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Elements of Vedic Mathematics


Swami Bharati Krishna Tirthaji

# Elements of Vedic Mathematics

**UDAYAN S. PATANKAR & SUNIL M. PATANKAR**

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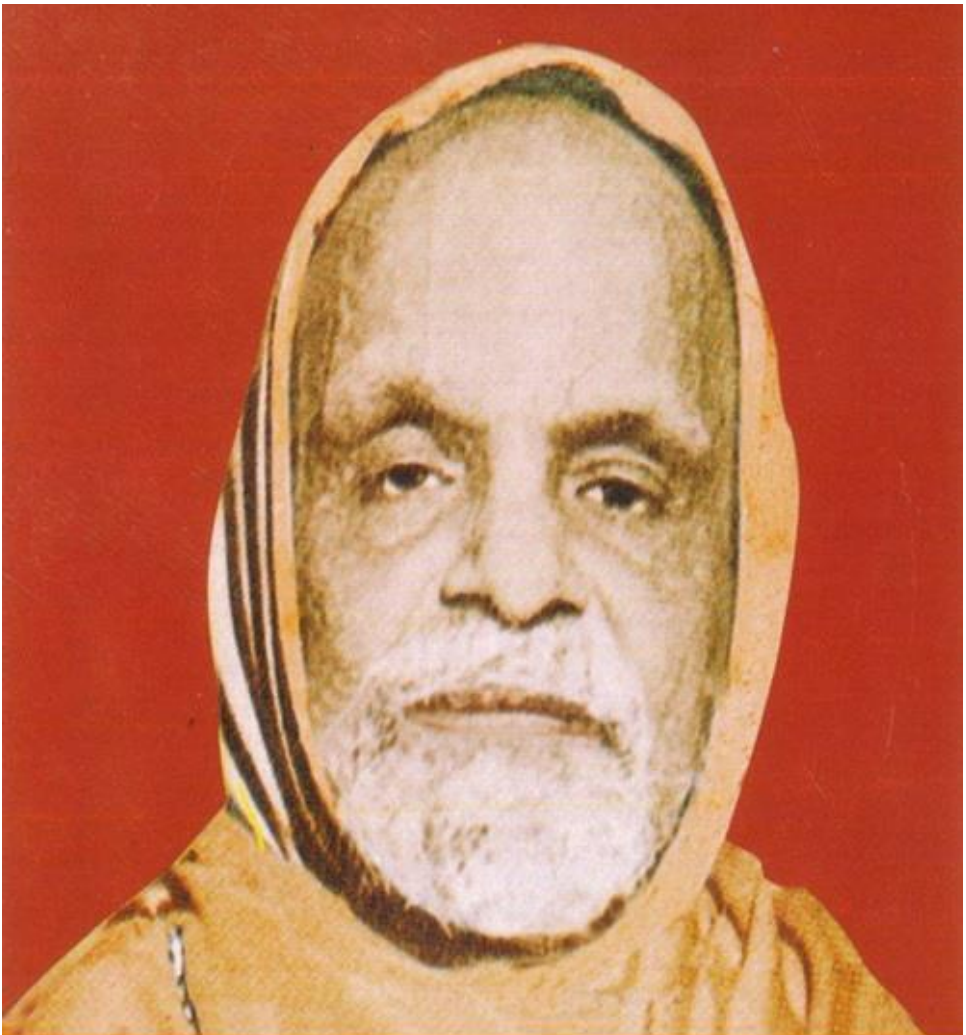
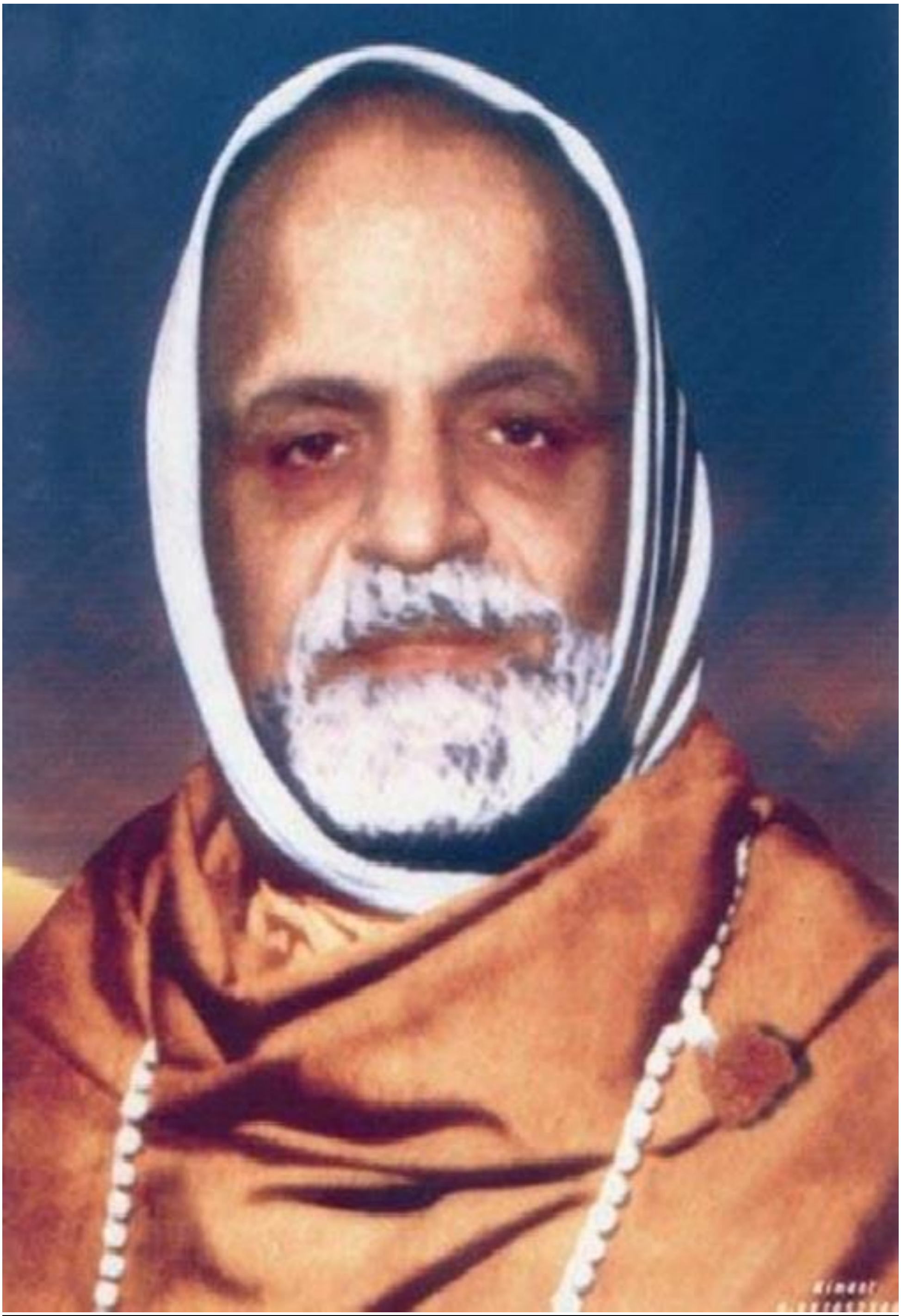


# Vedic Mathematics

Bharati Krsna Tirthaji



## SWAMI BHARATI KRISHANA TIRTH (1884 – 1960)

## (PIONEER OF VEDIC MATHEMATICS)

Swami Bharati Krishana Tirth was the author of the first available book "Vedic Mathematics". He was Jagadguru Shankaracharya of Govardhan Peeth, Puri (Odhisha). He was highly a learned person both in philosophy and in the field of science. He delivered many lectures and demonstrations on Vedic Maths in America and United Kingdom. He appeared at the M.A. examination of the American College of sciences, Rochester, New York, from Bombay Centre in 1903. He passed M.A. examination in seven subjects simultaneously securing highest honours in all, which is perhaps the all time world record of academic excellence and brilliance. His subjects included Sanskrit, Philosophy, English, Mathematics and Science.

During 1919 – 1927 he developed deep into Vedas and other scriptures. During this period, by meditating and by intuition, Swamiji formulated a concept of Vedic Maths. He postulated sixteen sutra of Vedic Mathematics. These sutras are easy to understand, apply and remember as well as the entire work can be summarized in one word as "MENTAL".

Swamiji had written in all sixteen volumes on Vedic Mathematics but at present only one introductory volume is available. Although he intended to rewrite all the previous volumes but his health did not permit it. A spiritual seer of highest order, Swami Bharati Krishana Tirth was also a philosopher, great mathematician and a scientist. His work on Vedic Mathematics is a gem of gift to the world of mathematics and to the entire mankind.

# PREFACE

Om Gurudevay Namha, I Am Thankful To God for Giving Me This Opportunity. I felt Mathematics like a GOD, No one has seen it, but it is there in everything. Being an Electronic Engineer I felt that Mathematics is the base of all sciences but thought to be a hard nut to crack by many students because of the monotonous, lengthy methods of conventional mathematics. We have witnessed that Vedic mathematics is already started gaining its spin in digital electronics. Now a days many researchers are working on how to use Vedic mathematic terms in different calculations, Computer Programs, Digital Designs etc. I had come across many interesting projects Like ALU based on Vedic Mathematics, which has given a very good possibilities of utilizing Vedic mathematics into various digital operations, securities etc.

Our great ancestors had a great knowledge about it and developed many easiest n simplest ways of doing it. So I felt necessity to have a single book for Vedic Mathematics which covers all the topics of three subjects- Vedic Arithmetic, Vedic Algebra and Vedic Geometry. While performing various critical operations I always thought to make it interesting. Vedic Maths encourages student's creativity. The element of choice and flexibility at each stage keeps his mind alert and lively. In this way holistic development of human brain automatically occurs. Vedic Mathematics will help slow learners to know the basic concepts while inducing creativity amongst intelligent.

Vedic Mathematics is a uniquely modern and extremely user friendly system. These Vedic Mathematics methods will be helpful in developing not only an insight into mathematics but also helpful to work in different subject. These methods are based on the scientific footings and practically ease the work without sacrificing anything. The credit to create utmost interest in mathematics goes to Swamiji Bharati Krishan Tirth – Shankaracharya of Govardhan Peeth (Puri). His blessings are always with us for learning Vedic Mathematics.

During my study I came across various personalities starting from my home my Mother, Father and Big Brother. They always supported me to explore new things. My Mentors from sports to study, Dr. Vilas Nitnaware, Prof. Toomas Rang, Dr. Ants Koel and many more who has taught me many things about how to keep myself motivated and focused. I am thankful for their valuable guidance and support.

Let us enjoy!

Udayan Sunil Patankar

# PREFACE

I am glad to present this text book "Elements of Vedic Mathematics" for certificate course in Vedic Mathematics affiliated to KAVIKULAGURU KALIDAS SANSKRIT VISHWAVIDYALAYA, RAMTEK (NAGPUR). I felt necessity to have a single book for certificate course in Vedic Mathematics which covers all the topics of three subjects- Vedic Arithmetic, Vedic Algebra and Vedic Geometry.

Regarding Vedic Mathematics Bharati Krishana Tirtha Shankaracharya Swamiji has said "It is magic until you understand it and mathematics thereafter". Mathematics is the base of all sciences but thought to be a hard nut to crack by many students because of unwillingness attitude to change the monotonous, lengthy methods of conventional mathematics.

Vedic Maths encourages student's creativity. The element of choice and flexibility at each stage keeps his mind alert and lively. In this way holistic development of human brain automatically occurs. Mathematics will be popular as an interesting subject to majority students by including easiest and quick methods of Vedic Mathematics in the syllabus from primary level itself. Vedic Mathematics will help slow learners to know the basic concepts while inducing creativity amongst intelligent. Vedic Mathematics is a uniquely modern and extremely user friendly system. These Vedic Mathematics methods will be helpful in developing an insight into mathematics first and other subjects thereafter. These methods will not only provide quicker calculation but also increase the creativity of mind, develop the brain and will provide the eagerness to learn more and more about mathematics. These methods are based on the scientific footings and practically ease the work without sacrificing anything.

The credit to create utmost interest in mathematics goes to Swamiji Bharati Krishan Tirth – Shankaracharya of Govardhan Peeth (Puri). His blessings are always with us for learning Vedic Mathematics. During my study towards Vedic Mathematics I came across various personalities in this field- Dr.A.W.Vyawahare (Nagpur), Dr.S.D. Mohogaonkar (Nagpur), Dr.Kailash Vishwakarma (Rath), V.G.Unkalkar (Bengaluru), M.Sitaramarao (Hydrabad), Shriram Chouthaiwale (Akola), Prof. Bachchubhai Rawal (Ahamadabad). I am thankful for their valuable guidance.

I am thankful to Mrs.Maya Madhukar Kondalwadikar for performing laborious proof reading task of the notes. I am thankful to Mrs.Maya Madhukar Kondalwadikar and Shri Umesh – K - Bhonsule for their valuable guidance and assistance in preparation of this book. Definitely, Vedic Mathematics methods will be a boon for various competitive examinations to save time, space and energy. Let us enjoy!

Sunil Manohar Patankar

**Special Thanks for Family, Guru's and Friends**

**For their Strong belief, Guidance and Blessings.**

**This milestone is the beginning of New Journey.**

**DON’T FEAR**

**JUST DO THE DARE**

**GOD IS HERE**

**TO TAKE CARE…**

**- UDAYAN S. PATANKAR**

# Contents

# PART
# ONE
# VEDIC ARITHMETIC

# CHAPTER ONE: ADDITION AND SUBTRACTION

## <u>ADDITION</u> (संकलन)

Addition is the simplest operation.

Upa-Sutra: - शुद्ध (Dot) Shudhda

Ex. 879 + 466 + 587

| | | | | |
|---|---|---|---|---|
| | 8 | *7 | *9 | |
| + | *4 | 6 | *6 | Starting from unit place of last number we have to |
| + | 5 | *8 | 7 | add to two unit place digits first so on. |
| 1 | 9 | 3 | 2 | ← Answer |

**Step – I** Adding unit place digits vertically. (7+6) = (7+3+3) = (10+3). For '10' put dot (*) on Left overhead of digit '6' and carry '3' upwards and add it to next digit'9'. (9+3) = (9+1+2) = (10+2). For '10' increase the digit in left column by one by marking a dot (*) Left over-head and write '2' as unit place of answer.

**Step – II** Count the dots of last column which are '2' and add (8+2) =10. For '10' increase the digit in left column by one by marking a dot (*) over-head of digit 8 as the addition is exact 10 there is no carry over for next digit. Add (6+7) = (7+3+3) = (10+3). For '10' increase the digit in left column by one by marking a dot (*) over-head and write '3' in tens place of answer.

**Step – III** Proceeding like this we get '9' in hundreds place of answer (4+5+2) = 11. Put a dot (*) on left side of digit '4' and add 1 upwards with 8. Then counting dot (*) write the '1' in thousands place of answer.

Illustration

| | | | | |
|---|---|---|---|---|
| | 8 | 7 | 9 | |
| + | 4 | 6 | *6 | starting from unit place of last number we have to |
| + | 5 | 8 | 7 | add to two unit place digits first and so on. |
| | ---- | ---- | ----- | |

| | | | | |
|---|---|---|---|---|
| | 8 | 7 | 9 | |
| + | 4 | 6 | *6 | 7+6 = 13=10+3 here 3 is overhead of sum first two |
| + | 5 | 8 | 7 | unit place digits. So to indicate that at 2nd unit place |
| | | | ---- | |

Addition result goes beyond 9 we put overhead (*) mark on the left side of 2nd unit place digit, which indicates us to add that much number in next 10's place addition.

8 7 * 9 Previous overhead i.e. 3 is added to $3^{rd}$ unit place

\+ 4 6 *6 digit. 3+9= 12 = 10+2 here 2 is overhead. So put

\+ 5 8 7 overhead (*) mark at left side of $3^{rd}$ unit place.

---- ---- 2 After this there is no more unit place digits left,

So write that overhead number at unit place result place.

8 7 * 9 Count the number of overhead marks present in unit

\+ 4 6 *6 place column then add that count in next place $1^{st}$ digit

\+ 5 *8 7 i.e. here 2 is the count that is added in $1^{st}$ Tens place

\+ 2 digit. So 2+8 = 10 = 10+0, it indicates that sum is

---- ---- 2 more than 9 but no overhead for next Tens place

Addition, so put overhead (*) mark on the left side of 8 and perform next Tens Place addition.

8 7 * 9 Previous overhead is 0 so 0+6 = 6

\+ 4 6 *6

\+ 5 *8 7

\+ 2

---- ---- 2

8 *7 * 9 6+ 7= 13 = 10 +3; it shows that sum is greater than 9

\+ 4 6 *6 and 3 is overhead of sum, so we put overhead (*)

\+ 5 *8 7 mark on the left side of $3^{rd}$ Tens place digit.

\+ 2

---- 3 2 After this there is no more Tens place digits left, so write that overhead number at Tens result place.

8 *7 * 9 Count of Tens place column overhead (*) marks

\+ 4 6 *6 which has to be added into $1^{st}$ hundreds place digit.

\+ 5 *8 7

\+ 2 2

---- 3 2

| | | | |
|---|---|---|---|
| | 8 | *7 | * 9 |
| + | 4 | 6 | *6 |
| + | 5 | *8 | 7 |
| + | 2 | 2 | |
| | ---- | 3 | 2 |

2+5 =7, addition is less than 9 so add this result in next hundreds place digit in next step

| | | | |
|---|---|---|---|
| | 8 | *7 | * 9 |
| + | *4 | 6 | *6 |
| + | 5 | *8 | 7 |
| + | 2 | 2 | |
| | ---- | 3 | 2 |

7+4 =11 = 10+1, it shows result is more than 9 and generates overhead so put overhead (*) mark on left side of 4. 1 is overhead of sum needs to be add to next hundreds place digit.

| | | | |
|---|---|---|---|
| | 8 | *7 | * 9 |
| + | *4 | 6 | *6 |
| + | 5 | *8 | 7 |
| + | 2 | 2 | |
| | 9 | 3 | 2 |

1+8 = 9 it shows no overhead and sum is not greater than 9.

After this there is no more hundreds place digits left, so write that sum result at Tens result place as no overhead generated in last step.

| | | | | |
|---|---|---|---|---|
| | | 8 | *7 | * 9 |
| + | | *4 | 6 | *6 |
| + | | 5 | *8 | 7 |
| + | | 2 | 2 | |
| | 1 | 9 | 3 | 2 |

Count of Overhead (*) marks present in hundreds place column. Place this digit in result place directly as there is no thousands place digit in actual numbers.

← Final Result

Thus we can say that,

879 + 466 + 587 = 1932

## **SUBTRACTION** (व्यवकलन)

Upa-Sutra: - शुध्द (Dot) Shudhda

Ex. 5000 – 3568

$$\begin{array}{rrrrr} & 5 & 0 & 0 & 0 \\ - & 3^* & 5^* & 6^* & 8 \\ \hline & 1 & 4 & 3 & 2 \end{array}$$

‘8’ is to be subtracted from zero i.e. bigger digit from smaller digit. Put a dot (*) to the right hand side of previous digit i.e.’6’. Then find compliment of ‘8’ i.e. to reduce ‘8’ from ‘10’ and write ‘2’ as answer at the unit place.

6 + (*) = 7

‘7’ is to be subtracted from zero i.e. bigger digit from smaller digit. Put a dot (*) to the right hand side of previous digit i.e.’5’. Then find compliment of ‘7’ from ‘10’ and write ‘3’ as answer at the tens place.

5 + (*) = 6

‘6’ is to be subtracted from zero i.e. bigger digit from smaller digit. Put a dot (*) to the right hand side of previous digit i.e.’3’. Then find compliment of ‘6’ from ‘10’ and write ‘4’ as answer at the hundreds place.

3+ (*) = 4

‘4’ is to be subtracted from ‘5’.

5-4 = 1 to be written as answer at thousands place.

*Note- We Are Representing Negative Numbers with line on that number i.e. -2 is represented as $\overline{2}$.

## COMBINED OPERATION OF ADDITION AND SUBTRACTION

Ex. 322 + 422 – 277

$$\begin{array}{rr} & 3\,2\,2 \\ + & 4\,2\,2 \\ + & \bar{2}\,\bar{7}\,\bar{7} \\ \hline & 5\,\bar{3}\,\bar{3} \end{array}$$

.

Vinculum number is converted to Conventional number by using sutra Eknyunena Purvena and Nikhilam Navatah Charamam Dashatah.

Applying Eknyunena Purvena sutra to positive digit (5) and Nikhilam Navatah Charamam Dashatah sutra

to negative digits of vinculum number 5 $\bar{3}\,\bar{3}$

(5 – 1) = 4 and (9-3), (10-3)

**Answer: -** 467

Vinculum number is discussed in next chapter

---

**Exercise: -** (1) 622 + 711 – 455 (2) 445 – 339 + 177 (3) 853 – 700 + 162

(4) 678 – 231 + 789 (5) 923 – 543 + 134

# CHAPTER TWO: VINCULUM NUMBER

**Definition: -** The number containing both positive and negative digits is called as **Vinculum number**. Vinculum number helps in carrying mathematical operations at ease. The Vedic system envisages simplification of operations by reducing steps. Here the use of complement (Purak-rekhank) from 10 or 9 is required.

**Beejank** (बीजांक) **(Digital root)** (मूलांक) **of a number: -** Conversion of any conventional number into a single digit by the addition of its all digits and if the addition contains more than one digit then repetitive addition of digits is performed till single digit is obtained.

Any number can be written in Vinculum form thereby avoiding digits greater than '5'. Thus if the number contain no digits greater than '5' the computation becomes so easier and faster. The Vinculum approach is useful for addition, subtraction, multiplication and division as well as in developing multiplication tables.

Vinculum number of single digits above '5'

$6 = 10 - 4 = 1\bar{4}$

$7 = 10 – 3 = 1\bar{3}$

$8 = 10 – 2 = 1\bar{2}$

$9 = 10 – 1 = 1\bar{1}$

Vinculum number of two digit numbers

$17 = 20 – 3 = 2\bar{3}$

$19 = 20 -1 = 2\bar{1}$

$38 = 40 – 2 = 4\bar{2}$

$77 = 80 – 3 = 8\bar{3}$

$96 = 100 – 4 = 10\bar{4}$

We can convert any large number

$188 = 190 - 2 = 19\bar{2}$

$188 = 200 – 12 = 2\,\bar{1}\,\bar{2}$

$971 = 1000 - 30 + 1 = 10\bar{3}1$

$971 = 980 - 9 = 98\bar{9}$

For single digit take the complement of it from 10 (निखिलं नवत: चरमं दशत:) and add 1 to the previous digit (एकाधिकेन पूर्वेण). For a group of digits, take the complement of the group using निखिलं नवत: चरमं दशत: sutra and add one to previous positive digit (i.e.एकाधिकेन पूर्वेण).

**Vinculum number to conventional number conversion**

To convert Vinculum number to conventional number take the compliment of vinculum portion(i.e. negative digits) using निखिलं नवत: चरमं दशत: sutra and subtract one from its previous positive digit (i.e. एकन्यूनेन पूर्वेण).

$1\bar{4} = 06 = (1-1), (10-4)$

$2\bar{3} = 17 = (2 - 1), (10 - 3)$

$3\,\bar{1}\,2\,\bar{3} = 2917 = (3 -1), (10 - 1), (2 - 1), (10 - 3).$

The vinculum numbers are subject to all operations.

**Addition and Subtraction**

$\bar{4} + \bar{3} = \bar{7}$

$\bar{4} - \bar{3} = \bar{1}$

$\bar{4} - \bar{6} = 2$

$5 + \bar{2} = 3$

$5 - \bar{2} = 7$

**Multiplication and Division**

$\bar{2} * \bar{3} = 6$

$\bar{2} * 3 = \bar{6}$

$\bar{6} \div \bar{2} = 3$

$\bar{6} \div 2 = \bar{3}$

$6 \div \bar{2} = \bar{3}$

In a number, if smaller and bigger digits are mixed up then accordingly different groups are to be formed and conversion of each group is performed in above manner.

**e.g. Convert 17187 to vinculum form**

17187 have two groups 17 & 187. Their vinculum form is

$17 = 2\bar{3}$

$187 = 2\ \bar{1}\ \bar{3}$

$17187 = 2\ \bar{3}\ 2\ \bar{1}\ \bar{3}$

**Convert 16792976399823 to vinculum form**

16792976399823 have four groups 1679, 2976, 3998 & 23. Their vinculum form is

$1679 = 2\ \bar{3}\ \bar{2}\ \bar{1}$

$2976 = 3\ 0\ \bar{2}\ \bar{4}$

$3998 = 4\ 0\ 0\ \bar{2}$

& 23

$16792976399823 = 2\ \bar{3}\ \bar{2}\ \bar{1}\ 3\ 0\ \bar{2}\ \bar{4}\ 4\ 0\ 0\ \bar{2}\ 2\ 3$

It is not only bigger digits are converted to vinculum form but as per requirement of problem, even any digit – positive or negative can be converted so that a single number could be expressed in various ways.

**e.g. 88 can be expressed as**

$90 - 2 = 9\bar{2}$

$1000 - 912 = 1\ \bar{9}\ \bar{1}\ \bar{2}$

$100 - 12 = 1\ \bar{1}\ \bar{2}$

$88 = 9\bar{2} = 1\ \bar{9}\ \bar{1}\ \bar{2} = 1\ \bar{1}\ \bar{2}$

**Conversion of negative number into vinculum form**

**Ex. 1** Convert (- 38) into Vinculum form.

**Steps** 1) We write $-38 = \bar{3}\ \bar{8}$

2) By Nikhilam Sutra we get,

$0\ \bar{3}\ \bar{8} = |\ 0 - 1\ |\ 9 - 3\ |\ 10 - 8\ | = \bar{1}\ 6\ 2$

$-100 + 62 = -38$

**Ex.2** Convert (- 5696) into Vinculum form.

We write $(-5\ 6\ 9\ 6) = 0\ \bar{5}\ \bar{6}\ \bar{9}\ \bar{6} = \bar{1}\ 4\ 3\ 0\ 4$

$(-10000) + 4304 = (-5696)$

Similarly,

$-04 = -10 + 6 = \bar{1}6$

$-01 = -10 + 9 = \bar{1}9$

$-24 = -100 + 76 = \bar{1}76$

$-968 = -1000 + 032 = \bar{1}032$

If the beejank of conventional number is same as that of its vinculum number then the conversion is correct. In case, if beejank of vinculum number is negative digit, it is to be converted to positive digit by adding 9 to negative digit.

(If 9 is added or subtracted, the beejank remains unchanged.)

**e.g.**

$19 = 2\bar{1}$

Beejank of $19 = 1 + 9 = 10 = 1 + 0 = 1$

Beejank of $2\bar{1} = 2 – 1 = 1$

As both have same beejank, the conversion is correct.

---

**<u>Multiplication Table and Vinculum</u>**

**Prepare a multiplication table for 19.**

**Steps** 1) Write 19 in Vinculum form: i.e. $19 = 2\ \bar{1}$. LHS is called as Conventional number and RHS is called as Vinculum and used as Operator.

2) The operator at unit place is $\bar{1}$ and digit at unit place is 9. Go on reducing this digit by 1.

3) The operator at Tens place is 2 and digit at this place is 1. Go on increasing this digit by 2.

Thus we get a table for 19.

**Prepare a multiplication table for 87.**

**Steps** [1] Here 0 8 7 = 1 $\bar{1}$ $\bar{3}$

[2] The operator at unit place is $\bar{3}$, at Tens place $\bar{1}$ and hundreds place 1. Thus multiplication table is:

| | 1 | $\bar{1}$ | $\bar{3}$ | |
|---|---|---|---|---|
| | 0 | 8 | 7 | |
| | 1 | 7 | 4 | |
| | 2 | 6 | 1 | |
| = | (3 | 5 | $\bar{2}$) | Converting to normal form |
| | 3 | 4 | 8 | |
| | 4 | 3 | 5 | |
| | 5 | 2 | 2 | |
| = | (6 | 1 | $\bar{1}$) | Converting to normal form |
| | 6 | 0 | 9 | |
| = | (7 | $\bar{1}$ | 6) | Converting to normal form |
| | 6 | 9 | 6 | |
| | 7 | 8 | 3 | |
| | 8 | 7 | 0 | |

**Prepare a multiplication table for 8 9 7.**

**Steps** 1) 8 97 = 1 $\bar{1}$ 0 $\bar{3}$

2) The operator at unit place is $\bar{3}$, at Tens place is 0, at hundreds place is 1 and at thousands place is 1. Thus multiplication table is:

| | 1 | $\bar{1}$ | 0 | $\bar{3}$ |
|---|---|---|---|---|
| | 0 | 8 | 9 | 7 |
| | 1 | 7 | 9 | 4 |
| | 2 | 6 | 9 | 1 |
| = | (3 | 5 | 9 | $\bar{2}$) |
| | 3 | 5 | 8 | 8 |
| | 4 | 4 | 8 | 5 |
| | 5 | 3 | 8 | 2 |
| | 6 | 2 | 7 | 9 |
| | 7 | 1 | 7 | 6 |
| | 8 | 0 | 7 | 3 |
| | 9 | $\bar{1}$ | 7 | 0 |
| | 8 | 9 | 7 | 0 |

**Use of Vinculum in subtraction**

$$\begin{array}{r} 4\,7\,8 \\ -\ 3\,2\,5 \\ \hline 1\,5\,3 \end{array}$$

If upper digit is greater, then write down the difference directly.

$$\begin{array}{r} 3\,2\,4 \\ -\ 2\,7\,6 \\ \hline 1\,\bar{5}\,\bar{2} \end{array} \quad = 0\,4\,8 = 48$$

$$\begin{array}{r} 5\,6\,3\,8\,1 \\ -\ 1\,9\,6\,7\,0 \\ \hline 4\,\bar{3}\,\bar{3}\,1\,1 \end{array} \quad = 3\,6\,7\,1\,1$$

**Note**: - By writing the subtraction in Vinculum form we avoid "borrowing the digit".

## ADDITION

Find 8985 + 2376 + 4889 + 3605 + 7512.

**Note:** As per the present method of the addition we begin to add digits from unit place downwards or upwards.

We write the carry digit (if any) over-head and then perform addition at Tens place etc. But in Vedic method as the sum exceeds nine, increase the digit in left column by one by marking a star (*) over-head.

The symbols used are

8* = 8 + 1 = 9, 0* = 0 + 1 = 1, 9* = 9 + 1 = 0 & carry 1

**Ans :**

| | | | | |
|---|---|---|---|---|
| | 8 | 9 | 8 | 5 |
| 0* | 2* | 3* | 7* | 6 |
| | 4* | 8* | 8* | 9 |
| | 3 | 6 | 0 | 5 |
| 0* | 7* | 5 | 1 | 2 |
| 2 | 7 | 3 | 6 | 7 |

**<u>Exercise</u>: - Convert conventional number into vinculum number**

(1) 198 (2) 887 (3) 13726 (4) 2599468 (5) 8613693 (6) - 125

(7) - 1236

**Convert Vinculum number into conventional number**

(1) $4\bar{6}$ (2) $2\,\bar{3}\,\bar{2}$ (3) $1\,0\,\bar{3}\,\bar{3}\,\bar{1}$ (4) $1\,0\,2\,\,\bar{2}\,0\,\bar{1}$ (5) $1\,6\,\bar{3}\,2\,\bar{4}\,3\,\bar{1}\,1$

# CHAPTER THREE: MULTIPLICATION

**Sutra: - “**एकाधिकेन पूर्वेण” (By one more than the previous)

**Condition: -** Unit place digit of a number must be 5 also other place digits of a number must be same.

**e.g. : - 35 * 35**

1. Bifurcate answer by vertical line as LHS & RHS
2. Multiply unit place digits of multiplier & multiplicand. Product obtained is RHS of answer i.e. 5*5 =25
3. Previous digit of multiplicand is 3. Add one to it. i.e. (3+1) multiply 4 by 3 i.e. (4*3) Product obtained is LHS of answer.

| | | |
|---|---|---|
| | 35 | |
| X | 35 | |
| | LHS | RHS |
| | 4 *3 | 5*5 |
| | 12 | 25 |

**Answer**: - 1225.

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (35) * B (35) = B (3+5) * B (3+5)

= 8 *8

= 64

= (6+4)

= 10

= (1+0)

= 1

B (1225) = B $(1+\cancel{2}+\cancel{2}+\cancel{5})$

= 1

As Beejank is same hence answer is verified.

**EXERCISE** –

1. 145 X 145
2. 65 X 65
3. 85 X 85
4. 1125 X 1125
5. 35 X 35

**Sutra: -** अन्त्ययोर्दशकेपि (Sum of unit place digit of numbers is ten)

**Condition: -** Sum of unit place digit of numbers is 10 & same digit must be at other places of numbers.

**e. g.: - 42 * 48**

1) Bifurcate answer by vertical line as LHS & RHS.
2) Multiply unit place digits of multiplier & multiplicand. Product obtained is RHS of answer i.e. 2*8 =16
3) Previous digit of multiplicand is 4. Add one to it i.e. (4+1), multiply 5 by 4 i.e. (5*4). Product obtained is LHS of answer.

| | | |
|---|---|---|
| | 42 | |
| X | 48 | |
| | LHS | RHS |
| | 5*4 | 2*8 |
| | 20 | 16 |

**Answer**: - 2016

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (42) * B (48) = B (4+2) * B (4+8)
= 6*12
= 6*(1+2)
= 6*3
= 18
= (1+8)
= 9

B (2016) = B (2+0+1+6)
= 9

As **Beejank** is same hence answer is verified.

## EXERCISE

1) 191 X 199
2) 22 X 28
3) 44 X 46
4) 203 X 207
5) 37 X 33

**Sutra: -** “एकन्यूनेन पूर्वेण” (By one less than the previous)

**Condition: -** Multiplier number consists of digit 9 only.

There are three cases.

**Case – I** ⟶ No. of digits in multiplier and multiplicand are same.

**e.g. 345 * 999**

(1) Bifurcate answer by vertical line as LHS & RHS. (2) Here previous means multiplicand i.e. 345. (3) Compliment 345 is found by “निखिलं नवत: चरमं दशत:” sutra

i.e. (9-3), (9-4), (10-5) = 655. It is RHS of answer. (4) Using एकन्यूनेन पूर्वेण sutra LHS of answer is found i.e. (345 – 1) = 344.

| | | |
|---|---|---|
| | 345 | |
| X | 999 | |
| | LHS | RHS |
| | 345-1 | 655 |
| | 344 | 655 |

**Answer** = 344655

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (345) * B (999) = B $(3+\cancel{4}+\cancel{5})$ * B $(9\cancel{+9+9})$

= 3*9

= 27

= (2+7)

= 9

B (344655) = B $(3\cancel{+4+4}+6+\cancel{5+5})$

= 9

As Beejank is same hence answer is verified.

**2nd Method: -** (345 x 999)

= 345 (1000 - 1)

= (345000 - 345)

= 344655

**Case – II** ⟶ No. of digits in multiplier are more than no. of digits in multiplicand.

**e.g. : - 45 * 999**

(1) Bifurcate answer by vertical line as LHS & RHS. (2) Here previous means multiplicand i.e. 045. (3) Compliment 045 is found by "निखिलं नवत: चरमं दशत:" sutra i.e. (9-0), (9-4), (10-5) = 955. It is RHS of answer. (4) Using एकन्यूनेन पूर्वेण sutra LHS of answer is found. i.e. (045 – 1) = 44.

| | | |
|---|---|---|
| | 45 | |
| X | 999 | |
| | LHS | RHS |
| | 45-1 | 955 |
| | 44 | 955 |

**Answer**: - 44955

Applying "**गुणित समुच्चय: समुच्चय गुणित:**" sutra answer is verified.

B (045) * B (999) = B $(0+\cancel{4}+\cancel{5})$ * B $(\cancel{9}+\cancel{9}+\cancel{9})$

= 0*0

= 0

= 0 + 9

= 9

B (44955) = B $(\cancel{4}+\cancel{4}+\cancel{9}+\cancel{5}+\cancel{5})$

= 0

= 0 + 9

= 9

As Beejank is same hence answer is verified.

**2nd Method: -** (45 x 999)

= 45 (1000 - 1)

= (45000 - 45)

= 44955

---------------------------------------------------------------------------------------------------

**Case – III** ⟶ No. of digits in multiplier are less than no. of digits in multiplicand.

**e.g.: - 345 * 99**

(1) Write Multiplicand & Multiplier one below the other. (2) Using "एकन्यूनेन पूर्वेण" sutra new multiplicand is found i.e. (345 -1) = 344. (3) Put multiplier 99 to its right side. We get 34499 as RHS (4) Subtract 344 from 34499. We get (34499 - 344) = 34155 as the answer.

| | | |
|---|---|---|
| | 345 | |
| X | 99 | |
| | LHS | RHS |
| | 344 | 34499 |
| | 34499 -344 | |
| | 34155 | |

**Answer**: - 34155

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

$B\ (345) * B\ (99) = B\ (3+\cancel{4+5}) * B\ (\cancel{9}+9)$

$= 3*9$

$= 27$

$= (2+7)$

$= 9$

$B\ (34155) = B\ (\cancel{3+4+1+5+5}) = 0$

$= 0+9 = 9$ as final result is 0 then add 9.

As **Beejank** is same hence answer is verified.

**2nd Method: -** (345 x 99)

= 345 (100 - 1)

= (34500 - 345)

= 34155

---

**Exercise**: -

1) 174 X 999
2) 22 X 99
3) 836 X 99
4) 212 X 99
5) 6967 X 99

**Sutra: -** "निखिलं नवत: चरमं दशत:" (All from nine and last from ten)
**Condition: -** Base is nearer to Multiplicand & Multiplier.

**Case** (I) When both the numbers are smaller than base.

**Steps** (1) Select a base, which is nearest to given numbers.

Here base =10.

(2) Subtract each of the numbers to be multiplied from base.

Here 10 - 9 = 1, 10 - 8 = 2. These deficiency numbers are written as shown.

| | | |
|---|---|---|
| | 9 | $\bar{1}$ |
| X | 8 | $\bar{2}$ |
| | LHS | RHS |
| | (9-2) or (8-1) | 2 |
| | 72 | |

(3) Here numbers to be multiplied are smaller than the base hence these deficiency numbers are treated as negative numbers or bar digits.

(4) We get the answer (product) in two parts.

1) Right-hand part of the answer is $\bar{1} * \bar{2}$.

The number of digits in the right-hand part should be equal to n where $10^n$ is base number. ‘n’ is the index of base ‘10’

(5) The cross addition gives left-hand part of the answer.

---

**Case** (II) When both numbers to be multiplied are greater than the base.

**e.g.** **Find 112 × 109** **<u>Steps</u>**

| | |
|---|---|
| 112: 12 | 1) Base = 100 = $10^2$. |
| 109: 09 | 2) 112 – 100 = 12, 109 - 100 = 09. |
| 121: 108 | 3) 12 × 09 = 108, 112 + 09 = 121 |
| 122: 08 | 4) 121 + 1 = 122 |

**Ans:** 12208.

---

**Case** (III) when one number is greater than the base and other is smaller one.

**e.g.** **Find 115 × 98** Steps

| | |
|---|---|
| 115: 1 5 | 1) Base = 100 = $10^2$ |
| 98: 0 $\bar{2}$ | 2)115-100 = 15, 98-100 = 0 $\bar{2}$ |
| 113: $\bar{3}$ 0 | 3) 15 × $\bar{2}$ = $\bar{3}$ 0 |
| | 98 + 15 = 113=115 - 2 |

**Ans:** 11300 + $\bar{3}$ 0 = 11270

---

**<u>Exercise</u>: -**
(1) 899 X 998 (2) 101 X 105 (3) 124 X 95 (4) 94 X 88 (5) 8 X 9
(6) 1212 X 996 (7) 21 X 12

---

**<u>Sutra</u>: -** “निखिलं नवत: चरमं दशात:” (All from nine and last from ten)

**<u>Condition</u>: -** Base is not nearer to Multiplicand & Multiplier but nearer to sub-base.

When multiplicand & multiplier are not nearer the base i.e. 10, 100, 1000 ... then proportionate sub-base is chosen by “आनुरूप्येण” sutra. A sub-base is a convenient multiple of '10' e.g. 20, 30, 40, 200, 300, 400.The relation between sub-base and base is found by their division & is called as a ratio.

**Ratio** = Sub-base ÷ Base.

**e.g.: - 52 * 54** **Base** (B) = 10

**Sub-base** (SB) = 50

**Ratio** (R) = SB ÷ B

= 50 ÷ 10

R = 5

**Deviation** = Number – SB = (52 - 50) =2, (54 – 50) = 4

(1) Two numbers to be multiplied are written one below the other. Their deviations are written in front of the respective numbers with sign convention. (2) Bifurcate answer in two parts as LHS & RHS by vertical line. (3) RHS of answer is the product of deviations of numbers and will contain digits equal to no. of zeroes in base. If RHS contains less no. of digits than no. of zeroes in base, then remaining digits are filled up by giving zero or zeroes on left hand side of RHS. (4) LHS of answer is cross addition of one number with deviation of other & then multiplying by the ratio. If RHS contains more no. of digits than no. of zeroes in base, the excess digit or digits are to be added after getting complete part of LHS of answer.

| | | |
|---|---|---|
| | *52* | *2* |
| X | 54 | 4 |
| | LHS | RHS |
| | (52+4) or (54+2) | 8 |
| | 56 * 5 | 8 |
| | 280 | 8 |

**Answer**: - 2808

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B (52) * B (54) = (5 + 2) * (5+4) = 7 * 9 = 63 = (6+3) = 9

B (2808) = (2 + 8 + 0 + 8) = 18 = (1 + 8) = 9

As **Beejank** is same hence answer is verified.

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e.g. 1004 X 1112 **Base**: 1000

**Deviation**: $(d_1)$ = (1004 – 1000) = 004

$(d_2)$ = (1112 – 1000) = 112

| | | |
|---|---|---|
| | 1004 | 004($d_1$) |
| X | 1112 | 112($d_2$) |
| | LHS | RHS |
| | (1004+112) or (1112+004) | ($d_1$) ($d_2$) |
| | 1116 | 448 |

**Answer**: 1116448

B (1116448) = $(1+\cancel{1}+\cancel{1}+6+\cancel{4}+\cancel{4}+\cancel{8})$

= (1+6)

= 7

B (1004) x B (1112) = (1+0+0+4) x (1+1+1+2)

= 5 x 5

= 25

= (2+5)

= 7

As **Beejank** is same hence verified.

---

**e.g.** 512 x 497

Base: 100

Deviation: ($d_1$) = (512 – 500) = 12

($d_2$) = (500 – 497) = - 03

Sub- base (SB): 500

Ratio (R) = (Sub-base ÷ Base)

= (500 ÷ 100)

(R) = 5

एकन्यूनेन पूर्वेण sutra is applied to LHS and निखिलं नवत: चरमं दशत: to RHS

| | | |
|---|---|---|
| | 512 | 12 |
| X | 497 | - 03 |
| | LHS | RHS |
| | (512- 03) or (497+12) | - 36 |
| | 509 | - 36 |
| | 509 * 5 | - 36 |
| | 2545 | - 36 |
| | 2545 - 1 | 64 |
| | 2544 | 64 |

**Answer**: 254464

B (254464) = $(2+\cancel{5}+\cancel{4}+4+6+4)$

= 16

= (1+6)

= 7

B (497) x B (512) = $(4+\cancel{9}+7)$ x (5+1+2)

= 11 x 8

= (1+1) x 8

= 16

= (1+6)

= 7

As **Beejank** is same hence verified.

---------------------------------------------------------------------------------------------------

**Second Method: -**

| | | | |
|---|---|---|---|
| | 512 ($N_1$) | $5\cancel{00}$ ($b_1$) | 12 ($d_1$) |
| X | 497 ($N_2$) | $5\cancel{00}$ ($b_2$) | - 03 ($d_2$) |
| | LHS | RHS | |
| | ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$) | ($d_1$) * ($d_2$) | |
| | (512*5+5*-3) or (497*5+12*5) | 12 * - 03 | |
| | 2545 | - 36 | |
| | 2545 - 1 | 64 | |
| | 2544 | 64 | |

एकन्यूनेन पूर्वेण sutra is applied to LHS and निखिलं नवत: चरमं दशत: to RHS.

Answer: 254464

RHS of answer will be product of deviations and will have no. of digits equal to no. of equal zero(s) cancelled from different bases or sub-bases.

LHS of answer will be ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$)

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B (512) * B (497) = (5 +1+ 2) * (4+ $\cancel{9}$ + 7)

= 8 x 11

= 88

= (8 + 8)

= 16

= (1+6)

= 7

B (254464) = (2+$\cancel{5}$+$\cancel{4}$+4+6+4) = 16

= (1 + 6)

= 7

As Beejank is same hence answer is verified.

**EXERCISE**: - (1) 348 X302 (2) 26 X 28 (3) 62 X 63 (4) 88 X 96 (5) 908 X 916

**Sutra: -** “निखिलं नवत: चरमं दशत:” (All from nine and last from ten)

**Condition: -** Base or Sub-base is not nearer to Multiplicand & Multiplier.

**e.g.: - 108 * 1007**

Multiplicand & Multiplier are not nearer to common base or sub-base hence different bases or sub-bases nearer to it are chosen. Therefore $b_1$= 100 & $b_2$=1000 are chosen respectively. Deviation of multiplicand & multiplier with different base or sub-base is found by निखिलं नवत: चरमं दशत: sutra and written in front of multiplicand & multiplier respectively. Now cancel equal no. of zeroes from different bases or sub-bases.

| | | | |
|---|---|---|---|
| | 108 ($N_1$) | 1$\cancel{00}$ ($b_1$) | 08 ($d_1$) |
| X | 1007 ( $N_2$) | 10$\cancel{00}$ ($b_2$) | 007 ($d_2$) |
| | LHS | RHS | |
| | ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$) | ($d_1$) * ($d_2$) | |
| | (108*10+7*1) or (1007*1+10*8) | 8 * 7 | |
| | 1087 | 56 | |

**Answer**: **-** 108756

RHS of answer will be product of deviations and will have no. of digits equal to no. of equal zero(s) cancelled from different bases or sub-bases.

LHS of answer will be ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$)

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (108) * B (1007) = (1 + 0 + 8) * (1+0 + 0 + 7)

= 9 * 8

= 72

= (7 + 2)

= 9

B (108756) = ($\cancel{1}$ + 0 + $\cancel{8}$ + 7 + 5 + 6)

= 18

= (1 + 8)

= 9

As Beejank is same hence answer is verified.

e.g. 1123 X 112

| | | | |
|---|---|---|---|
| | 1123 ($N_1$) | $10\cancel{00}$ ($b_1$) | 123 ($d_1$) |
| X | 112 ($N_2$) | $1\cancel{00}$ ($b_2$) | 12 ($d_2$) |
| | LHS | RHS | |
| | ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$) | ($d_1$) * ($d_2$) | |
| | (1123*1+12*10) or (112*10+123*1) | 123 * 12 | |
| | 1243 | $_{14}$76 | |
| | 1243 + 14 | 76 | |
| | 1257 | 76 | |

**Answer**: **-** 125776

RHS of answer will be product of deviations and will have no. of digits equal to no. of equal zero(s) cancelled from different bases or sub-bases.

LHS of answer will be ($N_1 * b_2 + d_2 * b_1$) or ($N_2 * b_1 + d_1 * b_2$)

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (112) * B (1123) = (1 + 1 + 2) * (1+1 + 2 + 3)

= 4 * 7

= 28

= (2 + 8)

= 10

= (1 + 0)

= 1

B (125776) = ($\cancel{1}$ + $\cancel{2}$ + 5 + 7 + 7 + $\cancel{6}$)

= 19

= (1 + 9)

= 10

= (1 + 0)

= 1

As Beejank is same hence answer is verified.
**Exercise**: - (1) 22 X256 (2) 52 X1012 (3) 598 X 698 (4) 442 X 998
(5) 697 X 997

**Sutra: -** “निखिलं नवत: चरमं दशात:” (All from nine and last from ten)

**Condition: -** Multiplication of three numbers having same base.

**e.g.: - 107 * 103 * 104**

If $N_1$, $N_2$, $N_3$ are 3 numbers near same base i.e.100 & $d_1$ (7), $d_2$ (3), $d_3$ (4) being their respective deviations from the base. The answer is divided in 3 parts as LHS, middle & RHS. RHS & middle will contain no. digits equal to no. zeroes in base i.e.2.

LHS = (Any one number + deviations of other two numbers)

Middle side = Product of two deviations at a time added together.

RHS = Product of all three deviations.

Base: - 100

Deviation: - $d_1$ = (107-100) =07, $d_2$ = (103-100) = 03

$d_3$ = (104-100) = 04

| | | |
|---|---|---|
| | 107 ($N_1$) | 07 ($d_1$) |
| X | 103 ($N_2$) | 03 ($d_2$) |
| X | 104 ($N_3$) | 04 ($d_3$) |
| LHS | Middle Side | RHS |
| ($N_1 + d_2 + d_3$) or ($N_2 + d_1 + d_3$) or ($N_3 + d_1 + d_2$) | ($d_1 * d_2 + d_2 * d_3 + d_3 * d_1$) | ($d_1 * d_2 * d_3$) |
| (107 + 3 + 4) | (7 * 3) + (3 * 4) + (4 * 7) | (7 * 3 * 4) |
| 114 | 61 | 84 |

**Answer**: - 1146184

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B (107) * B (103) * B (104) = (1+0+7) * (1+0+3) * (1+0+4)

= 8 * 4 * 5

= 160

= (1+6+0)

= 7

B (1146184) = $(1+\cancel{1}+\cancel{4}+6+\cancel{1}+\cancel{8}+\cancel{4}) = 16$

= (1 + 6)

= 7

As Beejank is same hence answer is verified.

**Sutra: -** "ऊर्ध्वतिर्यग्भ्याम्" (Vertically and crosswise)

**Condition: -** Multiplication of multiplicand & multiplier of 3 digits.
Total no. of steps involved in this method of multiplication of two numbers of 'n' digits each is (2n-1) and answer will contain 2n or (2n-1) no. of digits.
Total no. of steps (2n – 1) = No. of Inclusion phase steps 'n'+ No. of Exclusion phase steps

('n – 1')

| Step No. | Place value | Chart | Procedure | Operation |
|---|---|---|---|---|
| **Inclusion phase** | | | | |
| 1 | Unit | H T U<br>H T U | Vertical product of unit place digits | 1 * 5 = 5 |
| 2 | Tens | H T U<br>H T U | Cross product of first two digits & their addition | (2 * 5) + (4 * 1) = 14 |
| 3 | Hundreds | H T U<br>H T U | Cross product of $1^{st}$ & $3^{rd}$ digits, Vertical product of $2^{nd}$ digits & their addition | (5 * 3) + (2 * 1) + (2 * 4) = 25 |
| **Exclusion phase** | | | | |
| 4 | Tens | H T U<br>H T U | Unit place digits are excluded. Cross product of $2^{nd}$ & $3^{rd}$ digits & their addition | (4 * 3) + (2 * 2) = 16 |
| 5 | Hundreds | H T U<br>H T U | Unit place & Tens place digits are excluded. Vertical product of Hundreds place digits. | (3 * 2) = 6 |

**e.g.: - 321 *245**

| | H | T | U | |
|---|---|---|---|---|
| | 3 | 2 | 1 | |
| X | 2 | 4 | 5 | |

$$6 \leftarrow {}_{1}6 \qquad {}_{2}5 \leftarrow {}_{1}4 \qquad 5$$

| | | | | |
|---|---|---|---|---|
| 7 | 8 | 6 | 4 | 5 |

**Answer**: - 78645

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (321) * B (245) = (3+2+1) * (2+~~4~~+~~5~~)

= (6 * 2) = 12 = (1+2) = 3

B (78645) = (7+8+6+~~4~~+~~5~~) = 21 = (2+1) = 3

As Beejank is same hence answer is verified.

**<u>Exercise</u>**: - (1) 346 X 658 (2) 271 X 721 (3) 555 X 232 (4) 508 X 812 (5) 912 x 329

**<u>Ganesh Daivadna's 'यथाकोष्ठ' method of multiplication.</u>**

**Sutra: -** 'तिर्यग्गत्या यथाकोष्ठं संयोज्य जातं तद्वेव' (cross multiplication & summation of its multiplication products)

**e.g. 524 × 423**

| | 5 | 2 | 4 | |
|---|---|---|---|---|
| 2 | 2 0 | 0 8 | 1 6 | 4 |
| 1+1 | 1 0 | 0 4 | 0 8 | 2 |
| $_{1}0$ +1 | 1 5 | 0 6 | 1 2 | 3 |
| | $_{1}5$+1 | $_{1}5$ | 2 | |

**Answer**: - 221652

When multiplicand & multiplier have equal no. of digits then square matrix of those no. of digits is drawn. If multiplicand & multiplier have unequal no. of digits then maximum no. of digits from multiplicand or multiplier is selected for drawing square matrix. Divide each matrix block by the diagonal into two parts as Left side and Right side as shown in figure above.

Write each digit of multiplier on right side of matrix in a sequence of top to bottom but outside of it. Write each digit of multiplicand on top side of matrix in a sequence of left to right but outside of it. Cross multiplication of each digit of multiplier with each digit of multiplicand is performed. Then multiplication product is written on left side and right side of matrix block diagonal in the multiplier digit row in a sequence of right to left respectively. If the multiplication product is of two digit then Tens place digit to be written on left side of matrix block diagonal and unit place digit to be written right side of matrix block diagonal. If the multiplication product is of one digit then put zero as Tens place digit & to be written on left side of matrix block diagonal as well as unit place digit to be written right side of matrix block diagonal

1st row $\longrightarrow$ (4*4) = 16, (4*2) = 8, (4*5) = 20

2nd row $\longrightarrow$ (2*4) = 8, (2*2) = 4, (2*5) = 10

3 rd row $\longrightarrow$ (3*4) = 12, (3*2) = 6, (3*5) = 15

Digits between two parallel diagonal divider lines are added in a sequence right to left then turning left and moving upwards.

Answer is to be verified by Beejank method.

**<u>Bhaskacharya's 'तस्थ' method of multiplication</u>**

**e.g.** $12 \times 135$

Here Multiplicand = 12

Multiplier = 135

Multiplicand is written equal to the no. of digits in multiplier. Here as multiplier has 3 digits then multiplicand is written 3 times. Now write multiplier digits separately below these multiplicands and multiplication is performed.

$$\begin{array}{ccc} 12 & 12 & 12 \\ \times\ \ 1 & \times\ \ 3 & \times\ \ 5 \\ \hline 12 \longleftarrow & {}_{3}6 \longleftarrow & {}_{6}0 \end{array}$$

Answer: - 1620

Answer is to be verified by Beejank method.

## SQUARE

Square of a number is the product of the number with itself. A number having ‘n’ digits will have ‘2n’ or ‘2n-1’ digits in its square. Perfect square of any number can not have 2, 3, 7 or 8 at its unit place.

The numbers having 0, 1, 5 and 6 at its unit place reproduce itselves as unit place digits in the squares of these numbers.

Following Vedic sutra are used for finding square of a number.

1) एकाधिकेन पूर्वेण

2) यावदूनं तावदूनीकृत्य वर्गं च योजयेत्

3) आनुरूप्येण

4) द्वंद्व योग

5) निखिलं नवत: चरमं दशत:

6) ऊर्ध्वतिर्यग्भ्याम्

एकाधिकेन पूर्वेण sutra is used when unit place digit of a number is 5 & other digit (s) is same.

**e.g. $115^2$**

Answer is bifurcated as LHS and RHS

LHS of answer is the product of digit prefixed to 5 and its एकाधिका.

If prefixed pair of digits is bigger, then multiplication with its एकाधिका can be performed by any suitable Vedic sutra

RHS of answer will be always (5 * 5) = 25

Here multiplication of 11 * 12 is performed by ऊर्ध्वतिर्यग्भ्याम् sutra.

| | |
|---|---|
| 11 * (11+1) | 5 * 5 |
| 11 * 12 | 25 |
| LHS | RHS |
| 132 | 25 |

**Ans**: - 13225

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (13225) = $(1+3+\cancel{2}+\cancel{2}+\cancel{5})$

$= 4$

B ($115^2$) = $(1+1+5)^2$

$= 7^2$

$= 49 = (4+\cancel{9})$

$= 4$

As Beejank is same hence answer is verified.

**<u>Exercise</u>: - (**1) 45 X 45 (2) 85 X 875 (3) 85 X 85 (4) 125 X 125 (5) 95 X 95

<u>यावदूनं तावदूनीकृत्य वर्गं च योजयेत्</u>

Answer is bifurcated as **LHS** and **RHS**

Whatever the deviation, add it to the number for finding LHS of answer. Deviation may be positive or negative if number is greater or smaller than base respectively.

RHS of answer will contain same no. digits as the no. of zero in the base. Excess digit or digits if any, is to be carried over to LHS of answer and deficit digit or digits if any, to be filled up by giving zero or zeros to the left of RHS of answer digit or digits.

RHS of answer is found by setting up square of the deviation.

**e.g.: - $998^2$** **Base**: - 1000

**Deviation**: - $\bar{2}$ (by निखिलं नवत: चरमं दशत:)

| LHS | RHS |
|---|---|
| 998 + (-2) | $(-2)^2$ |
| 996 | 4 |
| 996 | 004 |

As base contains 3 zero, RHS of answer must have 3 digits.

**Ans**: - 996004

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (996004) = ($\cancel{9}+\cancel{9}$+6+0+0+4)

= (6+4)

= 10

= (1+0) = 1

B $(998)^2$ = ($\cancel{9}+\cancel{9}$+8) * ($\cancel{9}+\cancel{9}$+8)

= (8 * 8)

= 64

= (6+4)

= 10

= (1+0)

= 1

As Beejank is same hence answer is verified.

**<u>Exercise</u>**: - (1) $96^2$ (2) $104^2$ (3) $88^2$ (4) $111^2$ (5) $106^2$

If the number is away from the base then suitable sub-base is selected and deviation from sub-base is found.

RHS of answer will be square of deviation from sub-base.

LHS of answer is to be modified by ratio of sub-base and base.

**e.g.: - $32^2$** Base: - 10 Sub-base: - 3 Deviation = (32-30) = 2

Ratio = (Sub-base ÷ Base) = (30 ÷ 10) = 3 by आनुरूप्येण sutra

| LHS | RHS |
|---|---|
| 32 + (2) | $(2)^2$ |
| 34 | 4 |
| 34 * 3 | 004 |

**<u>Answer</u>**: - 1024

Applying “गुणित समुच्चय: समुच्चय गुणित:” sutra answer is verified.

B (1024) = (1+0+2+4)

= 7

B (32)2 = (3+2) * (3+2)

= 5 *5

= 25

= (2+5)

**Exercise**: - (1) $32^2$ (2) $49^2$ (3) $59^2$ (4) $75^2$ (5) $68^2$

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## आनुरूप्येण

Bifurcate the number to be squared into two parts and determine ratio between two digits.

As index of number is 2 therefore 3 digits are required for obtaining square of number. Write

square of left digit followed by obtaining two more digits maintaining the ratio. Write middle

digit in next row but in same column. Column-wise addition is performed.

**e.g.: - $12^2$**

Ratio: - 1: 2

| | | |
|---|---|---|
| 1 | 2 | 4 |
| | 2 | |
| 1 | 4 | 4 |

**Ans**: - $12^2$

**Exercise**: - (1) $24^2$ (2) $31^2$ (3) $18^2$ (4) $89^2$ (5) $72^2$

## द्वंद्व योग (Duplex)

In द्वंद्व योग sutra operations are performed left to right while in उर्ध्वतिर्यग्भ्याम् it is performed from right to left.

D (a) = $a^2$

D (ab) = 2ab

D (abc) = 2ac + $b^2$

D (abcd) = 2ad + 2bc

**e.g.: - $412^2$**

$412^2$ =

| D4 | D41 | D412 | D12 | D2 |
|---|---|---|---|---|
| $4^2$ | 2*(4*1) | 2(4*2) + 1*1 | 2*(1*2) | $2^2$ |
| 16 | 8 | 17 | 4 | 4 |
| 16 | 8+1 | 7 | 4 | 4 |
| 16 | 9 | 7 | 4 | 4 |

**Ans**: - 169744

**Square of any number**

| **Base (B)** | **Sub Base (SB)** | **Ratio (R) SB ÷ B** | **Number (N) nearer to SB** | **Deviation(d) = (N-SB)** | **LHS = (N+d)*R** | **RHS = $d^2$** |
|---|---|---|---|---|---|---|
| 10 | 80 | 8 | 84 | 4 | (84+4)*8<br>88*8 = 704 | $4^2$ = 16 |
| 100 | 400 | 4 | 402 | 2 | (402+2)*4<br>404*4<br>=1616 | $2^2$ = 04 |
| 1000 | 2000 | 2 | 2003 | 3 | (2003+3)*2<br>2006*2<br>= 4012 | $3^2$ =009 |

**Number of digit in RHS shall be equal to number of zero in the Base.**

$84^2$ = 7 0 4 / 1 6 = 7056

$402^2$ =1616 / 04 = 161604

$2003^2$ = 4012 / 009 = 4012009

| **Base (B)** | **Number (N) nearer to Base** | **Deviation (d) = (N-B)** | **LHS = (N+d)** | **RHS = $d^2$** |
|---|---|---|---|---|
| 10 | 12 | 2 | (12 +2) =14 | $2^2$ = 4 |
| 100 | 98 | -2 | [98+(-2)] =96 | $-2^2$ =04 |
| 1000 | 1003 | 3 | (1003+3) =1006 | $3^2$ = 009 |

**Number of digit in RHS shall be equal to number of zero in the Base.**

$12^2$ = 14 / 4 = 144

$98^2$ = 96 / 04 = 9604

$1003^2$ = 1006 / 009 = 1006009

**<u>Exercise</u>**: - (1) $79^2$ (2) $127^2$(3) $99^2$ (4) $48^2$ (5) 28

## SQUARE – ROOT

A number having ‘n’ digits will have ‘n/2’ (when ‘n’ is even) or ‘(n+1)/2’ (when ‘n’ is odd) digits in its square-root. Hence only by observation (विलोकनम्) no. of digits of any number, no. of digits in its square-root can be predicted. Make group of two digits from right of the number. Last group at far left will be of one or two digits. The total no. of groups indicates the no. of digits in square-root. Give the decimal point after getting square-root digits. If number is not a perfect square then square-root will be in decimal form.

| Number | Square | Digital-root (Beejank) |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 4 | 4 |
| 3 | 9 | 9 |
| 4 | 16 | 7 |
| 5 | 25 | 7 |
| 6 | 36 | 9 |
| 7 | 49 | 4 |
| 8 | 64 | 1 |
| 9 | 81 | 9 |
| 10 | 100 | 1 |
| 15 | 225 | 9 |
| 25 | 625 | 4 |

1) Digital-root (Beejank) of perfect square will be 1, 4, 7 or 9 only and unit place digit of perfect square will be 1, 4, 5, 6, 9 or 0 only.

2) If unit place digit of perfect square is 5 then its last two digits of number will be 25.

3) Relation between unit place digit of a number and unit place digit of its square-root.

4) If no. of zero on right hand side of a number is not in multiple of two then a given number is not a perfect square.

| | | | | | |
|---|---|---|---|---|---|
| Unit place digit of a number | 1 or 9 | 2 or 8 | 3 or 7 | 4 or 6 | 5 |
| Unit place digit of its square-root | 1 | 4 | 9 | 6 | 5 |

Square of complements from 10 will have same digit at its unit place.

**<u>Procedure to find out square-root of a number by 'विलोकनम्' sutra.</u>**

Ignore last digits (unit place digit & tens place digit) and find out greatest number whose square is less than the remaining part of a number. Now adjusting obtained unit place digit on its right hand side, two numbers are obtained. Find out unique number having '5' as its unit place digit which lies between these two obtained numbers and its square is obtained. If given number is smaller than this square then smaller number amongst obtained two numbers will be square-root of a given number otherwise another one will be required square-root. If square of supposed square-root is a given number then result is correct

**e.g.: - Find square-root of 14641**

**Digital-root** (Beejank) of given number is 7 and its unit place digit is 1. Hence given number is a perfect square. Unit place digit of given number is 1 so unit place digit of square-root will be 1 or 9. Ignore last two digits (unit place digit & tens place digit) of given number then remaining part of given number 146 is obtained. The greatest positive integer whose square is less than 146 is 12. Now suffixing obtained unit place to 12, two numbers 121 & 129 are obtained. The number between these two numbers having 5 in its unit place is 125. Square of 125 is obtained by 'एकाधिकेन' sutra and which is 15625. As given number 14641 is less than 15625 hence 121 is the required square-root.

By digital root (Beejank) square-root of 14641 can be confirmed amongst two numbers 121 & 129.

B $(121)^2 = (1+2+1)^2 = 16 = (1+6) = 7$

B $(129)^2 = (1+2+9)^2 = 9$

B $(14641) = (1+4+6+4+1) = 16 = (1+6) = 7$

$\therefore \sqrt{14641} = 121$ --------------- (Ans)

## General procedure of finding square-root of a number whether it is perfect or imperfect square.

**e.g.: - Find square-root of 222784**

**Step1.** We group the digits of the given number in pairs from right to left as 22 | 27 | 84.

**Note:** The left most group may contain one digit.

**Step2.** Find the number whose square is nearest to left most group.

Here $4^2$ = 16 is nearest to 22, hence '4' is the first digit in the answer line, 22 - 16 = 6 is the first digit in remainder line and (2 x 4) = 8 is divisor.

**Step3.** Considering remainder 6 and next digit of given number i.e. 2, we get gross dividend 62. This is1$^{st}$ gross dividend (GD) & it is 1$^{st}$ net dividend (ND). Division is performed on net dividend by divisor. This 62 / 8 gives Quotient Q = 7 and Remainder R = 6. Quotient = 7 will be 2$^{nd}$ digit of square-root ahead of 1st digit of square-root.

**Step4**. ND = GD - Duplex of 2$^{nd}$ digit of square-root.

Now ND = 67 - D (7) = 67 - 49 = 18. And 18/8 gives Q= 2 and R = 2. Quotient =2 is 3$^{rd}$ digit of square-root ahead of 2$^{nd}$ digit of square-root.

**Step5.** ND = GD - Duplex of 2$^{nd}$ & 3rd digits of square-root.

= 28 - D (72) = 28 - 28 = 0. And 0/14 gives Q = 0 and R = 0.

**Step6.** ND = GD - Duplex of (720)

= 04 - D (720) = 4 - 4 = 0. And 0 / 14 gives Q = 0 and R = 0.

**Step7.** There are three groups in the given number; hence decimal point in the answer is placed after three digits counted from left.

| | | | | |
|---|---|---|---|---|
| 22 | 2 | 7 | 8 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| | 22 | 2 | 7 | 8 | 4 |
| 8 | | 6 | | | |
| | | 62 | | | |
| | | 62 | | | |
| | 4 / | | | | |

| | | | | | |
|---|---|---|---|---|---|
| | 22 | 2 | 7 | 8 | 4 |
| 8 | | 6 | 6 | | |
| | | 62 | 67 | | |
| | | 62 | 18 | | |
| | 4 / | 7 | | | |

| | | | | | |
|---|---|---|---|---|---|
| | 22 | 2 | 7 | 8 | 4 |
| 8 | | 6 | 6 | 2 | |
| | | 62 | 67 | 28 | |
| | | 62 | 18 | 0 | |
| | 4 / | 7 | 2 | | |

| | | | | | |
|---|---|---|---|---|---|
| | 22 | 2 | 7 | 8 | 4 |
| 8 | | 6 | 6 | 2 | 0 |
| | | 62 | 67 | 28 | 04 |
| | | 62 | 18 | 0 | |
| | 4 / | 7 | 2. | 0 | |

| | | | | | |
|---|---|---|---|---|---|
| | 22 | 2 | 7 | 8 | 4 |
| 8 | | 6 | 6 | 2 | 0 |
| | | 62 | 67 | 28 | 04 |
| | | 62 | 18 | 0 | 0 |
| | 4 / | 7 | 2. | 0 | 0 |

**Ans:** The square root of 222784 is 472 .00

If the number is Imperfect Square then square-root is calculated in same way giving decimal point as per the no. of groups formed and till required decimal places – if required, giving zeros on the number.

**e.g.: - Find square-root of 151**

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 / | 5 | 1 | 0 | 0 | 0 |
| 2 | | 0 | 1 | 1 | 0 | 1 |
| | 1 / | 2. | 3 | $\bar{1}$ | $\bar{2}$ | 2 |

12. 3 $\bar{1}$ $\bar{2}$ 2 = 12. 2882

**<u>Exercise</u>**: - Find square-root of (1) 529 (2) 2116 (3) 7569 (4) 3969 (5) 1681

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## <u>CUBE</u>

A perfect cube is a product of the three equal numbers.

In general the cube of number ‘a’ is a $\times$ a $\times$ a which is represented by $a^3$

Vedic sutra applied for finding cube of a number are

1) निखिलं नवत: चरमं दशात:

2) आनुरूप्येण

When given number is nearer to base the निखिलं नवत: चरमं दशात: sutra is applied.

N = given number whose cube is to be found.

B = base

d = deviation of given number from the base = (N - B). It can be positive or negative.

1) Bifurcate answer in three parts as LHS, Middle side, RHS.

**LHS** = (N + 2*d)

**Middle side** = $(3*d^2)$

**RHS** = $d^3$

Middle side and RHS of answer will have no. of digits equal to the no. of zeros in the base. Extra digit, if any in these parts will be carried over or deficit digit to be filled by prefixing zero or zeros to that part to get final answer.

**e.g.: - Find cube of 96**

Base (B) = 100

Number (N) = 96

Deviation (d) = (96 -100)

= $0\bar{4}$

| LHS | Middle side | RHS |
|---|---|---|
| (N + 2*d) | $3 * d^2$ | $d^3$ |
| $(96 + 2*0\bar{4})$ | $3 * (0\bar{4})^2$ | $(0\bar{4})^3$ |
| 96 -8 | 3 * 16 | $\bar{6}\,\bar{4}$ |
| 88 | 48 | $\bar{6}\,\bar{4}$ |

88 / 48 / $\bar{6}\,\bar{4}$

As cube of bar digit (negative number) is negative the answer will be a vinculum number which requires conversion to conventional number. Conversion of vinculum number to conventional number is performed by applying ‘एकन्यूनेन पूर्वेण’ and ‘निखिलं नवत: चरमं दशत:’ sutra.

88 / (48-1) / (9-6), (10-4)

88 / 47 / 36

**Ans: - 884736**

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B $(96)^3 = (\cancel{9} + 6)^3$

$= 216$

$= (2+1+6)$

$= 9$

B $(884736) = (8+8+4+7+\cancel{3}+\cancel{6})$

$= 27 = (2+7) = 9$

As beejank is same the answer verified

**Exercise**: - (1) $98^3$ (2) $89^3$ (3) $111^3$(4) $103^3$ (5) $143^3$

If a given number is not nearer to base, then cube can be found by applying निखिलं नवत: चरमं दशत: sutra using sub-base concept with slight modification in LHS and Middle side of answer. Here deviation will be from sub-base.

**e.g.: - Find $52^3$**

**N** = given number whose cube is to be found = 52

**B** = base = 10

**SB** = Sub-base = 50

**R** = Ratio of SB and B = (SB ÷ B) = (50 ÷ 10)

**R** = 5

**d** = deviation of given number from the Sub- base = (N -SB). It can be positive or negative

= (52 – 50)

**d** = 2

1) Bifurcate answer in three parts as LHS, Middle side, RHS.

**LHS** = $(N + 2*d) * R^2$

**Middle side** = $(3*d^2) * R$

**RHS** = $d^3$

| LHS | Middle side | RHS |
|---|---|---|
| $(N + 2*d) * R^2$ | $(3 * d^{2)} *R$ | $d^3$ |
| $(52+2*2) * 5^2$ | $(3 * 2^2) * 5$ | $2^3$ |
| (56 * 25) | (12 * 5) | 8 |
| 1400 | $_6 0$ | 8 |
| 1400 + 6 | 0 | 8 |
| 1406 | 0 | 8 |

**Ans**: **-** 140608

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B $(52)^3 = (5+2)^3$

$= 7^3$

= 343

(3+4+3)

= 10

= (1+0)

1

B (140608) = $(\cancel{1}+4+0+6+0+\cancel{8})$

= 10

= (1+0) = 1

As beejank is same the answer is verified.

**<u>Exercise</u>**: - (1) $82^3$ (2) $204^3$ (3) $701^3$ (4) $73^3$ (5) $321^3$

आनुरूप्येण

Ratio of two digits of a given number is found. Write cube of left digit followed by three numbers having same proportion to maintain the ratio. These numbers will be in geometric proportions of left digit's cube as per the ratio.

Write double of middle-second & middle-third number in next row below it selves respectively. Addition is performed column-wise. More than one digit in these numbers will be carried over to next place on LHS.

**e.g.: - Find $12^3$**

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 8 |
| | 4 | 8 | |
| 1 | 6 | $_{1}2$ | 8 |
| 1 | 6 + 1 | 2 | 8 |
| 1 | 7 | 2 | 8 |

**Ans** = 1728

Applying "गुणित समुच्चयः समुच्चय गुणितः" sutra answer is verified.

B (1728) = $(1+\cancel{7}+\cancel{2}+8)$

= 9

B $(12)^3 = (1+2)^3$

= 27

= (2+7)

= 9

As beejank is same the answer is verified.

**<u>Exercise</u>**: - (1) $34^3$ (2) $52^3$ (3) $108^3$(4) $307^3$(5) $46^3$

# CUBE-ROOT

| Number | Cube of a number | Digital root(Beejank0 of cube |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 8 | 8 |
| 3 | 27 | 9 |
| 4 | 64 | 1 |
| 5 | 125 | 8 |
| 6 | 216 | 9 |
| 7 | 343 | 1 |
| 8 | 512 | 8 |
| 9 | 729 | 9 |
| 10 | 1000 | 1 |
| 11 | 1331 | 8 |
| 12 | 1728 | 9 |
| 15 | 3375 | 9 |

1) Unit place digit of cubes is distinct.

2) When unit place digit of a number is 1, 4, 5, 6, 9 or 0 then unit place digit of its cube will be same.

3) When unit place digit of a number is 2, 3, 7 or 8 then unit place digit of its cube will be their complement from 10 i.e. 8, 7, 3 & 2 respectively.

4) Digital root (Beejank) of a perfect cube will be 1, 8 or 9 only.

5) When no. of zeros on right hand side of a number are not in multiple of three then given number is not perfect cube.

6) Form the groups of three digits from right hand side of perfect cube which will give idea of no. of digits in its cube root.

7) The last group at far left may contain one, two or three digits.

This helps in finding out unit place digit of cube root of perfect cube only by observing (विलोकनम) its unit place digit.

8) Now ignore the three digits from right hand side of a number and find the greatest number whose cube is lesser than the remaining digits together of a number as well as adjust obtained unit place digit on its right hand side. The result may be required cube root. Verify the answer by digital root (Beejank).

**e.g.: - Find cube root of 1728**

Form groups of 3 digits From RHS of a number. Here two groups are 1 & 728 are formed hence cube root a given number will have two digits.

From far left group i.e.'1' subtract maximum cube of a digit and this digit will be left digit (left of unit place digit) of cube root. Here '1' is the maximum cube that can subtracted from far left group. Hence '1' will be left digit of cube root.

From right hand side group '728' observe only unit place digit i.e.8 here. As cube of 2 has 8 in its unit place therefore unit place digit of cube root will be'2'

$\sqrt[3]{1728} = 12$

Applying "गुणित समुच्चय: समुच्चय गुणित:" sutra answer is verified.

B (1728) = $(1+\cancel{7}+\cancel{2}+8)$

$= 9$

B $(12)^3 = (1 + 2)3$

$= (3)^3$

$= 27$

$= (2 + 7)$

$= 9$

As beejank is same the answer is verified.

$(a + b + c)^3 = a^3 + 3a^2b +$............................ $+ 3\,bc^2 + c^3$

**e.g.: - Find $\sqrt[3]{78953589}$**

This number has 3 groups.

78, 953, 589

From far left group (78), cube of 4 can be subtracted. Hence far left digit cube root is '4' i.e. 'a' = 4.

Unit place digit of RHS group (589) is '9' and cube of '9' has also '9' in its unit place hence RHS digit of cube root is '9' i.e. c = 9

a = 4

c = 9

Subtract $c^3 = 729$ from the number 78953589

$$\begin{array}{r} 7\,8\,9\,5\,3\,5\,8\,9 \\ -\quad 7\,2\,9 \\ \hline 2\,8\,6\,\cancel{0} \end{array}$$

Discard the zero obtained at unit place of result. From algebraic expression if $c^3$ is subtracted then $3\,bc^2$ will be last term. Comparing terms of arithmetic & algebraic expressions it can be concluded that $3\,bc^2$ has unit place digit as 6. Convert $3\,bc^2$ part in terms of 'b' & observe unit place digit of its coefficient. In this example unit place digit of coefficient of b is '3'. Now in the table of 3 the number having 6 as its unit place digit appears only at second level. Hence b = 2.

$3\,bc^2$ unit place digit as 6

$3 * b * 9^2$ unit place digit as 6

243 b unit place digit as 6

i. e. 3 b unit place digit as 6

$\therefore\ b = \dfrac{\ldots\ldots.\ 6}{\ldots\ldots.\ 3}$

$= 2 \qquad (3 * 2 = 6)$

$\sqrt[3]{78953589} = 429$

Applying "गुणित समुच्चयः समुच्चय गुणितः" sutra answer is verified.

B (78953589) = $(7+8+\cancel{9}+5+3+5+8+\cancel{9})$
= 36
= (3+6)
= 9

B $(429)^3 = (4+2+\cancel{9})^3$
$= 6^3$
= 216
= (2+1+6)
= 9

As beejank is same the answer is verified.

When unit place digit of perfect cube is even digit we get two possible values for ‘b’ then value of cube root is ascertained by beejank method.

**Second method: -**When unit place digit of perfect cube is even digit, it must be divisible by ‘2’. Hence divide the perfect cube by ‘$2^3$’ i.e.8 as it must be least factor in the cube root. After dividing, 4 to 6 digit number is obtained then find its cube root by ‘विलोकनम्’ sutra as explained above. Multiply by cube root of ‘8’ i.e. 2 to get cube root of original number.

**Exercise**: - Find cube-root of

(1) 132651

(2) 250047

(3) 405224

(4) 46656

(5) 13824

## CHAPETER FOUR: DIVISION

“निखिलं नवत: चरमं दशत:” (All from nine and last from ten) sutra is applied when the divisor is nearer to base and less than base. The conventional form of division have four terms-

(1) Dividend (E) (2) Divisor (D) (3) Quotient (Q) (4) Remainder (R).

Relation between these four terms is as below:

**By division algorithm**

(E) = (Q * D) + R

Write the dividend and divisor as in conventional method.

e.g. Base :- 100

| Divisor | Dividend |
|---|---|
| 88 | 10205 |

Deviation :- Base – Divisor

Deviation is calculated by applying "**निखिलं नवत: चरमं दशत:**" sutra.

Deviation: - (9-8), (10-8)

Modified Divisor (MD) => 12

| Divisor | Dividend |
|---|---|
| 88 | 10205 |
| MD=> 12 | |

Dividend is split by vertical line such that the number of digit(s) on RHS of vertical line equals the number of zero in the base. The vertical line also identifies the quotient digit(s) (Q) appearing on its LHS and the remainder digit(s) on its RHS in the final answer.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 102 | 05 |
| MD => 12 | | |

Far left digit of dividend is brought down as 1st quotient digit. Multiply 1st quotient digit with MD digit- wise and product digit is written in the next row under 2nd & 3rd column of dividend.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2 | |

1

Addition of 2nd column digits of dividend gives 2nd digit of quotient.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2 | |

1 1

2nd quotient digit is multiplied with MD digit- wise and product digit is written in the next row under 3rd & 4th column of dividend.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2<br>1 | <br>2 |

1 1

Addition of $3^{rd}$ column digits of dividend gives $3^{rd}$ digit of quotient.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2<br>1 | <br>2 |

1 1 5

$3^{rd}$ quotient digit is multiplied with MD digit-wise. Even though the product is of two digits write obtained number below appropriate column ($4^{th}$ & $5^{th}$) respectively in the next row.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2<br>1 | <br>2<br>5 10 |

1 1 5

As the unit place digit of dividend is reached, the work of MD is over. Now a horizontal line is drawn below last row and column-wise digits are added from right to left.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 88 | 1 0 2 | 0 5 |
| MD => 12 | 1 2<br>1 | <br>2<br>5 10 |
| Quotient | 1 1 5 | 7 $_{1}$5 |
| Quotient | 1 1 5 | 8 5 |

Remainder

**Remainder must be positive number less than divisor always.**

**Answer => Quotient => 115 and Remainder=> 85**

Correctness of answer is verified by "गुणित समुच्चय: समुच्चय गुणित:" sutra.

(The product of the sum of coefficients in the factor is equal to the sum of coefficients in the product)

Dividend (E) = Quotient (Q) X Divisor (D) + Remainder (R).

Beejank (10205) = Beejank (115) X Beejank (88) + Beejank (85)

(1+0+2+0+5) = (1+1+5) X (8+8) + (8+5)

8 = 7 X 16 + 13

= 7 X (1+6) + (1+3)

= 7 X 7 + 4

=49 +4

= (4+9) + 4

=17

= (1+7) = 8

LHS = RHS

## पराव्रर्त्य योजयेत्

### (Transpose & apply)

### (Reverse the sign & use)

When divisor is nearer to base and greater than base then " पराव्रर्त्य योजयेत् " sutra is applied.

**e.g.** **1247 ÷ 113**

**Base**: 100

**Divisor**: 113

**Deviation**: (Divisor - Base)

: (113-100)

**Deviation**: 13

**Transpose**: $\bar{1}\,\bar{3}$

**Modified divisor** (MD): $\bar{1}\,\bar{3}$

Write the dividend and divisor as in conventional method.

| Divisor | Dividend |
|---|---|
| 113 | 1247 |

| Divisor | Dividend |
|---|---|
| 113 | 1247 |
| MD=> $\bar{1}\,\bar{3}$ | |

Dividend is split by vertical line such that the number of digit(s) on RHS of vertical line equals the number of zero in the base. The vertical line also identifies the quotient digit(s) (Q) appearing on its LHS and the remainder digit(s) on its RHS in the final answer.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 113 | 12 | 4 7 |
| MD $\bar{1}$ $\bar{3}$ | | |

Far left digit of dividend is brought down as 1st quotient digit. Multiply 1st quotient digit with MD digit- wise and product digit is written in the next row under 2nd & 3rd column of dividend.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 113 | 1 2 | 4 7 |
| MD $\bar{1}$ $\bar{3}$ | $\bar{1}$ | $\bar{3}$ |

1

Addition of 2nd column digits of dividend gives 2nd digit of quotient.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 113 | 1 2 | 4 7 |
| MD $\bar{1}$ $\bar{3}$ | $\bar{1}$ | $\bar{3}$ |

1 1

2nd quotient digit is multiplied with MD digit- wise and product digit is written in the next row under 3rd & 4th column of dividend.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 113 | 1 2 | 4 7 |
| MD => $\bar{1}$ $\bar{3}$ | ↓ $\bar{1}$ | $\bar{3}$ <br> $\bar{1}$ $\bar{3}$ |

1 1

As the unit place digit of dividend is reached, the work of MD is over. Now a horizontal line is drawn below last row and column-wise digits are added from right to left.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 113 | 1 2 | 4 7 |
| MD => $\bar{1}$ $\bar{3}$ | ↓ $\bar{1}$ | $\bar{3}$ <br> $\bar{1}$ $\bar{3}$ |
| Quotient → | 1 1 | 0 4 |

Remainder

**Remainder must be positive number less than divisor always.**

**Answer => Quotient => 11 and Remainder => 4**

Correctness of answer is verified by "गुणितसमुच्चय: समुच्चय गुणित:" sutra. (The product of the sum of coefficients in the factor is equal to the sum of coefficients in the product)

Dividend (E) = Quotient (Q) X Divisor (D) + Remainder (R).

Beejank (1247) = Beejank (11) X Beejank (113) + Beejank (4)

$(1+\cancel{2}+4+\cancel{7}) = (1+1)$ X $(1+1+3) + (4)$

5 = 2 X 5 + 4

= 10 + 4

= (1+0) + 4

= 5

LHS = RHS

**<u>Importance of zero in modified divisor</u>**

While converting divisor to modified divisor (MD), except at unit place, we may get zero at all other places Tens, hundred's etc. but as these zeros have place value and they have their importance.

e.g. Divisor --> 9

MD --> 1

Divisor --> 99

MD --> 01

Divisor --> 999

MD --> 001

Bifurcation of dividend as LHS and RHS by vertical line for the purpose of demarcation of two sides. RHS of dividend shall contain number of digits equal to the number of zero in the base followed by multiplication alters.

**e.g. Divide 1322 by 9, 99 and 999**

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 9 | 1 3 2 | 2 |
| MD => 1 | 1<br>4 | <br><br>6 |
| Quotient → | 1 4 6 | 8 |

Remainder

Base = 10

Divisor = 9

Deviation = (9-10)

Deviation = $\bar{1}$

Transpose = 1

Modified divisor (MD) = 1

**Answer: - Quotient = 146 & Remainder = 8**

| Division | Dividend | |
|---|---|---|
| | LHS | RHS |
| 99 | 1 3 | 2 2 |
| MD => 0 1 | 0 | 1<br><br>0 3 |
| Quotient | 1 3 | 3 5 |

Remainder

Base = 100

Divisor = 99

Deviation = (99 – 100)

$= 0\bar{1}$

Transpose = 0 1

Modified Div (MD) = 01

**Answer: - Quotient = 13 and Remainder = 35**

| Division | Dividend | |
|---|---|---|
| | LHS | RHS |
| 99 | 1 | 3 2 2 |
| MD => 0 0 1 | | 0 0 1<br><br>0 3 |
| Quotient | 1 | 3 5 3 |

Remainder

Base = 1000

Divisor = 999

Deviation = (999 – 1000)

$= 00\bar{1}$

Transpose = 0 0 1

Modified Div (MD)

= 001

**Answer: - Quotient = 1 & Remainder = 323**

Sometimes remainder is greater than the divisor, to get remainder less than divisor, divide the remainder by divisor and the quotient obtained is carry over to the digit of Quotient i.e. LHS of dividend and then added to it. New remainder obtained is less than divisor hence it is considered as remainder i.e. RHS of dividend.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 8 | 1 1 | 3 |
| MD => 2 | 2 | 6 |
| Quotient | 1 3 | 9 |
| Quotient | 1 3 + 1 | -8 |

8*1

| Quotient | 14 | 1 |
|---|---|---|

Base = 10

Divisor = 8

Deviation = Base-Div.

= 10 - 8

= 2

Modified Div. (MD) = 2

**Answer: - Quotient = 14 and Remainder = 1**

If the remainder is negative then reduce the quotient by 1 and add divisor in the negative remainder to get positive remainder.

| Divisor | Dividend | |
|---|---|---|
| | LHS | RHS |
| 111 | 1 4 5 | 2 1 |
| MD => $\bar{1}$ $\bar{1}$ | $\bar{1}$ $\bar{1}$ | $\bar{3}$ $\bar{3}$ |
| Quotient → | 1 3 | **-1 - 2** |
| | 13 – 1 | + 111 |

Quotient → 12 99 ← Remainder

Base = 100 Divisor = 111 Deviation = (100-111) = $\bar{1}\,\bar{1}$ MD = $\bar{1}\,\bar{1}$

## आनुरूप्येण

If the divisor is not near the base and still want to use "निखिलं नवत: चरमं दशत:" or "परावर्त्य योजयेत्"sutra then the sub sutra "आनुरूप्येण" (proportionately) is used. Here the divisor is multiplied or divided by a suitable number and then "निखिलं नवत: चरमं दशत:" or "परावर्त्य योजयेत्" sutra is used. Finally only quotient part (LHS) is multiplied or divided by that suitable number to get the actual quotient. There will be no change in the remainder (RHS) part.

**e.g. 1624 ÷ 333**

The divisor 333 is not near the base. It can be converted to near the base in 3 different ways and then we can follow " निखिलं नवत: चरमं दशत: "or" परावर्त्य योजयेत् sutra.

1) 333 ÷ 3 = 111

2) 333 * 3 = 999

3) 333 *4 = 1332

If suitable number is used to bring the divisor nearer to base, the quotient part after multiplying or dividing with that suitable number will always be a whole number. But sometimes there is a possibility of getting a fraction in the quotient part. In such cases, the fraction portion of quotient part (LHS) is added to remainder part (RHS) after multiplying with original divisor which will give the final remainder.

**e.g. 1595 ÷ 202** Base = 100

New Divisor = 202 ÷ 2 = 101 (By आनुरूप्येण sutra)

Deviation = (100 – 101) = 0 $\bar{1}$

Modified Divisor (MD) = 0 $\bar{1}$

| Divisor | Dividend | | | |
|---|---|---|---|---|
| | LHS | | RHS | |
| 101 | 1 | 5 | 9 | 5 |
| MD => 0 $\bar{1}$ | ↓ | 0 | $\bar{1}$ | |
| | | | 0 | $\bar{5}$ |
| Modified Quotient → | 1 | 5 | 8 | 0 |

Actual Quotient = 15 / 2

$= 7\frac{1}{2}$ / 80

= 7 / 101 + 80 ←(1/2 * 202 = 101)

= 7 / 181

Quotient = 7 and Final Remainder = 181

## ध्वजांक (On the flag)

Write the dividend and divisor as shown below

e.g.

| Divisor$^{flag}$ | Dividend |
|---|---|
| $7^2$ | 9216 |

We don't have to divide by ‘72’, instead we write down only first digit ‘7’ in the divisor column and place ‘2’on the top (power place) of ‘7’ as 'flag' as shown above. Entire division is performed by ‘7’.

Dividend is split as LHS for quotient and RHS for remainder by vertical line for the purpose of demarcation of two sides. RHS of dividend shall contain number of digits equal to the number of digits on the top (power place) as flag.

Divide the first digit of dividend i.e. ‘9’ by divisor ‘7’ to get ‘1’ as first quotient digit and ‘2’ as the first remainder digit. Put down ‘1’ in quotient column and place the remainder (2) before next dividend digit ‘2’ to form gross dividend (GD) 22.

| Divisor$^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | | |
| Quotient | 1 | | | |

From this (GD) 22 subtract the product of flag digit ‘2’ and 1$^{st}$ quotient digit ‘1’.Net Dividend (ND) = GD - product of flag digit ‘2’ and 1$^{st}$ quotient digit.

= 22 - (2*1)

= 20

| Divisor$^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | | |
| ND | | 20 | | |
| Quotient | 1 | | | |

Divide ND (20) by '7' to get '2' as next quotient digit and '6' as remainder. Place remainder '6' before next dividend digit (1) to get GD (61).

| $Divisor^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | 6 | |
| ND | | 20 | | |
| Quotient | 1 | 2 | | |

Subtract product of flag digit '2' and 2nd quotient digit '2' from GD (61).

ND = 61 - (2* 2)

= 57

| $Divisor^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | 6 | |
| ND | | 20 | 57 | |
| Quotient | 1 | 2 | | |

Divide ND (57) by '7' to get '8' as next quotient digit and '1' as remainder. Place remainder (1) before next dividend digit (6) to get GD (16).

| $Divisor^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | 6 | 1 |
| ND | | 20 | 57 | |
| Quotient | 1 | 2 | 8 | |

As the vertical demarcation line is reached we put a decimal in the quotient at this point i.e. after 8. Subtract product of flag digit 2 and 3rd quotient digit from GD (16).

ND = 16 - (2*8)

= 0

| Divisor$^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | 6 | 1 |
| ND | | 20 | 57 | 0 |
| Quotient | 1 | 2 | 8. | |

Divide ND (0) by’7’, next quotient digit is '0' and remainder also '0'. Since there are no further digits in dividend, this completes the process of division.

| Divisor$^{flag}$ | Dividend | | | |
|---|---|---|---|---|
| | LHS | | | RHS |
| $7^2$ | 9 | 2 | 1 | 6 |
| GD | | 2 | 6 | 1 |
| ND | | 20 | 57 | 0 |
| Quotient | 1 | 2 | 8. | 0 |

Final Answer: - Quotient (Q) = 128 and Remainder (R) = 0

**Remainder must be positive number less than divisor always.**

**Answer => Quotient => 128 and Remainder=> 0**

Correctness of answer is verified by "गुणित समुच्चय: समुच्चय गुणित:" sutra.

(The product of the sum of coefficients in the factor is equal to the sum of coefficients in the product)

Dividend = Quotient X Divisor + Remainder.

Beejank (9216) = Beejank (128) X Beejank (72) + Beejank (0)

$(9+\cancel{2}+\cancel{1}+\cancel{6}) = (\cancel{1}+2+\cancel{8}) \text{ X } (7+2) + (0)$

9 = 2 X 9 + 0

= 18

= (1+8)
= 9

LHS = RHS

**EXERCISE:-**

1. 222 5 ÷ 112
2. 14916 ÷113
3. 1520 ÷14
4. 255407 ÷ 12
5. 1567243 ÷ 1437
6. 1011 ÷ 23
7. 7685 ÷ 672
8. 1111 ÷ 8
9. 21205 ÷98
10. 1213 ÷ 11

# CHAPTER FIVE: HCF OR GCD

## Meaning of HCF or GCD

Common maximum value of divisor from possible divisors of given numbers is called as HCF or GCD of the given numbers. Divisibility rules can be used to find GCD.

**Factorisation method: -**

**e.g.: - Find GCD of 671, 781, 1441.**

Possible divisors of 671 = 1, 11, 61, 671

Possible divisors of 781 = 1, 11, 71,781

Possible divisors of 1441 = 1, 11,131, 1441

Common maximum value of divisor for numbers 671, 781, 1441 is "11".

Hence GCD of 671, 781, 1441 = '11'

**Division algorithm method: -** Dividend (E) = Divisor (D) * Quotient (Q) + Remainder (R)

Considering greater number as dividend and smaller number as divisor, division operation is performed.

Dividend = 781

Divisor = 671

781 = 671 * 1 + 110

Now previous divisor i.e.' 671' will become dividend and remainder i.e. '110' will become divisor. 671 = 110 *6 + 11

Same procedure is continued till getting '0' as remainder.

110 = 11 * 10 + 0

Then remainder of previous step or divisor of last step will GCD of two numbers 671 & 781.

GCD of 781 & 671 = 11 ..................................... (1)

Similarly GCD of numbers 1441 & 781 is found.

1441 = 781 * 1 + 660

781 = 660 * 1 + 121

660 = 121 * 5 + 55

12 1= 55 * 2 + 11

55 = 11 * 5 + 0

GCD of 1441 & 781 = 11 ................................ (2)

From (1) & (2)

GCD of 671, 781, 1441 = 11

---

'संकलनव्यवकलनाभ्याम्' sutra.

By addition & subtraction GCD can be found.

**e.g.: - Find GCD of 30, 40 and 50**

30 + 40 = 70 संकलन

70 – 50 = 20 व्यवकलन

70 = 20 *3 + 10

20 = 10 * 2 + 0

GCD of 30, 40 50 = 10

---

**<u>Exercise</u>: - Find the GCD**

1. 30, 40, 50
2. 35, 45, 75
3. 65, 156, 221
4. 119, 238, 255
5. 63, 30, 87
6. 117, 195, 156
7. 156, 231, 99
8. 671, 781, 1430
9. 266, 456, 190

# LCM

**e.g. Find LCM of 15, 25, and 35**

Select any two numbers out of three numbers.

Ratio of these numbers is established by 'आनुरूप्येण' sutra.

25: 15

$$\frac{25}{15} = \frac{\cancel{5} * 5}{\cancel{5} * 3}$$

Cancel the common factor (s). Then cross multiply 'उर्ध्वतिर्यग्भ्याम्' sutra.

25 * 3 = 15 * 5 = 75

LCM of 15 & 25 = 75

Now consider 75 & remaining number i.e. 35. Their ratio is established by 'आनुरूप्येण' sutra and similar procedure is applied.

$$\frac{75}{35} = \frac{3 * \cancel{5} * 5}{\cancel{5} * 7}$$

Cancel the common factor (s). Then cross multiply 'उर्ध्वतिर्यग्भ्याम्' sutra.

75 * 7 = 35 * 3 * 5 = 525

LCM of 15, 25, 35 = 525

$LCM = \dfrac{N_1 * N_2}{\text{GCD or HCF}}$ where $N_1$ & $N_2$ are the numbers whose LCM is to be found.

---

**<u>EXERCISE</u> - Find LCM**

1. 15, 25, 35
2. 30, 20, 40
3. 112, 140, 168

## DIVISION ALGORITHM

**$a = b(q) + r$**

Where a = Dividend (Integer)

b = Divisor (Integer)

q = Quotient (Integer)

r = Remainder (Integer)

$|r| < b$

Find the GCD of 13 and 44. Express it as 13x + 44y where 'x' and 'y' ∑ z in two ways.

a = Dividend (44)

b = Divisor (30)

$44 = (13 * 3) + 5$

$13 = (5 * 2) + 3$

$5 = (3 * 1) + 2$

$3 = (2 * 1) + 1$

$2 = (1 * 1) + 0$

GCD of 13 & 44 = 1

Last divisor = 1

$1 = 3 - (2*1)$

$1 = 3 - [5 - (3*1)]*1$

$= 3 - (5*1) + (3*1)$

$= 3(2) - 5(1)$

$= [13 - 5(2)]\ 2 - 5(1)$

$= 13(2) - 5(4) - 5(1)$

$= 13(2) - 5(5)$

= 13(2) – [44 – 13(3)] 5

= 13(2) – 44(5) + 13(15)

= 13(17) – 44(5)

= 13(17) + 44(-5)

Comparing with 13x + 44y

x = 17 and y = -5 ------------------ (1)

We have

1 = 13(17) + 44(-5)

= 13(17) + 44(-5) + 13(44) + 44(-13)

= 13(61) + 44(-18)

x = 61 and y = -18------------------- (2)

---

**Find the GCD of 5 and 17. Express it as 5x + 17y where 'x' and 'y' ∑ z in two ways.**

17 = (5*3) + 2

5 = (2*2) + 1

2 = (1*2) + 0

1 = 5 – 2(2)

= 5 – [17- 5(3)] 2

= 5 – 17(2) + 5(6)

= 5(7) – 17(2)

= 5(7) + 17(-2)

**Comparing with 5x + 17y**

x = 7 and y = -2 --------- (1)

1 = 5(7) + 17(-2)

$= 5(7) + 17(-2) + 5(17) + 17(-5)$

$= 5\ (24) + 17(-7)$

$x = 24$ and $y = -7$ ----------- (2)

$1 = 5(24) + 17(-7)$

$= 5(24) + 17(-7) + 5(17) + 17(-5)$

$= 5(41) + 17(-12)$

$x = 41$ and $y = -12$ --------- (3)

# CHAPTER SIX: DIVISIBILITY

During division when no remainder is left then we say that dividend is divisible by the divisor. By direct division method it is much difficult to confirm divisibility. Some general rules to decide divisibility are furnished below

| Divisibility test for divisor | Divisibility norms |
|---|---|
| 2 | If unit place digit of dividend is even (0,2,4,6 or 8) |
| 3 | If digital root (Beejank) of dividend is 3, 6 or 9 |
| 4 | Number formed by tens place digit & unit place digit of dividend, if divisible by 4 |
| 5 | If unit place digit of dividend is 0 or 5 |
| 6 | If unit place digit of dividend is 0, 2, 4, 6 or 8 and digital root (Beejank) of dividend is 3, 6 or 9 |
| 7 | Cancel the unit place digit of dividend and subtract its double from the remaining part of dividend. Continue the procedure till subtraction result is multiple of 7. If the result of subtraction is divisible by 7 |
| 8 | Number formed by hundreds place digit, tens place digit & unit place digit of dividend, if divisible by 8 |
| 9 | If digital root (Beejank) of dividend is 9 |
| 10 | If unit place digit of dividend is zero |
| 11 | The difference of addition of alternate place digits (odd & even) of dividend is either zero or multiple of 11 |
| 12 | Digital root (Beejank) of dividend is 3, 6 or 9 and number formed by tens place digit & unit place digit of dividend, if divisible by 4 |

In conventional method there are no such easy rules to test for odd numbers like 13, 17, 19, 29, 31, ------ or for the numbers having these prime numbers as one of the factors. Vedic mathematics provides some easiest ways to test the divisibility of such numbers.

**<u>Vedic Sub – Sutra</u>: -वेष्टनम्  (osculation) is used.**

1) वेष्टनांक , वेष्टनांक क्रिया, वेष्टनांक क्रियाफल i.e. Osculator, osculation and osculation result.

2) Different types of osculators:-

P:- Positive Osculators.

Q:- Negative Osculators

U: digit in unit place.

T: digit in Tens place.

**About Osculators (Veshtanank).**

Osculator is a parameter obtained from divisor which is the basic requirement for divisibility.

**To determine positive Osculator of a given divisor.**

[1] Multiply the divisor by such a minimum number which yields nine at the unit place in the product

[2] Add one to the product.

[3] Omit zero appearing at unit place.

[4] The remaining digit(s) gives Osculator of that divisor.

**Ex.1** To find Osculator of 7 (the divisor)

$(7 \times 7) + 1 = 50$ Osculator of 7 is 5.

**Ex.2** Osculator of 23 is 7. Because $(23 \times 3) + 1 = 7\cancel{0} = 7$

**Ex.3** As $(47 \times 7) + 1 = 330$ Osculator is 33.

**To determine negative Osculator of a number (divisor).**

[1] Multiply the divisor by such a minimum number which yields one at the unit place in the product.

[2] Subtract one from the product

[3] Omit zero appearing at the unit place.

[4] The remaining digit(s) give Osculator of divisor.

**Ex.1** To find Osculator of 17.

We have $17 \times 3 = 51$ Osculator of 17 is $(51 - 1) = 5\cancel{0} = 5$

**Ex.2** The Osculator of 13 is 9 as $13 \times 7 = 91$, Osculator of 13 is $(91 - 1) = 9\cancel{0} = 9$

**List of positive Oscillators (P)**

1) P for 9, 19, 29, 39 etc. (ending in 9) is 1,2,3 and 4 respectively.

2) P for 3,13,23, 33 etc. (ending in 3) is 1, 4, 7 and 10 respectively.

3) P for 7,17,27,37 etc. (ending in 7) is 5, 12, 19 and 26 respectively.

**List of negative Osculators (Q)**

1) Q for 11,21,31,51 etc. (all ending in 1) is 1, 2, 3 and 5 respectively

2) Q for 7, 17,27, 37,47 etc. (all ending in 7) is 2,5,8,11,14 resp.

3) Q for 3,13,23,33,43 etc. (all ending in 3) is 2,9, 16,23 resp.

4) Q for 9,19,29,39,49etc. (all ending in 9) are 8, 17,26,35 etc.

**Important Features of Oscillators**

1) P + Q = D,  D: Divisor.

2) For divisors ending in 3 P < Q.

3) For divisors ending in 7 Q < P.

. If unit place digit of divisor is '3' or '9' then positive osculator (P) is suitable. If unit place digit of divisor is '1' or '7' then negative osculator (Q) is suitable.

When negative osculator (Q) is used even place digits(2nd, 4th, 6th, ---------) of dividend from right are considered as negative by giving bar on these digits before proceeding for osculation process.

Digit by digit osculation process involves multiplying unit place digit of dividend by positive osculator or negative osculator and adding this product to its previous (left) digit of dividend. This process is continued till highest place digit of dividend is osculated. If final sum is zero, divisor or multiple of divisor then dividend is divisible by divisor otherwise not. If the osculated sum so obtained at certain place using positive osculator is of two digits here also multiply unit place digit with osculator and add its tens place digit as well as add tens place digit of dividend to get next osculated sum. If the osculated sum so obtained at certain place using negative osculator is of two digits here also multiply unit place digit with osculator and add its tens place digit considering it as –ve digit since it is even place digit as well as add tens place digit of dividend to get next osculated sum.

**Method to find out quotient part of original divisor using positive osculator**

If unit place digit of final osculated sum is '9' then quotient of the division can be found by writing complements from '9' of unit place digit of osculated sums obtained followed by complement from '10' of unit place digit of dividend. (निखिलं नवत: चरमं दशत:) sutra is applied.

**If final osculated sum is not ending with '9' but it is divisible by divisor then add suitable multiple of divisor at that place so that unit place digit of final osculated sum will be'9' and finally from quotient so obtained subtract that multiple with its place value to get actual quotient.**

**Ex.** Test if 871 is divisible by 13

Multiply 13 by 3 to get 9 at unit place.

13 * 3 = 39 applying Vedic sutra Ekadhiken Purven osculator of 39 is '4' & it is positive osculator. Here multiplication and addition involves place to place work. Write the dividend with a little gap between each digit. Osculate unit place digit of

dividend & add the product to its previous (left) digit of dividend. Write the sum below this digit. Continue same procedure till last (far left) digit of dividend is added. If final osculated sum is same as divisor or its multiple then dividend is divisible otherwise not divisible.

| | | | |
|---|---|---|---|
| 8 | 7 | 1 | (* 4) |
| 13 | 11 | | |

As last osculated sum is '13' which is divisor itself hence '871' is divisible by '13'

**Alternative Way:-** 871 / 13

| | | | |
|---|---|---|---|
| | 8 | 7 | 1 |
| + | 0 | 4 | |
| | 9 | 1 | |
| + | 4 | | |
| | 13 | | |

1) P = 4

2) U × P = 1 × 4 = 4

Add to previous & continue

3) Osculation Results are:

$S_1$ = 9 1

$S_2$ = 1 3

Last Osculation Result is 13which is divisor itself hence '871' is divisible by '13'

As final osculated sum is not ending with '9' but it is divisible by divisor then suitable multiple of divisor is added at that place so that unit place digit of final osculated sum will be '9' and finally from quotient so obtained subtract that multiple with its place value to get actual quotient.

| | | | | |
|---|---|---|---|---|
| 8 | 7 | 1 | (* 4) | |
| | 13 | 11 | | |
| + | 26 | 0 | 0 | |
| | 39 | 11 | 1 | |
| | 0 | 8 | 9 | ← quotient for '39' |

The quotient (89) is for divisor 39.**$Q_{39}$ = 89**

To get $Q_{13}$ (quotient for original divisor 13), **multiply $Q_{39}$ by 3.**

**$Q_{13}$ = 89 * 3 = 267**

Here multiple 2 of 13 (i.e.26) was added at hundreds place. Hence to get final value of quotient subtract 200 from **$Q_{13}$ = 267.**

∴ **Actual quotient = 267 – 200 = 67**

**------------------------------------------------------------------------------------------------------**

**Ex. Test by osculation whether 209437 is divisible by 19**

Here osculator of '19' is '2' using Vedic sutra एकाधिकेन पूर्वेण& it is positive osculator. Here multiplication and addition involves place to place work. Write the dividend with a little gap between each digit. Osculate unit place digit of dividend & add the product to its previous (left) digit of dividend. Write the sum below this digit. Continue same procedure till last (far left) digit of dividend is added. If final osculated sum is same as divisor or its multiple then dividend is divisible otherwise not divisible.

| | | | | | |
|---|---|---|---|---|---|
| 2 | 0 | 9 | 4 | 3 | 7 |
| 19 | 18 | 28 | 19 | 17 | |

As last osculated sum is '19' which is divisor itself hence '209437' is divisible by '19'

**Alternative Way:-** 209437 / 19

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 0 | 9 | 4 | 3 | 7 |
| + | | | | 1 | 4 | |
| | 2 | 0 | 9 | 5 | 7 | |
| + | | | 1 | 4 | | |
| | 2 | 1 | 0 | 9 | | |
| + | | 1 | 8 | | | |
| | 2 | 2 | 8 | | | |
| + | 1 | 6 | | | | |
| | 3 | 8 | | | | |
| + | 16 | | | | | |
| | 19 | | | | | |

1) P = 2

2) U × P = 7 × 2 = 14

Add to previous & continue

3) Osculation Results are:

$S_1$ = 2 9 0 5 7

$S_2$ = 2 10 9

S3 = 2 2 8

S4 = 3 8

S5 = 1 9

Last Osculation Result is 19which is divisor itself hence ‘209437’ is divisible by ‘19’

As dividend is divisible by divisor, quotient part of this division is found by applying निखिलं नवत: चरमं दशत: sutra to unit place digits of osculated sum obtained and unit place digit of dividend. Write the compliments from ‘9’ of unit place digits of osculated sums obtained followed by complement from ‘10’ of unit place digit of dividend.

| | | | | | |
|---|---|---|---|---|---|
| 2 | 0 | 9 | 4 | 3 | 7 |
| 19 | 18 | 28 | 19 | 17 | ↓ |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| 0 | 1 | 1 | 0 | 2 | 3 |

Hence quotientof 209437 ÷ 19 = 11023

**<u>Method to find out quotient part of original divisor using negative osculator</u>**

When unit place digit of final osculated sum is ‘0’ then simply write the unit place digits of each osculated stage sum and of dividend as well as have to reverse the sign of even place digits from right ($2^{nd}$, $4^{th}$, ---------). Finally converting vinculum number to conventional number we get actual quotient.

**If unit place digit of final osculated sum is not ‘0’ or ‘1’ but it is divisible by the divisor. When no. of digits in dividend is odd (i.e. 3,5,7,9 ...) then add suitable multiple of divisor to final osculated sum to get unit place digit of final osculated sum as ‘1’. When no. of digits in dividend is even (i.e. 2,4,6,8 ...) then subtract suitable multiple of divisor from final osculated sum to get unit place digit of final osculated sum as ‘0’**

**Ex. Test by osculation whether 784 is divisible by 7**

(A) To bring the divisor ending with ‘1’ we have multiplied it by ‘3’ (i.e. 7 *3 = 21). Here osculator of ‘7’ is ‘2’ using Vedic sutra एकन्यूनेन पूर्वेण& it is negative osculator. Here multiplication and addition involves place to place work. Write the dividend with a little gap between each digit. Mark even place digits from right as ‘-ve’ digits. i.e. $2^{nd}$, $4^{th}$, $6^{th}$, -------- digits from right are considered ‘-ve’ digits. Osculate unit place digit of dividend & add the product to its previous (left) digit of dividend but care must be taken during addition of ‘-ve’ digits. Write the sum below this digit. If osculated sum so obtained at certain place is of two digits here also tens place digit is to be considered as ‘-ve’ Continue same procedure till last (far left) digit of dividend is added. If final osculated sum is ‘0’, divisor or its multiple then dividend is divisible otherwise not divisible. .

| | | | |
|---|---|---|---|
| 7 | $\overline{8}$ | 4 | * 2 |
| 7 | 0 | | |

As last osculated sum is ‘7’ which is divisor itself hence ‘784’ is divisible by ‘7’

**Alternative Way:-** 784 / 7

| | | | | |
|---|---|---|---|---|
| | 7 | $\overline{8}$ | 4 | * 2 |
| + | 0 | 8 | | |
| | 7 | 0 | | |
| + | 0 | | | |
| | 7 | | | |

1) Q = 2

2) U × Q = 4 × 2 = 8

Add to previous & continue

3) Osculation Results are:

$S_1$ = 7 0

$S_2$ = 7

Last Osculation Result is 7 which is divisor itself hence ‘784’ is divisible by ‘7’

As unit place digit of final osculated sum is not ‘0’ or '1' but it is divisible by divisor in such cases to find quotient we have to add suitable multiple of divisor to final osculated sum to get a number whose unit place digit will be '1’since no. of digits in a number are odd. Finally from quotient so obtained subtract that multiple with its place value to get actual quotient.

7 $\bar{8}$ 4 * 2

7 0

+14

21

1 0 4

$Q_{21}$ = 104

$Q_7$ = (104 * 3 - 200)

= (312 – 200)

= 112

(B) Here osculator of ‘7’ is ‘5’ using Vedic sutra एकाधिकेन पूर्वेण& it is positive osculator. Here multiplication and addition involves place to place work. Write the dividend with a little gap between each digit. Osculate unit place digit of dividend & add the product to its previous (left) digit of dividend. Write the sum below this digit. Continue same procedure till last (far left) digit of dividend is added. If final osculated sum is same as divisor or its multiple then dividend is divisible otherwise not divisible.

| | | | |
|---|---|---|---|
| 7 | 8 | 4 | * 5 |
| 49 | 28 | | |

As last osculated sum is ‘49’ which is divisible by divisor’7’ hence ‘784’ is divisible by ‘7’

**Alternative Way:-** 784 / 7

| | | | | |
|---|---|---|---|---|
| | 7 | 8 | 4 | * 5 |
| + | 0 | 20 | | |
| | 7 | 28 | | |
| + | 42 | | | |
| | 49 | | | |

1) P = 5

2) U × P = 4 × 5 = 20

Add to previous & continue

3) Osculation Results are:

$S_1 = 7 \quad 28$

$S_2 = 49$

Last Osculation Result is 49which is divisible by divisor'7' hence '784' is divisible by 7. As unit place digit of final osculated sum is '9', quotient part of this division is found by applying निखिलं नवत: चरमं दशत: sutra to unit place digits of osculated sum obtained and unit place digit of dividend. Write the compliments from '9' of unit place digits of osculated sums obtained followed by complement from '10' of unit place digit of dividend.

| | | | |
|---|---|---|---|
| 7 | 8 | 4 | * 5 |
| 49 ↓ | 28 ↓ | ↓ | |
| 0 | 1 | 6 | |

$Q_{49} = 16$

$Q_7 = 16 * 7$

$= 112$

**Ex.**114959 ÷ 37 (37 * 3) = 111, Negative Osculator = 11

| | | | | | | |
|---|---|---|---|---|---|---|
| $\bar{1}$ | 1 | $\bar{4}$ | 9 | $\bar{5}$ | 9 | * 11 |
| 37 | $\bar{6}4$ | $\bar{3}6$ | $\bar{4}4$ | $\bar{9}4$ | | |

As the last osculated number is divisor itself hence 114959 is divisible by 37

**Alternative Way:-** 114959 / 37

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\bar{1}$ | 1 | $\bar{4}$ | 9 | $\bar{5}$ | 9 | * 11 |
| + | | | | | 99 | | |
| | $\bar{1}$ | 1 | $\bar{4}$ | 9 | $\bar{9}4$ | | |
| + | | | | 35 | | | |
| | $\bar{1}$ | 1 | $\bar{4}$ | $\bar{4}4$ | | | |
| + | | | 40 | | | | |
| | $\bar{1}$ | 1 | $\bar{3}6$ | | | | |
| + | | 63 | | | | | |
| | $\bar{1}$ | $\bar{6}4$ | | | | | |
| + | 38 | | | | | | |
| | 37 | | | | | | |

1) P = 11

2) U × P = 9 × 11 = 9 9

Add to previous & continue

3) Osculation Results are:

$S_1 = \bar{1}\ \ 1\ \ \bar{4}\ \ 9\ \ \bar{9}4$

$S_2 = \bar{1}\ \ 1\ \ \bar{4}\ \ \bar{4}4$

$S_3 = \bar{1}\ 1\ \ \bar{3}6$

$S_4 = \bar{1}\ \ \bar{6}4$

$S_5 = 3\ 7$

Last Osculation Result is 37which is divisor itself hence ‘114959’ is divisible by ‘37’

To get quotient if we do not get unit place digit of final osculated sum as ‘0’ but divisible by divisor in such cases to find quotient we have to subtract suitable multiple of divisor at that place since no. of digits in dividend are even as well as have to reverse the sign of even place digits from right (2nd, 4th, ---------). Finally from quotient so obtained subtract that multiple with its place value to get actual quotient

| | | | | | |
|---|---|---|---|---|---|
| $\bar{1}$ | 1 | $\bar{4}$ | 9 | $\bar{5}$ | 9 |
| 37 | $\bar{6}4$ | $\bar{3}6$ | $\bar{4}4$ | $\bar{9}4$ | |
| -37 | — | — | — | — | — |
| 0 | 4 | $\bar{6}$ | 4 | $\bar{4}$ | 9 |

Converting vinculum number to conventional number = 34369 ⟶ quotient for ‘111’

Quotient for ‘37’ = (34369 * 3) – 100000

= (103107 – 100000) = 3107

114959 ÷ 37 = 3107

**Exercise**: - Confirm the divisibility of the examples.

(1) 17051 ÷ 17 (2) 26182 ÷ 13 (3) 28122 ÷ 129 (4) 93821 ÷ 91

(5) 213522 ÷ 412 (6) 158268 ÷ 242

## DIVISIBILITY -- EVEN DIVISOR

Divisibility test for the numbers ending with even digits 2, 4, 6 & 8.

**Determination of positive osculator (P)**

(a) Numbers ending with ‘8’: Omit 8 of unit place & one more of remaining number is positive osculator.
e.g. ‘28’ : omit ‘8’ of ‘28’ & one more of remaining ‘2’ which is 2 +1 = 3 is a positive osculator of 28.

(b) Numbers ending with ‘6’: Multiply the number by 3 to get number ending with 8.

e.g. 16: 16 x 3 = 48. Omit 8 of 48 & one more of remaining 4 which is 4 + 1 =5 is a positive osculator of 16.

(c) Numbers ending with ‘4: Multiply the number by 2 to get number ending with 8.

e.g. 14: 14 x 2 = 28. Omit 8 of 28 & one more of remaining 2 which is 2 + 1 = 3

is a positive osculator of 14.

(d) Numbers ending with ‘2’: Multiply the number by 4 to get number ending with 8.

e.g. 12: 12 x 4 = 48. Omit 8 of 48 & one more of remaining 4 which is 4 + 1 = 5 is a positive osculator of 12.

## DETERMINATION OF NEGATIVE OSCULATOR

(A) Numbers ending with ‘2’: Omit 2 of unit place. Remaining number itself is its negative osculator.
e.g. 32: omit 2 of 32. Remaining number 3 is negative osculator of 32.

(B) Numbers ending with 6: Multiply by 2 to get number ending with 2.

e.g. 16: 16 x 2 = 32. Omit 2 of 32. Remaining number 3 is negative osculator of 16.

(C) Numbers ending with 4: Multiply by 3 to get number ending with 2.

e.g. 14: 14 x 3 = 42. Omit 2 of 42. Remaining number 4 is negative osculator of 14.

(D) Numbers ending with 8 : Multiply by 4 to get number ending with 2.

e.g. 8: 8 x 4 = 32. Omit 2 of 32. Remaining number 3 is negative osculator of 8

Generally when numbers ends with 4 & 8 it is preferable to use positive osculator and when numbers ends with 2 & 6 preferably negative osculator is used for osculation process. Sum of positive osculator & negative osculator is half of divisor.

e.g. Divisor = 08. Positive osculator = 0 + 1 =1. Negative osculator =8*4 = 32 = 3.

Positive osculator + Negative osculator = 1 + 3 = 4 = ½(Divisor ‘8’)

## OSCULATION PROCESS

**Sutra**:**सोपान्त्यद्वयमन्त्यम्**(Ultimate and twice the penultimate)

**Meaning**: Ultimate means unit place digit

Penultimate means number excluding unit place digit i.e. remaining number. Multiply unit place digit of dividend by positive/negative osculator. Add/Subtract this product to/from twice of remaining number.Continue the process till last digit of dividend is osculated. If the last osculated number is zero,divisor or its multiple then original dividend is divisible by given divisor.

e.g. Whether 6748 is divisible by 28?

28 + 2 = 3$\cancel{0}$

Positive osculator of 28 = 3

Step I: (2 * 674) + (8 * 3) = 1372

Step II : (2 * 137) + (2 * 3) = 280

Step III: (2 * 28) + (0 * 3) = 56

Step IV: (2 * 5) + (6 * 3) = 28

As last osculated number (28) is divisor itself hence 6748 is divisible by 28 --- (Ans)

---

**e.g. Whether 208736 is divisible by 32?**

32 – 2 = 3$\cancel{0}$

Negative osculator = 3

Step I: (2 * 20873) – (6 * 3) = 41728

Step II: (2 * 2087) – (8 * 3) = 4150

Step III: (2 * 208) – (0 * 3) = 416

Step IV: (2 * 41) – (6 * 3) = 64

Step V: (2 * 6) – (4 * 3) = 0

As last osculated number is ‘0’ therefore 208736 is divisible by 32 ------------ (Ans).

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## <u>2<sup>ND</sup> PROCEDURE</u>

(1) Write multipliers above each place of dividend from right to left in order as 1,2,4,8, (1st row)
(2) Multiply unit place digit with positive or negative osculator.
(3) When negative osculator is used then consider alternate digits (2nd,4th,6th ----) from right of dividend as negative (bar) digits for osculation process.
(4) Product of tens place digit & its multiplier + product of unit place digit & positive or negative osculator. Write sum below tens place digit.
(5) Osculate the sum at tens place as well as add product of hundreds place digit & its multiplier. Write this sum below hundreds place digit.
(6) Continue this procedure till last (far left) digit of dividend is osculated.
(7) Always multiply penultimate digit (s) by ‘2’ at any stage,

If last osculated sum is zero,divisor or its multiple then original dividend is divisible by given divisor.

**e.g**. Whether 3834 is divisible by 18?

18 + 2 = $2\cancel{0}$ (सोपान्त्यद्वयमन्त्यम् sutra)

Positive osculator = 2

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 4 | 2 | 1 | ⟵ | 1st row | ⟵ | Multiplier |
| 3 | 8 | 3 | 4 | ⟵ | 2nd row | ⟵ | Dividend |
| 36 | 42 | 14 | | ⟵ | 3rd row | ⟵ | Osculated sum |

Step I: (4 * 2) + (3 * 2) = 14

Step II: (4 * 2) + (1 * 2) + (8 * 4) = 42

Step III: (2 * 2) + (4 * 2) + (3 * 8) = 36

As last osculated number (36) is a multiple of divisor (18) therefore dividend (3834) is divisible by given divisor (18).

---

**E.g.** Whether 6784 is divisible by 32?

32 - 2 = $3\cancel{0}$ (सोपान्त्यद्वयमन्त्यम्sutra)

Negative osculator = 3

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 4 | 2 | 1 | ⟵ | $1^{st}$row | ⟵ | Multiplier |
| $\overline{6}$ | 7 | $\overline{8}$ | 4 | ⟵ | $2^{nd}$ row | ⟵ | Dividend |
| $\overline{3}\,\overline{2}$ | 16 | $\overline{4}$ | | ⟵ | $3^{rd}$ row | ⟵ | Osculated sum |

Step I: $(4 * 3) + (\overline{8} * 2) = \overline{4}$

Step II: $(\overline{4} * 3) + (7 * 4) = 16$

Step III: $(6 * 3) + (\overline{1} * 2) + (\overline{6} * 8) = \overline{3}\,\overline{2}$

As last osculated number ($\overline{3}\,\overline{2}$) is divisible by divisor (32) therefore 6784 is divisible by given divisor (32)

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## OSCULATION BY DIVISION

Osculation can also be performed in reverse way i.e. from left (highest place digit of dividend) to right (unit place digit of dividend).

Routine osculation procedure is performed by multiplying unit place digit of dividend by osculator and adding the product (for '+'ve osculator) or subtracting the product (for '-'ve osculator) from its next left hand digit i.e. Tens place digit as well as this procedure continued till the highest place digit is osculated.

In osculation by division highest place digit of dividend is divided by osculator followed by adding $1^{st}$ quotient digit (for '+'ve osculator) or subtracting $1^{st}$ quotient from its next order digit on right hand side of dividend and this procedure is continued till unit place digit of dividend.

Osculation by division procedure proves following

(1) If dividend is not divisible by given divisor, it gives remainder at the end of osculation process.

(2) If dividend is divisible by given divisor then it gives quotient part.

This method gives quotient and remainder simultaneously which is similar to 'Flag digit' division method.

P → Positive osculator for numbers ending with odd digits (1, 3, 7, 9) → एकाधिकेन sutra.

Q → Negative osculator for numbers ending with even digits (2, 4, 6, 8) → एकन्यूनेन sutra.

**<u>OSCULATION BY DIVISION PROCEDURE FOR NUMBER ENDING WITH ODD DIGITS (1, 3, 7, 9) USING POSITIVE OSCULATOR</u>**

P ← Positive Osculator ← एकाधिकेन sutra.

Highest place digit of dividend is divided by '+'ve osculator in terms of quotient and remainder. Write 1st quotient digit below highest place digit in 4th row and prefix remainder to its next order digit of dividend on right hand side which will be next dividend. Add 1st quotient digit to this dividend which will give working dividend placed in 3rd row below 2nd digit of dividend. Now divide this working dividend by osculator to get 2nd quotient digit and write it in 4th row to right hand side of 1st quotient digit. This procedure is continued till unit place digit is osculated.

If the last osculated number is zero, divisor or multiple of divisor then dividend is divisible by given divisor.

**(a) If final osculated number is divisor itself which is equivalent to one quotient. Hence one is added to quotient part to get actual quotient.**

e.g. 7337÷29

$29 + 1 = 3\cancel{0}$

Positive osculator (P) = 3

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 ÷ | 7 | $_{1}3$ | $_{0}3$ | $_{2}7$ | ← Dividend | ← 1st row |
| | | +2 | +5 | +2 | ← Add quotient digit | ← 2nd row |
| | | 15 | 8 | 29 | ← Working dividend | ← 3rd row |
| | 2 | 5 | 2 | | ← Quotient | ← 4th row |

As last osculated number (29) is divisor itself hence 7337 is divisible by given divisor (29). Quotient = 252.

Remainder = 29 which is equivalent to one quotient and to be added to quotient part.

Actual Quotient = (252 + 1) = 253

Actual Quotient = 253 and Actual Remainder = 0. --------- (Ans)

B (253) * B (29) + B (0) = (2 + 5 + 3) * (2 + $\cancel{9}$) + (0)

= 10 * 2 + 0

= 20

= (2 + 0)

= 2

B (7337) = (7 + 3 + 3 + 7)

= 20

= (2 + 0)

= 2 Answer is verified.

**(b) If last osculated number is less than divisor then it indicates that dividend is not divisible by given divisor and gives actual quotient as well as actual remainder.**

e.g. 61273 ÷ 59

59 + 1 = 6$\cancel{0}$

Positive osculator (P) = 6

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 ÷ | 6 | $_0$1 | $_2$2 | $_4$7 | $_2$3 | ← | Dividend | ← | 1st row |
| | | +1 | +0 | +3 | +8 | ← | Add quotient digit | ← | 2nd row |
| | | 2 | 22 | 50 | 31 | ← | Working dividend | ← | 3rd row |
| | 1 | 0 | 3 | 8 | | ← | Quotient | ← | 4th row |

As last osculated number (31) is neither divisor nor its multiple hence 61273 is not divisible by given divisor (59).

Actual Quotient = 1038 and Actual Remainder = 31 ---------------- (Ans).

**(c) If last osculated number is greater than given divisor then subtract one time or more time divisor which is equivalent to one quotient or more quotientand add this one or more quotient to quotient part as well as difference after subtraction gives actual remainder.**

e.g. 5234157 ÷ 23

$$23 \times 3 = 69$$

$$69 + 1 = 7\cancel{0}$$

Positive osculator = 7

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 7 ÷ | 5 | $_{5}2$ | $_{3}3$ | $_{5}4$ | $_{3}1$ | $_{4}5$ | $_{1}7$ | ← Dividend | ← $1^{st}$ row |
| | | +0 | +7 | +5 | +8 | +5 | +7 | ← Add quotient digit | ← $2^{nd}$ row |
| | | 52 | 40 | 59 | 39 | 50 | 24 | ← Working dividend | ← $3^{rd}$ row |
| | 0 | 7 | 5 | 8 | 5 | 7 | | ← Quotient | ← $4^{th}$ row |

As last osculated number (24) is neither divisor nor its multiple hence 5234157 is not divisible by 23.

Quotient $_{(69)}$ = 75857

Quotient $_{(23)}$ = 75857 * 3 = 227571

Remainder = 24 which greater than divisor (23). Remainder must be less than divisor always.

Dividing Remainder (24) by Divisor (23) which gives ‘1’ as Quotient & ‘1’ as Remainder.

Actual Quotient (23) = 227571 + 1 = 227572

Actual Remainder = 1 --------------- (Ans)

B (227572) * B (23) + B (1) = $(2 + \cancel{2} + \cancel{7} + 5 + \cancel{7} + \cancel{2}) * (2 + 3) + 1$

$= 7 * 5 + 1$

$= 36$

$= (3 + 6)$

$= 9$

B (52341 57) = $(5 + \cancel{2} + 3 + \cancel{4} + 1 + \cancel{5} + \cancel{7})$

= 9 As the Beejank is same hence verified.

**<u>OSCULATION BY DIVISION PROCEDURE FOR NUMBER ENDING WITH ODD DIGITS (1, 3, 7, 9) USING NEGATIVE OSCULATOR</u>**

Q ⟵ Negative Osculator ⟵ एकन्यूनेन sutra.

Highest place digit of dividend is divided by '-'ve osculator in terms of quotient and remainder. Write 1st quotient digit below highest place digit in 4th row and prefix remainder to its next order digit of dividend on right hand side which will be next dividend. Subtract 1st quotient digit from this dividend which will give working dividend placed in 3rd row below 2nd digit of dividend. Now divide this working dividend by osculator to get 2nd quotient digit and write it in 4th row to right hand side of 1st quotient digit. This procedure is continued till unit place digit is osculated.

If the last osculated number is zero, divisor or multiple of divisor then dividend is divisible by given divisor.

e.g. 99665 ÷ 31

$$31 - 1 = 3\cancel{0}$$

Negative osculator (Q) = 3

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 ÷ 9 | | $_0 9$ | $_0 6$ | $_1 6$ | $_0 5$ | ⟵ Dividend ⟵ 1st row |
| | | -3 | -2 | -1 | -5 | ⟵ Subtract quotient digit ⟵ 2nd row |
| | | 6 | 4 | 15 | 0 | ⟵ Working dividend ⟵ 3rd row |
| | 3 | 2 | 1 | 5 | | ⟵ Quotient ⟵ 4th row |

As last osculated number is zero therefore 99665 is divisible by 31.

Quotient = 3215 & Remainder = 0

$B(3215) * B(31) + B(0) = (\cancel{3} + 2 + \cancel{1} + \cancel{5}) * (3 + 1) + 0$

$= 2 * 4$

$= 8$

$B(99665) = (\cancel{9} + \cancel{9} + 6 + 6 + 5)$

$= 17$

= (1 + 7)

= 8

Hence answer is verified.

**e.g.** 131260 ÷ 201

201- 1 = 2$\cancel{0}\cancel{0}$

Negative osculator = 2

As two zeros are cancelled therefore group of two digits from right hand side of dividend is formed for osculation procedure.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | ÷ | 13 | $_{1}$12 | $_{0}$60 | ← | Dividend | ← | 1st row |
| | | | -6 | -53 | ← | subtract quotient digit | ← | 2nd row |
| | | | 106 | 7 | ← | working dividend | ← | 3rd row |
| | | 6 | 53 | | ← | quotient | ← | 4th row |

As last osculated number (7) is neither divisor nor its multiple therefore dividend (131260) is not divisible by divisor (201).

Quotient = 653 and Remainder = 7 ----------------------- (Ans).

B (653) * B (201) + B (7) = ($\cancel{6}$ + 5 + $\cancel{3}$) * (2 + 0 + 1) + 7

= 5 * 3 +7

= 22

= (2 + 2)

= 4

B (131260) = (1 + 3 + $\cancel{1}$ + $\cancel{2}$ + $\cancel{6}$ + 0) = 4 Hence Verified

e.g. 116323 ÷ 17

17 * 3 = 51

51 – 1 = 5$\cancel{0}$

Negative osculator = 5

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5÷ | 1 | $_{1}1$ | $_{1}6$ | $_{4}3$ | $_{1}2$ | $_{4}3$ ← Dividend | ← $1^{st}$ row |
| | | -0 | -2 | -2 | -8 | -0 ← subtract quotient digit | ← $2^{nd}$ row |
| | | 11 | 14 | 41 | 4 | 43 ← working dividend | ← $3^{rd}$ row |
| | 0 | 2 | 2 | 8 | 0 | ← quotient | ← $4^{th}$ row |

As last osculated number (43) is neither divisor nor its multiple hence dividend (116323) is not divisible by divisor (17).

Quotient (51) = 2280

Quotient (17) = 2280 * 3 = 6840

Remainder = 43 which is greater than divisor (17), hence Remainder (43) is again divided by divisor (17).

43 ÷ 17

Quotient = 2 and Remainder = 9

Actual Quotient $_{(17)}$ = 6840 + 2 = 6842

Actual Remainder = 9 ---------------------- (Ans)

**e.g.** 197030148 ÷ 19999

19999 + 1 = 2$\cancel{0}\cancel{0}\cancel{0}\cancel{0}$

Positive osculator = 2

As '4' zeroes are cancelled therefore group of '4' digits from right hand side of dividend is formed for osculation procedure.

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | ÷ | 1 | $_{1}9703$ | $_{1}0148$ | ← Dividend | ← $1^{st}$ row |
| | | | + 0 | + 9851 | ← Add quotient digit | ← $2^{nd}$ row |
| | | | 19703 | 19999 | ← Working dividend | ← $3^{rd}$ row |
| | | 0 | 9851 | | ← Quotient | ← $4^{th}$ row |

As last osculated number (19999) is divisor itself therefore dividend (197030148) is divisible by dividend (19999).

Quotient = 9851

Remainder = 19999 which is divisor itself and equivalent to ‘1’ quotient. Add ‘1’ to quotient part.

Actual Quotient = 9851 + 1 = 9852 and Remainder = 0 ----------- (Ans).

B (197030148) = (1 + $\cancel{9}$ + 7 + 0 + 3 + 0 + $\cancel{1}$ + 4 + $\cancel{8}$)

= 15

= (1 + 5) = 6

B (9852) * B (19999) + B (0) = ($\cancel{9}$ + 8 + 5 + 2) * (1 + $\cancel{9}$ + $\cancel{9}$ + $\cancel{9}$ + $\cancel{9}$) + 0

= 15 * 1

= 15= (1 + 5) = 6 Hence answer is verified.

**OSCULATION BY DIVISION PROCEDURE FOR NUMBER ENDING WITH EVEN DIGITS (2,4,6,8) USING POSITIVE OR NEGATIVE OSCULATOR**

Add or subtract twice the quotient for positive osculator or negative osculator from the next order prefixed number. Remaining procedure is same.

e.g. Whether 96520 is divisible by 38?

38 + 2 = 4$\cancel{0}$

Positive osculator = 4

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 ÷ | 9 | $_1$6 | $_0$5 | $_3$2 | $_2$0 | ← Dividend ← $1^{st}$ row |
| | | +4 | +10 | +6 | +18 | ← Add (2*quotient digit) ← $2^{nd}$ row |
| | | 20 | 15 | 38 | 38 | ← Working dividend ← $3^{rd}$ row |
| | 2 | 5 | 3 | 9 | | ← Quotient ← $4^{th}$ row |

As last osculated number(38) is divisor itself hence 96520 is divisible by 38.

Quotient = 2539 and Remainder = 38

Remainder must be less than divisor always.

Actual Quotient = 2539 + 1 = 2540 and Remainder = 0 ------- (Ans).

e.g. Whether 10848 is divisible by 32?

$32 - 2 = 3\cancel{0}$

Negative osculator = 3

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 ÷ | 1 | $_{1}0$ | $_{1}2$ | $_{0}8$ | $_{1}4$ | $_{0}8$ | ← Dividend | ← $1^{st}$ row |
| | | -0 | -6 | -4 | -2 | -8 | ← Subtract (2*quotient digit) | ← $2^{nd}$ row |
| | | 10 | 6 | 4 | 12 | 0 | ← Working dividend | ← $3^{rd}$ row |
| | 0 | 3 | 2 | 1 | 4 | | ← Quotient | ← $4^{th}$ row |

As last osculated number is ‘0’ hence 10848 is divisible by 32.

Quotient = 3214 and Remainder = 0

## CHAPTER SEVEN: AWARENESS OF 1 – 5 VEDIC SUTRA

### 1. EKADHIKENA PURVENA

The Sutra Ekādhikena Pūrvena means: “By one more than the previous one”.

**i) Squares of numbers ending in 5:**

Now we relate the sutra to the ‘squaring of numbers ending in 5’. Consider the example $25^2$. Here the number is 25. We have to find out the square of the number. For the number 25, the last digit is 5 and the 'previous' digit is 2. Hence, 'one more than the previous one', that is, 2+1=3. The Sutra, in this context, gives the procedure **'to multiply the previous digit 2 by one more than itself, that is, by 3'**. It becomes the L.H.S (left hand side) of the result, that is, 2 X 3 = 6. The R.H.S (right hand side) of the result is$5^2$, that is, 25.

Thus $25^2$ = 2 X 3 / 25 = 625.

In the same way,

$35^2$= 3 X (3+1) /25 = 3 X 4/ 25 = 1225;

$65^2$= 6 X 7 / 25 = 4225;

$105^2$= 10 X 11/25 = 11025;

$135^2$= 13 X 14/25 = 18225;

**Algebraic proof:**

a) Consider $(ax + b)^2 = a^2. x^2 + 2abx + b^2$

This identity for x = 10 and b = 5 becomes

$$(10a + 5)^2 = a^2. 10^2 + 2.10a . 5 + 5^2$$

$$= a^2 . 10^2 + a.10^2 + 5^2$$

$$= (a^2 + a ) . 10^2 + 5^2$$

$$= a (a + 1) . 10^2 + 25.$$

Clearly 10a + 5 represents two-digit numbers 15, 25, 35, -------,95 for the values a = 1, 2, 3, ----,9 respectively. In such a case the number $(10a + 5)^2$ is of the form whose L.H.S is **a (a + 1)** and R.H.S is 25, that is, **a (a + 1) / 25**.

Thus any such two digit number gives the result in the same fashion.

b) Any three digit number is of the form $ax^2+bx+c$ for x =10, $a \neq 0$, a, b, c Є W.

Now $(ax^2+bx+c)^2 = a2\ x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2cax^2$
$= a^2\ x^4+2ab.x^3+(b^2+ 2ca)x^2+2bc\ .\ x+ c^2$.

This identity for $x = 10$, $c = 5$becomes $(a\ .10^2 + b\ .10 + 5)^2$

$= a^2.10^4+ 2.a.b.10^3 + (b^2+ 2.5.a)10^2+ 2.b.5.10 + 5^2$

$= a^2.10^4+ 2.a.b.10^3 + (b^2 + 10\ a)10^2 + b.10^2+ 5^2$

$= a^2.10^4+ 2ab.10^3+ b^2.10^2+ a\ .\ 10^3 + b\ 10^2+ 5^2$

$= a^2.10^4 + (2ab + a).10^3 + (b^2+ b)10^2 +5^2$

$= [\ a^2.10^2 +2ab.10 + a.10 + b^2 + b]\ 10^2+ 5^2$

$= (10a + b)\ (\ 10a+b+1).10^2 + 25$
$= P\ (P+1)\ 10^2+ 25$, where $P = 10a+b$.

Hence any three digit number whose last digit is 5 gives the same result as in **(a)** for P=10a + b, the ‘previous’ of 5.

**Example :** $165^2= (1\ .\ 10^2 + 6\ .\ 10 + 5)^2$.

It is of the form $(ax^2+bx+c)^2$for a = 1, b = 6, c = 5 and x = 10. It gives the answer P(P+1) / 25, where P = 10a + b = 10 X 1 + 6 = 16, the ‘previous’. The answer is 16 (16+1) / 25 = 16 X 17 / 25 = 27225.

**ii) Vulgar fractions whose denominators are numbers ending in NINE :**

We now take examples of 1 / a9, where a = 1, 2, -----, 9. In the conversion of such vulgar fractions into recurring decimals, Ekadhika process can be effectively used both in division and multiplication.

**a) Division Method:** Value of 1 / 19.

The numbers of decimal places before repetition is the difference of numerator and denominator, i.e.19 -1=18 places.

For the denominator 19, the ***purva*** (previous) is 1.

Hence ***Ekadhikena purva*** (one more than the previous) is 1 + 1 = 2.

The sutra is applied in a different context. Now the method of division is as follows:

***Step. 1*** : Divide numerator 1 by 20.

i.e.,, 1 / 20 = 0.1 / 2 = $._{1}0$ ( 0 times, 1 remainder) ***Step. 2*** : Divide 10 by 2

i.e.,, $0.0_05$( 5 times, 0 remainder )

***Step. 3*** : Divide 5 by 2

i.e.,, $0.05_12$ ( 2 times, 1 remainder )

***Step. 4*** : Divide $_12$ i.e.,, 12 by 2

i.e.,, 0.0526 ( 6 times, No remainder )

***Step. 5*** : Divide 6 by 2

i.e.,, 0.05263 ( 3 times, No remainder )

***Step. 6*** : Divide 3 by 2

i.e.,, $0.05263_11$(1 time, 1 remainder )

***Step. 7*** : Divide 11 i.e.,, 11 by 2

i.e.,, $0.052631_15$ (5 times, 1 remainder )

***Step. 8*** : Divide $_15$ i.e.,, 15 by 2

i.e.,, $0.0526315_17$ ( 7 times, 1 remainder )

***Step. 9*** : Divide $_17$ i.e.,, 17 by 2

i.e.,, $0.05263157\ _18$ (8 times, 1 remainder )

***Step. 10*** : Divide $_18$ i.e.,, 18 by 2

i.e.,, 0.0526315789 (9 times, No remainder )

***Step. 11*** **:** Divide 9 by 2

i.e.,, $0.0526315789\ _14$ (4 times, 1 remainder )

***Step. 12*** **:** Divide $_14$ i.e.,, 14 by 2

i.e.,, 0.052631578947 ( 7 times, No remainder )

***Step. 13 :*** Divide 7 by 2

i.e.,, 0.052631578947$_{1}$3 ( 3 times, 1 remainder )

***Step. 14 :*** Divide 13 i.e.,, 13 by 2

i.e.,, 0.0526315789473$_{1}$6 ( 6 times, 1 remainder )

***Step. 15 :*** Divide 16 i.e.,, 16 by 2

i.e.,, 0.052631578947368 (8 times, No remainder )

***Step. 16 :*** Divide 8 by 2

i.e.,, 0.0526315789473684 ( 4 times, No remainder )

***Step. 17 :*** Divide 4 by 2

i.e.,, 0.05263157894736842 ( 2 times, No remainder )

***Step. 18 :*** Divide 2 by 2

i.e.,, 0.052631578947368421 ( 1 time, No remainder )

Now from step 19, i.e. dividing 1 by 2, Step 2 to Step. 18 repeats thus giving

1 / 19 = $0.\overline{052631578947368421}$ or $0.\dot{0}5263157894736842\dot{1}$

Note that we have completed the process of division only by using '2'. Nowhere the division by 19 occurs.

**b) Multiplication Method:** Value of 1 / 19

First we recognize the last digit of the denominator of the type 1 / a9. Here the last digit is 9.

For a fraction of the form in whose denominator 9 is the last digit, we take the case of 1 / 19 as follows:

For 1 / 19, 'previous' of 19 is 1. And one more than of it is 1 + 1 = 2.

Therefore 2 is the multiplier for the conversion. We write the last digit in the numerator as 1 and follow the steps leftwards.

***Step. 1 :*** 1

***Step. 2 :*** 21(multiply 1 by 2, put to left)

***Step. 3 :*** 421(multiply 2 by 2, put to left)

***Step. 4 :*** 8421(multiply 4 by 2, put to left)

***Step. 5 :*** $_1$68421 (multiply 8 by 2 =16, 1 carried over, 6 put to left)

***Step. 6 :*** $_1$368421 ( 6 X 2 =12,+1 [carry over]

= 13, 1 carried over, 3 put to left )

***Step. 7 :*** 7368421 ( 3 X 2, = 6 +1 [Carryover]

= 7, put to left)

***Step. 8 :*** $_1$47368421 (as in the same process)

***Step. 9 :*** 947368421 ( Do – continue to step 18)

***Step. 10 :*** $_1$8947368421

***Step. 11 :*** $_1$78947368421

***Step. 12 :*** $_1$578947368421

***Step. 13 :*** $_1$1578947368421

***Step. 14 :*** 31578947368421

***Step. 15 :*** 631578947368421

***Step. 16 :*** $_1$2631578947368421

***Step. 17 :*** 52631578947368421

***Step. 18 :*** 1052631578947368421

Now from step 18 onwards the same numbers and order towards left continue.

Thus **1 / 19 = 0.052631578947368421**

It is interesting to note that we have i) not at all used division process

ii) Instead of dividing 1 by 19 continuously, just multiplied 1 by 2 and continued to multiply the resultant successively by 2.

**Observations:**

a) For any fraction of the form 1 / 9 i.e. in whose denominator 9 is the digit in the units place and a is the set of remaining digits, the value of the fraction is in recurring decimal form and the repeating block's right most digit is 1.

b) Whatever may be a9, and the numerator, it is enough to follow the said process with (a+1) either in division or in multiplication.

c) Starting from right most digit and counting from the right, we see ( in the given example 1 / 19)

Sum of $1^{st}$ digit + $10^{th}$ digit = 1 + 8 = 9

Sum of $2^{nd}$ digit + $11^{th}$ digit = 2 + 7 = 9

- - - - - - - - --- - - - - - - - - - - - - - - - - - -

Sum of 9th digit + 18th digit = 9+ 0 = 9

From the above observations, we conclude that if we find first 9 digits, further digits can be derived as complements of 9.

i) Thus at the step 8 in division process we have $0.052631{5}_{1}7$ and next step. 9 gives 0.052631578

Now the complements of the numbers

0, 5, 2, 6, 3, 1, 5, 7, 8 from 9

9, 4, 7, 3, 6, 8, 4, 2, 1 follow the right order

i.e. 0.052631578947368421

Now taking the multiplication process we have

***Step. 8 :*** $_{1}47368421$

***Step. 9 :*** 947368421

Now the complements of 1, 2, 4, 8, 6, 3, 7, 4, 9 from 9
i.e. 8, 7, 5, 1, 3, 6, 2, 5, 0 precede in successive steps, giving the answer.

***0.052631578947368421***.

d) When we get (Denominator – Numerator) as the product in the multiplicative process, half the work is done. We stop the multiplication there and mechanically write the remaining half of the answer by merely taking down complements from 9.

e) Either division or multiplication process of giving the answer can be put in a single line form.

***Algebraic proof:***

Any vulgar fraction of the form 1 / a9 can be written

as

1 / a9 = 1 / [(a + 1) x - 1] where x = 10

$$= \frac{1}{[(a+1)x[1 - 1/(a+1)x]}$$

$$= \frac{1}{(a+1)x}[1 - 1/(a+1)x]^{-1}$$

$$= \frac{1}{(a+1)x}[1 + 1/(a+1)x + 1/(a+1)x^2 + \text{----------}]$$

$$= 1/(a+1)x + 1/(a+1)^2x^2 + 1/(a+1)^3x^3 + \text{----}\textit{ad infinitum}$$

$$= 10^{-1}(1/(a+1)) + 10^{-2}(1/(a+1)^2) + 10^{-3}(1/(a+1)^3) + \text{---}\textit{ad infinitum}$$

This series explains the process of ***Ekadhika***.

Now consider the problem of 1 / 19. From above we get

$$1 / 19 = 10^{-1} (1/(1+1)) + 10^{-2} (1/(1+1)^2) + 10^{-3} (1/(1+1)^3) + ----$$

( since a=1)

$$= 10^{-1} (1/2) + 10^{-2} (1/2)^2 + 10^{-3} (1/3)^3 + ----------$$

$$= 10^{-1} (0.5) + 10^{-2} (0.25) + 10^{-3} (0.125)+ ----------$$

= 0.05 + 0.0025 + 0.000125 + 0.00000625+ - - - - - - -

= 0.052631 - - - - - - -

**2. Nikhilam navatascaramam Dasatah**

The formula simply means: “all from 9 and the last from 10”

The formula can be very effectively applied in multiplication of numbers, which are nearer to bases like 10, 100, 1000 i.e., to the powers of 10 . The procedure of multiplication using the Nikhilam involves minimum number of steps, space, time saving and only mental calculation. The numbers taken can be either less or more than the base considered.

The difference between the number and the base is termed as deviation. Deviation may be positive or negative. Positive deviation is written without the positive sign and the negative deviation, is written using Rekhank (a bar on the number). Now observe the following table.

| **Number** | **Base** | **Number – Base** | **Deviation** |
|---|---|---|---|
| 14 | 10 | 14 - 10 | 4 |
| 8 | 10 | 8- 10 | -2 or $\bar{2}$ |
| 97 | 100 | 97- 100 | -03 or $0\bar{3}$ |
| 112 | 100 | 112 - 100 | 12 |
| 993 | 1000 | 993- 1000 | -007 or $00\bar{7}$ |
| 1011 | 1000 | 1011 - 1000 | 011 |

**Some rules of the method (near to the base) in Multiplication**

a) Since deviation is obtained by Nikhilam sutra we call the method as Nikhilam multiplication.

**Eg :** 94. Now deviation can be obtained by 'all from 9 and the last from 10' sutra i.e. the last digit 4 is from 10 and remaining digit 9 from 9 gives 06.

b) The two numbers under consideration are written one below the other. The deviations are written on the right hand side.

**Case (i):** Both the numbers are lower than the base. We have already considered the example 7 x 8, with base 10.

Now let us solve some more examples by taking bases 100 and 1000 respectively.

**Ex.1:** Find 97 X 94. Here base is 100. Now following the rules, the working is as follows:

$$
\begin{array}{ll|l}
97 & & \overline{03} \\
94 & & \overline{06} \\
\hline
(97-06) \text{ or} & & 3\text{x}6 \quad = 9118 \\
(94-03) & &
\end{array}
$$

**Ex. 2:** 986 X 989. Base is 1000.

$$
\begin{array}{ll|l}
986 & & 0\overline{14} \\
989 & & 0\overline{11} \\
\hline
(986-11) \text{ or} & & 14\text{x}11 \quad = 975 / 154 = 975154 \\
(989-14) & &
\end{array}
$$

**Case ( ii):** Both the numbers are higher than the base.

The method and rules follow as they are. The only difference is the positive deviation. Instead of cross – subtract, we follow cross – add.

**Ex. 3:** 13X12. Base is 10

$$
\begin{array}{ll|l}
13 & & 3 \\
12 & & 2 \\
\hline
(13+2) \text{ or} & & 3\text{x}2 \quad = 15 / 6 = 156 \\
(12+3) & &
\end{array}
$$

**Ex. 4:** 104X102. Base is 100.

| | |
|---|---|
| 104 | 04 |
| 102 | 02 |
| (104 + 2) or (102 + 4) | 04x02 |

=106 /04x02 = 10608

**Ex. 5:** 1275X1004. Base is 1000.

| | |
|---|---|
| 1275 | 275 |
| 1004 | 004 |

1279 / 275x4 $= 1279 / {}_{1}100$

$= 1280100$

**Case (iii ):** One number is more and the other is less than the base.

In this situation one deviation is positive and the other is negative. So the product of deviations becomes negative. So the right hand side of the answer obtained will therefore have to be subtracted. To have a clear representation and understanding a vinculum is used. It proceeds into normalization.

**Ex.6:** 13X7. Base is 10

| | |
|---|---|
| 13 | 3 |
| 7 | $\bar{3}$ |
| (13 -3) or (7 + 3) | $3 \times \bar{3}$ |

$= 10 / \bar{9} = (10-1) / (10-9) = 91$

**Note:** Conversion of common number into vinculum number and vice versa.
**Eg :**

$9 = 10 - 1 \quad = 1\bar{1}$

$98 = 100 - 2 = 10\bar{2}$

**Algebraic Proof:**

**Case (i):**

Let the two numbers $N_1$ and $N_2$ be less than the selected base say x.

$N_1 = (x-a)$, $N_2 = (x-b)$. Here a and b are the corresponding deviations of the numbers$N_1$ and $N_2$ from the base x. Observe that x is a multiple of 10.

Now $N_1 \text{ X } N_2 = (x-a)(x-b) = x.x - x.b - a.x + ab$

$= x\,(x - a - b\,) + ab$

$= x\,[(x - a) - b] + ab = x\,(N_1 - b) + ab$

or $= x\,[(x - b) - a] = x\,(N_2 - a) + ab.$

$x\,(x - a - b) + ab$ can also be written as

$x[(x - a) + (x - b) - x] + ab = x[N_1+N_2 - x] + ab$

A difficulty can be faced, if the vertical multiplication of the deficit digits or deviations i.e., a.b yields a product consisting of more than the required digits. Then rule-f will enable us to surmount the difficulty.

**Case (ii):**

When both the numbers exceed the selected base, we have $N_1 = x + a$, $N_2 = x + b$, x being the base. Now the identity $(x+a)\,(x+b) = x(x+a+b) + a.b$ holds good, of course with relevant details mentioned in case (i).

**Case (iii):**

When one number is less and another is more than the base, we can use $(x-a)\,(x+b) = x(x-a+ b)-ab$ and the procedure is evident from the examples given.

**Nikhilam in Division**

Consider some two digit numbers (dividends) and same divisor 9. Observe the following example.

i) $13 \div 9$ The quotient (Q) is 1, Remainder (R) is 4.

$$\text{Since } 9)\ 13\ (1 \qquad \begin{array}{r} -\ 9 \\ \hline 4 \end{array}$$

ii) $34 \div 9$, Q is 3, R is 7.

iii) $60 \div 9$, Q is 6, R is 6.

iv) $80 \div 9$, Q is 8, R is 8.

Now we have another type of representation for the above examples as given hereunder:

i) Split each dividend into a left hand part for the Quotient and right - hand part for the remainder by a slant line or slash.

**Eg.** 13 as 1 / 3, 34 as 3 / 4, 80 as 8 / 0.

ii) Leave some space below such representation, draw a horizontal line.

**Eg.** 1 / 3     3 / 4     8 / 0

______ , ______ , ______

iii) Put the first digit of the dividend as it is under the horizontal line. Put the same digit under the right hand part for the remainder, add the two and place the sum i.e. sum of the digits of the numbers as the remainder.

**Eg.**

| 1 / 3 | 3 / 4 | 8 / 0 |
|---|---|---|
| 1 | 3 | 8 |
| 1 / 4 | 3 / 7 | 8 / 8 |

Now the problem is over. i.e.

13 ÷ 9 gives Q = 1, R = 4

34 ÷ 9 gives Q = 3, R = 7

80 ÷ 9 gives Q = 8, R = 8

Now consider the divisors of two or more digits whose last digit is 9, when divisor is 89.

We Know 113 = 1 X 89 + 24, Q =1, R = 24

10015 = 112 X 89 + 47, Q = 112, R = 47.

Representing in the previous form of procedure, we have

| 89 ) 1 / 13 | 89 ) 100 / 15 |
|---|---|
| / 11 | 12 / 32 |
| 1 / 24 | 112 / 47 |

But how to get these? What is the procedure?

Now Nikhilam rule comes to rescue us. The Nikhilam states “all from 9 and the last from 10”. Now if you want to find 113 ÷ 89, 10015 ÷ 89, you have to apply Nikhilam formula on 89 and get the complement 11.Further while carrying the added numbers to the place below the next digit, we have to multiply by this 11.

| | | |
|---|---|---|
| 89 ) 1 / 13 | 89 )100 /15 | |
| / 11 | 11   11 / | first digit 1 x 11 |
| 1/ 24 | 1 / 1 | total second is 1x11 |
| | / 22 | total of 3[rd] digit is 2 x 11 |
| | 112 /47 | |

What is 10015 ÷ 98? Apply Nikhilam and get 100 – 98 = 02. Set off the 2 digits from the right as the remainder consists of 2 digits. While carrying the added numbers to the place below the next digit, multiply by 02.

Thus

| | | |
|---|---|---|
| 98 )   100 / 15 | | |
| 02   02 / | | i.e., 10015 ÷ 98 gives |
| 0 / 0 | | Q = 102, R = 19 |
| / 0 4 | | |
| 102 / 19 | | |

In the same way

897 ) 11 / 422

103   1 / 03
/ 206

12 / 658

Gives 11,422 ÷ 897, Q = 12, R=658.

In this way we have to multiply the quotient by 2 in the case of 8, by 3 in the case of 7, by 4 in the case of 6 and so on. i.e., multiply the Quotient digit by the divisors complement from 10. In case of more digit numbers we apply Nikhilam and proceed. Anyhow, this method is highly useful and effective for division when the numbers are near to bases of 10.

### 3. Urdhva - tiryagbhyam

Urdhva – tiryagbhyam is the general formula applicable to all cases of multiplication and also in the division of a large number by another large number. It means

#### (a) Multiplication of two 2 digit numbers.

**Ex.1:** Find the product 14 X 12

i) The right hand most digit of the multiplicand, the first number (14) i.e. 4 is multiplied by the right hand most digit of the multiplier, the second number (12) i.e.2. The product 4 X 2
= 8 forms the right hand most part of the answer.

1 4
1 2
---
: 4x2

ii) Now, diagonally multiply the first digit of the multiplicand (14) i.e., 4 and second digit of the multiplier (12)i.e., 1 (answer 4 X 1=4); then multiply the second digit of the multiplicand i.e.,1 and first digit of the multiplier i.e., 2 (answer 1 X 2 = 2); add these two i.e.,4 + 2 = 6. It gives the next, i.e., second digit of the answer. Hence second digit of the answer is 6.

1 4
1 2
---
2 + 4 : 8

iii) Now, multiply the second digit of the multiplicand i.e., 1 and second digit of the multiplier i.e. 1 vertically, i.e. 1 X 1 = 1. It gives the left hand most part of the answer.

1 4
1 2
---
1x1: 6 : 8 which gives 168

Thus the answer is 168.
Symbolically we can represent the process as follows:

1 4
1 2

The symbols are operated from right to left.
Now in the same process, answer can be written as

23
13

---

2: 6 + 3: 9 = 299 (Recall the 3 steps)

**Ex.2**

41 X 41

---

16: 4 + 4: 1 = 1681.

What happens when one of the results i.e. either in the last digit or in the middle digit of the result, contains more than 1 digit? Answer is simple. The right – hand – most digit thereof is to be put down there and the preceding i.e. left –hand –side digit(s)should be carried over to the left and placed under the previous digit or digits of the upper row. The digits carried over may be written as in Ex. 3.

**Ex.3:** 32 X 24

**Step (i):** 2 X 4 = 8

**Step (ii):** 3 X 4 = 12; 2 X 2 = 4; 12 + 4 = 16.

Here 6 is to be retained. 1 is to be carried out to left side.

**Step (iii):** 3 X 2 = 6. Now the carried over digit 1 of 16 is to be added i.e. 6 + 1 = 7.

Thus 32 X 24 = 768

We can write it as follows

32
24

---

668
1

---

768.

Note that the carried over digit from the result (3X4) + (2X2) = 12+4 = 16 i.e.1 is placed under the previous digit 3 X 2 = 6 and added.

After sufficient practice, you feel no necessity of writing in this way and simply operate or perform mentally.

**Algebraic proof:**

a) Let the two 2 digit numbers be (ax+b) and (cx+d). Note that x = 10. Now consider the product

$$(ax + b)(cx + d) = ac.x^2 + adx + bcx + b.d = ac.x^2 + (ad + bc)x + b.d$$

**Observe that**

i) The first term i.e., the coefficient of $x^2$ (i.e., 100, hence the digit in the100th place) is obtained by vertical multiplication of a and c i.e. the digits in10th place (coefficient of x) of both the numbers;

ii) The middle term, i.e. the coefficient of x (i.e., digit in the 10th place) is obtained by cross wise multiplication of a and d; and of b and c; and the addition of the two products;

iii) The last (independent of x) term is obtained by vertical multiplication of the independent terms b and d.

b) Consider the multiplication of two 3 digit numbers.

Let the two numbers be $(ax^2 + bx + c)$ and $(dx^2 + ex + f)$. Note that x=10

Now the product is

$$\begin{array}{rl} & ax^2 + bx + c \\ \text{X} & \\ & dx^2 + ex + f \\ \hline \end{array}$$

$$ad.x^4+bd.x^3+cd.x^2+ae.x^3+be.x^2+ce.x+af.x^2+bf.x+cf$$

$$= ad.x^4 + (bd + ae).\ x^3 + (cd + be + af).x^2 + (ce + bf)\ x + cf$$

Note the following points:

i) The coefficient of $x^4$ i.e. ad is obtained by the vertical multiplication of the first coefficient from the left side:

$$\begin{array}{l} ax^2 + bx + c \\ \downarrow \\ dx^2 + ex + f \\ \hline adx^4 \end{array}$$

ii) The coefficient of $x^3$ i.e. (ae + bd) is obtained by the cross –wise multiplication of the first two coefficients and by the addition of the two products;

$$\begin{array}{rl} & ax^2 + bx + c \\ \text{X} & \quad\times \\ & dx^2 + ex + f \\ \hline & aex^3 + bdx^3 \\ & = (ae + bd)\ x^3 \end{array}$$

iii) The coefficient of $x^2$ is obtained by the multiplication of the first coefficient of the multiplicand($ax^2+bx+c$) i.e. a; by the last coefficient of the multiplier ($dx^2+ex+f$) i.e. f; of the middle one i.e. b of the multiplicand by the middle one i.e. e of the multiplier and of the last one i.e. c of the multiplicand by the first one i.e. d of the multiplier and by the addition of all the three products i.e., af + be +cd :

$$\begin{array}{l} ax^2 + bx + c \\ dx^2 + ex + f \\ \hline afx^2 + bex^2 + cdx^2 \\ (af + be + cd)\,x^2 \end{array}$$

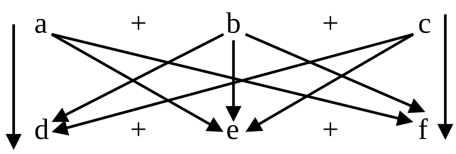
iv) The coefficient of x is obtained by the cross wise multiplication of the second coefficient i.e. b of the multiplicand by the third one i.e. f of the multiplier and conversely the third coefficient i.e. c of the multiplicand by the second coefficient i.e. e of the multiplier and by addition of the two products, i.e., bf + ce;

$$\begin{array}{l} ax^2 + bx + c \\ dx^2 + ex + f \\ \hline bf\,x + ce\,x \\ (bf + ce)\,x \end{array}$$

v) And finally the last (independent of x) term is obtained by the vertical multiplication of the last coefficients c and f i.e., cf

$$\begin{array}{r} ax^2 + bx + c \\ dx^2 + ex + f \\ \hline cf \end{array}$$

Thus the process can be put symbolically as (from left to right)



**Example 1:** Find the product of (a+2b) and (3a+b).

$$\begin{array}{ccccc} a & + & 2b \\ 3a & + & b \\ \hline 3a^2 & + & 7ab & + & 2b2 \end{array}$$

**Example 2:**

$$3a^2 + 2a + 4 \quad x \ \ 2a^2 + 5a + 3$$

i) 4 X 3 = 12

ii) (2 X 3) + (4 X 5) = 6 + 20 =26 i.e. 26a

iii) (3 X 3) + (2 X 5) + (4 X 2) = 9 + 10 + 8 = 27 i.e. $27a^2$

iv) (3 X 5) + (2 X 2) = 15 + 4 = 19 i.e. 19 $a^3$

v) 3 X 2 = 6 i.e.$6a^4$

Hence the product is $6a^4 + 19a^3 + 27a^2 + 26a + 12$

**Example 3 :** Find $(3x^2 + 4x + 7)(5x + 6)$

Now
$$\begin{array}{r} 3.x^2 + 4x + 7 \\ 0.x^2 + 5x + 6 \\ \hline \end{array}$$

i) 7 X 6 = 42

ii) (4 X 6) + (7 X 5) = 24 + 35 = 59 i.e., 59x

iii) (3 X 6) + (4 X 5) + (7 X 0) = 18 + 20 + 0 = 38 i.e., $38x^2$

iv) (3 X 5) + (0 X 4) = 15 + 0 = 15 i.e., $15x^3$

v) 3 X 0 = 0

Hence the product is $15x^3 + 38x^2 + 59x + 42$

**Urdhva – tiryak in converse for division process:**

As per the statement it is used as a simple argumentation for division process particularly in algebra.

Consider the division of $(x^3 + 5x^2 + 3x + 7)$ by $(x – 2)$ process by converse of urdhva – tiryak :

i) $x^3$ divided by x gives $x^2$. $\dfrac{x^3 + 5x^2 + 3x + 7}{x - 2}$

It is the first term of the Quotient.

Q = $x^2$ + - - - - - - - - - - -

ii) $x^2$ X – 2 = - $2x^2$. But $5x^2$ in the dividend hints$7x^2$ more since $7x^2 - 2x^2 = 5x^2$. This 'more' can be obtained from the multiplication of x by 7x. Hence second term of Q is 7x.

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2} \quad \text{gives } Q = x^2 + 7x + \text{- - - - - - - -}$$

iii) We now have – 2 X 7x = -14x. But the 3rd term in the dividend is 3x for which '17x more' is required since 17x – 14x =3x.

Now multiplication of x by 17 gives 17x. Hence third term of quotient is 17

Thus

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2} \quad \text{gives } Q = x^2 + 7x + 17$$

iv) Now last term of Q, i.e., 17 multiplied by –2 gives 17X–2 = -34 but the relevant term in dividend is 7. So 7 + 34 = 41 'more' is required. As there no more terms left in dividend, 41 remains as the remainder.

$$\frac{x^3 + 5x^2 + 3x + 7}{x - 2} \quad \text{gives } Q = x^2 + 7x + 17 \text{ and } R = 41.$$

**4. Paravartya Yojayet**

'Paravartya – Yojayet' means **'transpose and apply'**

(i) Consider the division by divisors of more than one digit, and when the divisors are slightly greater than powers of 10.

**Example 1:** Divide 1225 by 12.

**Step 1 :** (From left to right ) write the Divisor leaving the first digit, write the other digit or digits using negative (-) sign and place them below the divisor as shown.

1
2
-
2
-----

**Step 2:** Write down the dividend to the right. Set apart the last digit for the remainder.

**i.e.** 12 122 5
- 2
-----

**Step 3:** Write the 1st digit below the horizontal line drawn under the dividend. Multiply the digit by –2, write the product below the 2nd digit and add.

**i.e.** 12 122 5
-2 -2

10

Since 1 x –2 = -2and 2 + (-2) = 0

**Step 4:** We get second digits' sum as '0'. Multiply the second digit' sum thus obtained by –2 and write the product under 3rd digit and add.

12 122 5
- 2 -20

102

**Step 5:** Continue the process till the last digit.

**i.e.** 12 122 5
- 2 -20 -4

102 1

**Step 6:** The sum of the last digit is the Remainder and the result to its left is Quotient. Thus Q = 102 and R = 1

**Example 2:** Divide 239479 by 11213. The divisor has 5 digits. So the last 4digits of the dividend are to be set up for Remainder.

| | | | |
|---|---|---|---|
| 1 1213 | 2 3 | 9 4 7 9 | |
| -1-2-1-3 | -2 | -4 -2 -6 | with 2 |
| | | -1 -2-1 -3 | with 1 |
| | 2 1 | 4 0 0 6 | |

Hence Q = 21, R = 4006.

**Example 3 :** Divide 13456 by 1123

$$\begin{array}{r|rrrr|r} 1\ 12\ 3 & 1 & 3 & 4 & 5 & 6 \\ -1-2-3 & & -1 & -2 & -3 & \\ & & & -2 & -4 & -6 \\ \hline & 1 & 2 & 0 & -2 & 0 \end{array}$$

Note that the remainder portion contains –20, i.e. a negative quantity. To overcome this situation, take 1 over from the quotient column, i.e. 1123 over to the right side, subtract the remainder portion 20 to get the actual remainder.

Thus Q = 12 – 1 = 11, and R = 1123 - 20 = 1103.

Now let us consider the application of paravartya – yojayet in algebra.

**Example 1:** Divide $6x^2 + 5x + 4$ by $x - 1$

$$\begin{array}{c|l} x-1 & 6x^2 + 5x + 4 \\ \hline 1 & \quad\ \ 6 + 11 \\ \hline & 6x + 11 + 15 \end{array}$$

Thus Q = 6x+11, R=15.

**Example 2:** $2x^5 - 5x^4 + 3x^2 - 4x + 7$by $x^3 - 2x^2 + 3$.

We treat the dividend as $2x^5 - 5x^4 + 0.x^3 + 3x^2 - 4x + 7$ and divisor as $x^3 - 2x^2 + 0.x + 3$ and proceed as follows:

$$\begin{array}{ccc|cccccc} \multicolumn{3}{c|}{x^3 - 2x^2 + 0.x + 3} & 2x^5 & -5x^4 & +0.x^3 & +3x^2 & -4x & +7 \\ \hline 2 & 0 & -3 & & 4 & 0 & -6 & & \\ & & & & & -2 & 0 & +3 & \\ & & & & & & -4 & 0 & +6 \\ \hline & & & 2 & -1 & -2 & -7 & -1 & +13 \end{array}$$

Thus Q = $2x^2 - x - 2$, R = $-7x^2 - x + 13$.

You may observe a very close relation of the method paravartya in this aspect with regard to REMAINDER THEOREM and HORNER PROCESS of Synthetic division. And yet paravartya goes much farther and is capable of numerous applications in other directions also.

**Paravartya in solving simple equations:**

Recall that 'paravartya yojayet' means 'transpose and apply'. The rule relating to transposition enjoins invariable change of sign with every change of side. i.e., + becomes - and conversely; and X becomes ÷ and conversely. Further it can be extended to the transposition of terms from left to right and conversely and from numerator to denominator and conversely in the concerned problems.

**Type (i):**

Consider the problem $7x – 5 = 5x + 1$ $7x – 5x = 1 + 5$

i.e. $2x = 6$ $x = 6 \div 2 = 3$.

Observe that the problem is of the type $ax + b = cx + d$ from which we get by 'transpose' $(d – b)$, $(a – c)$ and

$$x = \frac{d - b}{a - c}$$

In this example $a = 7$, $b = - 5$, $c = 5$, $d = 1$

Hence

$$x = \frac{1 – (- 5)}{7 – 5} = \frac{1+5}{7-5} \quad \frac{6}{2} = 3$$

**Example 2:** Solve for x, $3x + 4 = 2x + 6$

$$x = \frac{d - b}{a - c} = \frac{6 - 4}{3 - 2} \quad \frac{2}{1} = 2$$

**Type (ii):** Consider problems of the type $(x + a)(x+b) = (x+c)(x+d)$. By paravartya, we get

$$x = \frac{cd - ab}{(a + b) – (c + d)}$$

It is trivial form the following steps

$$(x + a)(x + b) = (x + c)(x + d)$$

$$x^2 + bx + ax + ab = x^2 + dx + cx +$$

$$cd = bx + ax - dx - cx = cd - ab$$

$$x(a + b - c - d) = cd - ab$$

$$x = \frac{cd - ab}{a + b - c - d} \qquad x = \frac{cd - ab}{(a + b) - (c + d)}$$

**Example 1:** $(x - 3)(x - 2) = (x + 1)(x + 2)$.

By paravartya

$$x = \frac{cd - ab}{a + b - c - d} = \frac{1(2) - (-3)(-2)}{-3 - 2 - 1 - 2}$$

$$x = \frac{2 - 6}{-8} = \frac{-4}{-8} = \frac{1}{2}$$

**Type (iii):**

Consider the problems of the type $\dfrac{ax + b}{cx + d} = \dfrac{m}{n}$

By cross – multiplication,

$n(ax + b) = m(cx + d) = nax + nb = mcx + md = nax - mcx = md - nb$ $x(na - mc) = md - nb$

$$x = \frac{md - nb}{na - mc.}$$

Now look at the problem once again

$$\frac{ax + b}{cx + d} = \frac{m}{n}$$

paravartya gives md - nb, na - mc and

$$X = \frac{md - nb}{na - mc}$$

**Example 1:** $\dfrac{3x + 1}{4x + 3} = \dfrac{13}{19}$

$$x = \frac{md - nb}{na - mc} = \frac{13(3) - 19(1)}{19(3) - 13(4)} = \frac{39 - 19}{57 - 52} = \frac{20}{5}$$

$$= 4$$

**Example 2:** $\dfrac{4x + 5}{3x + 13/2} = \dfrac{7}{8}$

$$x = \frac{(7)\quad(13/2)-(8)(5)}{(8)(4) - (7)(3)}$$

$$= \frac{(91/2) - 40}{32 - 21} = \frac{(91 - 80)/2}{32 - 21} = \frac{11}{2 \text{ X } 11} = \frac{1}{2}$$

**Type (iv):** Consider the problems of the type $\dfrac{m}{x + a} + \dfrac{n}{x + b} = 0$

Take L.C.M and proceed.

$$\frac{m(x+b) + n(x+a)}{(x + a)(x + b)} = 0$$

$$\frac{mx + mb + nx + na}{(x + a)(x + b)} = 0$$

$$(m + n)x + mb + na = 0 \quad \therefore (m + n)x = -mb - na$$

$$x = \frac{-mb - na}{(m + n)}$$

Thus the problem $\dfrac{m}{x + a} + \dfrac{n}{x + b} = 0$, by paravartya process

gives directly

$$x = \frac{-mb - na}{(m + n)}$$

**Example 1 :**

$$\frac{3}{x+4} + \frac{4}{x-6} = 0$$

gives

$$X = \frac{-mb - na}{(m+n)}$$ Note that m = 3, n = 4, a = 4, b = - 6

$$= \frac{-(3)(-6) - (4)(4)}{(3+4)} = \frac{18 - 16}{7} = \frac{2}{7}$$

**Example 2:**

$$\frac{5}{x+1} + \frac{6}{x-21} = 0$$

gives

$$x = \frac{-(5)(-21) - (6)(1)}{5+6} = \frac{105 - 6}{11} = \frac{99}{11} = 9$$

**5. Sunyam Samya Samuccaye**

The Sutra 'Sunyam Samyasamuccaye' says the 'Samuccaya is the same, that Samuccaya is Zero.' i.e., it should be equated to zero. The term 'Samuccaya' has several meanings under different contexts.

i) We interpret, 'Samuccaya' as a term which occurs as a common factor in all the terms concerned and proceed as follows.

**Example 1:** The equation 7x + 3x = 4x + 5x has the same factor ‘ x ‘ in all its terms. Hence by the sutra it is zero i.e. x = 0.
Otherwise we have to work like this:

$$7x + 3x = 4x + 5x$$
$$10x = 9x$$
$$10x - 9x = 0$$
$$X = 0$$

This is applicable not only for ‘x’ but also any such unknown quantity as follows.

**Example 2:** 5(x+1) = 3(x+1)

No need to proceed in the usual procedure like

$$5x + 5 = 3x + 3$$
$$5x - 3x = 3 - 5$$
$$2x = -2 \text{ or } x = -2 \div 2 = -1$$

Simply think of the contextual meaning of 'Samuccaya'

Now Samuccaya is $(x + 1)$

$$x + 1 = 0 \quad \text{gives} \quad x = -1$$

ii) Now we interpret 'Samuccaya' as product of independent terms in expressions like $(x+a)(x+b)$

**Example 3:** $(x + 3)(x + 4) = (x - 2)(x - 6)$

Here Samuccaya is $3 \times 4 = 12 = -2 \times -6$
Since it is same, we derive $x = 0$

This example, we have already dealt in type (ii) of Paravartya in solving simple equations.

iii) We interpret ' Samuccaya 'as the sum of the denominators of two fractions having the same numerical numerator.

**Consider the example.**

$$\frac{1}{3x-2} + \frac{1}{2x-1} = 0$$

for this we proceed by taking LCM.

$$\frac{(2x-1) + (3x-2)}{(3x-2)(2x-1)} = 0$$

$$\frac{5x-3}{(3x-2)(2x-1)} = 0$$

$$5x - 3 = 0 ; \quad 5x = 3$$

Instead of this, we can directly put the Samuccaya i.e., sum of the denominators

i.e., $3x - 2 + 2x - 1 = 5x - 3 = 0$
Giving $5x = 3$ $\qquad x = 3 / 5$

It is true and applicable for all problems of the type

$$\frac{m}{ax+b} + \frac{m}{cx+d} = 0$$

Samuccaya is ax+b+cx+d and solution is ($m \neq 0$)

$$x = \frac{-(b+d)}{(a+c)}$$

iii) We now interpret 'Samuccaya' as combination or total.

If the sum of the numerators and the sum of the denominators be the same, then that sum = 0.

**Consider examples of type**

$$\frac{ax+b}{ax+c} = \frac{ax+c}{ax+b}$$

In this case, $(ax+b)(ax+b) = (ax+c)(ax+c) = a^2x^2 + 2abx + b^2$

$$= a^2x^2 + 2acx + c^2$$

$$2abx - 2acx = c^2 - b^2$$

$$= x(2ab - 2ac)$$

$$= c^2 - b^2$$

$$x = \frac{c^2-b^2}{2a(b-c)} = \frac{(c+b)(c-b)}{2a(b-c)} = \frac{-(c+b)}{2a}$$

As per Samuccaya $(ax+b) + (ax+c) = 0$

$$2ax+b+c = 0$$

$$2ax = -b-c$$

$$x = \frac{-(c+b)}{2a}$$ Hence the statement.

**Example 4:**

$$\frac{3x+4}{3x+5} = \frac{3x+5}{3x+4}$$

Since $N_1 + N_2 = 3x + 4 + 3x + 5 = 6x + 9$ , And $D_1 + D_2 = 3x + 4 + 3x + 5 = 6x + 9$

We have $N_1 + N_2 = D_1 + D_2 = 6x + 9$

Hence from Sunya Samuccaya we get $6x + 9 = 0$

$6x = -9$

$$x = \frac{-9}{6} \quad \frac{-3}{2}$$

**Example 5:**

$$\frac{5x + 7}{5x + 12} = \frac{5x + 12}{5x + 7}$$

Hence $N_1 + N_2 = 5x + 7 + 5x + 12 = 10x + 19$

And $D_1 + D_2 = 5x + 12 + 5x + 7 = 10x + 19$ $N_1 + N_2 = D_1 + D_2$ gives $10x + 19 = 0$

$10x = -19$

$$x = \frac{-19}{10}$$

Consider the examples of the type, where $N_1 + N_2 = K (D_1 + D_2)$, where K is a numerical constant, then also by removing the numerical constant K, we can proceed as above.

**Example 6:**

$$\frac{2x + 3}{4x + 5} = \frac{x + 1}{2x + 3}$$

Here $N_1 + N_2 = 2x + 3 + x + 1 = 3x + 4$

$D_1 + D_2 = 4x + 5 + 2x + 3 = 6x + 8$

$= 2 (3x + 4 )$

Removing the numerical factor 2, we get $3x + 4$ on both sides.

$3x + 4 = 0 \quad 3x = -4 \quad x = - 4 / 3.$

v) 'Samuccaya' with the same meaning as above, i.e., case (iv), we solve the problems leading to quadratic equations. In this context, we take the problems as follows;

If $N_1 + N_2 = D_1 + D_2$ and also the differences $N_1 \sim D_1 = N_2 \sim D_2$ then both the things are equated to zero, the solution gives the two values for x.

**Example 7:**

$$\frac{3x+2}{2x+5} = \frac{2x+5}{3x+2}$$

In the conventional text book method, we work as follows:

$$\frac{3x+2}{2x+5} = \frac{2x+5}{3x+2}$$

$$(3x + 2)(3x + 2) = (2x + 5)(2x + 5)$$
$$9x^2 + 12x + 4 = 4x^2 + 20x + 25$$
$$9x^2 + 12x + 4 - 4x^2 - 20x - 25 = 0$$
$$5x^2 - 8x - 21 = 0 \quad 5x^2 - 15x + 7x - 21 = 0$$
$$5x(x - 3) + 7(x - 3) = 0$$
$$(x - 3)(5x + 7) = 0$$
$$x - 3 = 0 \text{ or } 5x + 7 = 0$$
$$x = 3 \text{ or } -7/5$$

Now '**Samuccaya'** sutra comes to help us in a beautiful way as follows:

Observe $N_1 + N_2 = 3x + 2 + 2x + 5 = 5x + 7$
$D_1 + D_2 = 2x + 5 + 3x + 2 = 5x + 7$

Further $N_1 \sim D_1 = (3x + 2) - (2x + 5) = x - 3$
$N_2 \sim D_2 = (2x + 5) - (3x + 2) = -x + 3 = -(x - 3)$
Hence $5x + 7 = 0$, $x - 3 = 0$ $5x = -7$, $x = 3$
i.e., $x = -7/5$, $x = 3$

Note that all these can be easily calculated by mere observation.

**Example 8:** $\frac{3x+4}{6x+7} = \frac{5x+6}{2x+3}$

Observe that
$N_1 + N_2 = 3x + 4 + 5x + 6 = 8x + 10$ and
$D_1 + D_2 = 6x + 7 + 2x + 3 = 8x + 10$

Further $N_1 \sim D_1 = (3x + 4) - (6x + 7)$
$= 3x + 4 - 6x - 7$
$= -3x - 3 = -3(x + 1)$
$N_2 \sim D_2 = (5x + 6) - (2x + 3) = 3x + 3 = (3x + 1)$

By 'Sunyam Samuccaye' we have

$$8x + 10 = 0 \qquad 3(x + 1) = 0$$
$$8x = -10 \qquad x + 1 = 0$$
$$x = -10/8 \qquad x = -1$$
$$= -5/4$$

vi) ‘Samuccaya’ with the same sense but with a different context and application.

**Example 9:**

$$\frac{1}{x-4} + \frac{1}{x-6} = \frac{1}{x-2} + \frac{1}{x-8}$$

Usually we proceed as follows.

$$\frac{x-6+x-4}{(x-4)(x-6)} = \frac{x-8+x-2}{(x-2)(x-8)}$$

$$\frac{2x-10}{x^2-10x+24} = \frac{2x-10}{x^2-10x+16}$$

$$(2x-10)(x^2-10x+16) = (2x-1)(x^2-10x+24)$$
$$2x^3-20x^2+32x-10x^2+100x-160 = 2x^3-20x^2+48x-10x^2+100x-240$$

$$2x^3 - 30x^2 + 132x - 160 = 2x^3 - 30x^2 + 148x - 240$$
$$132x - 160 = 148x - 240$$
$$132x - 148x = 160 - 240$$
$$-16x = -80$$
$$x = -80/-16 = 5$$

Now ‘**Samuccaya’** sutra, tell us that, if other elements being equal, the sum-total of the denominators on the L.H.S. and their total on the R.H.S. be the same, that total is zero.

Now $D_1 + D_2 = x - 4 + x - 6 = 2x - 10$, and
$D_3 + D_4 = x - 2 + x - 8 = 2x - 10$

By Samuccaya, $2x - 10$ gives $2x = 10$

$$x = \frac{10}{2} = 5$$

**Example 10:**

$$\frac{1}{x-8} + \frac{1}{x-9} = \frac{1}{x-5} + \frac{1}{x-12}$$

$D_1 + D_2 = x - 8 + x - 9 = 2x - 17$, and
$D_3 + D_4 = x - 5 + x - 12 = 2x - 17$

Now $2x - 17 = 0$ gives $2x = 17$

$$x = \frac{17}{2} = 8\tfrac{1}{2}$$

**Sunyam Samya Samuccaye in Certain Cubes:**

Consider the problem $\mathbf{(x - 4)^3 + (x - 6)^3 = 2\,(x - 5)^3}$. For the solution by the traditional method we follow the steps as given below:

$$(x - 4)^3 + (x - 6)^3 = 2\,(x - 5)^3$$
$$x^3 - 12x^2 + 48x - 64 + x^3 - 18x^2 + 108x - 216$$
$$= 2\,(x^3 - 15x^2 + 75x - 125)$$
$$2x^3 - 30x^2 + 156x - 280 = 2x^3 - 30x^2 + 150x - 250$$
$$156x - 280 = 150x - 250$$
$$156x - 150x = 280 - 250$$
$$6x = 30$$
$$x = 30 / 6 = 5$$

But once again observe the problem in the Vedic sense

We have $(x - 4) + (x - 6) = 2x - 10$. Taking out the numerical factor 2, we have $(x - 5) = 0$, which is the factor under the cube on R.H.S. In such a case "Sunyam samya Samuccaye" formula gives that $x - 5 = 0$. Hence $x = 5$

Think of solving the problem $(x-249)^3 + (x+247)^3 = 2(x-1)^3$

The traditional method will be horrible even to think of.

But $(x - 249) + (x + 247) = 2x - 2 = 2\,(x - 1)$. And $x - 1$. on R.H.S. cube, it is enough to state that $x - 1 = 0$ by the 'sutra'. $x = 1$ is the solution. No cubing or any other mathematical operations.

**Algebraic Proof:**

Consider $(x - 2a)^3 + (x - 2b)^3 = 2\,(x - a - b)^3$ it is clear that $x - 2a + x - 2b$
$= 2x - 2a - 2b$
$= 2\,(x - a - b)$

Now the expression,

$$x^3 - 6x^2a + 12xa^2 - 8a^3 + x^3 - 6x^2b + 12xb^2 - 8b^3$$
$$= 2(x^3 - 3x^2a - 3x^2b + 3xa^2 + 3xb^2 + 6axb - a^3 - 3a^2b - 3ab^2 - b^3)$$

$$= 2x^3 – 6x^2a – 6x^2b + 6xa^2 + 6xb^2 + 12xab – 2a^3 – 6a^2b – 6ab^2 – 2b^3$$

cancel the common terms on both sides

$$12xa^2 + 12xb^2 – 8a^3 – 8b^3 = 6xa^2 + 6xb^2 + 12xab – 2a^3 – 6a^2b – 6ab^2 – 2b^3$$
$$6xa^2 + 6xb^2 – 12xab = 6a^3 + 6b^3 – 6a^2b – 6ab^2$$
$$6x (a^2 + b^2 – 2ab) = 6 [ a^3 + b^3 – ab ( a + b)]$$
$$x ( a – b )^2 = [ ( a + b ) ( a^2 + b^2 – ab ) – ( a + b )ab]$$
$$= ( a + b ) ( a^2 + b^2 – 2ab )$$
$$= ( a + b ) ( a – b )^2$$

$$\therefore x = a + b$$

**Example:**

$$\frac{(x + 2)^3}{(x + 3)^3} = \frac{x + 1}{x + 4}$$

with the text book procedures we proceed as follows

$$\frac{x^3 + 6x^2 + 12x + 8}{x^3 + 9x^2 + 27x + 27} = \frac{x + 1}{x + 4}$$

Now by cross multiplication,

$$(x + 4) (x^3 + 6x^2 + 12x + 8) = (x + 1) (x^3 + 9x^2 + 27x + 27)$$
$$x^4 + 6x^3 + 12x^2 + 8x + 4x^3 + 24x^2 + 48x + 32$$
$$= x^4 + 9x^3 + 27x^2 + 27x + x^3 + 9x^2 + 27x + 27$$
$$= x^4 + 10x^3 + 36x^2 + 56x + 32$$
$$= x^4 + 10x^3 + 36x^2 + 54x + 27$$

$$56x + 32 = 54x + 27$$
$$56x – 54x = 27 – 32$$
$$2x = - 5$$
$$x = - 5 / 2$$

Observe that $(N_1 + D_1)$ within the cubes on
L.H.S. is $x + 2 + x + 3 = 2x + 5$ and

$N_2 + D_2$ on the right hand side is $x + 1 + x + 4 = 2x + 5$.

By Vedic formula we have $2x + 5 = 0 \quad x = - 5 / 2$.

# CHAPTER EIGHT: CONTRIBUTION OF ANCIENT INDIAN MATHEMATICIANS IN ARITHMETIC

## BRAHMAGUPTA

Brahmagupta wrote 3 treatises - (1) Brahmasphuta-Sidhdanta (2) Khand-Khaddak (3) Uttar- Khand khaddak. He was the first mathematician who classified mathematics as Arithmetic and Bijaganit (Algebra).

**Contribution of Brahmagupta in arithmetic**

Brahmasphuta-Sidhdanta is the earliest known text to treat zero as a number in its own right,rather than as simply a place holder value digit in representing another number as was done by "Babylonian" or as a symbol for a lack of quantity as was done by "Ptolemy" and the "Romans". Hence Brahmagupta is considered the first to formulate the concept of zero as a number. He gave rules of using zero with negative and positive numbers. His rules for arithmetic on negative / positivenumbers and zero are quite close to modern understanding except that in modern mathematics division by zero is left undefined.

There are 20 operations like addition; subtraction etc. in mathematics was his postulate. He proved that addition & subtraction as well as multiplication & division are inverse operations. He gives rule facilitating the computation of square, square-root, cube & cube-root of an integer. He then gives rule for dealing with five types of combination of fractions, a/c + b/c, a/c *b/d,

a/1 + b/d, a/c +b/d *a/c = [a (d+b)]/ cd and a/c - b/d = [a(d-b)]/cd

He discovered formula to compute sum of the squares & cubes of first 'n' natural numbers.

Sum of squares of first 'n' natural numbers = [n (n+1)(2n+1)]/6

Sum of cubes of first 'n' natural numbers = $[n (n+1)/2]^2$

Formula for computing square ofany number

$X^2 = (X-Y)(X+Y) + Y^2$

$13^2 = (13-2)(13+2) + 2^2$

$= 11 * 15 + 4$

$= 169$

# MAHAVIRACHARYA

He was the author of the treatise "Ganitsarasangraha". His work was totally & purely on mathematics only. In his treatise Ganitsarasangrah he revised some intricacies of Brahmagupta's Brahma-sphuta sidhdanta and some additional information was incorporated.Ganitsarasangrah is one of the text books on modern mathematics.

## Contribution of Mahaviracharya in arithmetic

He was the first mathematician who invented different names for different places with its value in decimal numbersystem e.g. एक, दश, शत, सहस्त्रup to 24 places. The last place was named as "महाक्षोभ" whose value is $10^{24}$. He was the first mathematician to deal with least common multiple (LCM). He defined palindromes and their factorization by using multiplication as 15151 = 139* 109

12345654321 = 279946681 * 441 etc.

He discovered method to calculate square, square root& cube, cube root of a number.

$X^3 = 3\,\Sigma\,(Y^2 - Y) + X$

Here $\Sigma Y^2 = [X(X+1)(2X+1)]/6$

$\Sigma\,Y = [X(X+1)]/2$

He asserted that square root of a negative number did not exist. He invented systematic rule for expressing a fraction as the sumof many fractions whose numerator is one.

2/17 = 1/12 + 1/51 + 1/68

Mahaviracharya's one important formula

$1 = 1/2 + 1/3 + 1/3^2 + 1/3^3$ ....................... n, 'n' tending to 'Anant' i.e. infinity

He also obtained values of $n, n^2, n^3$, where n is a natural number.

Methods to solve continued fractions are available in his Ganitsarasangraha which are similar to Euler's methods of 1764.

Methods to solve puzzles are available in his Ganitsarasangraha which are similar to Euclid's algorithm method.

# SHRINIVAS RAMANUJAN

Shrinivas Ramanujan was a celebrated Indian mathematician. Subjects of his research were number theory, theory of partitions, arithmetical functions and hyper geometric series. His last diary is called as "Lost Note book" in which 600 theorems are incorporated. Efforts are made still today to prove these theorems. Every number has individual identity was his concept.

## Contribution of Shrinivas Ramanujan in arithmetic

Hardy-Ramanujan-Littlewoods circle method in number theory.

Roger-Ramanujan's identities in partition of numbers.

Work on continued fractions, partial sums and products of hyper geometric series.

**Properties of number 1729 invented by him are**

$1729 = 1^3 + 12^3 = 10^3 + 9^3$

$1729 = 1*7*13*19$

$1729^3 = 7*13*19*91*133*247$

$1729^4 = 7*13*19*91*133*247*1729$

$1729 = 1^3 + 6^3 + 8^3 + 10^3$

$1729 = 1^3 + 3^3 + 4^3 + 5^3 + 8^3 + 10^3$

$1729 = 37^2 + 19^2 - 1^2$

$1729 = 19 * 91$

$1729 = 865^2 - 864^2$

$1 \longrightarrow (9 - 7 - 2 + 1) \longrightarrow (1 + 7 + 2 - 9) \longrightarrow [7 - (\sqrt{9} + 2 + 1)]$

$2 \longrightarrow [\{(9 - 7) \div 2\} + 1]$

$3 \longrightarrow (9 + 2 - 7 - 1)$

$4 \longrightarrow [(2 \times 1) + 9 - 7] \longrightarrow [(9 + 2) - (1 \times 7)]$

$5 \longrightarrow (9 + 1 + 2 - 7) \longrightarrow [(7 \times 2) - (1 \times 9)]$

$6 \rightarrow [(7 \times 2) - 9 + 1]$

$7 \rightarrow (\sqrt{9} + 7 - 1 - 2)$

$8 \rightarrow [(\sqrt{9} \times 1) + 7 - 2]$

$9 \rightarrow [\{(9 + 7) \div 2\} + 1] \rightarrow (\sqrt{9} + 7 + 1 - 2)$

$10 \rightarrow [(9 \times 2) - 7 - 1]$

$11 \rightarrow (\sqrt{9} + 7 + 2 - 1)$

$12 \rightarrow [(9 \times 2) - 7 + 1] \rightarrow [(\sqrt{9} + 7 + 2) \times 1]$

$13 \rightarrow (9 + 7 - 2 - 1) \rightarrow (\sqrt{9} + 7 + 2 + 1)$

$14 \rightarrow [(\sqrt{9} \times 2) + 7 + 1]$

$15 \rightarrow (9 + 7 + 1 - 2)$

$16 \rightarrow [(7 \times 2) + \sqrt{9} - 1]$

$17 \rightarrow (9 + 7 + 2 - 1)$

$18 \rightarrow [(9 + 7) + (2 \times 1)] \rightarrow [(\sqrt{9} \times 7) - 2 - 1]$

$19 \rightarrow (9 + 7 + 2 + 1)$

$20 \rightarrow [(\sqrt{9} \times 7) + 1 - 2]$

$21 \rightarrow (9 + 7 + 1 + 2^2)$

$22 \rightarrow [(\sqrt{9} \times 7) + 2 - 1]$

$23 \rightarrow [\{(9 - 1) \times 2\} + 7] \rightarrow (9 + 7 + 2^3 - 1)$

$24 \rightarrow [(7 \times 2) + 9 + 1] \rightarrow [(9 \times 2) + 7 - 1]$

$25 \rightarrow [(7 \times 1) + (9 \times 2)] \rightarrow [\{(1 + 7) \times 2\} + 9] \rightarrow (9 + 7 + 2^3 + 1)$

$26 \rightarrow [(9 \times 2) + 7 + 1]$

27 $\longrightarrow$ $[\{(9 + 1) \times 2\} + 7]$

28 $\longrightarrow$ $[(9 \times 2^2) - 7 - 1]$

29 $\longrightarrow$ $[(9^2 - 7^2) - 2 - 1]$

30 $\longrightarrow$ $[\{(9^2 - 7^2) \times 1\} - 2]$ $\longrightarrow$ $[(9 \times 2^2) - 7 + 1]$ $\longrightarrow$ $[\{(\sqrt{9} + 1) \times 7\} + 2]$

31 $\longrightarrow$ $[\{(9+7) \times 2\} - 1]$ $\longrightarrow$ $[(9^2 - 7^2) + 1 - 2]$

32 $\longrightarrow$ $[(9 + 7) \times 2 \times 1]$

33 $\longrightarrow$ $[\{(9+7) \times 2\} + 1]$ $\longrightarrow$ $[\{(9+1) \times 2^2\} - 7]$ $\longrightarrow$ $[(9^2 - 7^2) + 2 - 1]$

34 $\longrightarrow$ $[\{(1+2) \times 9\} + 7]$ $\longrightarrow$ $[\{(2^2 - 1) \times 9\} + 7]$

35 $\longrightarrow$ $[\{(9+1) \div 2\} \times 7]$

36 $\longrightarrow$ $[\{(1+7) \div 2\} \times 9]$

37 $\longrightarrow$ $[\{(7 \times 2^2) + 9\} \times 1]$

38 $\longrightarrow$ $[\{(7 \times 2^2) + 9\} + 1]$

39 $\longrightarrow$ $[(9^2 - 7^2) + 2^3 - 1]$

40 $\longrightarrow$ $[\{(9^2 - 7^2) + 2^3\} \times 1]$

41 $\longrightarrow$ $[(9^2 - 7^2) + 2^3 + 1]$

42 $\longrightarrow$ $[\{(9 \times 2^2) + 7 - 1]$

43 $\longrightarrow$ $(7^2 + 2 + 1 - 9)$ $\longrightarrow$ $[(9 \times 2^2 \times 1) + 7]$

44 $\longrightarrow$ $[(9 \times 2^2) + 7 + 1]$

45 $\longrightarrow$ $(7^2 + 2^2 + 1 - 9)$

46 $\longrightarrow$ $[(7 \times 2^3) - 9 - 1]$

47 $\longrightarrow$ $[\{(9 + 1) \times 2^2\} + 7]$ $\longrightarrow$ $[(7^2 + 2^3) - 9 - 1]$

48 $\longrightarrow$ $[(7 \times 2^3) - 9 + 1]$

49 $\longrightarrow$ $[(7^2 - 2^3) + 9 - 1]$

$50 \longrightarrow [\{(7^2 - 2^3) + 9\} \times 1]$

$51 \longrightarrow (7^2 - 2^3 + 9 + 1]$

$52 \longrightarrow [(2^3 \times 7) - \sqrt{9} - 1]$

$53 \longrightarrow (7^2 + 9 - 2^2 - 1)$

$54 \longrightarrow [(9 \times 7) - 2^3 - 1]$

$55 \longrightarrow (7^2 + 9 - 2 - 1) \longrightarrow [\{(9 \times 7) - 2^3\} \times 1]$

$56 \longrightarrow [(9 \times 7) - 2^3 + 1]$

$57 \longrightarrow [(9^2 - 2^4) - 7 - 1]$

$58 \longrightarrow [\{(9^2 - 2^4) - 7\} \times 1]$

$59 \longrightarrow [(9^2 - 2^4) - 7 + 1]$

$60 \longrightarrow [(9 \times 7) - 2 - 1]$

$61 \longrightarrow [\{(9 \times 7) - 2\} \times 1] \longrightarrow (7^2 + 9 + 2 + 1)$

$62 \longrightarrow [(9 \times 7) + 1 - 2]$

$63 \longrightarrow [\{(9^2 - 7^2) \times 2\} - 1]$

$64 \longrightarrow [(9^2 - 7^2) \times 2 \times 1]$

$65 \longrightarrow [\{(9+7) \times 2^2\} + 1] \longrightarrow [\{(9^2 - 7^2) \times 2\} + 1]$

$66 \longrightarrow [(9 \times 7) + 1 + 2]$

$67 \longrightarrow [\{(9 \times 7) + 2^2\} \times 1] \longrightarrow (2^6 + 9 + 1 - 7)$

$68 \longrightarrow [\{(9 + 1) \times 7\} - 2] \longrightarrow [(2 \times 9) + 7^2 + 1]$

$69 \longrightarrow (9^2 - 2^2 - 7 - 1)$

$70 \longrightarrow [\{(9^2 - 2^2) - 7\} \times 1] \longrightarrow [\{(7 + 1) \times 9\} - 2] \longrightarrow [(9 \times 7) + 2^3 - 1]$

$71 \longrightarrow [\{(9 \times 7) \times 1\} + 2^3] \longrightarrow (9^2 - 2^4 + 7 - 1)$

$72 \longrightarrow [\{(9 + 1) \times 7\} + 2]$

$73 \longrightarrow (7^2 + 2^4 + 9 - 1) \longrightarrow (7^2 + 2^5 + 1 - 9)$

$74 \longrightarrow [\{(7 + 1) \times 9\} + 2]$

$75 \longrightarrow (7^2 + 2^4 + 9 + 1)$

$76 \longrightarrow [\{(9 + 2) \times 7\} - 1]$

$77 \longrightarrow [(9 + 2) \times 7 \times 1] \longrightarrow (9^2 + 2 + 1 - 7)$

$78 \longrightarrow [\{(9 + 2) \times 7\} + 1]$

$79 \longrightarrow (2^6 + 9 + 7 - 1)$

$80 \longrightarrow [(9 \times 7) + 2^4 + 1]$

$81 \longrightarrow (2^6 + 9 + 7 + 1)$

$82 \longrightarrow [(9^2 + 2^3 - 7) \times 1]$

$83 \longrightarrow (9^2 - 2^2 - 1 + 7)$

$84 \longrightarrow [(9 + 2 + 1) \times 1] \longrightarrow [(9^2 - 2^2 + 7) \times 1]$

$85 \longrightarrow (9^2 + 7 - 2 - 1)$

$86 \longrightarrow [(9^2 + 7 - 2) \times 1]$

$87 \longrightarrow (9^2 + 7 + 1 - 2)$

$88 \longrightarrow [(\sqrt{9} \times 2^5) - 7 - 1]$

$89 \longrightarrow (7^2 + 2^5 + 9 - 1) \longrightarrow (9^2 + 7 + 2 - 1)$

$90 \longrightarrow [(9^2 + 7 + 2) \times 1]$

$91 \longrightarrow (9^2 + 7 + 2 + 1)$

$92 \longrightarrow [(9^2 + 7 + 2^2) \times 1]$

$93 \longrightarrow (9^2 + 7 + 2^2 + 1)$

$94 \longrightarrow [(2 \times 7) + 9^2 - 1]$

$95 \longrightarrow (9^2 + 2^3 + 7 - 1)$

$96 \rightarrow [(2 \times 7) + 9^2 + 1]$

$97 \rightarrow [\{(1 + 7) \times 2\} + 9^2] \rightarrow (9^2 + 2^3 + 7 + 1)$

$98 \rightarrow [(\sqrt{9} \times 2^4) + 7^2 + 1]$

$99 \rightarrow [(7 + 2^2) \times 9 \times 1]$

$100 \rightarrow [(7^2 \times 2) + \sqrt{9} - 1]$

1729 is divisible by prime numbers 7, 13 & 19. The difference between 13 & 7 as well as 19 & 13 is same i.e. 6. When 1729 is divided by 19 (divisor), quotient obtained is 91. 19 & 91 are two numbers which are obtained by interchanging their digits.

Hence 1729 is called as Ramanujan number in his honour and memory.

He invented many best formulae

E.g. $184025^3 + 1119810^3 = 189209^3 + 1119666^3$

Formation of magic squares and rectangles was his hobby. He deriveda method for formation of magic squares and rectangles. He could calculate the value of $\sqrt{2}$ and $\pi$ upto anynumber of decimal places.

## ARYABHAT (5th century)

Aryabhat wrote his famous treatise "Aryabhatiya". "Aryabhatiya" is the reference treatise like Greek mathematician Euclid's treatise "Elements of mathematics". Maximum description in least words is the speciality of his "Aryabhatiya".

## Contribution of Aryabhat (5th century) in arithmetic

He invented a notation system consisting of alphabet numerals. Digits were denoted by alphabet numerals e.g.क = 1, ख = 2-------. Devnagari script contains varga and avarga letters. 1-25 are denoted by first 25 varga letters. He assigned numerical values to 33 consonants of Devnagari script to represent 1, 2, 3, -------------------25, 40, 50, ------------100. In only one Sanskrit verse he had explained his notation system and it is known as "क-च-ट-त-प" script.

He was familiar with place value system as well as numeral symbols and sign for zero. His calculation of square root & cube root would not have been feasible without the knowledge of place value system & zero. He invented rules for calculating square, square-root, cube, cube-root of a number along with their explanation. He invented for the first in India the formula for calculating interest, time & other related one in the problems of Interest. He invented formula for summation of series of squares & cubes.

$1^2 + 2^2 + 3^2 +$ ------------------------$n^2 = [n\ (n+1)(2n+1) / 6$

$1^3 + 2^3 + 3^3 +$-------------------------$n^3= [n^2\ (n+1)^2] / 4$

He inventedmethod of addition, subtraction, multiplication of simple and complex quantities. He formulated methods for solving puzzles.

e.g. A number when divided by 7509 produces 13 as a remainder and when divided by 5301 produces 25 as a remainder. Find out the number (dividend)?

(Ans: -21993874)

## D. R. KAPREKAR

D.R.Kaprekar was a recreational mathematician who describes several classes of natural numbers including the Kaprekar, Harshad& self numbers as well as discovered Kaprekar constant, named after him. He won the Wrangler R.P.Paranjape mathematical prize for an original piece of work in mathematics. He published writing about such topics as Recurring decimal, Magic squares and integers with special properties. He is also known as "Ganitanand". He discovered a number of results in number theory and described various properties of numbers.

### Contribution of D.R.Kaprekar in arithmetic

### Kaprekar constant for 3 digit and 4 digit number

He showed that 495 is reached in the limit as one repeatedly subtracts highest number and lowest number that can be constructed from a set of 3 digits that are not all identical.

e.g.:- 492

| | | | |
|---|---|---|---|
| 942 | 963 | 954 | 954 |
| - 249 | - 369 | - 459 | - 459 |
| 693 | 594 | 495 | 495 |

Repeating from this point onwards leaves the same number (495).

A similar constant for 4 digit number is 6174. In general, when the operation converges it does so in at most seven iterations. However, in base 10 a single such constant only exists for numbers of 3 or 4 digits; for more digits or 2 digits, the number enter into one of several cycles.

e.g.:- 1234

| | | | |
|---|---|---|---|
| 4321 | 8730 | 8532 | 7641 |
| -1234 | - 0378 | - 2358 | - 1467 |
| 3087 | 8352 | 6174 | 6174 |

**Kaprekar Number: -** It is a positive integer with the property that if it is squared then its representation can be partitioned into two positive integer parts whose sum is equal to the original number.

e.g.:- $45^2 = 2025$

$20 + 25 = 45$

Hence 45 is a Kaprekar number, also 9, 55, 99 etc.

However note the restriction that two numbers are positive; e.g. 100 is not a Kaprekar number even though $100^2 = 10000$ and $100+00 = 100$.

The operation of taking the rightmost digits of a square and adding it to the integer formed by leftmost digits is known as "Kaprekar operation". Some examples of Kaprekar number in base 10, besides numbers 9, 99,999.......... are

| Number | Square | Partition |
|---|---|---|
| 297 | $297^2 = 88209$ | 88 + 209 = 297 |
| 703 | $703^2 = 494209$ | 494 + 209 = 703 |
| 2223 | $2223^2 = 4941729$ | 494 + 1729 = 2223 |
| 2728 | $2728^2 = 7441984$ | 744 + 1984 = 2723 |
| 5292 | $5292^2 = 28005264$ | 28 + 005264 = 5292 |
| 7777 | $7777^2 = 60481729$ | 6048 + 1729 = 7777 |
| 857143 | $857143^2 = 734694122449$ | 734694 + 122449 = 857143 |

**<u>Kaprekar Number (1089)</u>**

| | | | |
|---|---|---|---|
| 741 | 594 | 603 | 297 |
| -147 | + 495 | - 306 | + 792 |
| 594 | 1089 | 297 | 1089 |

**<u>Harshad Number</u>: -** He defined Harshad Number by the property that they are divisible by the sum of their digits.

e.g.:- 12 which is divisible by (1+2) = 3 hence it is a Harshad Number.

476, 48, 70, 133, 209,247,308 are divisible by sum of their digits hence they are Harshad Number. Numbers which are Harshad in all bases are only 1, 2, 4& 6 called as **"All Harshad Number".**

**<u>Devalali or Self Number:-</u>**He defined the property which has come to be known as self numbers, which are integers that can not be generated by taking some other number and adding its own digits to it.

e.g. 21 is not a self number since it can be generated from 15 (15 + 1 + 5 = 21).

20 is a self number since it can not be generated from any other integer and adding its own digits to it.

**<u>Demlo Number: -</u>** The numbers 1, 121, 12321 which are the squares of the repunits 1, 11,111....... are called "Demlo Numbers".

**<u>Dattatraya Numbers: -</u>** He defined Dattatraya numbers with the property that if it is squared then its representation is partitioned in numbers equal to the number of digits in Dattatraya numbers and partitioned parts are also squares.

$13^2 = \underline{16}\ \ \underline{9}$

$57^2 = \underline{324}\ \ \underline{9}$

$1602^2 = \underline{256}\ \ \underline{64}\ \ \underline{0\,4}$

$40204^2 = \underline{16}\ \ \underline{16}\ \ \underline{36}\ \ \underline{16}\ \ \underline{16}$

Hence 13, 57,1602,40204 are Dattatraya Numbers.

**Some research work of D.R.Kaprekar**

$4204234125 = 4^3 + 20^3 + 42^3 + 3^3 + 41^3 + 25^3$

$5214324024 = 5^3 + 21^3 + 43^3 + 2^3 + 40^3 + 24^3$

$972^2 = 944784$

$= 944 + 784$

$= 1728$

$= 1 + 728$

$= 729$

27272 = 7436529

= 743 + 6529

= 7272

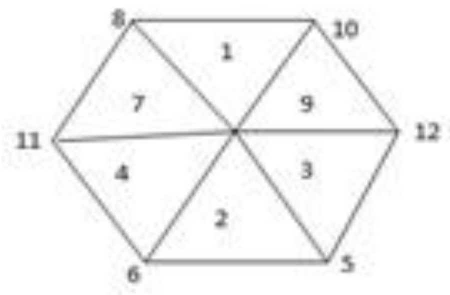
**Magic Square:-**Vertical, horizontal and diagonal sum = 89

| | | | |
|---|---|---|---|
| 15 | 8 | 19 | 47 |
| 48 | 18 | 11 | 12 |
| 9 | 14 | 46 | 20 |
| 17 | 49 | 13 | 10 |

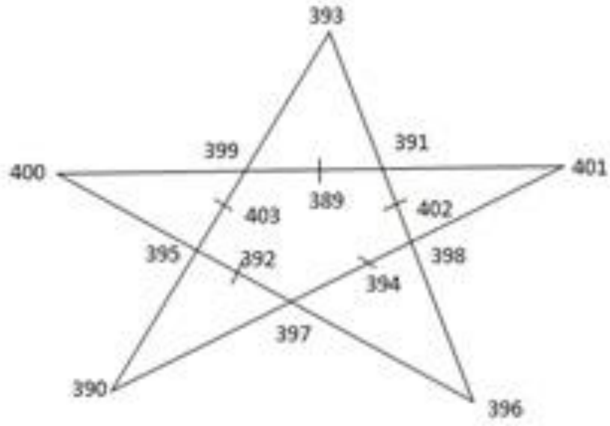
**Magic Hexagon: -** Number on each angle is the sum of numbers in two adjacent triangles. Numbers are 1 to 12 and each number is used once only.

e.g.:- 8 = 7 + 1

10 = 1 + 9



**<u>Kaprekar Star: -</u>** There are 5 numbers on each line and sum of these numbers is 1980.



# PART
# TWO
# VEDIC ALGEBRA

# CHAPTER NINE: ALGEBRIC MULTIPLICATION

उर्ध्वतिर्यग्भ्याम् sutra is applied same as applied in arithmetic. Each term of algebraic expression with sign convention is treated as a single digit of arithmetic and perform vertical as well as cross product of digits as in arithmetic. In algebra like terms (variables) are added only. If cross product results in different terms then they are written separately.

During such operations one has to consider the terms with their sign convention which will be same as vinculum numbers of arithmetic. Except these facts remaining procedure is same as in arithmetic.

**E.g.** Find $(2x^2 – 7x + 2)$ X $(x^2 + 2x -5)$

Multiplicand & Multiplier are written one below the other.

| | | |
|---|---|---|
| | $2x^2 – 7x + 2$ | as multiplier & multiplicand have 3 |
| X | | terms hence no. of steps will be (2n-1) |
| | $x^2 + 2x - 5$ | where 'n' is no. of terms in the expression |

| **V** | **IV** | **III** | **II** | **I** | ← **Step** |
|---|---|---|---|---|---|
| $x^4$ | $x^3$ | $x^2$ | x | const | |
| * * *<br>* * * | * * *<br>* * * | * * *<br>* * * | * * *<br>* * * | * * *<br>* * * | |
| 2*1 | (2*2)<br>+(-7*1) | (2*1) +<br>(-5*2)<br>+(-7*2) | (2*2)+<br>(-5*-7) | 2*-5 | |
| 2 | -3 | -22 | 39 | -10 | |

<u>**Answer**</u>: - 2 $x^4$ - 3 $x^3$ - 22 x2 + 39 x – 10

1) **Inclusion phase = 'n' terms**
   Step I to III
2) **Exclusion phase = 'n-1' terms**
   Step IV & V

B $(2x^4 -3x^3 -22 x^2 +39 x – 10)$ = (2 -3 -2-2 +3+9 -1-0) = 6

B $(2x^2 – 7x + 2)$ X B $(x^2 + 2x -5)$ = (2-7 + 2) (1 + 2 = (-3 * -2) = 6

As ‘**Beejank’** is same hence the answer is verified.
**Exercise**: -

1. $(8x + 9y) X (-2x + 5y)$

2. $(x + 1) X (3x^2 – 11)$

3. $(8x^2 – 7y) X (2x2 + 5y)$

4. $(2x^2 – 3y^2) X (4x^2 – 1 + y^2)$

5. $(8a^2 – 3ab + 6) X (2ab – 5a^2 – 3)$

6. $(3x + 5y + 6) X (4x + 2y – 7)$

**By निखिलम् नवत: चरमं दशत: sutra.** Any term of algebraic expression can be considered as base.
e.g. $(x + 3) (x + 4)$

Base (B): - ‘x’
Deviation = $(d_1)$ = +3
$(d_2)$ = + 4

| | | |
|---|---|---|
| | $x+ 3(N_1)$ | $3 (d_1)$ |
| X | $x + 4(N_2)$ | $4 (d_2)$ |
| | LHS | RHS |
| | $B*( N_1+ d_2)$ or $B*( N_2+ d_1)$ | $d_1 * d_2$ |
| | $x(x+3+4)$ | + 12 |
| | $x^2 + 7x$ | + 12 |

**Answer** = $x^2 + 7x + 12$

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By निखिलम् नवत: चरमं दशत: sutra.
$(x + 4y) X (x + 5y)$

Base: - ‘x’
Deviation: - $(d_1)$ = +4y
$(d_2)$ = +5y

| | | |
|---|---|---|
| | $x + 4y (N_1)$ | $+4y (d_1)$ |
| X | $x + 5y (N_2)$ | $+5y(d_2)$ |
| | LHS | RHS |
| | $B*( N_1+ d_2)$ or $B*( N_2+ d_1)$ | $d_1 * d_2$ |
| | $x*( x + 4y+5y)$ | $4y * 5y$ |
| | $x^2 + 9xy$ | $+ 20y^2$ |

**Answer** = $x^2 + 9xy + 20y^2$

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By निखिलम् नवत: चरमं दशत: sutra.

$(x + y + z) X (x + y - z)$

Base: - ‘x + y’

Deviation: - $(d_1) = + z$

$(d_2) = - z$

| | | |
|---|---|---|
| | $x + y + z (N_1)$ | $+ z (d_1)$ |
| X | $x + y - z (N_2)$ | $- z (d_2)$ |
| | LHS | RHS |
| | $B^*( N_1+ d_2)$ or $B^*( N_2+ d_1)$ | $d_1 * d_2$ |
| | $(x+y)^*( x + y+z -z)$ | $z * - z$ |
| | $(x + y)^2$ | $- z^2$ |

Answer = $(x +y)^2 – z^2$

# CHAPTER TEN: ALGEBRAIC DIVISION

Algebraic division operation is similar to arithmetic division operation and it is also performed by applying 'पञावर्त्य योजयेत्' sutra.

**Essential steps to perform algebraic division**

**1.** Write dividend (E) and divisor (D) in the descending order of power of variable. In case intermediate term or terms of any power is missing then that missing term is considered as zero (i.e. its numerical coefficient will be zero). If there are two variables in the expression then arrange dividend and divisor in descending order of power of one variable simultaneously ascending order of power of second variable considering zero as numerical coefficient for missing term or terms.

**e.g.** $3x^4 + 3x^2 – 2x + 3$

$x^3$ term is missing.

$\therefore$ $3x^4 + 0\,x^3 + 3x^2 – 2x + 3$

**e.g.** $a^4$, $a^3b$, $a^2b^2$, $ab^3$& $b^4$

Write their respective numerical coefficients below these terms in next row.

**2.** Calculate modified divisor (MD) as per procedure mentioned below

Select base.

**Deviation** = (divisor – base)

Change sign of deviation by applying 'पञावर्त्य योजयेत्' sutra.

**Modified divisor** (MD) = Deviation with sign change.

**3.** Bifurcate dividend coefficients as quotient part & remainder part by vertical demarcation line. No. of terms in remainder will be equal to power of selected base from RHS.

**e.g.Base = $x^2$, then remainder will have 2 terms from RHS.**

**Base = x, then remainder will have 1 term from RHS.**

**4.**Perform division process using Modified Divisor (MD) and dividend coefficients. Add column wise numbers pertaining to various terms (without carrying over)below each term to get coefficients of various terms of quotient and remainder.

**5.**Power of 1st quotient term is obtained by dividing 1st variable term of dividend with 1st variable term of divisor. Power of further quotient terms is obtained by continuously reducing the power of 1st quotient term by unity. Suffix these variables to respective coefficients of quotient part.

When expression contains '2' variables. For variables of quotient part decrease the power of 1st variable continuously and simultaneously increase the power of 2nd variable continuously. These quotient term variables are also obtained by dividing each dividend term by 1st variable term of divisor.

**6.** To obtain remainder suffix the variables of remainder part to respective remainder part coefficients.

**Note: -** If 1st term coefficient of divisor is not unity, then applying 'आनुरूप्येण'sutra divide all term or terms of divisor by its1st term coefficient to obtain unit coefficient for 1st term of divisor. Then proceed as above to obtain coefficients of quotient and remainder. Finally divide quotient part only by the same 1st term coefficient of divisor to obtain actual quotient. Remainder will remain unchanged.

**e.g. -** Find $(16a^4 - 28a^2b^2 + b^4) \div (4a^2 + 6ab + b^2)$

**Base**: - $a^2$

**Divisor**: - $(4a^2 + 6ab + b^2)$

**New divisor** : - $(4a^2 + 6ab + b^2) /4$

: $(a^2 + 3/2\ ab + 1/\ 4\ b^2)$

**Deviation**: - $(a^2 + 3/2\ ab + 1/\ 4\ b^2) - a^2$

: 3/2, 1/4

**Transpose**: - 3/2, - 1/4

**Modified divisor** (MD): - 3/2 - 1/4

**Actual quotient** = Modified quotient / 4

| $(4a^2 + 6ab + b^2) \div 4$ | $16a^4 + 0a^3b - 28a^2b^2$ | $+ 0ab^3 + b^4$ |
|---|---|---|
| MD -3/2 -1/4 | 16 0 -28<br>-24 - 4<br>36 | 0 1<br><br>6<br><br>-6 -1 |
| Modified quotient | 16 -24 4 | 0 0 |
| Actual quotient | 4 - 6 1 | 0 0 |

**Actual Quotient** = $(4\ a^2 - 6ab + b^2)$ **Remainder**= 0

$B(16a^4 - 28a^2b^2 + b^4) = (1+6-2-8+1)$

$= -2$

$= (-2 + 9)$

$= 7$

$B(4a^2 + 6ab + b^2) * B(4a^2 - 6ab + b^2) + 0 = (4+6+1)*(4 - 6 + 1) + 0$

$= 11 * -1$

$= (1+1)*(-1)$

$= -2$

$= (-2+9)$

$= 7$

As '**Beejank'** is same hence answer is verified.

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**Exercise**: -

1. $(8x^3 + 3x^2 + 2x + 4) \div (x - 2)$

2. $(7x^2 + 5x + 3) \div (x - 1)$

3. $(x^4 + 3x3 + 16x^2 + 8x + 4) \div (4x + 1)$

4. $(2x^4 - 3x^3 - 3x - 2) \div (x^2 + 1)$

5. $(x^4 - x^3 + x^2 + 3x + 5) \div (x^2 - x - 1)$

# CHAPTER ELEVEN: ALGEBRIC GCD OR HCF

**HCF: -** It is a factor of highest degree common to all given expressions. Thus HCF divides each one of given expressions exactly. Let 'U' and 'V' be two algebraic expressions. 'f' be a common factor to both. Then 'f' is a factor of (U ∩ V). Also if 'm' and 'n' be any multiples of 'U' & 'V' respectively then 'f' is a factor of (mU ∩ nV).

Consider **'U' = $2x^2 - 3x - 2$** and 'V' **= $3x^2 - 7x + 2$**

Their HCF = $(x - 2)$

Let m = 4 and n = 3 so
$mU = 8x^2 - 12x - 8$
$nV = 9x^2 - 1x + 6$
$mU + nV = 17x^2 - 33x - 2$ has (x-2) as one of its factors.
$mU - nV = -x^2 + 9x - 14$ has (x-2) as one of its factors.

Let 'U' be an algebraic expression & 'f' its factor. Then 'f' divides 'U' exactly. Also 'f' divides any multiple say 'm' of 'U' i.e. 'mU'.

Let $U = 2x^2 - 3x - 2$ 'f' = $(x - 2)$ is one of

its factors. Let m = 4

$mU = 8x^2 - 12x - 8$ of which (x-2) is a factor.

**GCD** – It is the greatest and common divisor of two or more numbers.

Greatest common divisor (GCD) and Highest common factor (HCF) has same meaning. Greatest common divisor is a number which divides given numbers with zero as remainder i.e. it divides numbers completely.

Procedure of finding GCD of algebraic expressions.

1. **Sutra: -**लोपनस्थापनाभ्याम्
2. **Process: -**संकलनव्यवकलनाभ्याम ्
3. **Rule: -**आद्यम आद्येन अन्त्यम् अन्त्येन

लोपनस्थापनाभ्याम् is used to eliminate the highest and lowest power from polynomials whose GCD is to be found. **'लोपनस्थापनाभ्याम्'** is performed by **'संकलनव्यवकलनाभ्याम्'**

sutra. While performing ‘संकलनव्यवकलनाभ्याम्’ आद्यम आद्येन अन्त्यमअन्त्येन sutra is applied.

**E.g**. Find HCF or GCD of $2x^2 – x – 3$ and $2x^2 + x – 6$.

| Elimination of highest power | Elimination of lowest power |
|---|---|
| $2\cancel{x^2} - x - 3$<br>$\mp \cancel{2}x^2 \mp x \mp 6$<br>$-2x + 3$<br>$-1\,(2x – 3)$ | $2x^2 + x – \cancel{6}$<br>$+ \quad -4x^2 + 2x + \cancel{6} \leftarrow -2 * (2x^2 – x – 3)$<br>$-2x^2 + 3x$<br>$-x\,(2x - 3)$ |

GCD or HCF = (2x -3) --------------- **(Answer)**

<u>**Least Common Multiple**</u>**: -** It is an algebraic expression which is exactly divisible by each of the given expression.

Three important properties of Least common multiple (LCM)

Two algebraic expressions ‘A’ and ‘B’ are given whose HCF is ‘H’ &

LCM ‘L’ then

1. ‘L’ contains ‘H’ as one of its factors
2. A * B = H * L ⟶ product of two expressions is equal to the product of their HCF & LCM
3. If three algebraic expressions A, B & C are given.

Let ‘X’ = LCM of A & B

‘Y’ = LCM of ‘X’ & ‘C’

Then ‘Y’ = LCM of A, B & C

**Find HCF & LCM of $x^4 + x^3 - 5x^2 - 3x + 2$ and $x^4 - 3x^3 + x^2 + 3x - 2$**

| Elimination of highest power | Elimination of lowest power |
|---|---|
| $\cancel{x^4} + x^3 - 5x^2 - 3x + 2$<br>- $\cancel{x^4} \mp 3x^3 \mp x^2 \mp 3x \mp 2$<br>$\cancel{4}x^3 - 6x^2 - 6x + 4$<br>- $\mp \cancel{4}x^3 \mp 4x^2 \mp 8x$<br>- $2x^2 + 2x + 4$<br>$-2(x^2 - x - 2)$ | $x^4 - 3x^3 + x^2 + \cancel{3}x - \cancel{2}$<br>+ $x^4 + x^3 - 5x^2 - \cancel{3}x + \cancel{2}$<br>$2x^4 - 2x^3 - 4x^2$<br>$2x^2(x^2 - x - 2)$<br>$x(2x^3 - 2x^2 - 4x) * 2$<br>$(4x^3 - 4x^2 - 8x)$ |

HCF = $(x^2 - x - 2)$ **--------------- (Answer)**

LCM by algebraic division (Paravartya Yojayet sutra)

**Base**: $x^2$

**Divisor**: $x^2 - x - 2$ (HCF)

**Deviation**: $(x^2 - x - 2) - x^2$
: -1 -2

**Transpose**: 1 2

**Modified Divisor**: 1 2

| $x^2$ | $x^4$ | $+x^3$ | $-5x^2$ | $-3x$ | $+2$ |
|---|---|---|---|---|---|
| MD 1 2 | 1 | 1 | -5 | - 3 | 2 |
| | | 1 | 2 | | |
| | | | 2 | 4 | |
| | | | | -1 | -2 |
| | 1 | 2 | -1 | 0 | 0 |

**Quotient** = $(x^2 + 2x - 1)$ **Remainder** = 0

**LCM** = $(x^2 + 2x - 1)$ $(x^4 - 3x^3 + x^2 + 3x - 2)$ **--------- (Answer)**

<u>**Exercise**</u>: **-** Find **HCF** and **LCM** of the following

1. $4x^4 + 11x^3 + 27x^2 + 17x + 5$ & $3x^4 + 7x^3 + 18x^2 + 7x + 5$
2. $2x^3 + x^2 - 9$ & $x^4 + 2x^2 + 9$

# CHAPTER TWELVE: FACTORISATION

**Factorisation** is just reverse of multiplication in the sense that algebraic expression of product of two or more factors is given then those factors are to be found i.e. multiplicand and multiplier (s).

Vedic sutra & sub-sutra used for factorisation are

(1) आनुरूप्येण
(2) आद्यम आद्येन अन्त्यम् अन्त्येन
(3) विलोकनम्
(4) लोपनस्थापनाभ्याम्
(5) ऊर्ध्वतिर्यग्भ्याम

The factors of a trinomial expression will have two binomial expressions. Vedic sutra आनुरूप्येण is used to split the coefficient of middle term in two parts such that ratio of coefficient of 1st term of expression with 1st part of middle term coefficient must be equal to ratio of 2nd part of middle term coefficient to the 3rd term coefficient of an expression as well as product of two parts of middle term coefficient must equal to 3rd term of expression.

**E.g.: -** $\mathbf{x^2 + 10x + 16}$

$$\underbrace{10}_{2\quad 8}$$

$$\frac{\text{Coefficient of 1st term}}{\text{1st part of middle term coefficient}} = \frac{\text{2nd part of middle term coefficient}}{\text{3rd term coefficient of expression}}$$

1: 2 **::** 8:16

2 * 8 = 16

1st factor by आनुरूप्येण sutra

Ratio = 1: 2

(x + 2)

2nd factor by आद्यम आद्येन अन्त्यम् अन्त्येन sutra

$$\left(\frac{x^2}{x}\right) + \left(\frac{16}{2}\right)$$

(x + 8)

$\therefore$ Factors of $x^2 + 10x = 16$ are (x+2) (x+8)

Answer is to be verified by **‘Beejank’**

**Factors of a polynomial expression of 2nd degree**

Here लोपनस्थापनाभ्याम~ sutra is applied.

First variables are eliminated one by one and finally variables are retained in factors. Most of the times, by eliminating any 2 of 3 variables serves the purpose of factorisation provided coefficient of that variable square is not unity. If it is unity, then eliminate that variable first to avoid confusion during factorisation of expression in two steps.

By **विलोकनम्** sutra their factors are directly linked considering common term in the brackets of two equations. Hence combining these common terms of the brackets results in the factors of given expression.

**E.g.: -** Find factors of $x^2 + 6y^2 - 2z^2 + 5xy - yz - zx$

Eliminate x by putting x = 0

$6y^2 - yz - 2z^2$

- 4 3

6 **:** - 4 **: :** 3 **:** - 2

1st factor by आनुरूप्येण sutra

Ratio = 3 **:** -2

(3y – 2z)

2nd factor by आद्यम आद्येन अन्त्यम् अन्त्येन sutra

$$\left(\frac{6y^2}{3y}\right) + \left(\frac{-2z^2}{-2z}\right)$$

(2y + z)

$\therefore$ Factors are $(3y - 2z)(2y + z)$ ---------------------- (1)

Eliminate y by putting $y = 0$

$x^2 - zx - 2z^2$

-2 1

1 : -2 : : 1 : -2

1st factor by आनुरूप्येण sutra

Ratio = 1:-2

$(x - 2z)$

2nd factor by आद्यम आद्येन अन्त्यम् अन्त्येन sutra

$$\left(\frac{x^2}{x}\right) + \left(\frac{-2z^2}{-2z}\right)$$

$(x + z)$

$\therefore$ Factors are $(x - 2z)(x + z)$ ---------------------- (2)
From (1) & (2)
Factors of expression are $(x + 3y - 2z)(x + 2y + z)$
Answer is to be verified by **'Beejank'**

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If no. of variables is still more than 2 variables at a time can be eliminated & trinomial is obtained for **Factorisation**.

**e.g.: -** $2p^2 - 2q^2 - 3r^2 - 3s^2 - 3pq + pr - 5ps - 7qr - 5qs$

Eliminate q & r by putting $q = 0$ & $r = 0$

$2p^2 - 5ps - 3s^2$

Factors are $(p - 3s)(2p + s)$ ----------------------------------- (1)

Eliminate p & r by putting $p = 0$ & $r = 0$

$-2q^2 - 5qs - 3s^2$

Factors are $(q + s)(-2q - 3s)$ ------------------------------------ (2)

Eliminate p & q by putting p = 0 & r = 0

$-3r^2 - 10rs - 3s^2$

Factors are $(-r - 3s)(3r + s)$ ------------------------------------------- (3)

From (1) (2) & (3)

Factors of the expression are $(p - 2q - r - 3s)(2p + q + 3r + s)$

Answer is to be verified by **'Beejank'**

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**<u>Exercise</u>: -**

1. $x^2 - 4xy - 45y^2$
2. $8x^2 + 26x + 21$
3. $12x^2 + x - 35$
4. $x^2y^2 + 15xy - 54$
5. $2x^2 - 12y^2 + 3z^2 + 5xy + 7xz - 5yz$
6. $4a^2 + 9b^2 - 12ab - 2a + 3b - 20$

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## CUBIC POLYNOMIAL

### Simple multiplication method (Urdhwatiryagbhyam i.e. vertically and cross wise)

Multiplication of 3 binomials of type (x+a) (x+b) (x+c) where a, b, c are numerical coefficients and 'x' is a variable. Cubic polynomials with unit coefficient of 1st term i.e. $x^3$ can be directly factorised.

$x^3 + (a+b+c)x^2 + (ab + ac + bc)x + abc = (x+a)(x+b)(x+c)$

Coefficient of $x^2$ is the sum of coefficients a, b & c whereas last term is their product i.e. (abc).

By विलोकनम् sutra three roots of last term are found such that their sum is the coefficient of $x^2$. Check can be done by comparing coefficients of x with the sum of product of these roots taken two at a time i.e. (ab + ac + bc). Write down directly the factors.

**E.g.**: - $x^3 + 6x^2 + 9x + 4$

$x^3 + (a+b+c)\,x^2 + (ab + ac + bc)\,x + abc$

Comparing coefficients of $x^2$ and constant term.

By विलोकनम् sutra

$a + b + c = 6$ $\quad a = 4$

$abc = 4$ $\quad b = 1$

$c = 1$

Factors of $x^3 + 6x^2 + 9x + 4 = (x+4)\,(x+1)\,(x+1)$

Answer is to be verified by 'Beejank'

**Rules: -**

<u>शून्यं साम्यसमुच्चये</u> (The Summation is equal to zero)

(1) A polynomial containing different powers of variable (say 'x'), if polynomial reduces to zero by putting $x = p$ (where 'p' is numerical value 1, 2, 3,---- $\longrightarrow$ then $(x-p)$ is a factor of that polynomial.

(2) A polynomial containing different powers of variable (say 'x'), if polynomial reduces to zero by putting $x = -p$ (where 'p' is numerical value 1, 2, 3,---- $\longrightarrow$ then $(x+p)$ is a factor of that polynomial.

Applying these rules if one of the factors is found then trinomial expression as quotient is obtained dividing the polynomial by this factor using परावर्त्य योजयेत् sutra and trinomial can be factorised using suitable Vedic sutra.

**E.g.: -**$x^3 + 9x^2 + 23x + 15$

Algebraic sum of coefficients = (-1) + 9 – 23 +15 = 0 (By putting $x = -1$ in the expression)

$\therefore$ $(x+1)$ is a factor of the polynomial.

Applying परावर्त्य योजयेत् sutra division is performed to obtain trinomial as a quotient.

Base ='x'

Divisor = (x+1)

Deviation = (x+1) – (x) = 1

Transpose = -1

Modified divisor (MD) = -1

| $x+1$ | $x^3$ | $+ 9x^2$ | $+ 23x$ | $+ 15$ |
|---|---|---|---|---|
| MD -1 | 1 | 9 | 23 | 15 |
| | | -1 | | |
| | | | -8 | |
| | | | | -15 |
| | 1 | 8 | 15 | 0 |

**Quotient** = $x^2 + 8x + 15$

Applying आनुरूप्येण & आद्यम आद्येन अन्त्यम् अन्त्येन sutra $(x^2 + 8x + 15)$ is factorised.

Factors are (x+3) (x+5)

$\therefore$ Factors of $x^3 + 9x^2 + 23x + 15 = (x+1)(x+3)(x+5)$

Answer is to be verified by comparing coefficients of 'x' with sum of product of the roots two taken at a time i.e. (ab+ac+bc).

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**e.g.: -** $x^3 - 6x^2 + 11x - 6$

Let (x + a) (x + b) (x + c) be the factors of the cubic expression

There is a following relation between coefficients of 3rd degree expression and its deviation.

Constant term (term without variable 'x') $\longrightarrow$ Product of three deviations (i.e. abc)

Coefficient of term '$x^2$' $\longrightarrow$ Summation of all three deviations (i.e. a + b + c)

Comparing coefficients of ‘$x^2$’ and constant term.

abc = -6
a + b + c = -6

By Vilokanam sutra
a = -1

b = -2
c = -3

∴ (x – 1)(x – 2)(x – 3) are the factors of given algebraic expression.

The answer is verified by comparing Coefficient of term ‘x’ ⟶ Summation of product of two deviations taken at a time out of three deviations (i.e. ab+ac+bc)

∴ (ab + ac + bc) = (-1*-2) + (-1 * -3) + (-2 * -3) = (2 + 3 + 6) = 11 which is equal to the coefficient of ‘x’ of given algebraic expression.

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**<u>Exercise</u>: -**

1. $x^3 - 39x + 70$
2. $x^3 - 17x^2 - x + 17$
3. $x^3 - 7x^2 + 11x - 5$
4. $x^3 + 9x^2 + 23x + 15$
5. $x^3 - 2x^2 - 5x + 6$

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## <u>ROOTS OF QUADRATIC EQUATION</u>

**$ax^2 + bx + c = 0$**

Eliminate ‘a’ from expression applying लोपनस्थापनाभ्याम् sutra.

Divide expression by ‘a’

$$\frac{\cancel{a}x^2}{\cancel{a}} + \frac{bx}{a} + \frac{c}{a} = 0$$

$x^2 + b/a\ x + c/a = 0$

Applying परावर्त्य योजयेत् sutra

∴ $x^2 + b/a*x = - c/a$

Applying पूरणापूरणाभ्याम् sutra

$x^2 + b/a*x + b^2/4a^2 = - c/a + b^2/4a^2$

$(x + b/2a)^2 = - c/a + b^2/4a^2$

$$(x + b/2a)^2 = \frac{- 4ac + b^2}{4a^2}$$

$$(x + b/2a)^2 = \frac{b^2 – 4ac}{4a^2}$$

Taking square-root of both sides

$$(x + b/2a) = \pm \frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$(x + b/2a) = \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ x = - b/2a \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ \frac{x}{1} = \frac{- b \pm \sqrt{b^2-4ac}}{2a}$$

Cross multiply

$$2ax = -b \pm \sqrt{b^2-4ac}$$

Transpose ‘b’ पशावर्त्य योजयेत् sutra

$$2ax + b = \pm \sqrt{b^2-4ac}$$

# VALUE OF ALGEBRAIC EXPRESSION

Applying आनुरूप्येण sutra

**Method: -** Draw a horizontal line. Write coefficients of expression in order above the line. From left most coefficient multiply by given value of ‘x’ and add to next order coefficient. Write sum below the line and repeat the procedure of multiplication & addition till extreme right coefficient is added. This final sum is the required value of algebraic expression.

e.g. Find value of $4x^5 + 3x^4 + 3x^3 – 2x^2 – 3x + 5$ when $x = 3$

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 x | 4 | 3 | 3 | -2 | -3 | 5 |
| | 12 | 15 | 48 | 142 | 423 | 1274 |

# CHAPTER THIRTEEN: PARTIAL FRACTION

पशावर्त्य योजयेत् Sutra is applied

A fraction of form f(x) ÷ g(x) is known as rational fraction. If in a rational fraction the degree of numerator is less than denominator then it is called as '**proper fraction**'. Resolving the proper fraction partially is known as '**partial fraction**'.

| Denominator of proper fraction | Type of proper fraction | Type of partial fraction |
|---|---|---|
| Non repeated factor | $\frac{f(x)}{(x-a)(x-b)}$ ----- | $\frac{A}{(x-a)} + \frac{B}{(x-b)} +$ ----- |
| Repeated factor | $\frac{f(x)}{(x-a)^n}$ | $\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{\text{--- fn}}{(x-a)^n}$ |

## NON REPEATED FACTOR

Let $\frac{f(x)}{g(x)}$ be proper fraction

Let g(x) = (x-a) * Φ (x) where (x-a) is a non-repeated factor.

$\therefore \frac{f(x)}{g(x)} = \frac{f(x)}{(x-a)*\Phi(x)} = \frac{A}{(x-a)}$ + ------- other partial fractions except (x-a)

Multiply both sides by (x-a)

$\therefore \frac{\cancel{(x-a)}*f(x)}{\cancel{(x-a)}*\Phi(x)}$ = A +(x-a) x other partial fractions

$\therefore \frac{f(x)}{g(x)}$ = A + (x-a) x other partial fractions

Put (x-a) = 0 then x = a by 'पशावर्त्य योजयेत्' sutra

$\therefore \frac{f(x)}{g(x)}$ = A + (a-a)

$\therefore \frac{f(x)}{g(x)}$ = A + 0

$\therefore \dfrac{f(x)}{g(x)} = A$

Value of constant 'A' as numerator for 1st partial fraction is obtained, similarly other constants 'B','C' ....... of further partial fraction are obtained in similar way.

Thus

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

Put x = a excluding (x-a) term

$$\frac{f(a)}{(a-b)(a-c)} = A$$

Similarly Put (x-b) = 0 then x = b excluding (x-b) term

$$\frac{f(b)}{(b-a)(b-c)} = B$$

Similarly Put (x-c) = 0 then x = c excluding (x-c) term

$$\frac{f(c)}{(c-a)(c-b)} = C$$

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

---

**E.g. Find partial fractions** of $\dfrac{x}{x^2 - 5x + 6}$

Factors of denominator are obtained by आनुरूप्येण and आद्यम आद्येन अन्त्यम् अन्त्येन sutras.

$x^2 - 2x - 3x + 6 = (x-2)(x-3)$

$$\frac{x}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Multiply both sides by (x-2) (x-3)

x = A (x-3) + B (x-2) -------------------------- (1)

Applying परावर्त्य योजयेत् Sutra ,

Put x=2 (as x-2 = 0) in (1)

2 = - A + 0

∴ A = -2 ------------------------ **(2)**

Applying परावर्त्य योजयेत् sutra,

Put x=3 (as x-3 = 0) in (1)

3 = 0 + B

B = 3 ------------------------ **(3)**

$$\therefore \quad \frac{x}{(x-2)(x-3)} = \frac{-2}{(x-2)} + \frac{3}{(x-3)} \quad \text{----- (Ans)}$$

**e.g. Find partial fractions of** $\dfrac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$

As power of numerator is greater than the power of denominator so division operation is performed applying परावर्त्य योजयेत् Sutra to get quotient and remainder in proper fraction form.

| $x^2 - 5x + 6$ | $x^3 - 6x^2$ | $+ 10x - 2$ | |
|---|---|---|---|
| MD 5 $\overline{6}$ | 1. -6<br>5 | 10 -2<br>-6<br>-5 6 | |
| Quotient → | 1 -1 | -1 + 4 | ← Remainder |

Base: - $x^2$
Divisor: - $x^2 - 5x + 6$

Deviation: - $(x^2 - 5x + 6) - x^2$

$= -5 \ 6$

Transpose $= 5 \ \overline{6}$

Modified divisor (MD) $= 5 \ \overline{6}$

$\therefore$ Quotient = (x-1) and Remainder = -x + 4

$$\therefore (x-1) + \frac{(-x+4)}{(x^2 - 5x + 6)}$$

Factors of denominator are obtained by आनुरूप्येण and आद्यम आद्येन अन्त्यम् अन्त्येन sutras.

$x^2 - 2x - 3x + 6 = (x-2)(x-3)$

$$\frac{(-x+4)}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$$

Applying परावर्त्य योजयेत् Sutra value of 'A' is obtained.

Put x=2 (as x-2 = 0) in LHS of expression excluding (x-2) term.

$$\therefore \frac{(-2+4)}{(2-3)} = A$$

$\therefore$ A = -2 ------------------------- (1)

Applying परावर्त्य योजयेत् Sutra value of 'B' is obtained.

Put x=3 (as x-3 = 0) in LHS of expression excluding (x-3) term.

$$\therefore \frac{(-3+4)}{(3-2)} = B$$

$\therefore$ B = 1 ------------------------- (2)

$$\frac{x^3 - 6x^2 + 10x - 2}{(x-2)(x-3)} = (x-1) - \frac{2}{(x-2)} + \frac{1}{(x-3)}$$ --------- (Ans)

## REPEATED FACTOR

Let proper fraction be $\dfrac{f(x)}{(x-a)^r}$

$$\therefore \frac{f(x)}{(x-a)^r} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \text{----} \frac{L}{(x-a)^{r-1}} + \frac{M}{(x-a)^r}$$

Applying परावर्त्य योजयेत् Sutra value of constants are obtained.

---

**e.g. Find partial fractions of** $\dfrac{3x+5}{(2x-1)^2}$

$$\frac{3x+5}{(2x-1)^2} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2}$$

Multiplying both sides by $(2x-1)^2$

$3x + 5 = A * (2x-1) + B$ ---------------------- **(1)**

Applying परावर्त्य योजयेत् Sutra

Put $x = 1/2$ in (1) (as $2x-1 = 0$)

$3 * 1/2 + 5 = A * (2*1/2 -1) + B$

$3/2 + 5 = A * (1-1) + B$

$13/2 = A * (0) + B$

$\therefore$ $B = 13/2$

Now put $x = -1/2$ and value of ‘B’ in (1)

$3 * -1/2 + 5 = A * (2* -1/2 -1) + 13/2$

$-3/2 + 5 = A*(-2) + 13/2$

$7/2 = -2A + 13/2$

$2A = 13/2 – 7/2$

$2A = 6/2$

$2A = 3$

$\therefore\ A = 3/2$

$$\therefore \frac{3x+5}{(2x-1)^2} = \frac{3}{2(2x-1)} + \frac{13}{2(2x-1)^2}$$ ---------------------- (Ans)

---

**e.g. Find partial fractions of** $\mathbf{\dfrac{x^2 + x - 1}{(x+1)^2 * (x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+2)}}$

Multiplying both sides by $(x+1)^2 * (x+2)$

$x^2 + x - 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$ -------------------------- **(1)**

Applying पशावर्त्य योजयेत् Sutra

Put $x = -1$ (as $x+1 = 0$) in (1)

$(-1)^2 + (-1) – 1 = A(-1+1)(-1+2) + B(-1+2) + C(-1+1)^2$

$1 – 1 – 1 = A(0) + B(1) + C(0)$

$-1 = B$

$\therefore\ B = -1$

Now put $x = -2$ (as $x+2 = 0$) in (1)

$(-2)^2 + (-2) – 1 = A(-2+1)(-2+2) + B(-2+2) + C(-2+1)^2$

$4 – 2 – 1 = A(0) + B(0) + C(-1)^2$

$1 = C$

$\therefore\ C = 1$

Now put $x = 1$ and value of constants 'B' as well as 'C' in (1)

$1 + 1 – 1 + A(1+1)(1+2) + B(1+2) + C(1+1)^2$

$1 = A(6) + (-1 * 3) + (1 * 4)$

$1 = 6A – 3 + 4$

$1 = 6A + 1$

$1 - 1 = 6A$

$0 = 6A$

$\therefore\ A = 0$

$$\therefore \frac{x^2 + x - 1}{(x+1)^2 * (x+2)} = \frac{0}{(x+1)} + \frac{-1}{(x+1)^2} + \frac{1}{(x+2)}$$

$$\therefore \frac{x^2 + x - 1}{(x+1)^2 * (x+2)} = \frac{1}{(x+2)} - \frac{1}{(x+1)^2}$$ --------------- (Ans)

---

**Exercise: -** Find partial fractions

(1) $\frac{x + 37}{x^2 + 4x - 21}$ (2) $\frac{x^2 - 10x + 13}{x^3 - 6x^2 + 11x - 6}$ (3) $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$

(4) $\frac{x^2 + x - 1}{(x+1)^2 (x+2)}$ (5) $\frac{5x - 11}{2x^2 + x - 6}$ (6) $\frac{3x + 5}{(2x - 1)^2}$

(7) $\frac{x}{(x+1)^2 (x+2)}$ (8) $\frac{3x + 5}{(x - 3)(x + 2)^2}$ (9) $\frac{x^3 + 7x^2 + 17x + 11}{x^2 + 5x + 6}$

(10) $\frac{x^2 + 9x + 19}{x^2 + 5x + 6}$

# CHAPTER FOURTEEN: DETERMINANT

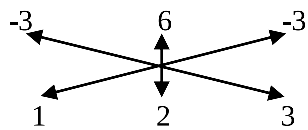
Determinant is an arrangement of elements in rows and columns which are in square form i.e. no. of rows and no. of columns are equal.

## SOME PROPERTIES OF DETERMINANT

1. If rows & columns are interchanged then value of determinant remains same. (परावर्त्य योजयेत्) sutra.

e.g.
$$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}\qquad \begin{vmatrix}4&5\\7&8\end{vmatrix}\quad \begin{vmatrix}4&6\\7&9\end{vmatrix}\quad \begin{vmatrix}5&6\\8&9\end{vmatrix}$$

(32-35) - (36-42) (45-48)




उर्ध्वतिर्यग्भ्याम् sutra

लोपनस्थापनाभ्याम्

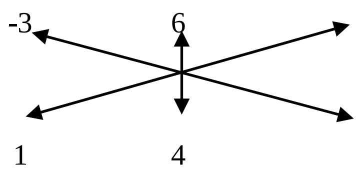
$D_1 = (-3 * 3) + (1 * -3) + (2*6)$

$= -9 - 3 + 12$

$D_1 = 0$

$$\begin{vmatrix}1&4&7\\2&5&8\\3&6&9\end{vmatrix}\qquad \begin{vmatrix}2&5\\3&6\end{vmatrix}\quad \begin{vmatrix}2&8\\3&9\end{vmatrix}\quad \begin{vmatrix}5&8\\6&9\end{vmatrix}$$

(12-15) - (18- 24) (45-48)




-3उर्ध्वतिर्यग्भ्याम् sutra

7 लोपनस्थापनाभ्याम्

$D_2 = (-3*7) + (1*-3) + (6*4)$

$= -21 -3 +24$

$D_2 = 0$

$D_1 = D_2$ (Value of determinant remains same)

2. If any two adjacent rows or columns are interchanged then value of determinant remains same with sign change. (परावर्त्य योजयेत्) sutra.

$$\begin{vmatrix}1 & 2 & 3\\ \hline 8 & 5 & 6\\ 7 & 8 & 10\end{vmatrix} \quad \begin{vmatrix}8 & 5\\ 7 & 8\end{vmatrix} \quad \begin{vmatrix}8 & 6\\ 7 & 10\end{vmatrix} \quad \begin{vmatrix}5 & 6\\ 8 & 10\end{vmatrix}$$

(64-35) - (80-42) (50-48)

29 -38 2 ऊर्ध्वतिर्यग्भ्याम्sutra

1 2 3 लोपनस्थापनाभ्याम्

$D_1 = (29*3) + (1*2) + (-38*2)$

$= (87 + 2 – 76)$

$D_1 = 13$

$$\begin{vmatrix}1 & 2 & 3\\ \hline 7 & 8 & 10\\ 8 & 5 & 6\end{vmatrix} \quad \begin{vmatrix}7 & 8\\ 8 & 5\end{vmatrix} \quad \begin{vmatrix}7 & 10\\ 8 & 6\end{vmatrix} \quad \begin{vmatrix}8 & 10\\ 5 & 6\end{vmatrix}$$

(35-64) - (42-80) (48-50)

-29 38 -2 ऊर्ध्वतिर्यग्भ्याम् sutra

1 2 3 लोपनस्थापनाभ्याम्

$D_2 = (-29*3) + (1*-2) + (38*2)$

$= (-87 -2 + 76)$

$D_2 = - 13$ (Value of determinant remains same but sign changes)

3. If two rows or columns of determinant are identical then value of determinant is zero.आनुरूप्ये शून्यमन्यत sutra.

e.g. $\begin{vmatrix} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix}$ $\quad \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \quad \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} \quad \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix}$

(8-5) - (12-6) (15 – 12)

ऊर्ध्वतिर्यग्भ्याम् sutra 3 -6 3

लोपनस्थापनाभ्याम् 1 2 3

D = (3*3) + (1*3) + (-6*2)

= 9 + 3 – 12

D = 0 (Value of determinant is zero)

---

If every element of a row or a column is multiplied by a constants ‘K’ then value determinant itself is multiplied by constant ‘K’. Hence constant ‘K’ is taken as common outside the determinant. आनुरूप्येण sutra.

**e.g.** $\begin{vmatrix} 1 & 2 & 3 \\ \hline 5 & 7 & 8 \\ 6 & 9 & 10 \end{vmatrix}$ $\quad \begin{vmatrix} 5 & 7 \\ 6 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} \quad \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix}$

(45-42) - (50 - 48) (70 - 72)

ऊर्ध्वतिर्यग्भ्याम् sutra 3 - 2 - 2

लोपनस्थापनाभ्याम् 1 2 3

$D_1$ = (3*3) + (1 * – 2) + (2* - 2)

= (9 - 2 – 4)

$D_1$= 3

$$\begin{array}{c} R_1 * K \\ \downarrow \\ 2 \\ \\ \end{array} \begin{vmatrix} 2 & 4 & 6 \\ \hline 5 & 7 & 8 \\ 6 & 9 & 10 \end{vmatrix} \qquad \begin{vmatrix} 5 & 7 \\ 6 & 9 \end{vmatrix} \quad \begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix} \quad \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix}$$

$$(45 - 42) \qquad - (50 - 48) \qquad (70 - 72)$$

उर्ध्वतिर्यग्भ्याम् sutra

$$\begin{array}{ccc} 3 & -2 & -2 \\ 2 & 4 & 6 \end{array}$$

लोपनस्थापनाभ्याम्

$D_2 = (3 * 6) + (2 * - 2) + (4 * - 2)$

$= (18 - 4 - 8)$

$D_2 = 6$

$\therefore\ D_2 = K * D_1$

$= 2 * 3$

$D_2 = 6$ (Value of determinant itself is multiplied by constant 'K')

---

4. If one row or column is a multiple of another row or column then value of determinant is zero. . आनुरूप्येण sutra and आनुरूप्ये शून्यमन्यत sutra.

e.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ \hline 2 & 4 & 6 \\ 5 & 7 & 8 \end{vmatrix} \qquad \begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix} \quad \begin{vmatrix} 2 & 6 \\ 5 & 8 \end{vmatrix} \quad \begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix}$$

$$(14 - 20) \qquad - (16 - 30) \qquad (32 - 42)$$

उर्ध्वतिर्यग्भ्याम् sutra

$$\begin{array}{ccc} -6 & 14 & -10 \\ 1 & 2 & 3 \end{array}$$

लोपनस्थापनाभ्याम्

$D = (- 6 * 3) + (1 * - 10) + (14 * 2)$

$= (- 18 - 10 + 28)$

$D = 0$ (Value of determinant is zero)

5. If each element of one row or column of determinant is sum of two terms then value of determinant is expressed as a sum of two terms. संकलनव्यवकलनाभ्याम् sutra.

6. If the elements of any row or column same multiple of corresponding elements of another row or column are added then value of determinant remains same. लोपनस्थापनाभ्याम्

$$\begin{vmatrix} 1 & 2 & 3 \\ \hline 2 & 4 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} \quad \begin{vmatrix} 2 & 6 \\ 7 & 9 \end{vmatrix} \quad \begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix}$$

$$(16 - 28) \qquad -(18 - 42) \qquad (36 - 48)$$

उर्ध्वतिर्यग्भ्याम् sutra $\quad -12 \qquad 24 \qquad -12$

लोपनस्थापनाभ्याम् $\quad 1 \qquad 2 \qquad 3$

$D_1 = (-12 * 3) + (1 * -12) + (24 * 2)$

$= (-36 - 12 + 48)$

$D_1 = 0$

$$\begin{vmatrix} 3 & 6 & 9 \\ \hline 2 & 4 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} \quad \begin{vmatrix} 2 & 6 \\ 7 & 9 \end{vmatrix} \quad \begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix}$$

$$(16 - 28) \qquad -(18 - 42) \qquad (36 - 48)$$

उर्ध्वतिर्यग्भ्याम् sutra $\quad -12 \qquad 24 \qquad -12$

लोपनस्थापनाभ्याम् $\quad 3 \qquad 6 \qquad 9$

$D_2 = (-12 * 9) + (3 * -12) + (24 * 6)$

$= (-108 - 36 + 144)$

$D_2 = 0$

$D_1 = D_2$ (Value of determinant remains same)

7. If ‘A’ & ‘B’ are square matrices of same order $\Delta$ ‘AB’ = $\Delta$ ‘A’ * $\Delta$ ’B’.

गुणित समुच्चय:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad B = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

$D_A = 1(45 – 48) – 2 (36 – 42) + 3 (32 – 35)$

$= (- 3 + 12 - 9)$

$D_A = 0$

$D_B = 1 (45 – 48) – 4 (18 – 24) + 7 (12 – 15)$

$= (- 3 + 24 – 21)$

$D_B = 0$

$$AB = \begin{vmatrix} 1*1, & 2*4, & 3*7 \\ 1*2, & 2*5, & 3*8 \\ 1*3, & 2*6, & 3*9 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 21 \\ 2 & 10 & 24 \\ 3 & 12 & 27 \end{vmatrix}$$

$D_{AB} = \quad 1 (270 – 288) – 8 (54 – 72) + 21 (24 – 30)$

$= - 18 + 144 – 126$

$D_{AB} = 0$

$\therefore\ D_{AB} = D_A * D_B$

(Value of determinant AB = Value of determinant A * Value of determinant B

8. If elements of rows or columns are to be added or subtracted as well as multiplied or divided by a constant then value of determinant remains same. . लोपनस्थापनाभ्याम् sutra.

When constant '1' is added to all row elements.

$$D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \qquad D_2 = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{vmatrix}$$

$D_1 = 1\,(45 - 48) - 2\,(36 - 42) + 3\,(32 - 35)$

$= (-3 + 12 - 9)$

$D_1 = 0$

$D_2 = 2\,(60 - 63) - 3\,(50 - 56) + 4\,(45 - 48)$

$= (-6 + 18 - 12)$

$D_2 = 0$

$\therefore\ D_1 = D_2$ (Value of determinant remains same)

---

9. If elements of any row or column are zero then value of determinant is zero.

e.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 0 & 0 & 0 \end{vmatrix}$$

$D = 1\,(0*6 - 0*7) - 2\,(5*0 - 0*7) + 3\,(5*0 - 6*0)$

$D = 0$ (Value of determinant is zero)

## PROCEDURE TO CALCULATE DETERMINANT

**Step – I** लोपनस्थापनाभ्याम् sutra

Eliminate 1st or 3rd row (1st or 3rd column) from other two rows or columns by dotted line.

$$\begin{vmatrix} \underline{A_1} & \underline{\mathbf{B_1}} & \underline{C_1} \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} A_1 & | & B_1C_1 \\ A_2 & | & B_2C_2 \\ A_3 & | & B_3C_3 \end{vmatrix}$$

**Step – II** उर्ध्वतिर्यग्भ्याम् sutra

From remaining two rows or columns 3 determinants of 2x2 order are formed
Row Elimination

$$\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}, \quad \begin{vmatrix} A_2 & C_2 \\ A_3 & C_3 \end{vmatrix}, \quad \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$$

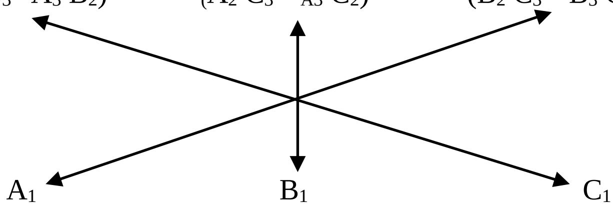
Evaluate these 2x2 order determinant and write the values as a new row or as a new column as the case may be giving ‘+’,’-‘, ‘+’ signs respectively.

$$+ (A_2 B_3 – A_3 B_2) \qquad – (A_2 C_3 – A_3 C_2) \qquad +(B_2 C_3 – B_3 C_2)$$

**Step – III** लोपनस्थापनाभ्याम् sutra

Elements of eliminated row or column are established below new row or RHS of new column as multiplier. Applying उर्ध्वतिर्यग्भ्याम् sutra multiplication is performed to get value of 3x3 order determinant.

$D \;=\; (A_2 B_3 – A_3 B_2) \qquad – (A_2 C_3 – A_3 C_2) \qquad (B_2 C_3 – B_3 C_2)$




$\therefore\; D = C1(A_2 B_3 - A_3 B_2) + A_1 (B_2 C_3 - B_3 C_2) – B_1 (A_2 C_3 - A_3 C_2)$

**Exercise:-**Evaluate the determinant

$$\begin{vmatrix} 5 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 3 & 3 \end{vmatrix} \qquad \begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

Solve determinant equation

$$\begin{vmatrix} 1 & -3 & x \\ 4 & -1 & x \\ 3 & 5 & x \end{vmatrix} = 40 \qquad \begin{vmatrix} x+1 & 1 & 5 \\ 1 & 1 & 5 \\ 1 & 5 & x+4 \end{vmatrix} = -20$$

# CHAPTER FIFTEEN: MATRICES

**Def**: - A system of 'mn' numbers arranged in a rectangular formation along 'm' rows and 'n' columns as well as bounded by brackets '[ ]' is called 'm' by 'n' matrix; which is also written as 'm x n' matrix. A matrix is also defined by a single capital letter.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \text{----------} a_{1n} \\ a_{21} & a_{22} & a_{23} \text{----------} a_{2n} \\ a_{31} & a_{32} & a_{33} \text{----------} a_{3n} \\ \vdots & & \\ a_{m1} & a_{m2} & a_{m3} \text{----------} a_{mn} \end{pmatrix}$$

It is a matrix of order 'mn'. **"m" rows and "n" columns.**

Each of 'mn' numbers is called an element of the matrix. To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes which specify row and column respectively.

e.g. $a_{ij}$ is a element of $i^{th}$ row and $j^{th}$ column of matrix 'A'.

Matrix 'A' is denoted by $[a_{ij}]$

A matrix should be treated as a single entity with a number of components rather than a collection of numbers.

e.g. The coordinates of a point in solid geometry is given by a set of three numbers which can be represented by the matrix [x, y. z]. A matrix can not be reduced to a single number like the determinant and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is the fact that an interchange of rows and columns does not alter the determinant but produces an entirely new matrix.

[1, 2, 3] is defined as Row matrix.

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is defined as Column matrix.

**Square Matrix**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Elements **1, 3, 5** of matrix 'B' is called **leading or principal diagonal**. The sum of diagonal elements of square matrix 'B' is called **the trace of matrix 'B'**. If determinant is zero then square matrix is called singular otherwise non singular.

**Diagonal Matrix**

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Elements of square matrix are zero except elements on leading diagonal is called 'Diagonal Matrix'.

**Scalar Matrix**

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

A square matrix whose all elements on leading diagonal are equal and other elements are zero is called 'Scalar Matrix'.

**Unit Matrix**

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A square matrix whose all elements on leading diagonal are unity and other elements are zero is called 'Unit Matrix'.

**Null Matrix**

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix whose all elements are zero is called ‘Null Matrix’.

**Symmetric Matrix**

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

In a square matrix when $a_{ij} = a_{ji}$ for all i and j then it is called ‘Symmetric Matrix’.

**Skew Matrix**

$$\begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$$

In a square matrix when $a_{ij} = -a_{ji}$ for all i and j so that all elements on leading diagonal are zero then it is called ‘Skew Matrix’.

**Triangular Matrix**

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

In a square matrix when all elements below leading diagonal are zero then it is called ‘Upper Triangular Matrix’.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

In a square matrix when all elements above leading diagonal are zero then it is called 'Lower Triangular Matrix'.

**Equality of Matrices**

Two matrices 'A' and 'B' are said to be equal if only if they are of same order and each element of 'A' is equal to the corresponding elements of 'B'.

**Addition of Matrices**

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

$$= (A + B)$$

Subtraction of Matrices

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} - B = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \\ a_3 - c_3 & b_3 - d_3 \end{bmatrix}$$

$$= (A - B)$$

Matrices of same order are added or subtracted.

Addition is commutative

$$A + B = B + A$$

Addition and Subtraction of matrices is associative.

$(A + B) - C = A + (B - C) = B + (A - C)$

**Multiplication of matrix by a Scalar**

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

K is a Scalar quantity.

$$KA = K\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \end{bmatrix}$$

Distributive law holds for such products

K (A + B) = KA + KB

e.g. Evaluate 3A – 4B

$$A = \begin{bmatrix} 3 & -4 & 6 \\ 5 & 1 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$3A = \begin{bmatrix} 9 & -12 & 18 \\ 15 & 3 & 21 \end{bmatrix} \qquad 4B = \begin{bmatrix} 4 & 0 & 4 \\ 8 & 0 & 12 \end{bmatrix}$$

$$3A - 4B = \begin{bmatrix} 9 - 4 & -12-0 & 18-4 \\ 15- 8 & 3-0 & 21-12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -12 & 14 \\ 7 & 3 & 9 \end{bmatrix}$$

**Multiplication of matrices**

When the number of columns in one matrix is equal to the number of rows in the second matrix then multiplication of both is feasible. Such matrices are said to be conformable.

$$A = \begin{bmatrix} a_{11} & a_{12}\text{---------} & a_{1n} \\ a_{21} & a_{22}\text{---------} & a_{2n} \\ a_{m1} & a_{m2}\text{---------} & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12}\text{---------} & b_{1n} \\ b_{21} & b_{22}\text{---------} & b_{2n} \\ b_{m1} & b_{m2}\text{---------} & b_{mn} \end{bmatrix}$$

‘A’ and ‘B’ be two (m*n) and (n*p) conformable matrices then their product is defined as (m*p) matrix.

$$AB = \begin{bmatrix} c_{11} & c_{12}\text{---------} & c_{1n} \\ c_{21} & c_{22}\text{---------} & c_{2n} \\ c_{m1} & c_{m2}\text{---------} & c_{mn} \end{bmatrix}$$

In product ‘AB’ matrix ‘A’ is said to be post multiplied by matrix ‘B’.

In product ‘BA’ matrix ‘A’ is said to be pre multiplied by matrix ‘B’

In one case product may exist and in the other case it may not. Also the product in both cases may exist yet may or may not be equal.

IA = AI = A

**Observation - 1: -**If AB = 0, it does not necessarily imply that either ‘A’ or ‘B’ is a null matrix.

e.g. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ although neither ‘A’ nor ‘B’ is null matrix.

**Observation - 2: -**Multiplication of matrices is associative

AB (C) = A (BC). Provided ‘A and ‘B’ are conformable for the product ‘AB’ as well as “B’ and ‘C’ are conformable for product ‘BC’.

**Observation - 3: -**Multiplication of matrices is distributive.

A (B + C) = AB + AC

‘A’ and ‘B’ are conformable for product ‘AB’ as well as ‘A’ and ‘C’ are conformable for product ‘AC’.

**Observation - 4: -**Power of Matrix

‘A’ is a square matrix.

Product $A * A = A^2$

$A * A^2 = A^3$

$A^2 * A^2 = A^4$

e.g.

$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$

No. of columns of matrix 'A' = No. of rows of matrix of 'B'= '3'.

Thus product of 'AB' is defined.

$$AB = \begin{bmatrix} 0*1 + 1*-1 + 2.2 & 0*-2 + 1*0 + 2*-1 \\ 1*1 + 2*-1 + 3*2 & 1*-2 + 2*0 + 3*-1 \\ 2*1 + 3*-1 + 4*2 & 2*-2 + 3*0 + 4*-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

No. of columns of matrix 'B $\neq$ No. of rows of matrix 'A'.

Therefore product of 'BA' is not possible.

---

Exercise: - (1) If $A=\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ $B=\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ $C=\begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$

Verify that (AB) C = A (BC) and A (B + C) = AB + AC

(2) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Find AB or BA whichever exists.

(3) If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A – B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$

Calculate the product AB.

## CHAPTER SIXTEEN: SIMULTANEOUS EQUATION WITH 3 VARIABLES

लोपनस्थापनाभ्याम, आनुरूप्येण, संकलनव्यवकलनाभ्याम,परावर्त्य योजयेत sutra applied.

**1st Type: –** A significant figure on RHS in only one equation and zero in other two equations.

**1st Method: -** From the homogeneous equation having zero, new equation is derived defining two of the unknowns in the terms third unknown. Then substituting these values in the third equation having significant value on its RHS and thus value of all three unknowns are found out.

**2nd Method: -** By addition and subtraction of proportionate multiples to bring the elimination of one of the unknowns and retention of other two unknowns.

**E.g.** $x + y - z = 0$ -------- (A)

$4x - 5y + 2z = 0$ -------- (B)

$3x + 2y + z = 10$ -------- (C)

**1st Method: -**

From **'A'**

$x = (z - y)$ ---------- (a)

$y = (z - x)$ ---------- (b)

From **'B'**

$4x - 5y = -2z$ -------- (c)

Put value of 'x' from (a) in (c)

$\therefore\ 4(z - y) - 5y = -2z$

$4z - 4y - 5y = -2z$

$-9y = -6z$

$\therefore\ y = (2/3)z$ ---------- (d)

Put value of 'y' from (b) in (c)

$\therefore\ 4x - 5(z - x) = -2z$

$4x - 5z + 5x = -2z$

$9x = 5z - 2z$

$9x = 3z$

$\therefore\ x = (3/9)\ z = (1/3)\ z$ --------- (e)

Substituting value of ‘x’ and ‘y’ from‘d’& ‘e’ in ‘C’

$\cancel{3} * (1/\cancel{3})\ z + 2 * (2/3)\ z + z = 10$

$z + (4/3)\ z + z = 10$

$$\frac{(3 + 4 + 3)\ z}{3} = 10$$

$\therefore\ 10\ z = 30$

$\therefore\ z = (30/10) = 3$

Put the value of ‘z’ in‘d’

$\therefore\ y = (2/\cancel{3}) * \cancel{3}$

$\therefore\ y = 2$

Put the value of ‘z’ in‘e’

$x = (1/3)\ z$

$x = (1/\cancel{3}) * \cancel{3}$

$\therefore\ x = 1$

$\therefore\ (x = 1, y = 2, z = 3)$ -------------- (**Answer**)

**2$^{nd}$ Method: -**

Adding (A) and (C)

$4x + 3y = 10$ ---------- (1)

Adding (2A) and (B)

$6x - 3y = 0$ ---------- (2)

Adding (1) and (2)

$10x = 10$

$\therefore x = (10/10) = 1$

Put value of 'x' either in (1) or (2)

$\therefore 4*1 + 3y = 10$

$\therefore 3y = 10 - 4 = 6$

$\therefore y = (6/3) = 2$

Put value of 'x' and 'y' in either (A) or (B) or (C)

$\therefore 1 + 2 - z = 0$

$\therefore z = 3 \therefore (x = 1, y = 2, z = 3)$ -------------- (**Answer**)

**2nd Type: -**

This is one where **RHS** of all three equations contain significant figure.

**E.g.** $2x - 4y + 9z = 28$ -------- (A)

$7x + 3y - 5z = 3$ ---------- (B)

$9x + 10y - 11z = 4$ ---------- (C)

C – (A + B)

$11y - 15z = -27$ --------------- (1)

9B – 7C

$-43y + 32z = -1$ -------------- (2)

From (1)

$$y = \frac{(-27 + 15z)}{11}$$

Putting this value of 'y' in (2)

$$-43\frac{(-27 + 15z)}{11} + 32z = -1$$

$\therefore 1161 - 645z + 352z = -11$

293 z = 1172

∴ z = (1172/293) = 4

Put value of ‘z’ in (1)

∴ 11y – 15*4 = -27

∴ 11y = -27 +60

∴ 11y = 33

∴ y = (33/11) = 3

Put value of ‘y’ and ‘z’ in (A)

2x – 4*3 + 9*4 = 28

∴ 2x = 28 -36 + 12

∴ 2x = 4

∴ x = (4/2) = 2

∴ (x = 2, y = 3, z = 4) -------------- (**Answer**)

---

**E.g.** x + 2y + 3z = 12

2x + 3y + 4z = 18

4x + 3y + 5z = 24

**By determinant method**

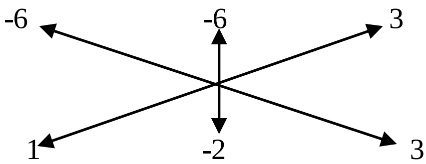
First determinant of the coefficients of the variables is found

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 3 & 5 \end{vmatrix}$$

Using उर्ध्वतिर्यग्भ्याम and लोपनस्थापनाभ्याम sutra

$$\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} \quad \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} \quad \begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix}$$

(6 – 12) (10 – 16) (15 – 12)



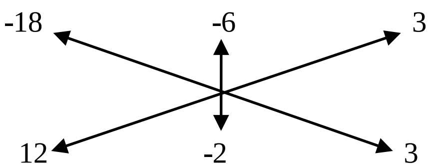

-18 + 3 +12

∴ D = -3

$$x = \begin{vmatrix} 12 & 2 & 3 \\ 18 & 3 & 4 \\ 24 & 3 & 5 \end{vmatrix} \div D$$

$$\begin{vmatrix} 18 & 3 \\ 24 & 3 \end{vmatrix} \quad \begin{vmatrix} 18 & 4 \\ 24 & 5 \end{vmatrix} \quad \begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix}$$

(54 – 72) (90 – 96) (15 – 12)



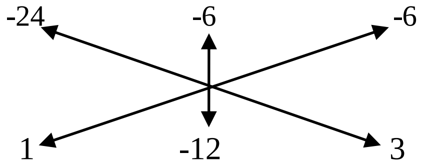

-54 +36 +12 = -6

∴ x = (-6) ÷ (-3) = 2

$$y = \begin{vmatrix} 1 & 12 & 3 \\ 2 & 18 & 4 \\ 4 & 24 & 5 \end{vmatrix} \div D$$

$$\begin{vmatrix} 2 & 18 \\ 4 & 24 \end{vmatrix} \qquad \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} \qquad \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix}$$

$(48 – 72)$ $\qquad$ $(10 – 16)$ $\qquad$ $(10 – 16)$



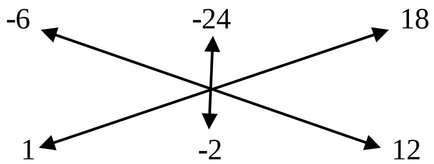

$-72 -6 +72 = -6$

$\therefore$ $y = (-6) \div (-3) = 2$

$$z = \begin{vmatrix} 1 & 2 & 12 \\ 2 & 3 & 18 \\ 4 & 3 & 24 \end{vmatrix} \div D$$

$$\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} \qquad \begin{vmatrix} 2 & 18 \\ 4 & 24 \end{vmatrix} \qquad \begin{vmatrix} 3 & 18 \\ 3 & 24 \end{vmatrix}$$

$(6 – 12)$ $\qquad$ $(48 – 72)$ $\qquad$ $(72 – 54)$




$-72 +18 +48 = -6$

$\therefore$ $z = (-6) \div (-3) = 2$

$\therefore$ $(x = 2, y = 2, z = 2)$------------- (**Answer**)

## CHAPTER SEVENTEEN: MERU – PRASTAR

**Meru** ⟶ means mountain **Prastar** ⟶ means expansion

Pingalacharya, an ancient Bharatiya mathematician invented Meru-Prastar concept which was used in the construction of meters (in prosody). Its shape and expansion is like a mountain. It is also known as 'Pascal's Triangle'. In fact many hundreds of years prior to 'Pascal', the method was known and used in India as "Meru-Prastar". It is a method of finding any power of any number without multiplying by itself. It is useful in finding coefficients of 'n' power of $(1+X)^n$, $(X+1)^n$ and $(X+Y)^n$. It helps in finding the value of different powers of integer numbers.

### DEVELOPMENT OF MERU-PRASTAR

First place '1' in the middle at 1st row and consider zeroes on its both sides. Now add adjacent integers to get '1' & '1' which is placed in the 2nd row. Now consider zeroes on both ends and add adjacent integers as well as place the addition in 3rd row as shown in figure below. Continuing the process of considering zeroes on both ends and adding an adjacent integer till last coefficient ‘1’ is obtained. Then mountain like structure obtained hence it is known as "**Meru-Prastar**". It is also known as Halayudh’s “**Pyramidal Structure**". It is also known as Bhaskaracharya’s “**Khand-Meru**".

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | **1** | | | | | | | | | | | **$(a+b)^0$** |
| | | | | | | | | | **1** | | **1** | | | | | | | | | | **$(a+b)^1$** |
| | | | | | | | | **1** | | **2** | | **1** | | | | | | | | | **$(a+b)^2$** |
| | | | | | | | **1** | | **3** | | **3** | | **1** | | | | | | | | **$(a+b)^3$** |
| | | | | | | **1** | | **4** | | **6** | | **4** | | **1** | | | | | | | **$(a+b)^4$** |
| | | | | | **1** | | **5** | | **10** | | **10** | | **5** | | **1** | | | | | | **$(a+b)^5$** |
| | | | | **1** | | **6** | | **15** | | **20** | | **15** | | **6** | | **1** | | | | | **$(a+b)^6$** |
| | | | **1** | | **7** | | **21** | | **35** | | **35** | | **21** | | **7** | | **1** | | | | **$(a+b)^7$** |
| | | **1** | | **8** | | **28** | | **56** | | **70** | | **56** | | **28** | | **8** | | **1** | | | **$(a+b)^8$** |
| | **1** | | **9** | | **36** | | **84** | | **126** | | **126** | | **84** | | **36** | | **9** | | **1** | | **$(a+b)^9$** |
| **1** | | **10** | | **45** | | **120** | | **210** | | **252** | | **210** | | **120** | | **45** | | **10** | | **1** | **$(a+b)^{10}$** |

### PROCEDURE TO FIND COEFFICIENT OF ANY ROW

In the first row write the integers from 'n' to '1' in descending order from column '3' and onwards. In 2nd row below these integers write the integers from '1'to 'n' in ascending order from column '3' and onwards. Place '1' as 1st coefficient in 4th row but in the previous column i.e. column '2'. Multiply this 1st coefficient by the integer from 1st row in 3rd column i.e. $R_1C_3$ as well as divide by the integer from2nd row in 3rd column i.e.

$R_2C_3$ to get 2nd coefficient. Now multiply this coefficient i.e. 2nd coefficient by the integer from 1st row in4th column i.e. $R_1C_4$ as well as divide by the integer from 2nd row in 4th column i.e. $R_2C_4$ to get 3rd coefficient. Continue this process till last coefficient '1' is obtained.

**e.g. Find the coefficients of row related to n = 5**

| | Column 1 | Column 2 | Column 3 | Column 4 | Column 5 | Column 6 | Column 7 |
|---|---|---|---|---|---|---|---|
| **Row 1** | × | | **5** | **4** | **3** | **2** | **1** |
| **Row 2** | ÷ | | **1** | **2** | **3** | **4** | **5** |
| **Row 3** | **Calculation** | | **$(1 \times 5) \div 1 = 5$** | **$(5\times4)\div 2 = 10$** | **$(10\times3)\div 3=10$** | **$(10\times 2)\div 4 = 5$** | **$(5\times 1)\div 5 = 1$** |
| **Row 4** | **Coefficients** | **1** | **5** | **10** | **10** | **5** | **1** |

In the expansion of $(a+b)^n$ the power of 1st letter component 'a' goes on decreasing by '1' till lowest power '$a^0$' and that of 'b' goes on increasing by '1' till the highest power '$b^n$' is obtained simultaneously. Then $(a+b)^n$ will have totally (n+1) terms.

The letter component pattern of $(a+b)^n$ expansion will be $a^n$, $a^{n-1} b$, $a^{n-2} b^2$, $a^{n-3} b^3$, ---------- $a^0b^n$ as total (n+1) terms. For evaluating expansion terms of binomial expression of type $(a+b)^n$ numerical coefficients are to be prefixed to its letter components then add all these terms to obtain its expansion terms. These numerical coefficients are found by "Halayudh's Pyramidal Structure" or Meru-Prastar.

Let letter 'c' be selected for these numerical coefficients so that $c_1$ be 1st numerical coefficient for $a^n$, $c_2$ for $a^{n-1} b$, $c_3$ for $a^{n-2} b^2$ and so on. Thus general equation for $(a+b)^n$ becomes

$(a+b)^n = c_1 a^n + c_2 a^{n-1} b + c_3 a^{n-2} b^2 +$ ----------------- $c_n a b^{n-1} + c_{n+1} b^n$.

| **Binomial Expression** | **Numerical coefficients in the expansion** | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $c_8$ | $c_9$ | $c_{10}$ | $c_{11}$ |
| $(a+b)^0$ | 1 | | | | | | | | | | |
| $(a+b)^1$ | 1 | 1 | | | | | | | | | |
| $(a+b)^2$ | 1 | 2 | 1 | | | | | | | | |
| $(a+b)^3$ | 1 | 3 | 3 | 1 | | | | | | | |
| $(a+b)^4$ | 1 | 4 | 6 | 4 | 1 | | | | | | |
| $(a+b)^5$ | 1 | 5 | 10 | 10 | 5 | 1 | | | | | |
| $(a+b)^6$ | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | |
| $(a+b)^7$ | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | |
| $(a+b)^8$ | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | |
| $(a+b)^9$ | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | |
| $(a+b)^{10}$ | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 |

**e.g. $(a+b)^5$**

Algebraic components are $a^5$, $a^4b$, $a^3b^2$, $a^2b^3$, $a^1b^4$, $a^0b^5$ and their numerical coefficients in sequence are 1, 5, 10, 10, 5, 1are taken from Meru-Prastar.

| | | | | | | |
|---|---|---|---|---|---|---|
| Numerical coefficients | 1 | 5 | 10 | 10 | 5 | 1 |
| × $b^n$ | $b^0$ | $b^1$ | $b^2$ | $b^3$ | $b^4$ | $b^5$ |
| × $a^n$ | $a^5$ | $a^4$ | $a^3$ | $a^2$ | $a^1$ | $a^0$ |
| Product | $1*1*a^5$ | $5*b*a^4$ | $10*b^2*a^3$ | $10*b^3*a^2$ | $5*b^4*a$ | $1*b^5*1$ |
| | $a^5$ | $5a^4b$ | $10a^3b^2$ | $10a^2b^3$ | $5ab^4$ | $b^5$ |

$\therefore\ (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

## PROCEDURE TO FIND OUT POWERS OF ANY NUMBER NEAR THE BASE OR SUB-BASE

**A. Numbers near the base i.e.10, $10^2$, $10^3$............ $10^n$**

Applying Vedic sutra 'यावदूनम'

1) The answer part is bifurcated in 'n' parts where 'n' denotes required power of a number.
2) The far left part of answer is obtained by formula: [X + (c2 - c1) d]

   Where

   'X' ⟶ Numerical number near the base.

   'd' ⟶ Deviation of number 'X' from the base.

   $c_1$, $c_2$ ⟶ Coefficients of 1st& 2nd terms of expansion respectively.

**If number 'X' is smaller than base then deviation'd' is '-ve' and if it is greater than base then deviation'd' is '+ve'.**

**e.g.** X = 98 & Base =100

Then deviation'd' = $0\bar{2}$ by 'निखीलं नवत: चरमं दशत:'

X = 104 & Base = 100

Then deviation'd' = 04 (104 - 100)

3) Each succeeding parts will contain no. of digit(s) equal to the no. of zeros in base. If no. of digit(s) is less than the no. of zeroes in base then place is filled by prefixing zero (es) to the digit(s) in that part. If no. of digit(s) is more than the no. of zero in base then excess digit(s) in that part is carried to left part of that group for addition.
4) $(c_2 - c_1)$ appears in 1st part of answer, Coefficient $c_3$ appears in 2nd part of answer as well as $c_4$ in the 3rd part and so on.
5) The numerical coefficients of these parts are obtained from 'Meru-Prastar' or 'Halayudh's Pyramid Structure' or Bhaskaracharya's 'Khand-Meru'.
6) Letter component i.e.'d' will be in its increasing power starting from 1st part (far left) of answer. Thus it will be '$d^2$' in 2nd part,'$d^3$' in 3rd part and finally '$d^n$' in last part of answer.

$X^2 \longrightarrow$ Answer in '2' parts

$X^3 \longrightarrow$ Answer in '3' parts

$X^4 \longrightarrow$ Answer in '4' parts

$X^n \longrightarrow$ Answer in 'n' parts

In general $X^n$: $- X + (c_2 - c_1)\, d \,/\, c_3\, d^2 \,/\, c_4\, d^3 \,/ \ldots\ldots\ldots\ldots\ldots\ldots\ldots / \, c_{n+1} d^n$.

Thus with the help of algebraic components and numerical coefficients, the expansion terms of binomial expression of any power can be calculated quickly.

**E.g. Find the value of ($99^4$)**

**<u>FIRST METHOD</u>**

X = 99

As power of number 'X' is '4' therefore answer will have 4 parts.

Base = 100

Deviation = $0\bar{1}$

As base contains 2 zeros hence succeeding parts will have 2 digits.

Numerical coefficients will be

$c_1 = 1$, $c_2 = 4$, $c_3 = 6$, $c_4 = 4$, $c_5 = 1$

| **Numerical number 'X'** | **Far left part** | **$2^{nd}$ part** | **$3^{rd}$ part** | **$4^{th}$ part** |
|---|---|---|---|---|
| 99 | $[X + (c_2 - c_1)d]$ | $c_3 d^2$ | $c_4 d^3$ | $c_5 d^4$ |
| | [99 + (4-1) (-1) | $6 * (-1)^2$ | $4 * (-1)^3$ | $1 * (-1)^4$ |
| | (99 - 3) | 06 | -04 | 01 |
| | 96 | 06 | -04 | 01 |

$99^4 = 96060\bar{4}01$

Answer is in vinculum form and it is converted into conventional number using Vedic sutras 'एकन्यूनेन पूर्वेण' and 'निखीलं नवत: चरमं दशत:'

$99^4 = 96059601$ (60-1) by 'एकन्यूनेन पूर्वेण' and (10 - 4) by 'निखीलं नवत: चरमं दशत:'

$B\,(99)^4 = 9$

$B\,(96059601) = B\,(\cancel{9} + 6 + 0 + 5 + \cancel{9} + 6 + 0 + 1)$

$= (18)$

$= (1 + 8)$

$= 9$

As both Beejank are same hence the answer is verified.

---

## SECOND METHOD

| | | | | | |
|---|---|---|---|---|---|
| | $\times$ | 4 | 3 | 2 | 1 |
| | $\div$ | 1 | 2 | 3 | 4 |
| calculation | | $(1\times4)\div1$ | $(4\times3)\div2$ | $(6\times2)\div3$ | $(4\times1)\div4$ |
| Numerical coefficients | 1 | 4 | 6 | 4 | 1 |
| $\times\ 9^n$ | $9^0$ | $9^1$ | $9^2$ | $9^3$ | $9^4$ |
| $\times\ 9^n$ | $9^4$ | $9^3$ | $9^2$ | $9^1$ | $9^0$ |
| Product | 1*1*6561 | 4*9*729 | 6*81*81 | 4*729*9 | 1*6561*1 |
| | 6561 | $_{2624}4$ | $_{3936}6$ | $_{2624}4$ | $_{656}1$ |

$99^4 = 6561 + {}_{2624}4 + {}_{3936}6 + {}_{2624}4 + {}_{656}1$

$= 96059601$

## THIRD METHOD

First Meru-Prastar is prepared for 'n' = 4

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | | $(a+b)^0$ |
| | | | 1 | | 1 | | | | $(a+b)^1$ |
| | | 1 | | 2 | | 1 | | | $(a+b)^2$ |
| | 1 | | 3 | | 3 | | 1 | | $(a+b)^3$ |
| 1 | | 4 | | 6 | | 4 | | 1 | $(a+b)^4$ |

Now taking numerical coefficient from the above Meru-Prastar.

| Numerical coefficients | 1 | 4 | 6 | 4 | 1 |
|---|---|---|---|---|---|
| $\times\ 9^n$ | $9^0$ | $9^1$ | $9^2$ | $9^3$ | $9^4$ |
| $\times\ 9^n$ | $9^4$ | $9^3$ | $9^2$ | $9^1$ | $9^0$ |
| Product | 1*1*6561 | 4*9*729 | 6*81*81 | 4*729*9 | 1*6561*1 |
| | 6561 | 26244 | 39366 | 26244 | 6561 |

$99^4$ = 6561 + 26244 +39366 + 26244 + 6561

= 96059601

---

### B. Numbers away from the base

When the number is not near to base then sub-base which is multiple of '10' i.e. 20, 30, 40................

Vedic formula remains same except multiplication of ratio.

Ratio (R) = Sub-base (SB) $\div$ Base (B)

Then Vedic formula will be

$X^n$: - $[X + (c_2 - c_1)\ d]\ R^{n-1}$ / $c_3\ d^2\ R^{n-2}$ / $c_4\ d^3 R^{n-3}$ /.................. / $c_n\ d^n$.

'X' $\longrightarrow$ Numerical number away from the base.

**e.g. Find the value of $52^5$**

X = 52, Base (B) = 10, Sub-base (SB) = 50, Ratio (R) = (SB $\div$ B) = (50 $\div$ 10) = 5

As power of number 'X' is '5' therefore answer will have 5 parts.

Deviation (d) = (52- 50) = 2

'n' = 5

As base contains 1 zero hence succeeding parts will have 1 digit.

Numerical Coefficients will be

$c_1 = 1$, $c_2 = 5$, $c_3 = 10$, $c_4 = 10$, $c_5 = 5$, $c_6 = 1$

| **Numerical number 'X'** | **Far left part** | **$2^{nd}$ part** | **$3^{rd}$ part** | **$4^{th}$ part** | **$5^{th}$ part** |
|---|---|---|---|---|---|
| 52 | $[X + (c_2 - c_1) d] R^{n-1}$ | $c_3 d^2 R^{n-2}$ | $c_4 d^3 R^{n-3}$ | $c_5 d^4 R^{n-4}$ | $c_6 d^5 R^{n-5}$ |
| | $[52+(5-1)2]5^4$ | $10*2^2*5^3$ | $10*2^3*5^2$ | $5*2^4*5$ | $1*2^5$ |
| | 60 * 625 | 40 *125 | 80 * 25 | 25 *16 | 32 |
| | 37500 | 5000 | 2000 | 400 | $_{3}2$ |
| | | $_{520}4$ | $_{204}0$ | $_{40}3$ | 2 |
| | 37500 + 520 | 4 | 0 | 3 | 2 |
| | 38020 | 4 | 0 | 3 | 2 |

$52^5 = 380204032$

$B(52^5) = (5+2)^5 = 4$

$B(380204032) = (\cancel{3}+8+0+\cancel{2}+0+\cancel{4}+0+3+2)$

$= 13$ $\qquad = (1+3)$ $\qquad = 4$

As both Beejank are same hence the answer is verified.

---

**SECOND METHOD**

| | | | | | | |
|---|---|---|---|---|---|---|
| | × | 5 | 4 | 3 | 2 | 1 |
| | ÷ | 1 | 2 | 3 | 4 | 5 |
| calculation | | $(1\times 5) \div 1$ | $(5\times 4) \div 2$ | $(10\times 3) \div 3$ | $(10\times 2) \div 4$ | $(5\times 1) \div 5$ |
| Numerical coefficients | 1 | 5 | 10 | 10 | 5 | 1 |
| $\times \quad 2^n$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ |
| $\times \quad 5^n$ | $5^5$ | $5^4$ | $5^3$ | $5^2$ | $5^1$ | $5^0$ |
| Product | 1*1*3125 | 5*2*625 | 10*4*125 | 10*8*25 | 5*16*5 | 1*32*1 |
| | 3125 | $_{625}0$ | $_{500}0$ | $_{200}0$ | $_{40}0$ | $_{3}2$ |

$52^5 = 38020403$

---

**THIRD METHOD**

Coefficients related to n = 5 in Meru-Prastar are 1 5 10 10 5 1. Here '5'is on tens place means on left, so we need to multiply the coefficients by the powers of '5' from right and '2' is on unit place means on right, so we will have to multiply the coefficients by the powers of '2' from left.

| Coefficients For n = 5 | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| $\times\ 2^n$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ |
| $\times\ 5^n$ | $5^5$ | $5^4$ | $5^3$ | $5^2$ | $5^1$ | $5^0$ |
| Product | 1*1*3125 | 5*2*625 | 10*4*125 | 10*8*25 | 5*16*5 | 1*32*1 |
| | 3125 | $_{625}0$ | $_{500}0$ | $_{200}0$ | $_{40}0$ | $_{3}2$ |

$52^5 = 38020403$

**EXPANSION** of $\mathbf{(x + y)^n}$

Write down the coefficients of the row of Meru-Prastar related to 'n'. Now multiply last coefficient on far right by '1' and then go on multiplying consecutive coefficients by x, $x^2$, $x^3$............as well as last by $x^n$. Now multiply coefficient on far left by '1' and then go on multiplying the consecutive coefficients by y, $y^2$, $y^3$............ as well as last by $y^n$.

$$(x+y)^n = x^n + n\,x^{n-1}\,y + \frac{n\,(n-1)}{\lfloor 2}\,x^{n-2}\,y^2 + \ldots\ldots + \frac{n\,(n-1)}{2}\,x^2\,y^{n-2} + n\,x\,y^{n-1} + y^n$$

**E.g.** Find the expansion of $\mathbf{(x+y)^4}$

| | $\times$ | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|
| | $\div$ | 1 | 2 | 3 | 4 |
| calculation | | $(1\times 4)\div 1$ | $(4\times 3)\div 2$ | $(6\times 2)\div 3$ | $(4\times 1)\div 4$ |
| Numerical coefficients for n = 4 | 1 | 4 | 6 | 4 | 1 |
| $\times x^n$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
| $\times y^n$ | $y^0$ | $y^1$ | $y^2$ | $y^3$ | $y^4$ |
| Product | $x^4$ | $4\,x^3\,y^1$ | $6\,x^2\,y^2$ | $4\,x^1\,y^3$ | $1\,x^0\,y^4$ |
| | $x^4$ | $4\,x^3\,y$ | $6\,x^2\,y^2$ | $4\,x\,y^3$ | $y^4$ |

$(x+y)^4 = x^4 + 4\,x^3\,y + 6\,x^2\,y^2 + 4\,x\,y^3 + Y^4$

**E.g.:** Find the expansion of$\mathbf{(x + y)^5}$

The coefficients of the row related to n = 5 in Meru-Prastar are 1  5  10  10  5  1.

| Coefficients For n = 5 | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| $\times \; y^n$ | $y^0$ | $y^1$ | $y^2$ | $y^3$ | $y^4$ | $y^5$ |
| $\times \; x^n$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
| Product | $1*1*x^5$ | $5*y*x^4$ | $10*y^{2}*x^3$ | $10*y^{3}*x^2$ | $5*y^{4}*x^1$ | $1*y^{5}*x^0$ |
| | $x^5$ | $5x^4y$ | $10x^3y^2$ | $10x^2y^3$ | $5xy^4$ | $y^5$ |

Now adding them we get -

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

---

**E.g.:** Find the expansion of $\mathbf{(3x + 2y)^4}$

The coefficients of the row related to n = 4 in Meru-Prastar are 1 4 6 4 1.

| Numerical coefficients for n=4 | 1 | 4 | 6 | 4 | 1 |
|---|---|---|---|---|---|
| $\times (3x)^n$ | $(3x)^4$ | $(3x)^3$ | $(3x)^2$ | $(3x)^1$ | $(3x)^0$ |
| $\times (2y)^n$ | $(2y)^0$ | $(2y)^1$ | $(2y)^2$ | $(2y)^3$ | $(2y)^4$ |
| Product | $1\times 81x^4\times 1$ | $4\times 27x^3\times 2y$ | $6\times 9x^2\times 4y^2$ | $4\times 3x\times 8y^3$ | $1\times 1\times 16y^4$ |
| | $81x^4$ | $216x^3y$ | $216x^2y^2$ | $96xy^3$ | $16y^4$ |

Now simplifying and adding we get -

$(3x + 2y)^5 = 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^5$

**E.g.:** Find the expansion of $\mathbf{(5x^2 + 7y^3)^3}$

The coefficients of the row related to n = 3 in Meru-Prastar are 1 3 3 1.

| Numerical coefficients for n= 3 | 1 | 3 | 3 | 1 |
|---|---|---|---|---|
| $\times (5x^2)^n$ | $(5x^2)^3$ | $(5x^2)^2$ | $(5x^2)^1$ | $(5x^2)^0$ |
| $\times (7y^3)^n$ | $(7y^3)^0$ | $(7y^3)^1$ | $(7y^3)^2$ | $(7y^3)^3$ |
| Product | $1\times 125x^6\times 1$ | $3\times 25x^4\times 7y^3$ | $3\times 5x^2\times 49y^6$ | $1\times 1\times 343y^9$ |
| | $125x^6$ | $525x^4y^3$ | $735x^2y^6$ | $343y^9$ |

Now simplifying and adding we get –

$(5x^2 + 7y^3)^3 = 125x^6 + 525x^4y^3 + 735x^2y^6 + 343y^9$

**Expansion** of **$(x + 1)^n$**

Write down the coefficients from the row of Meru-Prastar related to n. Now multiply last coefficient on right by 1 and then go on multiplying the consecutive coefficients by x, $x^2$ , $x^3$ , …………….. and last by $x^n$. Now adding the products we get the required expansion of

$$(x + 1)^n = x^n + n x^{n-1} + \frac{n (n-1)}{\lfloor 2} x^{n-2} + \ldots\ldots\ldots\ldots + \frac{n(n-1)}{2} x^2 + n x + 1$$

---

**E.g.:** Find the expansion of **$(x + 1)^5$**

The coefficients of the row related to n = 5 in **Meru-Prastar** are 1 5 10 10 5 1.

| **Coefficients For n = 5** | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| × $1^n$ | $1^0$ | $1^1$ | $1^2$ | $1^3$ | $1^4$ | $1^5$ |
| × $x^n$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
| Product | $1*1*x^5$ | $5*1*x^4$ | $10*1*x^3$ | $10*1*x^2$ | $5*1*x$ | $1*1*1$ |
| | $x^5$ | $5x^4$ | $10x^3$ | $10x^2$ | $5x$ | 1 |

Now adding them we get -

$$(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$$

---

**E.g.:** Find the expansion of**$(x + 1)^7$.**

The coefficients of row related to n = 7 in Meru-Prastar are 1 7 21 35 35 21 7 1

| **Coefficients For n = 7** | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |
|---|---|---|---|---|---|---|---|---|
| × $1^n$ | $1^0$ | $1^1$ | $1^2$ | $1^3$ | $1^4$ | $1^5$ | $1^6$ | $1^7$ |
| × $x^n$ | $x^7$ | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
| Product | $1*1*x^7$ | $7*1*x^6$ | $21*1*x^5$ | $35*1*x^4$ | $35*1*x^3$ | $21*1*x^2$ | $7*1*x$ | $1*1*1$ |
| | $x^7$ | $7x^6$ | $21x^5$ | $35x^4$ | $35x^3$ | $21x^2$ | $7x$ | 1 |

Now adding them we get$(x + 1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

**Expansion** of **$(1 + x)^n$**

Write down the coefficients of the row of Meru-Prastar related to n. Now multiply last coefficient on left by 1 and then go on multiplying the consecutive coefficients by x, $x^2$ , $x^3$ , …………….. and last by $x^n$. Now adding the products we get the required expansion of

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{\llcorner 2} + \ldots\ldots\ldots\ldots + \frac{n(n-1)}{2} x^{x-2} + n x^{n-1} + x^n$$

**E.g.:** Find the expansion of **(1 + x) $^4$**

The coefficients of the row related to n = 4 in Meru-Prastar are 1 4 6 4 1

| **Numerical coefficients for n = 4** | 1 | 4 | 6 | 4 | 1 |
|---|---|---|---|---|---|
| $\times 1^n$ | $1^4$ | $1^3$ | $1^2$ | $1^1$ | $1^0$ |
| $\times x^n$ | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ |
| Product | 1*1*1 | 4 *1 * $x^1$ | 6 *1* $x^2$ | 4 *1* $x^3$ | 1* 1* $x^4$ |
| | 1 | 4 x | 6 $x^2$ | 4 $x^3$ | $x^4$ |

Now adding them we get -

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

---

**E.g.:** Find the expansion of **(1 + mx) $^4$**

The coefficients of the row related to n = 4 in Meru-Prastar are 1 4 6 4 1.

| **Numerical coefficients for n = 4** | 1 | 4 | 6 | 4 | 1 |
|---|---|---|---|---|---|
| $\times 1^n$ | $1^4$ | $1^3$ | $1^2$ | $1^1$ | $1^0$ |
| $\times (mx)^n$ | $(mx)^0$ | $(mx)^1$ | $(mx)^2$ | $(mx)^3$ | $(mx)^4$ |
| Product | 1*1*1 | 4 *1 * $(mx)^1$ | 6 *1* $(mx)^2$ | 4 *1* $(mx)^3$ | 1* 1* $(mx)^4$ |
| | 1 | 4 mx | 6$m^2$ $x^2$ | 4 $m^3x^3$ | $m^4x^4$ |

Now adding them we get –

$$(1+mx)^4 = 1 + 4mx + 6m^2x^2 + 4m^3x^3 + m^4x^4$$

---

**E.g.:** Find the expansion of **$(1 + x)^6$.**

The coefficients of row related to n = 6 in Meru-Prastar are 1 6 15 20 15 6 1.

| Coefficients For n = 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 |
|---|---|---|---|---|---|---|---|
| $\times \quad x^n$ | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ |
| $\times \quad 1^n$ | $1^6$ | $1^5$ | $1^4$ | $1^3$ | $1^2$ | $1^1$ | $1^0$ |
| Product | 1*1* 1 | 6* x*1 | 15* $x^2$*1 | 20* $x^3$*1 | 15* $x^4$ | 6*$x^5$ | 1*$x^6$*1 |
| | 1 | 6x | 15 $x^2$ | 20 $x^3$ | 15 $x^4$ | 6 $x^5$ | $x^6$ |

Now adding them we get

$$(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

**E.g.:** Find the value of **$12^5$.**

We know $(x + y)^5 = x^5 + 5x^4y + 10x^3y^{2+} 10x^2y^3 + 5xy^4 + y^5$

Taking x = 10 and y = 2. $(x + y)^5 = (10 + 2)^5 = (12)^5$ so power of x =10 will turn into the place value. So we need to multiply the coefficients by power of y = 2 only.

| **Coefficients For n = 5** | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| $\times \quad 2^n$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ |
| $\times \quad 1^n$ | $1^5$ | $1^4$ | $1^3$ | $1^2$ | $1^1$ | $1^0$ |
| Products | 1*1*1 | 5*2*1 | 10*4*1 | 10*8*1 | 5*16*1 | 1*32*1 |

So $(12)^5$= 1| 10 | 40 | 80 | 80| 32 = 1| 10 | 40 | 80 | 83| (2)

= 1| 10 | 40 | 88 |( 32 ) = 1| 10 | 48 |( 832)

= 1| 14 | (8832) = 1 + 1 | (48832) = 248832

In short this can be easily calculated as

$(12)^5 = 1\times1 \quad 5\times2 \quad 10\times4 \quad 10\times8 \quad 5\times16 \quad 1\times32$

= 1 10 40 80 80 32 = 248832

**E.g.:** Find the value of $\mathbf{21^5}$

Here '2' is on Tens place means on left place so we need to multiply the coefficients by the powers of '2' from right and '1' is on Unit place means on right place, so we will have to multiply the coefficients by the powers of 1 from left.

Coefficients related to n = 5 in Meru-Prastar are 1 5 10 10 5 1. '2'is at tens place means on left and '1' is at unit place means on right.

| Coefficients For n = 5 | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| $\times\ 1^n$ | $1^0$ | $1^1$ | $1^2$ | $1^3$ | $1^4$ | $1^5$ |
| $\times\ 2^n$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
| Product | 1*1*32 | 5*16*1 | 10*8*1 | 10*4*1 | 5*2*1 | 1*1*1 |

So we can directly calculate in short as

$(2\ 1)^5 = 1\times32 \quad 5\times16 \quad 10\times8 \quad 10\times4 \quad 5\times2 \quad 1\times1$

= 32 80 80 40 10 1

= 4084101

---

**E.g.:** Find the value of $\mathbf{32^5}$**.**

Coefficients related to n = 5 in Meru-Prastar are 1 5 10 10 5 1. '3'is at tens place means on left and '2' is

at unit place means on right.

| **Coefficients For n = 5** | 1 | 5 | 10 | 10 | 5 | 1 |
|---|---|---|---|---|---|---|
| $\times\ 2^n$ | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ |
| $\times\ 3^n$ | $3^5$ | $3^4$ | $3^3$ | $3^2$ | $3^1$ | $3^0$ |
| Product | 1*1*243 | 5*2*81 | 10*4*27 | 10*8*9 | 5*16*3 | 1*32*1 |

So, $32^5 = 1\times1\times243 \quad 5\times2\times81 \quad 10\times4\times27 \quad 10\times8\times9 \quad 5\times16\times3 \quad 1\times32\times1$

= 243 810 1080 720 240 32 = 243 810 1080 720 (240+3)2

= 243 810 1080 (720+24) 32 = 243 810 (1080+74) 432

= 243 (810+ 115) 4432 = (243 + 92) 54432 = 33554432

---

**E.g.:** Find the value of **$53^4$.**

Coefficients related to n = 4 in Meru-Prastar are 1 4 6 4 1. '5' is at tens place means on left and 3 is at unit place means on right.

| **Coefficients For n = 4** | 1 | 4 | 6 | 4 | 1 |
|---|---|---|---|---|---|
| × $3^n$ | $3^0$ | $3^1$ | $3^2$ | $3^3$ | $3^4$ |
| × $5^n$ | $5^4$ | $5^3$ | $5^2$ | $5^1$ | $5^0$ |
| Product | 1*1*625 | 4*3*125 | 6*9*25 | 4*27*5 | 1*81*1 |

So $53^4$ = 1×1 ×625 4×3×125 6×9×25 4×27×5 1×81×1

= 625 1500 1350 540 81 = 7890481

---

**E.g.:** Find the value of **$74^3$.**

Coefficients related to n = 3 in Meru-Prastar are 1 3 3 1. '7' is at tens place means on left and 4 is at unit place means on right.

| **Coefficients For n = 3** | 1 | 3 | 3 | 1 |
|---|---|---|---|---|
| × $4^n$ | $4^0$ | $4^1$ | $4^2$ | $4^3$ |
| × $7^n$ | $7^3$ | $7^2$ | $7^1$ | $7^0$ |
| Product | 1*1*343 | 3*4*49 | 3*16*7 | 1*64*1 |

So $74^3$ = 1×1 ×343 3×4×49 3×16×7 1×64×1

= 343 588 336 64 = 405224.

# CHAPTER EIGHTEEN: AWARENESS OF 6-11 VEDIC SUTRA

## 6. ANURUPYE - SUNYAMANYAT

The Sutra Anurupye Sunyamanyat says: 'If one is in ratio, the other one is zero'.
We use this Sutra in solving a special type of simultaneous simple equations in which the coefficients of 'one' variable are in the same ratio to each other as the independent terms are to each other. In such a context the Sutra says the 'other' variable is zero from which we get two simple equations in the first variable (already considered) and of course give the same value for the variable.

**Example 1**

$$3x + 7y = 2$$
$$4x + 21y = 6$$

Observe that the y-coefficients are in the ratio 7: 21 i.e., 1: 3, which is same as the ratio of independent terms i.e., 2: 6 i.e., 1: 3. Hence the other variable x = 0 and 7y = 2 or 21y = 6 gives y = 2 / 7

**Example 2**

$$323x + 147y = 1615$$
$$969x + 321y = 4845$$

The very appearance of the problem is frightening. But just an observation and Anurupye Sunyamanyat give the solution x = 5, because coefficient of x ratio is
323: 969 = 1: 3 and constant terms ratio is 1615: 4845 = 1: 3. y = 0
and 323 x = 1615 or 969 x = 4845 gives x = 5

In solving simultaneous quadratic equations, also we can take the help of the 'sutra' in the following way:

**Example 3:**

Solve for x and y

$$x + 4y = 10$$
$$x^2 + 5xy + 4y^2 + 4x - 2y = 20$$

$x^2 + 5xy + 4y^2 + 4x - 2y = 20$ can be written as
$(x + y) (x + 4y) + 4x – 2y = 20$

$10 (x + y) + 4x – 2y = 20$ (Since $x + 4y = 10$)
$10x + 10y + 4x – 2y = 20$
$14x + 8y = 20$
Now $x + 4y = 10$
$14x + 8y = 20$ and 4: 8:: 10: 20

From the Sutra, x = 0 and 4y = 10, i.e. 8y= 20 y = 10/4 = 2½
Thus **x = 0** and **y = 2½** is the solution

## 7. SANKALANA – VYAVAKALANABHYAM

This Sutra means 'by addition and by subtraction'. It can be applied in solving a special type of simultaneous equations where the x - coefficients and the y - coefficients are found interchanged.

**Example 1:** $45x – 23y = 113$
$23x – 45y = 91$

In the conventional method we have to make equal either the coefficient of x or coefficient of y in both the equations. For that we have to multiply equation (1) by 45 and equation (2) by 23 and subtract to get the value of x and then substitute the value of x in one of the equations to get the value of y or we have to multiply equation (1) by 23 and equation (2) by 45 and then subtract to get value of y and then substitute the value of y in one of the equations, to get the value of x. It is difficult process to think of.

From Sankalana – Vyavakalanabhyam

**Add them,**

i.e., $(45x – 23y) + (23x – 45y) = 113 + 91$
i.e., $68x – 68y = 204$ $\therefore x – y = 3$

**Subtract one from other,**

i.e., $(45x – 23y) – (23x – 45y) = 113 – 91$
i.e., $22x + 22y = 22$ $\therefore x + y = 1$

And repeat the same sutra, we get x = 2 and y = - 1

Very simple addition and subtraction are enough, however big the coefficients may be.

**Example 2:** $1955x – 476y = 2482$
$476x – 1955y = -4913$

Oh! What a problem! And still

just add, $2431(x – y) = -2431$ therefore $x – y = -1$
subtract, $1479(x + y) = 7395$ therefore $x + y = 5$
once again add, $2x = 4$ therefore $x = 2$
therefore
subtract $-2y = -6$ $y = 3$

**HIGHEST COMMON FACTOR:**

To find the Highest Common Factor i.e. H.C.F. of algebraic expressions, the factorization method and process of continuous division are in practice in the conventional system. We now apply' Lopana - Sthapana' Sutra, the 'Sankalana vyavakalanakam' process and the 'Adyamadya' rule to find out the H.C.F in a more easy and elegant way.

**Example 1:** Find the H.C.F. of $x^2 + 5x + 4$ and $x^2 + 7x + 6$.

1. **FACTORIZATION METHOD**

$$x^2 + 5x + 4 = (x + 4)(x+1)$$

$$x^2 + 7x + 6 = (x + 6)(x +1)$$

**H.C.F.** is $(x + 1)$.

2. **CONTINUOUS DIVISION PROCESS**.

$$
\begin{array}{l}
x^2 + 5x + 4\ )\ x^2 + 7x + 6\ (\ 1 \\
\qquad\qquad\quad -x^2 - 5x - 4 \\
\qquad\qquad\quad \overline{\qquad\qquad\qquad} \\
\qquad\qquad\qquad\quad 2x + 2)\ x^2 + 5x + 4\ (\tfrac{1}{2}x \\
\qquad\qquad\qquad\qquad\qquad -x^2 - x \\
\qquad\qquad\qquad\qquad\qquad \overline{\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\quad 4x + 4)\ 2x + 2\ (\tfrac{1}{2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Thus $4x + 4$ i.e., $(x + 1)$ is H.C.F.

3. LOPANA - Sthapana process i.e. elimination and retention or alternate destruction of the highest and the lowest powers is as below:

$$
\text{Subtract} \left\{
\begin{array}{l}
x^2+5x+4 \\
\\
x^2+7x+6 \\
\text{----------------} \\
-2)\ -2x-2
\end{array}\right.
$$

$$
\begin{array}{l}
\text{----------} \\
\quad x + 1
\end{array}
$$

i.e. $(x + 1)$ is H.C.F

**Example 2:** Find **H.C.F.** of $2x^2 - x - 3$ and $2x^2 + x - 6$

$$\text{Subtract}\left\{\begin{array}{l} 2x^2 - x - 3 \\ 2x^2 + x - 6 \\ \hline -1)\ -2x + 3 \\ \quad\text{------------} \\ \quad\quad 2x + 3 \text{ is the H.C.F.} \end{array}\right.$$

**Example 3:** $x^3 - 7x - 6$ and $x^3 + 8x^2 + 17x + 10$.

Now by **Lopana - Sthapana** and **Sankalana – Vyavakalanabhyam**

$$\text{Subtract}\left\{\begin{array}{l} x^3 - 7x - 6 \\ x^3 + 8x^2 + 17x + 10 \\ \hline -8)\ -8x^2 - 24x - 16 \\ \hline \quad x^2 + 3x + 2 \end{array}\right.$$

**Example 4:** $x^3 + 6x^2 + 5x - 12$ and $x^3 + 8x^2 + 19x + 12$.

$$\text{Add}\left\{\begin{array}{l} x^3 + 6x^2 + 5x - 12 \\ x^3 + 8x^2 + 19x + 12 \\ \hline 2x^3 + 14x^2 + 24x \\ \hline 2x)\ x^2 + 7x + 12 \end{array}\right.$$

and

$$\text{Subtract}\left\{\begin{array}{l} x^3 + 6x^2 + 5x - 12 \\ x^3 + 8x^2 + 19x + 12 \\ \hline 2x^2 - 14x - 24 \\ \hline -2x)\ x^2 + 7x + 12 \end{array}\right.$$

**Example 5:** $2x^3 + x^2 - 9$ and $x^4 + 2x^2 + 9$.

By **Vedic sutras**:

Add: $(2x^3 + x^2 - 9) + (x^4 + 2x^2 + 9) = x^4 + 2x^3 + 3x^2$.

After performing $\div x^2$ it gives

$$x^2 + 2x + 3 \text{ ------ } (1)$$

Subtract after multiplying the first by x and the second by 2.

$$\text{Thus } (2x^4 + x^3 - 9x) - (2x^4 + 4x^2 + 18)$$
$$= x^3 - 4x^2 - 9x - 18 \text{ ------ } (2)$$

Multiply (1) by x and subtract from (2)

$x^3 - 4x^2 - 9x - 18 - (x^3 + 2x^2 + 3x) = -6x^2 - 12x - 18$

After performing $\div -6$ it gives $x^2 + 2x + 3$.

Thus $(x^2 + 2x + 3)$ is the H.C.F. of the given expressions.

**<u>ALGEBRAIC PROOF</u>:**

Let P and Q be two expressions and H is their H.C.F. Let A and B the Quotients after their division by H.C.F.

i.e., $\frac{P}{H} = A$ and $\frac{Q}{H} = B$ which gives $P = A.H$ and $Q = B.H$

$P + Q = AH + BH$ and $P - Q = AH - BH = (A+B).H = (A-B).H$

Thus we can write $P \pm Q = (A \pm B).H$

Similarly $MP = M.AH$ and $NQ = N.BH$ gives $MP \pm NQ = H(MA \pm NB)$

This states that the H.C.F. of P and Q is also the H.C.F. of $P \pm Q$ or $MA \pm NB$.

i.e. we have to select M and N in such a way that highest powers and lowest powers (or independent terms) are removed and H.C.F appears as we have seen in the examples.

## 8. PURANAPURANABHYAM

The Sutra can be taken as Purana - Apuranabhyam which means by the completion or Non - completion. Purana is well known in the present system. We can see its application in solving the roots for general form of quadratic equation.

We have: $ax^2 + bx + c = 0$

$x^2 + (b/a)x + c/a = 0$ ( dividing by a )

$x^2 + (b/a)x = -c/a$

Completing the square (i.e. purana) on the L.H.S.

$x^2 + (b/a)x + (b^2/4a^2) = -c/a + (b^2/4a^2)$

$[x + (b/2a)]^2 = (b^2 - 4ac) / 4a^2$

Proceeding in this way we finally get $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Now we apply purana to solve problems.

**Example 1**

$x^3 + 6x^2 + 11x + 6 = 0$
Since $(x + 2)^3 = x^3 + 6x^2 + 12x + 8$
Add $(x + 2)$ to both sides
We get $x^3 + 6x^2 + 11x + 6 + x + 2 = x + 2$
i.e., $x^3 + 6x^2 + 12x + 8 = x + 2$ i.e.,$(x + 2)^3 = (x + 2)$
This is of the form $y^3 = y$ for $y = x + 2$ **Solution** $y = 0, y = 1, y = -1$
i.e., $x + 2 = 0, 1, -1$ **which gives** $x = -2, -1, -3$

**Example 2:**

$x^3 + 8x^2 + 17x + 10 = 0$
We know $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$
So adding on the both sides, the term $(x^2 + 10x + 17)$
We get $x^3 + 8x^2 + 17x + x^2 + 10x + 17 = x^2 + 10x + 17$
i.e.,, $x^3 + 9x^2 + 27x + 27 = x^2 + 6x + 9 + 4x + 8$
i.e.,$( x + 3 )^3 = ( x + 3 )^2 + 4 ( x + 3 ) - 4$
$y^3 = y^2 + 4y - 4$ for $y = x + 3$
$y = 1, 2, -2$ **Hence** $x = -2, -1, -5$
Thus purana is helpful in factorization.Further purana can be applied in solving Biquadratic equations also.

## 9. Calana - Kalanabhyam

In the book on **Vedic Mathematics** Sri Bharati Krishna Tirthaji mentioned the Sutra 'Calana - Kalanabhyam' at only two places. The Sutra means 'Sequential motion'. In the first instance it is used to find the roots of a quadratic equation$x^2 – 5x – 6 = 0$ Swamiji called the sutra as calculus formula. Its application at that point is as follows.

1. In every quadratic expression put in its standard form i.e. with '1' as the coefficient of $x^2$, the sum of its two binomial factors is its first differential ($D_1$). Thus quadratic expression $x^2 – 5x – 6$ has binomial factors (X-2) and (x-3). Sum of these two binomial factors i.e. (2x-5) is its first differential ($D_1$).

2. The first differential of each term is obtained by multiplying power with its coefficient and reducing the power by one.

3. Defining the discriminant as the square of the coefficient of the middle term minus the product of double the first coefficient and double the independent term. Then very important proposition that the first differential is equal to square-root of the discriminant.

Now by calculus formula we say: $2x - 5 = \pm\sqrt{25-24} = \pm 1$ thus given quadratic equation is divided into two simple equations i.e.2x-5=1 and 2x-5 = -1 $\therefore$ x = 2 or 3
At the Second instance under the chapter 'Factorization and Differential Calculus' for factorizing expressions of 3rd, 4th and 5th degree, and the procedure is mentioned as **'Vedic Sutras relating to Calana – Kalana – Differential Calculus'**.

Factorisation and differentiation are closely connected with each other. This relationship is of immense practical help with regard to the use of successive differentials for the detection of repeated factors.

**<u>Factorise</u>** - $x^3 – 4x^2 + 5x - 2$

$\therefore\ D_1 = 3x^2 - 8x + 5 = (x-1)(3x-5)$

Judging from the first and last coefficient of given expression (E), rule out (3x-5), keeping (x-1)

$\therefore\ D_2 = 6x - 8 = 2(3x – 4)$

$\therefore$ We have $(x-1)^2$

Applying 'Adyamadyena Antyamaantyen' sutra to (E)

$\therefore\ E = (x-1)^2 (x-2)$

**Factorise: -** $x^4 – 6x^3 + 13x^2 – 24x + 36$

$\therefore$ $D_1 = 4x^3 – 18x^2 + 26x -24 = 2(2x^3 – 9x^2 + 13x -12)$

$\therefore$ $D_1 = 2(x-3) (2x^2 -3x +4)$

$\therefore$ $D_2 = 12x^2 -36x + 26$ (which has no rational factors)

Applying '**Adyamadyena Antyamaantyen**' sutra to (E)

$\therefore$ $E = (x-3)^2 (x^2 + 4)$

**Factorise: -** $2x^4 – 23x^3 + 84x^2 – 80x + 64$

$\therefore$ $D_1 = 8x^3 – 69x^2 + 168x – 80$

$\therefore$ $D_2 = 24x^2 -138x + 168 = 6(4x^2 -23x + 28) = 6(x-4) (4x – 7)$

$\therefore$ $D_3 = 48x – 138 = 6(8x – 23)$

$\therefore$ $D_2 = 6(x-4) (4x – 7)$

Applying 'Adyamadyena Antyamaantyen' sutra

$\therefore$ $D_1 = (x-4)^2 (8x-5)$

Applying 'Adyamadyena Antyamaantyen' sutra to (E)

$\therefore$ $E = (x-4)^3 (2x + 1)$

## 10. YAVADUNAM

The meaning of the Sutra is 'whatever the deficiency subtract that deficit from the number and write alongside the square of that deficit'.This Sutra can be applicable to obtain squares of numbers close to bases of powers of 10.

**Method-1:** Numbers near and less than the bases of powers of 10.

**Example 1**:

$9^2$Here base is 10
The answer is separated in to two parts by a'/'
Note that deficit is 10 - 9 = 1
Multiply the deficit by itself or square it
$1^2 = 1$. As the deficiency is 1, subtract it from the number i.e., 9–1 = 8
Now put 8 on the left and 1 on the right side of the vertical line or slash i.e., 8/1
Hence 81 is answer.

**Example2**:

$96^2$ here base is 100.

Since deficit is 100-96=4 and square of it is 16 and the deficiency subtracted from the number 96 gives 96-4 = 92, we get the answer 92 / 16 Thus $96^2 = 9216$.

**Example 3**:

$994^2$Base is 1000
Deficit is 1000 - 994 = 6. Square of it is 36.
Deficiency subtracted from 994 gives 994 - 6 = 988
Answer is 988 / 036 [since base is 1000]

**Example4**:

$9988^2$ **Base** is 10,000
Deficit = 10000 - 9988 = 12
Square of deficit = $12^2 = 144$
Deficiency subtracted from number = 9988 - 12 = 9976
Answer is 9976 / 0144 [since base is 10,000]

**Example5**:

$88^2$ base is 100

Deficit = 100 - 88 = 12

Square of deficit = $12^2$ = 144

Deficiency subtracted from number =88 - 12 = 76

Now answer is 76 / $_1$44 =7744 [since base is 100]

**ALGEBRAIC PROOF**:

The numbers near and less than the bases of power of 10 can be treated as (x-y), where x is the base and y, the deficit.

Thus

(1) 9 = (10 -1)

(2) 96 = ( 100-4)

(3) 994 = (1000-6)

(4) 9988 = (10000-12)

(5) 88 = (100-12)

$(x – y)^2 = x^2 – 2xy + y^2$

$= x (x – 2y) + y^2$

$= x (x – y – y) + y^2$

= Base (number – deficiency) + $(\text{deficit})^2$

Thus

$985^2 = (1000 – 15)^2$

$= 1000 (985 – 15) + (15)^2$

= 1000 (970) + 225

= 970000 + 225

= 970225

=

or we can take the identity $a^2 - b^2 = (a + b) ( a - b)$ and proceed as

$a^2 - b^2 = (a + b) (a - b)$

Gives$a^2 = (a + b) (a - b) +b^2$

Thus for a = 985 and b = 15

$a^2= (a + b) (a - b) + b^2$

$985^2 = (985 + 15) (985 - 15) + (15)^2$

= 1000 (970) + 225

= 970225.

**Method. 2**:

Numbers near and greater than the bases of powers of 10

**Example 1**: $\mathbf{13^2}$

Instead of subtracting the deficiency from the number we add and proceed as in Method-1.

For $13^2$, base is 10, surplus is 3.Surplus added to the number = 13 + 3 = 16.

Square of surplus = $3^2 = 9$

Answer is 16 / 9 = 169.

**Example 2**: $\mathbf{112^2}$

Base = 100, Surplus = 12,

Square of surplus = $12^2 = 144$

Add surplus to number = 112 + 12 = 124.

Answer is 124 / $_1$44 = 12544

Or think of identity $a^2 = (a + b)(a – b) + b^2$ for a = 112, b = 12:

$112^2 = (112 + 12)(112 – 12) + 12^2$

$= 124 (100) + 144$

$= 12400 + 144$

$= 12544.$

$(x + y)^2 = x^2 + 2xy + y^2$

$= x ( x + 2y ) + y^2$

$= x ( x + y + y ) + y^2$

= Base ( Number + surplus ) + ( surplus)$^2$

**Gives**

$112^2 = 100 (112 + 12) + 12^2 = 100 (124) + 144$

$= 12400 + 144$

$= 12544.$

**Example 3:** $\mathbf{10025^2}$

= ( 10025 + 25 ) / $25^2$

= 10050 / 0625 [ since base is 10,000 ]

= 100500625.

**Method - 3:** This is applicable to numbers which are near to multiples of 10,100, 1000 Etc. For this we combine the upa-Sutra 'anurupyena' and 'yavadunam tavadunikritya varganca yojayet' together.

**Example 1:** **$388^2$ nearest base = 400**.

We treat 400 as 4 x 100. As the number is less than the base we proceed as follows.

Number 388, deficit = 400 - 388 = 12

Since it is less than base, deduct the deficit i.e. 388 - 12 = 376.

Multiply this result by 4 since base is 4 X 100 = 400.

376 x 4 = 1504

Square of deficit = $12^2$ = 144.

Hence answer is 1504 / $_1$44 = 150544 [since we have taken multiples of100].

**Example 2:** **$485^2$nearest base = 500.**

Treat 500 as 5 x 100 and proceed

**$485^2$**= (485 – 15) / 15 [since deficit is 15]

= 470 x 5 / $_2$25 since 500 base is taken as 5 x 100 and 2 of 225 is carried over.

= 2350 / $_2$25

= 235225

**Example 3:** **$67^2$nearest base = 70.**

$67^2$ = (67 – 3) / $3^2$deficit is 3

= 64 x 7 / 9 [since 7 x 10 = 70]

= 448 / 9

= 4489

**Example 4:** **4162 nearest (lower) base = 400**

Here surplus = 16 and 400 = 4 x 100

$416^2$ = (416 + 16) / $16^2$

= 432 x 4 / $_2$56 since base is multiple 100 and 2 of 256 is carried over

= 1728 / $_2$56

= 173056

**Example 5:** **$5012^2$ nearest lower base is 5000 = 5 x 1000**

Surplus = 12

$5012^2 = (5012 + 12) / 12^2$

= 5024 x 5 / 144

= 25120 / 144

= 25120144

So far we have observed the application of yavadunam in finding the squares of number. Now with a slight modification yavadunam can also be applied for finding the cubes of numbers.

**<u>CUBING OF NUMBERS</u>:**

**Example :** Find the cube of the number 106.

We proceed as follows:

1. For 106, Base is 100. The surplus is 6. Here we add double of the surplus i.e. 106+12 = 118.(Recall in squaring, we directly add the surplus)This makes the left-hand -most part of the answer. i.e. answer proceeds like 118 /- - -

2. Put down the new surplus i.e. 118-100=18 multiplied by the initial surplus i.e. 6=108. Since base is 100, we write 108 in carried over form 108 i.e.As this is middle portion of the answer, the answer proceeds like 118 / $_1$08 /....

3. Write down the cube of initial surplus i.e. $6^3$ = 216 as the last portion i.e. right hand side last portion of the answer. Since base is 100, write 216 as $_2$16 as 2 is to be carried over. Answer is 118 / $_1$08 / $_2$16

Now proceeding from right to left and adjusting the carried over, we get the answer

119 / 10 / 16 = 1191016.

**Example 1:**

1023 = (102 + 4) / 6 X 2 / 23

= 106 = 12 = 08 = 1061208

Observe initial surplus = 2, next surplus =6 and base = 100

**Example 2:** $94^3$

Observe that the nearest base = 100. Here it is deficit contrary to the above examples.

1. Deficit = -6. Twice of it -6 X 2 = -12
   Add it to the number = 94 -12 =82.
2. New deficit is -18.
   Product of new deficit x initial deficit = -18 x -6 = 108
3. Deficit$^3$ = $(-6)^3$ = -216.
   Hence the answer is 82 / $_1$08 / -$_2$16

Since 100 is base 1 and -2 are the carried over. Adjusting the carried over in order, we get the answer

(82 + 1)/ (08 – 03) / (100 – 16)

= 83 / = 05 / = 84 = 830584

$\overline{16}$ become 84 after taking1 from middle most portion i.e. 100. (100-16=84).

Now 08 - 01 = 07 remains in the middle portion, and 2 or $\overline{2}$ carried to it makes the middle as 07 - 02 = 05. Thus we get the above result.

**Example 2:9983 Base = 1000; initial deficit = - 2.**

9983 = (998 – 2 x 2) / (- 6 x – 2) / (- 2)3

= 994 / = 012 / = -008

= 994 / 011 / 1000 - 008

= 994 / 011 / 992

= 994011992.

11. **Vyashtisamashtih** (whole as one and one as whole)

This sutra is applicable in algebraic equation for finding the value of the variables.

e.g. x + y = 7

Here (x + y) is Vyashti i.e. whole as one. 7 is Samashtih i.e. one as whole.

Sum of the value of variables ‘x’ and ‘y’ is such that it equal to ‘7’. It means that whole is one.

Value ‘7’ is equal to sum of the value of two variables. In means that one as whole.

## CHAPTER NINTEEN: CONTRIBUTION OF ANCIENT INDIAN MATHEMATICIANS IN ALGEBRA

### MAHAVIRACHARYA

He was the author of the treatise “**Ganitsarasangraha**". His work was totally & purely on mathematics only. In his treatise **Ganitsarasangrah** he revised some intricacies of Brahmagupta's Brahma-sphuta sidhdanta and some additional information was incorporated. Ganitsarasangrah is one of the text books on modern mathematics.

**Contribution of Mahaviracharya in algebra**

He discovered algebraic identity like $a^3 = a\,(a+b)\,(a-b) + b^2\,(a-b) + b^3$

He also discovered formula ${}^{n}C_{r}$ = [n (n-1) (n-2)-------(n-r+1)]/r(r-1)(r-2)----2*1

He explained many methods through puzzles, simple examples

e.g. If $P + X = 2\,(Y + Z)$

$P + Y = 3\,(Z + X)$

$P + Z = 5\,(X + Y)$

Then find the values of P, X, Y, Z

**Mahaviracharya's answer**: P=15, X=1, Y=3, Z=5. This example has infinite answers.

He had knowledge of ratio and proportion.

$a/b = c/d = e/f = (a+c+e)/(b+d+f)$

Using $(a+b)^2$, $(a+b+c)^2$, $(a+b+c)^3$ formulae he solved many algebraic problems.

He also solved simultaneous equation $X + Y = a$, $XY = b$.

Methods of finding area, perimeter, circumference of circle, measurement of ellipse, area of irregular figures are available in his treatise **'Ganisarsangraha'**.

Perimeter of Ellipse = $\sqrt{24A^2 + 16\,B^2}$

A & B = 1/2 * length of longitudes.

## NILKANTH SOMAYAJI

"Tantra-Sangraha" is Nilkanth Somayaji's important treatise out of his 10 treatises. He also presented commentary on 'Aryabhat's ( $5^{th}$century) work named as "Aryabhatiya-Bhashya". He made minor corrections/alterations in many formulae & theorems of Aryabhat ($5^{th}$ century) & presented in simplified manner. He also wrote "Sidhdantadarpan" (Laws of astronomy), Golasara (Spherical astronomy) and Chandrachhaya Ganitam (Calculation of the time during night from a measurement of the shadow cast by the Moon).

### Contribution of Nilkanth Somayaji in algebra

He gave excellent approximation of ($\Pi$) as (104348 $\div$33 215) = 3.14159265392 which is correct upto '9' decimal places. He explicitly proved that it is impossible to express the value of ($\Pi$) as the ratio of two integers. He stated that it is impossible to measure the circumference and the diameter by same unit. Ancient Indians knew the irrationality of ($\Pi$) since sulbsutra period. He was the first who explained it in exact terms of modern mathematics. Johann Heinrich Lambert, a Swiss mathematician rediscovered the same in 1761 AD. Nilkanth Somayaji's infinite series for ($\Pi$/4) has some special features which were not considered by Leibniz. He gave some rational approximations for error correction on taking only the first 'n' terms of series.

$$(\Pi/4) = 1 - (1/3) + (1/5) - (1/7) + (1/9) - (1/11) + \text{----------} \pm (n/2)^2 + \frac{1}{(n^2+4+1)(n/2)}$$

Thus the transformed series is as follows

$$(\Pi/4) = \frac{3}{4} - \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} \;\_\;\_\;\_\;\_$$

And

$$(\Pi/4) = (3/4) + (4/1^5) + 4*1 - (4/3^5) + 4*3 + (4/5^4) + 4*5 - (4/7^5) + 4.7 + \text{---------}$$

He also discussed the Madhav's infinite series of trigonometric functions

$$\text{Sin } x = x - (x^3/3!) + (x^5/5!) - (x^7/7!) + (x^9/9!) - \text{----------------------}$$

$$\text{Cos } x = 1 - (x^2/2!) + (x^4/4!) - (x^6/6!) + (x^8/8!) - \text{----------------------}$$

$$\text{Tan}^{-1} x = x - (x^3/3) + (x^5/5) - (x^7/7) + (x^9/9) - \text{----------------------}$$

He gave formula for computation of the length of a smaller circular arc ‘a’ in terms of chord ‘c’ (Jya = Rsine) and height ‘h’ (Sara = Rversine). Nilkantha’s formula in modern notations.

$1a = \sqrt{[1+(1/3)]}h^2 + c^2$

He applied the summation of an infinite convergent series in derivation of the above approximation formula. He knew that an infinite convergent series has a finite sum.

In this context he also explained how to sum the following infinite geometric series.

(1/4) + (1/4) + ----------- (1/4) -------- = (1/3)

He presented results as

(1/3) = [(1/4) + (1/4*3)], (1/4*3) = (1/4*4) + (1/4*4*3) and so on

Thus general result is

$(1/3) – [(1/4) + (1/4)^2 +$ -------------- $(1/4)^n] = (1/4)^n (1/3)$

He discovered some formulae as below.

$\Sigma r = [n (n+1)]/2$

$\Sigma r^2 = [n (n+1)(2n+1)]/6$

$\Sigma r3 = [n^2 (n+1)^2]/4$

Proof of above formulae are available in his commentary "**Aryabhatiya-Bhashya**".

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## BHASKARACHARYA(12thcentury)

He has been called greatest mathematician of medieval India. His main work "Sidhdant-Shiromani" is divided in four parts called Lilavati, Bijaganita, Grahaganita and Goladhyay which are also sometimes considered four independent works.

### Contribution of Bhaskaracharya (12thcentury) in Algebra

In **Lilavati**, solutions of quadratic, cubic and quartic indeterminate equations are explained. Solutions of indeterminate quadratic equations of type $\mathbf{ax^2 + b = y^2}$. Integer solutions of linear and quadratic indeterminate equations (Kuttaka).

The rules he gives are (in effect) the same as those given by the Renaissance, European mathematician of 17th century. A cyclic chakravala method for solving indeterminate equations of form.

$\mathbf{ax^2 + bx + c = y}$. The solution to this equation was traditionally attributed to William Brouncker in 1657, though his method was more difficult than the chakravala method.

The first general method of finding solutions of problem $x^2 - ny^2 = 1$ (so called "**Pell's equation”**) was given by Bhaskaracharya (12thcentury). Solutions of Diophantine equations of the second order, such as $\mathbf{61\, x^2 + 1 = y^2}$. This very equation was posed as a problem in 1657 by the French mathematician Pierre de Fermat, but its solution was unknown in Europe until the time of Euler in the 18th century.

Solved quadratic equations with more than one unknown and found negative as well as irrational solutions.

**Surds** (includes evaluating surds)

He was the first mathematician to recognize that a positive number has two square roots (a positive and negative square root) & also invented sign for square root as ' Õ '. He was the first mathematician who brought the idea of infinity (∞) while dividing any number by zero.

# PART

# THREE

# VEDIC GEOMETRY

# CHAPTER TWENTY: BODHAYAN GEOMETRY

The Hindu geometry has its history since the days of construction of the altars for the Vedic sacrifices of various kinds. Some of it were Nitya (indispensable) and some were Kamya (optional) sacrifices. The ancient scriptures mention that each sacrifice must be made in altars of specific size & shape. It was also stressed that even a slight change in form & size would not only nullify the object of the whole ritual but might even lead to an adverse effect. Hence there arose a need of geometry as well as arithmetic & algebra.

There were altars of the shape of square, equilateral, isosceles triangle, trapezium, falcon, wheel, tortoise etc. Units of measurement like 1sq. Vyama = 96 angulis (finger width) = 2 sq.yard, 1 Purusha =120 angulis = 2.5 sq.yards etc. Over a period of time this geometry developed from only the sacrificial importance to the scientific one. This happened in Vedic age when different schools of geometry were formed like Baudhayana, Apastamba, Katyana, Manava, Maitrayana, and Varaha& Vadhula.

Recognizing the need of manuals which will help in construction of correct/desired altars, the Vedic priests have composed a class of text called Sulba - sutras. These texts which are dated which are dated prior to 800 BC form a part of Kalpsutra that include

1. श्रौत- Deals in rituals associated with social welfare
2. गृह्य- Ritual related to household
3. धर्म- Duties & general code of conduct
4. शुल्ब- Geometry of the construction of fire altars.

The word "**sulba**" means "**to measure**". This word can be derived (presented) in many ways:

1. भावव्युतत्ति शुल्बनम् शुल्ब: (refer to act of measuring)
2. कर्मव्युत्पत्ति शुल्बयते इति शुल्ब: (derived from the result of measuring)
3. करणाव्युत्पत्ति शुल्बयत्यनेन इति शुल्ब: (refers to instrument of measuring)

There are **7 sulba** sutras which are available

1) Baudhayan SS
2) Apastamba SS
3) Katyayana SS
4) Manava SS
5) Maitrayana SS
6) Varaha SS
7) Vadhula SS.

Out of these 7 SS, the first four are more popular. According to scholars, Bodhayan SS is considered to be the most ancient one i.e. prior to 800 BC. This assessment is based upon the style, completeness & certain archaic images that are not frequently found in later texts. Boudhayan SS contains a very systematic and detailed explanation of several topics which are skipped in later texts. It is made up of three chapters 520 sutras (113+83+323). One may be tempted to say what is great about Boudhayan, anyone can make a square with scale & 'T' square etc. But remember in those days 3000 years back there was no such tool available. Any discovery/invention is great till it is given by somebody, after that understanding of the same looks simple.

The SS can be understood from the teachings of a teacher or the commentaries of various scholars on SS. Following is the list of some of the important commentaries on 3 earlier SS.

| Shulba Sutra | Name | Commentator |
|---|---|---|
| Bodhayan | Sulbadipika | Dwarkanath Yajva |
| Bodhayan | Sulba - Mimansa | Venkateswara Dikshit |
| Apastamba | Sulba Vyakhya | Kapardisvamin |
| Apastamba | Sulba Pradipika | Karavindaswamin |
| Apastamba | Sulba Pradipa | Sundararaja |
| Apastamba | Sulba Bhasya | Gopala |
| Katyayana | Sulba Sutra Vivrtti | Rama Ramchandra |
| Katyayana | Sulba sutravivcarana | Mahidhara |
| Katyayana | Sulba Sutra Bhasya | Karka |

- **SULBHAKARA’S ROLE**

Sulbakara is one who assists Vedic priests in the construction of yagyashala/altars. Mahidhara (17$^{th}$cent) mention following qualities of a Sulbakara.

A Sulbakara must be well versed in arithmetic, mensuration, must be an inquirer, knowledgeable in one's own discipline, must be enthusiastic in learning other disciplines, willing to learn from sculptures & architects & always industrious. This means a Sulbakara is far more than mere geometer. For example in today's world a civil engineering has many specialists like structure, town planner, environment etc. Unlike earlier Sulbakara qualities.

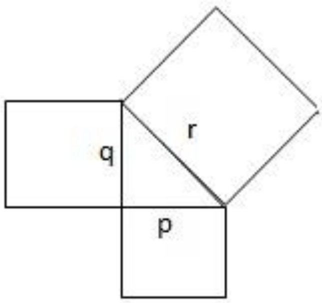
- **Bodhayan** SS belongs to Krishna Yajurved School whereas Katyayan SS belongs to Shukla Yajurved School.

**Sulva Sutra**: ढीर्घस्याअक्षण्यारज्जु: पाश्र्वमानी तिर्यङ्मानी कमानि च यत् पृथकभूते कुरूतस्त ढुभयम् करोति

ढीर्घस्या= Rectangle , अक्षण्या= Diagonal , रज्जु: = Rope ,पाश्र्वमानी = Lateral/horizontal side , तिर्यङ्मानी कमानि= Vertical side ,च = and ,पृथकभूते = Separately ,कुरूतस्तढुभयम् करोति = Added together.

**Meaning**: - A rope stretched along the diagonal of a rectangle produces an area which the vertical and the horizontal sides make (areas) together.

**Figure 1**




**SulvaSutra** - ढीर्घस्याअक्षण्यारज्जु: पाश्र्वमानी तिर्यङ्मानी कमानि च यत् पृथकभूते कुरूतस्त ढुभयम् करोति

**KatyayanaSS** - इति क्षेत्र ज्ञानम् has been added at the end => that this is the most fundamental theorem in geometry to be known which knowledge cannot be dispensed with.

**PythagoreanTheorem** came much later in 5th century (BC). **PythagoreanTheorem** notations => Horizontal side =x, Vertical side = y, Diagonal = z

Then $x^2 + y^2 = z^2$

**Bodhayan triplet (x, y, z) -** Three nos. x, y & z satisfying the equation

$x^2 + y^2 = z^2$. In the very next sutra following the statement of the theorem, Bodhayan illustrates it with a few examples –

तासां त्रिकचतुष्कयो:, द्वादशिकपञ्चिकयो:, पञ्चदशिकाष्टिकयो:, सप्तिकचतुर्विंशिकयो:, द्वादशिकपञ्चत्रिंशिकयो:, पञ्चदशिकषटत्रिंशिकयो: इत्येतासु उपलब्धि: |

(3, 4, 5) => $3^2 + 4^2 = 5^2$ --- तासां त्रिकचतुष्कयो:

(5, 12, 13) => $5^2 + 12^2 = 13^2$ --- द्वादशिकपञ्चिकयो:

(15, 8, 17) => $15^2 + 8^2 = 17^2$ --- पञ्चदशिकाष्टिकयो:

(7, 24, 25) => $7^2 + 24^2 = 25^2$ --- सप्तिकचतुर्विंशिकयो:

(12, 35, 37) => $12^2 + 35^2 = 37^2$ --- द्वादशिकपञ्चत्रिंशिकयो:

(15, 36, 39) => $15^2 + 36^2 = 39^2$ --- पञ्चदशिकषटत्रिंशिकयो:

**Datta in 'science of sutras' has given algebraic proof**

$na^2 = [(n+1)/2]^2 a^2 - [(n-1)/2]^2 a^2$

Putting $n = m^2$ & $a = 1$, we get

$m^2 + [(m^2-1)/2]^2 = [(m^2+1)/2]^2$

Here by putting m = 3, 5, 7.......... we get (3,4,5), (5,12,13), (7,24,25)

Multiplying the equation by 2 & then rewriting the equation we get

$(2m)^2 + (m^2 - 1)^2 = (m^2 + 1)^2$

By putting m = 2, 4, 6..... we get (4,3,5) , (8,15,17) , (12,35,37)

Apastamba used the principle that if (x,y,z) satisfies the relation $x^2 + y^2 = z^2$ , then (mx,my,mz) also satisfy the same relation , where 'm' is arbitrary rational number.

**E.g.**

If $3^2 + 4^2 = 5^2$ then,

$12^2 + 162^2 = 20^2$ (here m = 4)

$15^2 + 20^2 = 25^2$ (here m = 5)

If $5^2 + 12^2 = 13^2$ then

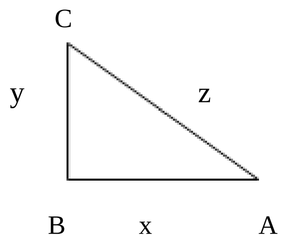
$15^2 + 36^2 = 39^2$ (here m = 3)

**BODHAYAN** also conjectured that if one of the x, y, z sides is divisible by 4, then z = (7/8) x + (y/2), if x is divisible by 4. But this was found to be correct only for some specific triplets.

**BODHAYAN TRIANGLE -** As we have seen the Indians had complete knowledge & clear understanding of geometry. We will now study Triples/Triplets which were known as "Tribhujank" in Sanskrit. This contains 3 words namely - 'tri' means three, 'bhuja' means side and 'ank' means digit or number.

Overall it stands for a set of three numbers representing the sides of a right angle triangle. These triplets are Bodhayan triplets for an angle.

Consider a right angle triangle ABC with base AB =x, perpendicular BC = y & hypotenuse AC = z such that




$$x^2 + y^2 = z^2$$

Then triplet of angle A = T (A) = (x, y, z)

Triplet of angle C = T(C) = (y, x, z)

Triplet of angle -A = T (-A) = (x,-y, z)

sin A = y/z, cos A = x/z & tan A = y/x

Note that the Vedic formula 'proportionately' tells that if one triplet is a multiple or sub-multiple of another triplet , then they are called' equal triplets' as the triangles of these triplets are of the same shape with all angles same ,

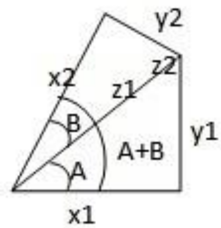
**E.g.** 5, 12, 13 = 10, 24, 26 = 25, 60, 65

When we transpose first two elements of a triplet , it gets converted into 'complementary triplet' e.g. 3,4,5 & 4,3,5 are complementary triplet after transposing 3,4 of one triplet to 4,3 of another

## Addition of triplet

The addition of triplets by Vedic method can be done by using "Urdhva-tiryak" i.e. "Vertically & Crosswise".

The new triplet so obtained represents the sides of right angled triangle with required angle obtained by adding the original triplets




Let triplet of angle A be ($x_1$, $y_1$, $z_1$)

Let triplet of angle B be ($x_2$, $y_2$, $z_2$)

Then the addition of these two triplets is given by

T (A+B) =($x_1 * x_2 - y_1 * y_2$, $y_1 * x_2 + x_1 * y_2$, $z_1 * z_2$)

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| T (A) = | $x_1$ | $y_1$ | $z_1$ | $x_1$ | $y_1$ | $z_1$ | $x_1$ | $y_1$ | $z_1$ |
| T (B) = | $x_2$ | $y_2$ | $z_2$ | $x_2$ | $y_2$ | $z_2$ | $x_2$ | $y_2$ | $z_2$ |

T (A+B) = $x_2 * x_1 - y_2 * y_1$ $\quad$ $x_2 * y_1 + y_2 * x_1$ $\quad$ $z_2 * z_1$

1st element $\quad$ 2nd element $\quad$ 3rd element

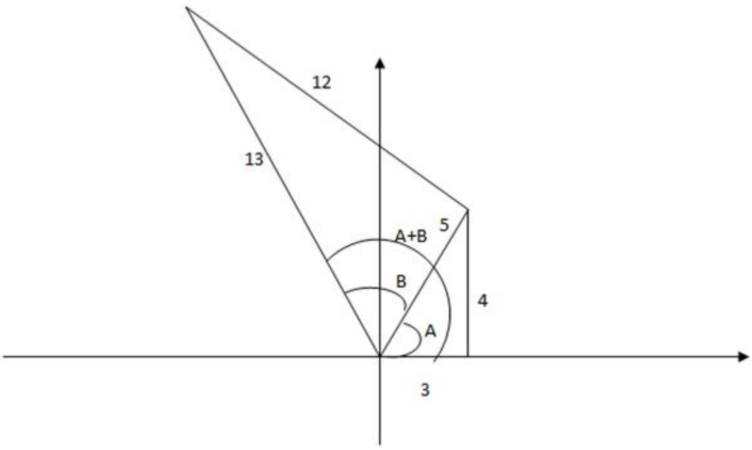
Above is the diagrammatic representation for the addition of two triple T (A) & T (B). First element is obtained by the vertical product of first 2 elements of T (A) & T (B) with their difference (subtraction) i.e. $x_1 * x_2 - y_1 * y_2$. The second element is obtained by the cross product of the first 2 elements of T (A) & T (B) with addition i.e. $y_1 x_2 + x_1 y_2$. Third element is the vertical product of last elements of T (A) & T (B) i.e. $z_1 * z_2$.

1. **Find Triplet T (A+B) if T (A) = (3,4,5) & T(B) = (5,12,13)**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| T (A) = | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |
| T (B) = | 5 | 12 | 13 | 5 | 12 | 13 | 5 | 12 | 13 |

| | | | |
|---|---|---|---|
| T (A+B) = | (3*5-4*12) | (4*5+3*12) | (5*13) |
| | (15-48) | (20+36) | (65) |
| | - 33 | 56 | 65 |

T (A+B) = (-33, 56, 65)




The '-'ve sign for the 1st element (base) indicates that (A+B) is a obtuse angle. Hence resultant triangle is extended in opposite direction i.e. 2nd quadrant.

**SUBTRACTION OF TRIPLET** - Similar to addition, the subtraction of triplets by Vedic method can be done by using same sutra i.e. "Urdhva-tiryak" i.e. "Vertically & Crosswise". Here the new triplet so obtained represents sides of a right angled triangle with required angle obtained by subtracting the original triplets.

This is shown as:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| T (A) = | $x_1$ | $y_1$ | $z_1$ | $x_1$ | $y_1$ | $z_1$ | $x_1$ | $y_1$ | $z_1$ |
| T (B) = | $x_2$ | $y_2$ | $z_2$ | $x_2$ | $y_2$ | $z_2$ | $x_2$ | $y_2$ | $z_2$ |

| | | | |
|---|---|---|---|
| T (A-B) = | $x_2 * x_1 + y_2 * y_1$ | $x_2 * y_1 - y_2 * x_1$ | $z_2 * z_1$ |
| | 1st element | 2nd element | 3rd element |

First element is obtained by the vertical product of first 2 elements of T (A) & T (B) with their sum (addition) i.e. $x_1 * x_2 + y_1 * y_2$. The second element is obtained by the cross product of the first 2 elements of T (A) & T (B) with their difference (subtraction) i.e. $y_1x_2 - x_1y_2$. Third element is the vertical product of last elements of T (A) & T (B) i.e. $z_1 * z_2$.

**1. Find triplet T (A-B) if T (A) = (3, 4, 5) &T (B) = (5, 12, 13)**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| T (A) = | 3 | 4 | 5 | 3 | 4 | 5 | 3 | 4 | 5 |
| T (B) = | 5 | 12 | 13 | 5 | 12 | 13 | 5 | 12 | 13 |

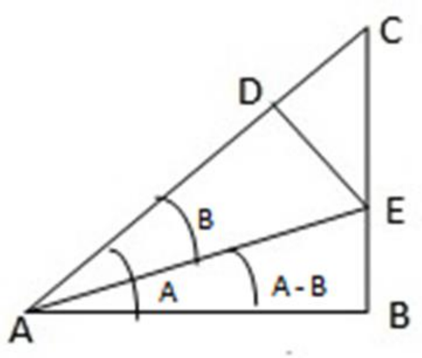
| | | | |
|---|---|---|---|
| T (A-B) = | (3*5+4*12) | (4*5-3*12) | (5*13) |
| | (15+48) | (20-36) | (65) |
| | 63 | -16 | 65 |

T (A-B) = (63, -16, 65)

The '-'ve sign for $2^{nd}$ element (height) indicates that the angle being subtracted (B) is larger than that of angle 'A', hence the new triplet is in the $4^{th}$ quadrant.

$\angle A = \angle CAB$, $\angle B = \angle DAE$

$\angle A\text{-}B = \angle EAB$ if $\angle A > \angle B$

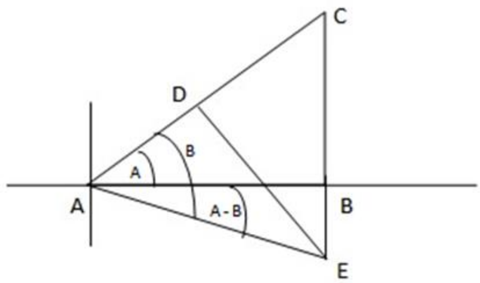



T (A) = ($x_1$, $y_1$, $z_1$)
T (B) = ($x_2$, $y_2$, $z_2$)
T (A-B) = (AB, BE, AE) will lie in $1^{st}$ quadrant **E.g.** if $\angle A = 60^0$ & $\angle B = 25^0$

Then $\angle A\text{-}B = 60^0 - 25^0 = 35^0$ will lie in $1^{st}$ quadrant.

$\angle A = \angle CAB$, $\angle B = \angle DAE$

$\angle A\text{-}B = \angle BAE$ if $\angle A < \angle B$




T (A) = (AB, BC, AC)
T (B) = (AD, DE, AE)
T (A-B) = (AB, BE, AE), 4th quadrant.

**E.g.** Let $\angle A = 50^0$ & $\angle B = 65^0$ then we get $\angle A\text{-}B = -15^0$ which will be in 4th quadrant.

**Note** –

1. If the three elements of new triplet have any common factor, it can be divided (taken) out, this does not make any difference in the angle (A-B) as the new triplet from division is nothing but the equal triplet of one before division.

2. If A & B are complementary triplet then automatically $\angle B = \angle 90\text{-}A$ as 3rd angle of triangle is $90^0$

**E.g. If T (A) = (3, 4, 5), find the sum of T (A) & its complementary triplet.**

As T (A) = (3, 4, 5) its complementary triplet will be T (B) = (4, 3, 5)

(i.e. transpose of first 2 elements)

| | | |
|---|---|---|
| T (A) = 3 | 4 | 5 |
| T (B) = 4 | 3 | 5 |

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T (A+B) = 12-12  16+9  25

T (A+B) = (0, 25, 25)

Or T (A+B) = (0, 1, 1) (dividing by 25)

Note here (1) as the base (1st element) is zero it must triplet of $90^0$ (2) Also as

$\angle A$ & $\angle B$ are complementary; their sum (A+B) must be $90^0$.

3. (0, 25, 25) & (0, 1, 1) are equal triplets and hence make no change in the angle (A+B).

**TRIPLETS OF STANDARD ANGLES**

### 1. Triplet of $0^0$

How do you obtain $0^0$ in a triangle when its height is zero (i.e.2nd element). This is possible only when x and z coincide. Let us consider right angled triangle whose base is 1 i.e. x=1, height (vertical side) y=0, then z=1. Hence we have x=1, y=0, z=1 as three nos. such that $x^2+y^2 = z^2$ or $1+0 = 1$.

$T(0^0) = (1, 0, 1)$

This can also be shown by subtraction of T(A) – T(B) taking A=B, then

$T(A-B) = (x^2+y^2, 0, z^2) = (z^2, 0, z^2)$ since $x^2+y^2 = z^2$, dividing by $z^2$,

$T(A-B) = (1, 0, 1) = T(0^0)$

### 2. Triplet of angle $90^0$

When base of right angled triangle (i.e.1st element) is zero means y & z coincide and is possible at $90^0$.

Hence we have base x=0, height (vertical side) y =1, then z = 1 as $x^2+ y^2 = z^2$

Or 0+1 = 1

$T(90^0) = (0, 1, 1)$

This can also be show by adding two triplets whose sum is $90^0$ i.e. When $A+B = 90^0$. Hence taking T (A) =(x, y, z) & T (B) = (y, x, z)

$T(A+B) = (xy-xy, y^2 +x^2, z^2) = (0, z^2,z^2)$, Dividing by $z^2$ we have

$T(A+B) = (0, 1, 1)$

**<u>Triplet for double angle</u>**: - Let T (A) = (x, y, z)

| | | | |
|---|---|---|---|
| T (A) | x | y | z |
| T (A) | x | y | z |
| T (A+A) | $(x^2-y^2)$ | $(xy+xy)$ | $z^2$ |

$T(2A) = (x^2-y^2, 2xy, z^2)$

### 3. Find triplet for $180^0$

$T(180^0) = T(2*90^0)$

But $T(90^0) = (0, 1, 1)$ & $T(2A) = (x^2-y^2, 2xy, z^2)$

Hence $T(180^0) = [0^2 - (1)^2, 0, 1^2]$

$= (-1, 0, 1)$

**4. If T (A) = (-7, 0, 7) then find $\angle A$**

$T(A) = (7*-1, 7*0, 7*1)$

Dividing by 7, we have

$T(A) = (-1, 0, 1)$ which is triplet for $180^0$ hence $\angle A = 180^0$

---

## EXERCISE

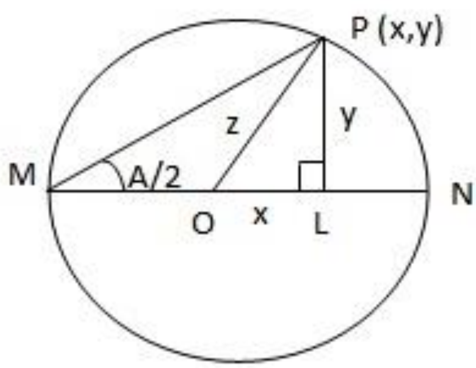
1. Find triplet for $270^0$ & $360^0$
2. Using $T(A) = (3, 4, 5)$ find triplet for $0^0$

## TRIPLET OF HALF ANGLE

Consider a circle with radius 'z' and centre at 'O'. Let MN be the diameter & P(x, y, z) be any point on the circle. Draw PL perpendicular to MN. Then length (OL) = x, length (PL) = y, length (OP) = z $\triangle$OLP being right angled triangle, $x^2 + y^2 = z^2$.

Let $\angle POL = A$, $T(A) = (x, y, z)$. Join PM. Now in $\triangle$ PLM, $\angle L = 90^0$

$\angle PML = A/2$ (angle subtended by an arc PN at centre is double the angle by this arc at a point on the circle) PLM.



Base ML = (z+x), Vertical (height) side PL = y,

$PM^2 = (ML)^2 + (LP)^2 = (z+x)^2 + y^2$

$PM = \sqrt{(x+z)^2 + y^2}$

$T(A/2) = (\ x+z,\ \ y,\ \ \sqrt{(x+z)^2+y^2}\ )$

**Note: -** The half angle triplet does not apply to quadrant triplets (except $90^0$)

**Derive triplet of angle $45^0$**

$T(45^0) = T(90^0/2)$ But we know that $T(A) = (x, y, z)$, then

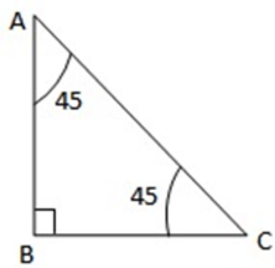
$T(A/2) = (x+z, y, \sqrt{(x+z)^2+y^2})$

Hence $T(45^0) = T(90^0/2) = (0+1, 1, \sqrt{(1+0)^2+1^2})$

Because $T(90^0) = (0, 1, 1)$, $T(45^0) = (1, 1, \sqrt{2})$

**<u>Or</u>**

Let ABC be right angled triangle with $\angle A = 45^0$ & $\angle C = 45^0$




Hence $\triangle$ ABC is an isosceles triangle, AB = BC

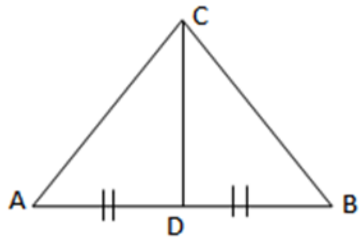
Let AB = BC =1unit, then $AC^2 = 1^2 + 1^2 = 2$ or $AC = \sqrt{2}$

$T(45^0) = (1, 1, \sqrt{2})$

**Derive triplet for angle $30^0$ and angle $60^0$**

Let $\triangle$ ABC be equilateral triangle with sides of 2 units each. Draw CD perpendicular from C to base AB which bisects the base, hence

AD =BD = 1 unit




AD =BD = 1 unit

$CD^2 = 2^2 - 1^2$ or $CD = \sqrt{3}$

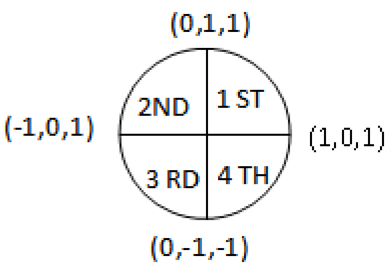
$T(30^0) = (CD, AD, AC) = (\sqrt{3}, 1, 2)$

$T(60^0) = (AD, CD, AC) = (1, \sqrt{3}, 2)$

**TRIPLETS FOR QUADRANTS**

We know triplets for $0^0$ & $90^0$ from this we can get $180^0$ ($90^0 + 90^0$),

$270^0$ ($180^0 + 90^0$) & $360^0$ ($180^0 + 180^0$)




From above we can say that $1^{st}$ quadrant will have all '+' ve elements , $2^{nd}$ quadrant will have '-' ve $1^{st}$ element , $3^{rd}$ quadrant will have both $1^{st}$& $2^{nd}$ elements '-' ve , $4^{th}$ quadrant will have $2^{nd}$ element '-' ve.

**EXERCISE**

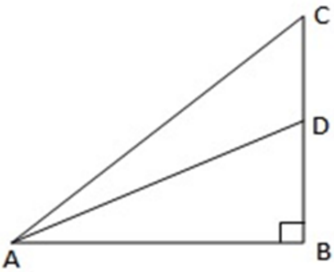
**A)** Find triplets for following angles

(1) $75^0$ (2) $105^0$ (3) $300^0$ (4) $15^0$ (5) $-15^0$ (6) $150^0$ (7) $195^0$ (8) $780^0$ (9) $675^0$

**B)** Find the triplet for T (A+B) & T (A-B)
(1) T (A) = (7, 24, 25) & T (B) = (3, 4, 5)
(2) T (A) = (12, 5, 13) & T (B) = (5, 12, 13)

**C)** Find 2A & A/2 for following triplets

(1) T (A) = $(2, 1, \sqrt{5})$
(2) T (A) = $(4, 1, \sqrt{17})$




**D)** In adjoining figure AB=3units, BC=4 units & CD=2 units

**Find the triplet for angle DAC, given that** $\angle$**ABD = $90^0$**

**Note: -** Once we obtain the triplet, we can calculate all trigonometrically functions

**E.g.**

Find $\sin\theta$, $\cos\theta$, $\tan\theta$ for $T(\theta) = (0, 1, 1)$
We know $\sin\theta$ = height / hypotenuse = 1 / 1 = 1

$\cos\theta$ = base / hypotenuse = 0 / 1 = 0

$\tan\theta$ = height / base = 1 / 0 = $\infty$

$T(\theta) = (0, 1, 1)$ = is a triplet for $90^{0,}$ hence $\theta = 90^0$. Hence $\sin 90^0 = 1$,

$\cos 90^0 = 0$ & $\tan 90^0 = \infty$

---

<u>**EXERCISE**</u>

I. If sin A = 12/13 & sin B = 3/5, find sin (A + B)
II. Show that $\sin(105^0) + \cos(105^0) = \cos(45^0)$
Here $T(90^0) = (0, 1, 1)$ & $T(15^0) = (\sqrt{3} + 1, \sqrt{3} - 1, 2\sqrt{2})$
III. Find triplets for $675^0$& write down its all trigonometric functions.

## Comparison between Bodhayan and Pythagoras

| Bodhayan | Pythagoras |
|---|---|
| (1) Figure : Rectangle | (1) Figure: Right angle triangle |
| (2) Proof : Proved | (2) Proof : Conjecture |
| (3) Application: In Altars construction | (3) Application : No |
| (4) Unknown value: If one value is known then other two can be calculated | (4) Unknown value: If two values are known then third can be calculated |
| (5) Circular function is related to angles (Anglometry) | (5) Trigonometric ratio is related to sides of figure (Trigonometry) |
| (6) Circular function | (6) Trigonometric function |
| (7) 800 to 900 BC | (7) 400 BC |

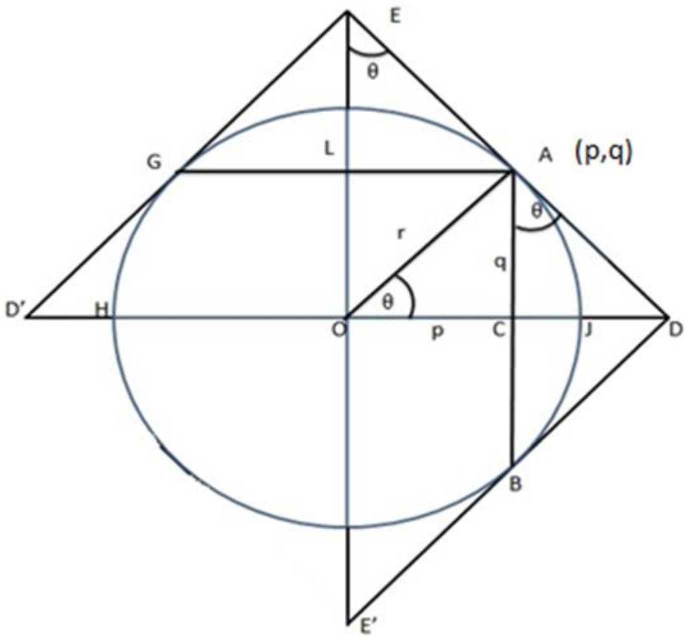



**Figure - 2**

Draw a circle with radius 'r'. Take any point A (p, q) on the circle. DD' & EE' are two perpendicular diameters as axes and 'O' as centre of circle. Draw AB a vertical chord from 'A'. Draw tangents ED at 'A', E'D at 'B' & ED' at 'G'.

Let $\angle COA = \theta$

Bodhayan defined all functions in Sanskrit with reference to chord & diameter of the circle.

1. **Sin θ** = ज्या θ = AB / HJ
   = 2 AC / 2 OA = AC / OA = q / r
2. **Cos θ** = कोज्या θ = AG / HJ
   = 2 OC / 2OA = OC / OA = p / r
3. **Tan θ** = स्पर्शज्या θ = (AD+BD) / HJ

△ ACD ~ △OCA

AD / AC = OA / OC

AD = (AC * OA) / OC = qr / p

Similarly, BD = qr / p

**Tan θ** = (qr / p + qr/p) / 2r = 2qr / 2pr = q / p

4. **Sec θ** = चेदिका θ =DD' / HJ
   = 2OD / 2OJ = 2OD / 2OA = OD / OA

△OAD ~ △ OCA

OA / OC = AD / AC = OD /OA
Sec θ = OD /OA = OA / OC = r / p
Since Sec θ = r / p = 1 / (p / r) = 1 / Cos θ

5. **Cosec θ** = को चेदिका θ = EE' / HJ
   = 2OE / 2OJ = 2OE / 2OA = OE / OA

△ EAO ~ △ OCA

EA / OC = OA / AC = EO / AO
Cosec θ = EE' / HJ = OE / OA = OA / AC = r / q
Since Cosec θ = r / q = 1 / (q / r) =1 / Sin θ

6. **Cot θ** = को स्पर्शज्या θ = HJ / (AD +BD) ⟶ same as tan θ proved above

**<u>SECOND METHOD</u>**

VI. **Cot θ** = (EA + EG) / HJ

△EAO ~ △ OCA

EA / OC = OA / AC

EA = (OA*OC) / AC = (r*p) / q

Similarly, EG = (r*p)/q & HJ = 2r

Cot θ = [(r*p)/q + (r*p)/q] / 2r = [2r *(p/q)]/2r = p/q

From the figure (ii) we have

OJ = OA = r = radii of circle.

HJ = 2r, OC = p

AC = BC = q

AD = BD

$CD^2 = AD^2 - AC^2 = [(r^2q^2)/p^2 - q^2]$ [AD = (OA *AC)/OC = (rq)/p]

$= q^2 * (r^2-p^2)/p^2 = q^2*(q^2)/p^2 = q^4/p^2$

$CD = q^2/p$

$OD^2 = OA^2 + AD^2 = r^2 + (rq/p)^2 = r^2 + (r^2q^2)/p^2$

$= r^2 (p^2 + q^2)/p^2 = (r^2 * r^2)/p^2 = r^4/p^2$

$OD = r^2/p$

---

**Prove identities by using circular function i.e. Bodhayan geometry & using triplets. Use same figure II**

**(1) $\sin^2 \theta + \cos^2 \theta = 1$**

**(a)By BODHAYAN -**

$LHS = AB^2/HJ^2 + AG^2/HJ^2$

$= 4AC^2/4OJ^2 + 4OC^2/4OJ^2$ (AB=2AC, AG=2OC, HJ = 2OJ)

$= (AC^2/OJ^{2)} + (OC^2/OJ^2)$

$= (AC^2 + OC^2)/OJ^2$

$= (AC^2 + OC^2)/OA^2$ (OJ = OA radii of circle)

$= OA^2/OA^2$ $(AC^2 + OC^2 = OA^2)$

$\sin^2 \theta + \cos^2 \theta = 1$ = RHS (proved)

**(b)By PYTHAGORAS -**

LHS = $\sin^2\theta + \cos^2\theta = q^2/r^2 + p^2/r^2$

$= (q^2+p^2)/r^2$

$= r^2/r^2 \quad (p^2 + q^2 = r^2)$

$\sin^2\theta + \cos^2\theta = 1$ = RHS (proved)

**(2) $1 + \tan^2\theta = \sec^2\theta$**

**(a)By BODHAYAN -**

$\tan\theta = (AD+BD) / HJ$

$\tan\theta = 2AD/2OJ \quad (AD=BD \;\&\; HJ=2OJ)$

$= AD/OJ$

LHS = $1 + AD^2/OJ^2$

$= 1 + (OD^2-OA^2)/OJ^2$

$= 1 + (OD^2/OJ^2) - (OA^2/OJ^2)$

$= 1 + (OD^2/OJ^2) -1 \quad$ (OJ = OA radii of circle)

$= OD^2/OJ^2$

But $OD / OJ = \sec\theta$ by definition

$1 + \tan^2\theta = \sec^2\theta$ = RHS (proved)

**b) By PYTHAGORAS**

LHS = $1 + \tan^2\theta$

$= 1 + q^2/p^2$

$= (p^2 + q^2)/p^2$

$= r^2/p^2$

$= (r/p)^2$

$= \sec^2\theta$ = RHS (proved)

**(3) $1 + Cot^2\theta = Cosec^2\ \theta$**

**(a) By <u>BODHAYAN</u> –**

$Cot\ \theta = (EA+EG)/HJ$

$Cot\ \theta = 2EA/2OJ$ (EA =EG & HJ = 2OJ)

$LHS = 1 + EA^2 / OJ^2$

$= 1 + (OE^2 - OA^2)/OA^2$ ($EA2 = OE^2 - OA^2$ & OJ = OA)

$= 1 + OE^2/OA^2 - OA^2/OA^2$

$= 1 + (OE / OA)^2 -1$

$= (OE / OA)^2$ But $OE / OA = Cosec\ \theta$

$1 + Cot^2\theta = Cosec^2\ \theta$ = RHS (Proved)

**(b) By <u>PYTHAGORAS</u> –**

$LHS = 1 + Cot^2\ \theta$

$= 1+ p^2 /q^2$
$= (p^2 + q^2)/q^2$
$= r^2 / q^2$ ($p2 + q^2 = r^2$)
$= (r/q)^2$
$= Cosec^2\ \theta$ = RHS (proved)

**(4) $Sec\ \theta - Cos\ \theta = Tan\ \theta * Sin\ \theta$**

**(a)By <u>BODHAYAN</u> –**

$LHS = Sec\ \theta - Cos\ \theta = (DD'/HJ) - (AG/HJ)$

$= (2OD / 2OJ) - (2OC/2OJ)$

$= (OD - OC)/OJ$

$= CD/OA$

$\triangle ACD \sim \triangle OCA$

$CD/AC = AC/OC$

$CD = AC^2/OC$

LHS = Sec LHS = Sec θ - Cos θ = $(AC^2/OC)/OA$

= (AC/OC)* (AC/OA)

= Tan θ * Sec θ = RHS (proved)

**(b)By <u>PYTHAGORAS</u>**

LHS = Sec θ - Cos θ

= r/p -p/r

= $(r^2 - p^2)/pr$

= $q^2 / pr$ $(r^2-p^2 =q^2)$

= q/p* q/r

= Tan θ * Sec θ = RHS (proved)

**(5) $Cosec^2 \theta + Sec^2 \theta = (Tan\ \theta + Cot\ \theta)^2$**

**(a)By <u>BODHAYAN</u>**

LHS = $Cosec^2 \theta + Sec^2 \theta = (EE'/HJ)^2 + (DD'/HJ)^2$

= $(2OE/2OJ)^2 + (2OD/2OJ)^2$

= $OE^2/OA^2 + OD^2/OA^2$ (OJ = OA radii of circle)

= $(OE^2 + OD^2)/OA^2$

= $ED^2/OA^2$ $(OE^2 + OD^2 = ED^2)$

= $[(EA+AD)/OA]^2$

= $[(EA/OA) + (AD/OA)]^2$

= $(Cot\ \theta + Tan\ \theta)^2$ = RHS (proved)

**(b)By <u>PYTHAGORAS</u>**

LHS = $Cosec^2 \theta + Sec^2 \theta = r^2/q^2 + r^2/p^2$

= $r^2 (p^2+q^2)/q^2p^2$

= $(p^2+q^2)^2/q^2p^2$ $(p^2+q^2 = r^2)$

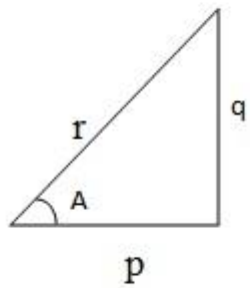
$= [(p^2+q^2)/pq]^2$

$= [(p^2/pq) + (q^2/pq)]^2$

$= (p/q + q/p)^2$

$= (\text{Cot}\ \theta + \text{Tan}\ \theta)^2 = \text{RHS}$ (proved)

**6) $\sqrt{\text{Sec}^2\theta + \text{Cosec}^2\theta} = \text{Tan}\ \theta + \text{Cot}\ \theta$**




Squaring both sides

$\text{Sec}^2\theta + \text{Cosec}^2\theta = (\text{Tan}\ \theta + \text{Cot}\ \theta)^2$

Further method is as problem no.5

**(7) $\text{Cos}^4A - \text{Sin}^4A + 1 = 2\ \text{Cos}^2A$**

LHS $= \text{Cos}^4A - \text{Sin}^4A + 1 = p^4/r^4 - q^4/r^4 + 1$

$= (p^2/r^2+q^2/r^2)(p^2/r^2-q^2/r^2) + (p^2/r^2 + q^2/r^2)$ $(\text{Sin}^2\theta + \text{Cos}^2\theta = 1)$

$= 1 * (p^2/r^2-q^2/r^2) + (p^2/r^2 + q^2/r^2)$ $(p^2+q^2=r^2)$

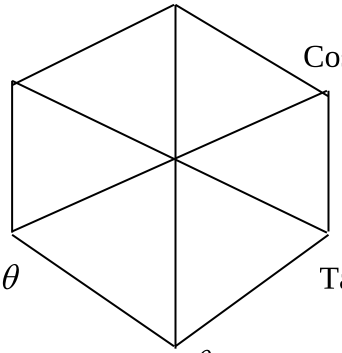
$= p^2/r^2 + p^2/r^2$

$= 2\ p^2/r^2$

$= 2\ \text{Cos}^2A = \text{RHS}$ (proved)

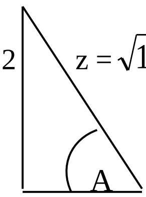
## EXERCISE

1. $1/(\sec A - \tan A) = \sec A + \tan A$
2. $(1+\tan^2A)/(1+\cot^2A) = \sin^2A / \cos^2A$
3. $\sqrt{(1-\sin A)/(1+\sin A)} = \sec A - \tan A$
4. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A$
5. $$\frac{1}{\text{Sec}\ A - \tan A} = \sec A + \tan A$$




**Using triplet of an angle prove that**

**2 $\tan^{-1}$ (2/3) = $\tan^{-1}$(12/5)**




Let $\tan^{-1}(2/3) = A$ --------- (1)

$\therefore$ $\tan A = (2/3)$

$(z^2 = x^2 + y^2)$

T (A) = (x, y, z) = (3, 2, $\sqrt{13}$ )

T (A) = (x, y, z) = (3, 2, $\sqrt{13}$ )

T (A + A) = (9 – 4, 6 + 6, 13)

T (2A) = (5, 12, 13)

$\therefore$ $\tan 2A = (12/5)$

$\therefore$ $2A = \tan^{-1}(12/5)$

Putting value of A from (1)

$\therefore$ $2\tan^{-1}(2/3) = \tan^{-1}(12/5)$ ------------------- (Ans)

**Exercise: -** **Prove that**

(1) $\sin^{-1} 3/5 \sin^{-1} 8/17 = \sin^{-1} 77/85$

(2) $\cos^{-1} 4/5 + \tan^{-1} 3/5 = \tan^{-1} 27/11$

(3) $\tan^{-1} 1/4 + \tan^{-1} 2/9 = 1/2 \cos^{-1} 3/5$

(4) $\cos^{-1} 4/5 + \cos^{-1} 12/13 = \cos^{-1} 33/65$

(5) $2 \tan^{-1} 1/3 + \tan^{-1} 1/7 = \pi/4$

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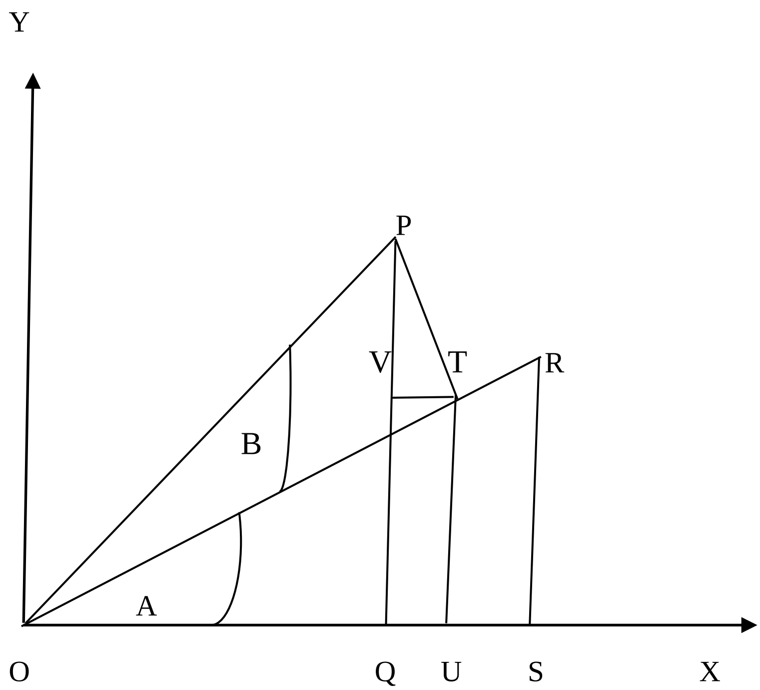
## ADDITION OF TRIPLETS




VT || OS meeting QP at 'V'

PQ $\perp$ OS

TU $\perp$ OS

VT $\perp$ QP

In $\Delta$ SOR $\angle$SOR = $\angle$A = $\angle$VTO ($\because$ Alternate angles)

$\angle$OTP = $90^0$

$\angle$PVT = $90^0$

$\angle$VPT = $\angle$A ($\because$ $\angle$VTP = $90^0 - A$)

T (A) = (OS, RS, OR) = $(x_1, y_1, z_1)$

In $\Delta$ OTP $\angle$TOP = $\angle$B

T (B) = (OT, PT, OP) = $(x_2, y_2, z_2)$

PQ $\perp$ to OS

$\therefore$ $\angle$PQO = $90^0$

$\angle$POQ = $\angle$(A + B)

$\therefore$ T (A+B) = (OQ, QP, PO)

OQ = OU – QU = OU – VT ($\because$ QU = VT opposite sides of rectangle VTUQ)

$\Delta$ OSR is similar to $\Delta$ OUT

$\therefore$ $(OU / x_2) = (x_1 / z_1)$

$$\therefore\ OU = \frac{x_1\, x_2}{z_1} \qquad \text{------- (1)}$$

$(VT / y_2) = (y_1 / z_1)$

$$\therefore\ VT = \frac{y_1\, y_2}{z_1} \qquad \text{---------- (2)}$$

From (1) and (2)

OQ = $[\{(x_1\, x_2) / z_1\} - \{(y_1\, y_2) / z_2\}]$ -------------------- (3)

QP = QV + VP = UT + VP ($\because$ QV = UT opposite sides of rectangle VTUQ)

$(UT / x_2) = (y_1 / z_1)$

$\therefore$ UT = $(x_2y_1 / z_1)$ ---------------------------- (4)

$(VP / y_2) = (x_1 / z_1)$

$\therefore$ VP = $(x_1y_2 / z_1)$ ------------------------------ (5)

From (4) and (5)

QP = $(x_2y_1 / z_1) + (x_1y_2 / z_1)$ ---------------- (6)

OP = $z_2$ --------------------------- (7)

From (3), (6) & (7)

T (A + B) = $\{(x_1 x_2) / z_1\} - \{(y_1 y_2) / z_2\}, \{(x_2y_1 / z_1) + (x_1y_2 / z_1)\}, z_2$

Multiplying by $(z_1)$

T (A + B) = $\{(x_1 x_2) - (y_1 y_2)\}, \{(x_2y_1 + x_1y_2)\}, z_1z_2$

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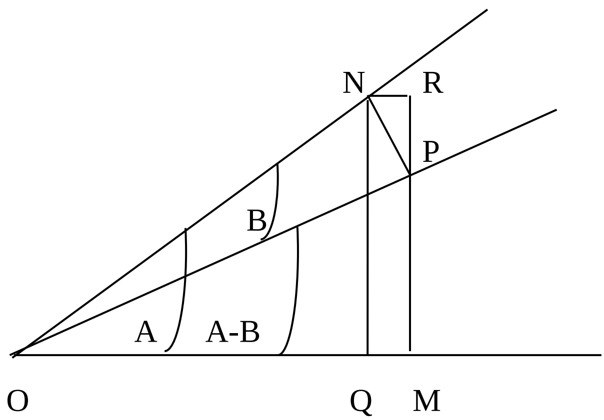
**SUBTRACTION OF TRIPLETS**




$\angle$A = $\angle$QON

T (A) = (OQ, QN, ON) = $(x_1, y_1, z_1)$

$\angle$B = $\angle$NOP

T (B) = (ON, NP, OP) = $(x_2, y_2, z_2)$

$\angle A - B = \angle MOP$

T (A - B) = (OM, MP, OP)

OM = OQ + QM = OQ + NR ($\because$ QM = NR opposite sides of rectangle QMRN)

$\angle ONP = 90^0$

$\therefore$ $(OQ / x_2) = (OQ / ON) = (x_1 / z_1)$

$\therefore$ $OQ = [(x_1x_2) / z_1)]$ ------------- (1)

$(NR / y_2) = (y_1 /z_1)$

$\therefore$ $NR = [(y_1y_2 / z_1)]$ ------------- (2)

From (1) and (2)

$OM = [(x_1x_2) / z_1)] + [(y_1y_2 / z_1)]$ ----------- (3)

MP = (MR – PR) = (NQ – PR) ($\because$ MR = NQ opposite sides of rectangle QMRN)

$(NQ / x_2) = (y_1 / z_1)$

$\therefore$ $NQ = [(x_2 y_1) / z_1)]$ --------------------- (4)

$(PR / y_2) = (x_1 / z_1)$

$PR = [x_1y_2 / z_1)]$ ------------------------- (5)

From (4) and (5)

$MP = [(x_2 y_1) / z_1] - [x_1y_2 / z)]$ --------------- (6)

$OP = z_2$

$\therefore$ $T (A – B) = [\{(x_1x_2 / z_1) + (y_1y_2 / z_1)\}, \{(x_2 y_1 / z_1) – (x_1y_2 / z_1)\}, z_2]$

Multiply by $(z_1)$

$\therefore$ $T (A – B) = [\{(x_1x_2) + (y_1y_2)\}, \{(x_2 y_1) – (x_1y_2)\}, z_1z_2]$

# CHAPTER TWENTY ONE: COMPLEX NUMBER

A number of the form ‘x +iy ‘where x & y are any real numbers as well as $i = \sqrt{-1}$ or $i^2 = -1$ is called a complex number.

x+ iy

Imaginary part of complex number

Real part of complex number

i = is called imaginary unit.

A pair of complex numbers x + iy and x – iy are said to be conjugate of each other.

**PROPERTIES**–

1. If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$.
2. Two complex numbers $x_1 + iy_1 = x_2 + iy_2$ are said to be equal when $x_1 = x_2$ and $y_1 = y_2$
3. Sum, difference, product and quotient of any two complex numbers is itself a complex number.

$x_1 + iy_1$ and $x_2 + iy_2$ are two given complex numbers, then

(a) **Sum** = $(x_1 + iy_1) + (x_2 + iy_2)$

$$= (x_1 + x_2) + (y_1 + y_2)\,i$$

(b) **Difference** = $(x_1 + iy_1) - (x_2 + iy_2)$

$$= (x_1 - x_2) - (y_1 - y_2)\,i$$

(c) **Product** = $(x_1 + iy_1) * (x_2 + iy_2)$

$$= x_1x_2 + (y_1y_2)i^2 + (x_1y_2 + x_2y_1)\,i$$

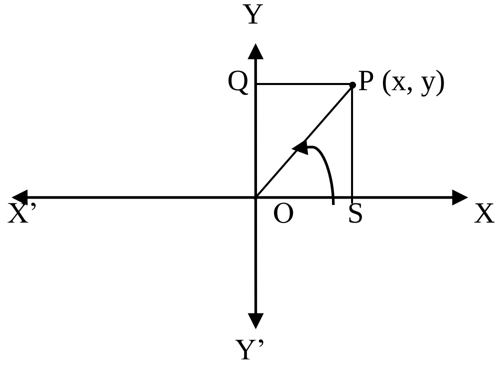
$$= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)\,i \quad (\text{since } i^2 = -1)$$

(d) **Quotient** = $\dfrac{(x_1 + iy_1)}{(x_2 + iy_2)} = \dfrac{(x_1 + iy_1) * (x_2 - iy_2)}{(x_2 + iy_2) * (x_2 - iy_2)}$

$$= \frac{x_1 x_2 + y_1y_2}{(x_2^2 + y_2^2)} + \frac{(x_2y_1 - x_1y_2)\,i}{(x_2^2 + y_2^2)} \quad (\text{since } i^2 = -1)$$

$$= \frac{(x_1 x_2 + y_1y_2) + (x_2y_1 - x_1y_2)\,i}{(x_2^2 + y_2^2)}$$

**4.** Every complex number z = (x + iy) can be expressed in the form of
$z = r(\cos\theta + i\sin\theta)$




OS = x units

PS = y units

OP = r units

$\angle POS = \theta$

Let horizontal line XOX' be the axis of real numbers and vertical line YOY' be the axis of pure imaginary numbers. Let direction along X'OX and Y'OY be positive. Corresponding to any complex number x + y*i*, point P(x,y) is plotted as shown in above figure. Thus any complex number can be shown by point(x,y) in plane by measuring 'a' units on real axis X'OX and 'b' units on imaginary axis Y'OY then drawing lines parallel to the axes. Plane determined by two axes is called complex plane.

Join OP

$OS / OP = x / r = \cos\theta$

$\therefore\ x = r\cos$ --------- (1)

$PS / OP = y / r = \sin\theta$

$\therefore\ y = r\sin\theta$ -------- (2)

Squaring both sides of (1) and (2); then adding them.

$x^2 = r^2\cos^2\theta$

$y^2 = r^2\sin^2\theta$

$x^2 + y^2 = r^2(\sin^2\theta + \cos^2\theta)$

$x^2 + y^2 = r^2$ ( since $\sin^2\theta + \cos^2\theta = 1$)

Taking square-root of both sides

$r = \sqrt{x^2 + y^2}$

The length of line joining origin to the point which represents complex number is called as modulus or absolute value of complex number z = (x + iy).

$\therefore$ Modulus = r = $\sqrt{x2 + y2}$

$|z| = \sqrt{x2 + y2}$

Modulus is represented by $|z|$

Angle made by the line joining origin to the point representing complex number with positive real axis is called amplitude or argument of complex number.

$\therefore$ Amplitude of complex number z = x + y*i*

$\therefore \tan\theta$ = (y/x)

$\therefore \theta = \tan^{-1}$ (y/x)is called as amplitude or argument of complex number

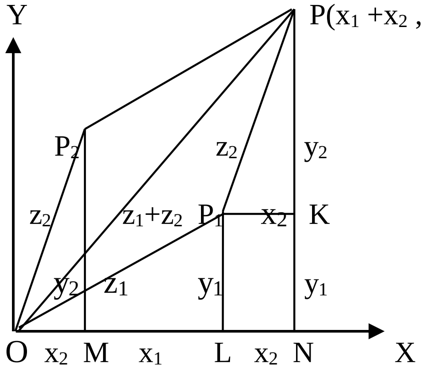
(x + iy) and written as amp (x + iy) or arg (x + iy)

(5) If z = (x +iy) then conjugate $\overline{z}$ = (x - iy)

$z\,\overline{z} = (x + iy) * (x - iy) = x^2 + y^2 = z^2$ (since $i^2 = -1$)

$\sqrt{z\,\overline{z}} = \sqrt{z^2} = |z|$

Complex numbers obey the four fundamental operations: - addition, subtraction, multiplication and division. **Geometric representation of ($z_1 + z_2$)**




Let $P_1$, $P_2$ be the complex numbers.

$P_1 \longrightarrow z_1 = x_1 + iy_1$

$P_2 \longrightarrow z_2 = x_2 + iy_2$

Complete the parallelogram $OP_1PP_2$. Draw $P_1L$, $P_2M$ and PN perpendicular to 'OX'. Draw $P_1K$ perpendicular to PN, $P_1K \parallel$ to OX

ON = OL + LN

LN = $P_1K$ (Opposite sides of rectangle.)

In $\triangle OMP_2$ & $\triangle P_1KP$

$OP_2 = P_1P = z_2$ (opposite sides of parallelogram $P_2OP_1P$)

$\angle P_2OM = \angle PP_1K$

$\angle PKP_1 = P_2MO = 90^0$

$\therefore \triangle OMP_2 \approx \triangle P_1KP$

$OL = x_1$

$P_1K = OM = x_2$

$ON = OL + P_1K$

$= x_1 + x_2$

$KP = MP_2 = y_2$

$NP = NK + KP$

$NK = LP_1 = y_1$

$NP = LP_1 + MP_2$

$NP = y_1 + y_2$

The coordinates of 'P' are $(x_1 + x_2, y_1 + y_2)$ and it represents complex number.

$z = x_1 + x_2 + i\,(y_1 + y_2)$

$\quad = (x_1 + iy_1) + (x_2 + iy_2)$

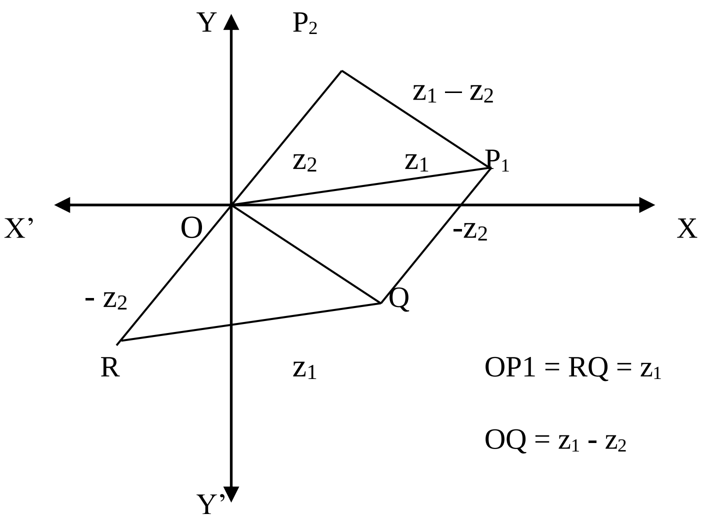
$\quad = z_1 + z_2$

$OP = z_1 + z_2$ (By law of parallelogram)

mod $(z_1 + z_2) = OP$ and amp = $\angle XOP$

---

**GEOMETRIC REPRESENTATION OF ($Z_1$ - $Z_2$)**




$OP1 = RQ = z_1$

$OQ = z_1 - z_2$

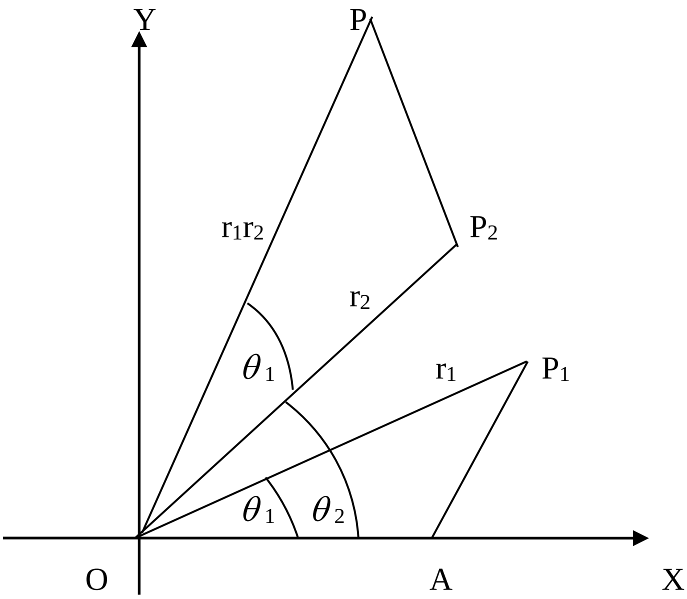
Let $P_1$, $P_2$ represent complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then subtraction of $z_2$ from $z_1$ will be considered as addition of $z_1$& - $z_2$. Produce $P_2Q$ backward to 'R' so that $OR = OP_2$. Coordinates of 'R' are $(-x_2, -y_2)$ and corresponds to complex number

$-z_2 = - x_2 - iy_2$

Complete parallelogram $ORQP_1$. Then sum of $z_1$ and $-z_2$ is represented by OQ.

$z_1 - z_2 = \overrightarrow{OQ} = \overrightarrow{P_2P_1}$ (By law of parallelogram)

## GEOMETRIC REPRESENTATION OF ($Z_1$ * $Z_2$)




Let $P_1$, $P_2$ represent complex numbers

$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

Measure off OA = 1 unit along 'OX'. Construct $\triangle$ $OP_2P$ on '$OP_2$' directly similar to $\triangle$ $OAP_1$

In $\triangle OP_2P$ & $\triangle$ $OAP_1$

$\angle POP_2 = \angle P_1OA = \theta_1$

So that $\dfrac{OP}{OP_1} = \dfrac{OP_2}{OA}$

i.e. $OP * OA = OP_1 * OP_2$

$OP = OP_1 * OP_2 = r_1 * r_2$ (since OA = 1 by construction) and $\angle AOP = \angle AOP_2 + \angle P_2OP$

$$= \angle AOP_2 + \angle AOP_1$$

$$= \theta_2 + \theta_1$$

$\therefore$ P represents the number $r_1 * r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

Hence product of two complex numbers $z_1$, $z_2$ is represented by point 'P' such that

1) $|z_1 z_2| = |z_1 * z_2|$

2) amp $(z_1 z_2)$ = amp $(z_1)$ + amp $(z_2)$

---

**Geometric representation of ($z_1$ / $z_2$)**

Let $P_1$, $P_2$ represent complex numbers

$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

Measure off OA = 1 unit along 'OX'. Construct $\triangle$OAP on 'OA' directly similar to $\triangle OP_2P_1$

So that $\dfrac{OP}{OA} = \dfrac{OP_1}{OP_2} = \dfrac{r_1}{r_2}$ (since OA = 1 by construction)

and $\angle XOP = \angle P_2OP_1$

$$= \angle AOP_1 - \angle AOP_2$$

$$= \theta_1 - \theta_2$$

$\therefore$ P represents the number $r_1 / r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

Hence product of two complex numbers $z_1$, $z_2$ is represented by point 'P' such that

1) $|z_1 / z_2| = |z_1| / |z_2|$

2) amp $(z_1 / z_2)$ = amp $(z_1)$ - amp $(z_2)$

**To find square-root complex number (x + iy)**

Let $\sqrt{x+iy}$ = a + ib where a & b are real numbers. ----------- (1)

Squaring both sides

$\therefore$ x + iy = $(a + ib)^2$

= $a^2 + 2abi + i^2b^2$

= $a^2 - b^2 + 2abi$ (since $i^2 = -1$)

Comparing real and imaginary parts (By आनुरूप्येण sutra)

$a^2 - b^2 = x$ ------------------------------- (2)

2ab = y -------------------------------- (3)

Value of a & b is found by विलोकनम् sutra from (2) and (3).

Putting value of a & b in (1) Square-root of complex number is obtained.

---

**e.g.** Express complex number $\dfrac{1}{3 - 4i}$ in the form of a + *i*b and hence find its modulus and amplitude

Using conjugate concept

Let z = 3 – 4*i* & its conjugate will be $\overline{z}$ = 3 + 4*i*

Multiplying Numerator & Denominator by $\overline{z}$ = 3 + 4*i*

$$\frac{1 * (3+4i)}{(3+4i)\,(3 - 4i)}$$

T (z) = (3, - 4)

T ($\overline{z}$) = (3, 4)

T (z * $\overline{z}$) = [(3*3 – 4*- 4), (3*4 + 3*- 4), ____ ]

= [(9 + 16), (12 – 12), _____]

= (25, 0, _____)

$(z * \overline{z}) = 25 + 0i$

$= 25$

$$\frac{1}{3-4i} = \frac{3 + 4i}{25}$$

$$\frac{1}{3-4i} = \frac{3}{25} + \frac{4}{25} i$$

$z = \sqrt{a^2 + b^2}$ a = 3/25 & b = 4/25

$= \sqrt{(3/25)^2 + (4/25)^2}$

$z = \sqrt{(9/625) + (16/625)}$

$= \sqrt{(25/625)}$

$= (1/5)$

Modulus (z) = 1/5 = 0.2

$\tan\theta = b/a$

$$= \frac{(4/25)}{(3/25)}$$

$\tan\theta = 4/3$

$\theta = \tan^{-1}(4/3)$

Amplitude ($\theta$) = $\tan^{-1}(4/3)$

---

# CUBE – ROOT OF UNITY

Show that $w = \dfrac{-1 + \sqrt{3}\,i}{2}$ is a cube root of unity

Let $w = \sqrt[3]{1}$ then $w^3 = 1$

Transposing by पशावर्त्य योजयेत् sutra

$w^3 - 1 = 0$

$(w - 1)(w^2 + w + 1) = 0$

$\therefore\ (w - 1) = 0$ or $(w^2 + w + 1) = 0$

$\therefore\ w = 1$ or $w = \dfrac{-1 \pm \sqrt{-3}}{2}$

Thus cube roots of Unity are 1, $\dfrac{-1 + \sqrt{-3}}{2}$ and $\dfrac{-1 - \sqrt{-3}}{2}$

---

**Exercise: -**

Using triplets of complex number evaluate the following where $i = \sqrt{-1}$

1. $(6 + 7\,i) \times (4 - 3\,i)$
2. $(2 + 11\,i) \times (1 + 3\,i)$

Find $Z_1 * Z_2$ if $Z_1 = 2 + 3\,i$ $Z_2 = 1 - 2\,i$ where $i = \sqrt{-1}$

Find $(Z_1 + Z_2)$ and $(Z_1 - Z_2)$ if $Z1 = 3 - 4\,i$ $Z_2 = 1 + i$ where $i = \sqrt{-1}$

Find modulus and amplitude of a complex number $1 + \sqrt{3}\,i$

Put $\dfrac{1}{3 - 4\,i}$ in the form of a + ib and hence find its modulus and amplitude.

Express the number $\dfrac{1 + i}{2 - 3\,i}$ in the form of a + ib

Find reciprocal of the complex number 7 + 4 i

Find positive square root of 7 + 24 i

If x = 1 + i, find the value of x2 - 2x + 3

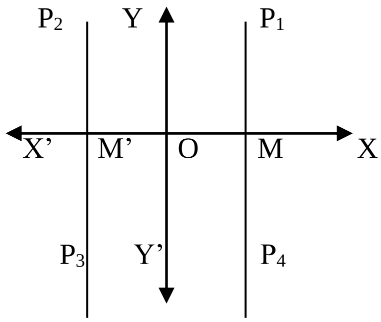
Find the value of $i^{16} - i^{20} + i^{24}$ where $i = \sqrt{-1}$

Find the square root of 15 + 8 i

Value of sum $i + i^2 + i^3 + \ldots\ldots\ldots\ldots + i^{10}$

Show that $w = \dfrac{-1 + \sqrt{3}\,i}{2}$ is a cube root of unity

# CHAPTER TWENTY TWO: COORDINATE GEOMETRY




Line XX' $\longrightarrow$ 'X' axis

Line YY' $\longrightarrow$ 'Y' axis

Two together is called the axes of Coordinates.

'O' $\longrightarrow$ origin

OX $\longrightarrow$ + ve quantity

OX' $\longrightarrow$ - ve quantity

OY $\longrightarrow$ + ve quantity

OY' $\longrightarrow$ - ve quantity

From any point in the plane a straight line parallel to 'Y' axis and meeting 'X' axis at 'M' is drawn.

Distance OM $\longrightarrow$ Abscissa of point '$P_1$'.Distance $MP_1$ $\longrightarrow$ Ordinate of point '$P_1$'. Abscissa and Ordinate together is called coordinates of point '$P_1$'. Distance measured parallel to 'X' axis is called 'x' with or without suffix and distance measured parallel to 'Y' axis is called 'y' with or without suffix.

$MP_1$ = 'y'

OM = 'x'

When coordinates of a point '$P_1$' is known then its position from origin is known.

OM = 'x' units and $MP_1$ = 'y' units

Then coordinates of point '$P_1$' = (x, y)

OM' = '- x' units and $M'P_2$= 'y' units

Then coordinates of point '$P_2$' = (-x, y)

OM' = '- x' units and $M'P_3$= '- y' units

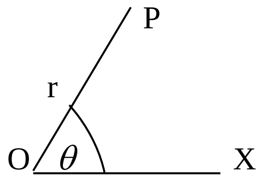
Then coordinates of point '$P_3$' = (-x, - y)

OM = 'x' units and $MP_4$ = '- y' units

Then coordinates of point '$P_4$' = (x, - y)

This system of coordinates is known as "Cartesian System"

<ins>**POLAR COORDINATE**</ins>



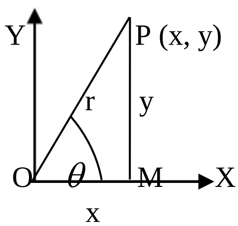

Let 'O' be the fixed point called origin or pole and 'OX' a fixed line called initial line. Position of point 'P' is clearly known when $\angle$XOP and length 'OP' are known. Two taken together is called 'Polar Coordinates of point 'P'.

$\angle$XOP = $\theta$ and distance 'OP' = 'r'

$\therefore$ Polar coordinates of point 'P' = (r, $\theta$)

---

**To change from Cartesian coordinates to polar coordinates and vice-versa.**




Cartesian coordinates of point 'P' = (x, y)

Polar coordinates of point 'P' = (r, $\theta$)

'O' is origin or pole.

'OX' initial line.

Draw 'PM' perpendicular to 'OX'.

$\therefore$ OM = x

PM = y

OP = r

$\angle$MOP = $\theta$

In $\Delta$ MOP

$\cos \angle MOP = (OM / OP) = (x / r) = \cos \theta$

$x = r \cos \theta$

$OM = x = OP * \cos \angle MOP = r \cos \theta$

$\sin \angle MOP = (PM / OP) = (y / r)$

$y = r \sin \theta$

$MP = y = OP * \sin \angle MOP = r \sin \theta$

But

$OP^2 = OM^2 + MP^2$

$r^2 = x^2 + y^2$

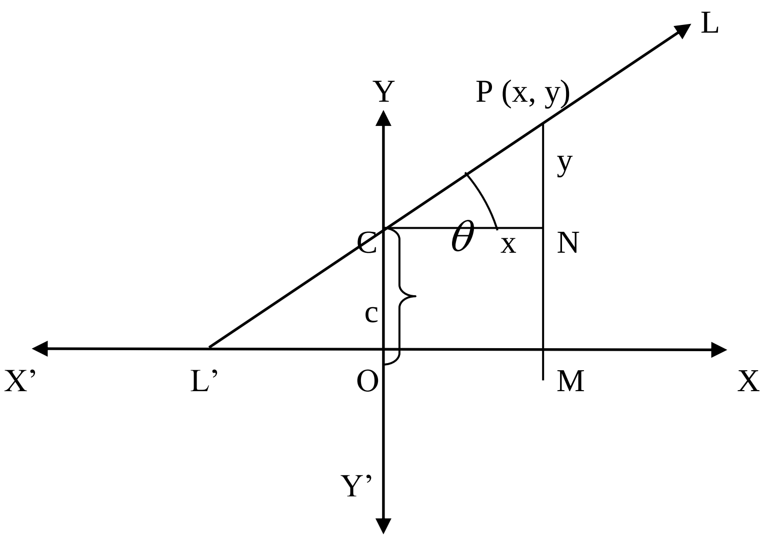
$\therefore\ r = \sqrt{x^2 + y^2}$

$\tan \theta = (y/x) = (MP/OM)$

| | |
|---|---|
| $x = r \cos \theta$ | Cartesian coordinates are expressed |
| $y = r \sin \theta$ | in terms of Polar coordinates. |

| | |
|---|---|
| $r = \sqrt{x^2 + y^2}$ | Polar coordinates are expressed |
| $\tan \theta = (y/x)$ | in terms of. Cartesian coordinates |

**Equation of a straight line which cuts off a given intercept on 'Y' axis and is inclined at a given angle to 'X' axis.**




OC $\longrightarrow$ 'c' intercept on 'Y' axis.

$\theta \longrightarrow$ angle subtended with 'X' axis.

Through 'C' draw straight line LCL' inclined at $\angle \theta$ to 'X' axis.
$\therefore$ tan$\theta$ = 'm' = slope of line LL'
$\therefore$ LCL' is a straight line whose equation is to be found.

Draw 'PM' perpendicular to' OX' and a line parallel to 'OX' from 'C' to meet 'PM' at 'N'.

$\therefore$ OM = x

MP = y

MP = MN + NP

MN = OC = c (since opposite sides of rectangle are equal)

CN = OM = x (since opposite sides of rectangle are equal)

$\therefore$ tan$\theta$ = NP / CN

$\therefore$ NP = CN * tan $\theta$

= CN * m (since tan $\theta$ = 'm')

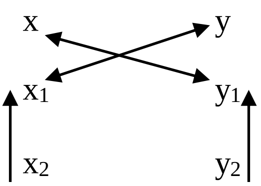
NP = m * CN
NP = m * x (since CN = x)
$\therefore$ MP = c + mx (since MN = c)
$\therefore$ y = mx + c (since MP = y) ------------- (General equation of a straight line)
When a straight line passes through origin 'O' i.e. its intercept on 'Y' axis is zero and equation becomes "y = mx".

**Gradient: -** It is the slope of given straight line. A triplet (x, y, z) will have gradient = (y / x) which is the ratio of 2$^{nd}$ element to 1$^{st}$ element.

## Equation of a straight line passing through two points

General equation of a straight line $\longrightarrow$ $y = mx + c$

Write coordinates of two points one below the otherCoefficients of x, y and constant are determined by applying ऊर्ध्वतिर्यग्भ्याम् sutra.

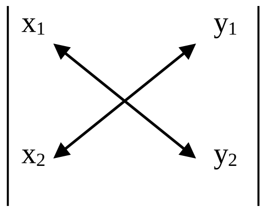



Coefficient of ‘y’ = $(x_2 - x_1)$

Coefficient of ‘x’ = $(y_2 - y_1)$

Constant ‘c’ = $(x_2y_1 - y_2x_1)$

c =



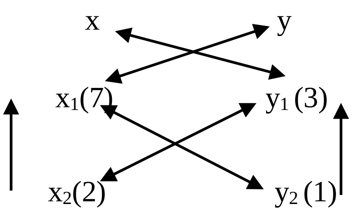

Putting value of Coefficient of ‘y’, Coefficient of ‘x’ & Constant ‘c’ in general equation of a straight line.

$\therefore\ (x_2 - x_1)\, y = (y_2 - y_1)\, x + (x_2y_1 - y_2x_1)$

---

**E.g. Find the equation of line joining the points (7, 3) and (2, 1)**

Applying ऊर्ध्वतिर्यग्भ्याम् sutra.




$(2 - 7)\, y = (1 - 3)\, x + (2*3 - 1*7)$

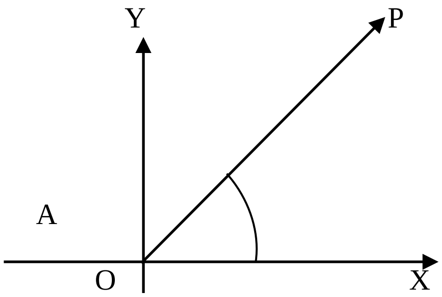
$-5y = -2x - 1$

$5y = 2x + 1$

Transposing (परावर्त्य योजयेत्)

$2x - 5y + 1 = 0$ ----------- (Required equation of a line)

## EQUATION OF A STRAIGHT LINE MAKING A REQUIRED ANGLE TO THE GIVEN LINE




"**OP**' is a straight line making $\angle A$ with 'X' axis. When required angle is subtended through a given point in anti-clockwise direction of line then addition of triplets of line & point is performed to get equation of required line. When required angle is subtended through a given point in clockwise direction of line then subtraction of triplets of line & point is performed to get equation of required line. Only first two elements of triplet are require to determine equation of straight line hence need not write third element of triplet.

---

**E.g. Find equation of a straight line through (4, 1) making an angle $45^0$ to the line**

**y =** $3x + 4$

As the line makes $\angle 45^0$ in anti-clockwise direction of line $y = 3x + 4$. Addition of line & angle is performed.

Triplet of line (l) $\longrightarrow$ (1, 3, ----)

+

Triplet of $\angle 45^0$ (A) $\longrightarrow$ (1, 1, ----)

---

T (l + A) = (1*1 – 3*1, 1*3 + 1*1, ---------)

= (- 2, 4, ----------)

$\therefore$ Equation of a straight line $\longrightarrow$ $-2y = 4x + c$ --------------- (1)

Value of constant 'c' is to be obtained as line passes through (4, 1) & putting these coordinate values in equation (1)

$\therefore$ -2* 1 = 4*4 + c

-2 = 16 + c

$\therefore$ c = -18

$\therefore$ Equation of a straight line -2y = 4x – 18

Dividing both sides by 2

-y = 2x – 18

Transposing (**परावर्त्य योजयेत**)

2x + y – 18 = 0 (Required equation of a straight line)

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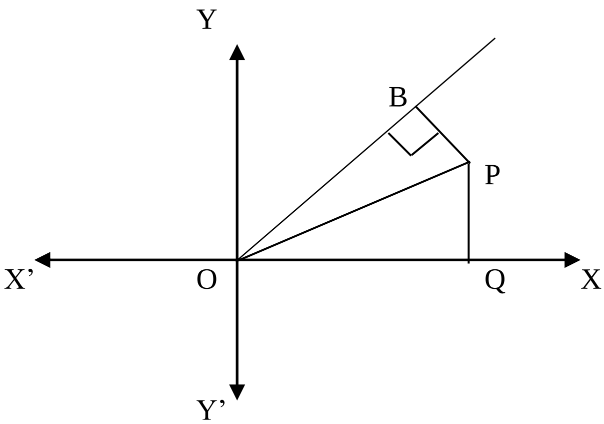
## LENGTH OF PERPENDICULAR




When a distance from a given point on given line is require then triplets of line and point are involved in determination of perpendicular distance on the line from given point.

The difference between triplets of line i.e. $\angle$BOQ and point i.e. $\angle$POQ will determine triplet for $\angle$BOP from which the perpendicular distance on given line from given point can be determined. Perpendicular distance is calculated by dividing 2nd element of new triplet with 3rd element of line triplet. As it is a distance its value will be modulus i. e. always '+ve'.

OP is the common hypotenuse of $\Delta$ POQ and $\Delta$ BOP

---

**e.g.** **Find the length of perpendicular from point (3, 4) on the line 2y = x**

T (line) = T (l) = (2, 1, $\sqrt{5}$

T (point) =T (p) = (3, 4, -------)

T (l – p) = (2*3 + 1*4, 3*1 – 4*2, -----------)

T (l – p) = (10, -5, -------)

Divide 2nd element of new triplet i.e. (- 5) by 3rd element of line triplet i.e. $\sqrt{5}$ to get length of perpendicular. As it is distance it is always '+ve'.

$$\therefore \ p = \left| \frac{-5}{\sqrt{5}} \right|$$

p = $\sqrt{5}$ units

---

**e.g.** **Find the length of perpendicular from point (-2, 3) on the line 3x – 4y – 5 = 0**

Equation of line → 4y = 3x – 5

Coordinates of point = (-2, 3)

As intercept of line on 'Y' axis is (-5) hence it is not passing through origin.

If x = 3 is put in the equation then y = 1 i.e. point (3, 1) lies on the line. Transpose origin to (3, 1) applying **'परावर्त्य योजयेत्**sutra.

$\therefore$ Point (-2, 3) is transposed to (-2-3, 3-1) i.e. (-5, 2)

Equation of line → 4y = 3x – 5

New coordinates of point = (-5, 2)

T (l) = (4, 3, 5)

T (p) = (-5, 2, -----)

---

T (l – p) = (4*-5 + 3*2, -5*3 – 2*4, -------)

= (-14, -23, --------)

Length of perpendicular (p) = $\left|\dfrac{-23}{5}\right|$

$$p = (23 / 5) \text{ units}$$

## <u>ANGLE BETWEEN TWO LINES</u>

By subtracting triplets of two given lines, triplet for angle between these lines is obtained. As slope (tan $\theta$ or m) is considered in obtaining angle between these lines, the absolute terms are ignored. If value of slope (tan $\theta$ or m) is a positive quantity then angle between two lines is acute angle, if value of slope (tan $\theta$ or m) is a negative quantity then angle between two lines is obtuse angle. If value of slope (tan $\theta$ or m) is ($\infty$) then angle between two lines is a right angle.

**e.g. Find the angle between lines**

$3x - y + 2 = 0$ and $3y + x = 7$

Equations are brought to standard form y = mx + c

Transposing (**पशावर्त्य योजयेत्** sutra)

($l_1$) $y = 3x + 2$ --------------------------- (1)

($l_2$) $3y = -x + 7$ --------------------------- (2)

Applying **उर्ध्वतिर्यग्भ्याम्** sutra

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| T ($l_1$) | 1 | 3 | $\sqrt{10}$ | 1 | 3 | $\sqrt{10}$ | 1 | 3 | $\sqrt{10}$ |
| T ($l_2$) | 3 | -1 | $\sqrt{10}$ | 3 | -1 | $\sqrt{10}$ | 3 | -1 | $\sqrt{10}$ |

T ($l_{1}$ - $l_2$) = [(1*3 + 3*-1), (3*3 – 1*-1), 10)]

= (0, 10, 10)

Dividing by 10

T ($l_{1}$ - $l_2$) = (0, 1, 1)

tan ($l_{1}$ - $l_2$) = 1/0 = $\infty$

($l_{1}$ - $l_2$) = $\tan^{-1}(\infty)$ = $90^0$ Angle between two lines = $90^0$

**EXERCISE**–

Using triplets of a line find equation of line passing through following two points.

(1) (0, 0) & (4, 5)  (2) (2,-1) & (-2,-3)  (3) (5, 12) & (15, 8)

Find the equation of line passing through a point (3, -2) and parallel to y-axis.

Find the equation of a line parallel to x-axis and passing through a point (3,4)

Find an equation of a line passing through origin and making angle of $30^0$ with x-axis

Find acute angle between the lines $x + y – 5 = 0$ and $2x – y + 7 = 0$

Find the distance between two parallel lines $x + y + 5 = 0$ and $3x + 3y + 11 = 0$

Show that lines $3y = 4x -3$ and $12y = -9x + 4$ are perpendicular

Show that lines $4x + 3y = 4$ and $16x + 12y = 7$ are parallel

Find the length of perpendicular from a point (2, 3) on line $y = 4x$

Find the length of perpendicular from origin on line $3x + 4y + 5 = 0$

The acute angle made by the line $x + y = 9$ with x-axis.

Find the equation of a line joining origin and a point (2, -5)

---

General Equation of a line => $Ax +By+ C = 0$

$By = - Ax –C$

Dividing by 'B'

$y = - A/B *x – C/B$

But $y = mx +c$

$m = - A/B$ and $c = - C/B$

The equation $Ax +By+ C = 0$ represents a straight line cutting off an intercept $– C/B$ from 'Y' axis and inclined at an angle $\tan^{-1}(- A/B)$ to 'X' axis.

If intercept on 'x' axis is '0' then y = constant

$y = 0x +c$

$1 = 0 + 1$

T (l) parallel to 'x' axis = (1, 0, 1)

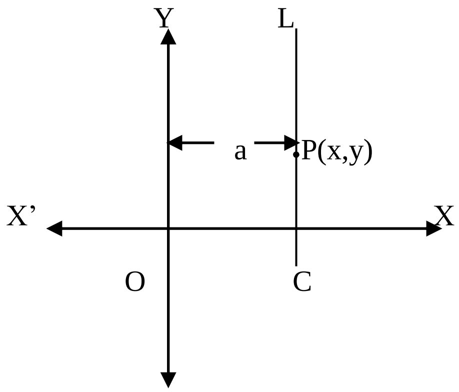
If intercept on 'y' axis is '0' then x = constant

$0 = -1 + 1$

T (l) parallel to 'y' axis = (0, -1, 1)

## EQUATION OF A STRAIGHT LINE WHICH IS PARALLEL TO ONE OF THE COORDINATE AXES.

### 1. Straight line parallel to 'Y' axis




Let CL be any straight line parallel to 'Y' axis and passing through a point 'C' on 'X' axis such that OC = 'a'.

Let 'P' be any point on this line whose coordinates are (x,y).

Distance of point 'P' on 'X' axis is always 'a'.

$\therefore$ x = a ----------------------------- (1)

This is true for every point on line CL (produced infinitely both sides). The equation does not contain coordinate 'Y'.

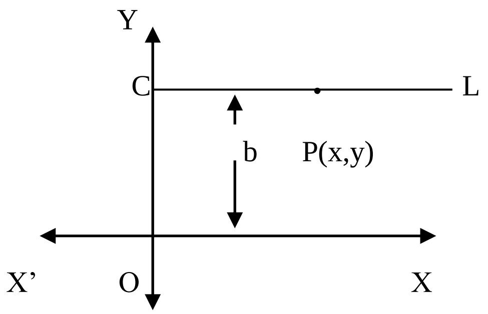
$\therefore$ Equation of straight line parallel to 'Y' axis

$$1x + 0y + 0 = 0$$

$\therefore$ 0y = -1x + 0

$\therefore$ $T(L_Y) = (0. -1, 1)$

**Straight line parallel to 'X' axis**




Let CL be any straight line parallel to 'X' axis and passing through a point 'C' on 'Y' axis such that OC = 'b'.

Let 'P' be any point on this line whose coordinates are (x,y)'

∴ Distance of point 'P' on 'Y' axis is always 'b'. This is true for every point on line 'CL' (produced infinitely both sides).

The equation does not contain coordinate 'x'.

Equation of straight line parallel to 'X' axis.

$0x + 1y + 0 = 0$

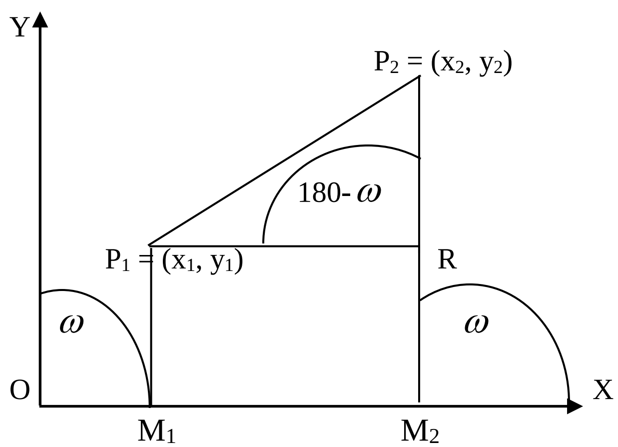
∴ $1y = 0x + 0$

∴ $T(L_X) = (1, 0, 1)$

---

**TO FIND THE DISTANCE BETWEEN TWO POINTS WHOSE COORDINATES ARE GIVEN.**

$P_1 = (x_1, y_1)$
$P_2 = (x_2, y_2)$



$OM_1 = x_1$ $\quad\quad$ $M_1P_1 = y_1$

$OM_2 = x_2$ $\quad\quad$ $M_2P_2 = y_2$

$P_1R = M_1M_2$ ($\because$ Opposite sides of rectangle $P_1M_1M_2R$)

$P_1R = OM_2 - OM_1 = (x_2 - x_1)$

$RP_2 = (M_2P_2 - M_2R) = (M_2P_2 - M_1P_1)$ ($\because M_2R = M_1P_1$)

($\because$ Opposite sides of rectangle$P_1M_1M_2R$)

$\therefore$ $RP_2 = y_2 - y_1$

$\angle P_1RP_2 = \angle OM_2P_2 = (180^0 - \angle P_2M_2X) = (180^0 - \omega)$ ($\because \angle YOM_1 = \angle P_2M_2X$)

($\because$ Corresponding angles)

$\therefore$ By Law of Parallelogram

$(P_1P_2)^2 = (P_1R)^2 + (RP_2)^2 - 2\, P_1R * RP_2 * \cos(180^0 - \omega)$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2\,(x_2 - x_1) * (y_2 - y_1) \cos \omega$

When $\angle \omega = 90^0$ $\quad$ $\cos 90^0 = 0$

$\therefore (P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2\,(x_2 - x_1) * (y_2 - y_1) * 0$

$\therefore (P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 0$

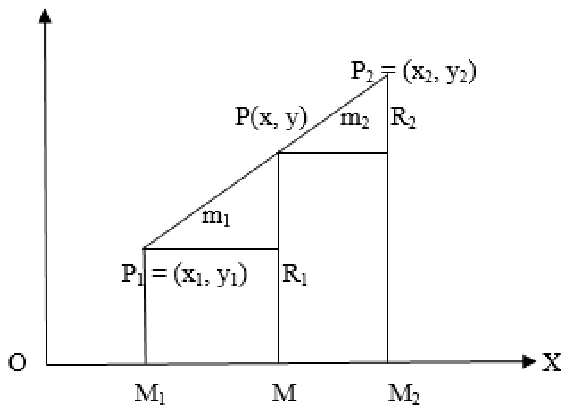
$\therefore (P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\therefore (P_1P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

---

**To find the coordinates of the point which internally divides the line joining two given points $(x_1, y_1)$ and $(x_2, y_2)$ in given ratio $(m_1 : m_2)$**

$P_1 = (x_1, y_1)$
$P_2 = (x_2, y_2)$



$P_1P : PP_2 = m_1 : m_2$

$OM_1 = x_1$ $\quad$ $M_1P_1 = y_1$

$OM_2 = x_2$ $\quad$ $M_2P_2 = y_2$

$OM = x MP = y$

$P_1P = m_1$ $\quad$ $PP_2 = m_2$

$P_1R_1 = M_1M$ $\quad$ ($\because$ Opposite sides of rectangle$P_1M_1MR_1$)

$P_1R_1 = OM – OM_1 = (x – x_1)$ -------------- (1)

$PR_2 = MM_2$ $\quad$ ($\because$ Opposite sides of rectangle$PMM_2R_2$)

$PR_2 = OM_2 – OM = (x_2 – x)$ ----------- (2)

$PR_1 = MP – MR_1 = MP – M_1P_1$ ($\because MR_1 = M_1P_1 \because$ opposite sides of rectangle $P_1M_1MR_1$)

$\therefore$ $PR_1 = (y – y_1)$ ----------------- (3)

$P_2R_2 = M_2P_2 – M_2R_2 = M_2P_2 – MP$ = ($M_2R_2 = MP$ $\because$ opposite sides of rectangle $PMM_2R_2$)

$\therefore$ $PR_1 = (y_2 – y)$ ----------------- (4)

$\Delta\, P_1R_1P$ and $\Delta\, PR_2P_2$ are similar.

$\therefore (P_1P / PP_2) = (P_1R_1 / PR_2) = (m_1 / m_2)$

From (1) and (2)

$$\therefore \frac{m_1}{m_2} = \frac{(x – x_1)}{(x_2 – x)}$$

$m_1(x_2 – x) = m_2(x – x_1)$

$(m_1x_2 – m_1 x) = (m_2x - m_2x_1)$

$\therefore (m_1x_2 + m_2x_1)= (m_1x + m_2x)$

$\therefore x (m_1 + m_2) = (m_1x_2 + m_2x_1)$

$$\therefore x = \frac{(m_1x_2 + m_2x_1)}{(m_1 + m_2)}$$

$(P_1P / PP_2) = (m_1 / m_2) = (PR_1 / P_2R_2)$

$\therefore \ (m_1 / m_2) = (PR_1 / P_2R_2)$

From (3) and (4)

$$\therefore \frac{m_1}{m_2} = \frac{(y - y_1)}{(y_2 - y)}$$

$m_1 (y_2 - y) = m_2(y - y_1)$

$(m_1y_2 - m_1y) = (m_2y - m_2y_1)$

$\therefore (m_1y_2 + m_2y_1) = (m_1y + m_2y)$

$\therefore y (m_1 + m_2) = (m_1y_2 + m_2y_1)$

$$\therefore y = \frac{(m_1y_2 + m_2y_1)}{(m_1 + m_2)}$$

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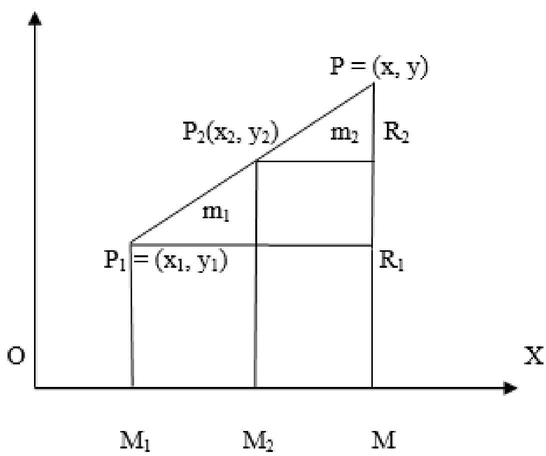
$$\begin{array}{ccc} x & & y \\ x_1 & m_1 & y_1 \\ x_2 & m_2 & y_2 \end{array}$$

$$\therefore x = \frac{(m_1x_2 + m_2x_1)}{(m_1 + m_2)}$$

---

**To find the coordinates of the point which externally divides the line joining two given points ($x_1$, $y_1$) and ($x_2$, $y_2$) in given ratio ($m_1$: $m_2$)**




$P_1 = (x_1, y_1)$
$P_2 = (x_2, y_2)$

$P_1P: P_2P = m_1: m_2$

$OM_1 = x_1$ $M_1P_1 = y_1$

$OM_2 = x_2$ $M_2P_2 = y_2$

$OM = x$ $MP = y$

$P_1P = m_1$ $P_2P = m_2$

$P_1R_1 = M_1M$ ($\because$ Opposite sides of rectangle$P_1M_1MR_1$)

$P_1R_1 = OM - OM_1 = (x - x_1)$ -------------- (1)

$P_2R_2 = M_2M$ ($\because$ Opposite sides of rectangle$P_2M_2MR_2$)

$P_2R_2 = OM - OM_2 = (x - x_2)$ ----------- (2)

$R_1P = MP - MR_1 = MP - M_1P_1$ ($\because MR_1 = M_1P_1 \because$ opposite sides of rectangle $P_1M_1MR_1$)

$\therefore$ $R_1P = (y - y_1)$ ----------------- (3)

$R_2P = MP - MR_2 = MP - M_2P_2 =$ ($MR_2 = M_2P_2 \because$ opposite sides of rectangle $P_2M_2MR2$)

$\therefore$ $R_2P = (y - y_2)$ ----------------- (4)

$\Delta\, P_1R_1P$ and $\Delta\, P_2R_2P$ are similar.

$\therefore (P_1P / P_2P) = (P_1R_1 / P_2R_2) = (m_1 / m_2)$

From (1) and (2)

$$\therefore \frac{m_1}{m_2} = \frac{(x - x_1)}{(x - x_2)}$$

$m_1(x - x_2) = m_2(x - x_1)$

$(m_1x - m_1 x_2) = (m_2x - m_2x_1)$

$\therefore (m_1x - m_2x) = (m_1x_2 - m_2x_1)$

$\therefore x\,(m_1 - m_2) = (m_1x_2 - m_2x_1)$

$$\therefore x = \frac{(m_1x_2 - m_2x_1)}{(m_1 - m_2)}$$

$(P_1P / P_2P) = (m_1 / m_2) = (R_1P / R_2P)$

$\therefore\ (m_1 / m_2) = (R_1P / R_2P)$

From (3) and (4)

$$\therefore\ \frac{m_1}{m_2} = \frac{(y - y_1)}{(y - y_2)}$$

$m_1\,(y - y_2) = m_2(y - y_1)$

$(m_1y - m_1y_2) = (m_2y - m_2y_1)$

$\therefore\ (m_1y - m_2y) = (m_1y_2 + m_2y_1)$

$\therefore\ y\,(m_1 - m_2) = (m_1y_2 - m_2y_1)$

$$\therefore\ y = \frac{(m_1y_2 - m_2y_1)}{(m_1 - m_2)}$$

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$$\begin{array}{ccc} x & & y \\ x_1 & m_1 & y_1 \\ x_2 & m_2 & y_2 \end{array}$$

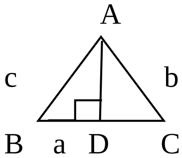
$$\therefore\ x = \frac{(m_1x_2 - m_2x_1)}{(m_1 - m_2)}$$

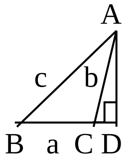
$$\therefore\ y = \frac{(m_1y_2 - m_2y_1)}{(m_1 - m_2)}$$

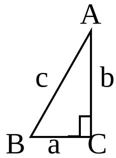
# CHAPTER TWENTY THREE: SOLUTION OF TRIANGLES

**In any $\Delta$ ABC**

**(Sin A / a) = (Sin B / b) = (Sin C / c)**




From ‘A’ AD is drawn perpendicular to BC.

In $\Delta$ ABD

(AD / AB) = Sin B

$\therefore$ AD = AB * Sin B = c * Sin B ------------ (1)

In $\Delta$ ACD

(AD / AC) = Sin C

$\therefore$ AD = AC * Sin C = b * Sin C ------------ (2)

From (1) and (2)

c * Sin B = b * Sin C

(Sin B / b) = (Sin C / c) --------------- (3)

If $\angle C$ is obtuse

$\therefore$ In $\Delta$ ACD

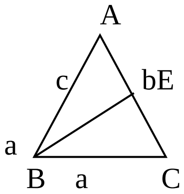
(AD / AC) = Sin ($180^0$ – C) = Sin C

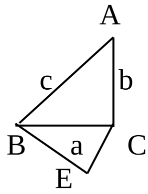
$\therefore$ AD = AC * Sin C = b * Sin C ------------- (4)

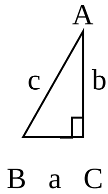
From (1) and (4)

c * Sin B = b * Sin C

(Sin B / b) = (Sin C / c) ------------ (5)




In similar manner from ‘**B**’ BE is drawn perpendicular to AC

In $\Delta$ ABE

(BE / AB) = Sin A

$\therefore$ BE = AB * Sin A = c * Sin A ----------- (6)

In $\Delta$ BCE

(BE / BC) = Sin C

$\therefore$ BE = BC * Sin C = a * Sin C ----------- (7)

From (6) and (7)

c * Sin A = a * Sin C

(Sin A / a) = (Sin C / c) --------------- (8)

From (5) and (8)

(Sin A / a) = (Sin B / b) = (Sin C / c)

In right angled $\Delta$ ABC

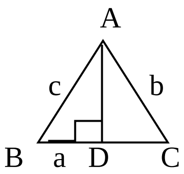
Sin A = (a / c)

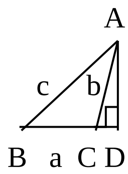
Sin B = (b / c)

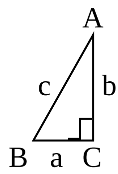
Sin C = (b / b) = 1

(Sin A / a) = (Sin B / b) = (1 / c) = (Sin C / c)

**IN ANY TRIANGLE TO FIND COSINE OF AN ANGLE IN TERMS OF THE SIDES.**




From ‘A’ AD is drawn perpendicular to BC

In $\Delta$ ABD

$AB^2 = AD^2 + BD^2$ ------------ (1)

In $\Delta$ ADC

$AC^2 = AD^2 + CD^2$

$\therefore\ AD^2 = AC^2 - CD^2$

Substituting this value of $AD^2$ in (1)

$AB^2 = AC^2 - CD^2 + BD^2$ ------------ (2)

BC = BD + CD

$\therefore\ BD = BC – CD = (a – CD)$

$BD^2 = (a – CD)^2$

$BD^2 = (a^2 – 2aCD + CD^2)$

Substituting this value of $BD^2$ in (2)

$\therefore\ AB^2 = AC^2 - \cancel{CD}^2 + a^2 – 2aCD + \cancel{CD}^2$

$AB^2 = AC^2 + a^2 – 2aCD = b^2 + a^2 – 2a\ CD$ -------- (3)

(CD / AC) = Cos C

$\therefore$ CD = AC * Cos C = b * Cos C

Substituting this value of CD in (3)

$\therefore\ AB^2 = b^2 + a^2 – 2ab \text{ Cos } C$

$\therefore\ c^2 = a^2 + b^2 – 2ab \text{ Cos } C$

$a^2 + b^2 - c^2 = 2ab \text{ Cos } C$

$$\therefore \text{ Cos } C = \frac{(a^2 + b^2 - c^2)}{2ab}$$

If $\angle C$ is obtuse

In $\Delta$ ABD

$AB^2 = AD^2 + BD^2$

BD = (BC + CD)

$\therefore AB^2 = AD^2 + (BC + CD)^2$

$AB^2 = AD^2 + BC^2 + CD^2 + 2 BC^* CD$ ---------- (4)

In $\Delta$ ACD

$AC^2 = CD^2 + AD^2$

Substituting this value of $CD^2 + AD^2$ in (4)

$AB^2 = AC^2 + BC^2 + 2 BC^* CD$ ------------------- (5)

In $\Delta$ ACD

$(CD / AC) = \text{Cos } (180^0 - C) = - \text{Cos } C$

$\therefore CD = AC * (-\text{Cos } C) = - b * \text{Cos } C$

Substituting this value of CD in (5)

$AB^2 = AC^2 + BC^2 + 2 BC^* (-b \text{ Cos } C)$

$\therefore AB^2 = a^2 + b^2 - 2ab \text{ Cos } C$

$\therefore c^2 = a^2 + b^2 - 2ab \text{ Cos } C$

$\therefore 2ab \text{ Cos } C = a^2 + b^2 - c^2$

$$\therefore \text{ Cos } C = \frac{a^2 + b^2 - c^2}{2ab}$$

In similar manner it can be proved that

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{Cos B} = \frac{a^2 + c^2 - b^2}{2ac}$$

If $\angle C = 90^0$

$$\text{Cos C} = \frac{a^2 + b^2 - c^2}{2ab}$$

But Cos $90^0 = 0$

$$\therefore\ 0 = \frac{a^2 + b^2 - c^2}{2ab}$$

Cross multiplying

$a^2 + b^2 - c^2 = 0$

$\therefore\ a^2 + b^2 = c^2$

**Cos A = Cos (A/2)*Cos (A/2) -Sin (A/2) * Sin (A/2) = Cos$^2$ (A/2) –Sin$^2$ (A/2) =1 – 2 Sin$^2$ (A/2)**

**[$\because$ Sin$^2$ (A/2) + Cos$^2$ (A/2) = 1]**

$\therefore$ 2 Sin$^2$ (A/2) = 1 - Cos A

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore\ 2\,\text{Sin}^2\,(A/2) = 1 - \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}$$

$$\therefore\ 2\,\text{Sin}^2\,(A/2) = \frac{(a + b - c) * (a - b + c)}{2bc}$$

$2s = a + b + c$

$\therefore\ s = [(a + b + c) / 2]$

$$\therefore 2 \sin^2 (A/2) = \frac{[(a + b + c) - 2c] * [(a + b + c) - 2b]}{2bc}$$

$$= \frac{(2s - 2c) * (2s - 2b)}{2bc}$$

$$= \frac{\cancel{2} (s - c) * 2(s - b)}{\cancel{2}bc}$$

$$\therefore \cancel{2} \sin^2 (A/2) = \frac{\cancel{2} (s - c) * (s - b)}{bc}$$

$$\therefore \sin^2 (A/2) = \frac{(s - c) * (s - b)}{bc}$$

$$\therefore \sin(A/2) = \frac{\sqrt{(s-b)(s-c)}}{\sqrt{bc}}$$

Similarly

$$\sin(B/2) = \frac{\sqrt{(s-a)(s-c)}}{\sqrt{ac}}$$

$$\sin(C/2) = \frac{\sqrt{(s-a)(s-b)}}{\sqrt{ab}}$$

---

**To find cosine of half angle in terms of the sides**

**$\cos A = \cos (A/2)*\cos (A/2) - \sin (A/2) * \sin (A/2) = \cos^2 (A/2) - \sin^2 (A/2) = 2 \cos^2 (A/2) - 1$**

**$[\because \sin^2 (A/2) + \cos^2 (A/2) = 1]$**

$2 \cos^2 (A/2) = 1 + \cos A$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore 2\,\mathrm{Cos}^2(A/2) = 1 + \frac{(b^2 + c^2 - a^2)}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b^2 + 2bc + c^2) - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$\therefore 2\,\mathrm{Cos}^2(A/2) = \frac{(b + c + a) * (b + c - a)}{2bc}$$

$2s = a + b + c$

$\therefore\ s = [(a + b + c) / 2]$

$$\therefore 2\,\mathrm{Cos}^2(A/2) = \frac{(a + b + c) * [(a + b + c) - 2a]}{2bc}$$

$$= \frac{2s * (2s - 2a)}{2bc}$$

$$= \frac{\cancel{2}s * 2(s - a)}{\cancel{2}bc}$$

$$\therefore \cancel{2}\,\mathrm{Cos}^2(A/2) = \frac{\cancel{2}\,s * (s - a)}{bc}$$

$$\therefore \mathrm{Cos}^2(A/2) = \frac{s * (s - a)}{bc}$$

$$\therefore \mathrm{Cos}(A/2) = \frac{\sqrt{s(s-a)}}{\sqrt{bc}}$$

Similarly

$$\text{Cos}(B/2) = \frac{\sqrt{s(s-b)}}{\sqrt{ac}}$$

$$\text{Sin}(C/2) = \frac{\sqrt{s(s-c)}}{\sqrt{ab}}$$

---

**To find tangents of half angles in term of the sides.**

Tan (A/2) = [(Sin A/2) $\div$ (Cos A/2)]

$$= \frac{\dfrac{\sqrt{(s-b)(s-c)}}{\cancel{\sqrt{bc}}}}{\dfrac{\sqrt{s(s-a)}}{\cancel{\sqrt{bc}}}}$$

$$= \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}$$

Similarly,

$$\text{Tan}\ (B/2) = \frac{\sqrt{(s-a)(s-c)}}{\sqrt{s(s-b)}}$$

$$\text{Tan}\ (C/2) = \frac{\sqrt{(s-a)(s-b)}}{\sqrt{s(s-c)}}$$

---

In a triangle $\angle A \langle 180^0$

$\therefore \angle A/2 \langle 90^0$

$\therefore$ Sine, Cosine, Tangent of (A/2) are always positive.

---

**To express Sine of angle of a triangle in terms of the sides**

Sin A = 2 Sin (A/2) * Cos (A/2)

$$= \frac{2\sqrt{(s-b)(s-c)}}{\sqrt{bc}} * \frac{\sqrt{s(s-a)}}{\sqrt{bc}}$$

$$= \frac{2\sqrt{(s-b)(s-c)} * \sqrt{s(s-a)}}{bc}$$

$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Similarly

$$\text{Sin B} = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{Sin C} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

## CHAPTER TWENTY FOUR: AWARENESS OF 12 TTO 16 VEDIC SUTRA

**12. <u>SHESHANYANKEN CARAMENA</u>**(Remainder by last digit)

If we take any remainders and multiply it by caramen (i.e. last digit) then last Digit of the product is actually the quotient at that step. The sutra here is **Sheshanyanken caramena (**Sheshani anken caramena**)** which is therefore of the utmost Significance and practical utility in mathematical computation.

**E.g.**1 / 7 **Remainder**

10 – 7*1 = 3
30 – 7*4 = 2
20 – 7*2 = 6
60 – 7*8 = 4
40 – 7*5 = 5
50 – 7*7 = 1

Multiply by caramen i.e. 7, these remainders which give successive Products.

21, 14, 42, 28, 35, 7

Ignoring left hand side (tens place digit) digits simply and write down the last

Digit i.e. caramen (unit place digit) of each product.

$\therefore$ 1/7 = 0.12857

**E.g.** 1/13 **Remainder**

10 – 13*0 = 10
100 – 13*7 = 9
90 – 13*6 = 12
120 – 13*9 = 3
30 – 13*2 = 4
40 – 13*3 = 1

Multiply by caramen (last digit) i.e. 3, these remainders which give successive

Products.

30, 27, 36, 9, 12, 3

$\therefore$ 1/13 = 0.076923

**13. <u>SOPANTYADWAYAMANTYAM (</u>**Ultimate and Twice the Penultimate**)**

**Ultimate means unit place digit.**

**Penultimate means remaining number (excluding unit place digit)**

Divisibility by even divisors

Multiply unit place digit of dividend by positive/negative osculator. Add this

Product to twice of remaining number, continue this procedure on the new Number formed till a number is obtained by reaching left most digit of the Dividend. If this result of osculation is zero, divisor or its multiple, the original Dividend is divisible by original divisor otherwise not.

For even divisorsSopantyadwayamantyam sutra is applied. Ratio for digit to digit is to Be maintained. Thus multiplier comes into picture for all digits. As per Sopantyadwayamantyam sutra penultimate is multiplied by '2' at any stage. Hence ratio between any two consecutive digits is 2:1 like Hundreds place: Tens place Or Tens place: Unit place etc. If the multiplier at unit place is '1' then the Multipliers will be 2, 4, 8, 16, ----- etc. respectively for tens, hundreds, Thousands, ten thousands places.

e.g. Whether 3834 divisible by 18?

| | | | | |
|---|---|---|---|---|
| 8 | 4 | 2 | 1 | 18 + 2 = 2Ø |
| 3 | 8 | 3 | 4 | Positive Osculator = 2 |
| 36 | 42 | 14 | | Step – I = (4*1)2 + (2*3) = 14 |

Step – II = [(4*2) + (1*2)] + (4*8) = 42

Step – III = [(2*2) + (4*2)] + (8*3) = 36

As last osculated number (36) is multiple of Divisor (18). ∴ 3834 is divisible by 18

e.g. Whether 6784 divisible by 32? 32 – 2 = 3Ø

Negative osculator = 3

$2^{nd}$, $4^{th}$ ------ place digit as negative digit

| | | | | |
|---|---|---|---|---|
| 8 | 4 | 2 | 1 | |
| $\overline{6}$ | 7 | $\overline{8}$ | 4 | Step – I = [(1*4)3 + (2*$\overline{8}$)] = $\overline{4}$ |
| $\overline{32}$ | $\overline{1}6$ | $\overline{4}$ | | Step – II = [(4*3) + (4*$\overline{7}$)] = 16 |

-32 is divisible by 32 Step – III = [(6*3) + ($\overline{1}$*2) + (8*$\overline{6}$) = -32

Hence, 6784 is divisible by 32

## 14. EKANYUNENA PURVENA

The Sutra Ekanyunena purvena comes as a Sub-sutra to Nikhilam which gives the meaning 'One less than the previous' or 'One less than the one before'.

1) The use of this sutra in case of multiplication by 9, 99,999... is as follows .

**Method:**

1) The left hand side digit (digits) is (are) obtained by applying the ekanyunena purvena i.e. by deduction 1 from the left side digit (digits).

   e.g. (1) 7 x 9; 7 – 1 = 6 (L.H.S. digit)

b) The right hand side digit is the complement or difference between the multiplier and the left hand side digit (digits) i.e.7 X 9 R.H.S is 9 - 6 =3.

c) The two numbers give the answer; i.e. 7 X 9 = 63.

**Example 1:** 8 x 9
**Step (a)** gives 8 – 1 = 7 (L.H.S. Digit)
**Step (b)** gives 9 – 7 = 2 (R.H.S. Digit)
**Step (c)** gives the answer 72

**Example 2:** 15 x 99

**Step (a) :**15 – 1 = 14
**Step (b) :**99 – 14 = 85 (or 100 – 15)
**Step (c) :** 15 x 99 = 1485

**Example 3:** 24 x 99

**Answer**: (24 – 1) = 23 / (99 – 23) = 76 (or 100 – 24)

**Example 4:** 356 x 999

**Answer**: (356 – 1) = 355 / (999 – 335) = 644 = 355644

**Example 5:** 878 x 9999

**Answer:** (878 – 1) = 877 / (9999 – 887) = 9122(10000 – 876)= 8779122

**NOTE THE PROCESS -** The multiplicand has to be reduced by 1 to obtain the LHS and the right side is mechanically obtained by the subtraction of the L.H.S from the multiplier which is practically a direct application of Nikhilam Sutra.

Now by **Nikhilam**

| | | | |
|---|---|---|---|
| | 24 – 1 = 23 | L.H.S. | |
| x | 99 – 23 = 76 | R.H.S. (100–24) | |
| | 23 / 76 | | = 2376 |

Reconsider the **Example 4:**

| | | | |
|---|---|---|---|
| | 356 – 1 = 355 | L.H.S. | |
| X | 999 – 355 = 644 | R.H.S. | |
| | 355 / 644 | | = 355644 |

and in **Example 5:** 878 x 9999 we write

| | | | |
|---|---|---|---|
| | 0878 – 1 = 877 | L.H.S. | |
| X | 9999 – 877 = 9122 | R.H.S. | |
| | 877 / 9122 | | = 8779122 |

**ALGEBRAIC PROOF:**

As any two digit number is of the form (10x + y), we proceed

$$\begin{aligned} & (10x + y) \times 99 \\ = \; & (10x + y) \times (100 - 1) \\ = \; & 10x \,.\, 10^2 - 10x + 10^2 \,.y - y \\ = \; & x \,.\, 10^3 + y \,.\, 10^2 - (10x + y) \\ = \; & x \,.\, 10^3 + (y - 1) \,.\, 10^2 + [\, 10^2 - (10x + y)] \end{aligned}$$

Thus the answer is a four digit number whose $1000^{th}$ place is x,$10^{th}$ place is ( y - 1 ) and the two digit number which makes up the $10^{th}$ and unit place is the number obtained by subtracting the multiplicand from 100.(or apply Nikhilam).

Thus in 37 X 99. The $1000^{th}$ place is x i.e. 3

$100^{th}$ place is (y - 1) i.e. (7 - 1) = 6

Number in the last two places 100-37=63.Hence answer is 3663.

We have deal the cases

i) When the multiplicand and multiplier both have the same number of digits
ii) When the multiplier has more number of digits than the multiplicand.

In both the cases the same rule applies. But what happens when the multiplier has lesser digits?
i.e. for problems like 42 X 9, 124 X 9, 26325 X 99 etc.,

For this let us have a re-look in to the process for proper understanding.

**MULTIPLICATION TABLE OF 9.**

| | | a | b |
|---|---|---|---|
| 2 x 9 = | | 1 | 8 |
| 3 x 9 = | | 2 | 7 |
| 4 x 9 | = | 3 | 6 |
| | | | |
| 8 x 9 | = | 7 | 2 |
| 9 x 9 | = | 8 | 1 |
| 10 x 9 = | | 9 | 0 |

Observe the left hand side of the answer is always one less than the multiplicand (here multiplier is 9) as read from Column (a) and the right hand side of the answer is the complement of the left hand side digit from 9 as read from Column (b)

**Multiplication table when both multiplicand and multiplier are of 2 digits.**

a b
11 x 99 = 10 89 = (11–1) / 99 – (11–1) = 1089
12 x 99 = 11 88 = (12–1) / 99 – (12–1) = 1188
13 x 99 = 12 87 = (13–1) / 99 – (13–1) = 1287
-----------------------------------------------
18 x 99 = 17 82 ----------------------------
19 x 99 = 18 81
20 x 99 = 19 80 = (20–1) / 99 – (20–1) = 1980

The rule mentioned in the case of above table also holds good here

Further we can state that the rule applies to all cases, where the multiplicand and the multiplier have the same number of digits.

Consider the following **Tables**.

**(i)**

```
  a    b
11 x 9 =  9  9
12 x 9 = 10  8
13 x 9 = 11  7
-----------------------
18 x 9 = 16  2
19 x 9 = 17  1
20 x 9 = 18  0
```

**(ii)**

```
21 x 9 = 18  9
22 x 9 = 19  8
23 x 9 = 20  7
-----------------------
28 x 9 = 25  2
29 x 9 = 26  1
30 x 9 = 27  0
```

**(iii)**

```
35 x 9 = 31  5
46 x 9 = 41  4
53 x 9 = 47  7
67 x 9 = 60  3
-------------------------so on.
```

From the above tables the following points can be observed:

1) **Table (i)** has the multiplicands with 1 as first digit except the last one. HereL.H.S of products are uniformly 2 less than the multiplicands. So also with20 x 9

2) **Table (ii)** has the same pattern. Here L.H.S of products are uniformly 3 lessthan the multiplicands.

3) **Table (iii)** is of mixed example and yet the same result i.e. if 3 is first digitof the multiplicand then L.H.S of product is 4 less than the multiplicand; if 4 is first digit of the multiplicand then, L.H.S of the product is 5 less than the multiplicand and so on.

4) The right hand side of the product in all the tables and cases is obtained by subtracting the R.H.S. part of the multiplicand by Nikhilam.

Keeping these points in view we solve the problems:

**<u>EXAMPLE</u>1:** 42 X 9

1) Divide the multiplicand (42) of by a Vertical line or by theSign **:** into a right hand portion consisting of as many digits as the multiplier.
   i.e. 42 has to be written as **4/2** or **4:2**
2) Subtract from the multiplicand one more than the whole excess portion on the left. i.e. left portion of multiplicand is 4.

   one more than it 4 + 1 = 5.

   We have to subtract this from multiplicand i.e. write it as

```
 4 : 2 : -5
---------------
3 : 7
```

This gives the L.H.S part of the product.

This step can be interpreted as "take the ekanyunena and sub tract from the previous" i.e. the excess portion on the left.

3) Subtract the R.H.S. part of the multiplicand by Nikhilam process.

i.e. R.H.S of multiplicand is 2its Nikhilam is 8.
It gives the R.H.S of the product, i.e. answer is 3 : 7 : 8 = 378.
Thus 42 X 9 can be represented as

```
   4 : 2 :-5 : 8
------------------
3 : 7 : 8 = 378
```

**<u>EXAMPLE</u>2 :**124 X 9

Here Multiplier has one digit only.
We write 12 : 4Now

**Step 2 -** 12 + 1 = 13
i.e. 12 : 4 -1 : 3

**Step 3 -** R.H.S. of multiplicand is 4. Its Nikhilam is 6
124 x 9 is 12 : 4

```
   -1 : 3 : 6
----------------------
11 : 1 : 6  =  1116
```

The process can also be represented as

124 x 9 = [ 124 – ( 12 + 1 ) ] : ( 10 – 4 ) = ( 124 – 13 ) : 6 = 1116

**EXAMPLE 3:** 15639 x 99

Since the multiplier has 2 digits, the answer is

[15639 – (156 + 1)] : (100 – 39) = (15639 – 157) : 61 = 1548261

Ekanyunena Sutra is also useful in Recurring Decimals. We can take up this under a separate treatment.

Thus we have a glimpse of majority of the Sutras. At some places some Sutras are mentioned as Sub-Sutras. Any how we now proceed into the use of Sub-Sutras. As already mentioned the book on Vedic Mathematics enlisted 13 Upa-Sutras.

But some approaches in the Vedic Mathematics book prompted some serious research workers in this field to mention some other Upa-Sutras. We can observe those approaches and developments also.

## 15.GUNITSAMAUCHCHAYAH (The whole product is same)

It means that the operations are performed with the numbers have same effect When the same operations are performed with their Beejank.

**E.g.** 42 + 39

| Beejank of 42 | Beejank of 39 |
|---|---|
| ↓ | ↓ |
| 4 + 2 | 3 + $\cancel{9}$ |
| ↓ | ↓ |
| 6 | 3 |

Now 6 + 3 = 9 is the Beejank of sum of two numbers.Further 42 + 39 = 81

It's Beejank = 8 + 1 = 9

Consider a number and find its Beejank. Convert that number to vinculum number. Then find Beejank of that vinculum number.

If both Beejank are same then the conversion is correct.

Number = 196

Its Vinculum Number = $2\bar{1}6$

Beejank of 196 = $1 + \cancel{9} + 6 = 7$

Beejank of $2\bar{1}6 = 2 + (-1) + 6 = 7$

Number = 187

Its Vinculum Number = $2\,\bar{1}\,\bar{3}$

Beejank of 187 = $1 + 8 + 7 = 7$

Beejank of $2\,\bar{1}\,\bar{3} = 2 + (-1) + (-3) = -2$

$= -2 + 9$

$= 7$

Variation in Beejank can be easily understood

$\bar{2} = \bar{2} + 9 = 7$

**16. <u>GUNAK SAMUCHCHAYAH</u>**(Collectivity of multipliers)

Gunak-Samucchayah which postulates that if and when a quadratic expression is the Product of binomials (x+a) (x+b) then its first differential is the sum of the said two factors.

If y = uv   when u and v be the functions of x

$\therefore$ dy/dx = v (dy/dx) + u (dv/dx)

Gunak-Samucchayah sutra denote, connote and imply the same mathematical truth.

e.g.   $x^2 + 3x + 2 = (x + 1)(x + 2)$

$D_1$ (first differential) = $(2x + 3) = (x + 1) + (x + 2) = \sum a$

$X^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3)$

$$\underset{\downarrow}{(x+1)}\ \underset{\downarrow}{(x+2)}\ \underset{\downarrow}{(x+3)}$$

a   b   c

$D_1$ (first differential) = $(3x^2 + 12x + 11) = (ab + bc + ca)$

$= (x^2 + 3x + 2) + (x^2 + 5x + 6) + (x^2 + 4x + 3)$

## CHAPTER TWENTY FIVE: CONTRIBUTION OF ANCIENT INDIAN MATHEMATICIANS IN GEOMETRY

### BRAHMAGUPTA

Brahmagupta wrote 3 treatises - (1) Brahmasphuta-Sidhdanta (2) Khand-Khaddak (3) Uttar- Khand khaddak. He was the first mathematician who classified mathematics as Arithmetic and Bijaganit (Algebra).

**Contribution of Brahmagupta in geometry**

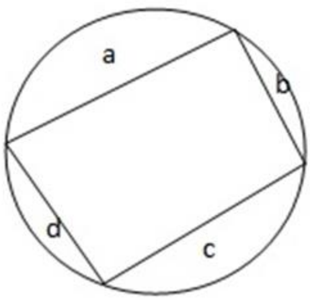
**Brahmagupta's formula for computing area of cyclic quadrilateral**

Let a, b, c, d are length of cyclic quadrilateral sides.

Approximate area of cyclic quadrilateral = $[(a+c)/2 * (b+d)/2]$

$S \longrightarrow$ Semi-perimeter = $(a+b+c+d)/2$

Exact area of cyclic quadrilateral = $\sqrt{(S-a)(S-b)(S-c)(S-d)}$




One theorem of Brahmagupta states that the base increased or decreased by the difference between the squares of the sides divided by the base and again divided by two, gives the lengths of two true segments.

Length of two true segments = $1/2[b \pm (c^2-a^2)/b]$

Where b = base of triangle

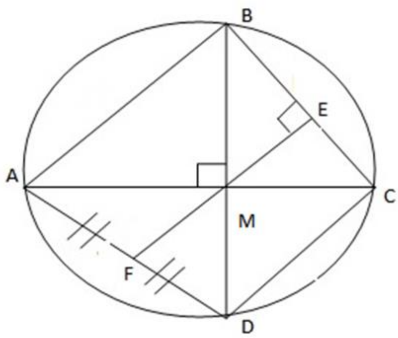
a & c = sides of triangle

**He further gives a theorem on rational triangles**

A triangle with rational sides a, b, c and rational area is of form

$a = 1/2[(u_2/v) + v]$, $b = 1/2[(u_2/w) + w]$, $c = 1/2[\{(u_2/v)-v\} + \{(u_2/w)-w\}]$ for some rational numbers u, v & w.

If a cyclic quadrilateral is orthodiagonal (that it has perpendicular diagonals) then the perpendicular to a side from the point of intersection of diagonals always bisects the opposite side.




$\overline{BD} \perp \overline{AC}, \overline{EF} \perp \overline{BC} \Rightarrow |\overline{AF}| = |\overline{FD}|$

To prove AF = FD

FAM = CBM (Inscribed angles that intercept the same arc of a circle)

FMA = CME (opposite angles)

FMD = BME (opposite angles)

FAM = FMA

△FMA is an isosceles triangle

AF = FM

Similarly DMF = BCM = BME = FDM

△DMF is an isosceles triangle.

FM = FD

From (1) & (2)

AF = FD

He found accurate value of $\pi$ as $\sqrt{10}$

He gives construction of various figures with arbitrary sides. He essentially manipulated right angles to produce isosceles triangles, scelene triangles, rectangles, isosceles trapezoids with three equal sides and a scelene cyclic quadrilateral.

He proved that cyclic quadrilateral is formed by taking two right angle triangles
($75^2+40^2 = 51^2 + 68^2 = 7225 =85^2$)

## NILKANTH SOMAYAJI

"**Tantra-Sangraha**" is Nilkanth Somayaji's important treatise out of his 10 treatises. He also presented commentary on 'Aryabhat's ($5^{th}$century) work named as "Aryabhatiya-Bhashya". He made minor corrections/alterations in many formulae & theorems of Aryabhat ($5^{th}$ century) & presented in simplified manner.

### Contribution of Nilkanth Somayaji in geometry

The speciality of 'Tantra-Sangraha' is that all formulae are derived from spherical geometry. He devised astro-science triangle's three points with Sun, North Pole & Zenith for the proof of his theorems.

Suppose 3 sides & 3 angles of such triangle are a, b, c & A, B, C respectively then his theorems are

Cos a = cos b*cos c + sin b*sin c*cos A

Sine rule = sin a/sin A = sin b / sin B = sin c / sin C

Nilkanth Somayaji's important formula

$\tan^{-1} X = X - X^3/3 + X^5/5 - X^7/7 + \ldots$

If X =1 is assumed then for $\pi$ infinite series can be derived

e.g. $\pi/4 = 1 - 1/3 +1/5 -1/7 + \ldots$

In 1670 Labaneitz & Gregory discovered formula $\tan^{-1}(1) = \pi/4$ but Nilkanth Somayaji invented formula $\tan^{-1}(1) = \pi/4$ and proved it 200years prior to them.

## PARMESHWARAN

Parmeshwaran presented some commentaries as - "Bhatdeepika-on Aryabhat's ($5^{th}$century) treatise Aryabhatiya", "Karmadeepika - on Bhaskaracharya's ($6^{th}$century) treatise Mahabhaskariya", "Parmeshwari - on Bhaskaracharya's ($6^{th}$century) treatise Laghubhaskariya", "Vivaran-2 on Bhaskaracharya's ($12^{th}$century) treatise Leelavati".

## Contribution of Parmeshwaran in geometry

Parmeshwaran was the first mathematician who invented method of calculating radius of circle with an inscribed quadrilateral.

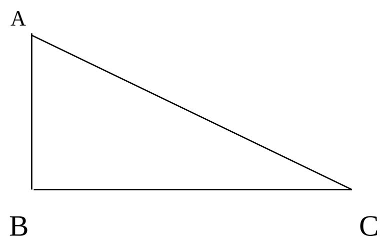
1. If a, b, c, d are the four sides of a cyclic quadrilateral then

   Formula for radius (R) of circle is

$$R = \sqrt{\frac{(ab+cd)\,(ac+bd)\,(ad+bc)}{(a+b+c-d)\,(b+c+d-a)\,(c+d+a-b)\,(d+a+b-c)}}$$

He invented this formula in the year1440 prior to french mathematician Simon Huliar in the year1782.

2. If ABC is a right angle triangle & B = $90^0$

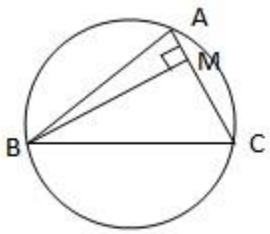



then $AB^2 + BC^2 = AC^2$

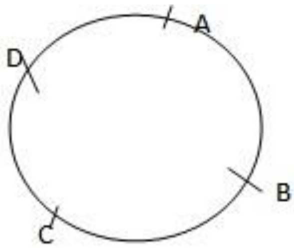
He proved this theorem. His proof is different than Pythagoras theorem.

3. He invented formula for Area of a triangle = 1/2 *base* height
4. $\triangle$ ABC is inscribed in a circle. BM is drawn perpendicular to AC, then formula for evaluating radius 'R' of a circle is

2R = (AB*BC)/BM

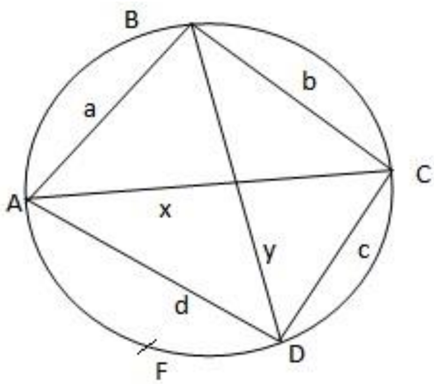



5. If A, D, B, C are four points on circle and arc AD = arc BC, then $AB^2 - BC^2 = AC * DB$




6. $Sin^2 A - Sin^2 B = Sin (A+B) * Sin (A-B)$
   $Sin A * Sin B = Sin^2 (A+B)/2 - Sin^2 (A-B)/2$

7. ABCD is a cyclic quadrilateral. AB = a, BC = b, CD = c, DA = d. 'F' is a point on circle such that arc AF = arc CB. If diagonal AC = x, diagonal BD = y, side DF = z




then length of these diagonals as per Permeshwaran's formula

$x^2 = [(ac+bd)(ad+bc)]/(ab+cd)$ , $y^2 = [(ab+cd)(ac+bd)]/(ad+bc)$

If a, b, c, d are four sides of a cyclic quadrilateral and $2s = (a+b+c+d)$

then formula for area of cyclic quadrilateral

$A^2 = (s-a)(s-b)(s-c)(s-d)$

He proved all the above mentioned formulae.

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## **BHASKARACHARYA** **(12$^{th}$century)**

He has been called greatest mathematician of medieval India. His main work "**Sidhdant-Shiromani**" is divided in four parts called Lilavati, Bijaganita, Grahaganita and Goladhyay which are also sometimes considered four independent works.

### Contribution of Bhaskaracharya (12$^{th}$century) in geometry

A proof of the Pythagorean Theorem by calculating the same area in two different ways and then cancelling out terms to get $a^2 + b^2 = c^2$.
He also discovered spherical trigonometry along with other trigonometrically results.

The Sidhdanta-Shiromani demonstrates his knowledge of trigonometry including the Sine table and relationships between different trigonometric functions.Among the many interesting results given by Bhaskaracharya , discoveries first found in his works include computation of sines of angles 18 and 36 degrees and now well-known formulae for sin (a + b) and sin (a - b).

## **JYESHTHADEVA**

He was the author of “Ganita Yuktibhasha” in Sanskrit to elucidate the Yuktis = rationales of the theories given in “Tantrasangraha” of Nilakantha Somayaji. Actually Yuktibhasha is considered to be a complete text book on Indian calculus. Yuktibhasha consists of two parts and first part is dedicated to mathematics as well as second part to astronomy.

Jyeshthadeva provided the proof, the rationale and derivation of all theories of Indian mathematical astronomy. He gave full proof of Madhav’s infinite series of ( $\prod$ ), sine, cosine, inverse tangent functions and so called Taylor’s series. He also elaborated the concepts of calculus such as term by term integration, differentiation, iterative methods for solution of nonlinear equations etc. He described the integration methods for finding area and volume of a sphere. He also provided detailed procedure and proof for a general method called “Samaghata Sankalita” for estimating the sum of powers of natural numbers.

He was the first who gave the following product of (pi) which was rediscovered by John Wallis in 17th century.

$(\prod / 2) = (2/1)*(2/3)*(4/3)*(4/5)*(6/5)*(6/7)*(8/7)*(8/9)*(10/9)$ ------------------

Yuktibhasha gives strong and indisputable evidence for the fact that Indians used the concepts of calculus, power series of sine, cosine & inverse tangent functions, second and third order Taylor series approximations of sine and cosine, geometrical derivations of infinite series and infinite series of ($\prod$), ($\prod / 4$) and its convergent approximations many centuries before Newton, Leibniz, Gregory, Demoivre, Taylor, Maclaurin and Euler.

Jyeshthadeva described a geometric method for finding the length of an arc by approximating it to a straight line to derive infinite series. He also described an iterative re-substitution method to derive infinite binomial series expansion for the expression $(1+x)^{-1} = 1 - x + x^2 +$ ------------- $(-1)^r x^r +$ ---------------- and arrived the asymptotic expansions through a number of repeated summations (Varamsamkalita) of series.

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## SHANKAR VARIYAR

He wrote a detailed commentary named “Kriyakramakari” on “Lilavati” of Bhaskaracharya (12th century). He elaborated the theories of mathematics discussed in “Lilavati”. He could complete his commentary upto 199 Sanskrit verses only out of 279 verses. Later Narayana completed this commentary.

He also wrote commentary “Laghuvivriti” on “Tantrasangraha” of Nilakantha Somayaji and “Karansara” on astronomy. He elaborated Madhav’s infinite series of ($\prod$), sine, cosine and inverse tangent functions in his “Kriyakramakari”. He also described Madhav’s geometric method for computing the value of the circumference of a circle.Actually “Kriyakramakari” also gives full details of Madhav’s contributions to mathematical astronomy.

## NARAYAN PANDIT

He wrote treatise “Ganita Kaumudi” and also treatise on algebra “Bijaganitavatamsa”. He worked on arithmetic, algebra, geometry, combinatorics and magic squares.

He gave seven different and innovative methods of squaring a number.

$A^2 = (a+b)^2 = (a-b)^2 + 4ab$

$25^2 = (15 + 10)^2 + (4*15*10) = 625$

$A^2 = (a+b)^2 = a^2 + b^2 + 2ab =$

$25^2 = (15 + 10)^2 = 15^2 + 10^2 + (2*15*10) = 225 + 100 + 300 = 625$

$n^2 = (n-a) * (n+a) + a^2$

$25^2 = (25-5) * (25+5) + 5^2 = (20*30) + 25 = 625$

He extensively worked on cyclic quadrilateral and he was the first who discovered the following theorems on cyclic quadrilateral.

(1) If the sides have been given then only three diagonals are feasible for a cyclic quadrilateral.

(2) Divide the sum of three diagonals of a cyclic quadrilateral by four times the radius.

Or

Divide the three diagonals of a cyclic quadrilateral by twice the circumdiameter to find the area of a cyclic quadrilateral.

He also gave formula for finding circumradius of a cyclic quadrilateral.

Area of triangle = Divide the sum of sides by four times the circumradius.

## MADHAVAN

Madhavan's work was in pure mathematics. He presented his every formula through infinite terms and this was his speciality.

### Contribution of Madhavan in geometry

Formula for length of an arc

r = radius of circle

s = sine chord

c = cosine chord

Length of an arc = $(s/c)r - 1/3(s/c)^3 r + 1/5 (s/c)^5 r - 1/7(s/c)^7 r + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

If $r = 1$ & $s/c = \tan \Theta$

then Length of an arc = $\tan \Theta - \tan^3\Theta/3 + \tan^5\Theta/5 - \tan^7\Theta/7 + \ldots\ldots\ldots\ldots\ldots\ldots$

Madhavan invented this formula in 14th century prior to James Gregory in 1671.

If a = length of an arc , r = radius of circle.

then cosine chord = r sin(a/r) = $a - a^3/(3!r^2) + a^5/(5!r^4) - a^7/(7!r^6) + \ldots\ldots\ldots\ldots\ldots\ldots$

If $r = 1$ & $a = r\Theta$ ($\Theta$ is an angle)

then $\sin\Theta = \Theta - \Theta^3/3! + \Theta^5/5! - \Theta^7/7! + \ldots\ldots\ldots\ldots\ldots\ldots$

similarly $\cos\Theta = 1 - \Theta^2/2! + \Theta^4/4! - \Theta^6/6! + \ldots\ldots\ldots\ldots\ldots\ldots$

He calculated two values for $\pi$

$= 104348/ 33215 = 3.1415926539211$

$= 2827433388233 / (9*10^{11}) = 3.141592653359$

## LIST OF SUTRAS (FORMULAE) AND REFERENCES

### LIST OF SUTRAS (FORMULAE)

- Ekadhiken Purvena (By one more than the previous one)
- Nikhilam Navatashcharamam Dashatah (All from nine and last from ten)
- Urdhva tirgbhyam (Vertically and crosswise)
- Paravartya Yojayet (Transpose and apply)
- Sunyam Samyasamuchchaye (The summation is equal to zero)
- Anurupye Sunyamanyat (If one is in ratio, the other is zero)
- Sankalana Vyavakalanabhyam (By addition and subtraction)
- Puranapuranabhyam (By the completion and non-completion)
- Chalana kalanabhyam (Sequential motion)
- Yavadunam (The deficiency)
- Vyashtisamashtih (Whole as one and one as whole)
- Sheshanyankena charamena (Remainder by the last digit)
- Sopantyadvayamantyam (Ultimate and twice penultimate)
- Ekanyunena Purvena (By one less than the previous one)
- Gunitasamuchchayah (The whole product is the same)
- Gunakasamuchchayah (Collectivity of multipliers)

### LIST OF UPA – SUTRAS (SUB-FORMULAE)

- आनुरूप्येण - (Anurupyen) (Propertionately)
- शिष्यते शेषसंज्ञः - (Shishyate Sheshasandyah) (The remainder remains constant)
- आद्यमाद्येनान्त्यमन्त्येन - ( Adyamadyenantyamantyena) (First by the first and last by the last)
- केवलैः सप्तकं गुण्यात् - (Kevalaih saptakam gunyat) (1/7 by the product)
- वेष्टनम् - (Veshtanam) By osculation
- यावदूनं तावदूनम् - (Yavadunam Tavadunam) (Whatever deficiency further lessen that much)

- यावदूनं तावदूनीकृत्य वर्गं च योजयेत् Yavadunam Tavadunikrity vargam cha yojayet (Lesser by the deficiency and use its square)
- अन्त्ययोर्दशकेपि - (Antyayordashkepi) (Sum of last digits is ten)
- अन्त्ययोरेव - (Antyayoreva) (Only by the last)
- समुच्चयगुणितः - (Samuchchaya gunitah) (Product of whole)
- लोपनस्थापनाभ्याम् - (Lopanasthapanabhyam) (By alternate elimination and retention)
- विलोकनम् - (Vilokanam) (By observation)
- गुणितसमुच्चयः समुच्चयगुणितः- (GunitasamuchchahSamuchchayagunitah) (Product of the whole is equal to whole of the product)
- द्वंद्वयोग - Dwandwayog (Duplex)
- शुद्धः - (Shuddhah) (Purify)
- ध्वजांक - Dhwajanka (Flag digit)

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॥ ॐ सिद्ध सिद्धेश्वरनाथाय नमः ॥