]$ implies $\widetilde{w} \in BV[-1,1]$.

[3] $BV[0,t_0]$ is a space of functions of bounded variation on $[0,t_0]$.

Since $\widetilde{w} \in L_2(-1,1)$, it can be expanded into the Fourier-Legendre series. Let us derive this expansion. It follows from (4.25) that

$$\int_{-1}^{1} \widetilde{s}^n \widetilde{w}(\widetilde{s}) d\widetilde{s} = \int_{-1}^{1} ((\widetilde{s}-b)+b)^n \widetilde{w}(\widetilde{s}) d\widetilde{s} = \sum_{m=0}^{n} \binom{n}{m} b^{n-m} \int_{-1}^{1} (\widetilde{s}-b)^m \widetilde{w}(\widetilde{s}) d\widetilde{s}$$
$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^{m+1} \Gamma(\alpha - m + 1) f^{(m)}(t_1), \quad n \in \{0\} \cup \mathbb{N}.$$

It implies that for the normalized Legendre polynomials

$$\int_{-1}^{1} P_n(\widetilde{s}) \widetilde{w}(\widetilde{s}) d\widetilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \int_{-1}^{1} \widetilde{s}^{n-2l} \widetilde{w}(\widetilde{s}) d\widetilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m}$$
$$\times b^{n-2l-m} a^{m+1} \Gamma(\alpha - m + 1) f^{(m)}(t_1) = A_n.$$

Then

$$\widetilde{w}(\widetilde{s}) = \sum_{n=0}^{\infty} A_n P_n(\widetilde{s}). \tag{4.26}$$

The series (4.26) converges in $L_2(-1,1)$ and for almost every $\widetilde{s} \in (-1,1)$ [71].

For $\widetilde{w} \in BV[-1,1]$ the series (4.26) is convergent pointwise for $\widetilde{s} \in (-1,1)$ and according to the Theorem 1 [7]

$$|\widetilde{w}(\widetilde{s}) - \sum_{n=0}^{N} A_n P_n(\widetilde{s})| \le \frac{c_1(\widetilde{s})}{N}, \quad \widetilde{s} \in (-1,1),$$

where $c_1(\widetilde{s})$ is a positive constant.

Since the change of variables $\widetilde{s} = \frac{a}{t_1 - t} + b$, $t \in [0, t_0]$, is diffeomorphic and

$$v(t) = (t_1 - t)^{-\alpha - 2} \widetilde{w}\left(\frac{a}{t_1 - t} + b\right),$$

all assertions of the theorem follow from the proved properties of the series (4.26). □

Remark 4.1. It follows from (4.22) that for f of form $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$, $t > t_0$, where $v \in L_2(0,t_0)$, the sum of series $V_\infty(\alpha, f, t_1, t_0)(t)$ is independent of $t_1 > t_0$.

The partial sums $V_N(\alpha, f, t_1, t_0)(t)$, $N < \infty$, however, still may depend on t_1.

For example, if $v = 1$ then $V_0(\alpha, f, t_1, t_0)(t) = \frac{\sqrt{0.5}}{\alpha+1} (t_1 - t)^{-\alpha-2} \left[t_1^{\alpha+1} - (t_1 - t_0)^{\alpha+1}\right]$.

4.6 Solution formulas for inverse problems in case of usual fractional derivatives

In this Section we consider IP1 and IP2 in case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$, $n \ge 1$. Then ${}^R D_0^{\{k\},n}$ and ${}^C D_0^{\{k\},n}$ are the Riemann-Liouville and Caputo fractional derivatives of the order $n + \beta - 1$ given by the formulas (4.1) and (4.2), respectively.

Theorem 4.3. *Let $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$, $n \ge 1$. Then the following assertions hold.*

(i) *If* $u \in L_2(0,T)$, $k * u \in W_1^n(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},n} = {}^R D_0^{n+\beta-1}$ *then*

$$u(t) = \mathscr{F}_{R,t_1}^{\beta,n}(g - {}^R D_{t_0}^{n+\beta-1}\varphi)(t), \quad a.e.\ t \in (0,t_0), \tag{4.27}$$

where the operator $\mathscr{F}_{R,t_1}^{\beta,n}$ *is given by the rule*

$$\mathscr{F}_{R,t_1}^{\beta,n}(f)(t) = V_\infty(-n-\beta, f, t_1, t_0)(t). \tag{4.28}$$

(ii) *If* $u \in W_2^n(0,T)$, *solves* IP1 *with* $D_0^{\{k\},n} = {}^C D_0^{n+\beta-1}$ *then*

$$u(t) = \mathscr{F}_{C,t_1}^{\beta,n}(\varphi; g - {}^C D_{t_0}^{n+\beta-1}\varphi)(t), \quad t \in (0,t_0), \tag{4.29}$$

where

$$\mathscr{F}_{C,t_1}^{\beta,n}(\varphi;f)(t) = \sum_{j=0}^{n-1} \lim_{\tau \to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{\Gamma(n)} V_\infty(-\beta, f, t_1, t_0)(\tau)d\tau. \tag{4.30}$$

The formulas (4.27) *and* (4.29) *are valid for any* $t_1 \in (t_0, T)$.

Proof. (i) Firstly, we represent the IP1 in form (4.19) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. That is identical to (4.21) with $\alpha = -n-\beta$, $v(t) = u(t)$ and $f(t) = g(t) - {}^R D_{t_0}^{n+\beta-1}\varphi(t)$. Thus, the Theorem 4.2 implies (4.27).

(ii) Similarly to the previous case we start from representing the problem in a form (4.20) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. This gives us the relation (4.21) with $\alpha = -\beta$, $v(t) = u^{(n)}(t)$ and $f(t) = g(t) - {}^C D_{t_0}^{n+\beta-1}\varphi(t)$. By applying Theorem 4.2 to it we obtain

$$u^{(n)}(t) = V_\infty(-\beta, f, t_1, t_0)(t), \ \ a.e.\ t \in (0,t_0), \quad f = g - {}^C D_{t_0}^{n+\beta-1}\varphi.$$

Since the condition $u|_{(t_0,T)} = \varphi$ implies

$$u^{(j)}(t_0) = \lim_{\tau \to t_0^+} \varphi^{(j)}(\tau), \quad j = 0 \ldots n-1,$$

the solution formula (4.29) is valid. □

Remark 4.2. Let us consider the approximations of the exact solutions to IP1 with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. In case (i) this is given by the formula

$$u_{N,t_1}(t) = V_N(-n-\beta, f, t_1, t_0)(t), \quad t \in (0,t_0), \quad N < \infty,$$
$$\text{where} \quad f(t) = g(t) - {}^R D_{t_0}^{n+\beta-1}\varphi(t).$$

In case (ii) the approximation is given the formula

$$u_{N,t_1}(t) = \sum_{j=0}^{n-1} \lim_{\tau \to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{\Gamma(n)} V_N(-\beta, f, t_1, t_0)(\tau)d\tau,$$
$$t \in (0,t_0), \quad N < \infty, \quad \text{where} \quad f(t) = g(t) - {}^C D_{t_0}^{n+\beta-1}\varphi(t).$$

Then Theorem 4.2 can be used to compare u_{N,t_1} with u in the process $N \to \infty$. In case (i), $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ in $L_2(0,t_0)$ and $u_{N,t_1}(t) \to u(t)$ a.e. $t \in (0,t_0)$. Similarly, in case (ii), $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ in $W_2^n(0,t_0)$ and $u_{N,t_1}^{(n)}(t) \to u^{(n)}(t)$ a.e. $t \in (0,t_0)$.

If in addition to the assumptions of (i), $u|_{(0,t_0)} \in BV[0,t_0]$ holds then $|u_{N,t_1}(t) - u(t)|$ is of the order $1/N$ for every $t \in (0,t_0)$. Similarly, if in addition to the assumptions of (ii), $u^{(n)}|_{(0,t_0)} \in BV[0,t_0]$ is valid then $|u_{N,t_1}^{(n)}(t) - u^{(n)}(t)|$ is of the order $1/N$ for every $t \in (0,t_0)$.

The computation of u_{N,t_1} is moderately ill-posed problem, because it contains the derivatives of the finite order of the data.

Corollary 4.2. *Let $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$, $n \geq 1$. Then the following assertions hold.*

(i) *If $(u,F) \in \{u \in \mathscr{U} : (k * Bu)(x,\cdot) \in W_1^n(0,T)\, \forall x \in \Omega\} \times C(\Omega \times (0,T))$ solves IP2 with $D_0^{\{k\},n} = {}^R D_0^{n+\beta-1}$ then*

$$u(x,t) = \left[B^{-1} \mathscr{F}_{R,t_1}^{\beta,n}\big(g(x,\cdot) - {}^R D_{t_0}^{n+\beta-1} \varphi(x,\cdot)\big)\right](t), \quad (x,t) \in \Omega \times (0,t_0).$$

(ii) *If $(u,F) \in \{u \in \mathscr{U} : Bu(x,\cdot) \in W_2^n(0,T)\, \forall x \in \Omega\} \times C(\Omega \times (0,T))$, solves IP2 with $D_0^{\{k\},n} = {}^C D_0^{n+\beta-1}$ then*

$$u(x,t) = \left[B^{-1} \mathscr{F}_{C,t_1}^{\beta,n}\big(\varphi(x,\cdot); g(x,\cdot) - {}^C D_{t_0}^{n+\beta-1} \varphi(x,\cdot)\big)\right](t), \quad (x,t) \in \Omega \times (0,t_0).$$

In both cases g is given by (4.7), t_1 is an arbitrary number in (t_0,T) and $F|_{\Omega \times (0,t_0)} = \left[D_0^{\{k\},n} Bu + D^l u - Au\right]\Big|_{\Omega \times (0,t_0)}$.

Proof. The proof follows from the Theorem 4.3 and the relations (4.6), (4.7) that describe the transition from IP2 to IP1. □

4.7 Solution formulas in case of tempered and Atangana-Baleanu derivatives

In this subsection we derive the solution formulas for particular subcases of the generalized fractional derivative of the order $n = 1$. They are based on solution formulas derived for the usual fractional derivative and involve the operators $\mathscr{F}_{R,t_1}^{\beta,1}$, $\mathscr{F}_{C,t_1}^{\beta,1}$. Again we assume that t_1 is an arbitrary number in the interval (t_0,T).

Firstly, let us consider case of tempered fractional derivative with the kernel (k5):

Theorem 4.4. *Let $k(t) = \frac{e^{-\lambda t} t^{-\beta}}{\Gamma(1-\beta)} + \lambda \int_0^t \frac{e^{-\lambda \tau} \tau^{-\beta}}{\Gamma(1-\beta)} d\tau$, $0 < \beta < 1$, $\lambda > 0$. Then the following assertions hold.*

(i) *If $u \in L_2(0,T)$, $k * u \in W_1^1(0,T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$ then*

$$u(t) = e^{-\lambda t} \mathscr{F}_{R,t_1}^{\beta,1}(e^{\lambda t} g - e^{\lambda t R} D_{t_0}^{\{k\},1} \varphi)(t), \quad a.e.\ t \in (0,t_0). \qquad (4.31)$$

(ii) *If $u \in W_2^1(0,T)$ solves IP1 with $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$ then*

$$u(t) = \lim_{\tau \to t_0^+} \varphi(\tau) - \int_t^{t_0} e^{-\lambda \tau} \mathscr{F}_{R,t_1}^{\beta,1}\left(e^{-\lambda \tau}(g - {}^R D_{t_0}^{\{k\},1} \varphi)'\right)(\tau) d\tau, \quad t \in (0,t_0). \quad (4.32)$$

Proof. Before starting the proof, let us point out that $k'(t) = \frac{e^{-\lambda t} t^{-1-\beta}}{\Gamma(-\beta)}$. Hence, for $t \in (t_0, T)$ and $v \in L_1(0, t_0)$:

$$\int_0^{t_0} k'(t-\tau) v(\tau) d\tau = e^{-\lambda t} \int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda \tau} v(\tau) d\tau. \tag{4.33}$$

(i) Firstly, the IP1 can be rewritten by means of (4.19) and then formula (4.33) leads us to the equation with the unknown term $e^{\lambda t} u(t)$

$$\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda \tau} u(\tau) d\tau = e^{\lambda t} g(t) - e^{\lambda t R} D_{t_0}^{\{k\},1} \varphi(t), \quad t \in (t_0, T).$$

Thus, by applying Theorem 4.2 and using the notation (4.28) we obtain (4.31).

(ii) Let us write IP1 in the form (4.20), differentiate it and obtain for $t \in (t_0, T)$

$$\int_0^{t_0} k'(t-\tau) u'(\tau) d\tau = \frac{d}{dt} (g(t) - {}^C D_{t_0}^{\{k\},1} \varphi(t)).$$

Then due to (4.33) we have

$$\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda \tau} u'(\tau) d\tau = e^{\lambda t} \frac{d}{dt} (g(t) - {}^C D_{t_0}^{\{k\},1} \varphi(t))$$

and similarly to (i) we deduce the formula (4.32) using the notation (4.30). □

Now we deal with the case (k4).

Theorem 4.5. *Let* $k(t) = \frac{1}{\Gamma(1-\beta)} e^{-\lambda t} t^{-\beta}, 0 < \beta < 1, \lambda > 0$. *Then the following assertions hold.*

(i) *If* ${}^R D_0^{\{k\},1} u \in L_2(0,T)$, $u \in W_1^1(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$ *then*

$$u(t) = M * e^{-\lambda t} \mathscr{F}_{R,t_1}^{1-\beta,1} (e^{\lambda t} \varphi' - e^{\lambda t R} D_{t_0}^{\{M\},1} g)(t), \quad t \in (0, t_0), \tag{4.34}$$

where $M(t) = \frac{1}{\Gamma(\beta)} e^{-\lambda t} t^{\beta - 1} + \frac{\lambda}{\Gamma(\beta)} \int_0^t e^{-\lambda \tau} \tau^{\beta-1} d\tau$ *is the associate kernel to* k.

(ii) *If* $u \in W_2^1(0,T)$ *solves* IP1 *with* $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$ *then*

$$u(t) = \lim_{\tau \to t_0^+} \varphi(\tau) - \int_t^{t_0} e^{-\lambda \tau} \mathscr{F}_{R,t_1}^{\beta,0} \left(e^{-\lambda \tau} (g - {}^C D_{t_0}^{\{k\},1} \varphi) \right)(\tau) d\tau, \quad t \in (0, t_0). \tag{4.35}$$

Proof. (i) Let us consider the dual problem for the function $v = {}^R D_0^{\{k\},1} u$, that is (4.8). Therefore, we apply Theorem 4.4(i) to the dual problem (4.8) (with replacing β by $1-\beta$ and k by M in the formulation) and obtain:

$$v = e^{-\lambda t} \mathscr{F}_{R,t_1}^{1-\beta,1} (e^{\lambda t} \varphi' - e^{\lambda t R} D_{t_0}^{\{M\},1} g)(t), \quad a.e. \ t \in (0, t_0).$$

Then by formula (4.9) we obtain the formula (4.34).

(ii) Though it is possible to handle this case similarly to (i) by reducing it to Theorem 4.4(ii), we treat this problem directly to derive a simpler solution formula.

The IP1 (4.3) is reduced to (4.20) with $n = 1$. Thus,

$$\int_0^{t_0} \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} e^{\lambda \tau} u'(\tau) d\tau = e^{\lambda t} (g - {}^C D_{t_0}^{\{k\},1} \varphi).$$

By applying Theorem 4.2 to the problem above and using the notation (4.28) we obtain

$$u'(t) = e^{-\lambda t}\mathscr{F}^{\beta,0}_{R,t_1} e^{\lambda t}(g - {}^{C}D^{\{k\},1}_{t_0}\varphi)(t), \quad t \in (t_0, T).$$

This implies the formula (4.35). □

To handle IP1 for derivatives that contain Mittag-Leffler functions, we need the following lemma.

Lemma 4.2. *Let $0 < \beta < 1$, $\lambda \in \mathbb{R}$ and $f \in W^1_1(0,T)$.*

Then the function $p(t) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(\lambda(t-\tau)^\beta)f(\tau)d\tau$ is a solution of the equation

$${}^{C}D^{\beta}_0 p(t) - \lambda p(t) = f(t), \quad t \in (0,T),$$

and the function $q(t) = \int_0^t E_\beta(\lambda(t-\tau)^\beta)f(\tau)d\tau$ is a solution of the equation

$${}^{C}D^{\beta}_0 q(t) - \lambda q(t) = I^{1-\beta}_0 f(t), \quad t \in (0,T).$$

Proof. The proof of the first assertion can be found e.g. in [20], p. 174, and the second assertion follows from the first one because $[t^{\beta-1}E_{\beta,\beta}(\lambda t^\beta)] * I^{1-\beta}_0 f = E_\beta(\lambda t^\beta) * f$ [37]. □

Next we consider the case of a tempered fractional derivative with the kernel (M6) denoted by k.

Theorem 4.6. *Let $k(t) = e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)$, $\frac{1}{2} < \beta < 1$, $\lambda > 0$. Then the following assertions are valid:*

(i) *If $u \in W^1_1(0,T)$ and u solves* IP1 *with $D^{\{k\},1}_0 = {}^{R}D^{\{k\},1}_0$ then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau}({}^{R}D^{\beta}_0 - \lambda^\beta I)\mathscr{F}^{\beta,1}_{R,t_1}\left(e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^{R}D^{\beta}_{t_0}e^{\lambda\tau}g\right)(\tau)d\tau + \lim_{\tau\to t_0}\varphi(\tau),$$
$$t \in (0, t_0). \tag{4.36}$$

(ii) *If $u \in W^2_1(0,T)$ and u solves* IP1 *with $D^{\{k\},1}_0 = {}^{C}D^{\{k\},1}_0$ then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau}({}^{C}D^{\beta}_0 - \lambda^\beta I)\mathscr{F}^{\beta,1}_{C,t_1}\left(e^{\lambda\tau}g; e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^{C}D^{\beta}_{t_0}e^{\lambda\tau}g\right)(\tau)d\tau + \lim_{\tau\to t_0}\varphi(\tau),$$
$$t \in (0, t_0). \tag{4.37}$$

Proof. Firstly we prove (ii). Let us define the function w as

$$w(t) = e^{\lambda t}\,{}^{C}D^{\{k\},1}u(t) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(\lambda^\beta(t-\tau)^\beta)(e^{\lambda\tau}u'(\tau))d\tau.$$

Due to Lemma 4.2, this function solves the equation

$${}^{C}D^{\beta}_0 w(t) - \lambda^\beta w(t) = e^{\lambda t}u'(t), \quad t \in (0,T). \tag{4.38}$$

Therefore, ${}^{C}D^{\beta}_0 w = e^{\lambda t}u' + \lambda^\beta w$ and in view of the conditions (4.3) we have the IP1 with usual fractional derivative

$$w|_{(t_0,T)} = e^{\lambda t}g, \quad {}^{C}D^{\beta}_0 w|_{(t_0,T)} = e^{\lambda t}\varphi' + \lambda^\beta e^{\lambda t}g. \tag{4.39}$$

In order to apply Theorem 4.3 (ii) to this problem, we must verify that $w \in W_2^1(0,T)$ is valid. Let us compute:

$$w'(t) = t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)u'(0) + [t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)] * (e^{\lambda t}u')'(t).$$

Due to the assumptions $\frac{1}{2} < \beta < 1$ and $u \in W_1^2(0,T)$ we have $t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta) \in L_2(0,T)$ and $(e^{\lambda t}u')' \in L_1(0,T)$. Using the Young's theorem for convolutions, we deduce $w' \in L_2(0,T)$. Thus $w \in W_2^1(0,T)$.

By applying Theorem 4.3 (ii) to (4.39) we obtain

$$w(t) = \mathscr{F}_{C,t_1}^{\beta,1}(e^{\lambda t}g; e^{\lambda t}\varphi' + \lambda^\beta e^{\lambda t}g - {}^C D_{t_0}^\beta e^{\lambda t}g)(t), \quad t \in (0,t_0).$$

Since by (4.38), $u' = e^{-\lambda t}({}^C D_0^\beta - \lambda^\beta I)w$, this implies formula (4.37).

Secondly we prove (i). Let us define $w(t) = e^{\lambda t R}D_0^{\{k\},1}u(t)$. Then $w(t) = (\frac{d}{dt} - \lambda)z(t)$, where

$$z(t) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(\lambda^\beta(t-\tau)^\beta)(e^{\lambda\tau}u(\tau))d\tau.$$

By Lemma 4.2 z solves the equation

$$ {}^C D_0^\beta z(t) - \lambda^\beta z(t) = e^{\lambda t}u(t), \quad t \in (0,T). \tag{4.40}$$

Let us differentiate the equation (4.40) to derive the equation for w:

$${}^R D_0^\beta(z' - \lambda z)(t) + {}^R D_0^\beta(\lambda z)(t) - \lambda^\beta z'(t) = \lambda e^{\lambda t}u(t) + e^{\lambda t}u'(t), \quad a.e.\ t \in (0,T).$$

That is

$${}^R D_0^\beta w(t) - \lambda^\beta w(t) + \lambda({}^R D_0^\beta z(t) - \lambda^\beta z(t)) = \lambda e^{\lambda t}u(t) + e^{\lambda t}u'(t), \quad a.e.\ t \in (0,T).$$

Since $z(0) = 0$, we have that ${}^R D_0^\beta z = {}^C D_0^\beta z$ and using (4.40) again we obtain

$${}^R D_0^\beta w(t) = \lambda^\beta w(t) + e^{\lambda t}u'(t), \quad a.e.\ t \in (0,T). \tag{4.41}$$

Based on (4.41) and (4.3) we formulate IP1 for w:

$$w|_{(t_0,T)} = e^{\lambda t}g, \quad {}^R D_0^\beta w|_{(t_0,T)} = e^{\lambda t}(\varphi' + \lambda^\beta g). \tag{4.42}$$

To apply Theorem 4.3 (i) we should prove that $w \in L_2(0,T)$, and $\left(\frac{t^{-\beta}}{\Gamma(1-\beta)}\right) * w = I_0^{1-\beta}w \in W_1^1(0,T)$, that is ${}^R D_0^\beta w \in L_1(0,T)$. Let us investigate

$$w(t) = \left(\frac{d}{dt} - \lambda\right)\left(t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * (e^{\lambda t}u(t)) = u(0)\left(t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)\right)$$
$$+ \left(t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t}u(t))' - \lambda e^{\lambda t}u(t)), \quad t \in (0,T).$$

Since $t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta) \in L_2(0,T)$ for $\beta \in (1/2,1)$ and $e^{\lambda t}u(t) \in W_1^1(0,T)$ we obtain that $\left(t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t}u(t))' - \lambda e^{\lambda t}u(t)) \in L_2(0,T)$, thus $w \in L_2(0,T)$. Due to (4.41) ${}^R D_0^\beta w \in L_1(0,T)$, because $w \in L_2(0,T)$ and $u \in W_1^1(0,T)$.

That enables us to apply Theorem 3 (i) to (4.42):

$$w(t) = \mathscr{F}_{R,t_1}^{\beta,1}\left(e^{\lambda t}(\varphi' + \lambda^\beta g) - {}^R D_{t_0}^\beta e^{\lambda t}g\right)(t), \quad a.e.\ t \in (0,t_0).$$

This in view of (4.41) implies the formula (4.36). □

Remark 4.3. It is possible to extend the range of β to $0<\beta<1$ in Theorem 4.6 assuming more regularity of u and the conditions $u(0)=0$ and $u'(0)=0$ in cases (i) and (ii), respectively.

Finally, we consider the case of Atangana-Baleanu fractional derivative.

Theorem 4.7. *Let* $k(t)=\frac{1}{1-\beta}E_\beta\left(-\frac{\beta t^\beta}{1-\beta}\right)$, $0<\beta<1$. *Then the following assertions hold:*

(i) *If* $u\in W_1^1(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},1}={}^RD_0^{\{k\},1}$ *then*

$$u(t)=\left(\frac{1-\beta}{\beta}\,{}^RD_0^\beta+I\right)\mathscr{F}_{R,t_1}^{\beta,1}\left(\beta g-{}^RD_{t_0}^\beta(\varphi-(1-\beta)g)\right)(t),$$
$$t\in(0,t_0). \tag{4.43}$$

(ii) *If* $u\in W_1^2(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},1}={}^CD_0^{\{k\},1}$ *then*

$$u(t)=\left(\frac{1-\beta}{\beta}\,{}^CD_0^\beta+I\right)\mathscr{F}_{C,t_1}^{\beta,1}\left(\varphi-(1-\beta)g;\beta g-{}^CD_{t_0}^\beta(\varphi-(1-\beta)g)\right)(t),$$
$$t\in(0,t_0). \tag{4.44}$$

Proof. (ii) Let us denote $w=(1-\beta){}^CD_0^{\{k\},1}u$. For this particular kernel type the relation holds:

$$w(t)=\int_0^t E_\beta\Big(-\frac{\beta(t-\tau)^\beta}{1-\beta}\Big)u'(\tau)d\tau.$$

By Lemma 4.2 and the identity $I_0^{1-\beta}u'={}^CD_0^\beta u$, w solves the equation

$${}^CD_0^\beta w(t)+\frac{\beta}{1-\beta}w(t)={}^CD_0^\beta u(t),\quad t\in(0,T). \tag{4.45}$$

Since the relation (4.3) is valid, $w|_{(t_0,T)}=(1-\beta)g$. It follows from (4.45) that ${}^CD_0^\beta(u-w)=\frac{\beta}{1-\beta}w$. Thus, we have the IP1 with usual fractional derivative

$$(u-w)|_{(t_0,T)}=\varphi-(1-\beta)g,\quad {}^CD_0^\beta(u-w)|_{(t_0,T)}=\beta g.$$

To apply Theorem 4.3 (ii), we have to show that $u-w\in W_2^1(0,T)$. Since $E_\beta'=\frac{1}{\beta}E_{\beta,\beta}$ and $E_\beta(0)=1$, we obtain

$$(u-w)'=\frac{\beta}{1-\beta}[t^{\beta-1}E_{\beta,\beta}(-\frac{\beta t^\beta}{1-\beta})]*u'.$$

Due to the assumptions of (ii), this belongs to $L_2(0,T)$, hence $u-w\in W_2^1(0,T)$. According to Theorem 4.3 (ii)

$$(u-w)|_{(0,t_0)}=\mathscr{F}_{C,t_1}^{\beta,1}\left(\varphi-(1-\beta)g;\beta g-{}^CD_{t_0}^\beta(\varphi-(1-\beta)g)\right). \tag{4.46}$$

The relation (4.45) implies $w=\frac{1-\beta}{\beta}\,{}^CD_0^\beta(u-w)$. Therefore,

$$w|_{(0,t_0)}=\frac{1-\beta}{\beta}\,{}^CD_0^\beta\mathscr{F}_{C,t_1}^{\beta,1}\left(\varphi-(1-\beta)g;\beta g-{}^CD_{t_0}^\beta(\varphi-(1-\beta)g)\right).$$

Hence, from (4.46) we obtain (4.44).

(i) Let us denote $w = (1-\beta)^R D_0^{\{k\},1} u$. Then

$$w = \frac{d}{dt} z, \quad \text{where} \quad z(t) = \int_0^t E_\beta \Big(-\frac{\beta (t-\tau)^\beta}{1-\beta} \Big) u(\tau) d\tau.$$

The function z solves the equation

$$^C D_0^\beta z(t) + \frac{\beta}{1-\beta} z(t) = I_0^{1-\beta} u(t), \quad t \in (0,T). \tag{4.47}$$

Next we differentiate the equation (4.47) and obtain

$$^R D_0^\beta w(t) + \frac{\beta}{1-\beta} w(t) = {}^R D_0^\beta u(t), \quad a.e.\ t \in (0,T). \tag{4.48}$$

Therefore $^R D_0^\beta (u-w)(t) = \frac{\beta}{1-\beta} w(t)$ that leads us to the IP1 with a usual fractional derivative

$$(u-w)|_{(t_0,T)} = \varphi - (1-\beta) g, \quad {}^R D_0^\beta (u-w)|_{(t_0,T)} = \beta g.$$

Now we have to show that $u - w \in L_2(0,T)$ and $^R D_0^\beta (u-w)(t) \in L_1(0,T)$. Firstly,

$$w(t) = \frac{d}{dt} \left(E_\beta \Big(-\frac{\beta}{1-\beta} t^\beta \Big) * u(t) \right) = u(0) E_\beta \Big(-\frac{\beta}{1-\beta} t^\beta \Big) + E_\beta \Big(-\frac{\beta}{1-\beta} t^\beta \Big) * u'(t).$$

Since $E_\beta \Big(-\frac{\beta}{1-\beta} t^\beta \Big) \in L_2(0,T)$ for any $\beta \in (0,1)$ we obtain that $w \in L_2(0,T)$. Due to the Sobolev embedding Theorem $u \in W_1^1(0,T) \subset L_2(0,T)$. Thus, $u - w \in L_2(0,T)$. Secondly, $^R D_0^\beta (u-w)(t) = \frac{\beta}{1-\beta} w(t) \in L_2(0,T)$.

We continue the proof by applying Theorem 3 (i) to the IP1 for $u - w$:

$$(u-w)|_{(0,t_0)} = \mathscr{F}_{R,t_1}^{\beta,1} \left(\beta g - {}^R D_{t_0}^\beta (\varphi - (1-\beta) g) \right).$$

It follows from (4.48) that $w = \frac{1-\beta}{\beta} {}^R D_0^\beta (u-w)$, thus, the formula (4.43) holds. □

Similarly to Corollary 4.2, formulas of solutions of IP2 can be derived in cases of tempered and Atangana-Baleanu derivatives.

Formulas of solutions of IP1 in case of multiterm and distributed fractional derivatives (kernels (k2) and (k3)) cannot be derived on the basis of Theorem 4.2. The problem of reconstruction of explicit representations for solutions in these cases remains open.

Conclusions

In the thesis we discuss some inverse problems that arise in generalized subdiffusion models. The problems considered in the thesis are new. In our treatment of subdiffusion models we do not address fractional diffusion, since it is rather well-known. In the thesis we focus on a generalized model, that maintain somewhat similar features and behaviour to the fractional diffusion, but is much richer in its applications. Therefore, the complement m to the fractional diffusion kernel was introduced in the Chapter 3 and the generalized fractional derivative $D_0^{\{k\}}$ was introduced in the Chapters 2 and 4. We try to keep the assumptions on the kernels k and M of the generalized fractional derivative as general as possible to make this research is compatible with wide range of potential applications . The content of the Chapters is based on the papers provided in the appendix.

In the Introduction we discuss the formulation of the problems and their history. We start from the discussion of the fractional diffusion and general concepts of inverse and ill-posedness. We provide the insight on how this ideas have developed over time: at first independent from each other and over time by merging into a separate field of studies.

Next we explain basic mathematical notations and concepts used in the thesis. Afterwards, we provide the insight into the underlying physical models and explain three approaches to derive the subdifusion equation. We deeply analyze the potential theoretical applications of the problems with a generalized fractional derivative. Therefore, we provide the list of subdiffusion kernels that fit into the description of the model and their Laplace transforms. We discuss the properties of the kernels and refer to their applications in the literature.

In the Chapters 2 and 3 two different theoretical methods of handling the inverse problems with final overdetermination are represented.

Firstly, in the Chapter 2 we discuss the problem of reconstruction of a space-dependent component f of a source term $F(t,x) = f(x)g(t,x) + h(t,x)$ in a subdiffusion equation with a generalised fractional derivative. The problem is considered in a setting of C- and Hölder spaces in time. The uniqueness of the solution is proved by means of the positivity principle and the existence and stability of the solution is shown by applying the Fredholm alternative. Thanks to the general problem formulation, namely to the assumption that $g = g(t,x)$, we are able to immediately apply this results to the inverse problem for reaction coefficient. We prove the global uniqueness and local existence and stability of solution to this problem. It is possible to apply similar approach to determine the higher order coefficients of the equation as well. Moreover, we show how to apply these results to the problem with the integral overdetermination condition. We show how the concrete kernels discussed before satisfy the conditions of the theorems of this Chapter, that clarifies further applications in the literature.

In Chapter 3 we consider the problem of reconstruction of a space-dependant term f of a source function $F(t,x) = f(x)g(t) + h(t,x)$ along with a state function u. We formulate the problems for the particular case of the kernel M that is a convolutional perturbation of a power function involved in the usual fractional derivative. This formulation is still sufficiently general, since it contains as particular cases most important Sonine kernels occurring in the practice (except for the continuously distributed kernels). Moreover, having a kernel of a usual fractional derivative as a principal part of perturbed kernel M, enables us to use the well-elaborated theory of Mittag-Leffler functions. Since $g = g(t)$ we are able to apply the Fourier method to the direct and the inverse problems, that cannot be applied in case $g = g(t,x)$. This methods enables to prove the results under rather weak assumptions on the data, in particular for g in L_p and the initial condition in L_2. By means of Fourier method we derive explicit solution formulas and obtain uniqueness, existence and stability results for the both direct and inverse problems. Finally, we consider an inverse problem backward in time that is solved in a similar manner.

The inverse source problems considered in the Chapters 2 and 3 are moderately ill-posed , since their solutions depend continuously on derivatives of the data of finite order. Therefore, in case approximate data are given with errors, regularization procedures can be successfully applied.

It is common to use a final time overdetermination data in order to reconstruct the space-dependent unknown. In practice, however, it is not possible to measure the state exactly at final time T. In fact a real measurement yields an average of u at some small neighborhood of T. In Chapter 4 we propose a new approach to the inverse problems with final overdetermination: we consider an inverse source problem with the extra data given in a neighbourhood of final time. Unlike problems with final overdetermination, such problem is severely ill-posed. Final data on a continuous interval makes it possible to reconstruct the source term $f(t,x)$.

It turns out that an inverse problem with the overdetermination condition on a final time subinterval for a subdiffusion equation and a wide variety of other fractional PDEs can all be treated in the same way. Among these PDEs are the equations governing fractional wave processes, equations with non-linear unbounded space operators. Precisely, this is done by reducing the original problem to the problem of backward continuation of function u, given the value of u and its generalized fractional derivative $D_0^{\{k\}}$ on a final time subinterval. We consider this backward continuation problem in details and prove the uniqueness result under very general assumptions on the kernel k. The uniqueness result for the inverse source problem is obtained as the consequence of the latter result. Afterwards, for some particular cases of the kernel k we deduce explicit solution formulas of the backward continuation problem . These formulas involve infinite series and contain all derivatives of the data. Approximation of the solutions by means of truncated series leads to moderately ill-posed problems (Remark 4.2).

Finally, we would like to present some open questions that arise from the research presented in the thesis. These are:

- the inverse coefficient problem for r with non-zero initial condition;
- derivation of solution formulas in case of multiterm and distributed order derivatives for the backward continuation problem studied in Chapter 4;
- effective numerical methods to the problems studied in Chapter 4;
- generalization of results of Chapter 4 to the case when PDE involves a fractional time derivative and a non-local space operator (e.g. fractional Laplacian), derivation of solution formulas for the inverse problems in Chapter 4 in cases of multiterm and distributed derivatives;. A problem consists in reconstruction of unknown source in $\Omega \times (0,T)$ provided measurements are given in $\Omega' \times (t_0,T)$ where $\Omega' \subset \Omega$, $0 < t_0 < T$.

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Abstract
Inverse Problems for Generalized Subdiffusion Equations

The thesis focuses on the inverse problems for subdiffusion equations with generalized fractional derivatives that generalize previously studied inverse problems for the time fractional diffusion equation. Since generalized fractional derivative incorporates the cases of usual fractional derivative, distributed order and tempered fractional derivatives, Atangana-Baleanu fractional derivative, the problems with generalized fractional derivatives have a lot more potential applications than their usual fractional analogues.

The thesis is started by considering two inverse problems for a generalized subdiffusion equation with the final overdetermination condition in a setting of C- and Hölder spaces. Firstly, a problem of reconstruction of a space-dependent component in a source term is studied. Existence, uniqueness and stability of the solution to this problem are proved. Based on these results, an inverse problem of identification of a space-dependent coefficient of a linear reaction term is considered. Thus, the uniqueness and local existence and stability of the solution to this problem are proved. This is done by means of the theory of evolutionary integral equations, positivity principle, Fredholm alternative and fixed point theorem.

The next object under consideration is an inverse problem to recover a space-dependent factor of a source term in a perturbed time fractional diffusion equation in a setting of Lebesgue spaces. Afterwards, backward in time problem for the same equation is investigated. Existence, uniqueness, and stability of solutions to these problems are proved, mainly by means of the Fourier method.

Finally, two inverse problems with a generalized fractional derivative with an overdetermination condition given in the neighbourhood of the final time are investigated. The first one is a problem of backward continuation of the function u based on its values and the values of its fractional derivative in the neighborhood of the final time. The uniqueness of the solution to this problem is proved by considering the problem in the Laplace domain. Afterwards, given measurements in a neighborhood of final time, the problem of reconstruction of a source term in an equation that generalizes fractional diffusion and wave equations is discussed. The source to be determined depends on time and all space variables. The uniqueness is proved based on the results for the backward continuation problem. In addition, the explicit solution formulas to the both problems for some particular cases of the generalized fractional derivative are derived.

Open problems that arise from this research are formulated.

Kokkuvõte
Pöördülesanded üldistatud subdifusioonivõrranditele

Väitekirjas tegeletakse pöördülesannetega subdifusioonivõrranditele, mis sisaldavad üldistatud murrulisi tuletisi, ja üldistavad varem vaadeldud pöördülesandeid murrulise ajatuletisega difusioonivõrrandile. Kuna üldistatud murruline tuletis hõlmab erijuhtudena tavalist murrulist tuletist, jaotatud ja tempereeritud tuletist ning Atangana-Baleanu tuletist, omavad taoliste tuletistega ülesanded palju suuremat rakenduspotentsiaali kui nende tavalised murrulised analoogid.

Kõigepealt käsitletakse kahte lõpptingimust kasutavat pöördülesannet üldistatud subdifusioonivõrrandile C- ja Hölderi ruumides. Esimeses ülesandes on eesmärgiks määrata ruumimuutujast sõltuv allikakomponent. Tõestatakse selle ülesande lahendi olemasolu, ühesus ja stabiilsus. Lähtudes neist tulemustest uuritakse pöördülesannet lineaarse reaktsiooniliikme ruumimuutujast sõltuva kordaja identifitseerimiseks. Tõestatakse selle ülesande lahendi ühesus ja lokaalne olemasolu ning stabiilsus. Analüüsimisel kasutatakse evolutsiooniliste integraalvõrrandite teooriat, positiivsusprintsiipi, Fredholmi alternatiivi ja püsipunktiprintsiipi.

Järgnevalt vaadeldakse pöördülesannet häiritud murrulise tuletisega difusioonivõrrandi allikafunktsiooni ruumimuutujast sõltuva komponendi määramiseks lõpptingimuse alusel Lebesgue'i ruumides. Seejärel uuritakse ajas pööratud ülesannet sama võrrandi jaoks. Tõestatakse nende ülesannete lahendite olemasol, ühesus ja stabiilsus peamiselt Fourier' meetodi abil.

Lõpuks käsitletakse kahte üldistatud murrulist tuletist sisaldavat pöördülesannet, mis kasutavad vaatlusandmeid lõpphetke ümbruses. Esimesene ülesanne seisneb funktsiooni u tahapoole jätkamises lähtudes funktsiooni u ja tema üldistatud murrulise väärtustest lõpphetke ümbruses. Tõestatakse selle ülesande lahendi ühesus Laplace'i teisenduse abil. Peale selle uuritakse ülesannet murrulisi difusioonivõrrandeid ja murrulisi lainevõrrandeid üldistavas võrrandis sisalduva allikafunktsiooni määramiseks lõpphetke ümbruses tehtud mõõtmiste alusel. Allikafunktsioon võib sõltuda nii aja- kui ruumimuutujatest. Tõestatakse selle ülesande lahendi ühesus kasutades ühesusteoreemi tahapoole jätkamise ülesande kohta. Lisaks tuletatakse mõlema ülesande lahendite jaoks ilmutatud lahendivalemeid teatud erijuhtudel.

Peale selle formuleeritakse mõned lahendamata probleemid, mis lähtuvad väitekirjas tehtud uuringutest.

Appendix 1

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Inverse problems for a perturbed time fractional diffusion equation with final overdetermination

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Inverse problems to recover a space-dependent factor of a source term and an initial condition in a perturbed time fractional diffusion equation containing an additional convolution term from final data are considered. Existence, uniqueness, and stability of solutions to these problems are proved.

KEYWORDS

fractional diffusion, fractional parabolic equation, inverse problem

1 | INTRODUCTION

Differential equations containing fractional time (and also space) derivatives of order less than 1 are extensively used to model slow diffusion (subdiffusion) processes in physics, chemistry, biology, nuclear power engineering, etc.[1-4]

Sometimes parameters of processes or models (coefficients of equations, source terms, initial or boundary conditions) are unknown. To determine unknown parameters, inverse problems that involve observation of states of processes are solved.[5-10]

Usually an observation of a state over a whole space-time domain is not possible or not practical. Depending on possibilities or unknowns to be recovered, measurements of the state in a subdomain, at a boundary of a space domain or at a final time moment, are used in the reconstruction.[6]

Problems with final overdetermination for diffusion equations containing single fractional time (and also space) derivatives were studied in papers[11-16] (reconstruction of source terms) and Sakamoto and Yamamoto[17] (reconstruction of an initial state). An inverse problem to recover a source term in an equation containing multiple Caputo time derivatives by means of local interior measurements was treated in Jiang et al.[18]

Recently, the second author[19] introduced a perturbed time fractional diffusion equation that contains an additional convolution term with a kernel m and generalizes diffusion models with multiple time fractional derivatives. The paper[19] studies reconstruction of m and an order of a derivative from measurements over the time.

In the present paper, we consider 2 inverse problems for the mentioned perturbed equation: a problem to reconstruct a space-dependent factor of a source function and a problem to determine an initial state. Additional data are given in the form of final measurements. We prove existence and uniqueness of the solutions of posed inverse problems and derive stability estimates. In addition, we consider a regularization and provide numerical examples.

We will establish the solvability in the L_2-space, which means that solutions of the inverse problems are allowed to have discontinuities. Proofs rely on positivity properties of solutions of involved direct problems. This brings along sign or smallness restrictions on m.

2 | FORMULATION OF DIRECT AND INVERSE PROBLEMS

Let us consider the generalized subdiffusion equation[19,20]

$$u_t(t,x) = \varkappa(M * \Delta u)_t(t,x) + Q(t,x), \tag{1}$$

where u is a physical state, t is the time, $x \in \mathbb{R}^n$ is a space variable, Δ is the Laplacian, subscript t stands for the time derivative, $\gamma > 0$ is a constant, Q is a source term, and $*$ denotes the time convolution, ie,

$$(v_1 * v_2)(t) = \int_0^t v_1(t-\tau)v_2(\tau)d\tau.$$

The kernel M is a memory function that is related to a waiting time density of an underlying random walk process going on in micro-level.[20]

In case of a power-type waiting time density, the kernel M has the form $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $0 < \beta < 1$, and (1) becomes the celebrated time fractional diffusion equation[4,20,21]

$$u_t = \varkappa D^{1-\beta}\Delta u + Q, \tag{2}$$

where $D^{1-\beta}v = \left(\frac{t^{\beta-1}}{\Gamma(\beta)} * v\right)_t$ is the Riemann-Liouville fractional derivative of the order $1-\beta$. More advanced fractional diffusion models contain multiple Riemann-Liouville derivatives. Then (refer to Mainardi et al and Sokolov et al[22,23]),

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{j=1}^{l} a_j \frac{t^{\beta_j-1}}{\Gamma(\beta_j)}, \quad 0 < \beta < \beta_j < 1, \tag{3}$$

or more generally, $M(t) = \int_0^1 a(s)\frac{t^{s-1}}{\Gamma(s)}d\mu$, where a is an integrable function and μ is a Borel measure.[22-24]

In the present paper, we assume that (cf Janno[19])

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + m * \frac{t^{\beta-1}}{\Gamma(\beta)}, \tag{4}$$

where m is an integrable function and $0 < \beta < 1$. The function (4) includes as particular cases the kernels of the form (3) (then $m(t) = \sum_{j=1}^{l} a_j \frac{t^{\beta_j-\beta-1}}{\Gamma(\beta_j-\beta)}$) and also $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \int_\beta^1 a(s)\frac{t^{s-1}}{\Gamma(s)}ds$, (then $m(t) = \int_\beta^1 a(s)\frac{t^{s-\beta-1}}{\Gamma(s-\beta)}ds$).

Plugging (4) into (1), we arrive at the perturbed time fractional diffusion equation

$$u_t = \varkappa D^{1-\beta}(\Delta u + m * \Delta u) + Q. \tag{5}$$

This equation enables an immediate integration. Applying the operator of fractional integration $I^{1-\beta} = \frac{t^{-\beta}}{\Gamma(1-\beta)} *$ of the order $1-\beta$ to (5), we obtain the equation

$$\partial^\beta u = \varkappa(\Delta u + m * \Delta u) + F \tag{6}$$

that contains the Caputo fractional derivative of the order β, ie, $\partial^\beta v = \frac{t^{-\beta}}{\Gamma(1-\beta)} * v_t$ and in case $m = 0$ is referred to as a normal form of the Equation 2.[23] There $F = I^{1-\beta}Q$.

We mention that the Equation (6) can be obtained by means of other considerations, too. For instance, it is an extension to the fractional case of the parabolic integro-differential equation $u_t = \varkappa(\Delta u + m * \Delta u) + F$ that describes heat processes with memory.[25] Moreover, it is a generalization of an equation with multiple Caputo derivatives $\partial^\beta u + \sum_{j=1}^{l} b_j \partial^{\mu_j} u = \varkappa\Delta u + z$, $0 < \mu_j < \beta < 1$, that was studied in previous works.[18,26,27] Rewriting the latter equation as

$$\partial^\beta u + k * \partial^\beta u = \varkappa\Delta u + z, \quad \text{where} \quad k(t) = \sum_{j=1}^{l} b_j \frac{t^{\beta-\mu_j-1}}{\Gamma(\beta-\mu_j)}, \tag{7}$$

defining m as a solution of the Volterra equation of the second kind $m + k * m = -k$ and applying the operator $\mathcal{I} + m *$, where $\mathcal{I}$ is the unity operator, to the Equation (7), we reach (6) with $F = z + m * z$.

For the sake of generality, let us transform the Caputo derivative $\partial^\beta u$ contained in (6) to the form $D^\beta(u - u(0,x))$ that does not contain the first order derivative of u. We obtain the following equation: $D^\beta(u - u(0,x)) = \varkappa(\Delta u + m * \Delta u) + F$.

Now we are in a situation to formulate problems to be treated in the present paper. Let $\Omega \in \mathbb{R}^n$ be an n-dimensional open bounded domain with sufficiently smooth boundary $\partial\Omega$. In *direct problem*, we have to find a function u that satisfies the differential equation, initial and homogeneous boundary conditions:

$$\left.\begin{array}{l} D^{\beta}(u-\varphi)(t,x) = \varkappa(\Delta u(t,x) + (m * \Delta u)(t,x)) + F(t,x), \quad x \in \Omega,\ t \in (0,T), \\ u(0,x) = \varphi(x), \quad x \in \Omega, \\ \mathcal{B}u(t,x) = 0, \quad x \in \partial\Omega,\ t \in (0,T). \end{array}\right\} \tag{8}$$

Here $\mathcal{B}$, is a boundary operator:

$$\mathcal{B}v(x) = v(x) \quad \text{or} \quad \mathcal{B}v(x) = \vartheta(x)\cdot\nabla v(x) + \theta v(x), \quad \theta > 0,$$

and $\vartheta(x)$ is the outer normal of $\partial\Omega$ at $x \in \partial\Omega$. A problem with nonhomogeneous boundary conditions can be transformed to a problem with homogeneous boundary conditions by means of a simple change of variables.

Moreover, we formulate 2 *inverse problems* that use the final overdetermination condition

$$u(T,x) = \psi(x), \quad x \in \Omega, \tag{9}$$

with a given observation function ψ.

IP1. Let F have the form

$$F(t,x) = g(t)f(x) + h(t,x), \tag{10}$$

where g and h are given functions. The aim is to find a function f such that the solution u of (8) with F of the form (10) satisfies the condition (9).

We mention that in terms of the physical source function Q occurring in (5), the formula (10) has the form $Q(t,x) = q(t)f(x) + H(t,x)$, where $g = I^{1-\beta}q$ and $h = I^{1-\beta}H$.

IP2. Find an initial state φ such that the solution u of (8) satisfies (9).

IP1 and IP2 in case $m = 0$ were studied in previous papers.[11-14,16,17]

3 | DEFINITIONS, NOTATION, AND AUXILIARY STATEMENTS

Firstly, we introduce some spaces of abstract functions that map the interval $(0,T)$ into a Banach space Y. As usual, $L_p((0,T);Y)$, $p \in [1,\infty]$, stands for the abstract Lebesgue space. The space $C([0,T];Y)$ consists of abstract functions that are continuous in the interval $[0,T]$. Next, let X be a Hilbert space.* We introduce the spaces

$$H_p^s((0,T);X) = \{w|_{(0,T)} \ :\ w \in H_p^s(\mathbb{R};X)\},\ \ p \in (1,\infty),\ s > 0,$$

where

$$H_p^s(\mathbb{R};X) = \{w \in L_p(\mathbb{R};X) \ :\ \mathcal{F}^{-1}|\xi|^s\mathcal{F}w \in L_p(\mathbb{R};X)\}$$

and $\mathcal{F}$ denotes the Fourier transform with the argument ξ (Prüss[29, p226]; Zacher[28, p28]). Moreover, we define

$${}_0H_p^s((0,T);X) = \{w|_{(0,T)} \ :\ w \in H_p^s(\mathbb{R};X),\ supp\, w \subseteq [0,\infty)\},\ \ p \in (1,\infty),\ s > 0.$$

In the particular case $X = \mathbb{R}$, we drop the symbol of value space, ie, write $H_p^s(0,T)$ instead of $H_p^s((0,T);\mathbb{R})$ and ${}_0H_p^s(0,T)$ instead of ${}_0H_p^s((0,T);\mathbb{R})$.

Next, we formulate a lemma that follows from discussions in Zacher.[28, p28-29]

Lemma 1. *Let $s \in (0,1), p \in (1,\infty)$. The operator of fractional integration of the order s, ie, $I^s = \frac{t^{s-1}}{\Gamma(s)} *$, is a bijection from $L_p((0,T);X)$ onto ${}_0H_p^s((0,T);X)$, the inverse of I^s is the Riemann-Liouville fractional derivative $D^s = \frac{d}{dt}I^{1-s}$ and $\|w\|_{{}_0H_p^{\beta}((0,T);X)} = \|D^s w\|_{L_p((0,T);X)}$ is a norm in the space ${}_0H_p^s((0,T);X)$. Moreover, in case $p \in (\frac{1}{s},\infty)$, it holds $H_p^s((0,T);X) \hookrightarrow C([0,T];X)$ and $w(0) = 0$ for $w \in {}_0H_p^s((0,T);X)$*

Another useful sentence is the *Young's theorem for convolutions* which states that for $m \in L_q(0,T)$ and $w \in L_p((0,T);Y)$ with $p,q \in [1,\infty]$, the convolution $m * w$ belongs to the space $m * w \in L_r((0,T);Y)$ where $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$ and the inequality $\|m * w\|_{L_r((0,T);Y)} \leq \|m\|_{L_q(0,T)}\|w\|_{L_p((0,T);Y)}$ is valid.†

*Or more generally, a Banach space of the class $\mathcal{HT}$ [28]

†Here $\frac{1}{s} = 0$ iff $s = +\infty$ (s is either p, q or r).

In treatment of convolutional terms, we will apply *norms with exponential weights*. Let us define these norms in the spaces of scalar functions $L_p(0,T)$, $p \in [1,\infty]$:

$$\|w\|_{p;\sigma} = \|e^{-\sigma t}w\|_{L_p(0,T)}, \quad \text{where } \sigma \geq 0.$$

If $\sigma = 0$, then $\|\cdot\|_{p;\sigma}$ becomes the usual norm in $L_p(0,T)$ and we denote it by $\|\cdot\|_p$. The following equivalence relations are valid:

$$e^{-\sigma T}\|w\|_p \leq \|w\|_{p;\sigma} \leq \|w\|_p.$$

Note that the weight can be easily brought into the convolution, ie, $e^{-\sigma t}m * w = (e^{-\sigma t}m) * (e^{-\sigma t}w)$ and the Young's inequality extended to the weighted norms:

$$\|m * w\|_{r;\sigma} \leq \|m\|_{q;\sigma}\|w\|_{p;\sigma}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{11}$$

Finally, in case $p < \infty$, $\|w\|_{p;\sigma} \to 0$ as $\sigma \to \infty$.

An important tool in the analysis of fractional differential equations is the family of Mittag-Leffler functions

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\gamma}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}, \quad z \in \mathbb{C}. \tag{12}$$

The function $E_{\alpha,\gamma}$ is entire in case $\alpha > 0, \gamma > 0$.[30] The formulas (12) immediately imply $E_{\alpha,1} = E_\alpha$ and

$$E_\alpha(0) = 1, \quad E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}, \quad E'_\alpha = \frac{1}{\alpha}E_{\alpha,\alpha}. \tag{13}$$

Let us point out some useful properties of $E_\beta(-z)$ and $E_{\beta,\beta}(-z)$ in case $\beta \in (0,1)$. The functions $E_\beta(-z)$ and $E_{\beta,\beta}(-z)$ are completely monotonic for $z \in [0,\infty)$ and satisfy the asymptotic relations (see Gorenflo et al[30])

$$zE_\beta(-z) = \frac{1}{\Gamma(1-\beta)} + O(z^{-1}) \quad \text{as} \quad z \to \infty, \tag{14}$$

$$z^2 E_{\beta,\beta}(-z) = -\frac{1}{\Gamma(-\beta)} + O(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{15}$$

Since $E_\beta(-z)$ is bounded for $z \geq 0$ and (14) holds, there exist $C_1, C_2 > 0$ such that

$$\frac{C_1}{1+z} \leq E_\beta(-z) \leq \frac{C_2}{1+z} \leq \frac{C_2}{z} \quad \text{for} \quad z \geq 0. \tag{16}$$

In addition to the Mittag-Leffler functions, we introduce the α-exponential function[31]:

$$e_\alpha^{\lambda t} = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha), \quad \alpha > 0. \tag{17}$$

The relations (13) and (17) yield the following formula:

$$\int_0^t \lambda e_\beta^{-\lambda\tau} d\tau = 1 - E_\beta(-\lambda t^\beta). \tag{18}$$

Moreover, the formula (18) in view of the relations

$$0 < E_\beta(-z) \leq 1, \quad z \geq 0, \tag{19}$$

following from the complete monotonicity of $E_\beta(-z)$ and $E_\beta(0) = 1$, implies

$$\|\lambda e_\beta^{-\lambda t}\|_1 \leq 1, \quad \lambda > 0. \tag{20}$$

We complete this section proving a technical lemma. It will be applied in proofs of Theorem 2 (ii), Theorem 3, and Lemma 4.

Lemma 2. *There exists a constant $C_3 > 0$ such that*

$$(\lambda e_\beta^{-\lambda t} *)^i E_\beta(-\lambda t^\beta) \leq \frac{C_2 C_3^i}{\lambda t^\beta}, \quad t > 0, \ \lambda > 0, \ i \in \mathbb{N}. \tag{21}$$

Proof. The convolution formula of Mittag-Leffler functions (see Haubold et al,[32] (11.12)) implies

$$\lambda e_\beta^{-\lambda t} * E_\beta(-\lambda t^\beta) = \lambda t^\beta E^2_{\beta,\beta+1}(-\lambda t^\beta) = \frac{\lambda t^\beta}{\beta}E_{\beta,\beta}(-\lambda t^\beta). \tag{22}$$

Here, $E^{\gamma}_{\alpha,\beta}$ is the 3-parametric Mittag-Leffler function and we used the formula $E^2_{\beta,\beta+1}(z) = \frac{1}{\beta}E_{\beta,\beta}(z)$ (Haubold et al,[32] (11.4)), too. The asymptotic relations (14), (15), and $\Gamma(1-\beta) = (-\beta)\Gamma(-\beta)$ yield

$$z^2 E_{\beta,\beta}(-z) = \beta z E_\beta(-z) + O(z^{-1}) \quad \text{as} \quad z \to \infty.$$

Thus, there exists $z_0 > 0$ such that $\frac{z}{\beta}E_{\beta,\beta}(-z) \le (1+\epsilon)E_\beta(-z)$ for $z > z_0$ where $\epsilon > 0$ is some fixed number. On the other hand, since $zE_{\beta,\beta}(-z) \in C[0, z_0]$ and $E_\beta(-z)$ is decreasing, we obtain $\frac{z}{\beta}E_{\beta,\beta}(-z) \le C_4 E_\beta(-z)$ for $0 \le z \le z_0$ where $C_4 = \frac{\max\limits_{0\le y\le z_0} yE_{\beta,\beta}(-y)}{\beta E_\beta(-z_0)}$. Therefore, $\frac{z}{\beta}E_{\beta,\beta}(-z) \le C_3 E_\beta(-z)$ for any $z \ge 0$, where $C_3 = \max\{1+\epsilon; C_4\}$ and from (22), we have

$$\lambda e_\beta^{-\lambda t} * E_\beta(-\lambda t^\beta) \le C_3 E_\beta(-\lambda t^\beta). \tag{23}$$

So we continue the iterations and obtain

$$(\lambda e_\beta^{-\lambda t} *)^i E_\beta(-\lambda t^\beta) \le C_3^i E_\beta(-\lambda t^\beta).$$

Finally, estimating $E_\beta(-\lambda t^\beta)$ by means of (16), we reach (21). □

4 | RESULTS CONCERNING DIRECT PROBLEM

Firstly, we put the direct problem (8) into a context of functional spaces. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ stand for the norm and the inner product in the space $L_2(\Omega)$, respectively. We define the operator $L = -\varkappa\Delta$ with the domain $\mathcal{D}(L) = \{z \in W_2^2(\Omega) : \mathcal{B}z = 0 \text{ in } \partial\Omega\}$ in the space $L_2(\Omega)$.

Let $0 < \lambda_1 \le \lambda_2 \le \dots$ be the eigenvalues and $v_1, v_2, \dots$ the corresponding orthonormal eigenvectors of the operator L. Then the system of functions $v_k, k \in \mathbb{N}$, forms a basis in the space $L_2(\Omega)$ and $\|z\|_{\mathcal{D}(L)} = \left[\sum_{k=1}^{+\infty} \lambda_k^2 \langle z, v_k\rangle^2\right]^{\frac{1}{2}}$ is a an equivalent norm in the space $\mathcal{D}(L)$.

In the sequel, we will search for the solution u of (8) from the following set:

$$\mathcal{U}_{r,\beta} = \{u \in L_r((0,T);\mathcal{D}(L)) \cap C([0,T];L_2(\Omega)) \ : \ u - u(0) \in {}_0H_r^\beta((0,T);L_2(\Omega))\}.$$

Let us introduce a notation for the Fourier coefficients of data functions involved in the direct problem:

$$u_k(t) = \langle u(t,\cdot), v_k\rangle, \quad F_k(t) = \langle F(t,\cdot), v_k\rangle, \quad \varphi_k = \langle \varphi, v_k\rangle, \qquad k \in \mathbb{N}.$$

Proposition 1. *Let $F \in L_p((0,T);L_2(\Omega))$ with some $p \in (1,\infty)$, $m \in L_1(0,T)$ and $\varphi \in L_2(\Omega)$. Then the following assertions are valid.*

(i) If $u \in \mathcal{U}_{r,\beta}$ with some $r \in (1,\infty)$ is a solution of the direct problem (8), then the Fourier coefficients u_k, $k \in \mathbb{N}$, belong to

$$\widehat{\mathcal{U}}_{r,\beta} = \{w \in C[0,T] \ : \ w - w(0) \in {}_0H_r^\beta(0,T)\}$$

and are solutions of the following sequence of problems for $k \in \mathbb{N}$:

$$D^\beta(u_k - \varphi_k)(t) + \lambda_k u_k(t) + \lambda_k(m * u_k)(t) = F_k(t), \quad t \in (0,T), \tag{24}$$

$$u_k(0) = \varphi_k. \tag{25}$$

(ii) If (24), (25) have solutions $u_k \in \widehat{\mathcal{U}}_{r,\beta}$, $k \in \mathbb{N}$, with some $r \in (1,\infty)$ such that $u = \sum\limits_{k=1}^{+\infty} u_k v_k \in \mathcal{U}_{r,\beta}$, then u is a solution of the direct problem (8).

Proof.

(i) Let $u \in \mathcal{U}_{r,\beta}$ with some $r \in (1,\infty)$ solve (8). Since $u - u(0) = u - \varphi \in {}_0H_r^\beta((0,T);L_2(\Omega))$, by Lemma 1 there exists $\hat{u} \in L_r((0,T);L_2(\Omega))$ such that $u - \varphi = I^\beta\hat{u}$ and $\hat{u} = D^\beta(u-\varphi)$. Let us denote $\hat{u}_k(t) = \langle \hat{u}(t,\cdot), v_k\rangle$. Due to $\hat{u} \in L_r((0,T);L_2(\Omega))$, we have $\hat{u}_k \in L_r(0,T)$. On the other hand, $u_k - \varphi_k = \langle u - \varphi, v_k\rangle = \langle I^\beta\hat{u}, v_k\rangle = I^\beta\langle\hat{u}, v_k\rangle = I^\beta\hat{u}_k$. This relation with Lemma 1 implies $u_k - \varphi_k \in {}_0H_r^\beta(0,T)$ and $D^\beta(u_k - \varphi_k) = \hat{u}_k$. Further, from $u \in \mathcal{U}_{r,\beta} \subset C([0,T];L_2(\Omega))$, we immediately have $u_k \in C[0,T]$. Moreover, taking the inner product of the initial condition $u(0,\cdot) = \varphi$ with v_k, we deduce (25). The relation $u_k - \varphi_k \in {}_0H_r^\beta(0,T)$ with (25) and

$u_k \in C[0,T]$ proves that $u_k \in \widehat{\mathscr{U}}_{r,\beta}$. The deduced equalities $\hat{u} = D^\beta(u-\varphi)$ and $D^\beta(u_k-\varphi_k) = \hat{u}_k$ imply $\langle D^\beta(u-\varphi), v_k\rangle = D^\beta(u_k-\varphi_k)$. Moreover, $\langle Lu, v_k\rangle = \langle u, Lv_k\rangle = \lambda_k\langle u, v_k\rangle = \lambda_k u_k$. Consequently, taking the inner product of the equation $D^\beta(u-\varphi) + Lu + m * Lu = F$ with v_k, we obtain the Equation (24).

(ii) Let the assumptions of (ii) hold for u_k. Denote $R = D^\beta(u-\varphi-\rho) + Lu + m * Lu - F$ and $\rho = u(0,\cdot) - \varphi$. Then $u \in \mathscr{U}_{r,\beta}$ solves the problem $D^\beta(u-\tilde{\varphi}) + Lu + m * Lu = \tilde{F}$, $u(0,\cdot) = \tilde{\varphi}$, where $\tilde{F} = F + R$ and $\tilde{\varphi} = \varphi + \rho$. Applying the proved statement (i) to this problem, we see that u_k, $k \in \mathbb{N}$, solve the problems $D^\beta(u_k - \widetilde{\varphi}_k) + \lambda_k u_k + \lambda_k(m * u_k) = \tilde{F}_k$, $u_k(0) = \tilde{\varphi}_k$, where $\tilde{F}_k = F_k + \langle R, v_k\rangle$ and $\tilde{\varphi}_k = \varphi_k + \langle \rho, v_k\rangle$. Comparing these problems with (24), (25), we see that $\langle R, v_k\rangle = 0$, $\langle \rho, v_k\rangle = 0$, $k \in \mathbb{N}$. This implies $R = 0$, $\rho = 0$. Consequently, u is a solution of (8). □

Theorem 1. *Let $m \in L_1(0,T)$ and $k \in \mathbb{N}$. Then the following statements hold.*

(i) (uniqueness) If $F_k = 0$, $\varphi_k = 0$ and $u_k \in \widehat{\mathscr{U}}_{r,\beta}$ with some $r \in (1,\infty)$ solves (24), (25) then $u_k = 0$.

(ii) If $F_k \in L_p(0,T)$ with some $p \in (\frac{1}{\beta},\infty)$ then the problem (24), (25) has a solution u_k in the space $\widehat{\mathscr{U}}_{p,\beta}$. This solution is represented by the uniformly in $[0,T]$ converging series

$$u_k(t) = \varphi_k\left(\sum_{i=0}^{+\infty}(M_k *)^i\right)E_\beta(-\lambda_k t^\beta) + \left(\sum_{i=0}^{+\infty}(M_k *)^i\right)\int_0^t e_\beta^{-\lambda_k(t-\tau)}F_k(\tau)d\tau, \tag{26}$$

$$\text{where } M_k(t) = -\lambda_k\int_0^t e_\beta^{-\lambda_k(t-\tau)}m(\tau)d\tau. \tag{27}$$

Proof.

(i) Let $F_k = 0$, $\varphi_k = 0$ and $u_k \in \widehat{\mathscr{U}}_{r,\beta}$ with some $r \in (1,\infty)$ solve (24), (25). Since $u_k(0) = \varphi_k = 0$, we have $u_k \in {}_0H_r^\beta(0,T)$. Denoting $y_k = D^\beta u_k$, we obtain $u_k = I^\beta y_k$ and $y_k \in L_r(0,T)$, by Lemma 1. Moreover, from the equation for u_k, we deduce the homogeneous Volterra equation of the second kind $y_k + K_k * y_k = 0$ with the kernel $K_k = \lambda_k\frac{t^{\beta-1}}{\Gamma(\beta)} + \lambda_k m * \frac{t^{\beta-1}}{\Gamma(\beta)} \in L_1(0,T)$. Such an equation has only the trivial solution. Consequently, $y_k = 0$ and $u_k = 0$.

(ii) Assume $F_k \in L_p(0,T)$ for some $p \in (\frac{1}{\beta},\infty)$. Let us consider the Volterra equation of the second kind $y_k + K_k * y_k = R_k$, where K_k is defined before and $R_k = F_k - \lambda_k\varphi_k - \lambda_k m * \varphi_k \in L_p(0,T)$. It has a solution $y_k \in L_p(0,T)$ (Gripenberg et al[33, Sect. 2.3]). Let us define $u_k = I^\beta y_k + \varphi_k$. By Lemma 1, $u_k - \varphi_k \in {}_0H_p^\beta(0,T)$ and $y_k = D^\beta(u_k - \varphi_k)$. From the equation of y_k, we deduce the Equation (24) for u_k. Since $p \in (\frac{1}{\beta},\infty)$, we obtain $u_k - \varphi_k \in C[0,T]$ and $u_k(0) - \varphi_k = 0$. This implies (25) and $u_k \in \widehat{\mathscr{U}}_{p,\beta}$. The existence assertion of (ii) is proved.
Finally, let us deduce the formula (26) with (27). To this end, we need a solution formula of the fractional differential equation $D^\beta w + \lambda w = z$. It can be found, eg, in Samko et al.[34, Example 42.2] Provided $z \in L_p(0,T)$, the solution $w \in {}_0H_p^\beta(0,T)$ of this equation is given by $w = e_\beta^{-\lambda t} * z$. Rewriting (24) in the form of the equation $D^\beta w_k + \lambda_k w_k = z_k$, where $w_k = u_k - \varphi_k$, $z_k = F_k - \lambda_k\varphi_k - \lambda_k m * u_k$ and solving, we obtain $u_k = e_\beta^{-\lambda t} * z_k + \varphi_k$. Using (18), (27), we transform the latter relation to the Volterra equation

$$u_k(t) = Q_k(t) + M_k * u_k(t), \quad t \in (0,T), \tag{28}$$

with $Q_k = \varphi_k E_\beta(-\lambda_k t^\beta) + e_\beta^{-\lambda_k t} * F_k \in C[0,T]$. Next, let us show that $M_k * \in \mathscr{L}(C[0,T])$. Since $m \in L_1(0,T)$, it holds that $M_k \in L_1(0,T)$, hence for any $w \in C([0,T])$, we have $M_k * w \in C[0,T]$. Due to (11) and (20), we obtain

$$\|M_k\|_{1;\sigma} \le \|m\|_{1;\sigma}\|\lambda_k e_\beta^{-\lambda_k t}\|_{1;\sigma} \le \|m\|_{1;\sigma}\|\lambda_k e_\beta^{-\lambda_k t}\|_1 \le \|m\|_{1;\sigma}. \tag{29}$$

For any $w \in C[0,T]$, we have $\|M_k * w\|_{\infty;\sigma} \le \|M_k\|_{1;\sigma}\|w\|_{\infty;\sigma} \le \|m\|_{1;\sigma}\|w\|_{\infty,\sigma}$. Consequently, $M_k * \in \mathscr{L}(C[0,T])$. Moreover, there exists sufficiently large σ such that $\|M_k * \|_{\mathscr{L}(C[0,T])} \le \|m\|_{1,\sigma} < 1$. Applying the theorem about the continuously inverse operator (see Trenogin[35, p140]), we express the solution of (28) by means of the uniformly convergent Neumann series (26). □

Theorem 2. *Assume $m \in L_1(0,T)$. Then*

(i) (uniqueness) if $F = 0$, $\varphi = 0$ and $u \in \mathscr{U}_{r,\beta}$ with some $r \in (1,\infty)$ solves the direct problem (8) then $u = 0$;

(ii) if $\varphi \in L_2(\Omega)$ and $F = 0$, then the direct problem (8) has a solution u that belongs to $\mathcal{U}_{r,\beta}$ for any $r \in (1, \frac{1}{\beta})$ and this solution has the form

$$u(t,x) = \sum_{k=1}^{+\infty} \varphi_k \left(\sum_{i=0}^{+\infty} (M_k \, *)^i \right) E_\beta(-\lambda_k t^\beta) v_k(x); \tag{30}$$

(iii) if $\varphi = 0$ and $F \in L_p((0,T); L_2(\Omega))$ with some $p \in (\frac{1}{\beta}, \infty)$ then direct problem has a solution $u \in \mathcal{U}_{p,\beta}$ and the solution has the form

$$u(t,x) = \sum_{k=1}^{+\infty} \left(\sum_{i=0}^{+\infty} (M_k \, *)^i \right) \int_0^t e_\beta^{-\lambda_k (t-\tau)} F_k(\tau) d\tau \, v_k(x). \tag{31}$$

Proof.

(i) is an immediate consequence of Proposition 1 (i) and Theorem 1 (i).

(ii) Let us consider the sequence of problems (24), (25) with $F_k = 0$. By Theorem 1 (ii), they have solutions $u_k \in \widehat{\mathcal{U}}_{p,\beta}$ for any $p \in \left(\frac{1}{\beta}, \infty\right)$. The aim is to show that $u = \sum_{k=1}^{+\infty} u_k v_k \in \mathcal{U}_{r,\beta}$ for any $r \in \left(1, \frac{1}{\beta}\right)$. Then the existence assertion follows by Proposition 1 (ii) and the formula (30) is obtained from (26). We start by showing $u \in C([0,T]; L_2(\Omega))$. Since $u_k - \varphi_k \subset C[0,T]$ and $v_k \in L_2(\Omega)$, it follows that $u_k v_k \in C([0,T]; L_2(\Omega))$. Now let us show that the series $u = \sum_{k=1}^{+\infty} u_k v_k$ is uniformly convergent in $[0,T]$ and therefore defines a continuous function. From (26) by means of Young's inequality (11), (19), and (29), we obtain

$$e^{-\sigma T} |u_k(t)| \le |\varphi_k| \left(\sum_{i=0}^{+\infty} \|M_k\|_{1;\sigma}^i \right) \|E_\beta(-\lambda_k t^\beta)\|_{\infty;\sigma} \le |\varphi_k| \left(\sum_{i=0}^{+\infty} \|m\|_{1;\sigma}^i \right) \le \frac{|\varphi_k|}{1 - \|m\|_{1;\sigma}}$$

provided σ is sufficiently large to guarantee $\|m\|_{1;\sigma} < 1$. In view of $\varphi \in L_2(\Omega)$, for any $\varepsilon > 0$, there exists $K_\epsilon \in \mathbb{N}$ such that $\sum_{k=K_\epsilon}^{+\infty} \varphi_k^2 < \frac{(1-\|m\|_{1;\sigma})^2}{e^{2\sigma T}} \varepsilon$. Thus,

$$\left\| \sum_{k=K_\epsilon}^{+\infty} u_k(t) v_k \right\|^2 = \sum_{k=K_\epsilon}^{+\infty} |u_k(t)|^2 \le \frac{e^{2\sigma T}}{(1 - \|m\|_{1;\sigma})^2} \sum_{k=K_\epsilon}^{+\infty} \varphi_k^2 < \epsilon \quad \forall t \in [0,T].$$

Therefore, this series is uniformly convergent and $u \in C([0,T]; L_2(\Omega))$. Secondly, we prove that $u \in L_r((0,T); \mathcal{D}(L))$. To this end, we investigate

$$\|u(t,\cdot)\|_{\mathcal{D}(L)} = \left\{ \sum_{k=1}^{+\infty} \lambda_k^2 \varphi_k^2 \left[\sum_{i=0}^{+\infty} (M_k \, *)^i E_\beta(-\lambda_k t^\beta) \right]^2 \right\}^{\frac{1}{2}}. \tag{32}$$

For each term of the inner series in view of (27), we get

$$|(M_k \, *)^i E_\beta(-\lambda_k t^\beta)| \le (|m| \, *)^i (\lambda_k e_\beta^{-\lambda_k t} \, *)^i E_\beta(-\lambda_k t^\beta).$$

Hence, Lemma 2 implies $|(M_k \, *)^i E_\beta(-\lambda_k t^\beta)| \le (|m| \, *)^i C_2 C_3^i [\lambda_k t^\beta]^{-1}$. Now, we use this inequality in (32). We reach the following estimate:

$$\|u(t,\cdot)\|_{\mathcal{D}(L)} \le \left\{ \sum_{k=1}^{+\infty} \varphi_k^2 \left[\sum_{i=0}^{+\infty} (|m| \, *)^i \frac{C_2 C_3^i}{t^\beta} \right]^2 \right\}^{\frac{1}{2}} = \left[\sum_{i=0}^{+\infty} (|m| \, *)^i \frac{C_2 C_3^i}{t^\beta} \right] \|\varphi\|.$$

Let us choose σ such that $C_3 \|m\|_{1;\sigma} < 1$. Since $\frac{1}{t^\beta} \in L_r(0,T)$ for $r \in (1, \frac{1}{\beta})$, due to (11), we obtain the estimate

$$\|u\|_{L_r((0,T);\mathcal{D}(L))} \le C_2 e^{\sigma T} \left[\sum_{i=0}^{+\infty} C_3^i \|m\|_{1;\sigma}^i \right] \|t^{-\beta}\|_{r;\sigma} \|\varphi\| < \infty.$$

This proves $u \in L_r((0,T);\mathcal{D}(L))$. Finally, we need to show that $u - u(0,\cdot) \in {}_0H_r^\beta((0,T);L_2(\Omega))$. Due to the proved uniform convergence of $u = \sum_{k=1}^{+\infty} u_k v_k$ and the initial conditions $u_k(0) = \varphi_k$, we have $u(0,\cdot) = \varphi$. Let $y_k = D^\beta(u_k - \varphi_k)$. From the Equation (24) (there $F_k = 0$), we obtain $y_k = -\lambda_k u_k - m * \lambda_k u_k$. Consider the function y defined by $y = \sum_{k=1}^{+\infty} y_k v_k$. In previous part of the proof of (ii), we showed that $Lu = \sum_{k=1}^{+\infty} \lambda_k u_k v_k \in L_r((0,T);L_2(\Omega))$. Thus, $y \in L_r((0,T);L_2(\Omega))$. Moreover, the series defining y is absolutely convergent for a.e $(t,x) \in (0,T) \times \Omega$. That allows us to use the Tonelli's theorem to deduce the relation

$$I^\beta y = I^\beta \sum_{k=1}^{+\infty} y_k v_k = \sum_{k=1}^{+\infty} I^\beta y_k v_k = \sum_{k=1}^{+\infty} (u_k - \varphi_k) v_k = u - \varphi.$$

In view of $y \in L_r((0,T);L_2(\Omega))$ and Lemma 1, we get that $u - \varphi \in {}_0H_r^\beta((0,T);L_2(\Omega))$.

(iii) Applying the operator I^β to the equation in (8), it is transformed to the following evolutionary integral equation in the space $L_2(\Omega)$:

$$u(t) + (a * Lu)(t) = -(a * m * Lu) + (a * F)(t), \quad t \in (0,T), \tag{33}$$

where $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. The assertion (iii) partially follows from Theorem 8.7 of Prüss[29] applied to this equation. However, in our case, the validity of assumptions of this theorem is not directly transparent and has to be verified. Let us list these assumptions with corresponding reasonings:

1. $L \in \mathcal{BIP}$,[‡] because L is a normal and sectorial operator, and $\theta_L = 0$, because L has positive real spectrum (cf Prüss[29, Sect. 8.7, comment c) (i)]);
2. a is 1-regular and θ_a-sectorial with $\theta_a = \pi/2$, because a is completely monotone (it follows from Proposition 3.3 of Prüss[29]);
3. $\theta_a + \theta_L < \pi$, because $\theta_a = \frac{\pi}{2}$ and $\theta_L = 0$;
4. $\lim_{\mu\to\infty} |\hat{a}(\mu)|\mu^\beta < \infty$ where $\hat{a}$ is the Laplace transform of a, because $\hat{a}(\mu) = \frac{1}{\mu^\beta}$.

Theorem 8.7 of Prüss[29] implies that (33) has a solution u in the space $L_p((0,T);\mathcal{D}(L))$. Bringing all terms of (33) except for u to the right-hand side, we obtain $u = I^\beta y$, where $y = -Lu - m * Lu + F \in L_p((0,T);L_2(\Omega))$. Thus, Lemma 1 implies $u \in {}_0H_p^\beta((0,T);L_2(\Omega))$. Since $p \in (\frac{1}{\beta}, \infty)$, we obtain $u \in C([0,T];L_2(\Omega))$ and the homogeneous initial condition $u(0,\cdot) = 0$. The proved properties of u show that $u \in \mathcal{U}_{p,\beta}$. Applying the operator D^β to (33), we reach the differential equation in (8). Finally, the formula (31) follows from (26). The proof is complete. □

Theorem 8.7 of Prüss[29] implies the existence of a solution of (8) in case $\varphi \neq 0$, too, but under the stronger assumption $\varphi \in \mathcal{D}(L)$. The assertion *(ii)* of Theorem 2 in the particular case $m = 0$ follows from Sakamoto and Yamamoto.[17, Theorem 2.1]

5 | RESULTS CONCERNING IP1

Let us introduce the notation for Fourier coefficients of functions involved in IP1: $f_k = \langle f, v_k \rangle$, $h_k(t) = \langle h(t,\cdot), v_k \rangle$, $\psi_k = \langle \psi, v_k \rangle$, $k \in \mathbb{N}$.

Proposition 2. *Assume that* $g \in L_p(0,T)$, $h \in L_p((0,T);L_2(\Omega))$ *with some* $p > \frac{1}{\beta}$, $m \in L_1(0,T)$ *and* $\varphi, \psi \in L_2(\Omega)$. *If* $f \in L_2(\Omega)$ *is a solution of IP1, then* f_k, $k \in \mathbb{N}$, *are solutions of the sequence of linear equations*

$$\begin{aligned} A_k f_k &= \psi_k - B_k, \quad A_k = \sum_{i=0}^{+\infty} \left((M_k *)^i e_\beta^{-\lambda_k t} * g \right)(T), \\ B_k &= \varphi_k \sum_{i=0}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta) \right)(T) + \sum_{i=0}^{+\infty} \left((M_k *)^i e_\beta^{-\lambda_k t} * h_k \right)(T). \end{aligned} \tag{34}$$

Conversely, if $f_k, k \in \mathbb{N}$, *are solutions of the Equations (34) and* $\sum_{k=1}^{+\infty} f_k^2 < \infty$ *then* $f = \sum_{k=1}^{+\infty} f_k v_k \in L_2(\Omega)$ *solves IP1 and the related solution of the direct problem (8) belongs to* $\mathcal{U}_{r,\beta}$ *for any* $r \in \left(1, \frac{1}{\beta}\right)$.

[‡] $\mathcal{BIP}$ is the space of operators with bounded imaginary powers.

Proof. Let $f \in L_2(\Omega)$ solve IP1. Then the functions φ and $F = gf + h$ satisfy the assumptions of Theorem 2 (ii) and (iii), respectively. Thus, the related solution of the direct problem (8) belongs to $\mathcal{U}_{r,\beta}$ for any $r \in \left(1, \frac{1}{\beta}\right)$. Using Proposition 1 and Theorem 1, we deduce the formula (26) with $F_k = gf_k + h_k$. Setting there $t = T$ and replacing $u_k(T)$ by ψ_k, we obtain (34). Conversely, let $f_k, k \in \mathbb{N}$, solve (34) and $\sum_{k=1}^{+\infty} f_k^2 < \infty$. By Theorem 2, the problem (8) with $F = gf + h$ and $f = \sum_{k=1}^{+\infty} f_k v_k$ has a solution $u \in \mathcal{U}_{r,\beta}$ for any $r \in \left(1, \frac{1}{\beta}\right)$. Again, by Proposition 1 and Theorem 1, we reach (26). Comparing it with (34), we see that $u_k(T) = \psi_k$, $k \in \mathbb{N}$. This implies (9). Thus, f solves IP1. □

Now, we prove a basic lower estimate of A_k in (34). We do it separately for the cases of negative, small and positive m.

Lemma 3. *Assume that $m \in L_1(0, T)$ and there exist $T_1 \in (0, T)$, $g_0 > 0$ such that one of the following conditions is valid:*

(A1) $m \le 0$, $g \in L_p(0, T)$ with some $p > \frac{1}{\beta}$, $g \ge 0$ and $g(t) \ge g_0$ a.e. $t \in (T_1, T)$;
(A2) $g \in L_\infty(0, T), g \ge 0, g(t) \ge g_0$ a.e. $t \in (T_1, T)$, and $\|m\|_1 < \frac{g_0 C_5}{g_0 C_5 + \|g\|_\infty}$, where $C_5 = 1 - E_\beta(-\lambda_1 (T - T_1)^\beta)$;
(A3) $m \ge 0$, $g \in W_1^1(0, T)$, $g - m * g \ge 0, g' \ge 0$ and $(g - m * g)(t) \ge g_0$ for $t \in (T_1, T)$.
Then $A_k \ge \frac{C_6}{\lambda_k}$, $k \in \mathbb{N}$, where $C_6 > 0$ is a constant independent of k.

Proof. Firstly, we consider the case (A1). Note that $m \le 0$ implies $M_k \ge 0$. Thus, due to (18), the properties of g and the monotonicity of $E_\beta(-z)$, we obtain

$$A_k \ge (e_\beta^{-\lambda_k t} * g)(T) \ge g_0 \|e_\beta^{-\lambda_k t}\|_{L_1(0,T-T_1)} = \frac{(1 - E_\beta(-\lambda_k (T - T_1)^\beta)) g_0}{\lambda_k} \ge \frac{C_6}{\lambda_k}, \tag{35}$$

where $C_6 = [1 - E_\beta(-\lambda_1 (T - T_1)^\beta)] g_0$.

In case (A2), by means of (11) and (35), we deduce

$$A_k \ge (e_\beta^{-\lambda_k t} * g)(T) - \left| \sum_{i=1}^{+\infty} \left((M_k *)^i e_\beta^{-\lambda_k t} * g \right)(T) \right| \ge \frac{(1 - E_\beta(-\lambda_1 (T - T_1)^\beta)) g_0}{\lambda_k} - \sum_{i=1}^{+\infty} \|M_k\|_1^i \, \|e_\beta^{-\lambda_k t}\|_1 \, \|g\|_\infty .$$

Using (20) and (29), we obtain $A_k \ge \frac{C_6}{\lambda_k}$, where $C_6 = [(1 - E_\beta(-\lambda_1 (T - T_1)^\beta))] g_0 - \frac{\|m\|_1}{1 - \|m\|_1} \|g\|_\infty > 0$.

Finally, we treat the case (A3). We point out that A_k can be represented as

$$A_k = \left(\sum_{i=0}^{+\infty} (M_k *)^{2i} \right) e_\beta^{-\lambda_k t} * \left(g - \lambda_k e_\beta^{-\lambda_k t} * g * m \right)(T). \tag{36}$$

By means of the integration by parts, we have

$$\left(\frac{d}{dt} E_\beta(-\lambda_k t^\beta) \right) * g = E_\beta(-\lambda_k t^\beta) g(0) - g + E_\beta(-\lambda_k t^\beta) * g'.$$

Since $-\lambda_k e_\beta^{-\lambda_k t} = \frac{d}{dt} E_\beta(-\lambda_k t^\beta)$, it holds that

$$g - \lambda_k e_\beta^{-\lambda_k t} * g * m = g + \left(\frac{d}{dt} E_\beta(-\lambda_k t^\beta) \right) * g * m = g(0) E_\beta(-\lambda_k t^\beta) * m + g' * E_\beta(-\lambda_k t^\beta) * m + g - m * g.$$

Therefore, in view of the assumptions (A3), we have $g - \lambda_k e_\beta^{-\lambda_k t} * g * m \ge g - m * g \ge 0$. Moreover, $M_k * M_k = (-m) * \lambda_k e_\beta^{-\lambda_k t} * (-m) * \lambda_k e_\beta^{-\lambda_k t} \ge 0$. We get from (36)

$$A_k \ge \left(e_\beta^{-\lambda_k t} * (g - m * g) \right)(T).$$

Then similarly to (35), $A_k \ge \frac{C_6}{\lambda_k}$, where $C_6 = [1 - E_\beta(-\lambda_1 (T - T_1)^\beta)] g_0$. □

From Proposition 2 and Lemma 3, we easily deduce the uniqueness assertion for IP1.

Corollary 1. *Let the assumptions of Lemma 3 be satisfied, $\varphi = 0, h = 0$ and $\psi = 0$. If $f \in L_2(\Omega)$ is a solution of the IP1, then $f = 0$.*

Proof. If $f \in L_2(\Omega)$ is a solution of the inverse problem, then by Proposition 2, the formulas (34) are valid and it yields from the assumptions of the corollary that $\psi_k = B_k = 0$, $k \in \mathbb{N}$. On the other hand, Lemma 2 implies $A_k > 0$, $k \in \mathbb{N}$. Therefore, the solution of (34) is $f_k = 0$, $k \in \mathbb{N}$. Thus $f = 0$. □

Remark 1. Recall that the physical source contains a time factor q that is connected with g by the formula $g(t) = I^{1-\beta}q(t) = \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}q(\tau)d\tau$ (see a remark after the formulation of IP1 in Section 2). The conditions for g in (A1) and (A2) may be satisfied in case of q that changes the sign or vanishes before T. For instance, defining $q(t) = \begin{cases} 1 & \text{if } t < T-\epsilon \\ -1 & \text{if } t > T-\epsilon \end{cases}$ where $0 < \epsilon < 2^{-\frac{1}{1-\beta}}T$ or $q(t) = \begin{cases} 1 & \text{if } t < \eta \\ 0 & \text{if } t > \eta \end{cases}$ where $0 < \eta < T$, (A1) and (A2) hold for g. Therefore, the assertion of Lemma 3 holds provided $m \in L_1(0,T)$ is negative or small enough, and the solution of IP1 is unique.

The case $m \le 0$ is comparable with results of the paper,[12] where a reconstruction of f in the heat equation with memory $u_t = Lu + m * Lu + g(t,x)f(x)$ from final data was considered. Uniqueness was proved under cone conditions that are even stronger than $m \le 0$.

The coupled conditions for m and g in (A3) cover all positive integrable m. This means that for any $m \in L_1(0,T)$, $m \ge 0$, it is possible to find a function g so that (A3) is valid. Let us construct such a g. Choose an arbitrary $z \in W_1^1(0,T)$ so that $z \ge 0$, $z' \ge 0$ and $z(t) \ge z_0 > 0$, $t \in (T_1, T)$ and define g as a solution of the Volterra equation of the second kind $g - m * g = z$. Then $g' - m * g' - g(0)m = z'$, hence $g' = \sum_{i=0}^{+\infty}(m\,*)^i(z' + g(0)m) \ge 0$. So the conditions (A3) are satisfied.

To formulate an existence theorem for IP1, we have to introduce fractional powers of L and related domains. The operator L^s, $s \ge 0$, can be defined by the relation $L^s z = \sum_{k=1}^{+\infty}\lambda_k^s\langle z, v_k\rangle v_k$ and has the domain

$$\mathcal{D}(L^s) = \left\{ z \in L_2(\Omega) \ : \ \|z\|_{\mathcal{D}(L^s)} := \left[\sum_{k=1}^{+\infty}\lambda_k^{2s}\langle z, v_k\rangle^2\right]^{\frac{1}{2}} < \infty \right\}$$

in the space $L_2(\Omega)$.[17] Evidently, $\mathcal{D}(L^0) = L_2(\Omega)$.

Theorem 3. *Let the assumptions of Lemma 3 be satisfied,* $\psi \in \mathcal{D}(L)$ *and* $h \in L_\rho((0,T);\mathcal{D}(L^\omega))$, $\sum_{k=0}^{+\infty}\lambda_k^{2\omega}\|h_k\|_\rho^2 < \infty$ *with some* $\rho \in (\frac{1}{\beta}, \infty]$, *where* $\omega > \frac{1}{\beta\rho}$ *for* $\rho \in \left(\frac{1}{\beta}, \infty\right)$ *and* $\omega = 0$ *for* $\rho = \infty$. *Moreover, let one of the following conditions be valid:*

(A4) $\varphi \in \mathcal{D}(L)$;
(A5) $\varphi \in \mathcal{D}(L^s)$ *for some* $s \in [0,1)$ *and* $m \in L_r(0,T)$ *for some* $r > \frac{1}{1-\beta(1-s)}$;
(A6) $\varphi \in L_2(\Omega)$ *and* $\exists c_m \ge 0$, $\gamma_m < 1 \ : \ |m(t)| \le \frac{c_m}{\Gamma(1-\gamma_m)}t^{-\gamma_m}$ *a.e* $t \in (0,T)$.

Then IP1 has a unique solution $f \in L_2(\Omega)$ *and the related solution of the direct problem (8) belongs to* $\mathcal{U}_{r,\beta}$ *for any* $r \in \left(1, \frac{1}{\beta}\right)$. *This solution satisfies the estimate*

$$\|f\| \le C_7\left\{ \|\psi\|_{\mathcal{D}(L)} + \|\varphi\|_{\mathcal{D}(L^\Theta)} + \left[\sum_{k=0}^{+\infty}\lambda_k^{2\omega}\|h_k\|_\rho^2\right]^{\frac{1}{2}} \right\}, \tag{37}$$

where the exponent Θ *equals 1, s and 0 in cases (A4), (A5), and (A6), respectively, and* C_7 *is a constant that depends on* C_6, m, T *and* β.

Proof. Let us consider the formula of B_k in (34). Firstly, we estimate the term containing h_k by means of (11) and (29):

$$\left|\sum_{i=0}^{+\infty}\left((M_k\,*)^i e_\beta^{-\lambda_k t} * h_k\right)(T)\right| \le e^{\sigma T}\sum_{i=0}^{+\infty}\|M_k\|_{1;\sigma}^i\|e_\beta^{-\lambda_k t} * h_k\|_\infty \le e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i\|e_\beta^{-\lambda_k t} * h_k\|_\infty. \tag{38}$$

In case $\rho = \infty$, by means of (20), we obtain $\|e_\beta^{-\lambda_k t} * h_k\|_\infty \le \|e_\beta^{-\lambda_k t}\|_1\|h_k\|_\infty \le \lambda_k^{-1}\|h_k\|_\infty$. Next, let $\rho \in \left(\frac{1}{\beta}, \infty\right)$. We note that the boundedness of $E_{\beta,\beta}(-z)$ for $z \ge 0$ and the asymptotical relation (15) imply the inequality $E_{\beta,\beta}(-z) \le \frac{C_8}{z^{1-\omega}}$ for $z \ge 0$ with some constant C_8. Thus,

$$e_\beta^{-\lambda_k t} = t^{\beta-1}E_{\beta,\beta}(-\lambda_k t^\beta) \le C_8\lambda_k^{\omega-1}t^{\beta\omega-1}.$$

Due to the assumed inequality $\omega > \frac{1}{\beta\rho}$, it holds $t^{\beta\omega-1} \in L_{\rho'}(0,T)$, where $\frac{1}{\rho}+\frac{1}{\rho'}=1$. We obtain $\|e_{\beta}^{-\lambda_k t} * h_k\|_\infty \leq C_8\lambda_k^{\omega-1}\|t^{\beta\omega-1}\|_{\rho'}\|h_k\|_\rho = C_9\lambda_k^{\omega-1}\|h_k\|_\rho$. Let us continue the estimation of (38). For any $\rho \in (\frac{1}{\beta},\infty]$, we have

$$\left|\sum_{i=0}^{+\infty}\left((M_k *)^i e_\beta^{-\lambda_k t} * h_k\right)(T)\right| \leq C_{10}e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i\lambda_k^{\omega-1}\|h_k\|_\rho = C_{11}\lambda_k^{\omega-1}\|h_k\|_\rho \tag{39}$$

with $C_{10}=\max\{1;C_9\}$, $C_{11}=\frac{e^{\sigma T}C_{10}}{1-\|m\|_{1;\sigma}}$, provided σ is large enough to guarantee $\|m\|_{1;\sigma}<1$.

Secondly, we estimate the factor of φ_k in (34). In the general case, when $m \in L_1(0,T)$ (it is so in the case (A4)), we have due to (19) and (29) that

$$\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i\|E_\beta(-\lambda_k t^\beta)\|_\infty \leq C_{12}. \tag{40}$$

In case (A5), by means of (16), (20), and (29), we obtain the estimate

$$\begin{aligned}\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| &= \left|E_\beta(-\lambda_k T^\beta) - \sum_{i=0}^{+\infty}\Big((M_k *)^i\lambda_k e_\beta^{-\lambda_k t}\right.\\ &\left.* m * E_\beta(-\lambda_k t^\beta)\Big)(T)\right| \leq E_\beta(-\lambda_k T^\beta) + e^{\sigma T}\sum_{i=0}^{+\infty}\|m\|_{1;\sigma}^i\|m * E_\beta(-\lambda_k t^\beta)\|_\infty.\end{aligned} \tag{41}$$

Since $E_\beta(-\lambda_k t^\beta) \leq \frac{C_2}{1+\lambda_k t^\beta} \leq \frac{C_2}{(\lambda_k t^\beta)^{1-s}}$, $s\in[0,1)$, we estimate

$$\|m * E_\beta(-\lambda_k t^\beta)\|_\infty \leq \lambda_k^{s-1}\|m\|_r\|t^{-\beta(1-s)}\|_{r'},$$

where $\frac{1}{r}+\frac{1}{r'}=1$. In this point, we have $\|t^{-\beta(1-s)}\|_{r'}<\infty$ because of the assumption $r > \frac{1}{1-\beta(1-s)}$. Thus, from (41), we obtain

$$\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq C_{14}\lambda_k^{s-1} \tag{42}$$

with $C_{14}=\frac{1}{T^{\beta(1-s)}}+\frac{e^{\sigma T}C_2}{(1-\|m\|_{1;\sigma})}\|m\|_r\|t^{-\beta(1-s)}\|_{r'}$. Finally, if the assumptions (A6) hold for m, we deduce

$$\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq \sum_{i=0}^{+\infty}\left(\left(\frac{c_m t^{-\gamma_m}}{\Gamma(1-\gamma_m)} *\right)^i(\lambda_k e_\beta^{-\lambda_k t} *)^i E_\beta(-\lambda_k t^\beta)\right)(T).$$

Using Lemma 2 and the formula $\frac{t^a}{\Gamma(1+a)} * \frac{t^b}{\Gamma(1+b)} = \frac{t^{1+a+b}}{\Gamma(2+a+b)}$ repeatedly, we continue the estimation:

$$\begin{aligned}\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| &\leq \sum_{i=0}^{+\infty}\left(\frac{c_m^i t^{i(1-\gamma_m)-1}}{\Gamma(i(1-\gamma_m))} * \frac{C_2C_3^i}{\lambda_k t^\beta}\right)(T)\\ &= \frac{1}{\lambda_k}C_2\Gamma(1-\beta)\sum_{i=0}^{+\infty}\frac{(C_3c_m)^iT^{i(1-\gamma_m)-\beta}}{\Gamma(i(1-\gamma_m)+1-\beta)} = C_{15}\lambda_k^{-1},\end{aligned} \tag{43}$$

where $C_{15}=C_2T^{-\beta}\Gamma(1-\beta)E_{1-\gamma_m,1-\beta}(C_3c_mT^{1-\gamma_m})$. Summing up, (40), (42), and (43) imply

$$\left|\sum_{i=0}^{+\infty}\left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq C_{16}\lambda_k^{\Theta-1} \tag{44}$$

with $C_{16}=\max\{C_{12};C_{14};C_{15}\}$ for all cases (A4)-(A6).

Now, we are able to estimate the quantity f_k in (34). Lemma 3 and the relations (39), (44) yield $|f_k| \leq C_7\{\lambda_k|\psi_k| + \lambda_k^\Theta|\varphi_k| + \lambda_k^\omega\|h_k\|_\rho\}$, where $C_7=\frac{1}{C_6}\max\{1;C_{11};C_{16}\}$. Assumptions of the theorem yield $\sum_{k=1}^{+\infty}f_k^2<\infty$. Therefore, Proposition 2 implies that $f = \sum_{k=1}^{+\infty} f_k v_k \in L_2(\Omega)$ solves IP1. Finally, plugging the deduced estimate for $|f_k|$ into the relation $\|f\| = \left[\sum_{k=1}^{+\infty} |f_k|^2\right]^{1/2}$ and using the triangle inequality in l_2-space, we obtain (37). □

The condition $|m(t)| \leq \frac{c_m}{\Gamma(1-\gamma_m)} t^{-\gamma_m}$, $c_m \geq 0, \gamma_m < 1$ in assumption (A6) holds for kernels of the special form $m(t) = \sum_{j=1}^{l} a_j \frac{t^{\beta_j-\beta-1}}{\Gamma(\beta_j-\beta)}$, $\beta_j > \beta, j = 1, \ldots, l$, that occur in models with multiple Riemann-Liouville derivatives (see Section 2). Moreover, this condition is valid for the kernel m in the Equation (6) that results from (7). Then the relation $m + k * m = -k$ is valid, and we have $m = \sum_{i=0}^{\infty} (-k\,*)^i(-k)$. The right formula in (7) implies $|k(t)| \leq \frac{c_k}{\Gamma(1-\gamma_k)} t^{-\gamma_k}$ with some $c_k \geq 0, \gamma_k < 1$. Estimating m by means of the formula $\frac{t^a}{\Gamma(1+a)} * \frac{t^b}{\Gamma(1+b)} = \frac{t^{1+a+b}}{\Gamma(2+a+b)}$, we obtain $|m(t)| \leq \frac{c_m}{\Gamma(1-\gamma_m)} t^{-\gamma_m}$ with $\gamma_m = \gamma_k$ and $c_m = c_k\Gamma(1-\gamma_k)E_{1-\gamma_k,1-\gamma_k}(c_k T^{1-\gamma_k})$.

6 | RESULTS CONCERNING IP2

Proposition 3. *Assume that $F \in L_p((0,T); L_2(\Omega))$ with some $p > \frac{1}{\beta}$, $m \in L_1(0,T)$ and $\psi \in L_2(\Omega)$. If $\varphi \in L_2(\Omega)$ is a solution of IP2, then φ_k, $k \in \mathbb{N}$, are solutions of the sequence of linear equations*

$$\begin{aligned} \hat{A}_k\varphi_k = \psi_k - \hat{B}_k, \quad \hat{A}_k &= \sum_{i=0}^{+\infty} \left((M_k\,*)^i E_\beta(-\lambda_k t^\beta)\right)(T), \\ \hat{B}_k &= \sum_{i=0}^{+\infty} \left((M_k\,*)^i e_\beta^{-\lambda_k t} * F_k\right)(T), \end{aligned} \tag{45}$$

where $\psi_k = \langle \psi, v_k\rangle$ as in the case of IP1. Conversely, if $\varphi_k, k \in \mathbb{N}$, are solutions of the Equations (45) and $\sum_{k=1}^{+\infty} \varphi_k^2 < \infty$ then $\varphi = \sum_{k=1}^{+\infty} \varphi_k v_k \in L_2(\Omega)$ solves IP2 and the related solution of (8) belongs to $\mathcal{U}_{r,\beta}$ for any $r \in \left(1, \frac{1}{\beta}\right)$.

The proof is similar to the proof of Proposition 2.

Next, we derive a basic lower estimate for $\hat{A}_k$ in case of negative or small m. We have no results in case of general positive m. Lack of an additional degree of freedom (as the function g in IP1) makes the study of the case $m \geq 0$ very complicated.

Lemma 4. *Let one of the following conditions hold:*

(A7) $m \in L_1(0,T)$, $m \leq 0$;
(A8) $m \in L_r(0,T)$ with some $r > \frac{1}{1-\beta}$ and $\|m\|_1 < 1$,

$$\frac{\|m\|_r}{1 - \|m\|_1} < \frac{C_1(1-\beta r')^{1/r'}}{C_2(1/\lambda_1 + T^\beta)T^{1/r'-\beta}},$$

where C_1 and C_2 are the constants from (16) and $\frac{1}{r} + \frac{1}{r'} = 1$;
(A9) $|m(t)| \leq \frac{c_m}{\Gamma(1-\gamma_m)} t^{-\gamma_m}$ a.e. $t \in (0,T)$ with some $\gamma_m < 1$ and a sufficiently small $c_m > 0$, such that

$$c_m E_{1-\gamma_m, 2-\gamma_m-\beta}\left(C_3 T^{1-\gamma_m} c_m\right) < \frac{C_1}{C_2 C_3 \Gamma(1-\beta) T^{1-\gamma_m-\beta}\left(1/\lambda_1 + T^\beta\right)}, \tag{46}$$

where C_3 is the constant from (21).
Then $\hat{A}_k \geq \frac{C_{17}}{\lambda_k}$, $k \in \mathbb{N}$, where $C_{17} > 0$ is a constant independent of k.

We remark that $E_{1-\gamma_m,2-\gamma_m-\beta}$ in the left hand side of (46) is locally bounded as an entire function.

Proof. In case (A7), we have $M_k \geq 0$ and by applying (16), we estimate

$$\hat{A}_k = \sum_{i=0}^{+\infty} \left((M_k\,*)^i E_\beta(-\lambda_k t^\beta)\right)(T) \geq E_\beta(-\lambda_k T^\beta) \geq \frac{C_1}{1 + \lambda_k T^\beta} \geq \frac{C_{17}}{\lambda_k},$$

where we take $C_{17} = \frac{C_1}{1/\lambda_1 + T^\beta}$.

Secondly, let us consider the case (A8). We have the relation

$$\hat{A}_k = \sum_{i=0}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T) \geq E_\beta(-\lambda_k T^\beta) - \left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \geq \frac{C_{17}}{\lambda_k} - \left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right|,$$

where we treat the series similarly to (41):

$$\left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| = \left|\sum_{i=0}^{+\infty} \left((M_k *)^i \lambda_k e_\beta^{-\lambda_k t} * m * E_\beta(-\lambda_k t^\beta)\right)(T)\right|$$

$$\leq \sum_{i=0}^{+\infty} \|m\|_1^i \|m * E_\beta(-\lambda_k t^\beta)\|_\infty \leq \frac{C_2 \|m * t^{-\beta}\|_\infty}{\lambda_k (1 - \|m\|_1)} \leq \frac{C_2 \|t^{-\beta}\|_{r'}}{\lambda_k (1 - \|m\|_1)} \|m\|_r.$$

Then

$$\hat{A}_k \geq \frac{C_{17}}{\lambda_k}, \quad \text{where } C_{17} = \frac{C_1}{1/\lambda_1 + T^\beta} - \frac{C_2 \|m\|_r}{1 - \|m\|_1} \left(\frac{T^{1-\beta r'}}{1 - \beta r'}\right)^{1/r'}.$$

Finally, the case (A9) can be treated similarly to (A8) in the sense that we start from the estimate

$$\hat{A}_k \geq \frac{C_1}{\lambda_k (1/\lambda_1 + T^\beta)} - \left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right|,$$

and estimate the series from above. As in (43) by means of Lemma 2, we obtain

$$\left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq \frac{1}{\lambda_k} C_2 \Gamma(1-\beta) \sum_{i=1}^{+\infty} \frac{(C_3 c_m)^i T^{i(1-\gamma_m)-\beta}}{\Gamma(i(1-\gamma_m) + 1 - \beta)}.$$

The series starts with $i = 1$; thus, we can extract the factor c_m and reach the estimate

$$\left|\sum_{i=1}^{+\infty} \left((M_k *)^i E_\beta(-\lambda_k t^\beta)\right)(T)\right| \leq \frac{1}{\lambda_k} C_2 C_3 c_m \Gamma(1-\beta) T^{1-\gamma_m-\beta}$$

$$\times \sum_{i=0}^{+\infty} \frac{(C_3 c_m)^i T^{i(1-\gamma_m)}}{\Gamma(i(1-\gamma_m) + 2 - \beta - \gamma_m)} = \frac{C_{18}}{\lambda_k} c_m E_{1-\gamma_m, 2-\gamma_m-\beta}\left(C_3 c_m T^{1-\gamma_m}\right),$$

where $C_{18} = C_2 C_3 \Gamma(1-\beta) T^{1-\gamma_m-\beta}$. We obtain the relation

$$\hat{A}_k \geq \frac{C_{17}}{\lambda_k}, \quad \text{where} \quad C_{17} = \frac{C_1}{1/\lambda_1 + T^\beta} - C_{18} c_m E_{1-\gamma_m, 2-\gamma_m-\beta}\left(C_3 c_m T^{1-\gamma_m}\right).$$

□

Corollary 2. *Let the assumptions of Lemma 4 be satisfied, $F = 0$ and $\psi = 0$. If $\varphi \in L_2(\Omega)$ is a solution of IP2, then $\varphi = 0$.*

The proof is similar to the proof of the previous corollary.

Theorem 4. *Let the assumptions of Lemma 4 be satisfied, $\psi \in \mathcal{D}(L)$ and $F \in L_\rho((0,T); \mathcal{D}(L^\omega))$, $\sum_{k=0}^{+\infty} \lambda_k^{2\omega} \|F_k\|_\rho^2 < \infty$ with some $\rho \in (\frac{1}{\beta}, \infty]$, where $\omega > \frac{1}{\beta\rho}$ for $\rho \in \left(\frac{1}{\beta}, \infty\right)$ and $\omega = 0$ for $\rho = \infty$. Then IP2 has a solution $\varphi \in L_2(\Omega)$ and the related solution of the direct problem (8) belongs to $\mathcal{U}_{r,\beta}$ for any $r \in \left(1, \frac{1}{\beta}\right)$. This solution satisfies the estimate*

$$\|\varphi\| \leq C_{19} \left\{ \|\psi\|_{\mathcal{D}(L)} + \left[\sum_{k=0}^{+\infty} \lambda_k^{2\omega} \|F_k\|_\rho^2\right]^{\frac{1}{2}} \right\},$$

where C_{19} is a constant that depends on C_{17}, m, T, and β.

Proof. Let us estimate $\hat{B}_k$ from above. As in (39), we deduce the relation

$$|\hat{B}_k| = \left|\sum_{i=0}^{+\infty} \left((M_k *)^i e_\beta^{-\lambda_k t} * F_k\right)(T)\right| \leq C_{11} \lambda_k^{\omega-1} \|F_k\|_\rho.$$

This estimate together with Lemma 4 yields $|\varphi_k| \leq C_{19}\{\lambda_k |\psi_k| + \lambda_k^\omega \|F_k\|_\rho\}$. Now the assertions of the theorem follow by means of arguments similar to the proof of Theorem 3. □

7 | REGULARIZATION

According to Theorem 3, the solution f of IP1 is stable with respect to ψ in the norm of $\mathcal{D}(L)$. However, if the function ψ is given approximately in a norm weaker than $\mathcal{D}(L)$, the stability of the solution is not ensured. That creates a necessity to incorporate regularization into the numerical computations.

Now let us assume that instead of exact ψ we are given $\psi^\delta \in L_2(\Omega)$, such that $\|\psi^\delta - \psi\| \le \delta$. For the sake of simplicity, we assume that the other data, ie, φ and h, are given exactly.

Since the solution has a closed form $f = \sum_{k=1}^{+\infty} \frac{\psi_k}{A_k} v_k - \sum_{k=1}^{+\infty} \frac{B_k}{A_k} v_k$, we propose a direct method of regularization that consists in truncation of the involved series and is known as a method of least error.[36] Similar approach in case $m = 0$ was exploited in Tuan and Dinh[16] and Zhang and Wei.[37] We define the approximate solution as

$$f^{N,\delta} = \sum_{k=1}^{N} \frac{\psi_k^\delta}{A_k} v_k - \sum_{k=1}^{+\infty} \frac{B_k}{A_k} v_k, \quad \psi_k^\delta = \langle \psi^\delta, v_k \rangle.$$

Here, the number N works as a regularization parameter and depends on the noise level δ. We skip the traditional part of *a priori* parameter choice and go directly to the *a posteriori* choice, because the latter one will be applied in numerical examples in next section. Namely, we choose N according to the discrepancy principle:

$$\|\mathcal{A}f^{N,\delta} - \psi^\delta\| \le c\delta < \|\mathcal{A}f^{N-1,\delta} - \psi^\delta\|, \tag{47}$$

where $c > 0$ is a constant and $\mathcal{A} : f \to \psi$ is the input-output mapping. It is defined by the formula

$$\mathcal{A}f = \sum_{k=1}^{+\infty} (A_k f_k + B_k) v_k,$$

whereas the inverse mapping is given by $f = \mathcal{A}^{-1}\psi = \sum_{k=1}^{\infty} \frac{\psi_k}{A_k} v_k - \sum_{k=1}^{+\infty} \frac{B_k}{A_k} v_k$, that implies $\mathcal{A}f^{N,\delta} = \mathcal{A}\mathcal{A}^{-1}\psi^{N,\delta} = \psi^{N,\delta}$ where $\psi^{N,\delta} = \sum_{k=1}^{N} \psi_k^\delta v_k$.

Practically, the approximate value of $N(\delta)$ can be found as follows:

- starting with $N = 1$ and increasing it, compute the integrals

$$R_N = \left\| \sum_{k=1}^{N} \psi_k^\delta v_k - \psi^\delta \right\|^2 = \int_\Omega \left[\sum_{k=1}^{N} \psi_k^\delta v_k(x) - \psi^\delta(x) \right]^2 dx; \tag{48}$$

- stop when R_N becomes smaller than or equal to $c^2\delta^2$.

The current goal is to derive an error estimate for this method, ie, to estimate $\|f^{N,\delta} - f\|$, given that (47) holds. For this reason, we impose the assumptions of the Theorem 3 on the data. Moreover, we assume that $\psi \in \mathcal{D}(L^{1+\mu})$ for some $\mu > 0$.

Firstly, let us deduce some auxiliary formulas. We obtain

$$c\delta < \|\psi^{N-1,\delta} - \psi^\delta\| \le \|(\psi^\delta - \psi) - (\psi^{N-1,\delta} - \psi^{N-1})\| + \|\psi^{N-1} - \psi\|,$$

$$\|(\psi^\delta - \psi) - (\psi^{N-1,\delta} - \psi^{N-1})\| = \left[\sum_{k=N}^{+\infty} (\psi_k - \psi_k^\delta)^2 \right]^{1/2} \le \|\psi - \psi^\delta\| \le \delta,$$

$$\|\psi^{N-1} - \psi\| \le \left[\frac{1}{\lambda_N^{2\mu+2}} \sum_{k=N}^{+\infty} \lambda_k^{2\mu+2} \psi_k^2 \right]^{1/2} = \frac{\|\psi\|_{\mathcal{D}(L^{1+\mu})}}{\lambda_N^{\mu+1}}.$$

Therefore we obtain an estimate of δ in terms of λ_N:

$$\delta < \frac{\|\psi\|_{\mathcal{D}(L^{1+\mu})}}{(c-1)\lambda_N^{\mu+1}}. \tag{49}$$

On the other hand,

$$\|\psi^{N,\delta} - \psi\| = \|\mathcal{A}f^{N,\delta} - \mathcal{A}f\| \le \|\mathcal{A}f^{N,\delta} - \psi^\delta\| + \|\psi - \psi^\delta\| \le \delta(c+1). \tag{50}$$

Now, let us introduce the norm

$$\|f\|_{(a)} = \left(\sum_{k=1}^{+\infty}\frac{1}{A_k^{2a}}f_k^2\right)^{1/2} = \left(\sum_{k=1}^{+\infty}\frac{\lambda_k^{2a}}{\overline{A}_k^{2a}}f_k^2\right)^{1/2}, \quad \overline{A}_k = A_k\lambda_k.$$

Here $\overline{A}_k \geq C_6 > 0$, by Lemma 3. It follows directly from the definition that $\|f\|_{(0)} = \|f\|_{L_2(\Omega)} = \|f\|$. Such a norm definition applied to the Hölder inequality results in the interpolation inequality

$$\|f\| \leq \|f\|_{(\mu)}^{1/(\mu+1)}\|f\|_{(-1)}^{\mu/(\mu+1)}. \tag{51}$$

Since $\overline{A}_k \geq C_6$, inequalities (49), (50) yield the estimate

$$\|f^{N,\delta} - f\|_{(\mu)} \leq \frac{\lambda_{N+1}^{\mu+1}}{C_6^{1+\mu}}\|\psi^{N,\delta} - \psi\| \leq \frac{\lambda_{N+1}^{\mu+1}\delta(c+1)}{C_6^{1+\mu}} < \frac{\|\psi\|_{D(L^{1+\mu})}(c+1)}{C_6^{1+\mu}(c-1)}\left(\frac{\lambda_{N+1}}{\lambda_N}\right)^{\mu+1}.$$

There exist $\underline{c}, \overline{c}$ and $a > 0$ such that $\underline{c}k^a \leq \lambda_k \leq \overline{c}k^a$, $k \in \mathbb{N}$.[38] Thus, $\frac{\lambda_{N+1}}{\lambda_N} \leq C_* = \underline{c}^{-1}\overline{c}2^a$ and we obtain $\|f^{N,\delta} - f\|_{(\mu)} \leq \frac{C_*^{\mu+1}\|\psi\|_{D(L^{\mu})}(c+1)}{C_6^{1+\mu}(c-1)}$. On the other hand, $\|f^{N,\delta} - f\|_{(-1)} = \|\mathcal{A}f^{N,\delta} - \mathcal{A}f\| \leq (c+1)\delta$, by (50). Hence, by means of the interpolation inequality (51), we derive the following error estimate for the discrepancy principle:

$$\|f^{N,\delta} - f\| \leq C_{20}\delta^{\mu/(\mu+1)}, \quad \text{where } C_{20} = \frac{C_*\|\psi\|_{D(L^{1+\mu})}^{1/(1+\mu)}(c+1)}{C_6(c-1)^{1/(1+\mu)}}.$$

A similar regularization scheme can be constructed for IP2 and the error estimate derived:

$$\|\varphi^{N,\delta} - \varphi\| \leq C_{21}\delta^{\mu/(\mu+1)}, \quad \text{where } C_{21} = \frac{C_*\|\psi\|_{D(L^{1+\mu})}^{1/(1+\mu)}(c+1)}{C_{17}(c-1)^{1/(1+\mu)}}.$$

8 | NUMERICAL EXAMPLES

Before we start, let us point out that in our computations, we strongly rely on the idea of the Fourier expansion and this is argumented by the theoretical part of the article. All the data are decomposed into Fourier series with respect to the eigenfunctions v_k of the operator L.

To provide the numerical examples for the IP1, we use the simulation scheme that works as follows:

- given the exact function f a numerical solution $u_{num}(t,x) = \sum_{k=1}^{N^*} u_k(t)v_k(x)$ to the direct problem is found, where N^* is a sufficiently high predefined order of approximation;
- the function $\psi(x) = u_{num}(T,x)$ is computed, based on that the synthetic noisy data ψ^δ is generated;
- the approximation order N for the inverse problem is computed by the parameter choice rule (47) (eg, computing R_N by (48) until it is small enough);
- given ψ^δ and the order of approximation N the solution to IP1 f^δ is computed.

For all the examples provided, we assume that the order of the fractional derivative is $\beta = 0.5$. The memory kernel is given by the formula $m = -\frac{t^{\gamma-1}}{\Gamma(\gamma)}$, eg, it is a fractional integral kernel. We fix $\gamma = 0.1$ in this relation.

The problems are considered in a domain $\Omega = (0,1)$ and the boundary conditions are $u(0,t) = u(1,t) = 0$. Thus, all the eigenfunctions v_k and eigenvalues λ_k, $k = 1, \ldots, \max\{N, N^*\}$ are found analytically. Fourier coefficients are computed by means of a trapezoidal rule. For this purpose, the grid $Grid = \cup_{i=0}^{M}\{x_i\}$, $x_i = \frac{i}{M}$, is defined.

Regarding the direct problem, we are looking for $u_{num}(t,x) = \sum_{k=1}^{N^*} u_k(t)v_k(x)$. As it has been shown before, for each $k = 1, \ldots, N^*$, the coefficient u_k is uniquely determined from (24), (25), that is equivalent to a weakly singular Volterra equation of the second kind

$$u_k(t) + \lambda_k\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}u_k(\tau)d\tau + \lambda_k\int_0^t \left[\frac{s^{\beta-1}}{\Gamma(\beta)} * m(s)\right]\Bigg|_{s=t-\tau} u_k(\tau)d\tau = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}F_k(\tau)d\tau + \varphi_k, \quad t \in (0,T).$$

Taking into account the particular form of m, each equation is solved by the linear spline collocation method on a uniform mesh.

Thus, we already know $\psi(x) = u_{num}(T, x)$ and continue with generation of the noisy data $\psi^{\delta}(x_i) = \psi(x_i) + r_i\delta, r_i \sim U(-1, 1), i = 0, \dots, M$. In its turn, it enables us to determine the order N of approximation to the inverse problem IP1.

In all the examples provided constant in the discrepancy principle is chosen as $c = 1.1$.

As for the inverse problem IP1, to find $f^{N,\delta}(x) = \sum_{k=1}^{N} f_k^{\delta} v_k(x)$, we reuse the approach for solving the direct problem. For this purpose, let us point out that

$$u_k(t) = u_k^1(t) + f_k^{\delta} u_k^2(t), \quad t \in [0, T],$$

where

- u_k^1 solves (24), (25) with $F_k(t) = h_k(t), t \in (0, T)$ and $u_k^1(0) = \varphi_k$,
- u_k^2 solves (24), (25) with $F_k(t) = g(t), t \in (0, T)$ and zero initial condition.

Finally, since $\psi_k^{\delta} = u_k(T)$, we compute the coefficients f_k^{δ} by the formula

$$f_k^{\delta} = \frac{\psi_k^{\delta} - u_k^1(T)}{u_k^2(T)}, \quad k = 1, \dots, N.$$

Actually, $u_k^1(T) = A_k$ and $u_k^2(T) = B_k$, where A_k, B_k are defined by (34).

Similar procedure is applied to provide the numerical examples for the IP2.

GNU Octave IDE has been used to run all the computations.

Example to the IP1. Given the input data $h = 0$, $\varphi(x) = x(1 - x)$, $g(t) = t$ and the exact solution to the inverse problem $f(x) = 10x(1 - x)(2/3 - x)$, we compute N under the level of noise δ and, afterwards, $f^{N,\delta}$. We also compute the maximal error on the grid

$$err = \max_{Grid} |f(x_i) - f^{N,\delta}(x_i)|.$$

Errors and N for different values of δ are listed in Table 1. Figure 1 illustrates the case $\delta = 0.001$.

Example 1 to the IP2. The inverse problem IP2 is investigated for the similar functions. Namely, the source term $F = 0$ and $\varphi(x) = 10x(1 - x)(2/3 - x)$. Errors and values of N are given in Table 2.

TABLE 1 Number N and errors in Example to IP1

δ	N	*err*
0.01	2	0.092959
0.001	4	0.032201
0.0001	8	0.010361

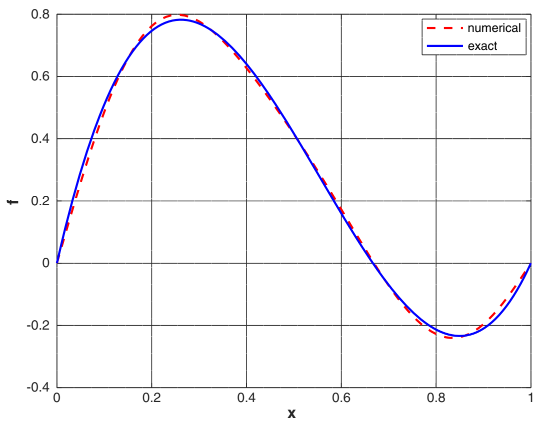

FIGURE 1 Example to IP1: f and $f^{N,\delta}$ for $\delta = 0.001$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Number N and errors in Example 1 to IP2

δ	N	*err*
0.01	2	0.094353
0.001	5	0.029403
0.0001	10	0.0066018

TABLE 3 Number N and errors in Example 2 to IP2

δ	N	*rmse*	err_1
0.01	5	0.27823	0.16459
0.001	17	0.18367	0.064002
0.0001	57	0.14417	0.020943

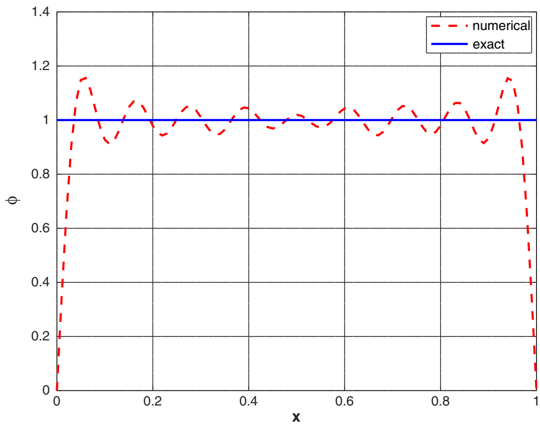

FIGURE 2 Example 2 to IP2: Oscillations of $\varphi^{N,\delta}$ near $\varphi = 1$ in case $\delta = 0.001$ [Colour figure can be viewed at wileyonlinelibrary.com]

Example 2 to the IP2. In another example, $F = 0$, $\varphi = 1$ was taken. (Then $\varphi \in \mathcal{D}(L^s), s < \frac{1}{4}$.) The numerical solution preserves constant error 1 at the boundary $\partial\Omega = \{0; 1\}$. Therefore, instead of *err* we employ the root mean square error:

$$rmse = \left[\frac{1}{M} \sum_{i=0}^{M} (\varphi(x_i) - \varphi^{N,\delta}(x_i))^2 \right]^{1/2}$$

and maximal error on a subgrid

$$err_1 = \max_{Grid \cap [0.2, 0.8]} |\varphi(x_i) - \varphi^{N,\delta}(x_i)|.$$

Results are given in Table 3 and an illustration in Figure 2.

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Appendix 2

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Inverse Problems for a Generalized Subdiffusion Equation with Final Overdetermination

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Abstract. We consider two inverse problems for a generalized subdiffusion equation that use the final overdetermination condition. Firstly, we study a problem of reconstruction of a specific space-dependent component in a source term. We prove existence, uniqueness and stability of the solution to this problem. Based on these results, we consider an inverse problem of identification of a space-dependent coefficient of a linear reaction term. We prove the uniqueness and local existence and stability of the solution to this problem.

Keywords: inverse problem, subdiffusion, final overdetermination, fractional diffusion.

AMS Subject Classification: 35R30; 35R11.

1 Introduction

Anomalous diffusion processes are described by different models [6]. Among them stands out the time (or space-time) fractional diffusion equation that is the most common way to represent a subdiffusion. For some situations such approach does not work [19]. Therefore, more general models that unify wider range of subdiffusion processes are introduced [19, 25].

In this paper we use an operator that is more general than the fractional time derivative:

$$D_t^{\{k\}} v = \frac{d}{dt} k * v, \tag{1.1}$$

where $*$ denotes the time convolution, i.e. $(v_1 * v_2)(t) = \int_0^t v_1(t-\tau) v_2(\tau) d\tau$. Taken $k = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, (1.1) transforms into a well-known Riemann-Liouville fractional derivative. The operator corresponding to the Caputo fractional derivative denoted as $\mathbb{D}^C_{(k)}v$ was introduced by Luchko and Yamamoto in [23] and also in [15].

The toolkit for treating such a type of derivative have been developed by Prüss et al. [5]. They have created a setting to introduce the operator inverse to $D_t^{\{k\}}$ through the concept of Completely Positive kernels [5]: a kernel $M \in L_{1,loc}(\mathbb{R}_+)$ is called completely positive if there are $k_0 \geq 0$ and nonnegative and nonincreasing $k_1 \in L_{1,loc}(\mathbb{R}_+)$ such that $M * (k_0\delta + k_1) = 1$ holds. The applications of this concept can be found in [1, 33, 34]. Another approach to this issue has been developed by Kochubei [19].

Often parameters of models are unknown. Then additional observations are performed and inverse problems solved to reconstruct unknown quantities [12, 13, 16, 17, 20, 21]. In the present paper we consider two inverse problems (IPs) that use final observation data: IP1 is to identify a space-dependent factor f of a source term $g(t, x)f(x)$; IP2 is to reconstruct a coefficient $r(x)$ of a linear reaction term.

IP1 for fractional and perturbed fractional diffusion equations is studied in several papers. Theoretical and numerical results are obtained in the particular case $g = g(t)$ [7,17,18,26] and in the case $g = g(t, x)$ [30,32]. In latter papers the existence and uniqueness of solutions are proved for almost all scalar diffusion coefficients. IP1 for a semilinear fractional diffusion equation is considered in [15]. Uniqueness of the solution is proved.

In this paper we consider IP1 for a more general diffusion equation that includes the operator (1.1) instead of the fractional derivative. We prove the uniqueness of the solution to IP1 by applying a modified version of the positivity principle from [15]. That falls into category of maximum principle results [13, 20, 22]. Similar approaches to the inverse problems are well-known in the domain of parabolic equations [2,12]. Next we prove the existence and stability of the solution of IP1 by means of the Fredholm alternative. The uniqueness of solution of IP2 follows from the IP1-results. Finally, we prove local existence and stability of the solution to IP2 by means of the contraction argument.

2 Formulation of direct and inverse problems

Let us consider the generalized subdiffusion equation

$$U_t(t, x) = (M * LU)_t(t, x) + Q(t, x), \tag{2.1}$$

where U physical state, t is the time, $x \in \mathbb{R}^n$ is a space variable, Q is a source term, the operator $L = L(x)$ is such that

$$L(x) = L_1(x) + r(x)I, \text{ where } L_1(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j}$$

and I is the unity operator. The kernel M is a memory function related to a non-locality of the diffusion process.

There are two ways to derive the equation (2.1) from physical laws. One method consists in modelling continuous time random walk processes in micro-level and taking a continuous limit in a macro-level [4] and another one uses conservative laws and specific constitutive relations with memory [27].

Real world applications of the equation (2.1) include diffusion in fractal and porous media, e.g. propagation of pollution, heat flow in media with memory, dynamics of protein in cells, transport in dielectrics and semiconductors, usage of optical tweezers, Hamiltonian chaos etc. [3,4,6,27,31].

Let us assume that there is a function k such that $k * M = M * k = 1$. Then if we apply $k*$ to (2.1), we obtain an equation that contains the explicit differential operator L and is called the normal form of (2.1): $k * U_t(t,x) = LU(t,x)+H(t,x)$, where $H(x,t) := k*Q(t,x)$. The term $k*U_t$ can be rewritten in the form $D_t^{\{k\}}(U-U(0,\cdot))$ that does not contain the 1st order derivative of U. Therefore, we get the equation

$$D_t^{\{k\}}(U-U(0,\cdot)) = LU(t,x) + H(t,x). \tag{2.2}$$

Conversely, in case of sufficiently regular U, the equation (2.1) follows from (2.2) by means of the application of the operator $\frac{\partial}{\partial t}M*$.

The equation (2.1) and its analogue (2.2) incorporate the following possibilities:

1. The kernel $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, $0 < \beta < 1$, represents a power-type memory. Then (2.1) becomes the celebrated time fractional diffusion equation $U_t = \varkappa D^{1-\beta} LU + Q$, where $D^{1-\beta}v = \left(\frac{t^{\beta-1}}{\Gamma(\beta)} * v\right)_t$ is the Riemann-Liouville fractional derivative of the order $1-\beta$ [4,17,20,26]. For such M, it holds $k = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ and $[k*(v-v(0)]_t = k*v_t = \partial_t^\beta v$ is the Caputo fractional derivative.

2. The kernel M or its associate k is a linear combination of power functions [25,31]:

$$M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} + \sum_{j=1}^{l} p_j \frac{t^{\beta_j-1}}{\Gamma(\beta_j)}, \quad 0<\beta<\beta_j<1, \quad p_j \ge 0,$$

$$k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)} + \sum_{j=1}^{l} q_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}, \quad 0<\beta_j<\beta<1, \quad q_j \ge 0.$$

3. The kernel M has the form $M(t) = \int_0^1 p(s)\frac{t^{s-1}}{\Gamma(s)}ds$ where $p \ge 0$ is a nonvanishing integrable function (cf. [3,25,31]). Such a kernel stands for the distributed order fractional derivative that is used for modeling diffusion with a logarithmic growth of the mean square displacement [19].

4. Tempered fractional calculus [29], that is another way to generalize a fractional calculus, falls into the case

$$M(t) = \frac{1}{\Gamma(\beta)} e^{-\lambda t} t^{\beta-1} + \frac{\lambda}{\Gamma(\beta)} \int_0^t e^{-\lambda\tau}\tau^{\beta-1}d\tau, \ \ \lambda > 0.$$

This type of kernel is used for modelling the transition from anomalous to normal diffusion.

Every presented example of M (or k) has a completely monotonic associate k (or M) that solves $k * M = 1$ (see Section 3).

Let $\Omega \in \mathbb{R}^n$ be an open bounded domain with the boundary $\partial\Omega$. In *direct problem* we have to find a function u that solves the initial-boundary value problem

$$\begin{aligned} &D_t^{\{k\}}(U - \Phi)(t,x) = LU(t,x) + H(t,x), \quad x \in \Omega,\, t \in (0,T), \\ &U(0,x) = \Phi(x), \quad x \in \Omega, \\ &\mathcal{B}(U - b)(t,x) = 0, \quad x \in \partial\Omega,\, t \in (0,T). \end{aligned} \tag{2.3}$$

Here Φ and b are given functions and

$$\mathcal{B}v(x) = v(x) \quad \text{or} \quad \mathcal{B}v(x) = \omega(x) \cdot \nabla v(x),$$

with $\omega \cdot \nu > 0$ and $\nu(x)$ denoting the outer normal of $\partial\Omega$ at $x \in \Omega$. An important particular case is $\omega = \left(\sum_{j=1}^n a_{ij}\nu_j|_{i=1,\dots,n}\right)$. Then the condition $\mathcal{B}(U - b)|_{(t,x)\in(0,T)\times\partial\Omega} = 0$ corresponds to the flux specified at $\partial\Omega$.

Let us proceed to inverse problems. To this end we introduce the condition

$$U(T,x) = \Psi(x), \quad x \in \Omega, \tag{2.4}$$

with a given observation function Ψ. Firstly, we formulate of an inverse source problem. Let

$$H(t,x) = g(t,x)f(x) + h_0(t,x), \tag{2.5}$$

where the components gf and h_0 may correspond to different sources or sinks. The factor f is unknown and to be reconstructed by means of the data (2.4). Since the whole function U is also unknown, the first inverse problem consists in determination a pair of functions (f, U) that satisfies (2.3), (2.4) and (2.5).

In the second inverse problem, our aim is to identify the coefficient r of the linear reaction term rU. In the mathematical formulation, the problem consists in finding a pair (r, U) that satisfies (2.3) and (2.4). We can handle the case of zero initial condition $\Phi = 0$ (for details, see the end of Section 6).

Methods to be used in this paper require homogeneous boundary conditions. Therefore, we perform the change of the second unknown $u = U - b$ in our problems. It brings along shifts of data by addends containing b.

Firstly, from (2.3) we obtain the following problem for $u = U - b$:

$$\begin{aligned} &D_t^{\{k\}}(u - \varphi)(t,x) = Lu(t,x) + F(t,x), \quad x \in \Omega,\, t \in (0,T), \\ &u(0,x) = \varphi(x), \quad x \in \Omega, \\ &\mathcal{B}u(t,x) = 0, \quad x \in \partial\Omega,\, t \in (0,T), \end{aligned} \tag{2.6}$$

where

$$\varphi(x) = \Phi(x) - b(0,x), \tag{2.7}$$

$$F(t,x) = H(t,x) + Lb(t,x) - D_t^{\{k\}}(b - b(0,\cdot))(t,x). \tag{2.8}$$

The overdetermination condition (2.4) in terms of u has the form

$$u(T,x) = \psi(x), \quad x \in \Omega, \tag{2.9}$$

where $\psi(x) = \Psi(x) - b(T,x)$. Plugging (2.5) into (2.8) we obtain

$$F(t,x) = g(t,x)f(x) + h(t,x), \tag{2.10}$$

where $h(t,x) = h_0(t,x) + Lb(t,x) - D_t^{\{k\}}(b - b(0,\cdot))(t,x)$.

In the reformulated first inverse problem (**IP1**), we seek for the pair of functions (f,u) that satisfies (2.6), (2.9) and (2.10).

Let us reformulate the second inverse problem, too. From the relations (2.3), (2.4) with $\Phi = 0$ by means of the change of variable $u = U - b$, we obtain the following problem for the pair (r,u):

$$\begin{aligned}
&D_t^{\{k\}}u(t,x) = L_1u(t,x) + r(x)(u+b)(t,x) + F_1(t,x) \quad x \in \Omega,\, t \in (0,T),\\
&u(0,x) = 0, \quad x \in \Omega, \qquad \mathcal{B}u(t,x) = 0, \quad x \in \partial\Omega,\, t \in (0,T),\\
&u(T,x) = \psi(x), \quad x \in \Omega,
\end{aligned} \tag{2.11}$$

where $b(0,x) = 0$, $x \in \Omega$, the function ψ is expressed by $\psi(x) = \Psi(x) - b(T,x)$ and $F_1(t,x) = H(t,x) + L_1b(t,x) - D_t^{\{k\}}b(t,x)$.

Thus, the reformulated second inverse problem (**IP2**) is to find the pair of functions (r,u) that satisfies (2.11).

3 Basic assumptions

In this section we collect basic conditions on the domain, operator L and kernels k and M that will be assumed throughout the paper.

We assume that $\partial\Omega$ is uniformly of the class C^2 and $\omega \in (C^1(\partial\Omega))^n$. Moreover, we assume that $a_{ij}, a_j, r \in C(\overline{\Omega})$ and the principal part of L is uniformly elliptic, i.e. $\sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \ \forall \xi \in \mathbb{R}^n$, $x \in \Omega$ for some $c > 0$.

Concerning the function k, we assume that

1. k belongs to $L_{1,loc}(0,\infty)$ and is a solution of the equation $M * k = 1$ with a kernel $M \in L_{1,loc}(0,\infty)$ that satisfies the conditions

$$\begin{aligned}
&M \in C^1(0,\infty), \quad \lim_{t\to 0^+} M(t) = \infty, \quad M > 0, \quad M' \leq 0,\\
&-M' \text{ is nonincreasing and convex;}
\end{aligned} \tag{3.1}$$

2. k has the following properties:

$$k \in C(0,\infty), \quad \lim_{t\to 0^+} k(t) = \infty, \quad k > 0, \quad k \text{ is nonincreasing,} \tag{3.2}$$

$$\exists t_k > 0 : k(t) \text{ is strictly decreasing in } (0,t_k). \tag{3.3}$$

The assumptions (3.1) ensure the existence of a sufficiently regular solution of the direct problem (see Lemma 3) and the assumptions (3.2), (3.3) are needed for the application of a positivity principle to this solution.

We mention that restricting generality a bit it is possible to reduce all conditions 1 and 2 to the single kernel M. Firstly, $M \in L_{1,loc}(0,\infty)$ and (3.1) imply the existence of a unique solution $k \in L_{1,loc}(0,\infty)$ of the equation $k * M = 1$ ([10], Ch. 5, Corollary 5.6). Secondly, all properties (3.2), (3.3) follow from conditions that are a bit stronger than (3.1). It is shown in the following lemma. Proof is in Appendix.

Lemma 1. *Let $M \in L_{1,loc}(0,\infty)$ satisfy* (3.1) *and $M' < 0$, $\log M$ - convex, $\log(-M')$ - convex. Then the solution of $M * k = 1$ satisfies* (3.2)*,* (3.3)*.*

The imposed assumptions on M and k hold for weakly singular completely monotonic kernels from

$$\mathcal{CM} = \{z \in L_{1,loc}(0,\infty) \bigcap C^\infty(0,\infty) : \lim_{t\to 0^+} z(t) = \infty,\ (-1)^i z^{(i)} > 0,\ i = 0,1,\ldots\}.$$

For M and k satisfying $M * k = 1$, it holds $M \in \mathcal{CM}$ if and only if $k \in \mathcal{CM}$ ([9], Theorem 3).

All examples of M and k given in Section 2 belong to $\mathcal{CM}$.

4 Preliminaries

4.1 Functional spaces

Let X be a Banach space. Since $k * M = 1$, we have

$$D_t^{\{k\}}(M * v) = \frac{d}{dt} k * M * v = \frac{d}{dt} 1 * v = v, \quad \forall v \in L_1((0,T);X), \tag{4.1}$$

where $L_1((0,T);X)$ is the space of functions $u : (0,T) \to X$ that are integrable in the Bochner sense on $(0,T)$. This means that the operator $M*$ is a one-to-one mapping from $L_1((0,T);X)$ to $\{M * v : v \in L_1((0,T);X)\}$ and $D_t^{\{k\}}$ is the inverse of $M*$.

As usual, let $C([0,T];X)$ stand for the Banach space of functions $u : [0,T] \to X$ that are continuous on $[0,T]$ with the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$ and $C_0([0,T];X) = \{u \in C([0,T];X) : u(0) = 0\}$. Based on the relation (4.1), we introduce the functional space

$$C_0^{\{k\}}([0,T];X) := M * C([0,T];X) = \{M * v : v \in C([0,T];X)\}.$$

It is a Banach space with the norm

$$\|u\|_{C_0^{\{k\}}([0,T];X)} = \|D_t^{\{k\}} u\|_{C([0,T];X)}.$$

Since $M* \in \mathcal{L}(C([0,T];X), C_0([0,T];X))$, it holds

$$C_0^{\{k\}}([0,T];X) \hookrightarrow C_0([0,T];X).$$

We also define the space

$$C^{\{k\}}([0,T];X) := C_0^{\{k\}}([0,T];X) + X$$
$$= \{u : u(t) = u_1(t) + u_2,\ u_1 \in C_0^{\{k\}}([0,T];X),\ u_2 \in X\} \tag{4.2}$$

that is a Banach space with the norm

$$\|u\|_{C^{\{k\}}([0,T];X)} = \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X.$$

Next we introduce the abstract Hölder spaces with corresponding norms

$$C_0^\alpha([0,T];X) = \Big\{u \in C_0([0,T];X) :$$
$$\|u\|_{C_0^\alpha([0,T];X)} := \sup_{0<t_1<t_2<T} \frac{\|u(t_2) - u(t_1)\|_X}{(t_2 - t_1)^\alpha} < \infty\Big\},$$
$$C^\alpha([0,T];X) = C_0^\alpha([0,T];X) + X,$$
$$\|u\|_{C^\alpha([0,T];X)} = \|u - u(0)\|_{C_0^\alpha([0,T];X)} + \|u(0)\|_X,$$

where $0 < \alpha < 1$, and define the Banach spaces with norms

$$C_0^{\{k\},\alpha}([0,T];X) = M * C_0^\alpha([0,T];X), \tag{4.3}$$
$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X)} = \|D_t^{\{k\}} u\|_{C_0^\alpha([0,T];X)},$$
$$C^{\{k\},\alpha}([0,T];X) = M * C^\alpha([0,T];X) + X,$$
$$\|u\|_{C^{\{k\},\alpha}([0,T];X)} = \|D_t^{\{k\}}(u - u(0))\|_{C^\alpha([0,T];X)} + \|u(0)\|_X.$$

Let us establish some connections between the spaces (4.2), (4.3) and the usual C, C^1- and Hölder spaces. For $C^{\{k\}}([0,T];X)$ the embeddings

$$C^1([0,T];X) \hookrightarrow C^{\{k\}}([0,T];X) \hookrightarrow C([0,T];X) \tag{4.4}$$

are valid. The right embedding follows from $M* \in \mathcal{L}(C([0,T];X))$[1] . To prove the left embedding, we choose some $u \in C^1([0,T];X)$. Then

$$\|u\|_{C^{\{k\}}([0,T];X)} = \|u - u(0)\|_{C_0^{\{k\}}([0,T];X)} + \|u(0)\|_X = \|k*u'\|_{C_0([0,T];X)} + \|u(0)\|_X$$

and since $k* \in \mathcal{L}(C([0,T];X), C_0([0,T];X))$, the left relation in (4.4) follows.

Analogous relations for the space $C_0^{\{k\},\alpha}([0,T];X)$ are

$$C_0^{1+\alpha}([0,T];X) \hookrightarrow C_0^{\{k\},\alpha}([0,T];X) \hookrightarrow C_0^\alpha([0,T];X) \tag{4.5}$$

where

$$C_0^{1+\alpha}([0,T];X) = \{u : u,\ u' \in C_0^\alpha([0,T];X)\}.$$

The right embedding in (4.5) is a consequence of the fact that $M* \in \mathcal{L}(C_0^\alpha([0,T];X))$ (see Lemma 4.2 in [14]) and the left embedding in (4.5) can be proved similarly to the left embedding in (4.4).

Under additional assumptions on M it is possible to show that the operator $M*$ increases the order of Hölder continuity of a function. Namely, the following lemma is valid. Its proof is deferred to Appendix.

[1] The symbol $\mathcal{L}$ stands for the space of linear and bounded operators.

Lemma 2. *If* $M(t) \leq c_1 t^{\beta-1}$, $|M'(t)| \leq c_2 t^{\beta-2}$, $t \in (0,T)$ *for some* $c_1, c_2 \in \mathbb{R}_+$, $0 < \beta \leq \alpha < 1$ *then* $M* \in \mathcal{L}(C_0^{\alpha-\beta}([0,T];X), C_0^{\alpha}([0,T];X))$.

Under conditions of Lemma 2, $C_0^{\{k\},\alpha-\beta}([0,T];X) \hookrightarrow C_0^{\alpha}([0,T];X)$. In the particular case $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$ (then $M*$ is the fractional integral of the order β), it holds the equality $C_0^{\{k\},\alpha-\beta}([0,T];X) = C_0^{\alpha}([0,T];X)$ [15].

By exchanging M and k in above relations, we obtain definitions and embeddings of spaces that contain $\{M\}$ instead of $\{k\}$ in the superscript.

4.2 Abstract Cauchy problem

Let $A : \mathcal{D}(A) \to X$ be a linear densely defined operator in a Banach space X. We say that A belongs to the class $\mathcal{S}(\eta,\theta)$ for $\eta \in \mathbb{R}$, $\theta \in (0,\pi)$ if

$$\rho(A) \supset \Sigma(\eta,\theta) = \{\lambda \in \mathbb{C} : \lambda \neq \eta,\ arg|\lambda-\eta| < \theta\} \quad \text{and}$$
$$\|(\mu - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\mu-\eta|} \quad \forall \mu \in \Sigma(\eta,\theta) \quad \text{for some constant } C > 0.$$

An operator $A \in \mathcal{S}(\eta,\theta)$ is closed. This implies that $X_A := \mathcal{D}(A)$ is a Banach space with the graph norm $\|w\|_{X_A} = \|w\|_X + \|Aw\|_X$.

Obviously, $\mathcal{S}(\eta,\theta_1) \subset \mathcal{S}(\eta,\theta_2)$ for $\theta_1 > \theta_2$. Operators of the class $\mathcal{S}(\eta,\theta)$, $\theta \in \left(\frac{\pi}{2},\pi\right)$, are the *sectorial* operators that generate analytic semigroups.

Now let us consider the Cauchy problem

$$D_t^{\{k\}}(u-\varphi)(t) = Au(t) + F(t), \quad t \in [0,T], \quad u(0) = \varphi, \tag{4.6}$$

with given $F : [0,T] \to X$ and $\varphi \in X$.

Lemma 3. *Let* $A \in \mathcal{S}(\eta, \frac{\pi}{2})$ *for some* $\eta \in \mathbb{R}$. *Then the following statements are valid.*
(i) (uniqueness) *Let* $u \in C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$ *solve* (4.6) *and* $\varphi = 0$, $F = 0$. *Then* $u = 0$.
(ii) *Let* $F \in C_0^{\alpha}([0,T];X)$ *and* $\varphi = 0$. *Then* (4.6) *has a solution* u *in the space* $C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^{\alpha}([0,T];X_A)$. *This solution satisfies the estimate*

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^{\alpha}([0,T];X_A)} \leq C_1 \|F\|_{C_0^{\alpha}([0,T];X)}. \tag{4.7}$$

(iii) *Let* $F \in C^{\alpha}([0,T];X)$ *and* $\varphi \in X_A$. *Then* (4.6) *has a solution* u *in the space* $C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$. *This solution satisfies the estimate*

$$\|u\|_{C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)} \leq C_2(\|F\|_{C^{\alpha}([0,T];X)} + \|\varphi\|_{X_A}). \tag{4.8}$$

The constants C_1 *and* C_2 *depend on* M *and* A.

Proof. The change of variable $v = D_t^{\{k\}}(u-\varphi) \ \Leftrightarrow\ u = M * v + \varphi$ reduces (4.6) of the integral equation

$$v(t) = A(M * v)(t) + F(t) + A\varphi, \quad t \in [0,T]. \tag{4.9}$$

Provided $F \in C([0,T];X)$, $\varphi \in X_A$, the function $u \in C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)$ solves (4.6) if and only if $v \in V := \{v \in C([0,T];X) : M * v \in C_0([0,T];X_A)\}$ solves (4.9). Similar one-to-one correspondence holds for $u \in C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^{\alpha}([0,T];X_A)$ and $v \in V^\alpha := \{v \in C_0^\alpha([0,T];X) : M * v \in C_0^\alpha([0,T];X_A)\}$ in the particular case $F \in C_0^\alpha([0,T];X)$, $\varphi = 0$.

Since M satisfies the conditions (3.1) and $A \in \mathcal{S}(\eta, \frac{\pi}{2})$, we can apply results of Ch. 3 of [28] to (4.9).

(i) Theorem 3.2 with Corollary 1.1 and Proposition 1.2 in [28] implies that there exists a family of operators $S : [0,\infty) \to \mathcal{L}(X)$ (called resolvent of (4.9)) so that a solution $v \in V$ (if it exists) is represented by the formula $v = \frac{d}{dt} S * F$. By assumptions of (i), (4.9) has a solution $v \in V$. Since $F = 0$, we have $v = 0$. Thus, $u = 0$.

(ii) Theorem 3.3 (i) [28] implies that for $F \in C_0^\alpha([0,T];X)$ there exists a solution $v \in V^\alpha$ of (4.9). This proves the existence of the solution $u \in C_0^{\{k\},\alpha}([0,T];X) \bigcap C_0^\alpha([0,T];X_A)$ of (4.6). The estimate (4.7) follows from the bounded inverse theorem.

(iii) It is sufficient to prove this assertion in case $F(t) \equiv \xi \in X$, because the problem with given pair of data (F,φ) can be splitted into two problems with the data $(F - F(0), 0)$ and $(F(0), \varphi)$, respectively. For the first problem, the assertion (ii) applies. Having proved (iii) for the second one, u is expressed as the sum of solutions of these two problems and satisfies (iii), too.

Thus, let us assume that $F(t) \equiv \xi \in X$. Due to Proposition 1.2 (ii) [28], (4.9) has the solution $v = S(\xi + A\varphi) \in V$. This implies the existence assertion of (iii). Due to the strong continuity of $S(t)$ [28], $\|S(t)\|_{\mathcal{L}(X)} \le C_3$, $t \in [0,T]$, where C_3 is a constant. Thus, $\|v\|_{C([0,T],X)} \le C_3 (\|\xi\|_X + \|A\varphi\|_X)$. Extracting the term $A(M*v)$ from (4.9) and estimating it we obtain $\|A(M*v)\|_{C_0([0,T],X)} \le (C_3+1)(\|\xi\|_X + \|A\varphi\|_X)$. Consequently,

$$\|u\|_{C^{\{k\}}([0,T];X) \bigcap C([0,T];X_A)} = \|v\|_V + \|\varphi\|_{X_A} \le C_4(\|\xi\|_X + \|\varphi\|_{X_A})$$

with a constant C_4. This implies (4.8). □

4.3 Statements on direct problem

In order to apply Lemma 3 to the direct problem (2.6), we must introduce appropriate Banach spaces of x-dependent functions and define realizations of the operator L in these spaces so that they belong to $\mathcal{S}\left(\eta, \frac{\pi}{2}\right)$.

Let us introduce the following spaces and operators:

1. $X_p = L_p(\Omega)$, $1 < p < \infty$,
   $A_p : X_{A_p} \to X_p$ with $X_{A_p} = \{z \in W_p^2(\Omega) : \mathcal{B}z|_{\partial\Omega} = 0\}$ and
   $A_p z = Lz, \ \ z \in X_{A_p}$.
2. $X_0 = \begin{cases} C_0(\overline{\Omega}) = \{z \in C(\overline{\Omega}) : z|_{\partial\Omega} = 0\} \text{ in case } \mathcal{B} = I, \\ C(\overline{\Omega}) \text{ in case } \mathcal{B} = \omega \cdot \nabla, \end{cases}$
   $A_0 : X_{A_0} \to X_0$ with $X_{A_0} = \{z \in \bigcap_{1<p<\infty} W_p^2(\Omega) : \mathcal{B}z|_{\partial\Omega} = 0,\ Lz \in X_0\}$
   and $A_0 z = Lz, \ \ z \in X_{A_0}$.

Corollary 1. Operators A_p, $p \in \{0\}\bigcup(1,\infty)$, are sectorial. Thus, Lemma 3 holds in cases $X = X_p$, $A = A_p$, $p \in \{0\}\bigcup(1,\infty)$ and applies to problem (2.6).

Proof. It follows from Theorems 3.1.2, 3.1.3 and Corollary 3.1.24 (ii) in [24]. □

Lemma 4. *Let $K \in L_1(0,T)\bigcap C^1(0,T)$, $\lim\limits_{t\to 0^+} K(t) = \infty$, $K > 0$, K be non-increasing and $\exists t_K > 0$: K is strictly decreasing in $(0,t_K)$. Moreover, let $F \in C([0,T]\times\overline{\Omega})$. Assume that u solves the problem*

$$D_t^{\{K\}}(u-\varphi)(t,x) = Lu(t,x) + F(t,x),\ t\in(0,T),\ x\in\Omega,$$
$$u(0,x) = \varphi,\ x \in \Omega$$

and satisfies the smoothness conditions $u \in C([0,T]\times\overline{\Omega})$, $u_{x_j} \in C((0,T]\times\overline{\Omega})$, $u \in C((0,T];W_p^2(\Omega))$ for some $p > n$, $L_1u \in C((0,T]\times\overline{\Omega}), D_t^{\{K\}}(u-\varphi) \in C((0,T]\times\overline{\Omega})$. Finally, let

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_0^\epsilon K(\tau)d\tau \sup_{0\le s\le\epsilon}|u(t-s,x)-u(t,x)| = 0, \quad \forall t\in(0,T],\ x\in\overline{\Omega}. \quad (4.10)$$

If $\varphi \ge 0$, $F \ge 0$ and $\mathcal{B}u|_{\partial\Omega} \ge 0$ then the following assertions are valid.

(i) *$u \ge 0$;*

(ii) *if $u(t_0,x_0) = 0$ in some point $(t_0,x_0) \in (0,T]\times\Omega_N$, where*

$$\Omega_N = \begin{cases} \Omega & \text{in case } \mathcal{B} = I \\ \overline{\Omega} & \text{in case } \mathcal{B} = \omega\cdot\nabla \end{cases},$$

then $u(t,x_0) = 0$ for any $t \in [0,t_0]$.

This lemma is a slight modification of a positivity principle that was proved in [15] for a semilinear equation in case of a more smooth solution $u \in C((0,T];C^2(\overline{\Omega}))$ and strictly decreasing in $(0,T)$ kernel K.

To prove Lemma 4, we need the following auxiliary result. It is proved in Appendix of the paper.

Lemma 5. *Let $w \in W_p^2(\Omega)$ for some $p>n$, $L_1w \in C(\overline{\Omega})$ and $x^* = \operatorname{argmin}\limits_{x\in\overline{\Omega}} w(x)$. In case $x^* \in \partial\Omega$ we also assume that $(\omega\cdot\nabla w)(x^*) \ge 0$. Then $L_1w(x^*) \ge 0$.*

Proof of Lemma 4. Without a restriction of generality we assume that $r \le 0$. Otherwise it is possible to define $\tilde{u} = e^{-\sigma t}u$ as in [15] and to consider the corresponding problem for $\tilde{u}$. Such a problem also satisfies the assumptions of Lemma 4 and has the coefficient $\tilde{r} = r - \sigma\int_0^T e^{-\sigma s}K(s)ds$ in place of r. Since $\lim\limits_{t\to 0^+} K(t) = \infty$, for sufficiently large σ, $\tilde{r} \le 0$.

Let us suppose that (i) does not hold. Then there exists $(t_1,x_1) \in (0,T]\times\overline{\Omega}$ such that $u(t_1,x_1) < 0$ and $(t_1,x_1) = \operatorname{argmin}\limits_{x\in\overline{\Omega},t\in[0,T]} u(t,x)$. It was shown in [15] (formula (37)) that the assumptions $D_t^{\{K\}}(u-\varphi) \in C((0,T]\times\overline{\Omega})$, (4.10),

$K > 0$ and K – nonincreasing together with the relations $u(t,x_1) \geq u(t_1,x_1)$ and $u(t_1,x_1) < 0$ imply $D_t^{\{K\}}(u-\varphi)(t_1,x_1) < 0$. On the other hand, Lemma 5 applies to the function $w = u(t_1,\cdot)$ at $x^* = x_1$. We obtain $L_1 u(t_1,x_1) \geq 0$. Also $r(x_1)u(t_1,x_1) \geq 0$ and $F \geq 0$. Thus, the left-hand side of the equation $D_t^{\{K\}}(u-\varphi)(t_1,x_1) = [Lu+F](t_1,x_1)$ is negative, but the right-hand side is nonnegative. We have reached a contradiction. The assertion (i) is valid.

Let us prove (ii). Let $u(t_0,x_0) = 0$ at $(t_0,x_0) \in (0,T] \times \Omega_N$. Define $\hat{t}_0 = \inf\{t : t \leq t_0,\ u(\tau,x_0) = 0 \text{ for } \tau \in [t,t_0]\}$. If (ii) is not valid, then $\hat{t}_0 > 0$ and $u(t,x_0) \geq \delta$, $t \in (t_2,t_3)$ for some $\delta > 0$ and $(t_2,t_3) \subset (0,\hat{t}_0)$ such that $\hat{t}_0 - t_2 < t_K$. Then, similarly to the proof in [15] p.138, from the assumptions $D_t^{\{K\}}(u-\varphi) \in C((0,T] \times \overline{\Omega})$, (4.10), $K > 0$, K – nonincreasing and relations $u \geq 0$, $u(t,x_0) \geq \delta > 0$, $t \in (t_2,t_3)$, we derive

$$D_t^{\{K\}}(u-\varphi)(\hat{t}_0,x_0) \leq \delta(K(\hat{t}_0-t_2) - K(\hat{t}_0-t_3)). \tag{4.11}$$

Since $0 < \hat{t}_0 - t_3 < \hat{t}_0 - t_2 < t_K$ and K is strictly decreasing in $(0,t_K)$, (4.11) implies $D_t^{\{K\}}(u-\varphi)(\hat{t}_0,x_0) < 0$. On the other hand, from $u(\hat{t}_0,x_0) = 0$ and $u(t,x) \geq 0$, $(t,x) \in (0,T] \times \Omega$, we conclude that $(\hat{t}_0,x_0) = \operatorname*{argmin}_{x\in\overline{\Omega}} u(\hat{t}_0,x)$. By Lemma 5, $L_1 u(\hat{t}_0,x_0) \geq 0$. Moreover, $(ru)(\hat{t}_0,x_0) = 0$ and $F \geq 0$. Left-hand side of the equation $D_t^{\{K\}}(u-\varphi)(\hat{t}_0,x_0) = [Lu+F](\hat{t}_0,x_0)$ is negative, but right-hand side is nonnegative. Again, we have reached the contradiction. Thus, (ii) holds. □

At this point we present somewhat more concrete assumptions on the input data of the direct problem (2.6) that imply the assumptions of Lemma 4 and Lemma 3.

Corollary 2. Let $F \geq 0$, $\varphi = 0$ and one of the assumptions (a1)–(a3) hold:
(a1) $F \in C^{\{M\},\alpha}([0,T];X_0)$ for some $0 < \alpha < 1$ and $F(0,\cdot) = 0$;
(a2) $F \in C_0^{\alpha}([0,T];X_0)$ and $M(t) \geq ct^{\gamma-1}$, $t \in (0,T)$ for some $c \in \mathbb{R}_+$, $0 < \gamma < \alpha < 1$;
(a3) $F \in C_0^{\alpha-\beta}([0,T];X_0)$ and $c_1 t^{\gamma-1} \leq M(t) \leq c_2 t^{\beta-1}$, $|M'(t)| \leq c_3 t^{\beta-2}$, $t \in (0,T)$, for some $c_1,c_2,c_3 \in \mathbb{R}_+$, $0 < \beta \leq \gamma < \alpha < 1$.
Then assertions Lemma 4 are satisfied by solution of the problem (2.6).

Proof. Defining $X = X_0$, Lemma 3 with Corollary 1 implies that the solution of (2.6) exists and satisfies the smoothness conditions of Lemma 4. It remains to show that (4.10) holds.

The case (a1). The relations $F \in C^{\{M\},\alpha}([0,T];X_0)$, $F(0,\cdot) = 0$ mean that $F = k * \hat{F}$, where $\hat{F} \in C^{\alpha}([0,T];X_0)$. Thus, it follows from Lemma 3 that the function $\hat{u}$ that solves (2.6) with F,φ replaced by $\hat{F}$, $\hat{\varphi} = 0$ belongs to the space $C^{\{k\}}([0,T];X_0)$. Next, after convolving equation for $\hat{u}$ with k it is easy to see that $u = k * \hat{u}$ solves (2.6) with $F = k * \hat{F}$. Therefore, $u \in k * C^{\{k\}}([0,T];X_0)$, that is $u = k * M * v = 1 * v$, $v \in C([0,T];X_0)$. This allows us to conclude that

$u \in C^1([0,T];X_0)$. Hence,

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^\epsilon k(\tau)d\tau \sup_{0\le s\le\epsilon} |u(t-s,x)-u(t,x)| = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^\epsilon k(\tau)d\tau \cdot O(\epsilon)$$
$$= 0, \quad \forall t \in (0,T],\ x\in\overline{\Omega}.$$

The case (a2). Again, by Lemma 3 (ii), $u \in C_0^{\{k\},\alpha}([0,T];X_0)$ and by (4.5), $u \in C_0^{\alpha}([0,T];X_0)$. The relation (4.10) follows from the estimate

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^\epsilon k(\tau)d\tau \sup_{0\le s\le\epsilon} |u(t-s,x)-u(t,x)| = \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\int_0^\epsilon k(\tau)d\tau \cdot O(\epsilon^\alpha)$$
$$\le \lim_{\epsilon\to 0^+} \frac{O(\epsilon^\alpha)}{\epsilon M(\epsilon)}\int_0^\epsilon M(\epsilon-\tau)k(\tau)d\tau = \lim_{\epsilon\to 0^+} O(\epsilon^{\alpha-\gamma}) = 0 \quad \forall t\in(0,T],\ x\in\overline{\Omega}.$$

The case (a3). According to Lemma 3 (ii), $F \in C_0^{\alpha-\beta}([0,T];X_0)$ implies that $u \in C_0^{\{k\},\alpha-\beta}([0,T];X_0) = M * C_0^{\alpha-\beta}([0,T];X_0)$. By Lemma 2 it holds $u \in C_0^{\alpha}([0,T];X_0)$. This enables us finish the proof as in case (a2). □

5 Results on IP1

We will study IP1 in context of Hölder spaces with respect to t. For the sake of generality, we will assume different orders of spaces related to g and h: for g we use α_1 and for h we use α_2.

Theorem 1. *Let one of the following assumptions be valid:*
(A1) $g \in C_0^{1+\alpha_1}([0,T];C(\overline{\Omega}))$ *for some* $0<\alpha_1<1$;
(A2) $g \in C_0^{\{k\},\alpha_1}([0,T];C(\overline{\Omega}))$ *and* $M(t)\ge ct^{\gamma-1}$, $t\in(0,T)$ *for some* $c\in\mathbb{R}_+$, $0<\gamma<\alpha_1<1$;
(A3) $g \in C_0^{\{k\},\alpha_1-\beta}([0,T];C(\overline{\Omega}))$ *and* $c_1t^{\gamma-1}\le M(t)\le c_2t^{\beta-1}$, $|M'(t)|\le c_3t^{\beta-2}$, $t\in(0,T)$, *for some* $c_1,c_2,c_3\in\mathbb{R}_+$, $0<\beta\le\gamma<\alpha_1<1$.
Additionally, we assume that $g\ge 0$, $g_1 := D_t^{\{k\}}g - Rg \ge 0$ *where* $R:=\max_{x\in\overline{\Omega}} r(x)$ *and*

$$a.e.\ x\in\Omega \quad \exists t_x\in(0,T]:\ g(t_x,x)>0. \tag{5.1}$$

In case $\mathcal{B}=I$ *we also assume that* $\forall x\in\partial\Omega$, *either* $g(T,x)>0$ *or* $g(\cdot,x)=0$.

Finally, let $(f,u)\in C(\overline{\Omega})\times\left(C_0^{\{k\}}([0,T];C(\overline{\Omega}))\bigcap C_0([0,T];W_p^2(\Omega))\right)$ *for some* $p>1$ *solve* IP1 *for* $\varphi=0$, $\psi=0$, $h=0$. *Then* $(f,u)=(0,0)$.

Proof. We start the proof by showing that in case $\mathcal{B}=I$, for any $x\in\partial\Omega$ such that $g(T,x)>0$, the equality $f(x)=0$ is valid. To show this, we consider the equality

$$D_t^{\{k\}}u(T,x) = f(x)g(T,x), \quad x\in\overline{\Omega},$$

that follows from equation (2.6) in view of $\psi=0$. If $x\in\partial\Omega$ and $\mathcal{B}=I$ then the left-hand side of this equality equals zero. Thus, $f(x)g(T,x)=0$ and provided $g(T,x)>0$ we obtain $f(x)=0$.

Let us introduce the functions $f^+ = \frac{|f|-f}{2}$ and $f^- = \frac{|f|+f}{2}$. Due to the definition, $f^\pm \in C(\overline{\Omega})$ and $f^\pm \geq 0$. Moreover,

$$\text{in case } \mathcal{B} = I, \text{ for any } x \in \partial\Omega \text{ such that } g(T,x) > 0, \text{ it holds } f^\pm(x) = 0. \quad (5.2)$$

Firstly, we consider the problems

$$\begin{aligned} &D_t^{\{k\}} u^\pm(t,x) = Lu^\pm(t,x) + g(t,x) f^\pm(x), \quad x \in \Omega,\, t \in (0,T), \\ &u^\pm(0,x) = 0, \quad x \in \Omega, \quad \mathcal{B}u^\pm(t,x) = 0, \quad x \in \partial\Omega,\, t \in (0,T). \end{aligned} \quad (5.3)$$

By assumptions of the theorem and (5.2), $g(t,\cdot) f^\pm \in X_0$, $t \in [0,T]$. Therefore, in cases (A1) and (A2) due to (4.5) we have $gf^\pm \in C_0^{\{M\},\alpha_1}([0,T];X_0)$ and $gf^\pm \in C_0^{\alpha_1}([0,T];X_0)$, respectively. Similarly, in case (A3) due to (4.5) and Lemma 2 we obtain $gf^\pm \in C_0^{\alpha_1}([0,T];X_0)$. Moreover, $gf^\pm \geq 0$. The assumptions of Corollary 2 are satisfied for the functions $F = gf^\pm$. Hence, the solutions $u^\pm$ of (5.3) satisfy the assertions of Lemma 4.

Secondly, let us consider the problems

$$\begin{aligned} &D_t^{\{k\}} v^\pm(t,x) = Lv^\pm(t,x) + g_1(t,x) f^\pm(x), \quad x \in \Omega,\, t \in (0,T), \\ &v^\pm(0,x) = 0, \quad x \in \Omega, \quad \mathcal{B}v^\pm(t,x) = 0, \quad x \in \partial\Omega,\, t \in (0,T). \end{aligned} \quad (5.4)$$

In case (A1) we have $g' \in C_0^{\alpha_1}([0,T];C(\overline{\Omega}))$. Thus, $g_1 = D_t^{\{k\}} g - Rg = k * g' - Rg \in C_0^{\{M\},\alpha_1}([0,T];C(\overline{\Omega}))$. From $g(t,\cdot)f^\pm \in X_0$, $t \in [0,T]$ we immediately get $g_1(t,\cdot) f^\pm \in X_0$, $t \in [0,T]$. Therefore, $g_1 f^\pm \in C_0^{\{M\},\alpha_1}([0,T];X_0)$.

Using similar reasoning, we deduce $g_1 f^\pm \in C_0^{\alpha_1}([0,T];X_0)$ and $g_1 f^\pm \in C_0^{\alpha_1-\beta}([0,T];X_0)$ in cases (A2) and (A3), respectively. Moreover, $g_1 f^\pm \geq 0$. Again, the assumptions of Corollary 2 are satisfied for $F = g_1 f^\pm$. The solutions $v^\pm$ of (5.4) satisfy the assertions of Lemma 4.

Let us point out that the problem for $M * v^\pm$ is equivalent to the problem for $u^\pm - RM * u^\pm$. Thus,

$$v^\pm = D_t^{\{k\}} u^\pm - Ru^\pm. \quad (5.5)$$

Moreover, since $f = f^+ - f^-$, we have $u = u^+ - u^-$. Thus, $\psi = u(T,\cdot) = 0$ implies that $u^+(T,\cdot) = u^-(T,\cdot)$. Let us denote $x^* = \operatorname{argmax}_{x\in\overline{\Omega}} u^+(T,x) = \operatorname{argmax}_{x\in\overline{\Omega}} u^-(T,x)$. By definition, either $f^+(x^*) = 0$ or $f^-(x^*) = 0$. Let us assume that $f^+(x^*) = 0$ (the situation when $f^-(x^*) = 0$ can be considered in a similar manner).

Let us suppose that either $x^* \in \Omega$ or $\mathcal{B} = \omega \cdot \nabla$ (the case $x^* \in \partial\Omega$ and $\mathcal{B} = I$ will be considered later separately). Then we can apply Lemma 5 to the function $w = -u^+(T,\cdot)$. We get $L_1 u^+(T,x^*) \leq 0$. Thus, from (5.3), (5.5) and $u^+ \geq 0$, $r \leq R$ it follows:

$$v^+(T,x^*) = L_1 u^+(T,x^*) + (r(x^+) - R) u^+(T,x^*) \leq 0. \quad (5.6)$$

Due to Lemma 4 (i),

$$v^+(t,x) \geq 0,\, (t,x) \in (0,T) \times \Omega. \quad (5.7)$$

Hence, (5.6) and (5.7) imply $v^+(T,x^*) = 0$. Thus, by Lemma 4 (ii), $v^+(t,x^*) = 0$, $t \in [0,T]$. By formula (5.5) it means $D_t^{\{k\}} u^+(t,x^*) - Ru^+(t,x^*) = 0$, $t \in [0,T]$. Applying $M*$ to to this equality, we obtain the following homogeneous Volterra equation of the second kind:

$$u^+(t,x^*) - RM * u^+(t,x^*) = 0, \quad t \in [0,T].$$

It has only the trivial solution $u^+(t,x^*) = 0$, $t \in [0,T]$. Hence, $u^+(T,x^*) = 0$.

Since x^* is a maximum point of $u^+(T,x)$ and $u^+(T,x) \geq 0$, we also get

$$u^+(T,x) = 0, \;\; x \in \Omega. \tag{5.8}$$

Now we consider the case $x^* \in \partial\Omega$, $\mathcal{B} = I$, too. Then by $\mathcal{B}u^+|_{\partial\Omega} = 0$, immediately $u^+(T,x^*) = 0$ and again we have (5.8).

Since $u = u^+ - u^-$ and $\psi = u(T,\cdot) = 0$ holds, from (5.8) we get $u^\pm(T,x) = 0$, $x \in \Omega$. Lemma 4 (ii) implies $u^\pm(t,x) = 0$, $(t,x) \in [0,T] \times \Omega$. Therefore, $u(t,x) = 0$, $(t,x) \in [0,T] \times \Omega$. From the differential equation for u we obtain $f(x)g(t,x) = 0$, $(t,x) \in [0,T] \times \Omega$. Finally, (5.1) yields $f = 0$. □

Next we provide simple sufficient conditions that imply the assumption $D_t^{\{k\}} g - Rg \geq 0$ in Theorem 1. For this we need the following lemma.

Lemma 6. *Let $w \in C^{\{k\}}([0,T];\mathbb{R})$ be nonnegative and nonincreasing. Then $D_t^{\{k\}} w \geq k(T)w$.*

Proof. The assertion follows from the estimate

$$\begin{aligned} D_t^{\{k\}} w(t) &= \lim_{\delta \to 0^+} \frac{1}{\delta}\Big[\int_t^{t+\delta} k(\tau)w(t+\delta-\tau)d\tau + \int_0^t k(\tau)(w(t+\delta-\tau) \\ &\quad - w(t-\tau))d\tau\Big] \geq \lim_{\delta\to 0^+} k(T+\delta)\frac{1}{\delta}\Big[\int_t^{t+\delta} w(t+\delta-\tau)d\tau + \int_0^t (w(t+\delta-\tau) \\ &\quad - w(t-\tau))d\tau\Big] = k(T)w(t), \quad 0 < t < T. \end{aligned}$$

□

Due to that Lemma 6, $D_t^{\{k\}} g - Rg \geq 0$ holds provided along with other assumptions on g in Theorem 1, g is nondecreasing in t and $k(T) \geq R$ in case $R > 0$.

Theorem 2. *Let g, M satisfy the assumptions of Theorem* 1 *and the inequality $g(T,x) > 0$, $x \in \overline{\Omega}$, hold. If $\varphi, \psi \in X_{A_p}$ and $h \in C^{\alpha_2}([0,T];X_p)$, where $p \in \{0\} \bigcup (1,\infty)$, $0 < \alpha_2 < 1$, then* IP1 *has a unique solution $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})$ and the following estimate holds:*

$$\begin{aligned} &\|f\|_{X_p} + \|u\|_{C^{\{k\}}([0,T];X_p) \bigcap C([0,T];X_{A_p})} \\ &\qquad \leq C_5 \left(\|\varphi\|_{X_{A_p}} + \|\psi\|_{X_{A_p}} + \|h\|_{C^{\alpha_2}([0,T];X_p)} \right). \end{aligned} \tag{5.9}$$

If additionally $\varphi = h(0,\cdot) = 0$, *then* $u \in C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})$ *where* $\alpha = \begin{cases} \alpha_2, & \textit{in case } (A1), \\ \min\{\alpha_1,\alpha_2\}, & \textit{in cases } (A2), (A3) \end{cases}$ *and the estimate*

$$\|f\|_{X_p}+\|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})}\le C_6\left(\|\psi\|_{X_{A_p}}+\|h\|_{C_0^{\alpha_2}([0,T];X_p)}\right) \quad (5.10)$$

is valid. The constants C_5 *and* C_6 *depend on the parameters* M, L, g, p, α_2.

Proof. Firstly, we are going to replace the overdetermination condition (2.9) by a fixed-point equation with respect to f.

Suppose that $(f,u) \in X_p \times C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})$ solves IP1. Then, since (2.9) holds, the equation (2.6) at $t = T$ with $F = fg + h$ yields

$$f(x) = \frac{\left(D_t^{\{k\}}(u-\varphi) - \eta u\right)(T,x) - (A_p - \eta)\psi(x) - h(T,x)}{g(T,x)}, \quad (5.11)$$

where η is chosen so that $0 \in \rho(A_p - \eta I)$.

Let us split u into the sum of two functions: $u = u_1 + u_2$, such that

$$\begin{aligned} &D_t^{\{k\}}u_1 = A_p u_1 + fg, \quad u_1(0,\cdot) = 0, \\ &D_t^{\{k\}}(u_2-\varphi) = A_p u_2 + h, \quad u_2(0,\cdot) = \varphi. \end{aligned} \quad (5.12)$$

In the context of IP1, u_2 is a known function. According to Lemma 3, the solution to (5.12) belongs to $u_2 \in C^{\{k\}}([0,T];X_p)$. Thus, $v_2 := D_t^{\{k\}}(u_2-\varphi) - \eta u_2 \in C([0,T];X_p)$. Next we formulate the following problem:

$$D_t^{\{k\}}v_1 = A_p v_1 + f(D_t^{\{k\}}g - \eta g), \quad v_1(0,\cdot) = 0. \quad (5.13)$$

Due to the assumptions (A1)–(A3) and (4.5), it holds $D_t^{\{k\}}g \in C_0^{\hat\alpha}([0,T];C(\overline{\Omega}))$ where

$$\hat\alpha = \begin{cases} \alpha_1, & \text{in cases (A1), (A2),} \\ \alpha_1 - \beta, & \text{in case (A3).} \end{cases} \quad (5.14)$$

Thus, $f(D_t^{\{k\}}g - \eta g) \in C_0^{\hat\alpha}([0,T];X_p)$. According to Lemma 3, (5.13) has a solution v_1 in $C_0^{\{k\},\hat\alpha}([0,T];X_p)\bigcap C_0^{\hat\alpha}([0,T];X_{A_p})$. It is easy to check that $v_1 = D_t^{\{k\}}u_1 - \eta u_1$.

The notations introduced allow us to rewrite (5.11) in the form

$$f = \mathcal{F}f + \mathcal{G}, \quad (5.15)$$

where

$$\mathcal{G}(x) = \frac{v_2(T,x) - (A_p-\eta)\psi(x) - h(T,x)}{g(T,x)}, \quad x \in \Omega, \quad (5.16)$$

$$(\mathcal{F}f)(x) = v_1[f](T,x)/g(T,x) \quad (5.17)$$

and $v_1[\cdot]$ stands for the operator that assigns to f the solution v_1 of (5.13). Thus, (2.6), (2.9), (2.10) imply (5.15). On the other hand, taking into account all the substitutions performed, we can move back from (5.15) to (5.11). Together with (2.6) at $t = T$ and (2.10) it implies $(A_p - \eta)u(T, x) = (A_p - \eta)\psi(x)$. Since $(A_p - \eta)$ is injective, it yields (2.9). Consequently, IP1 is in the space $X_p \times C^{\{k\}}([0, T]; X_p) \bigcap C([0, T]; X_{A_p})$ equivalent to the problem of finding the pair of functions (f, u) that solves (2.6), (2.10), (5.15).

We point out that (5.15) is an independent equation for the first component f of the solution of IP1. Let us analyse properties of the operator $\mathcal{F}$ involved in this equation. By Lemma 3, $v_1[\cdot] \in \mathcal{L}(X_p; C_0^{\hat{\alpha}}([0, T]; X_{A_p}))$. Thus, $v_1[\cdot](T, \cdot) \in \mathcal{L}(X_p, X_{A_p})$.

Furthermore, $X_{A_p} \hookrightarrow\hookrightarrow X_p$. In case $p \in (1, \infty)$ it is a direct consequence of $W_p^2(\Omega) \hookrightarrow\hookrightarrow L_p(\Omega)$. In case $p = 0$ it follows from the continuous embedding of X_{A_0} in $C^1_{\mathcal{B}}(\overline{\Omega}) := X_0 \bigcap C^1(\overline{\Omega})$ (see Theorems 3.1.19, 3.1.22 in [24]) and $C^1_{\mathcal{B}}(\overline{\Omega}) \hookrightarrow\hookrightarrow X_0$.

Therefore, $v_1[\cdot](T, \cdot) : X_p \to X_p$ is compact. Since $\frac{1}{g(T,\cdot)} \in C(\overline{\Omega})$ due to the assumptions of this theorem, $\mathcal{F} : X_p \to X_p$ is also compact.

Next, let us show that $1 \notin \sigma(\mathcal{F})$. Firstly, let us consider the case $p = 0$. Suppose that $1 \in \sigma(\mathcal{F})$. Then the equation $f = \mathcal{F}f$ has a solution $f \in X_0$, $f \neq 0$. This means that the problem (2.6), (2.10), (5.15) with homogeneous data $\varphi = 0, \psi = 0, h = 0$ has the nontrivial solution (f, u_1) in the space $X_0 \times C_0^{\{k\}}([0, T]; X_0) \bigcap C_0([0, T]; X_{A_0})$. But due to the Theorem 1, IP1 with a homogeneous data has only the trivial solution in such a space. We came to a contradiction. Consequently, $1 \notin \sigma(\mathcal{F})$.

Secondly, let us consider the case $p \in (1, \infty)$. We again suppose that $1 \in \sigma(\mathcal{F})$, hence the equation $f = \mathcal{F}f$ has a nontrivial solution $f \in X_p$. The idea is to show that this solution actually belongs to X_0. Then we can apply the arguments from the previous case to show that $1 \in \sigma(\mathcal{F})$ leads to a contradiction.

If $p > \frac{n}{2}$, then $v_1[f](T, \cdot) \in X_{A_p} \hookrightarrow X_0$. Thus, $f = \mathcal{F}f = \frac{1}{g(T,x)} v_1[f](T, \cdot) \in X_0$. If $p \le \frac{n}{2}$, then according to embedding theorems, $X_{A_p} \hookrightarrow X_{p_1} = L_{p_1}(\Omega)$, where $p_1 = \frac{np}{n-2p} > p$. Therefore, $v_1[f](T, \cdot) \in X_{p_1}$ and $f = \mathcal{F}f = \frac{1}{g(T,x)} v_1[f](T, \cdot) \in X_{p_1}$. After a finite number of iterations we obtain $f \in X_{p_i}$, where $p_i = \frac{np}{n-2ip} > \frac{n}{2}$ (works for $i > \frac{n}{2p} - 1$). Next iteration gives $f \in X_0$.

We have shown that the first case of Fredholm alternative is satisfied for the equation (5.15). Consequently, the solution to (5.15) exists and is unique for any $\mathcal{G} \in X_p$ and $(I - \mathcal{F})^{-1} \in \mathcal{L}(X_p)$.

Since $F = fg + h$ is Hölder-continuous with values in X_p, Lemma 3 implies that the problem (2.6), (2.10) has unique solution $u \in C^{\{k\}}([0, T]; X_p) \bigcap C([0, T]; X_{A_p})$. This completes the proof of the existence and uniqueness assertion of the theorem.

In the rest of the proof, $\widehat{C}$ stands for a generic constant depending on the parameters M, L, g, p, α_2. Let us deduce the stability estimate (5.9). We obtain

$$\|f\|_{X_p} \le \|(I - \mathcal{F})^{-1}\|_{\mathcal{L}(X_p)} \|\mathcal{G}\|_{X_p} \le \widehat{C} \Big(\|h(T, \cdot)\|_{X_p} + |\eta| \|\psi\|_{X_p} + \|\psi\|_{X_{A_p}}$$

$$+ \|D_t^{\{k\}}(u_2 - \varphi) - \eta u_2\|_{C([0,T];X_p)}\Big) \le \widehat{C}\Big(\|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\psi\|_{X_{A_p}} + \|\varphi\|_{X_{A_p}}\Big). \quad (5.18)$$

Further, we note that $g \in C_0^{\gamma}([0,T]; C(\overline{\Omega}))$ for any $\gamma \in (0,1)$ in case (A1) and for $\gamma = \alpha_1$ in cases (A2), (A3). Using Lemma 3 we have

$$\|u\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})} = \|u_1 + u_2\|_{C^{\{k\}}([0,T];X_p)\bigcap C([0,T];X_{A_p})}$$
$$\le \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^{\gamma}([0,T];C(\overline{\Omega}))} + \|h\|_{C^{\alpha_2}([0,T];X_p)} + \|\varphi\|_{X_{A_p}}).$$

Together with the estimate of f (5.18) it implies (5.9).

In case $\varphi = h(0,\cdot) = 0$, the solution of (2.6), (2.10) belongs to the space $C_0^{\{k\},\alpha}([0,T]; X_p)\bigcap C_0^{\alpha}([0,T]; X_{A_p})$ and can be estimated as

$$\|u\|_{C_0^{\{k\},\alpha}([0,T];X_p)\bigcap C_0^{\alpha}([0,T];X_{A_p})} \le \widehat{C}(\|f\|_{X_p}\|g\|_{C_0^{\gamma}([0,T];C(\overline{\Omega}))} + \|h\|_{C_0^{\alpha_2}([0,T];X_p)}).$$

This with (5.18) implies (5.10). □

We point out that in case $p = 0$ and $\mathcal{B} = I$, the assumptions of Theorem 2 allow to recover $f \in X_0 = C_0(\overline{\Omega})$ only. In order to fix that in the following theorem we provide some additional conditions that are sufficient to restore $f \in C(\overline{\Omega})$ in case $\mathcal{B} = I$.

Theorem 3. *Let g, M satisfy the assumptions of Theorem 2. If φ, ψ, $L\varphi \in X_{A_p}$ for some $p > \frac{n}{2}$, $L\psi \in C(\overline{\Omega})$, $h \in C^{\{k\},\alpha_2}([0,T]; X_p)\bigcap C([0,T]; C(\overline{\Omega}))$, where $0 < \alpha_2 < 1$ and $h(0,\cdot) \in X_{A_p}$ then* IP1 *has a unique solution $(f,u) \in C(\overline{\Omega}) \times C^{\{k\}}([0,T]; X_{A_p})$. Moreover, $Lu \in C([0,T]; C(\overline{\Omega}))$ and the estimate*

$$\|f\|_{C(\overline{\Omega})} + \|u\|_{C^{\{k\}}([0,T];X_{A_p})} + \|Lu\|_{C([0,T];C(\overline{\Omega}))} \le C_7\big(\|\varphi\|_{X_p} + \|L\varphi\|_{X_{A_p}} + \|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)\bigcap C([0,T];C(\overline{\Omega}))} + \|h(0,\cdot)\|_{X_{A_p}}\big) \quad (5.19)$$

holds. If additionally $\varphi = h(0,\cdot) = D_t^{\{k\}}h(0,\cdot) = 0$, then $u \in C_0^{\{k\},\alpha'}([0,T]; X_{A_p})$ and the estimate

$$\|f\|_{C(\overline{\Omega})} + \|u\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} + \|Lu\|_{C_0([0,T];C(\overline{\Omega}))}$$
$$\le C_8\big(\|\psi\|_{X_p} + \|L\psi\|_{C(\overline{\Omega})} + \|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)\bigcap C_0([0,T];C(\overline{\Omega}))}\big) \quad (5.20)$$

is valid where $\alpha' = \min\{\hat{\alpha}; \alpha_2\}$ and $\hat{\alpha}$ is given by (5.14)*. The constants C_7 and C_8 depend on M, L, g, p, α_2.*

Proof. Throughout the proof, $\widehat{C}$ denotes a generic constant depending on M, L, g, p, α_2 and RHS stands for the expression in brackets at the right-hand side of (5.19). By Theorem 2, IP1 has a unique solution $(f,u) \in X_p \times C^{\{k\}}([0,T]; X_p)\bigcap C([0,T]; X_{A_p})$. Let us consider the problem

$$D_t^{\{k\}}(w_2 - w_2(0,\cdot)) = A_p w_2 + D_t^{\{k\}}(h - h(0,\cdot)),\ w_2(0,\cdot) = L\varphi + h(0,\cdot). \quad (5.21)$$

Under the assumptions of this theorem, Lemma 3 implies that (5.21) has a unique solution $w_2 \in C^{\{k\}}([0,T];X_p) \cap C([0,T];X_{A_p})$. Moreover, due to (4.7) and (4.8), $\|w_2\|_{C([0,T];X_{A_p})} \le \widehat{C}(\|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)} + \|h(0,\cdot)\|_{X_{A_p}} + \|L\varphi\|_{X_{A_p}})$. It is easy to check that $w_2 = D_t^{\{k\}} * (u_2 - \varphi)$ and $u_2 = M * w_2 + \varphi$ where u_2 solves (5.12). Therefore, we have $u_2 \in C^{\{k\}}([0,T];X_{A_p}) \hookrightarrow C^{\{k\}}([0,T];C(\overline{\Omega}))$ and

$$\begin{aligned}&\|u_2\|_{C^{\{k\}}([0,T];X_{A_p})}\\ &\quad\le \widehat{C}\Big(\|h\|_{C^{\{k\},\alpha_2}([0,T];X_p)}+\|h(0,\cdot)\|_{X_{A_p}}+\|L\varphi\|_{X_{A_p}}\Big)+\|\varphi\|_{X_{A_p}}.\end{aligned} \tag{5.22}$$

Let us consider the function $\mathcal{G}$ given by (5.16). (Recall that there $v_2 = w_2 - \eta u_2$.) Due the proved properties of w_2 and u_2 and the assumptions of the theorem, it holds $\mathcal{G} \in C(\overline{\Omega})$ and $\|\mathcal{G}\|_{C(\overline{\Omega})} \le \widehat{C}\,\mathrm{RHS}$.

Now, let us provide an estimate for $\|f\|_{C(\overline{\Omega})}$ using the formulas (5.15) and (5.17). Since $1/g(T,\cdot) \in C(\overline{\Omega})$ and $v_1[\cdot](T,\cdot) \in \mathcal{L}(X_p, X_{A_p})$, we have

$$\begin{aligned}\|f\|_{C(\overline{\Omega})} &\le \|\mathcal{F}f\|_{C(\overline{\Omega})} + \|\mathcal{G}\|_{C(\overline{\Omega})} \le \widehat{C}\|v_1[f](T,\cdot)\|_{C(\overline{\Omega})} + \|\mathcal{G}\|_{C(\overline{\Omega})}\\ &\le \widehat{C}\|v_1[f](T,\cdot)\|_{X_{A_p}} + \|\mathcal{G}\|_{C(\overline{\Omega})} \le \widehat{C}\|f\|_{X_p} + \|\mathcal{G}\|_{C(\overline{\Omega})}.\end{aligned}$$

Since $(I - \mathcal{F})$ is invertible in X_p, the estimate holds

$$\|f\|_{X_p} \le \|(I-\mathcal{F})^{-1}\|_{\mathcal{L}(X_p)}\|\mathcal{G}\|_{X_p} \le \widehat{C}\|\mathcal{G}\|_{C(\overline{\Omega})}.$$

Thus, we obtain

$$\|f\|_{C(\overline{\Omega})} \le \widehat{C}\,\mathrm{RHS}. \tag{5.23}$$

Finally, let us derive an estimate for u and finish the proof of the first part of the theorem. We have $u = u_1 + u_2$, where $u_1 = M * w_1$, $w_1 = D_t^{\{k\}} u_1$ and w_1 solves the problem

$$D_t^{\{k\}} w_1 = A_p w_1 + f D_t^{\{k\}} g, \quad w_1(0,\cdot) = 0.$$

Since $f D_t^{\{k\}} g \in C_0^{\alpha'}([0,T];X_p)$, Lemma 3 implies $w_1 \in C_0^{\alpha'}([0,T];X_{A_p})$ and $\|u_1\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} = \|w_1\|_{C_0^{\alpha'}([0,T];X_{A_p})} \le \widehat{C}\|f\|_{C(\overline{\Omega})}\|D_t^{\{k\}} g\|_{C_0^{\alpha'}([0,T];X_p)}$. Using here (5.23) we have

$$\|u_1\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} \le \widehat{C}\,\mathrm{RHS}. \tag{5.24}$$

From (5.22) and (5.24) we obtain for $u = u_1 + u_2$ the estimate

$$\|u\|_{C^{\{k\}}([0,T];X_{A_p})} \le \widehat{C}\,\mathrm{RHS}. \tag{5.25}$$

It remains to estimate Lu in the space $C([0,T];C(\overline{\Omega}))$. Using (5.25) we deduce

$$\|D_t^{\{k\}}(u-\varphi)\|_{C([0,T];C(\overline{\Omega}))} \le \widehat{C}\|D_t^{\{k\}}(u-\varphi)\|_{C([0,T];X_{A_p})} \le \widehat{C}\,\mathrm{RHS}.$$

From the expression $Lu = D_t^{\{k\}}(u-\varphi) - fg - h$ due to the proved estimates for $D_t^{\{k\}}(u-\varphi)$ and f we obtain

$$\|Lu\|_{C([0,T];C(\overline{\Omega}))} \le \widehat{C}\,\mathrm{RHS}. \tag{5.26}$$

Summing up, (5.23), (5.25) and (5.26) imply (5.19).

Now let us focus on the second part of this theorem that is concerned with the particular case $\varphi = h(0,\cdot) = D_t^{\{k\}}h(0,\cdot) = 0$. Then RHS reduces to the expression in brackets at the right-hand side of (5.20). Lemma 3 implies that the function w_2 which solves (5.21) belongs the space $C_0^{\alpha'}([0,T];X_{A_p})$, the function $u_2 = M * w_2$ belongs to $C_0^{\{k\},\alpha'}([0,T];X_{A_p})$ and $\|u_2\|_{C_0^{\{k\},\alpha'}([0,T];X_{A_p})} \le \widehat{C}\|h\|_{C_0^{\{k\},\alpha_2}([0,T];X_p)}$. This relation by $u = u_1 + u_2$ and the estimates (5.23), (5.24) and (5.26) implies (5.20). □

Provided the assumptions of Theorem 3 hold and $\mathcal{B} = I$, an explicit expression of the unknown function f at the boundary can be derived. Namely, setting $t = T$ and $x \in \partial\Omega$ in (2.6) and taking the relations $F = fg + h$ and $u(T,\cdot) = \psi$ into account we obtain $f(x) = -\dfrac{1}{g(T,x)}[L\psi(x) + h(T,x)], \quad x \in \partial\Omega.$

6 Results on IP2

In the context of IP2 let us introduce the following sets for the coefficient r:

$$\mathcal{K}_R = \{r \in C(\overline{\Omega}) \,:\, r(x) \le R,\ x \in \overline{\Omega}\}, \qquad \text{where } R \in \mathbb{R}.$$

Theorem 4. *Let R be some real number and IP2 have two solutions (r,u), (r_1,u_1), such that*

$$r \in C(\overline{\Omega}), \quad r_1 \in \mathcal{K}_R, \quad u, u_1 \in C_0^{\{k\}}([0,T];L_1(\Omega)) \bigcap C_0([0,T];W_1^2(\Omega)),$$
$$u_1 - u \in C_0^{\{k\}}([0,T];C(\overline{\Omega})) \bigcap C_0([0,T];W_p^2(\Omega))$$

for some $p > 1$ and the function $U = u + b$ (and M) satisfy one of the following assumptions:
(A4) *$U \in C_0^{1+\alpha_1}([0,T];C(\overline{\Omega}))$ for some $0 < \alpha_1 < 1$;*
(A5) *$U \in C_0^{\{k\},\alpha_1}([0,T];C(\overline{\Omega}))$ and $M(t) \ge ct^{\gamma-1}$, $t \in (0,T)$ for some $c \in \mathbb{R}_+$, $0 < \gamma < \alpha_1 < 1$;*
(A6) *$U \in C_0^{\{k\},\alpha_1-\beta}([0,T];C(\overline{\Omega}))$ and $c_1t^{\gamma-1} \le M(t) \le c_2t^{\beta-1}$, $|M'(t)| \le c_3t^{\beta-2}$, $t \in (0,T)$, for some $c_1,c_2,c_3 \in \mathbb{R}_+$, $0 < \beta \le \gamma < \alpha_1 < 1$.*

Additionally, we assume that

$$U \ge 0, \quad D_t^{\{k\}}U - RU \ge 0, \tag{6.1}$$
$$\text{a.e. } x \in \Omega, \quad \exists t_x \in (0,T] \,:\, U(t_x,x) > 0.$$

In case $\mathcal{B} = I$ we also assume that $\forall x \in \partial\Omega$, either $U(T,x) > 0$ or $U(\cdot,x) = 0$. Then $(r_1,u_1) = (r,u)$.

Proof. The difference $(\hat{r},\hat{u}) = (r_1 - r, u_1 - u) \in C(\overline{\Omega}) \times \Big(C_0^{\{k\}}([0,T]; C(\overline{\Omega})) \bigcap C_0([0,T]; W_p^2(\Omega)) \Big)$ solves the problem

$$
\begin{aligned}
&D_t^{\{k\}} \hat{u}(t,x) = (L_1 + r_1)\hat{u}(t,x) + U(t,x)\hat{r}(x), \quad x \in \Omega,\ t \in (0,T),\\
&\hat{u}(0,x) = 0, \quad x \in \Omega, \quad \mathcal{B}\hat{u}(t,x) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \qquad (6.2)\\
&\hat{u}(T,x) = 0, \quad x \in \Omega.
\end{aligned}
$$

The inequalities (6.1) imply that $D_t^{\{k\}}U - R_r U \geq 0$, where $R_r := \max_{x\in\overline{\Omega}} r_1(x) \leq R$. Consequently, the assumptions of Theorem 1 are satisfied for the problem (6.2) and we obtain $\hat{r} = 0$, $\hat{u} = 0$. □

Let us formulate a problem that contains approximate data:

$$
\begin{aligned}
&D_t^{\{k\}}(\tilde{u}-\tilde{\varphi})(t,x) = L_1\tilde{u}(t,x) + \tilde{r}(x)(\tilde{u}+\tilde{b})(t,x) + \tilde{F}_1(t,x), \quad x \in \Omega,\ t \in (0,T),\\
&\tilde{u}(0,x) = 0, \quad x \in \Omega, \qquad \mathcal{B}\tilde{u}(t,x) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \qquad (6.3)\\
&\tilde{u}(T,x) = \tilde{\psi}, \quad x \in \Omega.
\end{aligned}
$$

We are going to prove an existence and approximation theorem for this problem in case its data vector $\widetilde{D} = (\tilde{b}, \tilde{F}_1, \tilde{\psi})$ is close to the data vector $D = (b, F_1, \psi)$ of the exact problem IP2.

Theorem 5. *Assume that* $R \in \mathbb{R}$ *and* IP2 *has a solution* $(r,u) \in \mathcal{K}_R \times C_0^{\{k\}}([0,T]; L_1(\Omega)) \bigcap C_0([0,T]; W_1^2(\Omega))$ *such that* $U = u + b$ *(and* M*) satisfy one of the assumptions* (A4)–(A6)*, the inequalities* (6.1) *and* $U(T,x) > 0$, $x \in \overline{\Omega}$. *Then the following statements are valid.*

(i) *Let* $p \in \{0\} \bigcup \left(\frac{n}{2}, \infty\right)$, $\alpha_2 \in (0,1)$. *There exist constants* $\delta_1 > 0$ *and* $K_1 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$
\tilde{D} - D \in \mathcal{D}_1 = C_0^{\alpha_2}([0,T]; C_{(p)}(\overline{\Omega})) \times C_0^{\alpha_2}([0,T]; X_p) \times X_{A_p}
$$

and $\|\tilde{D} - D\|_{\mathcal{D}_1} \leq \delta_1$, *where* $C_{(p)}(\overline{\Omega}) = \begin{cases} C(\overline{\Omega}), & \text{in case } p \in \left(\frac{n}{2}, \infty\right), \\ X_0, & \text{in case } p = 0, \end{cases}$ *then problem* (6.3) *has a unique solution in the set*

$$
\Big\{ (\tilde{r},\tilde{u}) \,:\, (\tilde{r} - r, \tilde{u} - u) \in \mathcal{X}_1 := X_p \times \Big(C_0^{\{k\},\alpha}([0,T]; X_p) \bigcap C_0^{\alpha}([0,T]; X_{A_p}) \Big),
\quad \|(\tilde{r} - r, \tilde{u} - u)\|_{\mathcal{X}_1} \leq K_1 \|\tilde{D} - D\|_{\mathcal{D}_1} \Big\},
$$

where $\alpha = \begin{cases} \alpha_2, & \text{in case (A4)}, \\ \min\{\alpha_1, \alpha_2\}, & \text{in cases (A5), (A6)}. \end{cases}$

(ii) *Let* $p \in \left(\frac{n}{2}, \infty\right)$, $\alpha_2 \in (0,1)$. *There exist constants* $\delta_2 > 0$ *and* $K_2 > 0$ *depending on* $M, L_1, r, U, p, \alpha_2$ *such that if*

$$
\tilde{D} - D \in \mathcal{D}_2 = \Big(C_0^{\{k\},\alpha_2}([0,T]; X_p) \bigcap C_0^{\alpha_2}([0,T]; C(\overline{\Omega})) \Big)^2 \times Y_p
$$

and $\|\tilde{D}-D\|_{\mathcal{D}_2} \le \delta_2$ *where* $Y_p = \{\psi : \psi \in X_{A_p}, L\psi \in C(\overline{\Omega})\}$, *then the problem* (6.3) *has a unique solution in the set*

$$\Big\{(\tilde{r},\tilde{u}) : (\tilde{r}-r,\tilde{u}-u) \in \mathcal{X}_2 := C(\overline{\Omega})\times \mathcal{U}_{p,\alpha'},\ \|(\tilde{r}-r,\tilde{u}-u)\|_{\mathcal{X}_2} \le K_2\|\tilde{D}-D\|_{\mathcal{D}_2}\Big\},$$

where $\mathcal{U}_{p,\alpha'} = \{v \in C_0^{\{k\},\alpha'}([0,T];X_{A_p}) : Lv \in C_0([0,T];C(\overline{\Omega}))\}$, $\alpha' = \min\{\hat{\alpha};\alpha_2\}$ *and* $\hat{\alpha} = \begin{cases} \alpha_1, & \textit{in cases } \text{(A4), (A5)},\\ \alpha_1-\beta, & \textit{in case } \text{(A6)}.\end{cases}$

We mention that in this theorem, the operator A_p and the space X_{A_p} defined on the basis of $L = L_1 + rI$ depend on the component r of the solution of the exact problem IP2.

Proof. Let us denote the difference $(\hat{r},\hat{u}) = (\tilde{r}-r,\tilde{u}-u)$. Then the problem for the pair $(\hat{r},\hat{u})$ reads

$$\begin{aligned} &D_t^{\{k\}}\hat{u} = (L_1+r)\hat{u} + \hat{r}(u+b) + \Big[\hat{r}\hat{u} + \tilde{F}_1 - F_1 + (\hat{r}+r)(\tilde{b}-b)\Big],\\ &\hat{u}(0,\cdot)=0,\quad \mathcal{B}\hat{u}|_{\partial\Omega}=0,\quad \hat{u}(T,\cdot)=\tilde{\psi}-\psi. \end{aligned} \tag{6.4}$$

This problem can be treated as IP1 with $f=\hat{r}$, $g=u+b$, $h=\hat{r}\hat{u}+\tilde{F}_1-F_1+(\hat{r}+r)(\tilde{b}-b)$. Therefore, applying the solution operator of IP1 $\mathcal{A}$ to (6.4), it is reduced to the operator equation

$$(\hat{r},\hat{u}) = \mathcal{F}_2(\hat{r},\hat{u}), \tag{6.5}$$

where $\mathcal{F}_2(\hat{r},\hat{u}) = \mathcal{A}(\hat{r}\hat{u}+\tilde{F}_1-F_1+(\hat{r}+r)(\tilde{b}-b),0,\tilde{\psi}-\psi)$.

We are going to show that $\mathcal{F}_2$ is a contraction in a ball $\|(\hat{r},\hat{u})\|_{\mathcal{X}_1} \le \rho$ with a suitable chosen $\rho>0$. Firstly, we have to prove that this ball remains invariant with respect to the operator $\mathcal{F}_2$. Let $\|(\hat{r},\hat{u})\|_{\mathcal{X}_1}\le\rho$. According to (5.10),

$$\|\mathcal{F}_2(\hat{r},\hat{u})\|_{\mathcal{X}_1} \le C_6\left(\|\tilde{\psi}-\psi\|_{X_{A_p}} + \|\hat{r}\hat{u}+\tilde{F}_1-F_1+(\hat{r}+r)(\tilde{b}-b)\|_{C_0^{\alpha_2}([0,T];X_p)}\right).$$

Let c_p be an embedding constant such that $\|w\|_{C(\overline{\Omega})} \le c_p\|w\|_{X_{A_p}}$. Then

$$\|\hat{r}\hat{u}\|_{C_0^{\alpha_2}([0,T];X_p)} \le \|\hat{r}\|_{X_p}\|\hat{u}\|_{C_0^{\alpha}([0,T];C(\overline{\Omega}))} \le \|\hat{r}\|_{X_p}c_p\|\hat{u}\|_{C_0^{\alpha}([0,T];X_{A_p})} \le c_p\rho^2.$$

Therefore,

$$\begin{aligned} &\|\mathcal{F}_2(\hat{r},\hat{u})\|_{\mathcal{X}_1} \le C_6\Big(\|\tilde{\psi}-\psi\|_{X_{A_p}} + c_p\rho^2 + \|\tilde{F}_1-F_1\|_{C_0^{\alpha_2}([0,T];X_p)}\\ &+(\rho+R_1)\|\tilde{b}-b\|_{C_0^{\alpha_2}([0,T];C_{(p)}(\overline{\Omega}))}\Big) \le C_6\left(c_p\rho^2 + (\rho+1+R_1)\|\tilde{D}-D\|_{\mathcal{D}_1}\right), \end{aligned}$$

where $R_1 = \|r\|_{X_p}$ in case $p \in \left(\frac{n}{2},\infty\right)$ and $R_1 = \|r\|_{C(\overline{\Omega})}$ in case $p=0$. Now let us take $\rho = K_1\|\tilde{D}-D\|_{\mathcal{D}_1}$ with a constant K_1. Then

$$\|\mathcal{F}_2(\hat{r},\hat{u})\|_{\mathcal{X}_1} \le C_6\left((c_pK_1^2+K_1)\|\tilde{D}-D\|_{\mathcal{D}_1} + 1 + R_1\right)\|\tilde{D}-D\|_{\mathcal{D}_1}.$$

In case $\|\tilde{D} - D\|_{\mathcal{D}_1} \le \delta_1$ we have

$$\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le C_6 \left((c_p K_1^2 + K_1)\delta_1 + 1 + R_1\right) \|\tilde{D} - D\|_{\mathcal{D}_1}.$$

Let us define the constants as follows: $K_1 = C_6(2 + R_1)$, $\delta_1 = \frac{1}{c_p K_1^2 + K_1}$. Then $\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le K_1 \|\tilde{D} - D\|_{\mathcal{D}_1}$. Consequently, for $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le \rho$ we have $\|\mathcal{F}_2(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le \rho$.

Secondly, inside the set $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le \rho = K_1 \|\tilde{D} - D\|_{\mathcal{D}_1}$ let us consider the difference of $\mathcal{F}_2$ at $(\hat{r}_1, \hat{u}_2)$ and $(\hat{r}_2, \hat{u}_2)$. Assuming $\|\tilde{D} - D\|_{\mathcal{D}_1} \le \delta_1$, we deduce the estimate

$$\begin{aligned}
&\|\mathcal{F}_2(\hat{r}_1, \hat{u}_1) - \mathcal{F}_2(\hat{r}_2, \hat{u}_2)\|_{\mathcal{X}_1} \le \|\mathcal{A}\| \|(\hat{r}_1 - \hat{r}_2)\hat{u}_1 + \hat{r}_2(\hat{u}_1 - \hat{u}_2) \\
&\quad + (\hat{r}_1 - \hat{r}_2)(\tilde{b} - b)\|_{C_0^{\alpha_2}([0,T];X_p)} \le C_6 \Big(c_p \rho \|\hat{r}_1 - \hat{r}_2\|_{X_p} \\
&\quad + c_p \rho \|\hat{u}_1 - \hat{u}_2\|_{C_0^{\alpha}([0,T];X_{A_p})} + \delta_1 \|\hat{r}_1 - \hat{r}_2\|_{X_p} \Big) \le C_6(c_p K_1 \delta_1 + \delta_1) \\
&\quad \times \|(\hat{r}_1 - \hat{r}_2, \hat{u}_1 - \hat{u}_2)\|_{\mathcal{X}_1} = \frac{1}{(2 + R_1)} \|(\hat{r}_1 - \hat{r}_2, \hat{u}_1 - \hat{u}_2)\|_{\mathcal{X}_1}.
\end{aligned}$$

It shows that the operator $\mathcal{F}_2$ is a contraction in the ball $\|(\hat{r}, \hat{u})\|_{\mathcal{X}_1} \le \rho$. According to the Banach fixed point theorem there exists a unique solution to the equation (6.5) in that ball. This proves the assertion (i).

(ii) The proof of (ii) repeats the proof of (i) with appropriate changes of spaces and norms. For $\mathcal{A}$, the estimate (5.20) is used instead of (5.10). □

Remark 1. In case the data of (6.3) are close to data of a process without reaction (i.e. $r = 0$), Theorem 5 implies the existence of the reaction coefficient $\tilde{r}$ in small.

Remark 2. Supposing the existence of a solution (r, u) of IP2, we ask: what are sufficient conditions on the data that guarantee the validity of inequality-type conditions (6.1) and $U(T, x) > 0$, $x \in \overline{\Omega}$ in Theorems 4, 5? To answer this question, we return to the problem (2.3) for U and set there $\Phi = H(0, \cdot) = 0$. Let us suppose that U is sufficiently smooth. Then constructing a corresponding problem for $D_t^{\{k\}} U - RU$ and assuming $D_t^{\{k\}} H - RH \ge 0$, $(D_t^{\{k\}} \mathcal{B}b - R\mathcal{B}b)|_{\partial\Omega} \ge 0$, Lemma 4 (i) implies the inequality $D_t^{\{k\}} U - RU \ge 0$. Next, we consider the conditions $U \ge 0$ and $U(T, x) > 0$, $x \in \overline{\Omega}$. Let us assume that

$$\begin{aligned}
&\exists \mu \in C[0, T],\ \mu \ge 0,\ \mu \ne 0,\ \mu - \text{nondecreasing}\ : \\
&H(t, x) \ge \mu(t),\ x \in \overline{\Omega},\ t \in [0, T], \quad \mathcal{B}b(t, x) \ge \mu(t),\ x \in \partial\Omega,\ t \in [0, T].
\end{aligned}$$

Define $V = U - \delta 1 * \mu$ with $\delta > 0$. The function V solves the problem

$$D_t^{\{k\}} V = LV + H_1, \quad V(0, \cdot) = 0\,, \quad \mathcal{B}(V - (b - \delta 1 * \mu))|_{\partial\Omega} = 0,$$

where $H_1 = H + \delta(r 1 * \mu - D_t^{\{k\}} 1 * \mu)$. Since $D_t^{\{k\}} 1 * \mu = k * \mu$, we get that for sufficiently small δ,

$$H_1(t, x) \ge \mu(t)[1 - \delta(\max_{x \in \overline{\Omega}} r(x) T + \|k\|_{L_1(0,T)})] \ge 0, \quad t \in [0, T], \ \ x \in \Omega$$

and $\mathcal{B}V|_{\partial\Omega} = \mathcal{B}(b - \delta 1 * \mu)|_{\partial\Omega} \geq 0$. Lemma 4 (i) yields $V \geq 0$. Thus, $U = V + \delta 1 * \mu \geq 0$ and $U(T,x) = V(T,x) + \delta \int_0^T \mu(\tau)d\tau > 0$, $x \in \overline{\Omega}$.

At the end of this section, we make some general remarks. We applied results on IP1 to analyze IP2. In a similar manner, results on IP1 can be applied to study inverse problems to determine other coefficients of L, too.

The basic set of assumptions (A1)–(A3) for g involves the restriction $g(0,\cdot) = 0$. This is due to the fact that in case $g(0,\cdot) \neq 0$ we cannot ensure sufficient regularity of u to apply the positivity principle in the proof of Theorem 1. In IP2, the function $u + b = U$ works as g. For that reason, we consider the case $\Phi = U(0,\cdot) = 0$ in IP2.

In the beginning of the proof of Lemma 4 we showed that the direct problem with $r > 0$ can be reduced to a problem with $r \leq 0$ by the change of unknown $\tilde{u} = e^{-\sigma t}u$, where $\sigma > 0$. This suggests a possible exponential growth of u and a related time limitation of the linear reaction model in case $r > 0$. For bigger T, nonlinear reaction models are more relevant [6].

Solutions of IP1 and IP2 depend continuously on derivatives of the data of finite order. This means that these problems are moderately ill-posed. In case approximate data are given with errors, regularization procedures can be effectively applied (cf. e.g. [17] for IP1 with $g = g(t)$).

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Appendix: Proofs of Lemmas 1, 2 and 5

Proof of Lemma 1. Theorems 3 and 4 of [9] guarantee that k is nonnegative, nonincreasing and convex. Convexity implies the continuity of k. From the equation $M * k = 1$ we easily deduce $\lim_{t\to 0^+} k(t) = +\infty$, because in the opposite case k is bounded from which it follows that $\lim_{t\to 0^+} (M * k)(t) = 0$.

Let us prove $k > 0$. Suppose that it is not true. Then in view of proved properties of k, $\exists t_0 :\ k(t) > 0$, $t < t_0$ and $k(t) = 0$, $t > t_0$. For $t > t_0$ from $M * k = 1$ we get $\int_0^{t_0} M(t-\tau)k(\tau)d\tau = 1$. Therefore, $\int_0^{t_0} M'(t-\tau)k(\tau)d\tau = 0$.

The last equality contradicts to the assumptions $k(t) > 0, t \in (0, t_0)$ and $M' < 0$. Thus, $k > 0$.

Finally, let us prove (3.3) Let us choose some $t_3 > 0$. Since $\lim_{t \to 0^+} k(t) = +\infty$, there exists an interval $(0, \delta)$, $\delta < t_3$, such that $k(t) > k(t_3)$ for $t \in (0, \delta)$. Suppose that (3.3) is not true. Then we can find two points $t_1 < t_2$ in $(0, \delta)$ so that $k(t_1) = k(t_2)$. Consequently, for $t_1 < t_2 < t_3$ we have $k(t_1) = k(t_2) > k(t_3)$. Obviously, it contradicts to the convexity of k. Therefore, (3.3) is valid. □

Proof of Lemma 2 is similar to proof of Theorem 14 in [11] that is concerned with the case $M(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$. Let $z \in C_0^{\alpha-\beta}([0, T]; X)$. Then $\|M * z(t)\|_X \leq \text{const}\, t^{\beta-1} * t^{\alpha-\beta} = O(t^\alpha)$. Secondly,

$$(M * z)(t) - (M * z)(t - h) = J_1 + J_2 + J_3,$$

where

$$J_1 = z(t) \int_{t-h}^{t} M(\tau) d\tau, \quad J_2 = -\int_0^h [z(t) - z(t - \tau) M(\tau) d\tau,$$
$$J_3 = -\int_h^t [z(t) - z(t - \tau)] \int_{\tau-h}^{\tau} M'(s) ds\, d\tau.$$

Immediately, $\|J_2\|_X \leq \text{const} \int_0^h \tau^{\alpha-\beta} \tau^{\beta-1} d\tau = O(h^\alpha)$. Moreover,

$$\|J_1\|_X \leq \text{const}\, t^{\alpha-\beta} \int_{t-h}^{t} \tau^{\beta-1} = \text{const}\, t^{\alpha-\beta} [t^\beta - (t - h)^\beta],$$
$$\|J_3\|_X \leq \text{const} \int_h^t \tau^{\alpha-\beta} \int_{\tau-h}^{\tau} s^{\beta-2} ds\, d\tau = \text{const} \int_h^t \tau^{\alpha-\beta} [(\tau - h)^{\beta-1} - \tau^{\beta-1}] d\tau.$$

Further estimation of J_1 and J_3 can be performed exactly as in [11]. As a result, we get $\|J_1\|_X, \|J_3\|_X = O(h^\alpha)$. This completes the proof. □

Proof of Lemma 5. Firstly, we point out that the assumption $w \in W_p^2(\Omega)$, $p > n$ implies $w \in C^1(\overline{\Omega})$. We will use maximum principles for elliptic equations in Sobolev spaces to prove the lemma. Let us consider the case $x^* \in \Omega$. Suppose that $L_1 w(x^*) < 0$. Then there exists a ball $B(x^*, \varepsilon) \subset \Omega$ and $\delta > 0$ such that $L_1 w(x) \leq -\delta < 0$ for $x \in B(x^*, \varepsilon)$. Let us define the auxiliary function

$$z(x) = \alpha |x - x^*|^2 \text{ with } \alpha > 0 \tag{7.1}$$

such that $L_1(w + z) \leq 0$ in $B(x^*, \varepsilon)$. Since $w(x^*) \leq w(x)$ and $z(x^*) < z(x)$ for $x \in \partial B(x^*, \varepsilon)$, we get

$$(w + z)(x^*) < (w + z)(x), \quad x \in \partial B(x^*, \varepsilon). \tag{7.2}$$

On the other hand, due to $L_1(w+z) \leq 0$ it follows from the Theorem 9.1 [8] that $\min_{x \in B(x^*, \varepsilon)} (w + z)(x) = \min_{x \in \partial B(x^*, \varepsilon)} (w + z)(x)$, that contradicts (7.2). Therefore, the supposition $L_1 w(x^*) < 0$ was wrong.

Next let us consider the case $x^* \in \partial\Omega$. Again, suppose $L_1 w(x^*) < 0$. Then there exists $B(x^*, \varepsilon)$ and $\delta > 0$ such that $L_1 w(x) \leq -\delta < 0$ for $x \in B(x^*, \varepsilon) \bigcap \Omega$. Similarly to the previous case we define z by (7.1) so that $L_1(w + z) \leq 0$ in $B(x^*, \varepsilon) \bigcap \Omega$. Then $(w + z)(x^*) < (w + z)(x)$ for $x \in B(x^*, \varepsilon) \bigcap \Omega$. Hence, Lemma 3.4 [8] is applicable and yields $\frac{\partial w}{\partial \nu}(x^*) = \frac{\partial (w+z)}{\partial \nu}(x^*) < 0$. That contradicts to $\frac{\partial}{\partial \nu} w(x^*) \geq 0$ following from the assumption $\frac{\partial}{\partial \omega} w(x^*) \geq 0$. Therefore, $L_1 w(x^*) \geq 0$ holds. $\square$

Appendix 3

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An Inverse Problem for a Generalized Fractional Derivative with an Application in Reconstruction of Time- and Space-Dependent Sources in Fractional Diffusion and Wave Equations

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Abstract: In this article, we consider two inverse problems with a generalized fractional derivative. The first problem, IP1, is to reconstruct the function u based on its value and the value of its fractional derivative in the neighborhood of the final time. We prove the uniqueness of the solution to this problem. Afterwards, we investigate the IP2, which is to reconstruct a source term in an equation that generalizes fractional diffusion and wave equations, given measurements in a neighborhood of final time. The source to be determined depends on time and all space variables. The uniqueness is proved based on the results for IP1. Finally, we derive the explicit solution formulas to the IP1 and IP2 for some particular cases of the generalized fractional derivative.

Keywords: inverse problem; source reconstruction; final overdetermination; subdiffusion; tempered subdiffusion; fractional wave equation; generalized fractional derivative; Atangana–Baleanu derivative

MSC: 35R30; 35R11

1. Introduction

Fractional derivatives are increasingly used in modeling various processes in physics, biology, economics, engineering sciences, etc. [1]. In addition to classical fractional derivatives, several generalizations have been introduced to better match the models to the reality in different situations. In this paper, we work with generalized fractional derivatives of Riemann–Liouville and Caputo type where the power-type kernel (fractional derivative case) is replaced by an arbitrary function k. Such a generalization was previously used in [2–5] and covers many specific cases that are important in applications (see Section 2.1).

Fractional derivatives of Riemann–Liouville and Caputo type are non-local: the derivative of a function $u(t)$ at $t = T$ depends on values of u at $t < T$. We consider an inverse problem (IP1) to recover a history of a function u at $0 < t < T$ by means of measurements of $u(t)$ and its generalized fractional derivative in a left neighborhood of T. To the authors' knowledge, such a problem has not yet been considered in the literature.

We use the results obtained for IP1 in order to investigate an inverse problem of reconstruction of a history of a source in a general PDE that includes as particular cases fractional diffusion and wave equations from the measurements in a left neighborhood of final time T (IP2).

Quite often in the inverse source problem, the goal is to determine a source that is either a space- or time-dependent function. The space-dependent source term is usually reconstructed based on the final time overdetermination condition [6–11]. The time-dependent source term can be recovered from additional boundary measurements [7] or from integral conditions [12,13]. In this paper [14], the source term dependent on time and part of the space variables has been determined. In this paper, we assume that the overdetermination condition is given not only at the final moment of time T, but in its neighborhood. This enables us to reconstruct the source term that depends on both time and all space variables.

In Section 2, we explain the concept of generalized fractional derivative with examples. Next, we formulate the inverse problems and give hints to their physical applications. In Section 3, we prove the uniqueness for a general class of kernels k and reduce IP1 to an integral equation that is further used to derive the solution formulas. Finally, in Section 4, we derive the solution formulas in some particular cases of k based on the expansion with the Legendre polynomials.

2. Problem Formulation

2.1. Generalized Fractional Derivatives

In this paper, $L_p(0,T)$ and $W_p^n(0,T)$ stand for real Lebesgue and Sobolev spaces.

We are solving problems with a generalized fractional derivative. This concept has been used in [2–5]. We utilize $D_a^{\{k\},n}$ as a *unified notation* that stands for the generalized fractional derivatives in Riemann–Liouville ${}^R D_a^{\{k\},n}$ and Caputo sense ${}^C D_a^{\{k\},n}$:

$$({}^R D_a^{\{k\},n} v)(t) = \frac{d^n}{dt^n}\int_a^t k(t-\tau)v(\tau)d\tau,\ ({}^C D_a^{\{k\},n} v)(t) = \int_a^t k(t-\tau)v^{(n)}(\tau)d\tau,$$
$$t > a,\ n \in \{0\}\cup\mathbb{N},\ k \in L_{1,loc}(0,\infty).$$

The notation of generalized fractional derivative incorporates the following possibilities.

The basic case is

(k1) $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$. Then, ${}^R D_a^{\{k\},n}$ and ${}^C D_a^{\{k\},n}$ are the Riemann–Liouville and Caputo fractional derivatives of the order $n+\beta-1$, i.e.,

$$({}^R D_a^{\{k\},n} v)(t) = ({}^R D_a^{n+\beta-1} v)(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}v(\tau)d\tau,$$
$$({}^C D_a^{\{k\},n} v)(t) = ({}^C D_a^{n+\beta-1} v)(t) = \int_a^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}v^{(n)}(\tau)d\tau.$$

Moreover, in case $k(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, ${}^R D_a^{\{k\},0}$ is the Riemann–Liouville fractional integral of the order $\beta > 0$, i.e.,

$$({}^R D_a^{\{k\},0} v)(t) = (I_a^\beta v)(t) = \int_a^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}v(\tau)d\tau.$$

Often a memory is not of power-type. A direct generalization of (k1) leads to **multiterm and distributed order fractional derivatives** [15–17]. These derivatives have the following kernels:

(k2) $k(t) = \sum_{j=1}^m p_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $\beta_j \in (0,1)$, $p_j \neq 0$, and

(k3) $k(t) = \int_0^1 p(\beta)\frac{t^{-\beta}}{\Gamma(1-\beta)}d\beta$, $p \in L_1(0,1)$, respectively.

Distributed order and multiterm derivatives enable to model accelerating and retarding sub(super) diffusion, since different powers of t dominate as $t \to 0^+$ and $t \to \infty$ in the kernel. A proper choice of p in (k3) allows modelling ultraslow diffusion [16].

The cases (k2) and (k3) can be unified to a form of Lebesque–Stiltjes integral $k(t) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)}d\mu(\beta)$, but we will treat them separately.

Tempered fractional derivatives are used to describe slow transition of anomalous diffusion to a normal one. There are two models of this type in the literature that differ in their mathematical

derivations. The corresponding kernels are:
(k4) $k(t) = \frac{e^{-\lambda t}t^{-\beta}}{\Gamma(1-\beta)} + \lambda \int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)}d\tau$, $0 < \beta < 1$, $\lambda > 0$ [18,19]; and
(k5) $k(t) = e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)$, $0 < \beta < 1$, $\lambda > 0$ [19,20].
We will call derivatives with kernels (k4) and (k5) tempered fractional derivatives of type I and II, respectively.

Removing the singularity of kernels at $t = 0$ allows to highlight memory effects better [21]. In this paper, we consider the following bounded kernels:
(k6) $k(t) = \frac{1}{1-\beta}e^{-\frac{\beta}{1-\beta}t}$, $0 < \beta < 1$ is the kernel of Caputo-Fabrizio derivative [21,22];
(k7) $k(t) = \frac{1}{1-\beta}E_\beta\left(-\frac{\beta t^\beta}{1-\beta}\right)$, $0 < \beta < 1$ is a kernel of Atangana–Baleanu fractional derivative [23,24].
Here, E_β and $E_{\beta,\beta}$ are one-parametric and two-parametric Mittag-Leffler functions, respectively, given by the formulas:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \operatorname{Re}\alpha > 0,$$
$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0.$$

2.2. Formulation of Inverse Problems

Let $0 < t_0 < T < \infty$. Our basic inverse problem consists in a reconstruction of a function in $(0, t_0)$ provided that this function and its derivative are given in (t_0, T).

IP1. Given $\varphi, g : (t_0, T) \to \mathbb{R}$, find $u : (0, T) \to \mathbb{R}$ such that

$$u|_{(t_0,T)} = \varphi \quad \text{and} \quad D_0^{\{k\},n}u|_{(t_0,T)} = g\,. \tag{1}$$

An example of IP1 is the reconstruction of physical quantities in constitutive relations involving fractional derivatives. In the Scott–Blair model of viscoelasticity, the stress is proportional to a time fractional derivative of the strain [25]. In this context, IP1 means the reconstruction of a history of the strain of a body by means of the measurement of strain and stress in a left neighborhood of a time value T. A similar meaning for IP1 can be given in the subdiffusion where the flux is proportional to a time fractional derivative of the concentration (temperature) gradient [26].

Next, we formulate IP2 that is an inverse source problem that can be reduced to IP1:

IP2. Given $\varphi, \Phi : \Omega \times (t_0, T) \to \mathbb{R}$, find $u, F : \Omega \times (0, T) \to \mathbb{R}$, such that

$$(D_0^{\{k\},n}Bu)(x,t) + D^l u(x,t) - Au(x,t) = F(x,t), \quad x \in \Omega,\ t \in (0,T) \tag{2}$$

is fulfilled and

$$u|_{\Omega\times(t_0,T)} = \varphi, \quad F|_{\Omega\times(t_0,T)} = \Phi.$$

Here, $\Omega \subseteq \mathbb{R}^N$ with some $N \in \mathbb{N}$, $D^l = \sum_{j=1}^{l} q_j \frac{\partial^j}{\partial t^j}$ with some $l \in \mathbb{N}$, $q_j \in \mathbb{R}$, and A and B are operators that act on functions depending on x. Throughout the paper, assume that A and B with their domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are such that $A : \mathcal{D}(A) \subseteq C(\Omega) \to C(\Omega)$, $B : \mathcal{D}(B) \subseteq C(\Omega) \to C(\Omega)$. We also assume that B is invertible.

Equation (2) generalizes the fractional wave equation ${}^C D_0^\beta u + \lambda(-\Delta)^\alpha u = F$, $\beta \in (1,2)$, $\alpha \in [0.5, 1]$, $\lambda > 0$ [13,27,28], the attenuated wave equation $\frac{\partial^2}{\partial t^2}u + \mu\, {}^R D_0^\beta u - \lambda\Delta u = F$, $\beta \in (0,1) \cup (1,2)$ [29,30] and different subdiffusion equations ${}^C D_0^{\{k\},1}u - \lambda\Delta u = F$ and $\frac{\partial}{\partial t}u - \lambda\, {}^R D_0^{\{k\},1}\Delta u = F$, where k has one of the above forms (k1)–(k7) [16–18,20,23,26,31]. In the latter equation, $B = -\lambda\Delta$ and, in order to guarantee the invertibility of B, proper boundary conditions must be specified in the domain $\mathcal{D}(B)$.

We point out that the operators A and B in (2) are not necessarily linear.

In case if $\Phi = 0$, IP2 means a reconstruction of a source that was active in the past using a measurement of the state of u in a left neighborhood of T. Such an inverse problem may occur in seismology, ground water pollution, etc.

Now, we reduce IP2 to IP1. Let (u, F) solve IP2. Then, Equation (2) restricted to $\Omega \times (t_0, T)$ has the form $(D_0^{\{k\},n} Bu)(x,t) + D^l \varphi(x,t) - A\varphi(x,t) = \Phi(x,t)$. Therefore, Bu is a solution of the following family of IP1:

$$Bu|_{\Omega\times(t_0,T)} = B\varphi \quad \text{and} \quad D_0^{\{k\},n} Bu|_{\Omega\times(t_0,T)} = g, \tag{3}$$

where

$$g(x,t) = \Phi(x,t) + A\varphi(x,t) - D^l\varphi(x,t), \quad x \in \Omega,\ t \in (t_0, T). \tag{4}$$

The solution of IP2 is expressed by means of Bu explicitly: $u = B^{-1}Bu$, $F = D_0^{\{k\},n} Bu + D^l u - Au$.

3. Results in Case of General k

3.1. Uniqueness Results

Lemma 1. *Let k be real analytic in $(0,\infty)$ and $v \in L_1(0, t_0)$. Then, $w(t) = \int_0^{t_0} k(t-\tau)v(\tau)d\tau$ is real analytic in (t_0,∞).*

Proof. The function k can be extended as a complex analytic function $k_{\mathbb{C}}$ in an open domain $D \subset \mathbb{C}$ containing the positive part of the real axis. Let us define $w_{\mathbb{C}}(z) = \int_0^{t_0} k_{\mathbb{C}}(z-\tau)v(\tau)d\tau$ for $z \in D_{t_0} = \{z : z = \xi + t_0,\ \xi \in D\}$. Using the analyticity of $k_{\mathbb{C}}$, it is possible to show that functions u and v involved in the formula $w_{\mathbb{C}}(t + is) = u(t,s) + iv(t,s)$, are continuously differentiable and satisfy Cauchy-Riemann equations in $\{(t,s) : t + is \in D_{t_0}\}$. This implies that $w_{\mathbb{C}}$ is complex analytic in D_{t_0}. On the other hand, its restriction to the subset $\{z = t + i0 : t \in (t_0,\infty)\}$ is the function w. Therefore, w is real analytic in (t_0,∞). □

We will denote the Laplace transform of a function $f : (0,\infty) \to \mathbb{R}$ by

$$\widehat{f}(s) = (\mathcal{L}_{t\to s} f)(s) = \int_0^\infty e^{-st} f(t)dt.$$

The symbol $*$ will stand for the time convolution, i.e., $(f_1 * f_2)(t) = \int_0^t f_1(t-\tau)f_2(\tau)d\tau$.

We prove a uniqueness theorem for IP1.

Theorem 1. *Assume that k satisfies the following conditions:*

$$\exists \mu \in \mathbb{R} : \int_0^\infty e^{-\mu t}|k(t)|dt < \infty, \tag{5}$$

$$k \text{ is real analytic in } (0,\infty), \tag{6}$$

$$\widehat{k}(s) \text{ cannot be meromorphically extended to the whole complex plane } \mathbb{C}. \tag{7}$$

Then, the following assertions hold.

(i) *If $u \in L_1(0,T)$, $k * u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^R D_0^{\{k\},n} u|_{(t_0,T)} = 0$, then $u = 0$.*

(ii) *If $u \in W_1^n(0,T)$ and $u|_{(t_0,T)} = {}^C D_0^{\{k\},n} u|_{(t_0,T)} = 0$, then $u = 0$.*

Proof. (i) Let us extend $u(t)$ by zero for $t > T$ and define the function $f : (0, \infty) \to \mathbb{R}$:

$$f = {}^R D_0^{\{k\},n} u.$$

Since $u(t) = 0, t > t_0$, it holds that

$$f(t) = \frac{d^n}{dt^n} \int_0^{t_0} k(t-\tau)u(\tau)d\tau = \int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau, \quad t > t_0.$$

The function k is real analytic, therefore, $k^{(n)}$ is also real analytic. Hence, Lemma 1 implies that f is real analytic in (t_0, ∞). Since $f(t) = 0,\ t \in (t_0, T)$, and f is real analytic, we obtain that $f(t) = 0,\ t > t_0$.

Due to (5) the $\widehat{k}(s)$ exists and is holomorphic for $\text{Re}s > \mu$. Moreover, in view the properties of f, the $\widehat{f}(s)$ also exists and is expressed by the formula

$$\widehat{f}(s) = s^n \widehat{k}(s)\widehat{u}(s) - p_0 s^{n-1} - \ldots - p_{n-1}, \quad p_j = \frac{d^j}{dt^j}(k * u)(t)\Big|_{t=0}, \quad \text{Re}s > \mu.$$

Therefore,

$$\widehat{k}(s) = \frac{\widehat{f}(s) + p_0 s^{n-1} + \ldots + p_{n-1}}{s^n \widehat{u}(s)} \quad \text{for any } s \text{ such that } \text{Re}s > \mu \text{ and } s^n\widehat{u}(s) \neq 0.$$

Since the values $f(t)$ and $u(t)$ vanish for $t > t_0$, $\widehat{f}$ and $\widehat{u}$ are entire functions. Thus, the function $\widehat{f}(s) + p_0 s^{n-1} + \ldots + p_{n-1}$ is also entire. Assume that $\widehat{u}$ does not vanish on $\mathbb{C}$. Then, by Identity theorem and the fact that $\widehat{u}$ is entire the set of zeros of $\widehat{u}$ does not contain accumulation points. This implies that the extension of $\widehat{k}$ is meromorphic on $\mathbb{C}$. This contradicts to the assumption (7) of the theorem. Therefore, the assumption $\widehat{u} \not\equiv 0$ is invalid, which implies $u = 0$ in $L_1(0, T)$.

(ii) At this part of the proof, let us use the notation $v := u^{(n)}$. Then, $v|_{(t_0,T)} = {}^R D_0^{\{k\},0} v|_{(t_0,T)} = 0$ and $v,\ k * v \in L_1(0, T)$. Therefore, by the assertion (i) of this theorem $v = 0$. Consequently, $u^{(n)} = 0$ and $u|_{(t_0,T)} = 0$ imply that $u = 0$ in $W_1^n(0, T)$. □

Let us compute the Laplace transform for the kernels from Section 1 to see if they satisfy the conditions of Theorem 1.

(k1) In the basic case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0, 1)$, it holds $\widehat{k}(s) = \frac{1}{s^{1-\beta}}$.

(k2) Similarly for $k(t) = \sum_{j=1}^m p_j \frac{t^{-\beta_j}}{\Gamma(1-\beta_j)}$, $0 < \beta_j < 1$, $p_j \neq 0$, we have $\widehat{k}(s) = \sum_{j=1}^m p_j \frac{1}{s^{1-\beta_j}}$.

(k3) For the distributed fractional derivative $k(t) = \int_0^1 p(\beta)\frac{t^{-\beta}}{\Gamma(1-\beta)}d\beta,\ p \in L_1(0, 1)$, the Laplace transform is $\widehat{k}(s) = \int_0^1 p(\beta)\frac{1}{s^{1-\beta}}d\beta$.

(k4) For the tempered fractional derivative of type I $k(t) = \frac{e^{-\lambda t}t^{-\beta}}{\Gamma(1-\beta)} + \lambda \int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)}d\tau,\ 0 < \beta < 1,\ \lambda > 0$, it holds $\widehat{k}(s) = \frac{(s+\lambda)^\beta}{s}$.

(k5) For the tempered fractional derivative of type II $k(t) = e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)$, $0 < \beta < 1$, $\lambda > 0$, we have that $\widehat{k}(s) = \frac{1}{(s+\lambda)^\beta - \lambda^\beta}$ [19].

(k6) The kernel of Caputo-Fabrizio fractional derivative $k(t) = \frac{1}{1-\beta}e^{-\frac{\beta}{1-\beta}t}$, $0 < \beta < 1$, has a Laplace transform $\widehat{k}(s) = \frac{1}{(1-\beta)s+\beta}$.

(k7) In case of Atangana–Baleanu fractional derivative $k(t) = \frac{1}{1-\beta}E_\beta\left(-\frac{\beta t^\beta}{1-\beta}\right)$, $0 < \beta < 1$, it follows from [32] that $\widehat{k}(s) = \frac{s^{\beta-1}}{(1-\beta)s^\beta+\beta}$.

The kernels (k1)–(k7) satisfy (5),(6). Moreover, it is evident that the kernels (k1), (k2), (k4), (k5), (k7) satisfy (7), because Laplace transforms of these functions have branch points. To guarantee that (k3) also satisfies (7) we assume additionally that $p \neq 0$, $p \geq 0$. Then,

$$\lim_{\substack{\mathrm{Arg}\, s \to \pm\pi, \\ |s|=1}} \mathrm{Im}\, \widehat{k}(s) = \int_0^1 p(\beta) \sin((\beta - 1) \times (\pm\pi)) d\beta \underset{>}{<} 0.$$

This shows that $\widehat{k}(s)$ has a jump at $s = -1$, hence (7) holds.

Summing up, the solution of IP1 for a derivative containing a kernel (k1)–(k5) or (k7) is unique.

The kernel of Caputo-Fabrizio fractional derivative (k6) does not satisfy (7) because it has the meromorphic in $\mathbb{C}$ Laplace transform. IP1 with this kernel has infinitely many solutions. Any function such that $\int_0^{t_0} e^{\frac{\beta}{1-\beta}\tau} u(\tau) d\tau = 0$, $u|_{(t_0,T)} = 0$ satisfies the homogeneous IP1 in case $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$ and any function such that $\int_0^{t_0} e^{\frac{\beta}{1-\beta}\tau} u^{(n)}(\tau) d\tau = 0$, $u|_{(t_0,T)} = 0$ satisfies the homogeneous IP1 in case $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$.

Now, we proceed to IP2. We define the following set related to operators A, B and D^l:

$$\mathcal{U} = \{u : \Omega \times (0,T) \to \mathbb{R} : u(\cdot,t) \in \mathcal{D}(A) \cap \mathcal{D}(B)\ \forall t \in (0,T),$$
$$u, Au, Bu \in C(\Omega \times (0,T)) \text{ and } q_j \frac{\partial^j}{\partial t^j} u \in C(\Omega \times (0,T)),\ j = 1, \ldots, l\}.$$

From Theorem 1, we can immediately deduce a uniqueness statement for IP2.

Corollary 1. *Let k satisfy (5)–(7). Then, the following assertions hold.*

(i) *If $(u_j, F_j) \in \{u \in \mathcal{U} : (k * Bu)(x,\cdot) \in W_1^n(0,T)\ \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, solve (2) with $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$ and $(u_1, F_1)|_{\Omega \times (t_0,T)} = (u_2, F_2)|_{\Omega \times (t_0,T)}$, then $(u_1, F_1) = (u_2, F_2)$.*

(ii) *If $(u_j, F_j) \in \{u \in \mathcal{U} : Bu(x,\cdot) \in W_1^n(0,T)\ \forall x \in \Omega\} \times C(\Omega \times (0,T))$, $j = 1,2$, solve (2) with $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$ and $(u_1, F_1)|_{\Omega \times (t_0,T)} = (u_2, F_2)|_{\Omega \times (t_0,T)}$, then $(u_1, F_1) = (u_2, F_2)$.*

Proof. Proof is technically the same in cases (i) and (ii). After considering the formulation of IP2 in terms of IP1 (3) and subtracting the corresponding equations for (u_1, F_1) and (u_2, F_2), we obtain that

$$(Bu_1 - Bu_2)|_{\Omega \times (t_0,T)} = 0 \quad \text{and} \quad D_0^{\{k\},n}(Bu_1 - Bu_2)|_{\Omega \times (t_0,T)} = 0.$$

Then, it follows from Theorem 1 that $(Bu_1 - Bu_2)|_{\Omega \times (0,T)} = 0$ and, consequently, since the operator B is invertible it holds $u_1(x,t) = u_2(x,t)$, $(x,t) \in \Omega \times (0,T)$. Finally, the Equation (2) implies $F_1(x,t) = F_2(x,t)$, $(x,t) \in \Omega \times (0,T)$. □

3.2. Reduction to Integral Equations

In this subsection, we reduce IP1 to integral equations. Let us assume that k satisfies (6).

Firstly, we consider the case $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$. Assume that $u \in L_1(0,T)$ solves IP1 and $k * u \in W_1^n(0,T)$. Then,

$$\int_0^t k(t-\tau) u(\tau) d\tau = \int_0^{t_0} k(t-\tau) u(\tau) d\tau + \int_{t_0}^t k(t-\tau) \varphi(\tau) d\tau \tag{8}$$

for $t \in (t_0, T)$, where the left hand side belongs to $W_1^n(t_0,T)$ and the first addend in the right-hand side belongs to $C^\infty(t_0, T]$. Thus, the data φ necessarily satisfies $\int_{t_0}^t k(t-\tau)\varphi(\tau) d\tau \in W_1^n(t_0 + \delta, T)$,

$\forall\delta \in (t_0, T)$. Applying $\frac{d^n}{dt^n}$ to (8), using the second condition in (1) and rearranging the terms, we obtain the following integral equation of the first kind for $u|_{(0,t_0)}$:

$$\int_0^{t_0} k^{(n)}(t-\tau)u(\tau)d\tau = f(t), \;\; t \in (t_0, T), \quad \text{where} \quad f = g - {}^{R}D_{t_0}^{\{k\},n}\varphi. \tag{9}$$

Secondly, let us consider the case $D_0^{\{k\},n} = {}^{C}D_0^{\{k\},n}$, $n \geq 1$. If $u \in W_1^n(0,T)$ solves IP1, then $u^{(n)}|_{(0,t_0)}$ is a solution of the integral equation

$$\int_0^{t_0} k(t-\tau)u^{(n)}(\tau)d\tau = f(t), \;\; t \in (t_0, T), \quad \text{where} \quad f = g - {}^{C}D_{t_0}^{\{k\},n}\varphi. \tag{10}$$

Since $\lim\limits_{\tau\to t_0^-} u^{(j)}(\tau) = \lim\limits_{\tau\to t_0^+} \varphi^{(j)}(\tau)$, $j = 0, \ldots, n-1$, the function $u|_{(0,t_0)}$ is obtained from $u^{(n)}|_{(0,t_0)}$ by the integration:

$$u(t) = \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{(n-1)!} u^{(n)}(\tau)d\tau + \sum_{j=0}^{n-1} \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!}, \quad t \in (0, t_0).$$

Due to Lemma 1, the integral operators involved in (9),(10) map $L_1(0, t_0)$ into the space of functions that are real analytic in $t > t_0$. This means that IP1 is severely ill-posed and necessarily, f is real analytic in (t_0, T). In the next section, we will derive solution formulas for IP1 that contain the quantities

$$f^{(m)}(t_1), \quad m \in \{0\} \cup \mathbb{N},$$

where t_1 is an arbitrary point in (t_0, T).

4. Solution Formulas in Particular Cases of k

4.1. A Basic Theorem

Theorem 2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, $t_1 > t_0 > 0$ and $f \in C^\infty(t_0, \infty)$. Let us introduce the following family of sums that depend on a variable $t \in (0, t_0)$ and parameters α, f, t_1, t_0:*

$$V_N(\alpha, f, t_1, t_0)(t) = (t_1 - t)^{-\alpha-2} \sum_{n=0}^{N} A_n P_n\left(\frac{2t_1(t_1-t_0)}{t_0(t_1-t)} - \frac{2t_1 - t_0}{t_0}\right).$$

Here, $N \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, P_n are normalized in $L_2(-1,1)$ Legendre polynomials

$$P_n(t) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} t^{n-2l}, \quad \text{where } c_{n,l} = \sqrt{\frac{2n+1}{2}}\frac{1}{2^n}(-1)^l \binom{n}{l}\binom{2n-2l}{n},$$

and

$$A_n = A_n(\alpha, f, t_1, t_0) = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m}\left(\frac{t_0 - 2t_1}{t_0}\right)^{n-2l-m}$$
$$\times \left(\frac{2t_1(t_1-t_0)}{t_0}\right)^m \Gamma(\alpha - m + 1) f^{(m)}(t_1).$$

Assume that $v \in L_2(0, t_0)$ and f is given by $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau)d\tau$, $t > t_0$. Then, the series $V_\infty(\alpha, f, t_1, t_0)(t)$ converges almost everywhere in $(0, t_0)$ and

$$v(t) = V_\infty(\alpha, f, t_1, t_0)(t), \quad a.e. \;\; t \in (0, t_0). \tag{11}$$

Moreover, $V_N(\alpha, f, t_1, t_0) \to v$ in $L_2(0, t_0)$ as $N \to \infty$. If in addition, $v \in BV[0, t_0]$, then $V_\infty(\alpha, f, t_1, t_0)(t)$ converges pointwise in $(0, t_0)$ and the estimate is valid:

$$|v(t) - V_N(\alpha, f, t_1, t_0)(t)| \leq \frac{c(t)}{N}, \quad t \in (0, t_0),$$

where $c(t)$ is a positive constant depending on t.

Proof. For $t_1 > t_0$ we have

$$\frac{1}{\Gamma(\alpha - n + 1)} \int_0^{t_0} (t_1 - \tau)^{\alpha - n} v(\tau) d\tau = f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N}. \tag{12}$$

The substitution $s = \frac{1}{t_1 - \tau}$ under the integral takes (12) to the form

$$\int_{\frac{1}{t_1}}^{\frac{1}{t_1 - t_0}} s^n w(s) ds = \Gamma(\alpha - n + 1) f^{(n)}(t_1), \quad n \in \{0\} \cup \mathbb{N}, \tag{13}$$

where $w(s) = s^{-\alpha-2} v\left(t_1 - \frac{1}{s}\right)$.

We would like to expand our function into series by means of orthonormal Legendre polynomials; thus, we apply a linear substitution that takes us from $[\frac{1}{t_1}, \frac{1}{t_1 - t_0}]$ to the interval $[-1, 1]$, where such an expansion can be applied:

$$\tilde{s} = as + b, \text{ where } a = \frac{2t_1(t_1 - t_0)}{t_0}, \quad b = -\frac{2t_1 - t_0}{t_0}.$$

We also denote $\tilde{w}(\tilde{s}) = w(s)$. Since the performed changes of variables under the integrals are diffeomorphic, $v \in L_2(0, t_0)$ implies $w \in L_2(\frac{1}{t_1}, \frac{1}{t_1 - t_0})$ and $\tilde{w} \in L_2(-1, 1)$ (cf. [33] Section 16.4). Similarly, $v \in BV[0, t_0]$ implies $\tilde{w} \in BV[-1, 1]$.

Since $\tilde{w} \in L_2(-1, 1)$, it can be expanded into the Fourier-Legendre series. It follows from (13) that for $n \in \{0\} \cup \mathbb{N}$

$$\int_{-1}^{1} \frac{1}{a^{n+1}} (\tilde{s} - b)^n \tilde{w}(\tilde{s}) d\tilde{s} = \Gamma(\alpha - n + 1) f^{(n)}(t_1)$$

and, therefore,

$$\int_{-1}^{1} \tilde{s}^n \tilde{w}(\tilde{s}) d\tilde{s} = \int_{-1}^{1} ((\tilde{s} - b) + b)^n \tilde{w}(\tilde{s}) d\tilde{s}$$
$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} \int_{-1}^{1} (\tilde{s} - b)^m \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^m \Gamma(\alpha - m + 1) f^{(m)}(t_1).$$

It implies that for the normalized Legendre polynomials

$$\int_{-1}^{1} P_n(\tilde{s}) \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \int_{-1}^{1} \tilde{s}^{n-2l} \tilde{w}(\tilde{s}) d\tilde{s} = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,l} \sum_{m=0}^{n-2l} \binom{n-2l}{m}$$
$$\times b^{n-2l-m} a^m \Gamma(\alpha - m + 1) f^{(m)}(t_1) = A_n.$$

Then, $\tilde{w}(\tilde{s}) = \sum_{n=0}^{\infty} A_n P_n(\tilde{s})$. This series converges in $L_2(-1, 1)$ and for almost every $\tilde{s} \in (-1, 1)$ [34].

For $\tilde{w} \in BV[-1,1]$, the series for $\tilde{w}$ is convergent pointwise for $\tilde{s} \in (-1,1)$ and according to Theorem 1 [35]

$$|\tilde{w}(\tilde{s}) - \sum_{n=0}^{N} A_n P_n(\tilde{s})| \leq \frac{c_1(\tilde{s})}{N}, \quad \tilde{s} \in (-1,1),$$

where $c_1(\tilde{s})$ is a positive constant.

Since the change of variables $\tilde{s} = \frac{a}{t_1 - t} + b$, $t \in [0, t_0]$, is diffeomorphic and $v(t) = (t_1 - t)^{-\alpha-2}\tilde{w}(\frac{a}{t_1-t} + b)$, all assertions of the theorem follow from the proved properties of the series $\tilde{w}(\tilde{s}) = \sum_{n=0}^{\infty} A_n P_n(\tilde{s})$. □

Remark 1. *It follows from* (11) *that for f of form $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$, $t > t_0$, where $v \in L_2(0, t_0)$, the sum of series $V_\infty(\alpha, f, t_1, t_0)(t)$ is independent of $t_1 > t_0$. The partial sums $V_N(\alpha, f, t_1, t_0)(t)$, $N < \infty$, however, still may depend on t_1 in case of such f. For example, if $v = 1$, then $V_0(\alpha, f, t_1, t_0)(t) = \frac{\sqrt{0.5}}{\alpha+1}(t_1 - t)^{-\alpha-2}\left[t_1^{\alpha+1} - (t_1 - t_0)^{\alpha+1}\right]$.*

4.2. Solution Formulas in Case of Usual Fractional Derivatives

In this subsection, we consider the case $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $\beta \in (0,1)$, $n \geq 1$. Then, ${}^R D_0^{\{k\},n}$ and ${}^C D_0^{\{k\},n}$ are the Riemann–Liouville and Caputo fractional derivatives of the order $n + \beta - 1$, respectively.

Theorem 3. *Let $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $0 < \beta < 1$. Then, the following assertions hold.*

(i) *If $u \in L_2(0,T)$, $k * u \in W_1^n(0,T)$ and u solves* IP1 *with $D_0^{\{k\},n} = {}^R D_0^{\{k\},n}$, then*

$$u(t) = \mathcal{F}_{R,t_1}^{\beta,n}(g - {}^R D_{t_0}^{\{k\},n} \varphi)(t), \quad a.e.\ t \in (0, t_0), \tag{14}$$

where the operator $\mathcal{F}_{R,t_1}^{\beta,n}$ is given by the rule

$$\mathcal{F}_{R,t_1}^{\beta,n}(f)(t) = V_\infty(-n - \beta, f, t_1, t_0)(t). \tag{15}$$

(ii) *If $u \in W_2^n(0,T)$, $n \geq 1$, solves* IP1 *with $D_0^{\{k\},n} = {}^C D_0^{\{k\},n}$, then*

$$u(t) = \mathcal{F}_{C,t_1}^{\beta,n}(\varphi; g - {}^C D_{t_0}^{\{k\},n} \varphi)(t), \quad t \in (0, t_0), \tag{16}$$

where

$$\mathcal{F}_{C,t_1}^{\beta,n}(\varphi; f)(t) = \sum_{j=0}^{n-1} \lim_{\tau \to t_0^+} \varphi^{(j)}(\tau) \frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{\Gamma(n)} V_\infty(-\beta, f, t_1, t_0)(\tau) d\tau. \tag{17}$$

The Formulas (14), (16) *are valid for any $t_1 \in (t_0, T)$.*

Proof. (i) Firstly, we represent the IP1 in form (9) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. That is identical to $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$ with $\alpha = -n - \beta$, $v(t) = u(t)$ and $f(t) = g(t) - {}^R D_{t_0} \varphi(t)$ and Theorem 2 implies (14).

(ii) Similarly to the previous case we start from representing the problem in a form (10) with $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$. This gives us the relation $f(t) = \int_0^{t_0} \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} v(\tau) d\tau$ with $\alpha = -\beta$, $v(t) = u^{(n)}(t)$ and $f(t) = g(t) - {}^C D_{t_0}^{\beta,n} \varphi^{(n)}(t)$. By applying Theorem 2 to it, we obtain

$$u^{(n)}(t) = V_\infty(-\beta, f, t_1, t_0)(t), \ \ a.e.\ t \in (0, t_0), \quad f = g - {}^C D_{t_0}^{\{k\},n} u.$$

Since the condition $u|_{(t_0,T)} = \varphi$ implies $u^{(j)}(t_0) = \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)$, $\quad j = 0\dots n-1$, the solution Formula (16) is valid. □

Remark 2. *Let us consider the approximations of the exact solutions defined by* $u_{N,t_1}(t) = V_N(-n-\beta, f, t_1, t_0)(t)$, $t \in (0,t_0)$, $N < \infty$, *in case (i) and* $u_{N,t_1}(t) = \sum_{j=0}^{n-1} \lim_{\tau\to t_0^+} \varphi^{(j)}(\tau)\frac{(t-t_0)^j}{j!} + \int_{t_0}^{t} \frac{(t-\tau)^{n-1}}{\Gamma(n)} V_N(-\beta, f, t_1, t_0)(\tau)d\tau$, $t \in (0,t_0)$, $N < \infty$, *in case (ii). Then, Theorem 2 can be used to compare* u_{N,t_1} *with* u *in the process* $N \to \infty$. *In case (i),* $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ *in* $L_2(0,t_0)$ *and* $u_{N,t_1}(t) \to u(t)$ *a.e.* $t \in (0,t_0)$. *Similarly, in case (ii),* $u_{N,t_1}|_{(0,t_0)} \to u|_{(0,t_0)}$ *in* $W_2^n(0,t_0)$ *and* $u_{N,t_1}^{(n)}(t) \to u^{(n)}(t)$ *a.e.* $t \in (0,t_0)$. *If in addition to the assumptions of (i),* $u|_{(0,t_0)} \in BV[0,t_0]$ *holds, then* $|u_{N,t_1}(t) - u(t)|$ *is of the order* $1/N$ *for every* $t \in (0,t_0)$. *Similarly, if in addition to the assumptions of (ii),* $u^{(n)}|_{(0,t_0)} \in BV[0,t_0]$ *is valid, then* $|u_{N,t_1}^{(n)}(t) - u^{(n)}(t)|$ *is of the order* $1/N$ *for every* $t \in (0,t_0)$.

Corollary 2. *Let* $k(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $0 < \beta < 1$. *Then, the following assertions hold.*

(i) *If* $(u,F) \in \{u \in \mathcal{U} : (k * Bu)(x,\cdot) \in W_1^n(0,T)\, \forall x \in \Omega\} \times C(\Omega\times(0,T))$ *solves* IP2 *with* $D_0^{\{k\},n} = {}^{R}D_0^{\{k\},n}$, *then*

$$u(x,t) = \left[B^{-1}\mathcal{F}_{R,t_1}^{\beta,n}(g(x,\cdot) - {}^{R}D_{t_0}^{\{k\},n}\varphi(x,\cdot))\right](t), \;\; a.e.\ (x,t) \in \Omega\times(0,t_0).$$

(ii) *If* $(u,F) \in \{u \in \mathcal{U} : Bu(x,\cdot) \in W_2^n(0,T)\, \forall x \in \Omega\} \times C(\Omega\times(0,T))$, $n \geq 1$, *solves* IP2 *with* $D_0^{\{k\},n} = {}^{C}D_0^{\{k\},n}$, *then*

$$u(x,t) = \left[B^{-1}\mathcal{F}_{C,t_1}^{\beta,n}(\varphi(x,\cdot); g(x,\cdot) - {}^{R}D_{t_0}^{\{k\},n}\varphi(x,\cdot))\right](t), \;\; (x,t) \in \Omega\times(0,t_0).$$

In both cases g *is given by* (4), t_1 *is an arbitrary number in* (t_0,T) *and* $F|_{\Omega\times(0,t_0)} = \left[D_0^{\{k\},n}Bu + D^l u - Au\right]\Big|_{\Omega\times(0,t_0)}$.

Proof. The proof follows from Theorem 3 and the relations, (3), (4), that describe the transition from IP2 to IP1. □

4.3. Solution Formulas in Case of Tempered and Atangana–Baleanu Derivatives

In this subsection, we derive the solution formulas for particular subcases of the generalized fractional derivative of the order $n = 1$. They are based on solution formulas derived for the usual fractional derivative and involve the operators $\mathcal{F}_{R,t_1}^{\beta,1}$, $\mathcal{F}_{C,t_1}^{\beta,1}$. Again, we assume that t_1 is an arbitrary number in the interval (t_0,T).

Firstly, let us consider the tempered fractional derivatives of type I.

Theorem 4. *Let* $k(t) = \frac{e^{-\lambda t}t^{-\beta}}{\Gamma(1-\beta)} + \lambda\int_0^t \frac{e^{-\lambda\tau}\tau^{-\beta}}{\Gamma(1-\beta)}d\tau$, $0 < \beta < 1$, $\lambda > 0$. *Then, the following assertions hold.*

(i) *If* $u \in L_2(0,T)$, $k * u \in W_1^1(0,T)$ *and* u *solves* IP1 *with* $D_0^{\{k\},1} = {}^{R}D_0^{\{k\},1}$, *then*

$$u(t) = e^{-\lambda t}\mathcal{F}_{R,t_1}^{\beta,1}(e^{\lambda t}g - e^{\lambda t}\,{}^{R}D_{t_0}^{\{k\},1}\varphi)(t), \quad a.e.\ \ t \in (0,t_0). \tag{18}$$

(ii) *If* $u \in W_2^1(0,T)$ *solves* IP1 *with* $D_0^{\{k\},1} = {}^{C}D_0^{\{k\},1}$, *then*

$$u(t) = \lim_{\tau\to t_0^+}\varphi(\tau) - \int_t^{t_0} e^{-\lambda\tau}\mathcal{F}_{R,t_1}^{\beta,1}\left(e^{-\lambda\tau}(g - {}^{R}D_{t_0}^{\{k\},1}\varphi)'\right)(\tau)d\tau,$$
$$t \in (0,t_0). \tag{19}$$

Proof. Before starting the proof, let us point out that $k'(t) = \frac{e^{-\lambda t}t^{-1-\beta}}{\Gamma(-\beta)}$. Hence, for $t \in (t_0, T)$ and $v \in L_1(0, t_0)$:

$$\int_0^{t_0} k'(t-\tau)v(\tau)d\tau = e^{-\lambda t}\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau}v(\tau)d\tau. \tag{20}$$

(i) Firstly, the IP1 can be rewritten by means of (9), and then Formula (20) leads us to the equation with the unknown term $e^{\lambda t}u(t)$

$$\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau}u(\tau)d\tau = e^{\lambda t}g(t) - e^{\lambda t\,R}D_{t_0}^{\{k\},1}\varphi(t), \quad t \in (t_0, T).$$

Thus, by applying Theorem 2 and using the notation (15), we obtain (18).

(ii) Let us write IP1 in the form (10), differentiate it and obtain for $t \in (t_0, T)$

$$\int_0^{t_0} k'(t-\tau)u'(\tau)d\tau = \frac{d}{dt}(g(t) - {}^C D_{t_0}^{\{k\},1}\varphi(t)).$$

Then, due to (20) we have $\int_0^{t_0} \frac{(t-\tau)^{-1-\beta}}{\Gamma(-\beta)} e^{\lambda\tau}u'(\tau)d\tau = e^{\lambda t}\frac{d}{dt}(g(t) - {}^C D_{t_0}^{\{k\},1}\varphi(t))$ and similarly to (i) we deduce Formula (19) using the notation (17). □

To handle IP1 for derivatives that contain Mittag-Leffler functions, we need the following lemma.

Lemma 2. *Let $0 < \beta < 1$, $\lambda \in \mathbb{R}$ and $f \in W_1^1(0, T)$. Then, the function $p(t) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(\lambda(t-\tau)^\beta)f(\tau)d\tau$ is a solution of the equation ${}^C D_0^\beta p(t) - \lambda p(t) = f(t)$, $t \in (0, T)$, and the function $q(t) = \int_0^t E_\beta(\lambda(t-\tau)^\beta)f(\tau)d\tau$ is a solution of the equation ${}^C D_0^\beta q(t) - \lambda q(t) = I_0^{1-\beta}f(t)$, $t \in (0, T)$.*

Proof. The proof of the first assertion can be found e.g., in [32], p. 174, and the second assertion follows from the first one because $[t^{\beta-1}E_{\beta,\beta}(\lambda t^\beta)] * I_0^{1-\beta}f = E_\beta(\lambda t^\beta) * f$ [6]. □

Next, we consider the case of a tempered fractional derivative of type II.

Theorem 5. *Let $k(t) = e^{-\lambda t}t^{\beta-1}E_{\beta,\beta}(\lambda^\beta t^\beta)$, $\frac{1}{2} < \beta < 1$, $\lambda > 0$. Then, the following assertions are valid:*

(i) *If $u \in W_1^1(0, T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$, then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau}({}^R D_0^\beta - \lambda^\beta \mathcal{I})\mathcal{F}_{R,t_1}^{\beta,1}\left(e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^R D_{t_0}^\beta e^{\lambda\tau}g\right)(\tau)d\tau + \lim_{\tau\to t_0}\varphi(\tau), \quad t \in (0, t_0). \tag{21}$$

(ii) *If $u \in W_1^2(0, T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$, then*

$$u(t) = \int_{t_0}^t e^{-\lambda\tau}({}^C D_0^\beta - \lambda^\beta \mathcal{I})\mathcal{F}_{C,t_1}^{\beta,1}\left(e^{\lambda\tau}g; e^{\lambda\tau}(\varphi' + \lambda^\beta g) - {}^C D_{t_0}^\beta e^{\lambda\tau}g\right)(\tau)d\tau + \lim_{\tau\to t_0}\varphi(\tau), \quad t \in (0, t_0). \tag{22}$$

Here, $\mathcal{I}$ is the unity operator.

Proof. Firstly, we prove (ii). Let us define the function w as

$$w(t) = e^{\lambda t\,C}D^{\{k\},1}u(t) = \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(\lambda^\beta(t-\tau)^\beta)(e^{\lambda\tau}u'(\tau))d\tau.$$

Due to Lemma 2, this function solves the equation

$$^C D_0^\beta w(t) - \lambda^\beta w(t) = e^{\lambda t} u'(t), \quad t \in (0, T). \tag{23}$$

Therefore, $^C D_0^\beta w = e^{\lambda t} u' + \lambda^\beta w$ and, in view of the condition (1), we have the IP1 with usual fractional derivative:

$$w|_{(t_0,T)} = e^{\lambda t} g, \quad ^C D_0^\beta w|_{(t_0,T)} = e^{\lambda t} \varphi' + \lambda^\beta e^{\lambda t} g. \tag{24}$$

In order to apply Theorem 3 (ii) to this problem, we must verify that $w \in W_2^1(0, T)$ is valid. Let us compute:

$$w'(t) = t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta) u'(0) + [t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)] * (e^{\lambda t} u')'(t).$$

Due to the assumptions $\frac{1}{2} < \beta < 1$ and $u \in W_1^2(0, T)$ we have $t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta) \in L_2(0, T)$ and $(e^{\lambda t} u')' \in L_1(0, T)$. Using the Young's theorem for convolutions, we deduce $w' \in L_2(0, T)$. Thus, $w \in W_2^1(0, T)$.

By applying Theorem 3 (ii) to (24), we obtain

$$w(t) = \mathcal{F}_{C,t_1}^{\beta,1}(e^{\lambda t} g; e^{\lambda t} \varphi' + \lambda^\beta e^{\lambda t} g - {}^C D_{t_0}^\beta e^{\lambda t} g)(t), \quad t \in (0, t_0).$$

Since by (23), $u' = e^{-\lambda t}({}^C D_0^\beta - \lambda^\beta \mathcal{I}) w$, this implies Formula (22).

Secondly we prove (i). Let us define $w(t) = e^{\lambda t R} D_0^{\{k\},1} u(t)$. Then, $w(t) = (\frac{d}{dt} - \lambda) z(t)$, where

$$z(t) = \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(\lambda^\beta (t - \tau)^\beta)(e^{\lambda \tau} u(\tau)) d\tau.$$

By Lemma 2, z solves the equation

$$^C D_0^\beta z(t) - \lambda^\beta z(t) = e^{\lambda t} u(t), \quad t \in (0, T). \tag{25}$$

Let us differentiate Equation (25) to derive the equation for w:

$$^R D_0^\beta (z' - \lambda z)(t) + {}^R D_0^\beta (\lambda z)(t) - \lambda^\beta z'(t) = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0, T).$$

That is

$$^R D_0^\beta w(t) - \lambda^\beta w(t) + \lambda({}^R D_0^\beta (z)(t) - \lambda^\beta z(t)) = \lambda e^{\lambda t} u(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0, T).$$

Since $z(0) = 0$, we have that $^R D_0^\beta z = {}^C D_0^\beta z$ and using (25) again, we obtain

$$^R D_0^\beta w(t) = \lambda^\beta w(t) + e^{\lambda t} u'(t), \quad a.e. \ t \in (0, T). \tag{26}$$

Based on (1),(26), we formulate IP1 for w:

$$w|_{(t_0,T)} = e^{\lambda t} g, \quad ^R D_0^\beta w|_{(t_0,T)} = e^{\lambda \tau}(\varphi' + \lambda^\beta g). \tag{27}$$

To apply Theorem 3 (i), we should prove that $w \in L_2(0,T)$, and $\left(\frac{t^{-\beta}}{\Gamma(1-\beta)}\right) * w = I_0^{1-\beta} w \in W_1^1(0,T)$, that is ${}^R D_0^\beta w \in L_1(0,T)$. Let us investigate

$$w(t) = \left(\frac{d}{dt} - \lambda\right) \left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * (e^{\lambda t} u(t)) = u(0) \left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right)$$
$$+ \left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t} u(t))' - \lambda e^{\lambda t} u(t)), \quad t \in (0,T).$$

Since $t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta) \in L_2(0,T)$ for $\beta \in (1/2,1)$ and $e^{\lambda t} u(t) \in W_1^1(0,T)$, we obtain that $\left(t^{\beta-1} E_{\beta,\beta}(\lambda^\beta t^\beta)\right) * ((e^{\lambda t} u(t))' - \lambda e^{\lambda t} u(t)) \in L_2(0,T)$; thus, $w \in L_2(0,T)$. Due to the (26) ${}^R D_0^\beta w \in L_1(0,T)$, because $w \in L_2(0,T)$ and $u \in W_1^1(0,T)$.

That enables us to apply Theorem 3 (i) to (27):

$$w(t) = \mathcal{F}_{R,t_1}^{\beta,1} \left(e^{\lambda t}(\varphi' + \lambda^\beta g) - {}^R D_{t_0}^\beta e^{\lambda t} g\right)(t), \quad a.e.\ t \in (0,t_0).$$

This in view of (26) implies Formula (21). □

Remark 3. *It is possible to extend the range of β to $0 < \beta < 1$ in Theorem 5 assuming more regularity of u and the conditions $u(0) = 0$ and $u'(0) = 0$ in cases (i) and (ii), respectively.*

Finally, we consider the case of Atangana–Baleanu fractional derivative.

Theorem 6. *Let $k(t) = \frac{1}{1-\beta} E_\beta\left(-\frac{\beta t^\beta}{1-\beta}\right)$, $0 < \beta < 1$. Then, the following assertions hold:*

(i) *If $u \in W_1^1(0,T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^R D_0^{\{k\},1}$, then*

$$u(t) = \left(\frac{1-\beta}{\beta}\ {}^R D_0^\beta + \mathcal{I}\right) \mathcal{F}_{R,t_1}^{\beta,1} \left(\beta g - {}^R D_{t_0}^\beta (\varphi - (1-\beta) g)\right)(t),$$
$$a.e.\ t \in (0,t_0). \tag{28}$$

(ii) *If $u \in W_1^2(0,T)$ and u solves IP1 with $D_0^{\{k\},1} = {}^C D_0^{\{k\},1}$, then*

$$u(t) = \left(\frac{1-\beta}{\beta}\ {}^C D_0^\beta + \mathcal{I}\right) \mathcal{F}_{C,t_1}^{\beta,1} \left(\varphi - (1-\beta)g;\ \beta g - {}^C D_{t_0}^\beta (\varphi - (1-\beta) g)\right)(t),$$
$$t \in (0,t_0). \tag{29}$$

Proof. (ii) Let us denote $w = (1-\beta)^C D_0^{\{k\},1} u$. For this particular kernel type the relation holds:

$$w(t) = \int_0^t E_\beta\Big(-\frac{\beta(t-\tau)^\beta}{1-\beta}\Big) u'(\tau) d\tau.$$

By Lemma 2 and the identity $I_0^{1-\beta} u' = {}^C D_0^\beta u$, w solves the equation

$${}^C D_0^\beta w(t) + \frac{\beta}{1-\beta} w(t) = {}^C D_0^\beta u(t), \quad t \in (0,T). \tag{30}$$

Since the relation (1) is valid, $w|_{(t_0,T)} = (1-\beta) g$. It follows from (30) that ${}^C D_0^\beta (u - w) = \frac{\beta}{1-\beta} w$. Thus, we have the IP1 with usual fractional derivative

$$(u-w)|_{(t_0,T)} = \varphi - (1-\beta) g, \quad {}^C D_0^\beta (u-w)|_{(t_0,T)} = \beta g.$$

To apply Theorem 3 (ii), we have to show that $u - w \in W_2^1(0,T)$. Since $E_\beta' = \frac{1}{\beta}E_{\beta,\beta}$ and $E_\beta(0) = 1$, we obtain $(u-w)' = -\frac{1}{1-\beta}[t^{\beta-1}E_{\beta,\beta}(-\frac{\beta t^\beta}{1-\beta})] * u'$. Due to the assumptions of (ii), this belongs to $L_2(0,T)$, hence $u - w \in W_2^1(0,T)$. According to Theorem 3 (ii)

$$(u-w)|_{(0,t_0)} = \mathcal{F}^{\beta,1}_{C,t_1}\left(\varphi - (1-\beta)g; \beta g - {}^C D^\beta_{t_0}(\varphi - (1-\beta)g)\right). \tag{31}$$

The relation (30) implies $w = \frac{1-\beta}{\beta}\, {}^C D_0^\beta(u-w)$. Therefore,

$$w|_{(0,t_0)} = \frac{1-\beta}{\beta}\, {}^C D_0^\beta \mathcal{F}^{\beta,1}_{C,t_1}\left(\varphi - (1-\beta)g; \beta g - {}^C D^\beta_{t_0}(\varphi - (1-\beta)g)\right).$$

Hence, from (31), we obtain (29).
(i) Let us denote $w = (1-\beta)^R D_0^{\{k\},1} u$. Then

$$w = \frac{d}{dt}z, \quad \text{where} \quad z(t) = \int_0^t E_\beta\Big(-\frac{\beta(t-\tau)^\beta}{1-\beta}\Big)u(\tau)d\tau.$$

The function z solves the equation

$$^C D_0^\beta z(t) + \frac{\beta}{1-\beta}z(t) = I_0^{1-\beta}u(t), \quad t \in (0,T). \tag{32}$$

Next, we differentiate Equation (32) and obtain

$$^R D_0^\beta w(t) + \frac{\beta}{1-\beta}w(t) = {}^R D_0^\beta u(t), \quad a.e.\ t \in (0,T). \tag{33}$$

Therefore, ${}^R D_0^\beta (u-w)(t) = \frac{\beta}{1-\beta}w(t)$ that leads us to the IP1 with a usual fractional derivative

$$(u-w)|_{(t_0,T)} = \varphi - (1-\beta)g, \quad {}^R D_0^\beta(u-w)|_{(t_0,T)} = \beta g.$$

Now, we have to show that $u - w \in L_2(0,T)$ and ${}^R D_0^\beta(u-w)(t) \in L_1(0,T)$. Firstly,

$$w(t) = \frac{d}{dt}\left(E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big)*u(t)\right) = u(0)E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big)+E_\beta\Big(-\frac{\beta}{1-\beta}t^\beta\Big)*u'(t)$$

Since $E_\beta\left(-\frac{\beta}{1-\beta}t^\beta\right) \in L_2(0,T)$ for any $\beta \in (0,1)$, we obtain that $w \in L_2(0,T)$. Due to the Sobolev embedding Theorem $u \in W_1^1(0,T) \subset L_2(0,T)$. Thus, $u - w \in L_2(0,T)$. Secondly, ${}^R D_0^\beta(u-w)(t) = \frac{\beta}{1-\beta}w(t) \in L_2(0,T)$.
We continue the proof by applying Theorem 3 (i) to the IP1 for $u - w$:

$$(u-w)|_{(0,t_0)} = \mathcal{F}^{\beta,1}_{R,t_1}\left(\beta g - {}^R D^\beta_{t_0}(\varphi - (1-\beta)g)\right).$$

It follows from (33) that $w = \frac{1-\beta}{\beta}{}^R D_0^\beta(u-w)$; thus, the Formula (28) holds. □

Similarly to Corollary 2, formulas of solutions of IP2 can be derived in cases of tempered and Atangana–Baleanu derivatives.

5. Conclusions

In this paper, two inverse problems were considered . The goal of IP1 was to reconstruct the history of a function based on its value and the value of its generalized fractional derivative on a final time subinterval. Afterwards, the obtained results were applied to IP2 that includes reconstruction of a source term in a fractional PDE based on the final time subinterval measurements. Defining the overdetermination condition on a final time subinterval, not pointwise, enabled us to treat the problem of the reconstruction of a source term (IP2) in a different manner than usual.

In this article, we have proved the uniqueness of the solution to IP1 and IP2 in case the derivative contains general kernel k and derived the solution formulas for some particular cases of k. Namely, these are the cases of usual fractional derivative, tempered, and Atangana–Baleanu fractional derivatives.

In the case of multiterm and distributed fractional derivatives, solution formulas cannot be derived by means of the method presented in this paper. The problem of reconstruction of explicit representations for solutions in these cases remains open.

Since the IP1 and IP2 are severely ill-posed the solution formulas cannot be applied to the real-life applications without prior regularization. Thus, the numerical analysis of the problems is another non-trivial open question to be considered.

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kevad, 2018	harjutustundide läbiviimine õppeaines "Matemaatilise füüsika võrrandid", TTÜ.

8. Kaitstud lõputööd

- 2014, Inverse problem for the heat equation with nonlocal conditions, MSc, juhendaja Prof. Ivanchov M., Ivan Franko nimeline Lvivi Rahvuslik Ülikool, mehaanika- ja matemaatikateaduskond.

9. Teadustöö põhisuunad

- pöördülesanded

10. Teadustegevus
Teadusartiklite ja konverentsiettekannete loetelu on toodud ingliskeelse elulookirjelduse juures.