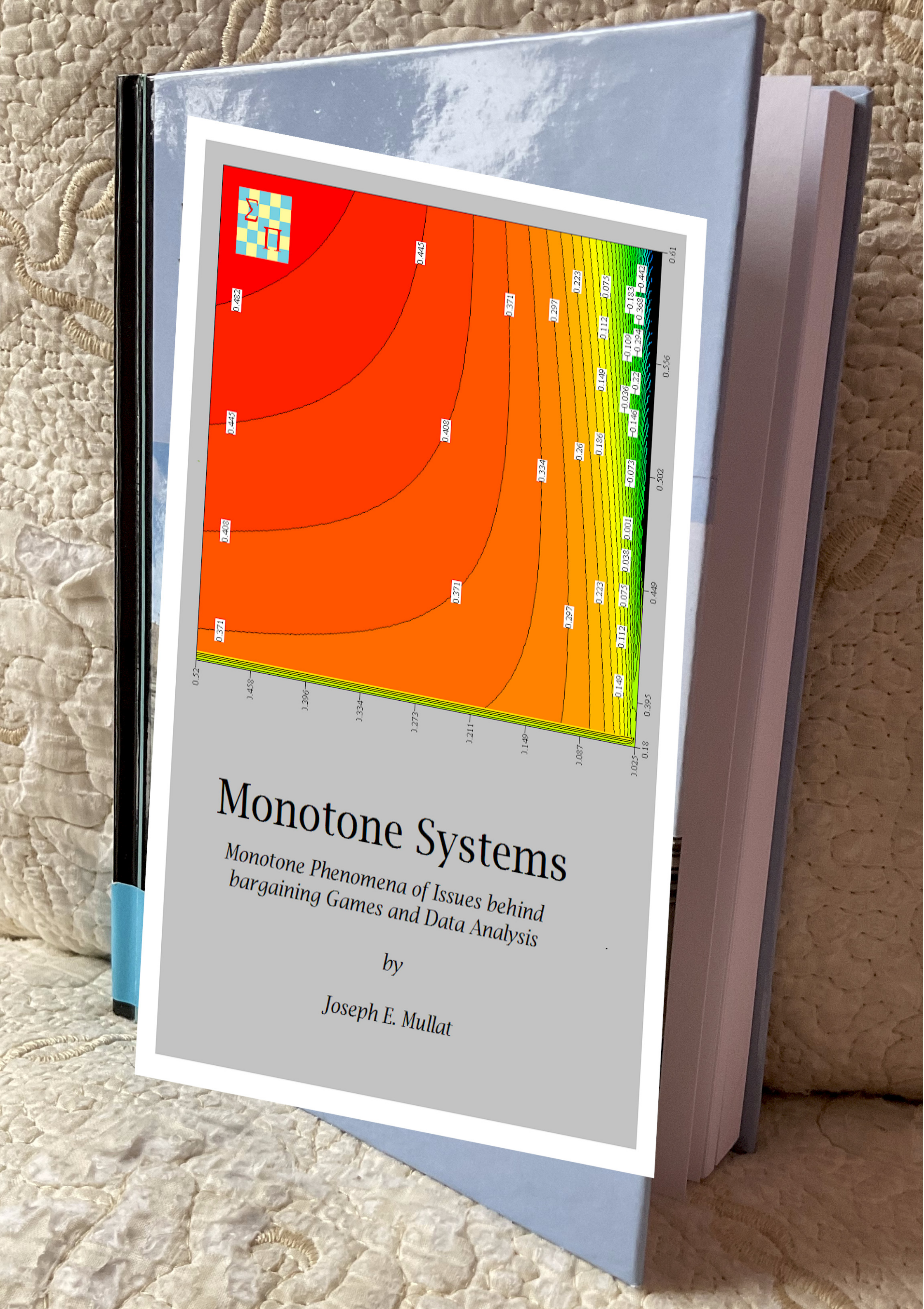
Monotone Systems
Monotone Phenomena of Issues behind bargaining Games and Data Analysis
by
Joseph E. Mullat

I

Monotone Systems

The implementation of the concept of a monotone (monotonic) system was discussed in two contexts. Firstly, the concept of monotony was introduced to reflect the adjustment of negotiating power in bargaining situations, in particular in negotiations between left and right political parties, that is, to clarify the structure of the political mechanism design. And secondly, the same concept was applied to data analysis. Although the application of monotonic systems in these two different areas may seem unexpected, they are united by the same idea or what is called stable or "stable lists" of elements in sets or topologies. Stable sets, by assigning certain certificates to their elements, provide a unifying perspective for virtual experiments. These virtual experiences create a basis for stability or equilibrium, as opposed to volatility in the economy or fuzziness in empirical research.

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Monotone Systems

MONOTONE PHENOMENA OF ISSUES BEHIND BARGAINING GAMES AND DATA ANALYSIS

By Joseph E. Mullat

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THE MONOTONE PHENOMENA

In this collection of articles, by a monotone or monotonic system, we understand a set of subsets ordering some indicators. Indicators or credentials of subset elements have a monotonic property synchronizing the dynamic nature of indicators. Indicators in the form of real numbers increase or decrease along with the partial ordering caused by the inclusion of subsets taken from some general set of indicators. Hereby, the Monotone Systems formalizes and generalizes the intuitive notion of ordering, sequencing, or arrangement of the elements in subsets. The theory was initiated by the author in 1971, and since then was further developed and published in Russian periodical of MAIK in 1976. In English it was originally distributed by Plenum Publishing corporation.

Concise Glossary of Mathematical Nomenclature

W — A common or general set of indicators, elements, objects, etc

Γ_j, X_i, H_i, H^j, H_1, H_2 … — Subsets of the General Set W

For $i, j = 1,2,..,n$ instead we sometimes use short notation $i, j = \overline{1,n}$

α, β, γ, μ, τ,… — Greek letters as elements of W, H_i, Γ,…

Credential $\pi(\alpha, H_j)$ assigned to an element $\alpha \in H_j$ of the subset H_j

Type $\oplus$ and type $\ominus$ operations on elements $\alpha, \beta, ...$

$\overline{\alpha}$, $\overline{\beta}$, — Sequences or sets $\langle \alpha_i \rangle$, $\langle \beta_j \rangle$, of ordered elements α_i, β_j,…

$H = \varnothing$, $\subset$, $\supset$, $\subseteq$, $\supseteq$, $H \subseteq W$, $W \supseteq \Gamma$,… – Pairwise relations

$H_1 \cup H_2$, $H_1 \cap H_2$, $W \setminus \overline{\alpha}$,… — Pairwise operations

Π^+H, Π^-H — Collections or arrays of general set W subsets

$\{\Pi^-H \mid H \subseteq W\}$ — This means that $\{\Pi^-H \mid$, where $H \subseteq W\}$, etc.

$\overline{X}$ — Denotes the complement $W \setminus X$ of a set X to the set W

$\forall$ — Generality quantifier and $\exists$ is existential quantifier

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PREFACE

MONOTONE PHENOMENA OF ISSUES BEHIND BARGAINING GAMES AND DATA ANALYSIS

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Introduction

In social sciences, natural language is used to describe the phenomena pertaining to numbers. This approach may be the reason for the problems that often emerge in predictions that do not align well with the reality. In natural sciences, converse is true, as numbers are used to describe and predict phenomena of various origins, natural or artificial. Yet again, applying mathematical assumptions or postulates is rarely adequate for depicting the complexity of the phenomena in question.

The problem of prediction, perhaps, is not rooted in mathematics. Rather, it likely stems from the issue of whether the actual mathematical approach used is adequately defined. This can be likened to window-shopping instead of visiting a store when purchasing an item of interest. Thus, to truly establish what mathematics really predicts, instead of relying on numbers, we must first try to explicate the subject under study using words. This approach will allow the subject to be well understood, precluding a move in the wrong direction, through incorrect use of mathematics. Still, in practice, this strategy can be protracted, as it can take years, or even decades, of exploring known or unknown mathematical schemes before we can portray the phenomenon is a sufficiently understandable form. It should also be noted that, we don't generally require mathematics in order to initiate seminal exploration of the phenomena of interest for us humans.

Having said that, what direction should research take? This question is very difficult to answer when the subject under study is diffused, the path ahead is unknown, and "a suitable vehicle" for the journey is difficult to identify. Is there a way to discover something hidden that can take us out of this uncomfortable situation? How can we find among these seemingly disparate subjects the one that could make the future for the researcher more appealing? While none of these questions have a definitive answer, it can be stated with certainty that the subject must be normatively challenging and comply with the coherence inherent in natural language. Moreover, the words used to describe phenomena under study must be sufficiently simple to be merged together. As the Danish philosopher Søren Kirkegård observed in his master's thesis in 1840, any subject should be described in a way that can be understood by a child. When considering this assertion, it should be noted that, in his time, the master degree thesis presentation and defense in an open session used to take about 7-8 hours. Thus, to gain their degree, the candidates had to be quite well prepared to answer the panel's questions regarding a wide range of phenomena. We will try to follow in their footsteps.

To do so we start with a "visual" or "pedagogical exhibit". At first glance, our exhibit may seem frivolous, but it is much easier to suggest something new if the essence of the matter is presented in the form of an allegory, which can be interpreted in such a way as to reveal the hidden meaning of reality. We preface everything that concerns theory with this simple example of further reasoning. The reader will find this passage again, but in a more precise mathematical form, in a separate article later in the text.

Wine Menu

The need for order is all-encompassing and we encounter it in everyday life. We seamlessly form orderly queues while waiting at a checkout counter, we take for granted the chronological or lexicographical order that makes our iPhone contact lists easier to use, we peruse tables of contents to explore books and catalogs at glance, etc. In academic literature, cited works (also known as a bibliography or references) are usually ordered or numbered chronologically. Some journals or periodicals require lexicographic citation order for the same purpose. These are all examples of events and word orderings. The next one might be more appealing.

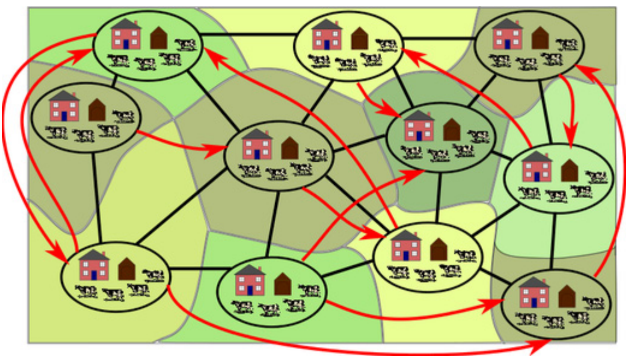
When accepting an order at a restaurant, the sommelier explains that at the moment some relatively inexpensive, and in other cases the cheapest wines indicated by the guest on the list as possible attractive ones are temporarily absent. The lack of comparatively inexpensive wines on the list will likely encourage guests to expand, or at least maintain, the list of cheap wines that at first glance are already approved. On the contrary, the lack of approved, at first glance, cheap wines may induce the sommelier to suggest more expensive wines in favor of others available for order, also cheaper, but quite good and better wines.

Yes, indeed, the world of wine is exciting, and price is definitely a parameter that many pay attention to. Taste "unfortunately" is individual and, of course, there is not always a connection between price and "good wine". However, we will use price as our primary parameter.

The wine list is ordered in descending order of price, and 1 multiplies the price of the most expensive wine, 2 multiplies the next local price, then 3 the next, and so on. We call these numbers as price credentials. The local maximum of credentials and the price of wine are selected when this peak location from the top of the ordered list — the maximum is reached. The guest decides to accept the price of the wine at the local credential's maximum as an acceptable level of price significance when choosing wines with a higher or equal price level. We call this ordering the defining sequence of credentials. The defining sequence is single-peaked, where the peak denotes the kernel (Mullat, 1971–1995) of a monotonic system. By definition, a credential is nothing more than the price multiplied by the number of different wines in the particular sub-list of wines to which a particular wine belongs. In fact, credentials defined this way organize nested subsets of wines from a wine list.

Graphs

We will continue our exploration by depicting various phenomena through graphs. A graph is a visual representation of relations between points connected by lines. They are akin to picture books aimed at young children, who are required to join numbered points to reveal the final image. In natural language, we also encounter nodes even if we are not aware of it. When their order is unimportant, they are connected by lines/edges on the graph; otherwise arcs are used as illustrated below. The other form of graph representation is given by quadrangle matrices, i.e., matrices with an equal number of rows and columns comprising items with either 0 or 1 value, thus denoting Boolean tables. In such case, rows represent arcs pointing from vertices/nodes, i.e., out from nodes into other vertices, while columns pertain to arcs pointing into the nodes. A graph given in a Boolean table form is also a binary relation. In the discussions that follow, graphs will be explained in terms of rows and columns.



Summing up all 1-s in each row and all 1-s in each column allows forming so-called “credentials” of rows and columns in graphs. In other words, credentials represent the frequencies of 1-s in rows and columns, as they are equivalent to the total number of incoming and outgoing arcs from any particular node within the graph. Credentials can also be assigned to cells in binary tables by summing up or multiplying credentials of rows and columns in a pair wise fashion. Alternatively, using various types of arithmetic composites can further extend these credentials. These composites, as combined credentials, may characterize graphs, allowing analysis to progress in a desirable direction. This approach is particularly useful for emphasizing the dynamic nature of graph architecture — its monotone phenomena. Indeed, simply eliminating an item assigned a value of 1 from a Boolean table representing the graph would always result in decreasing our credentials values. In other words, it is irrelevant whether we employ composite or simple credentials. Similarly, replacing 0 with 1 would result in increasing credentials, creating reverse dynamics. While this may seem rather complex, in essence, credentials of graph elements are nothing but frequencies of items filled with 1-s. This is the foundation of the theory of Monotone Systems orderings.[1]

Indicators

Indicators are the preferred tools for statisticians, physicists, natural scientists and economists. Think of different metrics, average incomes, taxes, and many other areas where numbers and values are helpful. Nevertheless, despite the apparent diversity, all of these examples obey the same lexicographic or chronological ordering rules. Indeed, upon closer examination, it becomes apparent that any part, subset or sub-list of the lexicographic ordering, regardless of whether they are in ascending or descending order, again, regardless of the original, so-called general ordering, are subject to the same ordering lexicographic or chronological rule.

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[1] It is was originally published by Mullat (1971) in the article of Tallinn Technical University Proceedings, Очерки по Обработке Информфции и Функциональному Фнализу, Seria A, No. 313, pp. 37-44 (in Russian), and (1972) in the article extension “Ühest Neelavate Markovi Ahelate Klassist,” On Absorbing Class of Markov Chains in EESTI NSV Teaduste Akadeemia Toimetised, Füüsika Matemaatika, vol. 21, No. 3, in Russian.

Let us examine an example of Grand Ordering of items and select two items from the list, denoting them as Item A and Item B. We can always establish that either $A \prec B$ or $B \prec A$, otherwise $A \approx B$. It is very easy to form these relations when the Grand Ordering is available. However, attempting to organize the Grand Ordering with the knowledge of relations between only a various items is problematic. Indeed, suppose that given a line of items $A, B, C, \ldots$ we can only say which one of these three relations $\prec, \succ, \approx$ holds for any pair. Is it possible to arrange the items in this list using some numeric indicator in harmony with these rules? This was the question that von Neumann and Morgenstern [2] attempted to answer. In their pioneering work, they provided some very strong formal axioms for rules allegedly applicable to pairs of items, denoted as the axioms of pairwise relations between items. The authors further posited that these rules must be obeyed to guarantee the desired ordering property of some numerical indicators, or what they referred to as utilities. Von Neumann and Morgenstern rigorously proved that the existence of such orderings confirmed axioms' validity, and thus established that these can be applied to order the items in accordance with the increase or decrease in their corresponding utilities. Their work was complemented by the famous theorem put forth by John Forbes Nash Jr. He provided its proof in the form of axiomatic approach to the bargaining situations, confirming that the solution of the bargaining problem based on utility orderings, as a prerequisite, is unique given that the axioms reflect the phenomena of the bargaining adequately.[3]

All orderings discussed thus far followed some usual numerical rules. However, Arrow, relative to those proposed by Von Neumann and Morgenstern, suggested much simpler rules, in relation to voting schemes. Unfortunately, when ordering axioms presupposing democracy were applied separately, although seemingly reasonable approach, this resulted in a paradox, as it was not possible to satisfy the same axioms applied simultaneously. This led to the conclusion, expressed in barmaid language, that democracy does not exist. Still, it is worthwhile exploring these axioms using more complex examples in which obvious coherence is employed to explain various phenomena more precisely.

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[2] John von Neumann and Oscar Morgenstern, (1953) Theory of Games and Economic Behavior, Princeton University Press.

[3] Nash J.F. (1950) The Bargaining Problem, Econometrica, Vol. 18, No. 2, 155-162.

Surveys

Polls are a common form of gathering the views and opinions of large groups of people and are used in many contexts. Government agencies, commissions, product market analysts, etc., conduct surveys to identify the true incentives of people. Typically, research results are presented in tabular form because it is a convenient way to visualize data and store it in databases. In fact, overview tables are extensions of charts that range from a quadrilateral to a rectangle. The only difference is that instead of binary (1 and 0) inputs, the elements of such tables usually consist of codes ($A, B, C, \ldots$) called attributes, measured on a nominal scale. The nominal scale is nothing more than a coded form of words or sentences reflecting some properties of products, a predetermined attitude of respondents towards the media, etc., usually accompanied by some personal data.

When such data is analyzed, the results are usually displayed as pie charts because they allow you to visualize the frequency of different responses at a glance. When a dataset is complex and consists of many inputs, many such diagrams are created, as analysts want to study the same subject from different angles depending on their purpose. This form of presentation is, again, nothing more than a visual representation of the frequency density distribution associated with different responses. As already noted, the form of the nominal frequency scale makes it possible to present the order of respondents' answers in accordance with some classification using personal data (as a rule, gender, age, education, etc.). It should be noted, however, that placing responses on a nominal scale might lead to ranking of respondents themselves based on their response rates. This effect manifests itself in the ranking of universities, car manufacturers, rating scales, etc.

Some researchers believe that this implementation of the nominal scale leads to the so-called *conforming scale*, which actually provides the truth [4,5]. However, we can discover something new by implementing the nominal scale in the form of a *defining ordering/sequence*. [6]

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4 Karin Juurikas, Ants Torim and Leo Võhandu. (2000) "Mitmemõõtmeliste andmete visualiseerimine isoleeritud majandusruumis, kasutades monotoonsete süsteemide konformismiskaalat: Uurimus Hiiumaa näitel," (Article: Multivariate Data Visualization in Social Space using Monotone Systems conforming Scale: Case study on Hiiumaa Data).

5 Tõnu Tamme, Leo Võhandu, and Ermo Täks. (2014) A Method to Compare the Complexity of Legal Acts, NaiL, 2nd International WorkShop on "Network Analysis in Low," December 5, Amsterdam.

6 Joseph E. Mullat. (1976) Extremal Subsystems of Monotonic Systems, I, Translated from Avtomatica i Telemekhanika, No. 5, pp. 130 – 139.

To proceed with the discussion, it is prudent to first explain the defining ordering through an example. Let us assume existence of a Grand Ordering of items $A_1, B_2, A_3, A_4, C_5, D_6, C_7, E_8$. Our goal is to reorganize the sequence according to their frequencies, i.e., frequencies $3,1,2,1,1$ of A, B, C, D, E. The indices $1,2,3,4,5,6,7,8 \equiv \overline{1,8}$ assigned to the items A, B, C, D, E in the sequence above denote their respective occurrences. The lowest frequencies are associated with B_2, D_6 and E_8. Let us eliminate these items from the sequence. After eliminating B_2, D_6, E_8, we eliminate C_5, C_7, as these now have the lowest frequencies, and then A_1, A_3, A_4. This results in $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$, referred to as the Grand defining sequence, highlighting the frequencies of items in different order. Namely, in contrast to its original form, the new sequence lists items in increasing/decreasing order of frequencies $1,1,1,2,1,3,2,1$. We can immediately observe upward and downward changes in frequencies, e.g., from 2 to 1, but also sliding frequencies, such as $3,2,1$. In the collection of our papers, these hikes are designated by Greek letters $\Gamma_1, \Gamma_{2,} \ldots$ and are thus referred to as Γ -hikes, reflecting the dynamic nature of such lists. In fact, when subsets of respondents or their survey answers/attributes are explored, it is always possible to arrange them into such dynamic lists, reflecting decreasing/increasing order of their corresponding frequencies. As a consequence, in line with representing Monotone Systems through graphs, the frequencies scale is equivalent to the Indicator of matching responses to the survey questions. It is important to emphasize, however, a fundamental property of the defining sequence. Namely, irrespective of which subset, sub-list, or subsequence we take from the Grand Ordering, we have independently arranged the subsequence by applying our defining rule, whereby its defining properties are in harmony with the Grand defining sequence arrangement, from which the subsequence was initially extracted.

Indeed, let us extract a subsequence A_1, C_5, A_4, C_7 form the list given earlier. Arranging the items independently, in accordance with the defining sequence rule, we obtain the frequencies $2,1,2,1$. It is irrelevant whether we eliminated A_1, A_4 before C_5, C_7 or vice versa — C_5, C_7

first, followed by A_1, A_4. Whichever path we take, we arrive at $2,1,2,1$ as the order of the frequencies. This is equivalent to generating the sequence C_5, C_7, A_1, A_4 in accordance with the Grand defining sequence $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$ arrangement.

Many natural phenomena follow well-defined rules and sequences, such as Fibonacci series, in which any subsequent element is the sum of two previous items ($1,2,3,5,8,13,\ldots$), with 1.618 as its limit. This value is also known as the golden ratio, indicating that the relationship between two quantities is the same as the ratio of their sum to the greater of the two. Golden ratios are widespread in nature, from the proportions of the human body, to arrangements of leaves, spiraling shells, pinecones, etc. Hence, we can say that our defining sequence obeys the Fibonacci rule.

Using the information presented above, we can apply the Grand defining sequence to a lexicographical or chronological order of words. It is important to recall that, when some items have been eliminated, similar to the exercise above in which frequencies were presented on a nominal scale, the value of frequencies/credentials decreases. The process starts with searching for items that have the lowest credential values on the credentials scale, followed by those that are next in increasing/decreasing order, while recalculating the remaining credentials as we proceed with item replacement. This is a best-explained using survey table.

Usually, survey tables are used to present respondents' answers reflecting their attitudes or views on a specific topic. For the sake of simplicity, when answering survey questions, respondents are usually required to select one of the options provided, and can thus be represented by $A, B, C, \ldots$, denoting their choice. Now, instead of presenting these items in a straight line, we can proceed with elimination, taking two directions. Respondents, like nodes with outgoing arcs, are presented in the rows of survey tables, while columns, like ingoing arcs in graphs, denote their responses to the survey questions, coded as $A, B, C, \ldots$. Some credentials composed from the corresponding frequencies of items can characterize the rows related to individual respondents $A, B, C, \ldots$. Alternatively, credentials of columns can be characterized by the same or distinct compositions of frequencies using more sophisticated composites of credentials compiling, for example, arithmetic/numerical expressions as products. [7] In applying the compositions of credentials to rows and columns summing up matching answers, it is essential to ensure that the composition functions remain non-decreasing.

Now, aiming to build the defining sequence of the respondents, we can proceed in the same way with credentials of respondents, credentials of their answers, or even combining these two types of credentials (the row and column credentials). First, we must identify a cell with the lowest composition, indicating the most unreliable answer type, suggesting that the respondents are unwilling (for whatever reason) to answer the particular question truthfully. Such unreliable respondents should be eliminated, along with their unreliable answers, before recalculating the credentials of the remaining respondents and their answers. Once this is accomplished, we search for the cell that now has the lowest credentials composite and, in line with the above, remove the respondent (and his/her responses) from any further consideration. As before, we make adjustments in the credentials among all other frequencies of item ($A, B, C, \ldots$) occurrences. We proceed in the same manner until no items in the survey table remain, as all respondents and answers will be removed. Note that, due to the nature of credentials, the dynamic is always decreasing. It is rather intuitive to conclude that, as the removal procedure progresses, the remaining respondents and their answers will assume increasing positions on the credentials scale — with the lowest credentials presented first — just because we move upwards while building the defining sequence. However, once we reach the peak, the credentials start to decline, indicating that the scale is single peaked. Indeed, it can be demonstrated that the respondents' credentials values will first show the tendency to grow, and once they reach a certain point, their values will start to decline. This pattern corresponds to a typical single-peakedness of the defining sequence. Therefore, the defining sequence does not only provide an ordinary order of the respondents, but also allows identifying the conditions under which the credentials reach the peak — the highest point on the scale.

Owing to this property, the defining sequence of credentials is a double-folded order — as the values of its elements first increase until the peak is reached, after which they start decreasing. In this respect, the defining sequence formation is akin to the Greedy type algorithms, aimed to

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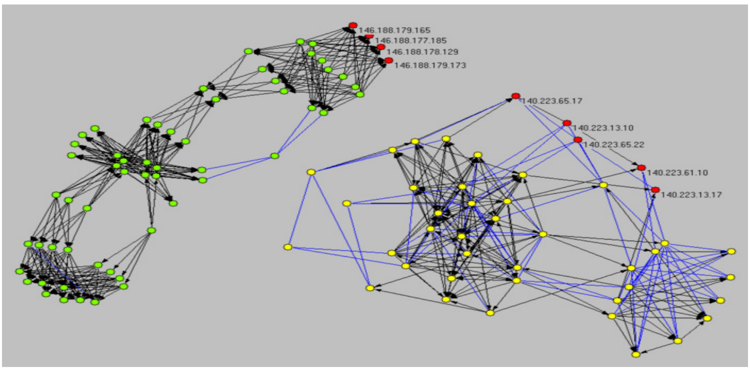
[7] An example of such type arithmetic may be found in L.K. Võhandu. (1980) Some Methods to Order Objects and Variables in Data Systems, Proceedings of Tallinn Technical University, No. 482, pp. 43 – 50..

improving some criteria.[8] Such algorithms are simple to use and are thus suitable for programming. However, it must be ascertained *a priori* that the result is an optimal solution, referred to as the Kernels. It is thus fortunate that the optimality of a defining sequence can be rigorously proved. This gives us confidence that we are not only proceeding in the right direction but have also chosen a suitable vehicle for our journey. This will be demonstrated through some significant examples below.

Cellular Networks

In particular, in the narrow sense of the term, *“A cellular network or mobile network is a communication network where the link to and from end nodes is wireless. The network is distributed over land areas called "cells", each served by at least one fixed-location transceiver (typically three cell sites or base transceiver stations). These base stations provide the cell with the network coverage, which can be used for transmission of voice, data, and other types of content…”* this paragraph is quoted from open sources.

In the broadest sense of the word, cellular networks promote ”media diversity” and are changing our reading habits. However, not many users are aware of the underlying processes that enable us to contact our friends via face book, “surf” various sites for the latest news, or obtain a response on queries on the subject of our interest. Cellular networks are a complex objects. Indeed, most cannot fathom how they function in practice. The following — keep in mind the picture below — may shed some light on this amazing technological invention.



[8] Advances in Greedy Algorithms, Edited by Witold Bednorz. Published by In-Tech (2008). In-Tech is Croatian branch of I-Tech Education and Publishing KG, Vienna, Austria, ISBN 978-953-7619-27-5.

In old days, when the personal computers were relative rarity, users could only interact with the system via the Disk Operating System (DOS). Some of the DOS commands can still be seen using the C:\ command prompt. If the user, for example, types “PING www.microsoft.com” command, the answer will usually be given in 25 ms, confirming that the site is active. If the response takes more than 25 ms to arrive, or we receive no response at all, this indicates that something has gone wrong with the Internet connection. Such commands will always confirm whether a data packet sent from our PC has reached the designated server. The PING command can be applied to make a connection between all websites — i.e., any two Internet locations. Likewise, for example, the "TRACERT www.microsoft.com" command will provide information pertaining to any packet delivery failure that occurs on route to its final destination. Their path is possible to trace, because all data packets proceed along the cells/locations to their final destination. In this path, the first cell is always occupied by the Gateway cell on the local subnet — the first router in the chain of routers responsible for packet delivery. Each router is a cell, akin to a post office, and is responsible for routing the packets passing through, stamping each one with receipts for delivery or transit. Therefore, if a direct communication cannot be established, it will be easy to identify the location at which the error has occurred. As cellular networks design allows for such malfunctions, whereby alternative paths are provided, any issues on one path/cell will have adverse effect on the total network throughput for other locations. The inverse situation is also true, as improving a direct connection somewhere on the cellular networks increases the overall throughput as well.

The process described above allows indicators to be assigned, corresponding to the average number of attempts made by packets on the Network (inclusive cells, which do not have direct connections) to reach the destination cell from the source cell. The number of cells within the network is extensive, and so is the total number of possible pairwise connections. Using our earlier nomenclature, it is equal to the number of items in the table of rows and columns — one of the standard forms of network representation. Some of the items in the table will be empty because there are no direct connections, which can be established between these cells.

Clearly, the main feature of the cellular networks is its dynamic nature. The average number of packet deliveries — the number of attempts to reach the destination — depends on current network structure, which can change these averages. At a more rigorous level of abstraction, the Markov

Chain, meets some postulates of packet deliveries, and can be employed when describing packet deliveries and processes required for these packets to reach their respective destinations. Some indicators, or credentials, formed by performing calculus on thus formed Markov Chains may help in elucidating this process. In fact, the following excerpt from Wikipedia may be useful:

> *A Markov chain (discrete-time Markov chain or DTMC), named after Andrey Markov, is a random process that undergoes transitions from one state to another on a state space. It must possess a property that is usually characterized as "memorylessness": the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it. This specific kind of "memory lessness" is called the Markov property. Markov chains have many applications as statistical models of real-world processes.*[9]

While the assumption that the pertinent information of the preceding states is implicitly included in the current state is an important property of Markov Chains is highly beneficial, its dynamic nature is of primary importance for the present discussion.

This principle can be applied to the cellular networks as the most common form of communication network. We will try to elucidate what the dynamics might represent in this context. In a real Web communication network, the cellular networks can be depicted as a collection of routers or switches that are “alive.” For the network to function, it is necessary to conduct periodic repairs, reconstruction or extensions, whereby some cells might be removed or replaced. Malfunctions are also a common occurrence due to the vastness and complexity of the network. So, what affect all these changes have on the network performance? Intuitively, malfunctions compromise the communication network abilities, while repairs enhance the quality of services. New communication units bring about better throughput, while removing the cells requires that the traffic be restructured. Similarly, traffic protocols are in place, allowing the packets along open routes to be rerouted in order to reach their destinations automatically.

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[9] Mullat J.E. (1979) An article was published on Markov Chain analysis in the spirit of this lines in Tallinn Technical University Proceedings, Data Processing, Compiler Writing, Programming, Анализ Данных, Построение Трансляторов, Вопросы Программирования, No. 464, pp. 71–84.

This is where the notion of "The Monotone System" is evident in its full power. In case of positive actions (repairs/extensions), network performance in enhanced, as the components and processes become more reliable. Conversely, negative actions (malfunctions) exert negative effects, whereby network performance worsens. However, in many cases, this level of abstraction is overly simplistic. In nature, we do not expect localized improvements to result in benefits to all elements and processes. Indeed, in any system, some elements will remain unaffected, or even experience worsening. As mathematics is an exact discipline, it is sometimes necessary to introduce some simplifications when describing such complex systems. Thus, for the sake of the discussions that follow, we will further postulate that the system performance as a whole is improving (worsening) when an improvement or worsening occurs locally.

This assumption prompts a very reasonable question. What does this view contribute to our understanding, explained above, of the communication networks functioning? It can, for example, allow us to proceed with optimal design of communication networks, as it renders the design process more precise.

Still, we will first revisit our Grand Ordering of items $A_1, B_2, A_3, A_4, C_5, D_6, C_7, E_8$ when constructing the main, i.e., the Grand defining sequence $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$ and its defining subsequence C_5, C_7, A_1, A_4. Let us examine the removed items B_2, D_6, A_3, E_8 more closely, in the context of constructing the sequence C_5, C_7, A_1, A_4 — as a result of which, the items B_2, D_6, A_3, E_8 and their credentials are removed. We can take an opposite approach and try to include these items back into the sequence C_5, C_7, A_1, A_4. We can first consider B_2 and then try with D_6, then with A_3 and finally E_8. In so doing, we can recreate the individual credentials for all items (B_2, D_6, A_3, E_8) even if they are not included in the existing sequence C_5, C_7, A_1, A_4. In fact, using this strategy would result in the following values: 1 for B_2, 1 for D_6, 3 for A_3 and 1 for E_8. If the objective was to increase credentials' values, we can conclude from the above that only the addition of item A_3 to the sequence

C_5, C_7, A_1, A_4 will have *a posteriori* a positive effect, as in all other cases the credentials decline below 2. In other words, inclusion of items B_2, D_6 and E_8 will worsen the situation, because the frequencies/credentials decrease from 2 to 1, whereas addition of A_3 does not change the value of credentials, which remain equal to 2. Formally, including items into subsequence can be viewed as a destabilization, or mapping of subsequences of items. It can be shown that, in spite of the destabilization factor, the defining sequence, however, at same point cannot be extended without worsening its quality. In that case, we can say that it has reached a stable or steady state condition.

This has beneficial implications for building a desirable network via some mappings explorations. The nomenclature of these mappings is very similar to the fixed-point approach.[10] It is also evident that, attempting to map a sequence C_5, C_7, A_1, A_4 to C_5, C_7, A_1, A_3, A_4, we have concluded that the sequence expanded by the addition of item A_3 has reached its most optimal condition. In other words, nothing can be added without worsening its state. Actually, in the discussions that follow, this fixed-point approach will be used to explain some mappings, rather than relying on a defining sequence. Thus, the communication networks analysis below will employ this fixed-point line of reasoning.

When designing a relatively simple communication network, one of the objectives might be to guarantee some throughput, such as stipulating that all packets must reach their destination in a 25 ms interval. As previously noted, the cells of the communication networks consist of routers or switches, responsible for redistributing and conducting packet movements from their source points, via temporary locations, to their final destinations. Switches are superior to routers as they learn about packets' temporary destinations, i.e., the path that must be taken when transmitting the packets, thereby significantly improving the throughput. A potential geographical layout of these extremely sophisticated and expensive devices is usually planned in the initial phase of the network design.

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[10] Mullat J.E. (October 1979) Fixed point searching was first introduced in "Stable Coalitions in Monotonic Games," Translated from Avtom i Telemekh., No. 10, pp. 84–94, in the form of sequences, in accordance with parameter values upon which the mapping was constructed. Later (July 1981), the mapping technique was explained in greater detail in "Counter Monotonic Systems in the Analysis of the Structure of multivariate Distributions," Translated from Avtom. i Telemekh., No. 7, pp. 167–175.

When deciding whether to place a router or a switch at the chosen geographical location, many factors must be taken into consideration.[11] While addition of a router or a switch will certainly improve the throughput, it also increases network maintenance, drift expenses become uncertain, and the costs of installation increase. In sum, not having an adequate number of these sophisticated devices will not provide sufficient throughput, whereas too many devices increase the costs. This dilemma is solved with a compromise that requires multilevel optimization while designing the communication networks.

It seems intuitive that the aforementioned fixed-point search can help to solve, at least in some cases, the problem. It is also advantageous to conduct Markov Chain analysis by building the net with a desirable property to maintain the throughput above a certain level. Thus, given a Markov Chain of potential network structure in tabular form, we can proceed by adding further cells or communication lines, and analyze the outcome. While it is likely that this process will improve the performance initially, at some point, further additions will be too costly for the benefits they provide. The problem thus reduces to finding the most optimal arrangement of lines and cells in the communication network, which guarantee the best throughput, such as 25 ms stipulated above. In doing so, we have the opportunity to convert the throughput credentials into some sort of effective credentials of packets' pass characteristics, representing average number of pair wise hits between cells within the communication network obeying the monotonic property in line with that applied to items $A, B, C, \ldots$.

Highly effective procedures already exist, the aim of which is to find the best stable solutions — the fixed points of Monotone Systems mappings. In these procedures, the defining sequence is constructed by means

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[11] In this direction, an extensive study, also based on the theory of "Monotone Systems" with cellular networks, was carried out by О. А. Шорин (2006), генеральный директор ЗАО «НИРИТ», д. т. н., профессор, кафедра радиотехнических систем, Московский Технологический Университет Связи и Информации; by Р. С. Токарь (2014), технический специалист ОАО «МТС», "Elektrosvjaz," No.1, pp. 45-48, , in Russian; Р.С. Аверьянов, директор по производственной деятельности ООО «НСТТ»; Г.О. Бокк (2017), директор по науке ООО «НСТТ», д.т.н., and А.О. Шорин, технический директор ООО «НСТТ»,, "*Optimizing the size of the ring antenna and the rule formation of territorial clusters for cellular network McWILL"*, "Elektrosvjaz," No.1, pp. 22-27,, *Method of "Adaptive Distribution of Bandwidth Resource"*, Russian Federation, Federal Service for Intellectual Property, RU 2 640 030 C1, Application 2017112131, in Russian,

other than those previously described. However, irrespective of the methodology applied, the outcome is still the defining sequence characterized by single peakedness. Most importantly, the point at which the maximum/minimum is reached will still represent our optimal solution. This is one of the examples of solving NP hard problems with polynomial P-NP complexity.

Economy

Next on our agenda is Monotone Systems implementation, this time in the context of retail networks. In the field of economy, this approach is typically applied in bilateral agreements between agents for goods delivery or production. This will mandate designing an economic network the structure of which can be visualized via graphs of potential agreements. The cells of such network represent agents, whereas connections represent contracts, i.e., bilateral goods delivery or requests, etc. It should be noted that, when requesting or delivering goods and commodities, expenses, prices and profit maintenance are the main consideration.

Let us consider this in an example of a client wishing to rent a car parking spot at the airport for some price during the vacation period. Given that, if the client is requesting a parking spot, this implies that he/she will drive to/from the airport, so the cost of petrol and any other charges (such as motorway tariffs) will have to be included in the overall cost of rental. This should be compared to the expenses incurred by traveling by a taxi or public transport and determine whether the option is viable. Which option the client will take will depend on any changes in prices, confirming that the structure of economic network is indeed dynamic. In addition, each agent has the right to decide with whom in the network to sign a contract. In terms of game theory, this can be represented by strategies in the form of lists of available agents, their corresponding services and costs.

Clearly, the structure of any economic network is dynamic — some new contracts will emerge, while some old ones will not be realized. This process is similar to that taking place in previously described communication networks. Thus, once again, we are under the jurisdiction of a Monotone System scheme. Indeed, in case of a bilateral agreement, certain action somewhere in the retail chain will not be realized and will have a negative impact on the performance of the entire chain. Forming new agreements, on the other hand, is likely to have a positive effect. However, in practice, addition of a new contract can also result in negative consequences, which some firms accept as they hope to cover those losses in future. Therefore, as was previously done, for simplification, we will postulate that, generally, new bilateral relations in the network always have a positive effect.

In analyzing the network, we might be interested in the abilities of the economic network to counteract so-called market volatility arising when prices of commodities and raw materials, or currency exchange rates, fluctuate. Volatility causes additional disturbing forces in the reconstruction of the network architecture. One of the known expenses affecting network functioning is transaction cost. Transaction cost parameter allows ordering all transactions in the network on the transaction costs scale. Most importantly, it enables us to apply the defining sequence of bilateral credentials — this time, performing calculus of profit indicators with regard to network architecture design.

Fixed Point Technique

The fixed point technique, when applied to economic network design, may be understood as a search for some equilibrium state when the network bilateral agreements are in stable condition, while the network as a whole is able to cope with economic volatility. When such a stable condition is achieved, it will be impossible to introduce new contracts without revising the entire network structure. Single peakedness of the defining sequence allows us to find the network parts that are most resilient to volatility. In addition, it allows making efficient decisions regarding delivery of commodities to their destinations and making requests for raw materials from producers. Such advantages are particularly relevant when attempting to attract new customers when trying to restructure existing networks with the aim of finding new possibilities to improve the services.

Thus far, we have considered Monotonic Systems consisting of atomic items. In other words, it was always possible to count how many items belong to the system, i.e., the number of items was finite. That was the case with lexicographical or chronological ordering of some items, whereby the credentials of items were chosen as frequencies. In such cases, the available items were presented sequentially and were clearly distinguished from others. The communication networks that were considered in the preceding discussions were also atomic, as the aim was to maximize the packet throughput from source to destination (i.e., minimize the delivery time). The same was the case in economic networks, where the network structure was only viable if it was profitable, as measured by transaction costs. In all these examples, our aim was to build a defining sequence in order to find the peak — the kernel of the ordering, because such a sequence was single peaked. It was also emphasized that the aim was to find a fixed point at which the structure design is optimal, whether we chose to design a communication or economic network.

Extending the defining sequence notion to analytical functions defined on various types of topologies is impossible because the resulting defining sequence will be infinite. Instead, we will apply the standard perspective when examining analytical single-peaked functions, aiming to find the peak of these functions. There is nothing new in this approach. The novelty, however, stems from the single-peaked phenomena, akin to the bargaining games. In such cases, one side has single-peaked preferences, and thus exhibits non-conforming behavior, while the second player, aims to maximize his/her benefits. In such scenario, the first player's preferences increase until they reach the peak, after which they start to decrease. In contrast, while the first player is moving along his/her single-peaked preferences, the second player's preferences always increase. The reader may benefit from exploring this further in the context of a sugar-pie game scheme, which is a suitable example of such analytical preferences.[12] In the present discussion, it is important to appreciate the extension of the single-peaked preferences representing the family of single-peaked functions, as this is the main advantage of this fixed-point approach. However, its application requires finding roots of some equations in order to identify stable states, inclusive of those credentials located at the peak of the credentials scale. A good example of such approach can be found in welfare economics, where the credentials of our scheme actually represent the level of transfer payments for those in need.

Mechanisms Design

Instead of analyzing and trying to predict the economic or political behavior of agents based on known standards of economic or political behavior, we will place agents in a desirable environment, expecting agents, as rational players, to come to reasonable decisions by virtue of their own rational behavior, or in the power of their own rational actions. In fact, such a scenario is explained by "The Sugar-Pie game" as an example where the trading model is reversed. In other words, the goal is not to find a solution as a result of the determination of the characteristics of the participants, but rather as a fair division of the cake among all players. In case of two players, dividing pie into two halves would be deemed fair,

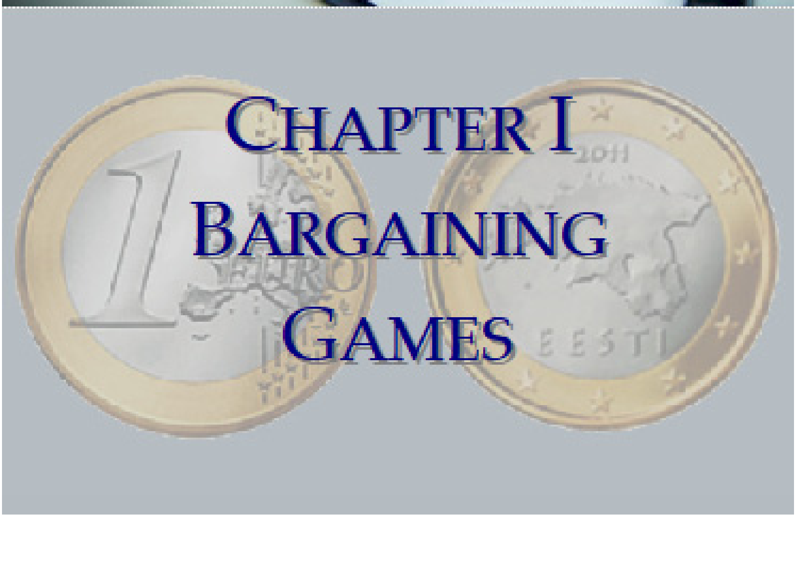
[12] Joseph E. Mullat. (2014) "The Sugar-Pie Game: The Case of Non-Conforming Expectations, Walter de Gruyter." Mathematical Economic Letters 2, 27–31..

and can thus be postulated as the desirable target. On the other hand, we may wish to predict the characteristics of participants *a posteriori*, i.e., after making this particular fair division, proclaimed as the best solution. This solution should also be understood as a design of partners' trading skills in such a way that the determination of the effective solution will be found to pursue this objective. However, it must be noted that this is the objective of the designer, rather than the goal of rational participants. Here, it must also be emphasized that we are not engaged in a symmetrical trading model, but rather the trading model characterized by so-called non-conforming interests of the participants. In fact, a standard economic situation involving company owners and company employees is not always 100% antagonistic with respect to wage negotiations. Frequently, the interests of the workers and the owners are not in conflict, even if this seems counterintuitive based on the well-known principle of scissors.

The solution to the problem of sweet pie division is also not straightforward if further costs are considered. If both parties hire lawyers, they will thereby charge a fee for service, which can in proportion of be set depending on their negotiating strength, cf., €230 and €770 in the Sugar-Pie game. If any of the negotiators wish to claim more of the pie, he/she will have to pay more for a lawyer who will have to work harder to achieve this fair but unequal position. In other words, some mappings on the credentials scale are necessary. In particular, the proposed sweet cake scheme can be used to create the desired political solutions in negotiations on the division of the tax pie collected by the state.

Indeed, citizens donate part of their salary as income tax. When new needy clients come along, their transfer payments must be funded. Thus, taxes increase, and after-tax incomes of citizens' decrease. With enough unemployed fined to work, the situation is reversed as tax revenues increase, ultimately leading to an after-tax benefit for all citizens. This situation can also be an example of what is now understood as the design of economic mechanisms.[13] It can thus be applied to design a political system that has desirable properties. One of these properties can be depicted as fixed points, reflecting the case when taxation reaches its absolute minimum, and when it is reached and the necessary adjustments to the rules and norms of taxation are made, the political system stabilizes with respect to economic volatility.

[13] The 2007 Nobel Memorial Prize in Economic Sciences was awarded to Leonid Hurwicz, Eric Maskin, and Roger Myerson *"for having laid the foundations of mechanism design theory"*.



Chapter I
Bargaining Games

The Sugar-Pie Game: The Case of Non-Conforming Expectations *

Abstract: Playing a bargaining game, the players with non-conforming expectations were trying to enlarge their share of a sugar-pie. The first player, who was not very keen on sweets, placed an emphasis on quality. In contrast, for the second player, all sweet options, whatever they might be, were open. Thus, this paper aims to determine the negotiating power of the first player, if equal division of the pie was desirable, i.e., both players aimed to get ½ of the available sweets.
Keywords: game theory; bargaining power; non-conforming expectations

1. INTRODUCTION

When bargaining, the players are usually trying to split an economic surplus in a rational and efficient manner. In the context of this paper, the main problem the players are trying to solve during negotiations is the slicing of the pie. Slicing depends upon characteristics and expectations of

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* Mullat J.E. (2014) The Sugar-Pie Game: The Case of Non-Conforming Expectations, Walter de Gruyter." *Mathematical Economic Letters* 2 , 27–31.

the bargainers. For example, while moving along the line at the z-axis (the size), the u-axis in Fig. 1 displays single-peaked expectations of player No. 1. In comparison, concave expectations of player No. 2 are shown in Fig. 2. The elevated single-peaked $5/6$ -slice curve in Fig. 1 corresponds to the lower, but adversely increasing, concave $1/6$ curve of expectations in Fig. 2, and for the other sugar-pie allotment $1/9$, $8/9$.

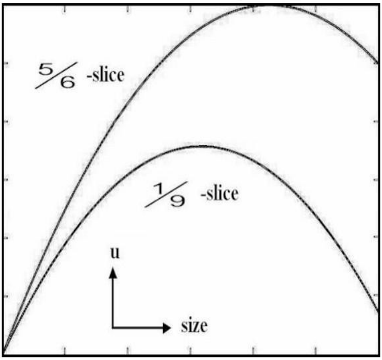

Figure 1. Player No. 1 expectations

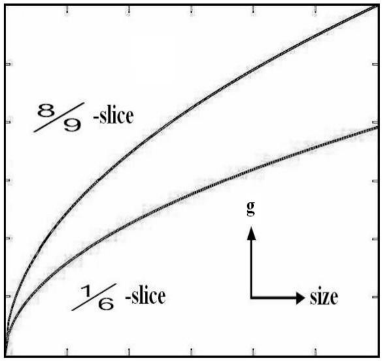

Figure 2. Player No. 2 expectations

Given that the players' expectations are non-conforming,[1] as shown in Fig 1., and Fig. 2, splitting a pie no longer represents any traditional bargaining procedure. Instead of dividing the slices, the procedures can be resettled. Thus, we can proceed at distinct levels of one parameter — parametrical interval of the size, which turns to be the scope of negotiations. In fact, Cardona and Ponsattí (2007, p. 628) noticed that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all the parties in the bargaining process have the same, conforming expectations. This is even true when the expectations of the second player are principally non-conforming, i.e., concave, rather than single-peaked. Indeed, in the case of non-conforming expectations, the scope of negotiations — also known as "*well defined bargaining problem*" or "*bargaining set*" related to individual rationality (Roth, 1977) — allows for dropping the axiom of "Pareto efficiency." Thus, combined with the *breakdown* point, the well-defined problem, instead of slices, can be solved inside parametrical interval of the pie size.

1 We say also interpersonally incompatible, i.e., impossible to match through a monotone transformation (Narens & Luce, 1983).

With these remarks in mind, any procedure of negotiating on slices accompanied by sizes can be perceived as two sides of the same bargain portfolio. Therefore, it is irrelevant whether the players are bargaining on slices of the pie, or trying to agree on their size. This highlights the main advantage of the parametric procedure — it brings about a number of different patterns of interpretations of outcomes in the game. For example, it can link an outcome of an economy to a suitable size of production, scarcity of resources, etc. — all of which are indicators of most desirable solutions. Indeed, our initiative could serve to unify the theoretical structure of economic analysis of productivity problem. Leibenstein (1979, p. 493) emphasized that "*...the situation need not be a zero sum game. Tactics, that determine the division can affect the size of the pie.*" Clarifying these guidelines, Altman (2006, p. 149) wrote:

> "*There are two components to the productivity problem: one relates to the determination of the size of the pie, while the second relates to the division of the pie. Looked upon independently, all agents can jointly gain by increasing the pie size, but optimal pie size is determined by the division of pie size.*"

2. THE GAME

The game demonstrates how a sugar-pie is fairly sliced between two players. The first player, denoted as HE, is a soft negotiator, not very keen on sweets, and would not accept a pie if the size of the pie is too small or too large. In HIS view, too small or too large sugar-pies are not of reasonable quality. The second player, hereafter referred to as SHE, is a tough negotiator and prefers obtaining sweets, whatever they are.[2]

The axiomatic bargaining theory finds the asymmetric Nash solution by maximizing the product of players' expectations above the disagreement point $d = \langle d_1, d_2 \rangle$:

$$\arg\max_{0 \le x+y \le 1} f(x, y, \alpha) = \left(u(x) - d_1\right)^{\alpha} \cdot \left(g(y) - d_2\right)^{1-\alpha},$$

the asymmetric variant (Kalai, 1977).

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[2] Note that, for the purpose of the game, we do not ignore the size of the pie but put this issue temporarily aside.

Although the answer may be known to the game theory purists, the questions often asked by many include: *What are* x, y, α, $u(x)$ *and* $g(y)$? *What does the point* $\langle d_1, d_2 \rangle$ *mean? How is the* $\arg\max$ *formula used?* The simple answer can be given as:

- x *is HIS slicing of the pie, and* α *is HIS bargaining power,* $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ *is HIS expectation, for example* $u(x) \equiv x$, *of HIS* x *slicing of the pie;*
- y *is HER slicing of the pie, and* $1-\alpha$ *is HER bargaining power,* $0 \le y \le 1$;
- $g(y)$ *is HER expectation, for example* $g(y) \equiv \sqrt{y}$, *of HER* y *slicing of the pie.*

Based on the widely accepted nomenclature, we call $s = \langle u(x), g(y) \rangle$ the utility pair. The disagreement point $d = \langle d_1, d_2 \rangle$ denotes what HE and SHE collect if they disagree on how to slice the pie. The sugar-pie disagreement point is $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$, whereby the players collect nothing. Further, we believe that expectations from the pie are more valuable for HER, indicating HER desire $g(\frac{1}{2}) = \sqrt{\frac{1}{2}} = 0.707$ for sweets, which is greater than HIS desire $u(\frac{1}{2}) = 0.5$.

Now, considering the $\arg\max$ formula $f(x, y, \alpha)$, one may ask a new question: *What is the standard that will help to redesign bargaining power* α *facilitating* HIS *negotiations to obtain a desired half of the pie?* SHE may only accept or reject the proposal. A technical person can shed light on the solution. We can start by replacing $u(x)$ with x, $y = 1 - x$, $g(y)$ with $\sqrt{1-x}$, and taking the derivative of the result $f(x, 1-x, \alpha)$ with respect to the variable x by evaluating $f'_x(x, 1-x, \alpha)$. Finally, with $x = \frac{1}{2}$, the equation $f'_x(\frac{1}{2}, \frac{1}{2}, \alpha) = 0$ can be solved for α; indeed, $\alpha = 1/3$ provides a solution to the equation $f'_x(\frac{1}{2}, \frac{1}{2}, \alpha) = 0$.

In general, one might feel comfort in the following judgment:

"*Even in the face of the fact that SHE is twice as tough a negotiator,*[3] *to count on the half of the pie is a realistic attitude toward HIS position of negotiations. Surely, rather sooner than later, since HE revealed that SHE prefers sweets whatever they are, HE would have HER agree to a concession.*"

This attitude might well be the standard of redesigning the power of HIS negotiation abilities if half of the pie is desirable as a specific outcome of negotiations.

Returning to the pie size issue, it will be assumed that, in the background of HIS judgment, the quality of the pie first increases, when the size is small. On the other hand, as the size increases, the quality eventually reaches the peak point, after which it starts to decline with the increasing size. Thus, the quality is single-peaked with respect to the size. For HER, the pie is always desirable. To handle the situation, we assume that HE possesses all the relevant skills of the pie slicing. Nonetheless, based on the aforementioned assumptions, for HIM, the slicing may, in some cases, not be worth the effort at all. If the slicing does not meet its goal, as just emphasized, HE can promote HIS own understanding of how to slice the pie properly. HE can enforce decisions, or effectively retaliate for breaches — recruiting for example "enthusiastic supporters," (Kalai, 1977, p.131). SHE, on the other hand, lacks slicing abilities, knowledge, skills or competence. Thus, if interests of both players in the final agreement are sometimes different or sometimes not, SHE cannot fully control HIS actions and intentions. In these circumstances, SHE might show a willingness to agree with HIS pie division, or at least not resist HIS privileges to make arrangements upon the size of the pie. Hence, from HER own critical point of view, by acting in common interest, SHE may admit HER lack of knowledge and skill. This clarifies HIS and HER asymmetric power dynamics.

Whether HE is committed or not is irrelevant for his decision to accept HER recommendation regarding the size z. HE is committed, however, only to slice x aligned in eventual agreement. The above can be restated, then, with the condition that HE seeks an efficient size z of the pie determined by the slice x. Let, as an example, the utility pair $\langle u, g \rangle$ of HIS and HER expectations be given by:

$$u(z, x) = z \cdot [(1 + x/2) - z] \;;$$

$$g(z, y) = z \cdot \sqrt{y}\,,\; z \in [0,1],\; x, y \in [0,1].$$

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[3] Let us say that SHE pays HER solicitor twice as much as HE does.

The root $z = \frac{1}{2}$ resolves $\left\langle u'_z(z,x)\big|_{x=0}\right\rangle = 0$ for z, and the root $z = \frac{3}{4}$ resolves $\left\langle u'_z(z,x)\big|_{x=1}\right\rangle$ accordingly. We can thus define efficient slices, relative to the size z, as a curve $x(z)$, which solves $u'_z(z,x) = 0$ for x. Evaluating x from $u'_z(z,x) = 0$ and subsequently replacing $x(z)$ into $u(z,x)$ and $g(z,x)$, yields $u(z) = z^2$ and $g(z) = z \cdot \sqrt{3 - 4 \cdot z}$. Now, given the scope $z \in \left[\frac{1}{2}, \frac{3}{4}\right] \subset [0,1]$ of the negotiations, the bargaining problem $\langle \boldsymbol{S}, d\rangle$ passes then into parametric space $\boldsymbol{S}_z = \langle u(z), g(z)\rangle$. In HIS view, the pie must fit the size requirements, since outside the interval $\left[\frac{1}{2}, \frac{3}{4}\right] \subset [0,1]$ the size z is inefficient — too small and thus not useful at all, or too large and of inferior quality. Therefore, the disagreement occurs at $d = \left\langle u(\frac{1}{2}), g(\frac{3}{4})\right\rangle$, $d = \left\langle \frac{1}{4}, 0\right\rangle$. The Nash symmetric solution to the game is found at $z = 0.69$, $x = 0.74$. On the other hand, HIS asymmetric power 0.21 is adequate for negotiating with HER about receiving half of the pie. The size $z = 0.62$, for example, in HIS view, fits the necessary capacities of a *stovetop* for provision of high quality sugar-pie.

Once again, to find the Nash symmetric solution, a technically minded person must resolve the equation $f'_z(z,\alpha) = 0$ *for* z, *where* $f(z,\alpha) = \left(u(z) - \frac{1}{4}\right)^{\alpha} \cdot g(z)^{1-\alpha}$ *when* $\alpha = \frac{1}{2}$; $z = 0.69$ *provides a solution to the equation. Thus, solving the equation* $u'_z(0.69, x) = 0$ *for* x *yields* $x = 0.74$. *To find the power of asymmetric solution, we first solve the equation* $u'_z(z, \frac{1}{2}) = 0$ *for* z, $z = 0.62$, $x = \frac{1}{2}$. *Then, we solve* $f'_z(0.62, \alpha) = 0$ *for* α *and find that HIS power matches* $\alpha = 0.21$, *which is adequate for negotiating with HER when an equal slicing of the pie is desirable, i.e., both HE and SHE receive ½ of the pie.*

3. BARGAINING PROCEDURE

The strategic bargaining game operates as a game of alternating offers. Given some light conditions, it is well known that, when players partaking in this type of game are willing to make concessions during the negotiations, they are likely to embrace the axiomatic solution. That is the reason why we continue our discussion in terms of a procedure similar to the strategic approach.

To recall, there are two players in our game — HE, with emphasis on quality, and SHE, with no specific preferences. A precondition for the agreement was that the expectations of negotiators solely depend on HIS framework of how to set the size parameter, rather than the slice. As a consequence of this dependence, efficient sizes provide a fundamental correspondence to crucial slices. Accepting the precondition, SHE will only propose efficient sizes, as HE will reject all other choices.

Nonetheless, it is realistic that SHE would — by negligence, mistake or some other reason — recommend an inefficient size, which HE would mistakenly accept. On the contrary, it is also realistic that HE has an intention to disregard an efficient recommendation. This will be irrational handling as, in any agreement, no matter what is going on, both players are committed by proposals to slices. Therefore, making a new proposal, HER recommendation on sizes makes a rational argument that HE must accept or reject in a standard way. Such an account, instead of an agreement upon slices, as we believe, explains that the outcome of the bargaining game might be a desirable size $z^{o} \in [z_1, z_2]$. Hereby, only the interval, named also the scope $[z_1, z_2]$ of negotiations, bids proposals, which now, by default, are binding efficient sizes with slices x. Consequently, the bargaining game performs exclusively in the interval $[z_1, z_2]$. Hence, $[z_1, z_2]$ is the scope of HIS efficient sizes of most trusted sugar-pie platforms for negotiations, where players would choose sizes, accepting or rejecting proposals. The negotiators' expectations, depending on $[z_1, z_2]$, arrange a bargaining frontier $\boldsymbol{S}_z$ as a way to assemble the bargain portfolio. Therefore, the negotiators may focus on making the size proposals. If rejected, the roles of actors change and a new proposal is submitted. The game continues in a traditional way, i.e., by alternating offers.

Observation. *In the alternating-offers sugar-pie game, the functions* $\left(u(z)-d_1\right)^{\alpha}$ *and* $\left(g(z)-d_2\right)^{1-\alpha}$ *imply HIS and HER expectations, respectively, over the pie size* $z \in \left[z_1, z_2\right]$. *With the risk* $1 >> q > 0$ *of negotiations to collapse prematurely into disagreement point* $d = \left[d_1, d_2\right]$, *the solution* z^o *of well-defined bargaining problem* $\left\langle \boldsymbol{S}_z, d \right\rangle$ *is enclosed into the interval* $\left[z', z''\right] \subset \left[z_1, z_2\right]$, $z^o \in \left[z', z''\right]$. *The margins* z', z'' *are solving the equations*

$$(1-q)\cdot\left(u(z^1)-d_1\right)^{\alpha} = \left(u(z^2)-d_1\right)^{\alpha},$$
$$(1-q)\cdot\left(g(z^2)-d_2\right)^{1-\alpha} = \left(g(z^1)-d_2\right)^{1-\alpha}$$

for variables z^1, z^2 (cf. Rubinstein 1998, p. 75).

In our example, when $x = \frac{1}{2}$ *(the half of the pie is a desirable (ex-ante) solution), HIS negotiation power* 0.21 *leads to the asymmetric solution* $z = 0.62$. *Let the risk factor of the premature collapse of negotiators be* $q = 0.05$. *Then, the interval* $\left[0.61, 0.64\right] \subset \left[0, 1\right]$ *sets up pie sizes providing the desirable solution, whereby the pie will be divided equally.*

4. Conclusion

In view of the above, a pretext for the analysis of the domain and the extent of bargain portfolio for two fictitious negotiators, denoted as HE and SHE, were established. The portfolio was supposed to account for the players having non-conforming expectations. Instead of slicing the sugar-pie, such an account allowed for the inclusion of a guide on how the eventual consensus ought to be analyzed and interpreted within the scope of negotiations upon the size of the pie. Players' bargaining power indicators specified by the bargaining problem solution were used in compliance with their respective desired visions and ambitions.

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A	B	C	D	X
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	1
1	1	1	0	0
1	1	1	1	0

Bargaining Solution on Boolean Tables

X	Y	Z	Y'	X.Y	Y'.Z	F= (X.Y) + (Y'.Z)
0	0	0	1	0	0	0
0	0	1	1	0	1	1
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	0	1	1	0	1	1
1	1	0	0	1	0	1
1	1	1	0	1	0	1

Abstract. This article reports not only a theoretical solution to the bargaining problem as used by game theoreticians, but also provides pertinent calculation. An algorithm that can produce the result within a reasonable time frame is proposed, which can be performed computationally. The aim is to increase the current understanding of one nontrivial case of bargaining.
Key words: coalition; game; bargaining; algorithm; monotonic system

"Rawls' second principle of justice: The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group. By simple extension, given that the worst-off is in his best position, the welfare of the second worst-off will be maximized, and so on. The difference principle produces a lexicographical ordering of the welfare levels of individuals from the lowest to highest." Cit. Public Choice III, Dennis C. Mueller, p.600

1. Introduction

Since the publication of "The bargaining problem" by John F. Nash, Jr. in 1950, the framework proposed within has been developed in different directions. For example, in their "Bargaining and Markets" monograph, Martin Osborn and Ariel Rubinstein (1990) extended the "axiomatic" concept initially developed by Nash to incorporate a "strategic" bargaining process pertinent to everyday life. The authors posited that the "time shortage" is the major factor encouraging agreement between bargainers. Various bargaining problem varieties emerged in the decades following Nash's pioneering work, prompting game theoreticians to seek their solutions, most of which did not necessarily comply with all Nash axioms. Beyond any doubt, the "Nonsymmetrical Solution" proposed by Kalai (1977); Hursanyi's (1967) "Bargaining under Incomplete Information"; "Experimental Bargaining", which was later proposed by Roth (1985); and the "Bargaining and Coalition" paper published by Hart (1985) are among some notable contributions to this field, confirming the fundamental importance of bargaining theory.

Bargaining and rational choice mechanisms are interrelated concepts and are treated as such in this work. In the context of general choice theory, the choice act can be formalized through internal and external descriptions, which requires use of binary relations and the theoretical approach, respectively. Thus, both description modes apply to the same object, albeit from different perspectives. The Nash Bargaining Problem and its solution express exactly the same phenomenon. Given a list of axioms, such as "Pareto Efficiency" or "Independence of Irrelevant Alternatives", in terms of binary relations the rational actors must follow, the solution is reached through scalar optimization applied to the set of alternatives. Indeed, the scalar optimization is at the core of the Nash's axiomatic approach and is the reason for its success in performing the bargaining solution calculation. In this respect, the motive of this work is to present a "calculation" of bargaining solution on large Boolean Tables and some theoretical foundations offered by the method. Unfortunately, in following Nash's scenario, numerous difficulties emerged.

Boolean Table representation transforms the real life "cacophonous" scenario into a relatively simple and understandable data format. However, allowing the scalar optimization not to be unique makes the picture more complex. Moreover, we are considering a purely atomic object that does not intuitively satisfy the "invariance under the change of scale of utilities" property typically assumed in the proofs. From the researcher's point of view, the issue stems from the incertitude pertaining to the most optimal choice of the scalar criteria. The Nash axiomatic approach, however, suggests that employing the product of utilities is the most appropriate, thus removing any uncertainty from further discussion. Nevertheless, in the context of the method presented here, it is posited that a reasonable solution might come into consideration, while game-analysts would be advised to include the method in a wider range of applicable game analysis tools.

In the next section, the main example of our bargaining game is introduced. In addition, in the appendix, we also illustrate a different bargaining between the coalition and its moderator applied to Boolean Tables using some conventional characteristic functions. It is worth noting that certain items in the main example, such as signals or misrepresentations, are not the primary topic of our discussion. These items must rather be understood as an illustration of the bargaining process complexity. In Section 3, we attempt to explain our intentions in a more rigorous manner. Here, we formulate our “Bargaining Problem on Boolean Tables” in pure strategies, thus providing the foundation for Section 4, where we exploit our pure Pareto frontier in terms of so-called Monotonic Systems chain-nested alternatives — the Frontier Theorem. In order to implement the Nash theorem for nonsymmetrical solution (Kalai, 1977), in Section 5, we introduce what we deem to be an acceptable, albeit complex, algorithm in general form. Even though lottery is not permitted in the treatment of Boolean Tables subsets representing pure strategies, as this approach does not necessary produce the typical convex collection of feasible alternatives, we claim that the algorithm will yield an acceptable solution. Finally, Section 6 presents an elementary attempt to formulate a regular approach of coalition formation under the coalition formation supervisor–the moderator structure. This attempt depicted in Figure 2, explaining the notation nomenclature of chain-nested alternatives adopted in our Monotonic Systems theory, discussed in Section 4. Section 7 summarizes the entire analysis, while also providing an independent heuristic interpretation, before concluding the study in Section 8.

2. EXAMPLE

Manager of the “Well-Being” company is determined to encourage employees to partake in health-promoting activities. The manger hopes to reduce company losses arising from disability compensations. To identify the employees’ preferences, the manager has initiated a survey. According to the survey responses, five health activities offered to the employees generated varying degrees of interest, as shown in Table 1.

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. 1		x	x			2
Em. 2	x	x		x	x	4
Em. 3		x	x	x		3
Em. 4	x	x		x	x	4
Em.. 5			x	x		2
Em. 6	x	x	x	x	x	5
Em. 7		x	x			2
Total	3	6	5	5	3	22

Table 1 Employee preferences pertaining to the company-sponsored health-promoting initiatives

The manager would like to treat the responses the employees have provided as an indication that they are willing to partake in the activities they selected. However, aware of unreliable human nature, he is not confident that they will keep their promises. The manager decides to reward all employees who actually participate in the wellness activities, and those who will be organized in the "Health Club". The manager has found a sponsor who has issued 12 Bank Notes in lieu of the project expenses. However, upon closer consideration of the rewards policy, the manager realized that many obstacles must be overcome in order to implement it in practice.

Fist, organizing activities that only a few employees would partake in is neither practical nor cost-effective. Thus, it is necessary to stipulate a minimum number of employees that must subscribe to each health activity. On the other hand, it is desirable to promote all activities, encouraging the employees to attend them in greater numbers. For this initiative to be effective, instructions (as a rule full of twists and turns) regarding the rewards regulations should be fair and concise. Usually, in such situations, someone (a moderator) must be in charge of the club formation and reward allocation. However, as the manager is responsible for financing health activities, he/she should retain control of all processes. Thus, the manager proposes to write down the **First Club Regulation**: *The manager rewards one Bank Note to an employee participating in at least* k *different activities* (where k is determined by the manager).

Determining the most optimal value of the parameter k is not a straightforward task, as it is not strictly driven by employees' preferences regarding specific activities to participate in. In fact, this task is in the moderator's jurisdiction, while also being dependent on the employees' decisions, as they act as the club members. The goal is to prohibit some club members to "spring over" health activities preferred by other members of the club by worsening, in the manager's view, the situation, thus requiring too many different activities to be organized. This issue can be avoided by the inclusion of the **Second Club Regulation**: *If a certain employee in favor of receiving rewards participated in fewer than* k *activities, no one will be rewarded.* By instituting this regulation, the manager aims to encourage the moderator to eliminate activities that would not have sufficient number of participants. Thus, the **Third Club Regulation**: *moderator's reward basket will be equal to the lowest number of participants per activity in the list of activities among all actually participating club members*. Indeed, to earn more rewards, the moderator might decide to organize a new club by excluding an activity with the lowest number of participants from the list of activities some of the members chose to attend as a part of the already organized club. This would effectively result in the lowest number of participants in the new and shorter list being higher than that in the previous list. It should also be noted that the reward regulation does not address the situation in which a club member declines an activity, allowing an individual outside the club to participate instead. In such a case, the club "activities list" may become shorter than that presented in Table 1, and would determine the size of the moderator's reward.

This scenario also provides the potential for the club members' preferences to be misrepresented to the company manager. Let us assume that the manager makes a decision $k = 1$, which has been, for whatever reason, made accessible to the moderator. Knowing that $k = 1$, the moderator actions can be easily predicted in accordance with the third club regulation. Indeed, using the employees' survey responses, the moderator can identify the most "popular" health activity, as well as the individuals that intend to participate in this activity. From the aforementioned regulations, it is evident that the moderator would receive the maximum reward if he manages to persuade other employees to participate in that particular activity only. Rational members would certainly agree to that proposal because, whether or not they take part in any other activity, their reward is still guaranteed.[1] The same logic obviously applies for $k > 1$ as well.

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1 We will disclose more complex misrepresentation opportunity later.

Thus, the essence of establishing fair rules pertains to determining the moderator's reward. If the moderator is not offered any rewards, the grand coalition formation is guaranteed, as all employees will become club members. This is the case, as participating in at least one activity would ensure that an employee receives a reward. However, due to the moderator actions, such grand coalition formation is not always feasible.

As previously noted, the moderator might receive a minor reward if a "curious" employee decides to take part in an "unpopular activity". Indeed, the third club regulation stipulates that the number of participants in the most "unpopular activity" governs the moderator reward size. Being aware of the potential manipulation of the regulations, and being a rational actor, the company manager will thus strive to keep the decision k a secret. It is also reasonable to believe that all parties involved — the club members, the moderator and the manager — will have their own preferences regarding the value of k. Therefore, an explanation based on the salon game principles is applicable to this scenario. Using this analogy, let us assume that the manager has chosen a card k and has hidden it from the remaining players. Let us also assume that the moderator and the club members have reached an agreement on their own card choice in line with the three aforementioned club regulations. The game terminates and rewards are paid out only if their chosen card is higher than that selected by the manager. Otherwise, no rewards will be paid out, despite taking into consideration the club formation.

However, not all factors affecting the outcome have been considered above. Indeed, the positive effect, f_k, which the manager hopes to achieve, depends on the decision k. We have to expect a single $\cap$-peakedness of the effect function for some reason. As a result, this function separates the region of k values into what we call prohibitive and normal range. In the prohibitive range, which includes the low k values, the effect has not yet reached its maximum value. On the other hand, when k value is high (i.e., in the normal range), the f_k limit is exceeded. Therefore, in the prohibitive range, the manager and the moderator interests compete with each other, making it reasonable to assume that the manager would keep his/her decision a secret. However, in the normal range, they might cooperate, as neither benefits from very high k values, given that both can lose their payoffs. Consequently, using the previous card game analogy, in the normal range, it is not in the manager's best interests to hide the k card.

Given the arguments presented above, the game scenario can be illustrated more precisely. Using the data presented in Table 1, and assuming that a reward will be granted at $k = 1,2$, the manager may count upon all seven employees to become the club members. If all employees participate in all activities, each would receive a Bank Note, and the moderator's basket size would be equal to 3. However, it would be beneficial for the moderator to entice to the club members to decline participation in *"No Smoking"* and *"Fattening Diet"* activities, as this would increase his/her own reward to 5. As all club members will still preserve their rewards, they have no reason not to support the moderator's suggestion, as shown in Table 2.

Table 2

Health activities	*Swimming Pool*	*BikeExercises*	*Moderate Alcohol*	*Total*
Em. 1	x	x		2
Em.. 2	x		x	2
Em.. 3	x	x	x	3
Em.. 4	x		x	2
Em. 5		x	x	2
Em.. 6	x	x	x	3
Em.. 7	x	x		2
Total	6	5	5	16

Table 3

Swimming Pool	*Total*
x	1
x	1
x	1
x	1
	0
x	1
x	1
6	6

In this scenario, the sponsor would have to issue 12 Bank Notes, which can be treated as expenses associated with organizing the club. The sponsor may also conclude that $k = 1$ is undesirable based on the previous observation that the moderator can deliberately misrepresent the members' preferences for personal gain.[2] The sponsor is aware that the moderator may offer one Bank Note to an employee that agrees to propose $k = 1$. Knowing that $k = 1$, the moderator may suggest to the club members to subscribe to the *"Swimming Pool"* activity only. However, in the sponsor's opinion, the moderator must compensate Employee No. 5 for the losses incurred by offering him/her one Bank Note. Otherwise, Employee No. 5, by participating in other activities distinct from *"Swimming Pool"* has the right to receive a reward and may report the moderator's fraud to the board. In this case, following the regulations in force (see Table 3), moderator's reward will be equal to 4 (1 would be deducted for

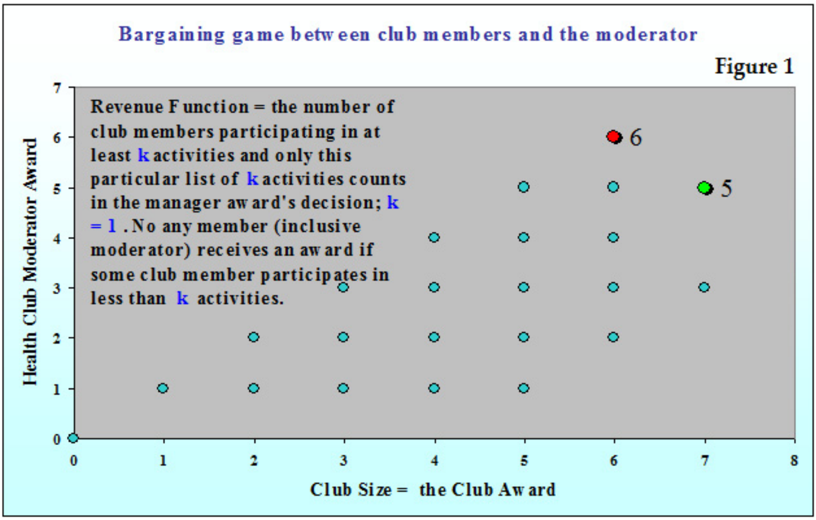
2 The more complex case of misrepresentation follows, as promised.

the signal and 1 for the compensation). However, this would still exceed the value indicated in Table 1. Thus, in order to decrease sponsor expenses or avoid misrepresentations, the company board may follow the sponsor's advice and propose $k \geq 3$.

It could be argued that $k \geq 3$ results in decreased participation in health activities because Employees No. 1, 5 and 7 will be excluded from the club and will immediately cease to partake in any of their initially chosen activities. However, based on Table 4, it can also be noted that, in such an event, the remaining employees (i.e., $2,3,4$ and 6) will still participate in heath activities and will still be rewarded.

Table 4

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. 2	x	x		x	x	4
Em. 3		x	x	x		3
Em. 4	x	x		x	x	4
Em. 6	x	x	x	x	x	5
Total	3	4	2	4	3	16



Now, the moderator's reward basket is equal to 2, since only Employees No. 3 and 6 would take part in *"Bike Exercises"*. Consequently, the sponsor expenses decrease from 10 to 6. In this case, the manager may decide to allow the moderator to retain his/her reward of 3 by eliminating *"Bike Exercises"* from the activity list, as organizing it for two participants only is not justified, as shown in Table 5. Note that Employee No. 3, due to this decision, must be excluded from the club list, in line with the second club regulation, cf. the suggestion above to eliminate *"No Smoking"* and *"Fattening Diet"* activities.

Table 5

Health activities	*No Smoking*	*Swimming Pool*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 2	x	x	x	x	4
Em. No. 4	x	x	x	x	4
Em. No. 6	x	x	x	x	4
Total	3	3	3	3	12

This decision does not seem reasonable, given that the aim of the initiative was to motivate the employees to exercise and improve their health. Thus, let us assume that $k = 5$ was the board proposal. This result would only concern Employee No. 6 being willing to participate in the health activities offered, see Table 6.

Table 6

Health activities	*No Smoking*	*Swimming Pool*	*Bike Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. 6	x	x	x	x	x	5
Total	1	1	1	1	1	5

The moderator may decide not to organize the club, as this would result in a reward equal to only one Bank Note. Similarly, the manager is not incentivized to promote all five activities if only one employee would take part in each one. As a result, at the board meeting, the manager would vote against the proposal $k = 5$. In sum, the manager's dilemma pertains to the alternative k choice based on the information given in Table 7.

Table 7.

	Club-members	*Clubmod-erator*	*Compen-sation*	*Signal*	*Bank Notes used*	*Bank Notes left*
T. 1, $k=2$	7	3	0	0	10	2
T. 2, $k=2$	7	5	0	0	12	0
T. 3, $k=1$	6	4	1	1	12	0
T. 4, $k=4$	3	1	0	0	4	8
T. 5, $k=4$	3	3	0	0	6	6
T. 6, $k=5$	1	1	0	0	2	10

To clarify the situation presented in tabular form, it would be helpful to visualize the manager's dilemma using the bargaining game analogy, where 12 Bank Notes are shared between the moderator and the club members.

The decision on the most optimal k value taken at the board meeting will be revealed later, using rigorous nomenclature, as only a closing topic is necessary to interrupt our pleasant story for a moment.[3]

Let us assume that three actors are engaged in the bargaining game: N employees, one moderator in charge of club formation, and the manager. Certain employees from $N = \{1,\ldots,i,\ldots n\}$ – the potential members of the club x, $x \in 2^N$, have expressed their willingness to participate in certain activities y, $y \in 2^M$, $W = \| a_{ij} \|_n^m$. Let a Boolean Table $W = \| a_{ij} \|_n^m$ reflect the survey results pertaining to employees' preferences, whereby $a_{ij} = 1$ if employee i has promised to participate in activity j, and $a_{ij} = 0$ otherwise. In addition, 2^M denotes of allegedly subsidized activities, whereby $y \in 2^M$ have been examined.

[3] Those unwilling to continue with the discussions on bargaining presented in the subsequent sections should nonetheless pay attention to this closing remark.

We can calculate the moderator payoff $F_k(H)$ using a sub-table H formed by crossing entries of the rows x and columns y in the original table W by further selection of a column with the least number $F_k(H)$ from the list y. The number of 1-entries in each column belonging to y determines the payoff $F_k(H)$. Characteristic functions family $\nu^k(x,y) \equiv \nu^k(H)$, $k \in \{1,\ldots,k,\ldots,k_{max}\}$, on N are known for the coalition games; in particular, for every pair $L \subset G$, $L, G \in 2^N \times 2^M$, we suppose that $\nu^k(L) \leq \nu^k(G)$. Further assuming that the manager payoff function $f_k(H)$ has a single $\cap$-peakedness, in line with the decisions $\langle 1,\ldots,k,\ldots,k_{max} \rangle$, $f_k(H)$ reflects some kind of positive effect on the company deeds. In this case, sponsor expenses will be equal to $\nu^k(H) + f_k(H)$.

Finally, it is appropriate to share some ideas regarding a reasonable solution of our game. The situation is similar to the Nash Bargaining Problem first introduced in 1950, where two partners — the club members and the moderator — are striving to reach a fair agreement. It is possible to find the Bargaining Solution $S_k \in \{H\} = 2^N \times 2^M$ for each particular decision k, see next sections. However, the choice of the number k is not straightforward, as previously discussed. For example, $k = 4, 5$ may be useful based on some *ex ante* reasoning, whereas maximum payoffs are guaranteed for the partners when $k = 1$. As that decision is irrational, because only one activity will be organized and, even though it will attract the maximum number of participants, it would fail to yield a positive effect $f(S_k)$ on the health deeds in general. The choice of higher k was previously shown to be counterproductive (too many activities will be offered, but would have only a few participants), yet the sponsor would benefit from issuing fewer rewards. For example, for $k = k_{max}$, an employee with the largest number of preferred k_{max} activities might become the only member of the club. This is akin to the median voter scheme (discussed by Barbera et al, 1993). However, a further consultation in this "white field" is necessary.

3. BARGAINING GAME APPLIED TO BOOLEAN TABLES

Suppose that employees who intend to participate in company activities have been interviewed in order to reveal their preferences. The resulting data can be arranged in $n \times m$ table $W = \|\alpha_{ij}\|$, where the entry $\alpha_{ij} = 1$ indicates that an employee i has promised to participate in activity j, otherwise $\alpha_{ij} = 0$. In this respect, the primary table W is a collection of Boolean columns, each of which comprises of Boolean elements related to one specific activity. In the context of the bargaining game, we can discuss an interaction between the health club and the moderator. The club choice x is a subset of rows $\langle 1,...,i,...,n \rangle$ denoting the newly recruited club members, whereby a subset y of columns $\langle 1,..., j,...m \rangle$ is the moderator's choice — the list of available activities. The result of the interaction between the club and the moderator can thus represent a sub-table H or a block, denoting the players' joint anticipation (x, y). The players are designated as Player No. 1 – the club, and Player No. 2 – the moderator, and both are driven by the desire to receive the rewards. Let us assume that all employees have approved our three reward regulations.[4] While both players are interested in company activities, their objectives are different. Player No. 1 might aim to motivate each club member to agree to partaking in a greater number of company-sponsored activities. Player No. 2, the moderator, might desire to subscribe maximum number of participants in each activity arranged by the company. Let the utility pair $(v(x), F(y))$ denote the players' payoff, whereby both players will bargain upon all possible anticipated outcomes (v, F).

Our intention in developing a theoretical foundation for our story was to follow the Nash's (1950) axiomatic approach. Unfortunately, as previously observed, some fundamental difficulties arise when adopting similar approach. Below, we summarize each of these difficulties, and propose a

4 We recall the main regulation that none of the club members, inclusive the moderator, receive their rewards if a certain club member participates in fewer than k activities.

suitable equivalent. When proceeding in this direction, we first formulate the Nash's axioms in their original nomenclature before reexamining their essence in our own nomenclature. This approach would allow us to provide the necessary proofs in the sections that follow.

As noted by Nash (1950), "... we may define a two-person anticipation as a combination of two one-person anticipation. ... A probability combination of two two-person anticipations is defined by making the corresponding combinations for their components" (p. 157). Readers are also advised to refer to Sen Axiom 8*1, p. 127, or sets of axioms, as well as Luce and Raiffa (1958), Owen (1968) and von Neumann and Morgenstern (1947), with the latter being particularly relevant for utility index interpretation. Rigorously speaking, the compactness and convexity of a feasible set $\boldsymbol{S}$ of utility pairs ensures that any continuous and strictly convex function on $\boldsymbol{S}$ reaches its maximum, while convexity guarantees the maximum point uniqueness.

Let us recall the other Nash axioms. The solution must comply with INV (invariance under the change of scale of utilities); IIA (independence of the irrelevant alternatives); and PAR (Pareto efficiency). Note that, following PAR, the players would object to an outcome s when an outcome s' that would make both of them better off exists. We expect that the players would act from a *strong individual rationality* principle SIR. An arbitrary set $\boldsymbol{S}$ of the utility pairs $s = (s_1, s_2)$ can be the outcome of the game. A disagreement arises at the point $d = (d_1, d_2)$ where both players obtain the lowest utility they can expect to realize – the *status quo* point. A *bargaining problem* is a pair $\langle \boldsymbol{S}, d \rangle$[5] and there exists $s \in \boldsymbol{S}$ such that $s_i > d_i$ for $i = 1,2$ and $d \in \boldsymbol{S}$. A *bargaining solution* is a function $f(\boldsymbol{S}, d)$ that assigns to every bargaining problem $\langle \boldsymbol{S}, d \rangle$ a unique element of $\boldsymbol{S}$. The bargaining solution f satisfies SIR if $f(\boldsymbol{S}, d) > 0$ for every bargaining problem $\langle \boldsymbol{S}, d \rangle$.

The advantage of our approach, which guarantees the same properties, lies in the following. We define a feasible set $\boldsymbol{S}$ of anticipations, or in more convenient nomenclature, a feasible set $\boldsymbol{S}$ of alternatives as a col-

5 We use the bold notifications $\boldsymbol{S}$ close to the originals. Notification S is preserved for stable point, see later.

lection of table W blocks: $\boldsymbol{\mathcal{S}} \subseteq 2^{\mathrm{W}}$. Akin to the disagreement event in the Nash scheme, we define an empty block $\varnothing$ as a *status quo* option in any set of alternatives $\boldsymbol{\mathcal{S}}$, which we call the refusal of choice. Given any two alternatives H and H' in $\boldsymbol{\mathcal{S}}$, an alternative $\mathrm{H} \cup \mathrm{H}'$ belongs to $\boldsymbol{\mathcal{S}}$. In other words, in our case, the set $\boldsymbol{\mathcal{S}}$ of feasible alternatives always forms an upper semi-lattice. Moreover, if an alternative $\mathrm{H} \in \boldsymbol{\mathcal{S}}$, it follows that all of its subsets $2^{\mathrm{H}} \subseteq \boldsymbol{\mathcal{S}}$. Although these arguments do necessitate further discussion, at this juncture, we will state that this is our equivalent to the convex property and will play the same role in proofs as it does in the Nash scheme.

The Nash theorem asserts that there is a unique bargaining solution $\mathrm{f}(\boldsymbol{\mathcal{S}}, \mathrm{d})$ for every bargaining problem $\langle \boldsymbol{\mathcal{S}}, \mathrm{d} \rangle$, which maximizes the product of the players' gains in the set $\boldsymbol{\mathcal{S}}$ of utility pairs $(\mathrm{s}_1, \mathrm{s}_2) \in \boldsymbol{\mathcal{S}}$ over the disagreement outcome $\mathrm{d} = (\mathrm{d}_1, \mathrm{d}_2)$. This is a so-called symmetric bargaining solution, which satisfies INV, IIA, PAR, and SYM – players symmetric identify, if and only if

$$\mathrm{f}(\boldsymbol{\mathcal{S}}, \mathrm{d}) = \arg\max_{(\mathrm{d}_1, \mathrm{d}_2) \le (\mathrm{s}_1, \mathrm{s}_2)} (\mathrm{s}_1 - \mathrm{d}_1) \cdot (\mathrm{s}_2 - \mathrm{d}_2).$$

It is difficult to make an *ad hoc* assertion regarding properties that can guarantee the uniqueness of similar solution on Boolean Tables. Nevertheless, in the next section, we claim that our bargaining problem on $\boldsymbol{\mathcal{S}} \subseteq 2^{\mathrm{W}}$ has the same symmetric or nonsymmetrical shape:

$$\mathrm{f}(\boldsymbol{\mathcal{S}}, \varnothing) \equiv \mathrm{f}(\boldsymbol{\mathcal{S}}) = \arg\max_{\mathrm{H} \in \boldsymbol{\mathcal{S}}} \mathrm{v}(\mathrm{H})^{\theta} \mathrm{F}(\mathrm{H})^{1-\theta}$$

for some $0 \le \theta \le 1$ provided that Nash axioms hold.

4. Theoretical Aspects of the Boolean game

Henceforth, the table $\mathrm{W} = \|\alpha_{ij}\|$ will denote the Boolean table discussed in the preceding section, representing employees' promises to attend company activities. It is beneficial to examine H rows x, symbolizing the arrival of new members to the club, committed to participating in at least k activities. Activities form, what we call here, a column's activity list y, $k = 2,3,...$, where k represents the reward decision. For each activity in the activity list y, at least $F(H)$ of club members intend to fulfill their promises. For example, let us consider the number of rows in H pertaining to the gain $v(H)$ of Player No. 1 (the club members), while the gain of Player No. 2 (the moderator's reward) is represented by $F(H)$.

Let us look at the bargaining problem in conjunction with players' preferences. The anticipations of the coming club members $i \in x$ towards the activity list y can easily be "raised" by $r_i = \sum_{j \in y} \alpha_{ij}$ if $r_i \geq k$, and $r_i = 0$ if $\sum_{j \in y} \alpha_{ij} < k$, $i \in x$, $j \in y$. Similarly, the moderator's anticipation towards the same activity list y can be "accumulated" by means of table H as $c_j = \sum_{i \in x} \alpha_{ij}$, $j \in y$.

We now consider this scenario in more rigorous mathematical form. Below, we use the notation $H \subseteq W$. The notation H contained in W will be understood in an ordinary set-theoretical nomenclature, where the Boolean Table W is a set of its Boolean 1-elements. All 0-elements will be dismissed from the consideration. Thus, H as a binary relation is also a subset of W. Henceforth, when referring to an element, we assume that it is a Boolean 1-element.

For an element $\alpha \equiv \alpha_{ij} \in W$ in the row i and column j, we use the similarity index $\pi_{ij} = c_j$, counting only on the Boolean elements belonging to H, $i \in x$ and $j \in y$. As the value of $\pi_{ij} = c_j$ depends on each subset $H \subseteq W$, we may write $\pi_{ij} \equiv \pi \equiv \pi(\alpha, H)$, where the set H represents the π-function parameter. It is evident that our similarity indices π_{ij} may only increase with the "expansion" and decrease with the "shrinking" of the parameter H. This yields the following fundamental definitions:

Definition 1. *Basic monotone property. Monotonic System will be understood as a family* $\{\pi(\alpha,H): H \in 2^W\}$ *of* π*-functions, such that the set* H *is a parameter with the following monotone property: for two particular values* $L, G \in 2^W$, $L \subset G$ *of the parameter* H*, the inequality* $\pi(\alpha, L) \leq \pi(\alpha, G)$ *holds for all elements* $\alpha \in W$*. In ordinary nomenclature, the* π*-function with the definition area* $W \times 2^W$ *is monotone on* W *with regard to the second parameter on* 2^W.

Definition 2. *Let* $V(H)$ *for a non-empty subset* $H \subseteq W$ *by means of a given arbitrary threshold* u *be the subset* $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq u\}$*. The non-empty* H*-set indicated by* S *is called a stable point with reference to the threshold* u *if* $S = V(S)$ *and there exists an element* $\xi \in S$*, where* $\pi(\xi, S) = u$*. See Mullat (1979, 1981) for a comparable concept. Stable point* $S = V(S)$ *has some important properties, which will be discussed later.*

Definition 3. By Monotonic System kernel we understand a stable point $S^* = S_{\max}$ with the maximum possible threshold value $u^* = u_{\max}$.

Libkin et al (1990); Genkin et al (1993); Kempner et al (1997); and Mirkin et al (2002) have investigated similar properties of Monotonic Systems and their kernels. With regard to the current investigation, it is noteworthy to state that, given a Monotonic System in general form, without any reference to any kind of “interpretation mechanism”, one can always consider a bargaining game between a coalition H – Player No. 1, with characteristic function $v(H)$, and Player No. 2 with the payoff function $F(H) = \min_{\alpha \in H} \pi(\alpha, H)$. Following Nash theorem, a symmetrical solution has to be found in form (1). In addition, we will prove below that our solution has to be found in the symmetrical or non-symmetrical form (2).

Definition 4. Let d be the number of Boolean 1's in table W. *An ordered sequence* $\overline{\alpha} = \langle \alpha_0, \alpha_1, ..., \alpha_{d-1} \rangle$ *of distinct elements in the table* W *is called a defining sequence if there exists a sequence of sets* $W = \Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *such that:*

A. Let the set $H_k = \{\alpha_k, \alpha_{k+1}, ..., \alpha_{d-1}\}$. *The value* $\pi(\alpha_k, H_k)$ *of an arbitrary element* $\alpha_k \in \Gamma_j$*, but* $\alpha_k \notin \Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$*,* $j = 0,1,...,p-1$.

B. *There does not exist in the set* Γ_p *a proper subset* L *that satisfies the strict inequality* $F(\Gamma_p) < F(L)$.

Definition 5. *A defining sequence is complete, if for any two sets* Γ_j *and* Γ_{j+1} *it is impossible to find* Γ' *such that* $\Gamma_j \supset \Gamma' \supset \Gamma_{j+1}$ *while* $F(\Gamma_j) < F(\Gamma') < F(\Gamma_{j+1})$*,* $j = 0,1,...,p-1$.

It has been established that, in an arbitrary Monotonic System, one can always find a complete defining sequence (see Mullat, 1971, 1976). Moreover, each set Γ_j is the largest stable set with reference to the threshold $F(\Gamma_j)$. This allows us to formulate our Frontier Theorem.

Frontier Theorem. *Given a bargaining game on Boolean Tables with an arbitrary set* $\boldsymbol{S}$ *of feasible alternatives* $H \in \boldsymbol{S}$*, the anticipations points* $(v(\Gamma_j), F(\Gamma_j))$*,* $j = 0,1,...,p$*, of a complete defining sequence* $\overline{\alpha}$ *arrange a Pareto frontier in* $\Re^2$.

Proof. Let $W^S \in \boldsymbol{S}$ be the largest set in $\boldsymbol{S}$ containing all other sets $H \in \boldsymbol{S}$: $H \subseteq W^S$. Let a complete defining sequence $\overline{\alpha}$[6] exist for W^S. Let the set H^c be the set containing all such sets $V(H)$, where $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq F(H)\}$. Note that $H \subseteq V(H^c)$ and $F(H^c) \geq F(H)$. Now, for accuracy, we must distinguish three situa-

[6] We are not going to use any new notifications to distinguish between Boolean Tables W and W^S.

tions: (a) in the sequence $\overline{\alpha}$ one can find an index j such that $F(\Gamma_j) \le F(H^c) < F(\Gamma_{j+1})$ $j = 0,1,...,p-1$; the case (b) $F(H^c) < F(W) = F(\Gamma_0)$; and (c) $F(H) > F(\Gamma_p)$. The case (c) is impossible because, on the set Γ_p, the function $F(H)$ reaches its global maximum. In case of (b), the anticipation $(v(\Gamma_0), F(\Gamma_0))$, $\Gamma_0 = W$, is more beneficial than $(v(H), F(H))$, which concludes the proof. In case of (a), let $F(\Gamma_j) < F(H^c)$, otherwise the equality $F(\Gamma_j) = F(H^c)$ is the statement of the theorem (when reading the sentence after the next, the index $j+1$ should be replaced by j). However, in this case, the set H^c must coincide with Γ_{j+1}, $j = 0,1,...,p-1$, otherwise the defining sequence $\overline{\alpha}$ is incomplete. Indeed, looking at the first element $\alpha_k \in H^c$ in the sequence $\overline{\alpha}$, it can be ascertained that, if $\Gamma_{j+1} = H^c$ does not hold, the set $H_k = H^c$ because it is the largest stable set up to the threshold $F(H^c)$. Hence, the set H_k represents an additional Γ-set in the sequence $\overline{\alpha}$ with the property A of a complete defining sequence. The inequalities $F(\Gamma_{j+1}) = F(H^c) \ge F(H)$, $v(\Gamma_{j+1}) = v(H^c) \ge v(H)$, due to $\Gamma_{j+1} = H^c \supseteq H$ and the basic monotonic property, are true. Thus, the point $(v(\Gamma_{j+1}), F(\Gamma_{j+1}))$ is more advantageous than $(v(H), F(H))$. ■

5. Calculation of the Bargaining Solution

To summarize, the discussion that follows is governed by the Nash bargaining scheme. Some reservations (see, for example, Luce and Raiffa, 6.6) hold as usual because our bargaining game on Boolean Tables is purely atomic, i.e., it does not permit lotteries (which are an important element of any bargaining scenario). Given this restriction, the uniqueness of the Nash solution cannot be immediately guaranteed. However, it is important to note that "...the Nash solution of $\langle \boldsymbol{S}, d \rangle$ depends only on disagreement point d and the Pareto frontier of $\boldsymbol{S}$. The compactness and convexity of $\boldsymbol{S}$ are important only insofar as they ensure that the Pareto frontier of $\boldsymbol{S}$ is well defined and concave. Rather than starting with the set $\boldsymbol{S}$, we could have imposed our axioms on a problem defined by a non-increasing concave function (and disagreement point d…Osborn and Rubinstein, 1990, p. 24). In our case, $\left(v(\Gamma_j), F(\Gamma_j)\right)$, $j = 0,1,\ldots,p$, represents the atomic Pareto frontier. Therefore, it is possible to provide the proof of non-symmetrical solution (see Kalai, 1977, p. 132), as well as perform the calculation with the product of utility gains in its asymmetrical form (2).[7] The problem of maximizing the product is primarily of technical nature. In the discussions that follow, we will introduce an algorithm for that purpose. We will first comment on the individual algorithm step in relation to the definitions.

The algorithm's last iteration, see below, through the step **T** detects the largest kernel $\overline{K} = S^{*}$[8] (Mullat, 1995). The original version (Mullat, 1971) of the algorithm aimed to detect the largest kernel and is akin to a greedy inverse serialization procedure (Edmonts, 1971). The original version of the algorithm produces a complete defining sequence, which is imperative for finding the bargaining solution aligned with the Frontier Theorem. In the context of the current version, however, it fails to produce a complete defining sequence. Rather, it only detects some thresholds u_j, and some stable set $\Gamma_j = S_j$. The sequence $u_0, u_1, \ldots$ is monotonically increasing: $u_0 < u_1 < \ldots$ while the sequence $\Gamma_0, \Gamma_1, \ldots$ is monotonically shrinking: $\Gamma_0 \supset \Gamma_1 \supset \ldots$, whereby the set $\Gamma_0 = W$ is stable towards the threshold $u_0 = F(W) = \min_{(i,j)\in W} \pi_{ij}$. Hence, the original algorithm is always characterized by higher complexity. However, for finding the bargaining solution, we can still implement an algorithm of lower complexity, which would require modifying the indices $\pi_{ij} = c_j$.

7 There are many techniques that guarantee the uniqueness of the product of utility gains. We are not going to discuss this matter here, because this case is rather an exemption than a rule.

8 It is possible that some smaller kernels exist as well.

Let us consider the problem of identifying the players' joint choice H_{max} representing a block $\arg\max_{H\in\mathcal{S}} v(H)^{\theta} F(H)^{1-\theta}$ of the rows x and columns y in the original table W satisfying the property $\sum_{j\in y}\alpha_{ij} \geq k$, $i \in x$.

Let an index $\pi_{ij} = r_i \cdot v^{\theta} \cdot c_j^{1-\theta}$[9]. The following algorithm solves the problem.

Algorithm.

I. Set the initial values.

1i. Assign the table parameter H to be identical with W, $H \Leftarrow W$. Set the minimum and maximum bounds a, b on the threshold u for $\pi_{ij} \in H$ values.

A. Establish that the next Step **B** produces a non-empty sub-table H. Remember the current status of table H by creating a temporary table H°: $H^{\circ} \Leftarrow H$.

1a. Test u as $(a+b)/2$ using Step **B**. If it succeeds, replace a by u, otherwise replace b by u and H by H°: $H \Leftarrow H^{\circ}$ - "regret action".

2a. Go to **1a**.

B. Test whether the minimum of $\pi_{ij} \in H$ over i, j can be equal or greater than u.

1b. Delete all rows in H where $r_i = 0$. This Step **B** fails if all rows in H must be deleted, in which case proceed to **2b**. The table H is shrinking.

2b. Delete all elements in columns where $\pi_{ij} \leq u$. This Step **B** fails if all columns in H must be deleted, in which case proceed to **3b**. The table H is shrinking.

3b. Perform Step **T** if no deletions were made in **1b** and **2b**; otherwise go to **1b**.

T. Test whether the global maximum is found. Table H has halted its shrinking.

1t. Among numbers $\pi_{ij} \in H$, find the minimum $\min \leftarrow \pi_{ij}$ and then perform Step **B** with new value $u = \min$. If it succeeds, set $a = \min$ and return to Step **A**; otherwise, terminate the algorithm.

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[9] This index obeys the basic monotone property as well.

6. BOOLEAN GAME COOPERATIVE ASPECTS

A cooperative game is a pair (N, v), where N symbolizes a set of players and v is the game characteristic function. Function v is called a super modular if $v(L) + v(G) \leq v(L \cup G) + v(L \cap G)$ whereas it is sub modular if the inequality sign $\leq$ is replaced by $\geq$, $L, G \in 2^N$. Among others (see Cherenin et al, 1948 and Shapley, 1971), where various properties of supermodular set functions are specified. In the appendix, we illustrate a game, which is neither supermodular nor submodular, but rather a mixture of the two, where single and pairwise players do not receive extra rewards. On the other hand, it is obvious that all properties of supermodular functions v remain unchanged for submodular $-v$ characteristic function or vice versa.

A marginal contribution into the coalition H of a player x (the player marginal utility) is given by $\pi(x; H) \equiv \frac{\partial H}{\partial x}$, where $\frac{\partial H}{\partial x} = v(H \cup x) - v(H)$ if $x \notin H$, the player x joins the coalition, and $\frac{\partial H}{\partial x} = v(H) - v(H \setminus x)$ if $x \in H$, the player x leaves the coalition, for every $H \in 2^W$. We denote in our nomenclature $H \cup x \equiv H + x$, and $H \setminus x \equiv H - x$, see later.

Suppose that the interest of player x to join the coalition equals the player's marginal contribution $\frac{\partial H}{\partial x}$. A coalition game is convex (concave) if for any pair L and G of coalitions $L \subseteq G \subseteq W$ the inequality $\frac{\partial L}{\partial x} \leq \frac{\partial G}{\partial x}$ $\left(\frac{\partial L}{\partial x} \geq \frac{\partial G}{\partial x}\right)$ holds for each player $x \in W$.[10]

---

[10] Shapley (1971) recognized this condition as equivalent, whereby similar derivatives in their investigation of some optimization problems (Nemhauser et al, 1978) have been proposed; Muchnik and Shvartser (1987) also pointed to the link between submodular set functions and the Monotonic Systems, see Mullat (1971).

Theorem. *For the coalition game to be convex (concave) it is necessary and sufficient for its characteristic function to be a supermodular (submodular) set function*. Extrapolated (1978) from Nemhauser et al.

Now, in view of the theorem, marginal utilities of players in the supermodular game motivate them in certain cases to form coalitions. In a modular game, where the characteristic function is both supermodular and submodular, marginal utilities are indifferent to collective rationality because entering a coalition would not allow anybody to win or lose their respective payments. In contrast, it can be shown that collective rationality is sometimes counterproductive in submodular games. Therefore, in supermodular games, formation of too many coalitions might be unavoidable, resulting in, for example, the grand coalition. In such cases, in Shapley's (1971) words, this leads to a "snowballing" or "band-wagon" effect. On the other hand, submodular games are less cooperative. In order to counteract these "bad motives" of players in both supermodular and submodular games, we introduce below a second actor – the moderator. Hence, we consider a bargaining game between the coalition and the moderator.

Convex game induces an accompanied bargaining game with the utility pair $\left(v(H), F(H)\right)$, where $F(H) = \min_{x \in H} \dfrac{\partial H}{\partial x}$; concave game induces utility pair with $F(H) = \max_{x \in H} \dfrac{\partial H}{\partial x}$. Here, the coalition assumes the role of Player No. 1 with the characteristic function $v(H)$. The coalition moderator, the Player No. 2, expects the reward $F(H)$.

Proposition. *The solution* $f(\boldsymbol{S}, \varnothing)$ *of a Nash's Bargaining Problem* $\left\langle \boldsymbol{S}, \varnothing \right\rangle$, *which accompanies a convex (concave) coalition game with characteristic function* v, *lies on its Pareto frontier* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *maximizing (minimizing) the product* $v(\Gamma_j)^{\theta} \cdot \dfrac{\partial \Gamma_j}{\partial \alpha}^{1-\theta}$ *for some* $j = 0,1,\ldots,p$, *and* $0 \le \theta \le 1$. This statement is an obvious corollary from the Frontier Theorem. ■

In accordance with the basic monotonic property, see above, given some monotonic function $\pi(x;H) \equiv \frac{\partial H}{\partial x}$ on $N \times 2^N$, it is not immediately apparent that there exists some characteristic function $\nu(H)$ for which the function $\pi(x;H)$ constitutes a monotonic marginal utility $\frac{\partial H}{\partial x}$. The following theorem, accommodated in line with the work of Muchnik and Shvartser (1987), addresses this issue.

The existence theorem. *For the function* $\pi(x,H)$ *to represent a monotonic marginal utility* $\frac{\partial H}{\partial x}$ *of some supermodular (submodular) function* $\nu(H)$, *it is necessary and sufficient that*

$$\begin{aligned}&\frac{\partial}{\partial y}\frac{\partial H}{\partial x} \equiv \pi(x;H) - \pi(x;H-y) = \\ &= \pi(y;H) - \pi(y;H-x) \equiv \frac{\partial}{\partial x}\frac{\partial H}{\partial y}\end{aligned}$$

holds for $x, y \in H \subseteq N$.

The interpretation of this condition is left for the reader.

7. Heuristic Interpretation

Only the last issue is relevant to our bargaining solution $\Gamma = f(\boldsymbol{S}, \varnothing)$ to the supermodular bargaining game. The coalition Γ is a stable point with reference to the threshold value $u = F(\Gamma) = \min_{x \in K} \frac{\partial \Gamma}{\partial x}$. This coalition guarantees a gain $u = F(\Gamma)$ to Player No. 2. Therefore, Player No. 2 can prevent anyone $x \notin \Gamma$ outside the coalition $\Gamma \in \boldsymbol{S}$ from becoming a new member of the coalition because the outsider's marginal contribution $\frac{\partial \Gamma}{\partial x}$ reduces the gain guaranteed to Player No. 2. The same incentive governing the behavior of Player No. 2 will prevent some members $x \in \Gamma$ from leaving the coalition. The unconventional interpretation given below might help elucidate this situation.

Let us observe a family of functions on $N \times 2^N$ monotonic towards the second set variable H, $H \in 2^N$. Let it be a function $\pi(x;H) \equiv \frac{\partial H}{\partial x}$. We already cited Shapley (1971), who introduced the convex games, with the marginal utility $\frac{\partial H}{\partial x} = \nu(H) - \nu(H - x)$, which is the one of many exact utilizations of the monotonicity $\pi(x, L) \le \pi(x, G)$ for $x \in L \subseteq G$. Authors of some extant studies, including this researcher, refer to these marginal $\nu(H) - \nu(H - x)$ set functions as the derivatives of supermodular functions $\nu(H)$. By inverting the inequalities, we obtain submodular set functions.

Convex coalition game, referring to Shapley(1971) once again, can have a "snowballing" or "band-wagon" effect of cooperative rationality; i.e., in a supermodular game, the cooperative rationality suppresses the individual rationality. In contrast, in submodular games with the inverse property $\pi(x, L) \ge \pi(x, G)$ (an extrapolation this time), the individual rationality suppresses the collective rationality. Hence, it is not beneficial in either case. On a positive note, if the moderator is in charge for coalition formation, the moderator reward will be equal to the least marginal utility $u = F(H) = \min_{x \in H} \frac{\partial H}{\partial x}$ of some weakest player in the coalition H under formation. Now, we can focus on a two-person cooperative drama to be played out between the moderator and the coalition.

We start this discussion with our heuristic interpretation. Following the apparatus of monotonic systems in terms of data mining (Mullat, 1971), it is reasonable to find the Pareto frontier in terms of the game theory as well. The potential moderator's bargaining strategy is presented next. First, in the grand coalition $N \equiv \Gamma_0$, the moderator identifies the players with the least marginal utility $u_0 = F(N) = \min_{x \in N} \frac{\partial N}{\partial x}$. The moderator will advise them to stay in line and wait for their rewards. All players that have joined the line will be temporarily disregarded in any coalition formation. Following the game convexity, one of the remaining players (i.e., those still remaining in the coalition formation process) must find themselves worse off owing to the players in line being excluded from the process. Moderator would thus suggest to these players to also join the line and wait for their rewards. The moderator continues the line construction in the same vein. This process will result in a scenario in which all remaining players Γ_1 (outside the line) are better off than u_0, i.e., better off than those waiting in line for their rewards. Now, the moderator repeats the entire procedure upon players $\Gamma_1, \Gamma_2, \ldots$ until all players from N are assigned to wait in line to obtain their rewards. Moderator, certainly, keeps a record of the events $0, 1, \ldots$ and is aware when the marginal utility thresholds increases from u_0 to u_1, etc. It is obvious that the increments are always positive: $u_0 < u_1 < \ldots < u_p$.

What is the outcome of this process? Players staying in line arrange a nested sequence of coalitions $\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \rangle$. The most powerful marginal players, those present when the last event p occurs, form a coalition Γ_p. The next powerful coalition will be Γ_{p-1}, etc., coming back once again to the starting event 0, when the players arrange the grand coalition $N = \Gamma_0$. Our Frontier Theorem guarantees that such a moderator bargaining strategy, in convex games, classifies a Pareto frontier $\langle (\nu(\Gamma_0), u_0), (\nu(\Gamma_1), u_1), \ldots, (\nu(\Gamma_p), u_p) \rangle$ for a bargaining game between the moderator and coalitions under formation.[11] Thus, the game ends when a bargaining agreement is reached between the moderator and the coalition. However, some players might still stay in line, waiting in

[11] This sequence of players/elements in line arranges so-called defining sequence in data mining process.

vain for their rewards, because the moderator might not agree to allow them to partake in coalition formation. Indeed, due to the existence of those marginal players, the moderator may lose a large portion of his/her reward $F(\Gamma_k)$, for some $k's \in \langle 1,\dots,p\rangle$. [12]

8. CONCLUSION

Nash bargaining solution being understood as a point on the Pareto frontier in Monotonic System might be an acceptable convention in the framework of "fast" calculation. The corresponding algorithm for finding the solution is characterized by a relatively few operations and can be implemented using known computer programming "recursive techniques" on tables. From a purely theoretical perspective, we believe that our technique is a valuable addition to the repertoire presently at the disposal of the game theoreticians. However, our bargaining solution is presently not fully grounded in validated scientific facts established in game theory. Consultations with specialists in the field are thus necessary to develop our work further. In our view, our coalition formations games are sufficiently clear and do not require specific economic interpretations. Nevertheless, they need to be confirmed by other fundamental studies.

APPENDIX. Illustration of a club formation bargaining game with neither supermodular nor submodular characteristic function.

Recall the health club formation game from Section 2. Given the characteristic function $\nu(H)$, although whether the club members actually arrive at individual payoffs or not is irrelevant, the club formation is still of our interest. Let the game participants $N = \{1,2,3,4,5,6,7\}$ try to organize a club. Let the characteristic (revenue) function comply with the promises of individual employees to participate in the offered health activities in accordance with their survey responses, see Table 1. However, we demand that all five-health activities be materialized.

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[12] We refer to similar behaviour of players in "Left- and Right-Wing Political Power Design: The Dilemma of Welfare Policy with Low-Income Relief" as political parties bargaining game agents registered under the social security administration.

Define $$\nu(H) = |H| + \sum_{x \in H} \sum_{j=1}^{5} a_{xj}, \text{ where}$$
$$H \subseteq N = \{1,2,3,4,5,6,7\}.$$

In other words, a promise fulfilled by the club member contributes a Bank Note to the player. In addition to all the promises fulfilled, a side payment per capita is available. According to this rule $\nu(\{1\}) = 3$, $\nu(\{2\}) = 5$,… Nonetheless, we are going to change the side payments rule, so that the game transforms into neither supermodular nor submodular game. Note that $\sum_{i}^{7} \nu(\{\,i\}) = \nu(N) = \nu(\{1,2,3,4,5,6,7\}) = 29$, which renders non-essential game.

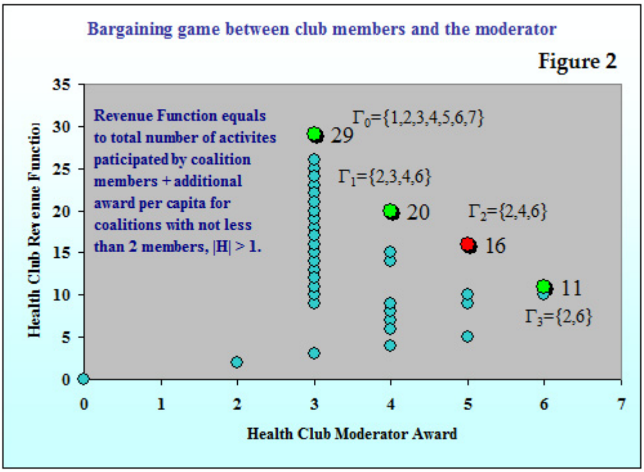
Yes, indeed, the employees, whether they choose to cooperate or not, will be discouraged from forming a club arriving at the same gains. To change the situation into that similar to "*the real life cacophonous*", let the side payment per capita be removed for single and pairwise players while keeping the rewards intact for all other coalitions for which the size exceeds 2. Thus $\nu(\{1\}) = 2$, $\nu(\{2\}) = 4$, $\nu(\{1,2\}) = 6$, $\nu(\{3,6\}) = 5$, $\nu(\{2,3,5\}) = 12$, etc. Moderator's gain, which was defined as $F(H) = \min_{x \in H} \dfrac{\partial H}{\partial x} \equiv \left(\nu(H) - \nu(H - x\right)$, see above, makes the employees' "cooperative behavior" close to grand coalition less profitable for the moderator.

Therefore, the moderator would benefit from encouraging the employees to enter the club of a "reasonable size". In Table 8, we examine this phenomenon using different moderator gain $F(H)$ values.

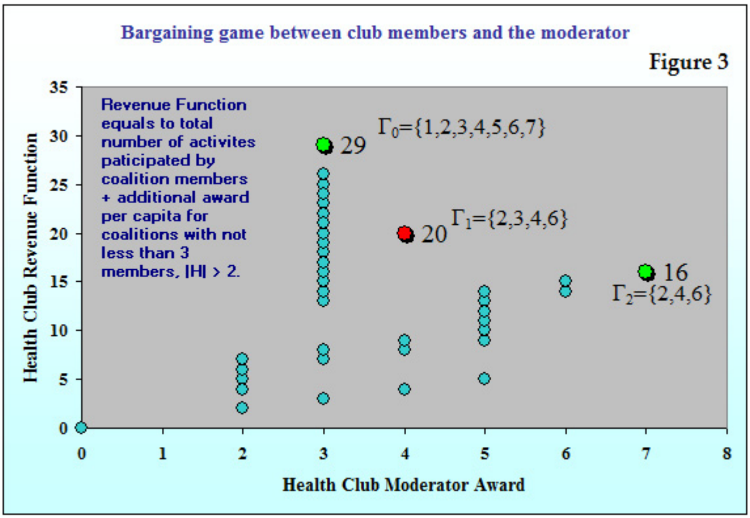
Table 8.

Health Clubs List							Marginal Utilities p/capita							x	y
1	2	3	4	5	6	7	1	2	3	4	5	6	7	ν(H)	F(H)
*							2							2	2
	*							4						4	4
*	*						2	4						6	2
		*							3					3	3
*		*					2		3					5	2
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
		*		*					3		2			5	2
*		*		*			5		6		5			10	5
	*	*		*				7	6		5			12	5
*	*	*		*			3	5	4		3			15	3
			*	*						4	2			6	2
*			*	*			5			7	5			11	5
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	.	*	*	*	*	*		.	4	5	3	6	3	21	3
*	.	*	*	*	*	*	3	.	4	5	3	6	3	24	3
.	*	*	*	*	*	*	.	5	4	5	3	6	3	26	3
*	*	*	*	*	*	*	3	5	4	5	3	6	3	29	3

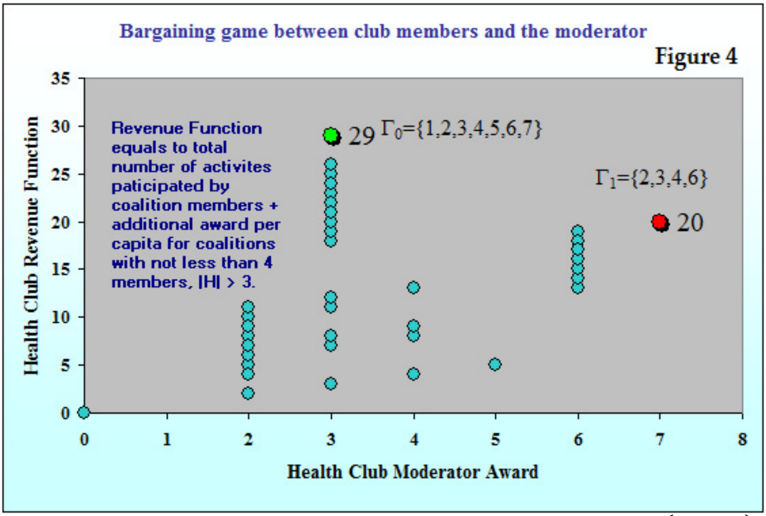
At last, we illustrate the bargaining game in the graph below and make some comments.



N.B. Observe that utility pairs $(29,3)$, $(20,4)$, $(16,5)$ and $(11,6)$ constitute the Pareto frontier of bargaining solutions for the bargaining problem involving the moderator as Bargainer No. 1 and coalitions as Bargainer No. 2. Accordingly, given the grand coalition $N = \Gamma_0 = \{1,2,3,4,5,6,7\}$, three proper coalitions $\Gamma_1 = \{2,3,4,6\}$, $\Gamma_2 = \{2,4,6\}$ and $\Gamma_3 = \{2,6\}$ exist. Solutions $\nu(\Gamma_1) = 20$, $F(\Gamma_1) = 4$ and $\nu(\Gamma_2) = 16$, $F(\Gamma_2) = 5$, maximize the product of players' gains over the disagreement point $(0,0)$ at $20 \cdot 4 = 16 \cdot 5 = 80$. More specifically, as noted at the beginning of the paper, the solution might not be unique and some external considerations may help. For example, the sponsor expenses for $(20,4)$ are equal to 24, while those pertaining to $(16,5)$ are equal to 21, which might be decisive. That is the case when the bargaining power $\theta = ½$ of the coalitions Γ_1, Γ_2 and the moderator are in balance. Otherwise, choosing the coalition bargaining power $\theta < ½$, the moderator will be better off materializing the solution $(5,16)$. Conversely, coalition Γ_2 will be better off if $\theta > ½$.



NB. Comparison with Fig. 2 reveals that coalition $\Gamma_3 = \{2,6\}$ is no longer located on the Pareto frontier.



N.B. Comparison with Fig. 3 indicates that coalition $\Gamma_2 = \{2,4,6\}$ no longer lies on the Pareto frontier.

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How to arrange a Singles' Party: Coalition Formation in Matching Game *

Abstract: The study addresses important issues relating to computational aspects of coalition formation. Finding payoffs — imputations belonging to the core — is, while almost as well known, an overly complex, NP-hard problem, even for modern supercomputers. The problem becomes vague because, among other issues, it is unknown whether the core is non-empty. In the proposed cooperative game, under the name of singles, the presence of non-empty collections of outcomes (payoffs) similar to the core (that being said quasi-core) is fully guaranteed. Quasi-core is defined as a collection of coalitions minimal by inclusion among non-dominant coalitions induced through payoffs similar to super-modular characteristic functions. As claimed, the quasi-core is identified via a version of P-NP problem that utilizes the branch and bound heuristic and the results are visualized by Excel spreadsheet.

Keywords: stability; game theory; coalition formation.

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1. INTRODUCTION

It is almost a truism that many university and college students abandon schooling soon after starting their studies. While some students opt for incompatible education programs, the composition of students following particular programs may not be optimal; in other words, students and programs are mutually incompatible. Indeed, so-called mutual mismatches of priorities were among the reasons (Võhandu, 2010) behind the unacceptably high percentage of students in Estonian universities and colleges dropping out of schools, wasting their time and the entitlement to government support. However, matching students and education programs more optimally could mitigate this problem.

Similar problems have been thoroughly studied (Roth, 1990; Gale, 1962; Berge, 1958) leading, perhaps, L. Võhandu (LV) to propose a way, in this wide area of research, to solve the problem of students and programs mutual incompatibility by introducing “matching total” as the sum of duplets — priorities selected within two directions — horizontal priorities of students towards programs, and vertical priorities of programs towards students. The best solution found among all possible horizontal and vertical duplet assignments, according to LV, is where the sum reaches its minimum.

Finding the best solution, however, is a difficult task. Instead, LV proposed a greedy type workaround. In LV’s words, the best solution to the problem of matching students and programs will be close enough (consult with Carmen et al, 2001) to a sum of duplets accumulated while moving along direction of duplets in non-decreasing ordering. It seems that LV’s proposal to the solution is a typical approach in the spirit of classical utilitarism, when the sum of utilities has to be maximized or minimized (Bentham, The Principals of Morals and Legislation, 1789; Sidgwick, The Methods of Ethics, London 1907).

As noted by Rawls in "Theory of Justice", the main weakness of utilitarian approach is that, when the total **max** or **min** has been reached, those members of society at the very low utility levels will still be receiving very low compensations for incapacity, such as transfer payments to the poor. Arguing for the principal of "***maxima of the lowest***", referred to as the "Second Principal of Justice", Rawls suggested an alternative to the utilitarian approach. The motive driving this study is utilitarian approach. The motive driving this study is similar. We address by example an alternative to conventional core solution in cooperative games, along the lines of monotonic game (Mullat, 1979), whereby the lowest incentive/compensation should be maximized. The reader studying matching problems can also find useful information about these issues, where a number of ways of constructing an optimal matching strategy have been discussed (Veskioja, 2005).

Learning by example is of high value because the conventional core solution in cooperative games cannot be clearly explained unless the readers are sufficiently familiar with utopian reality — a reality that sometimes does not exist. Thus, a rigorous set up of a simple game will be presented here; aiming to explain the otherwise rather complicated intersection of interests. More specifically, we hope to shed light on what we call a Singles-Game. It should be emphasized that, even though the game primitives represent an independent mathematical object in a completely different context, we have still "borrowed" the idea of LV duplets to estimate the benefits of matching. For this reason, we changed the nomenclature of duplets to mutual credentials in order to justify the scale of payoffs — the incentives and compensations.

The rest of the paper is organized as follows. We start with the preliminaries, where the game primitives are explained. In Section 3, we introduce the core concept of conventional stability in relation to the Singles-game. In Section 4, the reader will come across an unconventional theory of kernel coalitions, and kernel coalitions, minimal by inclusion in accordance with the formal scheme. In Section 5, we continue explaining our techniques and procedures used to locate stable outcomes of the game. The study ends with conclusions and suggestions for future work, which are presented in Section 6. Appendix contains a visualization, which brings to the surface the theoretical foundation of coalition formation. Finally, interested readers would benefit from exploring the Excel spreadsheet, which helps visualize a "realistic" intersection of interests of 20 single women and 20 single men. The addendum provides a sketched outline for the evidence of some propositions.

2. PRELIMINARIES

Five single women and five single men are ready to take part in the party. It is assumed that all participants are not very inclined to take risks when looking for new acquaintances. All guests were asked to take part in a survey to determine the qualities they are looking for in their potential partner. Those who choose to provide this information have been promised to collect a box of goodies and are henceforth referred to as participants, while others are marked as dummies by default and cannot participate in the game. In addition to the goodies promised to those who want to reveal their priorities, we continue to set the rules for payments in the form of encouraging successful participants in the game or compensation in case of failure to find the right partner. The game is played in stages: the players enter into agreements (duplets or matches) in pairs, after which they no longer participate in the game. Gradually, the list of duplets expands, and the list of potential or new possible duplets narrows. At the request of the participants, the game can end at any stage or continue until all possible matches are created. Conventionally, we can also say that the game ends either with a partial match in most cases, or with a complete match. In order to cover the expenses of the dating agency, such as soft drinks, rewards, etc., the entry fee is set at -50 €. Thus, the amount of 500 € will be at the disposal of the cashier.

We use index i for the women, and an index j for the men taking part in the dating party. Assuming that all the guests have agreed to participate in the game, there are $\{1,...,i,...5\}$ women and $\{1,...,j,...5\}$ men, resulting in $2\times5\times5$ combinations. Indeed, when priorities have been revealed, they can form two 5×5 tables, $W=\left\|w_{i,j}\right\|$, and $M=\left\|m_{i,j}\right\|$, indicating that each woman i, $i=\overline{1,5}$ revealed her priorities positioned in the rows of table W towards men as horizontal permutations w_i of numbers $\langle1,2,3,4,5\rangle$. Similarly, each man j, $j=\overline{1,5}$, revealed his priorities positioned in columns of the table M towards women as vertical permutations m_j. As can be seen below in W Table-1, now it is convenient to nominate as priorities $w_{i,j}$ (numbers $\langle\overline{1,5}=1,2,3,4,5\rangle$) might repeat in both the columns of the table W and in the rows of the table M. To be sure, more than one man may prefer the same woman at priority level $w_{i,j}$, and many women, accordingly, may prefer the same man at the level $m_{i,j}$. Thus, duplets or mutual credentials $r_{i,j}=w_{i,j}+m_{i,j}$ occupy the cells in table $R=\left\|r_{i,j}\right\|$.

Table 1 W

	$M1$	M_2	M_3	M_4	M_5
W_1	1	5	3	2	4
W_2	5	4	1	2	3
W_3	3	5	4	2	1
W_4	2	5	3	1	4
W_5	4	3	1	2	5

= **Women Priorities**

Table 2 + M

	M_1	M_2	M_3	M_4	M_5
W_1	3	4	2	1	2
W_2	1	3	4	2	4
W_3	5	2	3	4	3
W_4	4	5	1	3	1
W_5	2	1	5	5	5

Men Priorities

Table 3 = R

	M_1	M_2	M_3	M_4	M_5
W_1	4	9	5	3	6
W_2	6	7	5	4	7
W_3	8	7	7	6	4
W_4	6	10	4	4	5
W_5	6	4	6	7	10

Mutual Duplets/Credentials

Noting the assumption that all participants are risk-averse, some lucky couples with lower level of mutual credentials start dating. These lucky couples will receive an incentive, such as a prepaid ticket to an event, free restaurant meal, etc. On the other hand, unlucky participants — i.e., those that did not find a partner — may claim compensation, as only high-level mutual risk partners remained, given that the eligible participants at the low level of mutual risk have been matched.

If no one has found a suitable partner, the question is — should the party continue? Apparently, given that the original data that failed to produce matches might have not been completely truthful, it would be unwise to offer compensation in proportion to mutual credentials $r_{i,j}$. Nonetheless, let us assume that the compensation equals $\frac{1}{2} r_{i,j} \cdot 10€$. In that case, couple's $[5,5]$ profit may reach 50€! Instead, the dating bureau decides to organize the game, encouraging the players to follow Rawls second principle of justice. In Table-3, the minimum — the lowest **mutual risk** among all participants — is $r_{1,4} = 3$. Following the principle, the compensation to all unlucky participants will be equal to $\frac{1}{2} r_{1,4} \cdot 10 = 15€$. This setting is also fiscally reasonable from the cashier's point of view. The balance of payoffs for all participants, will be –25€, as –50€ paid as entrance fee will be reduced by 15€ compensation amount, and additionally by 10€, i.e., inclusive of the cost of collected goodies. Further on, we assume that each member of a dating couple will receive an incentive that is offered to all dating couples and is equal to double the compensation amount.

What happens when the couple $[1,4]$ decides to date? The entire table R should be dynamically transformed to reflect the fact that the participants $[1,4]$ are matched. Indeed, as the women $\{2,3,4,5\}$ and men $\{1,2,3,5\}$ can no longer count on $[1,4]$ as their latent partners, the priorities will decline, whereby the scale $\langle 1,2,3,4,5 \rangle$ dynamically shrinks to $\langle 1,2,3,4 \rangle$ [1]. To reflect this, Tabs.1-3 transform into Tabs.4-6:

Table 4 W

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	4	3	1		2
W_3	2	4	3		$\boxed{1}$
W_4	1	4	$\boxed{2}$		3
W_5	3	2	1		4

Women Priorities

1 To highlight theoretical results of mutual credentials, incitements or compensations, or whatever the scales we use, the dynamic quality of monotonic scales is the only feature fostering the birth of MS — the "monotone system." Otherwise, the MS terminology, if used in any type of serialization methods applied for data analysis, will remain sterile.

Table 5 $+M$

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	1	3	3		3
	4	2	2		2
W_4	3	4	1		1
W_5	2	1	4		4

Men Priorities

Table 6 $=R$

	M_1	M_2	M_3	M_4	M_5
W_1					
W_2	5	6	4		5
W_3	6	6	5		3
W_4	4	8	3		4
W_5	5	3	5		8

Mutual Duplets/Credentials

The ***minimum*** mismatch compensation did not change and is still equal to 15€. However, couple's $[1,4]$ potential balance $-50€+10€+2\cdot 15€=-10€$ of payoffs improves (**W_1** and **M_4** each receive 30€ as an incentive to date, based on the rule that it is equal to twice the mismatch compensation). For those not yet matched, the balance remains negative (in deficit) and equals 15€. On the other hand, if, for example, the couple $[3,5]$ decides to date, the balance of payoffs improves as well.

The party is over and the decisions have been made about who will date and who will leave the party without a partner. The results are passed in writing to the dating bureau. What would be the best collective decision of the participants based on the principle of "***maxima of the lowest***" in accord with the rules of singles-game?

2. CONVENTIONAL STABILITY[2]

In this section, the aim is to present the well-established solution to the singles-game by utilizing the conventional concept, called the core. First, without any warranty, it is helpful to focus on the core stability.

2 Terminology, which we shall use below, is somewhat conventional but mixed with our own.

In order to meet this aim, the original dating party arrangement is expanded to a more general case. The game now has $n \times m$ participants, of who n are single women $\langle 1,\dots,i,\dots n \rangle$ and m are single men $\langle 1,\dots,j,\dots m \rangle$. Some of the guests expressed their willingness to participate in the game and have revealed their priorities. Those who refused, in line with the above, are referred to as *dummy players*. All those who agreed to play the game will be arranged by default into the Grand Coalition $\mathscr{P}$, $|\mathscr{P}| \le n + m$. Thus, indices i, j and labels $\alpha,\dots,\sigma \in \mathscr{P}$ are used to annotate the guests participating in the game. Only the guests in $\mathscr{P}$ are regarded as *participants*, whereas couples $[i, j]$ are referred to as $\alpha,\dots,\sigma$. This differentiation not only helps make notations short, when needed, but can also be used in reference to an eventual matching or a couple without any emphasis on gender.

In the singles-game, we focus on the participants $D \subseteq \mathscr{P}$ who are matched. Having formed a coalition, we suppose that coalition D has the power and is in a position to enforce its priorities. It is assumed that participants in D can persuade all those in $X = \mathscr{P} \setminus D$, i.e., participants that are not yet matched, to leave the party without a partner and thus receive compensation. It is realistic to assume that the suppression of interests of participants' in X is not always possible. It is conceivable that, those in the coalition $D' \subseteq X$, whose interests would be affected (suppressed), will still be capable to receive as much as the participants in D. However, we exclude this opportunity, as it is better that no one expects that coalition D' can be realized concurrently with D and act as its direct competition.

Insisting on this restriction, however, we note that the coalition D are those participants who are involved in partial matching at some final stage of the game, while X are those others who were unlucky and left the party without suitable partners since the game ended. Participants in X by the rules of the game receive 50% of the incentives in D. A realistic situation may occur when all participants in $\mathscr{P}$ are matched, $D = \mathscr{P}$, or, in contrast, no one decides to date, $D = \varnothing$. It is also reasonable that, after revealing their priorities, some individuals might decide not to proceed with the game and will, thus, be labeled as a *dummy* player $\delta \notin \mathscr{P}$.

Among all coalitions D, we usually distinguish rational coalitions. Couple α, joining the coalition D, extracts from the interaction in the coalition a benefit that satisfies $\alpha \in \mathrm{D}$. In the singles-game, we anticipate that the extraction of benefits, i.e., the incentives and mismatch compensations, strictly depend on the membership — couples in D or participants of coalition X. Using the coalition membership $\mathrm{D} \subseteq \mathscr{P}$, we can always construct a payoff x to all participants $\mathscr{P}$, i.e., we can quantify the positions of all participants. The inverse is also true. Given a payoff x, it is easy to establish which couple belongs to the coalition D and identify those belonging to the coalition $\mathrm{X} = \mathscr{P} \setminus \mathrm{D}$. We label this fact as D_{x}. Recall that couples of the coalition D_{x} receive an incentive to date, which is equal to the double amount of the mismatch compensation. Thus, the allocation D_{x} may provide an opportunity for some participants $\sigma \in \mathscr{P}$ to start, or initiate, new matches, thus moving to better positions. We will soon see that, while the best positions induced by special coalitions $\mathscr{N}$, called the kernel, have been reached, this movement will be impossible to realize. [3]

The inability of players to move to better positions by "pair comparisons" is an example of stability. In the work "Cores of Convex games", convex games have been studied (Shapley, 1971); these are so-called games with a non-empty core, where similar type of stability exists. The core forms a convex set of end-points (*imputations*) of a multidimensional octahedron, i.e., a collection of available payoffs to all players. Below, despite the players' asymmetry with respect to $\mathrm{D}_{\mathrm{x}} = \mathscr{P} \setminus \mathrm{X}$, we focus on their payoffs driving their collective behavior as participants $\mathscr{P}$ to form a coalition D_{x}, $\mathrm{D}_{\mathrm{x}} \subseteq \mathscr{P}$; here, $\overline{\mathrm{X}} = \mathrm{D}_{\mathrm{x}}$ is called an anti-coalition to X.

In contrast to individual payoffs improving or worsening the positions of participants, when playing a coalition game, the total payment to a coalition X as a whole is referred to the characteristic function $\mathrm{v}(\mathrm{X}) > 0$. In classical cooperative game theory, the payment $\mathrm{v}(\mathrm{X})$ to coalition X is known with certainty, whereby the variance

3 Our terminology is unconventional in this connection.

$v(X) - v(X \setminus \{\sigma\})$ provides a marginal utility $\pi(\sigma, X)$. Inequality $\pi(\alpha, X \setminus \{\sigma\}) \le \pi(\alpha, X)$ of the scale of credentials expresses a monotonic decrease (increase) in marginal utilities of the membership for $\alpha \in X$. The monotonicity is equivalent to the supermodularity $v(X_1) + v(X_2) \le v(X_1 \cup X_2) + v(X_1 \cap X_2)$ (Nemhauser et al, 1978). Thus, any characteristic function $v(X)$, payment on which is built according to the scale of credentials, is supermodular. The inverse submodularity was used to find solutions of many combinatorial problems (Edmonds, 1970; Petrov and Cherenin, 1948). In general, such a warranty cannot be given.

Recall that we eliminated all rows and columns in tables $W = \|w_{i,j}\|$, $M = \|m_{i,j}\|$ in line with $\overline{X} = D_x$. Table $R = \|\pi(\alpha, X) = w_{i,j}(X) + m_{i,j}(X)\|$, $\alpha = [i, j] \in X$ reflects the outcome of shrinking priorities $w_{i,j}$, $m_{i,j}$ when some $\sigma \in \overline{X}$ have found a match and have formed a couple. Priorities $w_{i,j}$, $m_{i,j}$ are consequently decreasing. Given in the form of characteristic function, e.g., the value $v(X) = \sum_{\alpha \in X} \pi(\alpha, X)$ sets up a coalition game.[4] An imputation for the game $v(X)$ is defined by a $|\mathscr{P}|$-vector fulfilling two conditions: **(i)** $\sum_{\alpha \in \mathscr{P}} x_\alpha = v(\mathscr{P})$, **(ii)** $x_\alpha \ge v(\{\alpha\})$, for all $\alpha \in \mathscr{P}$. Condition (ii) clearly stems from repetitive use of monotonic inequality $\pi(\alpha, X \setminus \{\sigma\}) \le \pi(\alpha, X)$.

A significant shortcoming of the cooperative theory given in the form of the characteristic function stems from its inability to specify a particular imputation as a solution. However, in our case, such imputation can be defined in an intuitive way. In fact, the concept of risk scale determines a popularity index of players. More specifically, the lower the risk of engagement $\pi(\alpha, X)$ of $\sigma \in X$, the more reliable the couple's α coexistence is. Therefore, we set up a popularity index p_i of a woman i

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4 $v(X) = |X|^2 \cdot (|X| + 1)$. Check that $v(\mathscr{P}) = 150$ for 5×5-game, or use the Table-1.

among men in the coalition X as number $p_i(X)=\sum_{j\in X} m_{i,j}$. The index number p_j of a man j among women, accordingly, is given by $p_j(X)=\sum_{i\in X} w_{i,j}$. We intend to redistribute the total payment $v(X)$ in proportion to the components of the vector $p(X)=\langle p_i(X),p_j(X)\rangle$, or as the vector $p(X)$. Hereby we can prove, owing to monotonic scale of priorities, that the payoffs in imputation $p(\mathscr{P})$ cannot be improved by any coalition $X\subset\mathscr{P}$. Therefore, the game solution, among popularity indices, will be the only imputation $p(\mathscr{P})$. In other words, popularity indices core of the cooperative game $v(X)$ consists of only one point $p(\mathscr{P})$.

In line with the terminology used above, we draw the readers' attention to the fact that the singles-game considered next is not a game given in the form of a characteristic function. The above discussion was presented as the foundation for the course of further investigation only.

3. CONCEPT OF A KERNEL

In the view of "monotone system" (Mullat, 1971-1995) exactly as in Shapley's convex games, the basic requirement of our model validity emerges from an inequality of monotonicity $\pi(\alpha,X\setminus\{\sigma\})\le\pi(\alpha,X)$. This means that, by eliminating an element/match σ from X, the utilities (weights) on the rest will decline or remain the same. In particular, a class of monotone systems is called **p**-monotone (Kuznetsov et al, 1982, 1985), where the ordering $\langle\pi(\alpha,X)\rangle$ on each subset X of utilities (weights) follows the initial ordering $\langle\pi(\alpha,\mathcal{W})\rangle$ on the set $\mathcal{W}$. The decline of the utilities on **p**-monotone system does not change the ordering of utilities on any subset X. Thus, serialization (greedy) methods on **p**-monotone system might be effective. Behind a **p**-monotone system lays the fact that an application of Greedy framework can simultaneously accommodate the structure of all subsets $X\subset\mathcal{W}$. Perhaps, for different reasons, many will argue that **p**-monotone systems are rather simplistic and fail to compare to the serialization method. Nonetheless, many economists, including Narens and Luce (1983), almost certainly, will point out that subsets X of **p**-monotone systems *perform* on interpersonally compatible scales.

An inequality $F(X_1 \cup X_2) \geq \min\langle F(X_1), F(X_2)\rangle$ holds for real valued set function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$, referred to as quasi-convexity (Malishevski, 1998). We observed monotone systems, which we think is important to distinguish. The system is non quasi-convex when two coalitions contradict the last inequality. We consider such systems as non-quasi-convex, which applies to the singles-game case.

The ordering of priorities in singles-games — i.e., what men look for in women, and vice versa — remain intact within an arbitrary coalition X. However, in these systems, the ordering of mutual credentials $\|r_{i,j}\|$ on Grand Coalition $\mathscr{P}$ does not necessarily hold for some $X \subset \mathscr{P}$. Contrary to initial ordering on $\langle \mathscr{P} \rangle = \|\pi(\alpha, \mathscr{P}) = r_{i,j}\|$, the ordering of mutual credentials on $\langle X \rangle = \|\pi(\alpha, X)\|$ may be inverse of the ordering on $\langle \mathscr{P} \rangle$ for some couples. In that case, e.g., the ordering of two couples' mutual credentials can turn "upside down" while the risk scale is shrinking compared to the original ordering on the Grand Coalition $\mathscr{P}$. Thus, in general, the mutual credentials scale is not necessarily interpersonally compatible. In other words, interpersonal incompatibility of this risk scale radically differs from the **p**-monotone systems. This difference became apparent when it was no longer possible to find a solution using Greedy type framework of so-called defining chain algorithm — i.e., the monotone system was non-quasi-convex. Before proceeding with the formal side of these processes, it is informative to understand the nature of the problem.

Definition 1 By kernel coalition we call a coalition $\mathscr{K} \in \arg\max_{X \subseteq \mathscr{P}} F(X)$; $\{\mathscr{K}\}$ is the set of all kernels.

Recalling the main quality of defining a chain — a sequence of elements of a monotone system — it is possible to arrange the elements $\alpha \in \mathscr{W}$, i.e., the couples $\alpha \in \mathscr{P}$ of players by a sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$, $k = \overline{1, n}$. The sequence follows the lowest risk ordering in each step k corresponding to sequence of coalitions $\langle H_k \rangle$, $H_1 = \mathscr{P}$, $H_{k+1} \leftarrow H_k \setminus \{\alpha_k\}$, $\alpha_k = \arg\min_{\alpha \in H_k} \pi(\alpha, H_k)$. Given any arbitrarily coalition $X \subseteq \mathscr{P}$, we say that the defining sequence obeys the left concurrence quality if there exists a superset H_t such that $H_t \supseteq X$, $t = \overline{1,k}$, where the first element $\alpha_t \in H_t$ to the left in the sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$ belongs to the set X, $\alpha_t \in X$ as well. On the condition that the element α_t is not a member of the superset $\mathscr{H} = \{\mathscr{K} \in \arg\max_{X \subseteq \mathscr{P}} F(X)\}$ including all kernels $\mathscr{K}$, $\alpha_k \notin \mathscr{H}$, we observe that $\pi(\alpha_t, X) < \pi(\alpha_t, H_t)$. Hereby, we can conclude that $F(X) \leq \pi(\alpha_t, H_t)$ is strictly less than the global maximum of the set function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$. The left concurrence quality guarantees that the sequence can potentially be used for finding the largest kernel $\mathscr{H}$. Due to non-quasi-concavity, the left concurrence quality is no longer valid. Eliminating a couple $\alpha_k = [i, j]$, see above, we delete the row i and the column j in the mutual credentials table $\| r_{i,j} \|$. Thus, the operation $H_{k+1} \leftarrow H_k \setminus \{\alpha_k\}$ is not an exclusion of a couple $\alpha_k \in H_k$, given that the couple $\alpha_k = [i, j]$ is about to start dating, but rather an exclusion of adjacent couples α in $[i,*]$-row and $[*, j]$-column. We annotate the engagement as $H_{k+1} \leftarrow H_k - \alpha_k$ or as an equal notation $D_{k+1} \leftarrow D_k + \alpha_k$.

In conclusion, note, once again, that, despite the properties of monotone system remaining intact, the chain algorithm, assembling the defining sequence of elements $\alpha \in \mathscr{P}$, cannot guarantee the extraction of the supposedly largest kernel $\mathscr{H}$, particularly in the form given by Kempner et al (2008). Thus, we need to employ special tools for finding the solution. To move further in this direction, we are ready to formulate some propositions for finding kernels $\mathscr{K}$ by branch and bound algorithm types.

The next step will require a modified variant of imputation (Owen, 1982). We define an imputation as the outcome connected to the singles-game in the form of a $|\mathscr{P}|$-vector of payoffs to all participants. More specifically, the outcome is a $|\mathscr{P}|$-vector, where each partner in a couple $\sigma \in \mathrm{X}$ receives the ***lowest*** mismatch compensation $\mathrm{F(X)}$, whereas each partner in the couple $\sigma \notin \mathrm{X}$ belonging to the anti-coalition $\overline{\mathrm{X}} = \mathrm{D}_{\mathrm{x}}$ receives the incentive to date, which is equal to twice that amount, i.e., $2 \cdot \mathrm{F(X)}$, cf. Tables 3 and Table 6. The concept of outcome (imputation) in this form is not common because the amount to be claimed by all participants is not fixed and equals $|\mathscr{P}| + \mathrm{F(X)} \cdot \left(|\mathrm{X}| + 2 \cdot |\overline{\mathrm{X}}|\right)$. Thus, it is likely that participants will fail to reach an understanding, and will claim payoffs obtaining less than available total amount $(\mathrm{n} + \mathrm{m}) \cdot 50$€. The situation, in contrast, when participants will claim more than total amount, is also conceivable.

Any coalition X induces a $|\mathscr{P}|$-vector $\mathrm{x} = \langle \mathrm{x}_\sigma \rangle$ as an outcome x :[5]

$$\mathrm{x}_\sigma = \begin{cases} 2 + \mathrm{F(X)} \text{ if } \sigma \in \mathrm{X}, \\ 2 \cdot \left(1 + \mathrm{F(X)}\right) \text{ if } \sigma \notin \mathrm{X}. \end{cases} \rightarrow$$

$$\sum_{\sigma \in \mathscr{P}} \mathrm{x}_\sigma = |\mathscr{P}| + \mathrm{F(X)} \cdot \left(|\mathrm{X}| + 2 \cdot |\overline{\mathrm{X}}|\right).$$

In this case, x_σ is a quasi-imputation.

This definition of *outcome* is used later, adapting the concept of the quasi-imputation for the purpose of the singles-game. We say that an arbitrary coalition X induces an outcome x. Computed and prescribed by coalition X, the components of x consist of two distinct values $\mathrm{F(X)}$

[5] Further, we follow the rule that capital letters represent coalitions $\mathrm{X}, \mathrm{Y}, \ldots, \mathscr{K}, \mathscr{H}, \ldots$ while lowercase letters $\mathrm{x}, \mathrm{y}, \ldots, \boldsymbol{k}, \boldsymbol{h}, \ldots$ represent outcomes induced by these coalitions.

and $2 \cdot \mathrm{F}(\mathrm{X})$. Participants $\sigma \in \mathrm{X}$ could not form a couple, while participants $\sigma \in D_x$ were able to match. Recall that the notation for $\overline{\mathrm{X}}$ is also used as a mixed notation for dating couples D_x.

Before we move further, we will try to justify our mixed notation $\overline{\mathrm{X}}$. Although a coalition $\overline{\mathrm{X}} = \mathrm{D}_\mathrm{x}$ uniquely defines both those D_x among participants $\mathscr{P}$ who went on dating, and those $\mathrm{X} = \mathscr{P} \setminus \mathrm{D}_\mathrm{x}$ who did not, the coalition $\overline{\mathrm{X}}$ does not specifically indicate matched couples. In contrast, using the notation D_x, we indicate that all participants in D_x are matched, whereas a couple $\sigma \in \mathrm{D}_\mathrm{x}$ also indicates an individual decision how to match. More specifically, this annotation represents all men and all women in D_x standing in line facing one member of the opposite sex, with whom they are matched. However, any matching or engagement among couples belonging to D_x, or whatever matches are formed in D_x, does not change the payoffs x_σ valid for the outcome x. In other words, each particular matching D_x induces the same outcome x. Decisions in D_x with respect to how to match provide an example of individual rationality, while the coalition D_x formation, as a whole, is an example of collective rationality. Therefore, in accordance with payoffs x, the notation D_x subsumes two different types of rationality — the individual and the collective rationality. In that case, the outcome x accompanying D_x represents the result of a partial matching of participants $\mathscr{P}$. Propositions below somehow bind the individual rationality with the collective rationality.

One of the central issues in the coalition game theory is the question of the possible formation of coalitions or their accessibility, i.e., the question of coalition feasibility. While it is traditionally assumed that any coalition $\mathrm{X} \subseteq \mathscr{P}$ is accessible or available for formation, such an approach is generally unsatisfactory. We will try to associate this issue with a similar concept in the theory of monotone systems. The issue of accessibility of subsets $\mathrm{X} \subset \mathcal{W}$ in the literature of monotone systems has been considered not only in the context of the totality $2^{\mathcal{W}}$ of its subsets $X \in 2^{\mathcal{W}}$ but also with respect to special collections of subsets $\mathcal{F} \subset 2^{\mathcal{W}}$. A singleton chain α_t adding elements step-by-step, starting with the empty set $\varnothing$, can, in principle, access any set $X \in \mathcal{F}$, or access the set X by removing the elements starting with the grand ordering $\mathcal{W}$ — so called upwards or downwards accessibility.

Definition 2 *Given coalition* $X \subseteq \mathcal{P}$, *where* $\mathcal{P}$ *is the Grand Coalition, we call the collection of pairs* $C(X) = \{\arg\min_{\alpha \in X} \pi(\alpha, X)\}$ *naming* $C(X)$ *as best latent couples, capable of matching with the lowest mutual risk, within the coalition* X.

Consider a coalition D_x, generated by the formation by a chain of steps $D_{k+1} \leftarrow D_k + \langle \alpha_k \rangle$. Let $X_1 = \mathcal{P}$, $X_k = \mathcal{P} \setminus D_k$, where D_k are participants trying to match during the step k; $C(X_k)$ are couples in X_k with the lowest mutual risk/credential among couples not yet matched in steps $k = \overline{1, n}$, $X_{n+1} = \varnothing$. Coalitions in the chain $X_{k+1} = X_k - \alpha_k$ are arranged after the rows and columns, indicated by couple α_k, have been removed from W, M and R. Mutual credentials $R = \|r_{i,j}\|$ have been recalculated accordingly.

Definition 3 *Given the sequence* $\langle \alpha_1, \ldots, \alpha_k \rangle$ *of matched couples, where* $X_1 = \mathcal{P}$, $X_{k+1} = X_k - \alpha_k$, *we say that coalition* $D_x = \overline{X} = \mathcal{P} \setminus X$ *of matched (as well as* X *of not yet matched) participants is feasible, when the chain* $\langle X_1, \ldots, X_{k+1} = X \rangle$ *complies with the rational succession* $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$. *We call the outcome* x, *induced by sequence* $\langle \alpha_1, \ldots, \alpha_k \rangle$, *a feasible payoff, or a feasible outcome.*

Proposition 1 *The rational succession rationality necessarily emerges from the condition that, under the coalition* D_x *formation a couple* α_k *does not decrease the payoffs of couples* $\langle \alpha_1, \ldots \alpha_{k-1} \rangle$ *formed in previous steps.*

The accessibility or feasibility of coalition D_x formation offers convincing interpretation. Indeed, the feasibility of coalition D_x means that the coalition can be formed by bringing into it a positive increment of utilities to all participants $\mathscr{P}$, or by improving the position of existing participants having already formed a coalition when new couples enter the coalition in subsequent steps. We claim that, in such a situation, coalitions are formed by rational choice. The rational choice $C(X)$ satisfies so-called heritage or succession rationality described by Chernoff (1954), Sen (1970), and Arrow (1959). Below, we outline the heritage rationality in the form suitable for visualization.

The proposition states that, in matches, the individual decisions are also rational in a collective sense only when all participants in D_x individually find a suitable partner. We can use different techniques to meet the individual and collective rationality by matching all participants only in D_x, which is akin to the stable marriage procedure (Gale & Shapley, 1962). In contrast, the algorithm below provides an optimal outcome/payoff accompanied by partial matching only—i.e., only matching some of participants in $\mathscr{P}$ as participants of D_x; once again, this is in line with the Greedy type matching technique.

Proposition 2 *The set* $\{\mathcal{K}\}$ *of kernels in the singles-game arranges feasible* coalitions. *Any outcome* κ *induced by a kernel* $\mathcal{K} \in \{\mathcal{K}\}$ *is feasible.*

At last, we are ready to focus on our main concept.

Definition 4 *Given a pair of outcomes* x *and* y, *induced by coalitions* X *and* Y, *an outcome* y *dominates the outcome* x, $x \prec y$:

(i) $\exists S \subseteq \mathscr{P} \,|\, \forall \sigma \in S \rightarrow x_\sigma < y_\sigma$,

(ii) the outcome y is feasible.

Condition (i) states that participants/couples $\sigma \in \mathrm{S} \subset \mathscr{P}$ receiving payoffs x_σ can break the initial matching in D_x and establish new matches while uniting into D_y. Alternatively, some members of X, i.e., not yet matched participants in S, can find suitable partners amid participants in D_y, or, even their compensations in Y may be higher than their incentives in x. Thus, by receiving y_σ instead of x_σ the participants belongings to S are guaranteed to improve their positions. The interpretation of the condition (ii) is obvious. Thus, the relation $\mathrm{x} \prec \mathrm{y}$ indicates that participants in S can cause a split (bifurcation) of D_x, or are likely to undermine the outcome x.

Definition 5 *A kernel* $\mathcal{N} \in \{\mathcal{K}\}$ *minimal by inclusion is called a kernel — it does not include any other proper kernel* $\mathcal{K} \subset \mathcal{N}$ *:* $\mathcal{K} \not\subset \mathcal{N}$ *is true for all* $\mathcal{K} \neq \mathcal{N}$.

Proposition 3 *The set* $\{\boldsymbol{n}\}$ *of outcomes, induced by kernel* $\{\mathcal{N}\}$*, arranges a quasi-core of the singles-game. Outcomes in* $\{\boldsymbol{n}\}$ *are non-dominant upon each other, i.e.,* $\boldsymbol{n} \prec \boldsymbol{n}'$*, or* $\boldsymbol{n} \succ \boldsymbol{n}'$ *are false.* Thus, the quasi-core is internally stable.

The proposition above clearly indicates that the concept of internal stability is based on "pair comparisons" (binary relation) of outcomes. The traditional solution of coalition games recognizes a more challenging stability, known as *NM* solution, which, in addition to the internal stability, demands external stability. External stability ensures that any outcome x of the game outside *NM*-solution cannot be realized because there is an outcome $\boldsymbol{n} \in \{\mathcal{N}\}$, which is not worse for all, but it is necessarily better for some participants in x. Therefore, most likely, only the outcomes $\boldsymbol{n}$ that belong to *NM*-solution might be realized. The disadvantage of this scenario stems from the inability to specify how it can occur. In contrast, in the singles-game, we can define how the transformation of one coalition to another takes place, namely, only along feasible sequence of couples. However, it may happen that for some coalitions X outside the quasi-core $\{\mathcal{N}\}$, feasible sequence may stall unable to reach any kernel $\boldsymbol{n} \in \{\mathcal{N}\}$, whereby starting at X the quasi-core is feasibly unreachable. This is a significant difference with respect to the traditional *NM*-solution.

4. FINDING THE QUASI-CORE

In general, when using Greedy type algorithms, we gradually improve the solution by a local transformation. In our case, a contradiction exists because nowhere is stated that local improvements can effectively detect the best solution — the best outcome or payoffs to all players. The set of best payoffs, as we already established above, arranges a quasi-core of the game. Usually, finding the core in the conventional sense is a NP-hard task, as the number of "operations" increases exponentially, depending on the number of participants. In the singles-game, or in almost all other types of coalition games, we observe an extensive family of subsets constituting traditional core imputations. Even if it is possible to find all the payoff vectors in the core, it is impractical to do so. We thus posit that it is sufficient to find some feasible coalitions belonging to the quasi-core and the payoffs induced by these coalitions.

This can be accomplished by applying a procedure of *strong improvements* of payoffs, and several *gliding procedures*, which do not worsen the players' positions under coalition formation. Indeed, based on rationality, known as the rational succession, Definition 3, it is not rational in some situations to use the procedure of strong improvements, as these do not exist. However, using gliding procedures, we can move forward in one of the promising directions to find payoffs not worsening the outcome. Experiments conducted using our polynomial algorithm show that, while using a mixture of improvement procedure and gliding procedures, combined with the succession condition, one can take the advantage of backtracking strategy, and might find feasible payoffs of the singles-game belonging to the quasi-core.

We use five procedures in total — one improvement procedure and four variants of gliding procedures. Combining these procedures, the algorithm below is given in a more general form. While we do not aim to explain in detail how to implement these five procedures, in relation to rational succession, it will be useful to explain beforehand some specifics of the procedures because a visual interaction is best way to implement the algorithm.

In the algorithm, we can distinguish two different situations that will determine in which direction to proceed. The first direction promises an improvement in case the couple $\alpha \in X$ decides to match. We call the situation when $C(X-\alpha) \cap C(X) = \varnothing$ as a latent improvement situation. Otherwise, when $C(X-\alpha) \cap C(X) \neq \varnothing$, it is a latent gliding direction. Let $CH(X)$ be the set of rows $C(X)$, the horizontal routes in R Tables 3 & 6, which contain the set $C(X)$. By analogy $CV(X)$ represents the vertical routes, the set of columns, $C(X) \subseteq CH(X) \times CV(X)$. To apply our strategy upon X, we distinguish four cases of four non-overlapping blocks in the mutual risk $R = \left\| r_{i,j} \right\|$ Tables 3 & 6: $CH(X) \times CV(X)$; $CH(X) \times \overline{CV(X)}$; $\overline{CH(X)} \times CV(X)$; $\overline{CH(X)} \times \overline{CV(X)}$.

Proposition 4 *An improvement in payoffs for all participants in the singles-game may occur only when a couple* $\alpha \in X$ *complies with the latent improvement situation in relation to the coalition* X*, the case of* $C(X-\alpha) \cap C(X) = \varnothing$. *The couple* $\alpha \in X$ *is otherwise in a latent gliding situation.*

The following algorithm represents a heuristic approach to finding a kernel $\boldsymbol{n}$ among kernel $\{\mathcal{N}\}$ of the singles-game.

Input Build the mutual credentials Tables 3 & 6, $R = W + M$—a simple operation in Excel spreadsheet. Recall the notation $\mathcal{P}$ of players as the game participants. Set $k \leftarrow 1$, $X \leftarrow \mathcal{P}$ in the role of not yet matched participants, i.e., as players available for latent matching**.** In contrast to the set X, allocate indicating by $D_x \leftarrow \varnothing$ the initial status of matched participants.

Do Step up: **S**, Find a match $\alpha_k \in CH(X) \times CV(X)$,
$D_x \leftarrow D_x + \alpha_k$, such that $F(X) < F(X-\alpha_k)$,
$X \leftarrow X - \alpha_k$, $X_k = X$, $k = k+1$, otherwise *Track Back.*

Gliding: **D**, Find a match $\alpha_k \in \mathrm{CH(X)} \times \mathrm{CV(X)}$,

$\mathrm{D_x} \leftarrow \mathrm{D_x} + \alpha_k$, such that $\mathrm{F(X)} = \mathrm{F(X} - \alpha_k)$,

$\mathrm{X} \leftarrow \mathrm{X} - \alpha_k$, $\mathrm{X_k} = \mathrm{X}$, $\mathrm{k} = \mathrm{k} + 1$, otherwise *Track Back.*

Jump **F**, Find a match $\alpha_k \in \mathrm{CH(X)} \times \overline{\mathrm{CV(X)}}$,

$\mathrm{D_x} \leftarrow \mathrm{D_x} + \alpha_k$, such that $\mathrm{F(X)} = \mathrm{F(X} - \alpha_k)$,

$\mathrm{X} \leftarrow \mathrm{X} - \alpha_k$, $\mathrm{X_k} = \mathrm{X}$, $\mathrm{k} = \mathrm{k} + 1$, otherwise *Track Back.*

Jump **G**, Find a match $\alpha_k \in \overline{\mathrm{CH(X)}} \times \mathrm{CV(X)}$,

$\mathrm{D_x} \leftarrow \mathrm{D_x} + \alpha_k$, such that $\mathrm{F(X)} = \mathrm{F(X} - \alpha_k)$,

$\mathrm{X} \leftarrow \mathrm{X} - \alpha_k$, $\mathrm{X_k} = \mathrm{X}$, $\mathrm{k} = \mathrm{k} + 1$, otherwise *Track Back.*

Jump **H**, Find a match $\alpha_k \in \overline{\mathrm{CH(X)}} \times \overline{\mathrm{CV(X)}}$,

$\mathrm{D_x} \leftarrow \mathrm{D_x} + \alpha_k$, such that $\mathrm{F(X)} = \mathrm{F(X} - \alpha_k)$,

$\mathrm{X} \leftarrow \mathrm{X} - \alpha_k$, $\mathrm{X_k} = \mathrm{X}$, $\mathrm{k} = \mathrm{k} + 1$, otherwise *Track Back.*

Loop Until no couples to match can be found in accordance with cases **S, D, F, G** and **H.**

Output The set $\mathrm{D_x}$ has the form $\mathrm{D_x} = \langle \alpha_1, \ldots, \alpha_k \rangle$. The set $\mathcal{N} = \mathcal{P} \setminus \mathrm{D_x}$ represents a kernel of the game while the pay-off $\boldsymbol{n}$ induced by $\mathcal{N}$ belongs to the quasi-core.

In closing, it is worth noting that a technically minded reader would likely observe that coalitions $\mathrm{X_k}$ are of two types. The first case is $\mathrm{X} \leftarrow \mathrm{X} - \alpha_k$ operation when the mismatch compensation increases, i.e., $\mathrm{F(X_k)} < \mathrm{F(X_k} - \alpha_k)$. The second case occurs when gliding along the compensation $\mathrm{F(X_k)} = \mathrm{F(X_k} - \alpha_k)$. In general, independently of the first or the second type, there are five different directions in which a move ahead can proceed. In fact, this poses a question — in which sequence couples α_t should be selected in order to facilitate the generation of the *sequence* $\mathrm{D_x} = \langle \alpha_1, \ldots, \alpha_k \rangle$? We solved the problem for singles-games underpinning our solution by backtracking. It is often clear in which direction to move ahead by selecting improvements, i.e., either a strict improvement by **s)** or gliding procedures though **d)**, **f)**, **g)** or **h)**. However, a full explanation of backtracking is out of the scope of our current investigation. Thus, for more details, one may refer to similar techniques, which effectively solve the problem (Dumbadze, 1989).

5. CONCLUSIONS

The uniqueness of singles-game lies in the dynamic nature of pair wise priorities. As the construction of the matching sequence proceeds, priorities dynamically shrink, and finally converge at one point. Dynamic transformation, or the monotonic (dynamic) nature of priorities, enabled constructing a game based on so-called monotone system, or MS. One disadvantage behind the use of the MS-system is its drawback in the respective interpretation of the analysis results. More specifically, when the process of extracting the core terminates, the interpretation requires further corrections. However, with regards to the choice of the best variants, i.e., the choice of the best matches in the singles-game, the paper reports a scalar optimization in line with "***maxima of the lowest***" principle, or rather an optimal choice of partial matching. This view opens the way to consider the best partial matching as the choice of the best variants (alternatives), and to explore the matching process from the perspective of a choice problem.

Usually, when trying to analyze the results, a researcher must rely on the common sense. Therefore, applying the well known and well thought out concepts and categories that have been successfully applied in the past, we can move forward in the right direction. Our advantage was that this relation was found, and was transformed into a shape similar to the core, which is known concept in the theory of stability of collective behavior, e.g., in the theory of coalitional games.

Irrespective of the complexity of intersections in the interests of players, deftly twisted rules for compensations in unfortunate circumstances, incitements, etc., singles-game, as it seems, makes a point. However, this is not enough in social sciences, especially in economics, when a formal scheme rarely depicts the reality, e.g. the difference in political views and positions of certain groups of interest, etc. Perhaps, the individual components of the game will still be helpful in moving closer to answering the question of what is right or wrong, or what is good and what is bad, which would be a fruitful path to explore in future studies of this type.

APPENDIX

Visualization

Recall that, in the singles-game, the input to the algorithm presented in the main body of the paper contains three tables (cf. Tables 1-6): $W = \left\| w_{i,j} \right\|$ — priorities table w_i the women specify with the respect to the characteristics the men should possess, in the form of permutations of numbers $\overline{1,n}$ in rows; $M = \left\| m_{i,j} \right\|$ — priorities m_j the men specify with the respect to the characteristics the women should possess, in the form of permutations of numbers $\overline{1,m}$ in columns; and $R = \left\| w_{i,j} + m_{i,j} \right\|$. These tables, and tabular information in general, are well suited for use in Excel spreadsheets that feature calculation, graphing tools, pivot tables, and a macro programming language called VBA — Visual Basic for Applications.

A spreadsheet was developed in order to present our idea visually, i.e., the search for kernel of the singles-game, and the stable coalitions with outcomes belonging to the quasi-core induced by these coalitions. The spreadsheet takes for granted the Excel functions and capabilities can be downloaded from http://datalaundering.com/download/singles_game.xls (Accessed 23.12.2021)

SPREADSHEET LAYOUT

There are 20 single women and 20 single men attending the party, i.e., $n, m = 20$. Three tables are thus available: The Pink table W — women's priorities; The Blue M — men's priorities, and the Yellow R — the mutual credentials table. The column to the right of the R lists all women $i = \overline{1,20}$ showing $\min_{j=\overline{1,20}} r_{i,j}$, $r_{i,j} = w_{i,j} + m_{i,j}$: the level of risk of couples $[i,*]$. The row down of the bottom of R lists all men $j = \overline{1,20}$ showing $\min_{i=\overline{1,20}} r_{i,j}$ level of risk of couples $[*, j]$. In the right hand bottom corner cell, the lowest $\min_{i=\overline{1,20}, j=\overline{1,20}} r_{i,j} = F(X)$

level of risk over the whole R is given. Notice that the green cells in the R table visually represent the effect of $\arg\min_{i=\overline{1,20},\, j=\overline{1,20}} r_{i,j}$ operation. Actually, the green cells visualize the choice operator $\mathrm{C}(\mathrm{X})$. Arrays **V24:AO25** and **V26:AO26** will be implemented in the process of generating the matching sequence together with the levels of risk associated by this sequence. The players' balance of payoffs occupies the array **V31:AO32**. Some cells reflecting the *state of finances* of cashier are located below, in the array **AP34:AP44**. Cells in row-**1** and column-**A** contain the guests' labels. We use these labels in all macros.

EXTRACTING KERNEL OF THE GAME

We came closer to the goal of our visualization, where we visually demonstrate the main features of the theoretical model of the game by example. Generating the matching sequence, which is performed in a stepwise fashion, constitutes the framework of the theory. At each step, to the right side of the sequence generated in the preceding steps, we add a couple found by one of the macros CaseS, CaseD, CaseH, i.e., a couple that has decided to date. This process is repeated until the mutual regrets (i.e., credentials) sequence reached the level 6. When using these macros one can easily verify that, the levels of risk initially increase, and decline towards the end in case we proceed further with these macros. This single $\cap$-peakedness is a consequence of the levels of mutual credential monotonicity $\pi(\alpha, \mathrm{H} \setminus \{\sigma\}) \leq \pi(\alpha, \mathrm{H})$. Indeed, recall that credential levels are recalculated after each match. With the proviso of recommendations in our heuristic algorithm, see above, due to the recalculation, the priority scales will "shrink" or "pack together", as only not yet matched participants remain. Let us try to generate a Matching Sequence using macros: CaseS, CaseD, CaseF,...The data, e.g., will occupy the array **V24:O28**. It is possible to come back to the initial status of the spreadsheet by using macros: Ctrl+o, Ctrl+b and Ctrl+l. As an example of these macros we have prepared the result in cells **B51:L52**. Just copy the contents of these cells into **V24:F25** and then use the macro Ctrl+n, which will visualizes a kernel of 11 matches of the game.

Table 7

Match No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Women	19	10	1	6	4	11	17	9	5	2	15
Men	5	9	10	17	15	6	13	11	7	14	2
Duplets	3	3	4	5	6	6	6	6	6	6	6

Table 8

Participant No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Women' payoffs	70 €	40 €	40 €	70 €	70 €	70 €	40 €	40 €	40 €	70 €
Mens' payoffs	70 €	40 €	70 €	70 €	70 €	70 €	40 €	40 €	70 €	70 €
Participant No.	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Women' payoffs	70 €	40 €	70 €	40 €	70 €	40 €	40 €	70 €	70 €	70 €
Mens' payoffs	40 €	70 €	70 €	70 €	70 €	40 €	40 €	40 €	40 €	40 €

Let us look at Tables 7, where only 11 matches are accomplished, i.e., all columns to right starting at from the match [19,5] till [15,2] visualize the kernel of our single game. Table 7 marks those participants who decided to date, while all the rest but on this particular list are not yet taken their decisions or have been, perhaps, unlucky to find a partner.

Table 8 will note the payoffs, that is, the imputation caused by the kernel coalition, i.e., the size of payoffs as incentives, or compensation for non-compliance, to all 40 participants — 20 women and 20 men. Payoffs of 40€ and 70€ correspond to the kernel makes up the result in cash. The result is the total amount of 2000€ in the form of participation fees minus payoffs, what makes 2260€ not in favor of the cashier.

We can continue creating the sequence with macros using m**A**tch [ctrl + a], pointing to the cell in the top box: pink on the left (or yellow on the right), until all participants have been matched. Please note this, starting with pair **No.12**; we can no longer use the macros of our heuristic algorithm. There are no couples with increasing duplets — 1-11 represents the maximum point — a kernel $\boldsymbol{\mathcal{n}}$ of the game.

In the Tables 9-10 below, the Matching Sequence consists of length 20, $k = \overline{1,20}$; we labeled couple $[\mathrm{i}, \mathrm{j}]$ by α using notation α_{k}. Together with levels of mutual Duplets in row 3, the rows 1,2 correspond to the sequence $\langle \alpha_{\mathrm{k}} \rangle$. Compensations and incentives for dating are not payable at all, and only the costs of goodies (each worth 10€) occupy rows

4,5. Notice that, in accordance with single $\cap$-peakedness, the ***lowest*** levels of risk first increase starting at 3, and after reaching 6, starting at couple **No.12**, they start declining down to 0. For couple **No.3**, Duplets jump from 4 to 5, while, for couple **No.4**, they increase from 5 to 6.

Table 9

Match No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Women	19	10	1	6	4	11	17	9	5	2
Men	5	9	10	17	15	6	13	11	7	14
Duplets	3	3	4	5	6	6	6	6	6	6
W-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
M-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

Table 10

Match No.	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Women	15	18	20	7	13	16	8	14	3	12
Men	2	1	4	12	20	18	19	3	16	8
Duplets	6	5	5	4	0	3	3	3	2	0
W-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
M-payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

The list of macros used.

- **CaseS.** Ctrl+**s,** Trying to move by improvement along the block $\mathrm{CH}(X)\times \mathrm{CV}(X)$ of cells $\sigma=[i,j]$ by"<" operator in order to find a new match at the strictly higher level. [6]
- **CaseD.** Ctrl+**d,** Trying to move while gliding along the block $\mathrm{CH}(X)\times \mathrm{CV}(X)$ of cells $\sigma=[i,j]$ by "<=" operator in order to find a new match at the same or higher level.
- **CaseF.** Ctrl+**f,** Trying to move while gliding along the block $\mathrm{CH}(X)\times \overline{\mathrm{CV}(X)}$ of cells $[i,j]$ by "<=" operator in order to find a new match at the same or higher level.
- **CaseG.** Ctrl+**g**, Trying to move while gliding along the block $\overline{\mathrm{CH}(X)}\times \mathrm{CV}(X)$ of cells $\sigma=[i,j]$ by "<=" operator in order to find a new match at the same or higher level.
- **CaseH.** Ctrl+**h**, Trying to move while gliding along the block $\overline{\mathrm{CH}(X)}\times \overline{\mathrm{CV}(X)}$ of cells $\sigma=[i,j]$ by "<=" operator in order to find a new match at the same or higher level.

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6 CH — cells in horizontal direction, CV — cells in vertical direction

FUNCTIONAL TEST

The spreadsheet users are invited first to perform a functional test, in order to become familiar with the effects of **ctrl-keys** attached to different macros. Calculations in Excel can be performed in two modes, ***automatic*** and ***manual***. However, it is advisable to choose properties and set the calculus in the manual mode, as this significantly speeds up the performance of our macros. The steps one can take if something goes wrong are listed below.

- **Originate.** [Ctrl+**o**], Perform the macro by Ctrl+**o**, and then use Ctrl+**b**. This macro restores the original status of the game saved by the Bac**K**up, i.e., saved by ctrl-**k**.
- **RandM.** [Ctrl+**m**], Perform the macro by Ctrl+m. It randomly rearranges columns of **M**en's priority **M** table by random (permutations).
  Notice the effect upon men's priority **M**.
- **RandW.** [Ctrl+**w**], Perform the macro by Ctrl+w. It randomly rearranges rows of **W**omen's priority table **W** by random (permutations). Notice the effect upon women's priority table **W**.
- **Proceed.** [Ctrl+**e**], While proc**E**eding with macros Rand**M** and Rand**W,** the macro is using random permutations for men and women until it generates the priority tables **M** and **W** with minimum mutual risk equal to 4.
- **Dummy.** [Ctrl+**u**], This macro is removing from the list of participants those guests that do not wish to play the game, or who decide not to pursue the dating. We call them d**U**mmy players. Activate the row-1, or column-A by pointing at man $\mathbf{m}_{\#\#}$**,** or woman $\mathbf{w}_{\#\#}$ and then perform Ctrl+**u** excluding the chosen guests from playing the game.
- **MCouple.** [Ctrl+**a**]**,** Try to m**A**tch [ctrl+**a**] a couple by pointing at the cell in the upper block: pink color to the left (or yellow to the right) in the row $\mathbf{w}_i$ (corresponding to a woman) and the column $\mathbf{m}_j$ (corresponding to a man).
- **TrackR.** [Ctrl+**r**]**,** Visualizes Tracking forwa**R**d. Memorizes the status of *Women*-**W** and *Men*-**M** priorities to be restored by **TrackB** macro. The effect of this macro is invisible. It can be used whenever it is appropriate to save the active status of all tables and all the arrays necessary to restore the status by **TrackB** macro. Only when the search for quasi-core coalitions is performed manually, the effect of macro is visible.
- **TrackB.** [Ctrl+**b**] Visualizes Tracking **B**ack. Restores the status of *Women*-**W** and *Men*-**M** priorities memorized by **TrackR** macro.
- **Happiness** [Ctrl+**p**]**,** The macro calculates an index of ha**P**piness using the initial Duplets table.
- **Coalition** [Ctrl+**n**], The macro rebuilds the matching coalitio**N** following the coalition matching list previously transferred into area "AV24:AO25".
- **Chernoff** [Ctrl+**q**], Useful when indicating **by red font** in Excel the status of the Choice Operator $C(X) = \{\arg\min\}$. Using this macro will help to confirm the validity of the Succession Operator. To clear the status, use Ctrl+**l**.

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ADDENDUM

If one wants to understand the essence of the proposition below in a simpler way, then one can outline the situation in the following passage. The situation is such that the game is viewed as a dynamic transformation of the preferences of the participants in the formation of our coalitions X_k, step by step from k to $k+1$, when a couple of participants leave the game, shortening the coalitions $X_k \supset X_{k+1}$ in the chain, thereby depriving the players in X without partners from adhering to their preferences. However, in the case when the best preferences of some pairs of the participants in X_{k+1} who have not yet been brought together remain the same as in X_k, then in the new situation X_{k+1} these latent pairs of participants are still present in their previous role of the best choice in the new situation.

One circumstance must be kept in mind here. On the one hand, we are dealing with matchings, but on the other hand, the considered matchings are also a certain set of cells or blocks X embedded into $n \times m$ tables in our coalition game, and therefore it is quite appropriate here to consider matchings from the point of view of set theory, where the usual operations of inclusion, intersection of table cells as elements, etc. are allowed.

As the coalition-formation chain X_k shrinks into $X_k \supset X_{k+1}$ blocks the proposition below can be explained by latent least-risk function $F(X_k) = \min_{\sigma \in X_k} \pi(\sigma, X_k)$ generating choices $C(X_k)$ in the form of a list $\langle \alpha = \arg\min_{\sigma \in X_k} \pi(\sigma, X_k) \rangle$ of couples $\alpha \in X_k$.

The list $C(X_k)$ represents matchings $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle$ that couples $\overline{\alpha}$ decide to date. A couple $\sigma \in X_{k+1}$ now in the role of $\alpha_{k+1} = \sigma$ will try to realize their latent relations. In the new situation all participants in X_{k+1}, must reconsider to whom they prefer to date, as their favored σ, while the chain X_k is under formation, due to the fact that all participants in $\overline{\alpha}$ no longer will be available.

Based on the remarks above, the following can be stated.

Proposition 5. For function $F(X) = \min_{\sigma \in X} \pi(\sigma, X)$ the statement $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$ is true if $F(X_k) \le F(X_{k+1})$. The set $C(X_k) \cap X_{k+1} = \varnothing$ in case when $F(X_k) < F(X_{k+1})$. It revises rational choice succession postulate: *"..., which is the same as Postulate 4 of Chernoff, 1954; or condition α of Sen, 1971; or the axiom C2 of Arrow-Uzawa, Arrow 1959,"* cf. Malishevski 1981. [7]

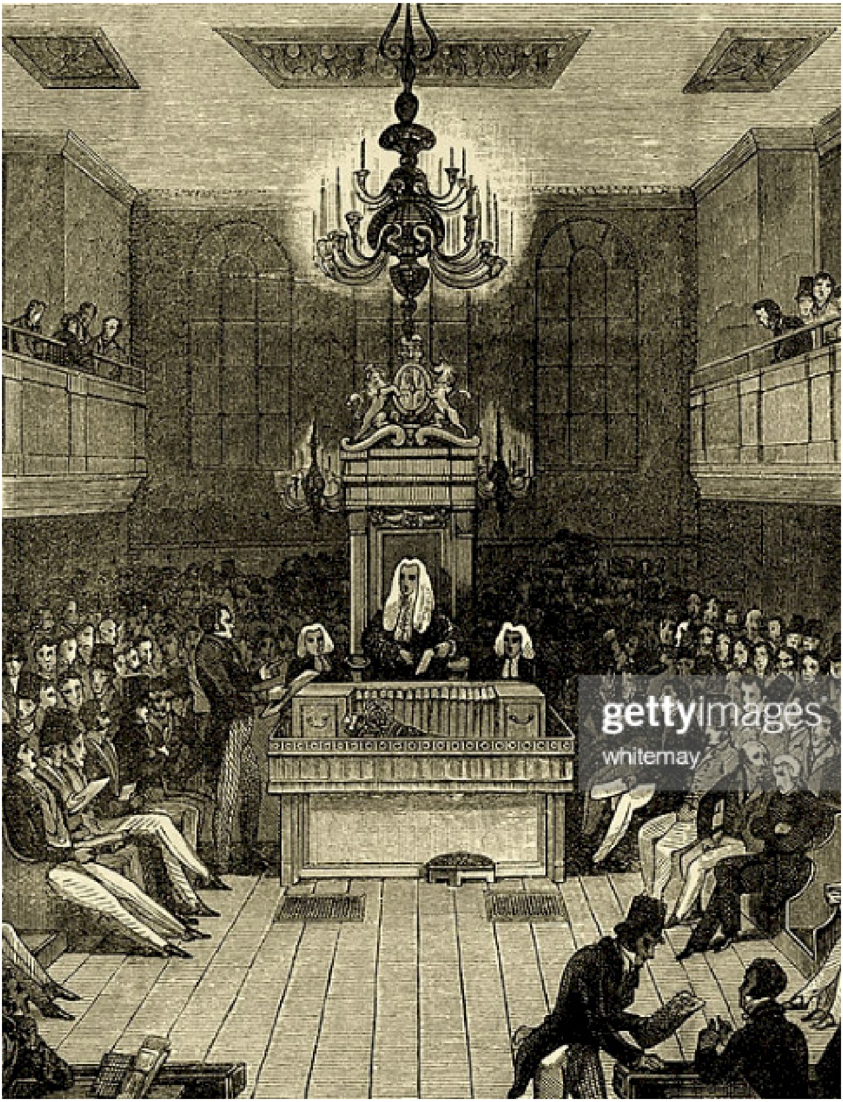
The proof may be explained in the basic terms. The idea is to apply a mathematical induction scheme. We claim that, starting from the initial state $\mathscr{P}$ of the game, where nobody has been matched yet, it is possible to

[7] In the language of the barmaid, this statement reads: "Old love does not rust."

reach an arbitrary coalition X not yet matched participants by sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle$, $\mathrm{X}_1 = \mathscr{P}$, $\mathrm{X}_{k+1} = \mathrm{X}_k - \alpha_k$ $\mathrm{X} = \mathrm{X}_{k+1}$, $\overline{1,k}$. The sequence will improve payoffs x_k on previous steps $\langle \alpha_1, \ldots, \alpha_{k-1} \rangle$ in accordance with non-decreasing values $\mathrm{F}(\mathrm{X}_k)$.

First, the statement of the proposition can be verified by observation of all preference tables and all coalitions X that emerged from all $n \times m$ tables, when both n and m are small integers. For higher n and m values, it is NP-hard problem. Second, consider an arbitrary coalition X of the $n \times m$-game. While the coalition $\overline{\mathrm{X}} = \mathrm{D}_x$ includes all matched couples, in order to arrange a new couple, all participants in X are still unmatched. We can thus always find a couple $\alpha_0 \in \overline{\mathrm{X}}$ such that $\mathrm{F}(\mathscr{P}) \le \mathrm{F}(\mathscr{P} - \alpha_0)$. Consider $(n-1) \times (m-1)$-game, which can be arranged from $n \times m$-game by declaring the partners of the couple α_0 as dummy players $\delta \notin \mathscr{P}$.

By the induction scheme, there exists a sequence of pairs $\langle \alpha_1, \ldots, \alpha_k \rangle$ with required quality of improving the payoffs x_k starting from $\mathrm{X}_1 = \mathscr{P} - \alpha_0$. Restoring the dummy couple α_0 to the role of players in the $n \times m$-game, we can, in particular, ensure the required quality of the sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_k \rangle$. The statement of the proposition is obviously the corollary of the claim above. However, it is clear that, ensured by its logic, the claim is a more general statement than the statement of the proposition. The first part of the statement is self-explanatory. The coalition $\mathscr{N}$ stops being a proper subset among kernels $\{\mathscr{K}\}$ as soon as the payoff function $\mathrm{F}(\mathscr{N})$ allows improving the outcome $\boldsymbol{n}$. The second part of the proposition is the same statement, worded differently. Nonetheless, we consider it necessary to provide complete proofs of all statements, since proofs are presented here only in a concise form.



The Left- and Right-Wing Political Power Design:* The Dilemma of Welfare Policy with Low-Income Relief

Abstract: Findings from this experiment contributed novel insights into the theoretical field of welfare policy, addressing fundamental questions about wealth redistribution rules and norms. The expenses of the redistribution pertaining to basic goods, as well as those associated with public (non-basic) but vital goods, are separately estimated by transforming the expenses into functions of the poverty line. The findings reveal that, along the poverty line that treats all citizens equally, the politicians representing opposing ideologies decide how the redistribution of basic and vital goods should be financed. Politicians should come to an agreement, subject to an approval of their decisions by voters-citizens. However, in the absence of such approval, politicians have no alternative but to continue the negotiations. Based on this premise, we concluded that political decisions with an elevated poverty line, as a parameter, would give rise to inverse working incentives of benefits claimants. This may result in unbalanced books, due to the expenditure on the delivery of basic and non-basic goods to their respective destinations. By keeping the books in balance, we postulate that ½ of median income μ, which is recognized as Fuchs point, it may be used in the form of poverty line as ½ μ for just and fair wealth redistribution in resolving the ideological controversies between left- and right-wing politicians. As a result of modeling the rules and norms of compensation payments, which have been known since 1962 as the Negative Friedman Income Tax (NIT), the wealth redistribution exclusion rule by income level ½ μ, has reduced the Gini coefficient.

Keywords: bargaining; welfare policy; public goods; taxation; voting

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* Becker Friedman Institute, Working Paper Series, No. 2012-13, October 19, 2012; *The Conference of Economic Design*, Palma de Mallorca, June 30, 1-2 July 2004, SED2004, *The 3rd International Conference on Public Economics* of the Association for Public Economic Theory (APET), Paris, Pantheon-Sorbonne, July 4-6, 2002, Research Announcements, Economics Bulletin, Vol. 28, No. 22, 2001; *First World Congress of the Game Theory* (Games 2000), July 24-28, 2000, http://at.yorku.ca/c/a/e/z/12.htm (Accessed 23.12.2021).

1. INTRODUCTION

Political competition related to wealth redistribution often fosters debate regarding what the state "should" or "should not" deliver. Wider and more substantial welfare benefits and relief payments could be problematic, as they might encourage certain behaviors, such as low savings or productivity when economic security is guaranteed. Similarly, they may lead to high wage demands, as an incentive to remain in employment, given that unemployment benefits are substantial and are compensated by high tax rates τ. In addition, high taxes are an incentive for entering a black labor market that avoids paying taxes, or moonlighting, i.e., holding multiple jobs. Finally, high benefits typically undermine social and geographical mobility. Evidence also shows that, under these conditions, a few would opt for working just because financially they would not be tempting, while many will be wondering why studying is worth the efforts and sacrifices. In sum, excessive benefits might result in human capital not developing quickly and well enough, *e.g.,* "*…implicit support to those waiting on benefits looking for the 'right type of job' or a job that pays well enough,*" as noted by Oakley and Saunders (2011).

As the welfare policy of the state presupposes the existence of both a functioning market economy and a democratic political system, its hallmark is that the distribution of public goods and services is governmental responsibility and obligation. The term *public* in this context refers solely to wealth redistribution. In particular, an obligation to ensure that those on low incomes are awarded appropriate levels of social benefits and relief payments results in a more egalitarian allocation of wealth than can be provided by the free market. In this scenario, politicians face a dilemma of whether such allocation is just and fair to all citizens. The solution depends on many factors, including the characteristics and views of the main benefactors of wealth redistribution. In the absence of a universal definition, in this work, we use the term "wealth" in the scholarly sense, delivered through tax channels and distributed by the state. Under this premise, the average taxable income per capita represents the wealth $\mathbf{W}$.

The primary goal of this experiment is to demonstrate fallacy of arguments advocating in favor of higher benefits and relief payments. Beyond the negative perception of higher benefits, it is also reasonable to believe that distribution of citizens' incomes σ is, perhaps, the only target for control and an exclusive source of information for assessing the amount of benefits available. Our goal is to highlight a hidden side of public interests to welfare issues (Flora, ed., 1987), its geographical, historical justification and broad experimental support in analyzing credible income distributions (Huber et al, 2008). Since we approach welfare redistribution from a more theoretical perspective, we need to have a different emphasis compared to these issues. However, apart from this key aspect, the solution of the welfare policy dilemma, based on numerical simulations, yields the benefits to the needy that are sufficiently close to be considered a realistic match (see Table 1), as noted by Bowman in 1973, to *"what amounts to a moving poverty line at ½ of median income."* In support of this approach, it is worth noting that Rawls (1971, 2005) pronounced the Fuchs (1965) point as an alternative to the measurement of poverty with no reference to social position. The motive of the experiment presented here is thus to provide — while acknowledging that a few examples clearly cannot make a trend — a theoretical confirmation for the claim recognizing the poverty line, defined as ½μ of the median income μ, as a realistic political consensus.

Table 1. Numerical experiment behind the welfare policy dilemma of income redistribution; SA—Social Agencies, PA—Public Agencies

Obtained by means of income distribution density (Fig. 3); personal allowance $\phi = 4.03$; $\theta = 61.9$; $h = -0.11$; $m = 2.07$; *subsidy function* $s(\xi) = 0.83\xi$.		Policy of equal, symmetric power of negotiators η	*SA proposal accepted by PA* λ_1, $q = 5\%$	*Proposal minimizing wealth tax* λ, $q \approx 0\%$	Income floor. 50% of median income $\frac{1}{2}\mu$	*PA proposal accepted by SA* λ_2, $q = 5\%$	Policy of disagreement, the breakdown δ
Income floor–welfare policy	$\xi =$	65.94	*34.03*	*38.40*	45.32	*42.81*	6.64
Poverty rate: percentage of agents below the income floor		31.91%	*10.51%*	*13.04%*	17.39%	*15.77%*	0.44%
Negotiating power of social agencies	$\alpha(\xi)$	0.50	*0.14*	*0.17*	0.24	*0.22*	Not defined
Guaranteed social minimum	$u(\xi)$	47.57	*25.49*	*28.63*	33.55	*31.77*	7.07
Account for public, goods expenses	$g(\xi)$	16.15	*30.15*	*28.72*	26.18	*27.15*	−19.75
Account for subsidies transfers	$B(\xi)$	17.53	*2.98*	*4.17*	6.57	*5.62*	0.02
Account for public spending, the size of the welfare-pie	$z(\xi)$	33.68	*33.14*	*32.89*	32.75	*32.77*	−19.73
Average taxable income—the wealth amount	$W(\xi)$	113.52	*116.38*	*115.73*	114.84	*115.14*	121.59
Wealth-tax, marginal tax rate	$\tau(\xi)$	29.67%	*28.47%*	*28.42%*	28.52%	*28.46%*	−16.22%

In our scheme, citizens earning low incomes (below a certain level, in this case the poverty line ξ) receive relief payments, whereas those with higher incomes (above the aforementioned level) do not. In this regard, it should be noted that, in 1962, Milton Friedman (2002) proposed a similar scheme of wealth redistribution, combined with flat tax, called the negative income tax — the NIT. According to the rules and norms of the NIT, low-income earners receive a relief payment proportional to the difference between their earnings and the predetermined NIT poverty line. Most importantly, the total — the sum of the key income and the NIT relief payment — is not subject to taxation. We argue that levying taxes in compliance with the tax rules and norms in force for all, inclusive of low-income citizens, would have the same result. Although the total income of low-income citizens is now taxable, they would, even so, still be eligible for the relief in line with NIT, similar to the widely adopted low-income — LI relief. The known drawback of such an approach, and the relief, in particular, stems from the issue of social abuse by those earning low income. In order to mitigate these undesirable effects, in this work, we introduce the so-called hazard of working incentives, referred to as the h-effect.

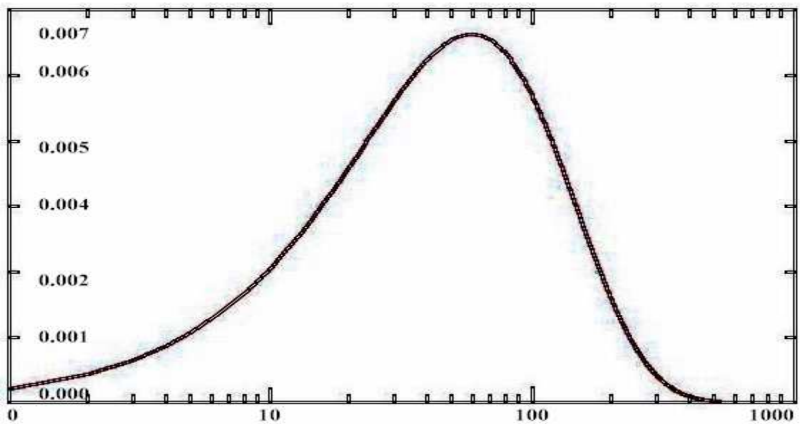

Figure 1. At the sample $P(\sigma, \theta + h \cdot ½\mu)$ of the income density distribution, μ solves the equation $\int_0^{\xi} P(\sigma, \theta + h \cdot \xi) d\sigma = 0.5$ for ξ, $\mu = 82.30$.

Appendix A1 contains the analytical form for the sample expression in Figure 1.

We thus present a theoretical model of visionary politicians, whereby we consider a masquerade of life or a scenario of *realistic utopia*. In this scenario, two actors/politicians, akin to two political coalitions, are playing a bargaining game, each attempting to implement his/her own wealth redistribution policy. *Left-wing politicians* tend to oppose the disproportion in private consumption, unjust wealth redistribution, profit motive, and private property as the main sources of socioeconomic evil. *Right-wing politicians*, owing to a different ideology, tend to focus on regulating business and financial risks, thus encouraging the government's use of its powers in combating corruption, criminal violence and commercial fraud. While left-wing politicians prefer immediate and equitable sharing of the available stock of goods and services, both sides are aware of the citizens' sacrifices — in terms of direct contribution of a part of their income to the funding of welfare benefits and public goods. We posit that applying the rules and norms of wealth redistribution pertaining to the reliance on the elevated relief would increase the quantity of the relief payments to be delivered. Consequently, citizens will have to meet a greater tax burden. This outcome is not ideal, given that lower tax burden and greater private consumption always lie at the heart of citizens' economic and political aspirations. These private objectives prompt majority of voters, who hold power in electing political parties, to oppose increasing the tax burden. As a result, they are instrumental in the competition between the left- and right-wing politicians and their views on tax policies.

Political consensus is rarely possible in reality. Consequently, we aim to design an experiment capable of predicting an appropriate political division between interest groups for desirable implementation of the welfare policy. This approach does not require analysis of the voting system or a scheme by which voters-citizens express their arguments. In adopting this approach, we analyze political power indicators as replications $(\alpha, 1-\alpha)$, $0 < \alpha < 1$, in line with Kalai's bargaining game (1977) in which division of \$1 is attempted. In this scenario, among other assumptions, it is posited that a power α is appropriate to adopt the ability to negotiate, or be in the position to request financial support to a greater extent than the opposite side. Similar interpretation of players' power dynamic may be found in the recent work of Mullat (2014). In short, we adopted the view of Roberts who noted in 1977, "*The point is not whether choices in the public domain are made through a voting mechanism but whether choice procedures mirror some voting mechanism.*"

These brief remarks should be sufficient to elucidate some goals of the state, allowing us to conclude that welfare policy in a representative democracy always faces ideological controversies of politicians and citizens. A further aim of this experiment is to shed light on how a political consensus is reached and whether it reflects a criterion of tax policy that results in the least burden to the citizens. To address this issue, as already stated, we focus our analysis on two visionary politicians. For the purpose of the experiment, we assume that these politicians are granted a political mandate to initiate proposals ensuring that the relief payments are allocated to citizens who are in need. We thus assume that, in balancing the books accounting for finance of relief payments and for vital public goods and services, expenses are constrained. This premise ensures that the citizens control the negotiations, forcing the politicians to act within the imposed budget constraints in order to pledge safe funding for their proposals. While trying to reduce the after-tax income inequality, the politicians in their respective roles of left- and right-wing actors are committed to ensuring that the wealth is redistributed fairly.

At this point, it is essential to state the assumptions/limitations underpinning the analysis of a hypothetical behavior of those occupying three distinct roles in the negotiations — those of left- and right-wing politicians and voters-citizens. Throughout this work, we emphasize the incomparability between the aims of the left-wing politicians struggling to ensure adequate access to basic goods and the right-wing politicians advocating for availability of non-primary but vital goods and services. In the analysis, we implicitly assume that politicians do not have adequate knowledge of citizens' needs in a more primitive environment. Hence, they can only work with the monetary payoff specification. Given this limitation, politicians are unaware that the provision of equivalently valued public services is not a perfect substitute. For example, we assume that politicians do not have any information on how household income is assembled and used to buy private health insurance or services of nursing housing, etc. Thus, we do not merit the debate on what is right or wrong in the economic or political environment involving left- and right-wing politicians and voters-citizens. In short, our work does not extend to the democratic context of voters' prototypes/characteristics. While acknowledging the significance of prototypes, in this work, we view voters' behavior as a binary process, allowing support for either left- or right wing politicians. This, however, introduces a risk $q > 0$ of premature political breakdown of negotiations. In addition, we refer to the tax revenue in accord with voters' preferences as the "wealth-pie" $\tau \cdot W$, which is divided into two parts (x, y), whereby x denotes various social benefits or relief payments, and y pertains to public goods, so that $x + y = 1$. We posit that any further enrichment of voters' characteristics would disrupt the delicate balance between the motives of our experiment and the theoretical framework, which is already technically sophisticated.

Roadmap. Because of the narrative complexity, it is possible that the reader would find proceeding with the content of the paper in chronological order difficult. Thus, to mitigate this potential issue, Section 3 presents the most relevant problems, in particular, the pre-equity condition of political breakdown of the negotiations. In our view, it is prudent to master the material presented in Section 3.1 before moving to Section 4. Similarly, Section 3.2 aims to assist with understanding of the content of Section 5, while Section 3.4 supports Section 6. On the other hand, those not wishing to delve deeply into the technical aspects of this work could simply move onto Section 7. Nonetheless, Section 3.3 provides a scheme pertaining to the pre-equity of breakdown of the negotiations and, in our view, does not require further clarification.

2. PRELIMINARIES

Before delving deeper into our work, we specify the category of the game payoffs functions $u(\xi, x)$, $g(\xi, y)$ and taxes $\tau(\sigma, x)$ required for the model validity. As noted above, Section 3 provides background information that assists in understanding material given in Section 4-6. In Section 4, we disclose fiscally safe welfare policy in amalgamation with imposed *budget constraints* for financing relief payments. Referred to as volatility constraint, the amalgamation dynamically restricts the h-effect — an inverse working incentives phenomenon of low-income citizens. In Section 5, citizens' ambivalence and multifaceted welfare policy perceptions are discussed from the perspective of the alternating-offers game. The policy on poverty associates the left- and right-wing politicians with payoffs functions $u(\xi, x)$ and $g(\xi, y)$. Under these conditions, it is possible to obtain an analytical solution to the game with incomes σ density distribution $P(\sigma, \xi)$. Indeed, as will be shown, the calculus of indicators $(\alpha, 1-\alpha)$ complies with the political power design given in Section 6. The results are discussed in Section 7, followed by concluding remarks, presented in Section 8.

In the current experiment, an income σ equal to the poverty line ξ, $\xi \in [\xi_1, \xi_2]$ parameterizes all arguments and functions. In this vein, we adopt quantitative measurement, whereby we utilize a scale quantum as an average income with the income σ density $P(\sigma, \xi)$ distribution, $0 \le \sigma < \infty$. The average establishes the ratio scale. Hence, we suggest that $u(\xi, x) = (1 - \tau(\xi, x) \cdot (\xi - \phi) + \phi$ (the after-tax residue of income $\sigma = \xi$) signifies the 1st actor's social position at the specified scale, i.e., the left-wing political aims. We apply the residue formula based on Malcomson's (1986) model, with a personal allowance parameter ϕ, $0 < \phi < \xi$, determined by the tax bracket $[\phi, \infty)$. The 2nd actor's aim — the right-wing political objective $g(\xi, y)$ — is ensuring sufficient amount of the non-basic goods per capita. Here, we refer to the citizen $\sigma = \xi$ as *marginal citizen*. While, for the minority of voters, the relief is more attractive than lower taxes, the 3rd actor is the implicit partaker embodying the majority of voters whose preference is minimizing tax obligation $\tau(\sigma, x)$. This is a typical public finance dilemma of efficient division (x, y) of the tax-revenue into shares $x + y = 1$. In this work, the dilemma is represented by the alternating-offers bargaining game $\Gamma(q)$ with premature risk q, $0 < q << 1$, of political breakdown. When $q \to 0$, the solution converges into Nash axiomatic approach (1950). The relationship between the one that suggests the alternating-offers bargaining and axiomatic solution is well known from the work of Osborn and Rubinstein (1990). As this game is thoroughly described by Osborn and Rubinstein, for brevity, no further elaboration is offered here.

When negotiating on finance issues, under the guise of a "*wealth-pie workshop,*" politicians will allegedly try to divide the wealth-pie in a rational and efficient manner. As a result, the tax $\tau(\sigma, x)$ will increase as will the wealth-pie, when increasing the poverty line ξ. Logically, a decrease in taxes would yield the reverse effect. While taxes vary, the division will depend upon the characteristics and expectations of the bargainers involved. Indeed, the left- and right-wing political aims $u(\xi, x)$ pertaining to basic goods, as well as the objective $g(\xi, y)$ related to the non-basic goods, are controversial. We illustrate this tax controversy by elevated single-peaked frontier of $u(\xi, x)$, the $\frac{2}{5}$-share/slice in Figure 2, which corresponds to the lower, but progressively increasing, concave frontier of $g(\xi, y)$, the $\frac{3}{5}$-share/slice in Figure 3, as well as for another division of the pie, into shares/slices $\left(x = \frac{1}{8}, y = \frac{7}{8}\right)$. We believe, that, while $\left(x = \frac{2}{5}, y = \frac{3}{5}\right)$ highlights the left-wing political aspirations, the share/slice $\left(\frac{1}{8}, \frac{7}{8}\right)$ elucidates those of the right-wing political objective. This premise appears to be crucial for understanding our primary goal in resolving the welfare policy dilemma.

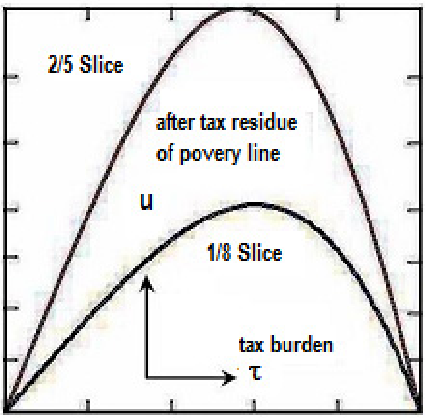

Figure 2. Left-wing politicians' emphases.

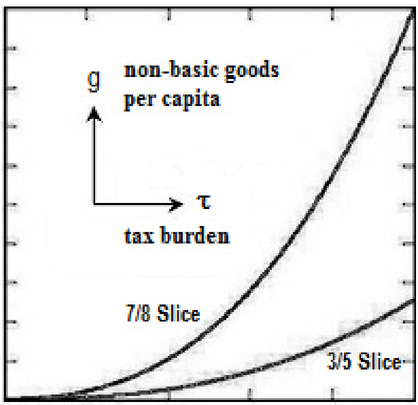

Figure 3. Right-wing politicians' emphases.

In support of the aforementioned assumption, the political payoffs in general, as shown in Figure 2 and Figure 3, emerge within a two-man economy endowed by citizens' income abilities marginalized at the level of poverty line. According to Black (1948), single peakedness plays the key role in collective decision making when the decision is reached by voting. The payoffs for the two actors, shaped in this way, are non-conforming/incomparable, and are thus impossible to match through a monotone transformation, as established by Narens and Luce (1983). The single peakedness is nonetheless in line with Malcomson's tax residue $u(\xi, x)$, when the terms of contract commit the actors to shares (x, y). This, however, requires that the expenses covered by flat taxes will balance the books, while accounting for relief payments, as shown in Figure 2. Clearly, increasing the poverty line requires an excessive increase in taxes, which in turn provides a greater amount of non-basic goods $g(\xi, y)$, as shown in Figure 3. An opposite scenario of increasing the available amount of non-basic goods $g(\xi, y)$ equally requires an excessive tax increase, which may lead to the possibility of increasing poverty line.

Following the traditional procedure for division of the wealth-pie in the alternating-offers game, when the pie is desirable at all the times, the politicians (bargainers) — changing roles — commit to shares (x, y), $x + y = 1$. According to the shares (x, y), the valid rules and norms of wealth redistribution, which guarantee a desirable level of relief payments, require establishing a poverty line ξ parameter. However, an efficient division of the wealth-pie — as a result of single-peaked $\cap$-curves depicted in Figure 2 — no longer represents any traditional bargaining procedure. This is the case as, instead of division, the procedure can be resettled. Indeed, we can proceed at distinct levels of one parameter — within the poverty line interval $[\xi_1, \xi_2]$ — reflecting the scope of negotiations. In fact, Cardona and Ponsattí (2007), also noted that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all the parties in the bargaining process have the same, conforming expectations. This argument applies even when the expectations of the first player are principally non-conforming, i.e., single-peaked, rather than excessively concave in regard to the second player. In our experiment, the scope of negotiations on the "contract curve" of non-conforming expectations allows for omitting the "Pareto efficiency" and replacing the axiom by "well defined bargaining problem," as posited by Roth (1977). The well-defined problem (x, y) of the wealth-pie division can now be solved (resettled) inside the poverty line interval $[\xi_1, \xi_2]$.

Settings

In accordance with Friedman's NIT system, in this work, we assume that, for the unfair subsistence of the less fortunate citizen $\sigma < \xi$, the relief amount $r \cdot (\xi - \sigma)$, $0 < r \le 1$, serves as a monetary compensation designated for purchasing an eligible "poverty basket" of food, clothing, shelter, fuel, etc. According to Rawls, "*primary goods are things which it is supposed a rational man wants whatever he wants.*" In defining the parameter ξ in this manner, it becomes contingent on financing the relief. This can be achieved by assuming that elevating the poverty line ξ requires an increased marginal tax rate $\tau(\sigma, x)$. In increasing the wealth-pie through tax channels, we assume an acceleration $\tau''_\sigma(\sigma, x) > 0$ of the tax rate $\tau(\sigma, x)$; $\tau'_\sigma(\sigma, x) > 0$ inclusive all of those citizens who indicate the marginal income ξ denoted by $\sigma = \xi$.

As noted previously, the marginal citizen $\sigma = \xi$ must bear the cost of the left-wing political aims using tax residue $u(\xi, x)$, as well as the right-wing political objective $g(\xi, x)$, referred to as "public or non-basic goods." With the proviso that politicians commit to the shares (x, y), we conclude that $u(\xi, x)$ is a single $\cap$-peaked curve, due to the tax rate $\tau(\xi, x)$ increase upon ξ. While objective $g(\xi, x)$ of right-wing politicians decreases with an increase in x, the reverse is true with elevating ξ due to $\tau(\xi, x)$ acceleration. Here, payoffs $\langle u, g \rangle$ are considered analytic functions $u(\xi, x)$, $g(\xi, x)$. Given the interval $[\xi_1 \le \xi \le \xi_2]$, referred to as the scope of negotiations, $u(\xi, x)$ reflects single $\cap$-peakedness — $u''_\xi < 0$ upon ξ increase, $u'_\xi(\xi_1, x) > 0$, $u'_\xi(\xi_2, x) < 0$. Following an increase in x, the payoffs $u(\xi, x)$ become convex, $u''_x > 0$, $u'_x > 0$, whereas an increase in ξ would produce concave payoffs $g(\xi, x)$, with $g'_\xi > 0$, $g''_\xi > 0$. It can be shown that, with increasing x, payoffs g always decrease; in other words, in both circumstances, either $g''_x > 0$ is convex, or $g''_x < 0$ is concave.

3. RELEVANT TRENDS AND ISSUES

In the extant literature (Espring-Andersen, 1990; Iversen, 2005; Swank, 2002) the welfare, economic, and political issues are usually addressed in reference to specific questions. In our view, a much deeper analysis is achieved when addressing them more generally, adopting well-established knowledge discovery methodologies. In particular, our wealth-pie workshop concept, jointly adopting four issues — (a) public finance, (b) alternating-offers game, (c) negotiations' collapse analysis, and (d) political power design — leads to a more informative point of departure.

To explain the root cause of the results in order to bring the welfare, economic, and political content to the surface in a rigorous analytical form, and to find bilaterally acceptable solutions to the game, we will visit all of the classrooms in our workshop. Our goal is to lay the foundation for a more constructive welfare policy comprehending the meaning of following four narratives:

Fiscal policy	During the delivery to its final destinations, provided that the books accounting for the relief payments finance have been balanced *a priori*, the wealth-pie must remain balanced throughout and in spite of volatility in the economy;
Negotiations	The left- and right-wing political bargaining on how to share the wealth-pie complies with the rules and norms of the alternating-offers bargaining game;
Pre-equity of breakdown	Political breakdown, or threat point, directly affects the bargaining solution. Pre-equity guarantees equal conditions for players before the bargaining game commences;
Political power design	Bringing a motion to a vote is necessary to address the majority opposition to high taxes and excessive public spending. Whether it is viewed as positive or negative, or whether it ought to be acknowledged or not, rejected or accepted, this motion must be politically designed in advance.

In our wealth-pie workshop, these four narratives can be understood as obligations/constraints to be met by welfare policy rules and norms, akin to ″Rational man″ deliberation of Rubinstein (1998). This interpretation allows us to provide a scenario under which the narratives are embedded into the welfare policy of the state. In addition, evaluating the welfare policy from this perspective reveals that the analysis can be subject to and performed by computer simulations, as shown in Appendix A2. Our initiative could also serve to unify the theoretical structure of economic analysis of public spending. It can be used to evaluate the political power design of left- and right-wing politicians, or to launch systematic inquiry into impacts of governmental decisions and actions on wealth redistribution.

As the state has the duty to help the less fortunate, our experiment approaches wealth redistribution in a two-fold manner. First, it addresses the provision of basic necessities or goods, such as shelter and heating, clean and fresh water, nutrition, etc., before focusing on non-basic goods, including national defense, public safety and order, roads and highway systems, and so on. Welfare policy issues, according to Boix (1998), "*...There is wide agreement in the literature that governments controlled by conservative or social democrats parties have distinct partisan economic objectives that they would prefer to pursue in the absence of any external constrains.*" Meeting this challenge, based on income σ density distribution $P(\sigma,\xi)$, we identify an effective approach to the division $\left(x^{\circ},y^{\circ}\right)$ into shares $x^{\circ}+y^{\circ}=1$ pertaining to basic x° and non-basic goods y°. Fundamentally, the efficient division $\left(x^{\circ},y^{\circ}\right)$ of the wealth-pie aims at just and fair delivery of all aforementioned goods, traditionally perceived as public goods. In our experiment, we refer to public goods as non-basic but vital goods, whereas basic goods are deemed fundamental. Incidentally, during the delivery of basic and non-basic goods to their end destinations, we treat both as public goods.

We assume that the left-wing politicians have the necessary political influence — when an offer is made, irrespective of its legitimacy — to control the redistribution of basic goods independently. Given the single-peaked aspirations of the left-wing, in contrast to the objective of their right-wing counterparts, the influence the left-wing politicians enjoy, is supposed to be adequate enough to reach the peak of these expectations. In particular, we believe that, beyond some peak position, inefficient usage of basic goods would lead to an excessive decline in the quality of welfare services, as well as cause deterioration in access to public goods for all citizens. In making these suppositions, we agree with Rawls's statement, about the precepts of perfect justice: "*The sum of transfers and benefits [...] from essential public goods should be arranged so as to enhance the emphases of the least favored consistent with the required saving and the maintenance of equal liberties.*"

An efficient usage of public resources implies that a consensus between left- and right-wing politicians might be reached. Despite some views to the contrary (Rothstein, 1987), we posit that the bargaining aimed at finding a just and fair division of basic vs. non-basic goods is an acceptable path to the bargaining dynamics. Based on this premise, we can identify relevant connections in extant works on economic and political behavior that analyze the sociological and political aims of ensuring adequate welfare by using public finance. This is likely being the best starting point for visiting our wealth-pie workshop.

*3.1. **Fiscally safe welfare policies**, to be continued in Section 4*

Public finance focuses on the revenue side of tax policy. In particular, it pertains to the budget formation, as noted by Formby and Medema (1995), aiming to provide a guaranteed level of welfare to citizens endowed by poor productivity. While the welfare policy is a separate issue, it should be considered on the grounds of legal and moral rights of citizens. Empirical evidence confirming that such policy is government's legal obligation can be found in pertinent literature. For example, as noted by Saunders (1997), "*...poverty line. The line was initially set (in 1966) equal to the level of the minimum wage plus family benefits for one-earner couple with two children.*" Similarly, a hypothesis consistent with moral obligations can be found in the literature of economic politics (Eichenberger, 1996; Feld, 2002).

In 1959, Musgrave examined two basic approaches to taxation — the "*benefit approach*" and "*ability-to-pay,*" which put taxation into efficiency and equity context, respectively. In this work, we utilized the benefit approach in order to augment the existing standard of welfare policy, whereby we allocate a guaranteed amount of income for minimum taxes. We posit that a flat tax system — based on injecting optimal equity according to the ability-to-pay principle of "proportional sacrifice" — ensures that taxes remain *fairly levied*.

Taxation is a principal funding source of social costs and benefits. Thus, the first postulate in our welfare policy workshop (see above) discloses an obvious paradigm in social policy. According to the ability-to-pay principle commonly adopted in public finance, in order to stabilize the distortion of tax polices, the known terms of warranty must rely on exogenous taxes enforced on the productivity of citizens. The concept, proposed in 1996 by Berliant and Page Jr., is a variant of the classic public finance and similar approaches, applicable when an agent characterized by a specific level of productivity does not shift his/her labor supply after all adjustments to the tax formula have been implemented. In other words, under this paradigm, optimal taxation enforces optimal labor supply.

Yet another "treatment of policies," closely related to societal instability, entails equity of pre- and post-tax positions of citizens. Such a view demarcates between citizens and has attracted the attention of economists and tax policy makers. In the view of Kesselman and Garfinkel (1978), credit tax-scheme analysis opposes the income-tested program in the rich-and-the-poor, also known as two-man economy. Poverty measurements have also been addressed in the works of Sen (1976), Atkinson, (1987), Ebert (2009), and Hunter (2002) et al. According to Tarp (2002) et al: "*The poverty line acts as a threshold with households falling below the poverty line considered poor and those above poverty line considered nonpoor.*" García-Peñalosa (2008) investigated wealth redistribution as a form of social insurance in relation to economic growth. On the other hand, Stewart et al (2009) attempted to reduce horizontal inequalities, proposing "*a reallocation in the production, operation and consumption of publicly funded services.*"

In the attempt to assess and control the circulation of wealth through tax channels, we argue that, unless fiscal stabilization is not a required condition when justifying public spending, it will be difficult to explain how the citizens eligible for relief gain access to the benefits and relief payments. Thus, while we continue to rely on fiscal stabilization, in order to highlight a particular type of the dynamics stability, we refer to welfare policy as idempotent. For clarity, a choice operation (or decision) applied multiple times is deemed idempotent if, beyond the initial application, it yields the same result (Malishevski, 1998). Thus, based on this dynamic definition, idempotent scheme allows the politicians to honor the pledges made during the election campaign as, once the political decision is taken, it eliminates the need for further stabilization. While visiting the workshop, the circulation of wealth is supposed to be dynamically stable, i.e., it is idempotent.

3.2. Bargaining the Welfare State rules and norms, to be continued in Section 5

Bargaining is the key element of economics and is at the core of politics. On the other hand, as pointed out by North (2005), "*The interface between economics and politics is still in a primitive state in our theories but its development is essential if we are to implement policies consistent with intentions.*" More recently, Feldstein (2008) noted, "*Unfortunately, there is no reason to be pleased about the analysis in policy discussions of the efficiency effects…of the welfare consequences of proposed tax changes.*" Similarly, in a review on "Handbook of New Institutional Economics," Richter (2006) stressed, "*…that the sociological analysis…and large institutional structures in economic life is still at an early stage…game theory, and computer simulation could help to further develop the new institutional approach…game theory might be a defendable heuristic device of NIE.*" Indeed, the left- and right-wing politicians, like actors in the game, strive to implement their vision of the state welfare institutions. This is succinctly explained by Ostrom (2005), who noted, "*These flimsy structures, however, are used by individuals to allocate resource flows to participants according to rules that have been devised in tough constitutional and collective-choice bargaining situations over time.*"

In order to achieve the aforementioned vision of collective choice, it is appropriate to consider a scenario in which the actors/voters play the "bargaining drama" of economic and political issues. Bargaining has been a theme of a wide range of publications, including the work of Alvin E. Roth (1985). Despite the simplification, the binary behavior of voters remains at the root of the democratic transformation of public institutions. In this regard, binary position fits particularly well into the bargaining game with exogenous risk q, $0 < q << 1$, of breakdown (Osborn and Rubinstein). Actually, bargaining can be risky for all interested actors because they may lose voters to the competition if their terms are not met. Thus, it is essential to first clarify political power dynamics of both the left-wing and the right-wing politicians. Henceforth, they are respectively referred to as LWP, the 1st actor, benefiting from a power α, $0 < \alpha < 1$, and RWP, the 2nd actor, benefiting from a power $1-\alpha$.

Numerous factors — such as economic growth, decline or stagnation, demographic shift or pit, political change or change in scarcity of resources, skills and education of the labor force, etc., — might create fiscal imbalance in a desirable welfare policy due to the transfers of benefits and relief payments. As a result, the size of the wealth-pie might be too small (i.e., not worth the effort required for its redistribution), or too large (introducing mutual traps) to achieve a stabilized public spending mechanism. In either case, the actors may decide not to share the pie at all. To address this controversy, as previously underlined, we assume that politicians participate in relevant public institutions. If the institutions cannot or do not want to follow RWP's policy of wealth redistribution, RWP — in order to promote their own understanding — can be sufficiently legitimate to deliver the wealth "properly." In doing so, RWP can enforce vital decisions by several means, including resource mobilization, retaliation for breaches and criminal fraud, recruiting political volunteers and managing public service commissions, soliciting private contributions, etc. In other words, as Kalai pointed out, RWP would rely on an "*enthusiastic supporter.*" On the other hand, as LWP face decay in political legitimacy for perfect justice, they cannot fully control RWP's actions and intentions when their political interests in the final agreement are incomparable. In these circumstances, RWP are aware that their abilities and access to information might necessitate agreeing with, or at least not resisting, LWP's privileges to make arrangements upon the size of the pie. Hence, from the RWP's critical point of view, whether acting politically in common interest or not, it might be prudent to acknowledge LWP's welfare activities. This elucidates the asymmetric dynamics of political power division between the LWP and RWP.

Returning to the main points of asymmetric bargaining, we will illustrate an efficient solution $\left(x^{\circ}, y^{\circ}\right)$ by division of \$1 aimed at maximizing the product of actors' payoffs above the disagreement point $d = \langle d_1, d_2 \rangle$:

$$\left(x^{\circ}, y^{\circ}\right) = \arg\max_{0 \le x+y \le 1} f(x, y, \alpha) = \\ = \left(u(x) - d_1\right)^{\alpha} \cdot \left(g(y) - d_2\right)^{1-\alpha} .$$

Although game theory purists might find the solution clear, the questions asked by many often include: *What are* x*,* y*,* α*,* $u(x)$*, and* $g(y)$ *? What does the point* $\langle d_1, d_2 \rangle$ *mean, and how is the* $\arg\max$ *formula used?* The simple answer, as initially provided by Kalai as an asymmetric variant of Nash problem, is as follows:

- x is the 1st actor's share of \$1, with α as the 1st actor's asymmetric power indicator, $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ denotes the 1st actor's payoffs of the 1st actor's \$1 share x;
- y is the 2nd actor's share of \$1, where $1-\alpha$ is the 2nd actor's asymmetric power indicator, $0 \le y \le 1$;
- $g(y)$ denotes the 2nd actor's payoffs of the 2nd actor's \$1 share y.

Based on the widely accepted nomenclature, we refer to $s = \langle u(x), g(y) \rangle$ as to the utility or payoffs pair. Thus, the disagreement/threat point $d = \langle d_1, d_2 \rangle$ represents the payoffs the two actors obtain if they cannot agree on how to share the wealth-pie. In the same vein, $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$ represents the disagreement or breakdown point, whereby the players collect nothing.

In the subsequent sections, we will provide an analytical solution exploiting payoffs in the form $\langle u(\xi), g(\xi) \rangle$ and taxes in the form $\tau(\xi)$ within the scope of negotiations $[\xi_1, \xi_2]$ comprising the endpoints of the interval $[\xi_1, \xi_2]$. According to the analytical solution, implicitly hiding the variables x, y, it follows that any negotiation of shares (x, y) can be perceived as two sides of the same bargain's portfolio, as the shares (x, y) are accompanied by poverty lines $\xi \in [\xi_1, \xi_2]$. While hiding the variables x, y, $x + y = 1$, we may respond to the question of whether solution $\xi^\circ \in [\xi_1, \xi_2]$ is efficient in a traditional sense. Indeed, akin to the above, political bargaining can now be expressed by poverty line ξ° maximizing the product of political payoffs above the threat point $d = \langle d_1 = u(\xi_1), d_2 = g(\xi_2) \rangle$:

$$\xi^\circ = \operatorname{argmax}_{\xi \in [\xi_1, \xi_2]} f(\xi, \alpha) = (u(\xi) - d_1)^\alpha \cdot (g(\xi) - d_2)^{1-\alpha}.$$

On the other hand, unlike the traditional threat point $d=(d_1,d_2)$, the public/vital goods amount d_2 in the game — the d_2 component of the point d — might be negative. This will apply in our experiment of a breakdown of negotiations, whereby funds need to be borrowed or acquired through other means in order to balance the books and account for the welfare expenses — a situation of "genuine negative taxes." It is important to note that, while this may seem counterintuitive to some readers, in the theory of public finance, the use of genuine negative taxes is not prohibited.

Finally, we conclude that, all these remarks notwithstanding, it is irrelevant whether the players are bargaining on shares (x, y) or trying to agree on the poverty line level. This assertion highlights the main advantage of hiding the variables x, y. In particular, it brings about a number of different patterns of outcome interpretations in the game, such as linking an outcome to the lowest tax rate, which is the most desirable sacrifice of voters' majority. In consideration of alternative approaches — which describe outcomes of collective bargaining in the form of voting, or partaking in any voting scheme in the form of bargaining — the scope of negotiations $[\xi_1, \xi_2]$ brings the voting and bargaining schemes into the same context, as both can be enriched by adopting this approach. Our insight is forward-looking in the sense that it aims to identify an alternative-offers game solution, whereby both actors accept at once the proposals (moves) made by the other side. Our initiative could also serve to unify the theoretical structure of economic analysis of productivity problem. Indeed, when referring to Leibenstein's work (1979), Altman (2006) noticed:

> *Leibenstein (1979, p.493) argued that there are two components to the productivity problem: one relates to the determination of the size of the pie, while the second relates to the division of the pie. Looked upon independently, all agents can jointly gain by increasing the pie size…"the situation need not be a zero-sum game. Tactics that determine pie division can affect the size of the pie. It is this latter possibility that is especially significant.*

3.3. Pre-equity of political breakdown

Beyond the asymmetric dynamics, the game inherits a premature disagreement or breakdown point, similar to that discussed by Osborn and Rubinstein:

> *We can interpret a breakdown as the result of the intervention of a third party, which exploits the mutual gains. A breakdown can be interpreted also as the event that a threat made by one of the parties to halt the negotiations is actually realized. This possibility is especially relevant when a bargainer is a team (e.g., government), the leaders of which may find them unavoidably trapped by their own threats.*

In our game, the asymmetric solution incorporates the left- and right-wing political power indicators $(\alpha, 1-\alpha)$ into a breakdown policy. In order to be addressed properly, the indicators cannot be given exogenously. To overcome this obstacle, we introduce a policy of endogenously extracted breakdown $d = \langle d_1, d_2 \rangle$ into the game, based on a condition referred to as the *pre-equity of political breakdown.*

Traditionally, in the alternating-offers game, the breakdown corresponds to two standard pairs of payoffs $\{\langle 1,0 \rangle, \langle 0,1 \rangle\}$, or in the words of Osborn and Rubinstein, *"to the worst outcome."* In the left- and right-political bargaining, due to the implicit pressure from the voters, as both politicians aim to find — at least from their perspective — a just and fair solution, there will always be a temptation for binary voters to defect to the other side. This puts the negotiations at risk $0 < q << 1$ of a premature collapse. Even under the worst circumstances, in the event of collapse, the quality and the size of the wealth-pie should be equal for both politicians. This premise holds in these unfavorable circumstances, as the entire pie will be decided upon by one of the politicians. Thus, when the premature collapse occurs, it is important to arrange the terms of contract in such a way that neither politician can exploit or misuse these adverse circumstances to his/her own advantage. To meet this condition, when normalizing the standard breakdown under the description valid for the alternating-offers game $\Gamma(q)$, we are working toward an endogenous form for equity in accordance with political non-conforming expectations.

As stated, the standard case of breakdown in the alternating-offers game corresponds to two pairs $\{\langle 1,0\rangle,\langle 0,1\rangle\}$ of payoffs. In this form, the breakdown is generally found using ex-ante linear transformation, namely the exogenous normalization of utilities. When the collapse is imminent, the political breakdown exposes equity condition pertaining to the actual event of breakdown. Unlike the standard case, once the most unfavorable result occurs, the resulting collapse must include additional parameters — the tax τ and the wealth W. In order to equalize — endogenously normalize — the breakdown, the politicians involved in negotiations can make *a priori* arrangements, or sign binding agreements upon these two parameters, i.e., τ and W. Without availability or warranty of such a pre-equity, an endogenous normalization is unrealistic. In the view of the voters' electoral maneuvering (discussed in the next subsection), even if the *pre-equity normalization* is not always achievable, it is more constructive to determine the breakdown according to some rational context.

Before proceeding further with a detailed assessment of the aforementioned definition, we recall the concept of wealth amount W, redistributed by the state as the average taxable income per capita, scholarly defined as "prosperity or a commodity." Next, according to the conditions characterizing the collapsed environment, at the start of the negotiations, the draft of a contract includes both taxes τ and — in line with our nomenclature — the wealth amount W. The product $\tau(\xi)\cdot W(\xi)$ identifies the size z of the wealth-pie within an interval $[\xi_1,\xi_2]$ within the scope of negotiations, thus establishing the boundary for the two politicians. The lower limit ξ_1 denotes the initial proposal, which is the most attractive for RWP, while being the most unattractive for LWP. In the same but inverse order $u_2 = u(\xi_2)$ can be paired with $g_2 = g(\xi_2)$. Having set these limits, we can proceed with examining how the breakdown $\{\langle u_1,g_1\rangle,\langle u_2,g_2\rangle\}$ might be conditionally, albeit endogenously, encoded into the game.

Indeed, we now contribute to implementing our wealth definition of how the breakdown can be established endogenously. To do so, we consider a situation driving the welfare policy in the context of cost-benefit equity. When the collapse of negotiations is imminent, the differences in the amounts of wealth and taxes for funding low-cost welfare policy ξ_1

against an expensive policy ξ_2, $\xi_1 < \xi_2$ — i.e., funding payoffs $\langle u_1, g_1 \rangle$ for ξ_1 against $\langle u_2, g_2 \rangle$ for ξ_2, $u_1 < u_2$, $g_1 > g_2$ — can amplify misunderstandings and contribute to traps. At the endpoints of the scope $[\xi_1, \xi_2]$, the wealth-pie sizes $z(\xi_1)$ and $z(\xi_2)$ at poverty lines ξ_1 and ξ_2 can require the delivery of wealth amounts $W(\xi_1)$ and $W(\xi_2)$, albeit at different prices, represented as taxes $\tau(\xi_1)$ and $\tau(\xi_2)$, Buchanan (1967). Hence, prior to the start of the game, and in line with the cost-benefit equity, in the most adverse circumstances, the payoffs $s_1 = \langle u_1, g_1 \rangle$ and $s_2 = \langle u_2, g_2 \rangle$ should preserve equal prices τ for the delivery of equal amounts W of wealth. Such a market-driven interpretation of *commodities delivery to the end destinations* relies heavily on the size of the wealth-pie, which is equal to $\tau \cdot W$. It should be noted that this interpretation is only relevant to the case of flat (proportional) taxes.

To explicate the interpretation of reasoning in previous lines, it is worth examining the "well defined bargaining problem," depicted as the contract curve in Figure 4. Based on the discussion presented thus far, our goal is to set an interval $[\xi_1, \xi_2]$ solving two non-linear equations, $\tau(\xi_1) = \tau(\xi_2)$ and $W(\xi_1) = W(\xi_2)$, by attempting to find a cross-point (τ^*, W^*) where the curve crosses its own contour, as *YX*-axis coordinates, on the plane with (τ, W), which is equivalent to the roots ξ_1^* and ξ_2^*. Although the calculus of the point (τ^*, W^*) does not extend beyond high school mathematics, it does not confirm the possibility of normalization in general. This, however, does not invalidate our discussion, as we do not claim that the equity condition can be achieved in all circumstances. It should still be pointed out that, in a number of examples where the validity of the condition was detected, we found a breakdown endogenously encoded into the game, indicating normalization in the form of

$$\{\langle u_1^*, g_1^* \rangle, \langle u_2^*, g_2^* \rangle\} = \{\langle u(\xi_1^*), g(\xi_1^*) \rangle, \langle u(\xi_2^*), g(\xi_2^*) \rangle\}.$$

The Swing of the Contract Curve within $[\xi_1, \xi_2]$

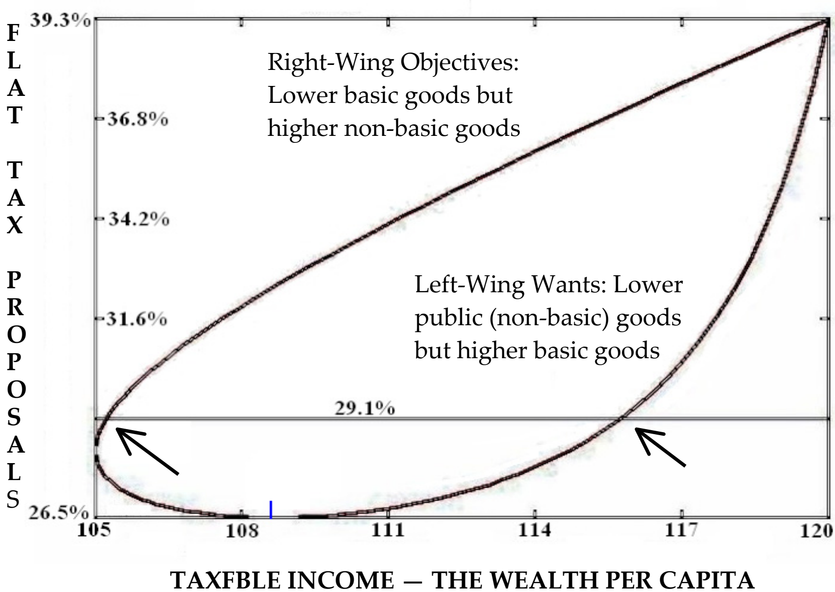

Figure 4. The graph depicts two different motions for a vote. For the higher tax $\tau = 29.1\%$, marked by the horizontal line, and the lowest tax $\tau = 26.52\%$, marked by the vertical dash. Indicated by $\rightarrow$, at cross-points of the contract curve with the horizontal line, we observe controversial expectations of voters. The shares of lower basic but higher public goods are shown on the left, while this payoff reverses towards the right side of the graph, as the shares of basic goods increase while those of public goods decrease. Thus, the higher tax $\tau = 29.1\%$ cannot lead to a political consent, in line with Observation 5.

In line with the above, as the aim is to bring the politicians, if possible, into just and equal positions prior to negotiations, equalizing taxes τ and wealth amounts W in the collapsed environments ξ_1 and ξ_2 might be a rational starting point. Under this premise, endogenously encoded into the game, we label the equity condition, as a *pre-equity of political break-*

down.: $\left[\tau(\xi_1)=\tau(\xi_2), W(\xi_1)=W(\xi_2)\right]$. If valid, this condition equalizes fiscally realistic and just demands for public spending prior to negotiations — in particular, the size of the wealth-pie $z(\xi_1) = z(\xi_2)$.

3.4. Voting and political power design, to be continued in Section 6

Only the voting results can reveal the true incentives of people that will give the democracy its final judgment. The voting process is the only avenue for the voters to assume the roles of current or upcoming politicians to whom the opportunity will be granted in line with population's aspirations to redesign the rules and norms of wealth redistribution. Voters' inequalities, life plans, background, social class and experience, native endowments, political capital, etc., determine the bulletin collected at the voting table. Consequently, incongruence in voters' views or interpretations of reality affects the individual choices and thus the voting results, thereby influencing political pre-election campaign. Voting results are not fully predictable due to the deviations in voters' views and opinions on how the wealth redistribution ought to be achieved. The problem stems from the fact that welfare policy proposals that benefit minority of citizens sometimes require higher taxes. On the other hand, majority of voters would be primarily guided by selfish attitudes toward lower taxes, which would implicitly affect the political bargaining positions. Such an attitude likely deserves a critical examination. Given these arguments, our question is — Why should the left- and right-wing politicians care about lower taxes?

It is timely to recall political outmaneuvering with an implicit risk q, $0 < q << 1$, upon negotiations suffering a premature collapse. Indeed, Figure 5 depicts the contract curve of efficient public policies/proposals ξ upon poverty lines in the bargaining game $\Gamma(q)$. Politically rational and economically effective proposals ξ, forming the curve, have been projected onto the two-dimensional space of the tax rate $\tau(\xi)$ and taxable income — the wealth amount $W(\xi)$. Although the payoffs $\langle u(\xi), g(\xi)\rangle$ are embedded in each point, they are not visible on the graph. These invisible/hidden payoffs in the upper part of the graph symbolize wealth-pie division (x, y) into lower basic $x(\xi)$, yet higher of public goods shares $y(\xi)$, as left-wing politicians aim for $u(\xi)$, whereas those in the right-wing political party aspire towards $g(\xi)$ accordingly. Similarly, the payoffs in the lower part symbolize a reverse situation — the higher basic, vs. lower public goods, as shown in Figure 4. Thus, once all views are represented, the political payoffs $\langle u(\xi), g(\xi)\rangle$ for pledged tax hikes $\tau(\xi)$ are more favorable for some coalitions of voters compared to others. As voters' preferences for the balance between basic and public goods vary, the approach to determining efficient poverty line resulting from eventual agreement between politicians is two-fold. Indeed, unless the tax hikes are excessively high, the *upper coalitions*' representatives will always try to outmaneuver the *lower coalitions*' representatives. The politicians are aware of this dynamic when taxes are high. As they feel trapped in negotiations, binary voters become more likely to defect to the other side, putting the negotiations at risk $q > 0$ of a premature collapse. In contrast, when taxes are sufficiently low, the range of eventual voters' electoral maneuvering will substantially reduce or even vanish. The lowest tax is likely the one that yields desirable outcomes for the majority of citizens.

In line of reasoning that concerns the majority of citizens, it is appropriate to address of the design of the political power indicators $(\alpha, 1-\alpha)$. Considering the bargaining outmaneuvering of left- and right-wing politicians according to the alternate-offers game $\Gamma(q)$, we state that the politicians on the opposite sides of the bargaining table might disagree with respect to the terms of outcomes. Consequently, they would delay the decision while consolidating a draft of a consensus document. This document might not necessarily yield the best outcome for the citizens, who represent the majority, and are of view that the policy that minimizes taxes is always the most desirable choice. Despite knowing that the majority will never endorse higher taxes, the minimum tax rate might not necessarily be a desirable outcome from the political perspective. Thus, politicians may choose to disregard the majority interests because political power of LWP or RWP, as rational actors/politicians, might be strong enough to negotiate selfish decisions that are beneficial only for them. In order to entice politicians not to act selfishly, as this would likely result in ultimate collapse in the negotiation process, their political power indicators $(\alpha, 1-\alpha)$ ought to represent a *natural power consensus* motivating them to choose a desirable outcome for themselves and for the majority of citizens — a platform that should ideally be designed in advance. This completed our preliminary investigation of the problem.

4. Analysis of Fiscally Safe Welfare Policies, *continued from Section 3.1*

Delivery of basic goods, which counteracts negative contingency, if it occurs, is the main political responsibility of the left-wing actors. Herewith, the left-wing political intervention is of the greatest political importance. It is universal in the sense that it pertains to all citizens, irrespective of individual situation before or after the contingency. Under this premise, basic goods that are available to citizens are of sufficiently high quality and poverty is not allowed, as stressed by Greve (2008). This course provides a relatively high level of welfare spending and taxes, creating misbalance in the books accounting for public finances, thereby introducing volatility conditions into the wealth-pie delivery. Hence, secured largely independently of market forces, the high level of basic goods might have a conflict-driven effect on the welfare policy, which should not be borne solely by citizens as, as already noted, the state has a duty to help the disadvantaged.

Assuming that the conflict-driven welfare policy guides our political actors in trying to reach an agreement, the left-wing politicians should aim to secure an efficient size of the wealth-pie. Thus, LWP prescribe the size of the pie and propose the division method, which the right-wing politicians accept or reject. If rejected, the RWP would suggest their preferred division, while only having the authority to recommend a size that the LWP might not be obligated to accept. We also assumed that, upon delivery to its end destinations, the wealth-pie remains fiscally safe, i.e., it does not change its size. Under the rules of the alternating-offers procedure (see later), the game will continue until a consensus is reached. This process presupposes that left-wing politicians are committed to the share of the pie, while not being committed to the size.

Let us now envisage a contrasting scenario, whereby the public spending increases. Hence, both actors know that, upon delivery, the size of the wealth-pie might change. This, in turn, leads to a misbalance between the relief payments, which can put the pie size in doubt or make it even more difficult to ascertain. As a result, the difficulty related to political pledges might force both sides to retreat. In such volatile conditions, the wealth-pie is no longer fiscally safe and might affect the expectations of both politicians. Consequently, a fiscally safe plan in spite volatile conditions for the delivery and division of the wealth-pie is needed. Otherwise, unless welfare policy fails to enforce fiscal safety, the rules and norms of the relief payments are not living up to their claims. In other words, having a criterion for determining whether a welfare policy is fiscally safe is necessary.

It is helpful to focus first on welfare policy without any warranty of fiscal safety. It could, for example, be determined by the poverty line ξ, identifying the recipients of wealth redistribution. When ξ is low, the variable σ, $0 < \sigma \leq \xi$, allocates the income of the needy or the benefit claimants. In this scenario, the benefit claimant $\sigma < \xi$ claims and receives a relief payment proportional to $\xi - \sigma$, i.e., $r \cdot (\xi - \sigma)$, as previously discussed. In this scenario, all other citizens — both the wealthy and those with marginal income, denoted as $\sigma > \xi$ and $\sigma = \xi$, respectively — receive no relief payment.

Next, we study a specific scheme highlighting the readiness of the society to fund welfare and public spending. For this analysis, we assume that the average cost B of the relief payments and the average taxable income W both depend on the poverty line parameter ξ, $B \equiv B(\xi)$, $W \equiv W(\xi)$ — this is realistic, as shown in Appendix A1. As previously scholarly defined, $W(\xi)$ can refer to the wealth amount. Based on our perception of income σ density $P(\sigma, \xi)$ distribution samples, the product $\tau \cdot W(\xi)$ estimates the average tax revenue. Let the average cost of public goods be $g(\xi)$, whereas the size $z(\xi)$ of the wealth-pie equals $\tau \cdot W(\xi)$, $z(\xi) = \tau \cdot W(\xi)$. We assume that welfare and public spending reached the intended recipients, whereby the total spending equals $\tau \cdot W(\xi) = B(\xi) + g(\xi)$. This suggests that the basic and non-basic goods have been delivered to their final destinations. In other words, the wealth collected through tax channels is spent.

Now, let us assume that politicians in the game preferred to commit to the shares fixing (x, y), and might agree to hold the balance $B(\xi) = x \cdot \tau \cdot W(\xi)$ of the books accounting for financing the relief payments B. That is, the left-wing politicians must be ready to finance the relief, i.e., to deliver $B(\xi)$ by dividing the wealth-pie $\tau \cdot W(\xi)$. In this scenario, the politicians pledge to retain the balance $B(\xi) = x \cdot \tau \cdot W(\xi)$ of the relief payments between credits $B(\xi)$ and debts $x \cdot \tau \cdot W(\xi)$ as a portion x of the wealth-pie $\tau \cdot W(\xi)$. The balance also specifies the welfare policy ξ — an implementation of the poverty line ξ, welfare reform, pact, program, etc. While the aforementioned balance is initially valid, it might not be in the future, putting the adjustment in ξ on the agenda either once or repeatedly. Thus, the policy ξ might represent a problem of fiscal imbalance. Almost all citizens, even if for different reasons, will prefer the opposite in the long run — a fiscally safe policy ξ. For this reason, we now shift the focus on examining a constraint that corresponds to fiscal safety of welfare policy ξ, identifying — what we called above as idempotent — the safe delivery of the wealth-pie to its end destinations.

Idempotent rules and norms of wealth redistribution

The delivery of basic and public (non-basic) goods does not necessarily safeguard the funding of the expenses. As the expenses neither match nor prevent taxation hikes, the size of the wealth-pie could vary too rapidly. This leads, as previously discussed, to numerous adjustments of welfare policy rules and norms. To mitigate this issue, we have to examine at the sequence $.,\xi',\xi'',.$ of multiple adjustments of the poverty line ξ. This highlights the fact that, on delivery, no adjustments of the wealth-pie are desirable. Consequently, it is better to keep the size of the pie unchanged, i.e., fiscally safe. In other words, when replacing the old policy ξ' with ξ'', the two must coincide. Similar schemes, known as *idempotent*, stem from bounded rationality mechanisms (Rubinstein, 1998; Malishevski, 1998). This premise suggests that, even if welfare policy rules and norms are subject to multiple adjustments, these adjustments should not change the *machinery* of relief payments. In particular, when implemented twice, the rules must produce the same outcome. To guarantee the fiscal safety of the poverty line, such an understanding requires that the poverty lines must coincide amid a sequence of pairs (ξ',ξ'') at some matching policy $(\xi'=\xi'')$.

The need to balance the books accounting for the delivery of relief payments $B(\xi)=x\cdot\tau\cdot W(\xi)$, in spite the wealth-pie volatility, can also be seen as immunity for financing the welfare policy. In particular, the immunity restricts, or at least realistically limits the h-effect of wealth redistribution. Given the *immune,* i.e., *fiscally idempotent*, composition

$\left[B(\xi), W(\xi)\right]$, the idempotent scheme is equivalent to implementing the policy ξ only once. For this reason, we assume that the rules and norms of the relief payments have been socially planned and redesigned accordingly.

In this idempotent mode that outlines the fiscal safety of public spending, the rules and norms must reflect idempotent policy ξ that brings the spending policy into focus. We conclude that the expenses $x \cdot \tau \cdot W(\xi)$ designated for welfare spending must be in balance not only for funding relief payments $B(\xi)$, when the particular policy ξ takes effect, but the policy ξ must also enforce the fiscal safety in the full spectrum of current and future events.

Clearly, the balance $B(\xi) = x \cdot \tau \cdot W(\xi)$ is a static relationship leading to functional dependency $\tau \equiv \dfrac{B(\xi)}{x \cdot W(\xi)}$ that links the arguments ξ and x. Hereby, the tax rate τ becomes a function of ξ and x, expressed as $\tau \equiv \tau(\xi, x)$. According to rules and norms in force of relief payments, the post-tax residue $\pi(\xi, \tau) = \left(1-\tau\right) \cdot \left(\xi - \phi\right) + \phi$ of the marginal citizens' $\sigma = \xi$ comprises fiscal limitations of wealth redistribution, while ϕ determines the personal allowance parameter, as shown above. The dependency $\tau \equiv \tau(\xi, x)$ transforms $\pi(\xi, \tau)$ into a fiscally realistic social position $\pi(\xi, \tau(\xi, x))$. Irrespective of the current expenditure on basic goods, the real cost of living does not necessarily match $\pi(\xi, \tau(\xi, x))$. Hence, ensuring realistic and fiscally idempotent rules and norms, and/or, in particular, attempting to avoid the h-effect of this mismatch or adopt rules to keep the effect tolerable at the least, an equation for a fiscally idempotent policy ξ should be solved.

Observation 1. *Constraint on left-wing political aims* $u = \pi(\xi, \tau(\xi, x))$ *is necessary for upholding idempotent fiscal rules and norms of imposed budget constraint* $B(\xi) = x \cdot \tau \cdot W(\xi)$.

According to this observation, whatever tax increase is implemented, the poverty line residue u of the marginal citizens' $\sigma = \xi$ is unfeasibly high and cannot be attained when the condition has been violated.

Corollary. *When* $u = \pi(\xi, \tau(\xi, x))$ *solves for* ξ*, the subsequent adjustments* ξ', ξ'',... *are unnecessary. An option to change their welfare positions is irrational for citizens with incomes* $\sigma < \xi$ *or* $\sigma > \xi$*; thus, the root* ξ *restricts (realistically limits) the h-effect.* All pertinent proofs are given in Appendix A3.

The fiscally idempotent policies ξ induce the basis for solutions in our game as fiscally idempotent compositions $[B(\xi), W(\xi)]$. A reasonable question thus emerges: *Which taxable income* $W(\xi)$ *characterizes fiscally idempotent welfare policies* ξ *for the delivery of relief payments* $B(\xi)$? The answer is provided in the form of the following three constraints: [1]

Delivery constraint by which the wealth-pie is spent — the basic and public goods have been delivered. This form of constraint makes sense only for proportional or flat taxes. Flat taxes will later substantially simplify the method of function minimization with constraints.

$$\tau \cdot W(\xi) = \\ = B(\xi) + g \qquad (1)$$

Budget constraint imposed on relief payments finance in accordance with the share x of the wealth-pie — the tax-revenue. The left-wing politicians pledge to credit/debit the account $B(\xi)$ that must be equal to the average of relief shifted by the policy ξ.

$$B(\xi) = \\ = x \cdot \tau \cdot W(\xi) \qquad (2)$$

[1] Below, we continue to refer to the average taxable income as "wealth."

Stability constraint that determines fiscally idempotent property of (2). In contrast to $(\sigma, \tau) \in \Re^2$, we distinguish poverty line residues $u = \pi(\xi, \tau)$ as one-dimensional curves $\pi(\xi, \tau) \in \Re \subset \Re^2$.

$$u = (1 - \tau) \cdot \cdot (\xi - \phi) + \phi \tag{3}$$

Taking the expression $\tau(\xi, x) \equiv \dfrac{B(\xi)}{x \cdot W(\xi)}$ out of the constraint (2) and replacing $\dfrac{B(\xi)}{x \cdot W(\xi)}$ into $u = \pi(\xi, \tau(\xi, x))$, the constraint given in (3) can be resolved with a fiscally idempotent policy for ξ, thus yielding:

$$L(\xi, x, u) = (\xi - \phi) \cdot B(\xi) - x \cdot (\xi - u) \cdot W(\xi) = 0 . \tag{4}$$

Referred to as the volatility constraint, the constraint (4) determines the fiscal safety module. It holds down the h-effect amalgamating the constraints (2) and (3) by balancing the books accounting for relief payments.

Summary. The outcome $\phi, \xi \Rightarrow z, x, \alpha, \tau, \langle u, g \rangle$ constitutes the citizens' bargaining shield for wealth redistribution that relates to a bundle of arguments or constants: ϕ, ξ are controls, and z, x, α, τ are status variables, [2] while $\langle u, g \rangle$ are the competing political proposals:

- ϕ – the personal allowance establishing the tax bracket $[\phi, \infty)$; it is an ex-ante control (tuning) variable, $0 < \phi = \text{const} < \xi$;
- ξ – the income frame, the poverty line; a policy determining who is living in poverty, as well as the choice or the control parameter;

[2] Status and control variables are the prerogatives of control theory.

z – the size $z = \tau \cdot W(\xi)$ of the wealth-pie; the amount of wealth-pie that is equal to public spending per capita when taxes are proportional;

x – the share of the wealth-pie of size z; a portion x of z to be deposited
in favor of the left-wing politicians for funding the relief payments, $0 \leq x \leq 1$;

α – the political power of the left-wing politicians, $0 < \alpha < 1$;

τ – the marginal tax rate, the rate $\tau(\xi, x)$ of the wealth amount $W(\xi)$ determined by (1);

u – the after-tax residue of the income frame equal to the poverty line ξ, the wants function $u(\xi, x)$ of the left-wing politicians, as determined by (2) and (3);

g – the objective function $g(\xi, x)$ of the right-wing politicians, determined by (1) and (2); the account for the refund of public goods expenses per capita.

5. ANALYSIS OF THE WELFARE STATE BARGAINING RULES AND NORMS, *continued from Section 3.2*

Suppose that politicians, in pursuit of their commitments to a fair division of the wealth-pie, agreed to play the alternating-offers bargaining game $\Gamma(q)$ (Osborn and Rubinstein). In doing so, rational politicians are motivated to align the procedure to participate in any eventual agreement. The risk $q > 0$ of a premature collapse during negotiations, especially early in the game, might be the driving force behind their commitment to reach the consensus. Once a consensus on division is reached, they must agree on who will determine the size of the pie. Politicians negotiate on such matters when there are equal and symmetric preconditions in place that guarantee their equal rights. Thus, both will play an equal role in the decision regarding the pie size. Considering the right-wing vital political objective of wealth redistribution, it will be realistic to reduce the scope of RWP's duties concerning welfare matters, while allowing them to retain their advisory rights. Our subsequent discussions are based on this premise.

5.1. *Left- and right-wing politicians' bargaining procedure*

Previously, we emphasized that, in a representative democracy, the division of the wealth-pie will always be subject to controversy. Recall that we consider two politicians — one acting in the role of LWP, who is aiming to provide basic goods to all citizens, and the other, representing RWP, advocating for availability of non-basic goods. A precondition for the bilateral agreement is that the expectations of these two politicians depend solely on efficient policies of the LWP within the framework aimed at setting the poverty line ξ. However, politicians are more concerned with shares (x, y) than they are with the size of the wealth-pie. As a consequence of this independence, efficient poverty line ξ° provides shares related to efficient divisions (x°, y°). Accepting this precondition, the RWP will only propose an efficient line ξ°, as failure to do so would result in all other shares being rejected with certainty by LWP. Nonetheless, it is realistic that the RWP would — by negligence, mistake or some other reason — recommend an inefficient poverty line ξ', which the LWP would mistakenly accept. It is also possible that, in a reverse scenario, the LWP would choose to disregard an efficient recommendation ξ°. This would be an irrational choice as, in any agreement, regardless of the underlying motives, both politicians are committed by proposals to shares (x, y). Indeed, within the scope of negotiations $[\xi_1, \xi_2]$, the recommendation ξ° concurs with RWP's efficient share proposal y°. Consequently, accepting $1-y^{\circ}$, while shifting LWP's ξ° mistakenly to $\xi' \neq \xi^{\circ}$, at which both politicians must be committed to (x°, y°), the shift ξ' becomes inefficient and thus superfluous. Hence, making a proposal, the RWP's recommendation on poverty lines makes a rational argument that the LWP must accept or reject in a standard way. Such an account, in our view, explains that the outcome of the bargaining game might be a desirable poverty line $\xi \in [\xi_1, \xi_2]$. Hereby, the interval is referred to as the scope $[\xi_1, \xi_2]$ of negotiations or bids proposals that are now, by default, linking efficient lines ξ° with shares (x°, y°). The bargaining occurs exclusively in the interval $[\xi_1, \xi_2]$ as a scope for efficient lines ξ° of most trusted policy platforms for negotiations, where both players would either accept or reject the proposals. Political competition, depending on $[\xi_1, \xi_2]$, arranges a contract curve $\boldsymbol{S}_b$ (shown in Figure 4 and Figure 5) as a way to assemble the bargain portfolio. Given that the portfolio "has changed its color from shares to lines," the politicians can now conceive themselves as making poverty line proposals. If a proposal is rejected, the roles of politicians change and a new proposal is submitted. The game continues in the traditional way by alternating offers.

The Contract Curve Projection within $[\xi_1, \xi_2]$

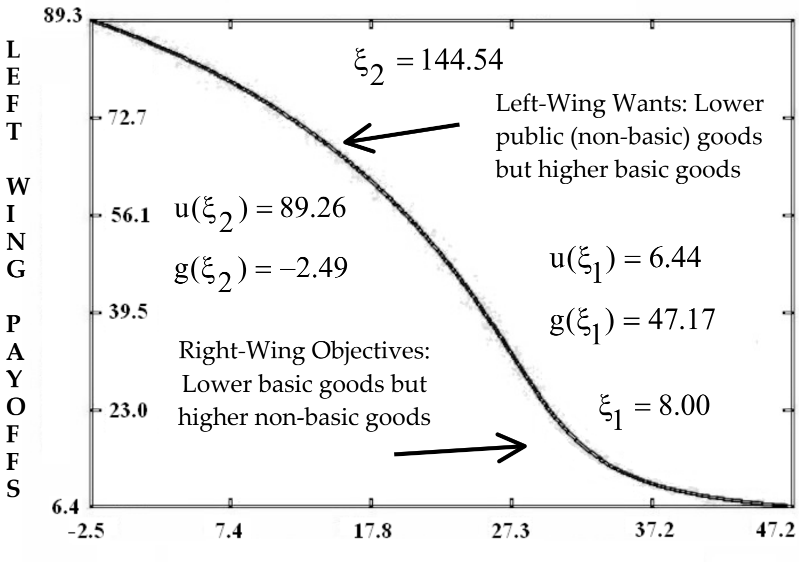

Figure 5. The aspirations of left-wing politicians expressed when opposing the right-wing political objectives are depicted on the vertical and horizontal axes, respectively. The graph shows the contract curve sloping from ξ_2 toward ξ_1, projected on the surface of basic goods *vs.* vital goods — the projection of efficient poverty lines $\xi \in [\xi_1, \xi_2]$ resolving the contract constraint (5).

5.2. Alternating-offers bargaining game analysis

We now proceed to a more accurate analysis of the game rules. Although the rules can be perceived as fiscally idempotent, the game itself contains a new challenge. The elevated poverty line ξ does not necessarily increase the marginal citizens' $\sigma = \xi$ after-tax residue $u(\xi, x)$. The low-income citizens — the benefit recipients — can claim relief payments, whereby an increased number of claims might have a reverse effect on $u(\xi, x)$, which would consequently decline. Indeed, in contrast to increasing poverty line ξ and despite the required unavoidable increase in taxes — as the hazard (h-effect) is still present — in this scenario, the residue $u(\xi, x)$ will decrease. With the proviso that the left-wing politicians commit to the share x, the right-wing politicians are left with $y = 1 - x$. Thus, the fiscally idempotent poverty line tax residues $u(\xi, x)$ correspond to a narrower set than $0 \le x \le 1$, $0 \le y \le 1$ — the set of shares $\langle x, y \rangle$ of what we refer to as a *contract curve* $\boldsymbol{S}_b$ of payoffs $\langle (u(\xi, x)), g(\xi, y)) \rangle$ with poverty line ξ as a parameter. [3]

Assuming that the maximum of a single $\cap$-peaked residue function $u(\xi, x)$ can be reached, the peak position $\xi^\circ = \arg\max_\xi u(\xi, x^\circ)$ indicates an efficient welfare policy. Although the bargain portfolio of left-wing politicians contains an efficient policy ξ° as a function of x°, the portfolio also includes the share $x = x^\circ$. The maximum value given by $u = u^\circ$, in the inverse situation, which corresponds to u°, consolidates an efficient policy $\xi^\circ \in [\xi_1, \xi_2]$. A unique share x°, which solves $u(\xi^\circ, x) = u^\circ$ and corresponds to $g(\xi^\circ, y^\circ) = g^\circ$, represents the nonconforming expectations of politicians. We can thus refer to the shares

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[3] We already highlighted the worsening quality of welfare services for all citizens when the LI level is "climbing" high.

(x°, y°) as an efficient division linked to the policy ξ°. This scenario is depicted in Figure 4 on wealth amount W and taxes τ — efficient peaks ξ°, which correspond to efficient shares (x°, y°), and in Figure 5 in various projections on payoffs $\langle u^\circ, g^\circ \rangle$ geometry. This geometry highlights the maximum values u° can take — namely, efficient policies of left-wing politicians at peaks ξ° that refer to the well-known result obtained by Canto et al (1981), also known as the Laffer curve:

> *The marginal tax-revenue raised decreases with increase in tax rates, finally reaching some point where the marginal tax-revenue raised is zero. Beyond this point, any tax rate increases will reduce revenue collection.*

Our result pertaining to the single-peaked aspirations of the left-wing politicians is similar. First, "*poverty line residue* u *being proposed in the normal range of poverty line parameter* ξ." Next,

> *…by passing through the top point of* u *as a function, the proposals* u *will be assessed and reviewed in the range of prohibited values of* ξ.

We previously introduced an idempotent composition $[B(\xi), W(\xi)]$ — the average $B(\xi)$ of the relief payments, and the average $W(\xi)$ of the taxable income, denoted as the wealth. The expectations of the two politicians, reflecting their preferred rules and norms pertaining to relief payments, can now be set using the composition $[B(\xi), W(\xi)]$. At the end of the subsection, the composition will lead to an appropriately settled bargaining problem that will associate the threat originating from the implicit partaker — in the form of the electoral maneuvering of voters — with an implicit risk of the negotiations collapsing prematurely. This requires augmenting the standard rules of the game we have already presented with two further rigorous suppositions. Let us first specify the payoffs.

Political payoffs of the 1st/2nd actor and the third partaker′s implicit risk factor are defined as follows:

Politician No. 1, Politician No. 1, u – the left-wing political aspirations, the marginal citizens' $\sigma = \xi$ after-tax residue, basic necessities of the needy, cost of living;

Politician No. 2, g – he right-wing political objective, expenses that benefit all citizens — expenses upon vital goods alone, without relief payments;

Third Partaker, q, τ – voters' electoral maneuvering facing higher taxes τ expressing an implicit risk $0 < q << 1$ of the negotiations collapsing prematurely.

As promised, we now assume that the rules and norms of the wealth redistribution that are efficient with respect to the wealth-pie division include the volatility constraint (4), which certifies the idempotent composition $[B(\xi), W(\xi)]$ for the policy ξ. In the game, the composition $[B(\xi), W(\xi)]$ could not be implemented without the volatility constraint $L(\xi, x, u) = 0$ (Observation 1). This assumption is contingent on the conclusions of the previously undertaken analysis.

When varying ξ under their own rules and norms, let us assume that LWP propose a fiscally idempotent policy $\xi = \xi^{\circ}$, which — for each share $x = x^{\circ}$ they commit to — links x° to ξ°, irrespective of who originates the proposals x° or y°. This ensures the efficient proposal of poverty line residue $u(\xi^{\circ}, x^{\circ}) = \max_{\xi} u(\xi, x^{\circ})$. Clearly, inefficient recommendation ξ', proposed by the RWP if $\xi' \neq \xi^{\circ}$ for share y°, will be rejected by the LWP. As a result, an efficient policy $\xi = \xi^{\circ}$ must occur on contract curve amid efficient shares x° at $\langle u^{\circ} = u(\xi^{\circ}, x^{\circ}),\ g^{\circ} = g(\xi^{\circ}, x^{\circ}) \rangle$ as an ongoing precondition for the agreement — as previously discussed. Indeed, LWP have no reason to reject efficient recommendation ξ°, as doing so, when $\xi' \neq \xi^{\circ}$, they cannot ultimately maintain the efficient commitment to x°. Below, we assume the efficiency by default when it is convenient.

Observation 2. *Idempotent policies* ξ *at the contract curve* $\boldsymbol{S}_b = \langle u(\xi,x), g(\xi,x)\rangle$, *which certifies the composition* $[B(\xi), W(\xi)]$, *must satisfy the constraint*

$$\begin{aligned} D(\xi,x,u) &= \frac{\partial}{\partial\xi} L(\xi,x,u) = \\ &= \frac{\partial}{\partial\xi}\left[(\xi-\phi)\cdot B(\xi) - x\cdot(\xi-u)\cdot W(\xi)\right] = 0 \end{aligned}. \qquad (5)$$

Particularly, when we collated sub-expressions and introduced some simplifications upon

$Q(\xi,\tau,g)=0$	$\rightarrow$ Delivery(1)	enforcing constraints on rules and norms of the wealth redistribution.
$L(\xi,x,u)=0$	$\rightarrow$ Volatility(4)	
$D(\xi,x,u)=0$	$\rightarrow$ Contract curve(5)	

These constraints, with the proviso of flat taxes, together with the previously detailed preliminary settings $\tau'_\xi > 0$, $\tau''_\xi > 0$, $u''_\xi < 0$, $u'_\xi > 0$, $u'_\xi < 0$, $u''_x > 0$, $u'_x > 0$, $g'_\xi > 0$, $g''_\xi > 0$, $g''_x \neq 0$, lead to an analytical solution: $u(\xi) = \xi - \dfrac{(\xi-\phi)}{\nu(\xi)}$, where

$$\nu(\xi) = 1 + (\xi-\phi)\cdot\left(\frac{\dot{B}(\xi)}{B(\xi)} - \frac{\dot{W}(\xi)}{W(\xi)}\right);^{4} \quad \tau(\xi) = \frac{1}{\nu(\xi)},$$

$$g(\xi) = \frac{W(\xi)}{\nu(\xi)} - B(\xi); \text{ the size of wealth-pie}$$

$$z(\xi) = B(\xi) + g(\xi) = \frac{W(\xi)}{\nu(\xi)}.$$

---

[4] $\pm$ rates $\dot{W}(\xi) \le 0$, $\dot{W}(\xi) \ge 0$ of the changes in the wealth amounts $W(\xi)$ are essential for the analysis, whereas the function $B(\xi)$ is valid only when $\dot{B}(\xi) > 0$, and $0 < \phi < u < \xi$.

Now it is evident that payoffs $\langle u, g\rangle$ at the contract curve $\boldsymbol{S}_b$ depend exclusively on policies ξ, $\langle u(\xi), g(\xi)\rangle \in \boldsymbol{S}_b$. We conclude that politicians are only concerned with making proposals that pertain to efficient policies ξ, since effective shares (x, y) have been linked to ξ. Contract curve $\boldsymbol{S}_b = u(g)$ in Figure 4 illustrates the payoffs. The functions $g(\xi)$ and $u(\xi)$ in the form presented above are, in fact, not a subject to any constraints. They are mathematically derived in Appendix A4.

Before proceeding with further line of analysis, let us recall the threat phenomenon created by voters that increases the implicit risk of the negotiations collapsing prematurely. As noted previously, if politicians reject their counterpart's proposal — knowing that it is risky to continue the bargain — they will likely consolidate a draft. This introduces the risk that the voters will reject the draft when politicians, without fulfilling the voters' terms, try to continue bargaining over costly and controversial proposals, thereby putting the negotiations at a risk of collapse, as previously discussed.

Suppose that politicians bargain over all fiscally idempotent policies $\xi \in [\xi_1, \xi_2]$ within the scope of negotiations $[\xi_1, \xi_2]$. We follow the alternating-offers game $\Gamma(q)$ with an exogenous risk $0 < q << 1$ of a premature collapse, as described previously (Osborn and Rubinstein). We posit that, each time the proposal ξ is rejected by one of the politicians, the momentary phase of the game results in a draft, which can be opposed by the voters, as just recalled. In these circumstances, the politicians might be uncertain on how to proceed, if the voters' terms are not met. As a result, they might choose to leave the bargaining table prematurely. Extracted from the endpoints $\xi_1 < \xi_2$ of contract curve $\boldsymbol{S}_b$, the outcome

$$\{\langle u_1, g_1\rangle, \langle u_2, g_2\rangle\} =$$
$$= \{\langle u(\xi_1), g(\xi_1)\rangle, \langle u(\xi_2), g(\xi_2)\rangle\}$$

naturalizes this risk q in the worst-case scenario.

What is known as the *well-defined bargaining problem,* first introduced by Roth, or the individual rationality associated with the Nash bargaining scheme $\langle \boldsymbol{S}, \mathrm{d} \rangle$, seems to be instructive for further analysis. Indeed, inequalities $g_1 > g_2$ and $u_1 < u_2$ hold for the pair $\mathrm{d} = \langle d_1 = u_1, d_2 = g_2 \rangle$. Synthesizing the unfavorable political outcome $\{\langle u_1, g_1 \rangle, \langle u_2, g_2 \rangle\}$ into a policy δ on poverty introduced below will naturalize the Nash disagreement point d into the problem $\langle \boldsymbol{S}_b, \mathrm{d} \rangle$, $\boldsymbol{S}_b \subset \Re^1$. Thus, compared to the traditional approach of compact convex set $\boldsymbol{S} \subset \Re^2$, inequalities $\mathrm{s} > \mathrm{d}$ are also true for any pair $\mathrm{s} \in \boldsymbol{S}_b$. The pair $\langle \boldsymbol{S}_b, d \rangle$ for the contract curve $\boldsymbol{S}_b$ becomes a well-defined bargaining problem. Given that it is not immediately apparent whether the policy δ is a fiscally idempotent outcome of the game, the following observation removes any doubt.

Observation 3. *To test whether the point* $\mathrm{d} = \langle d_1, d_2 \rangle = \langle u_1, g_2 \rangle$ *becomes a fiscally idempotent outcome of the left- and right-wing political bargaining, it is necessary and sufficient that there exists a policy* δ *on poverty, which solves the equation:*

$$(\delta - \phi) \cdot (B(\delta) + d_2) - (\delta - d_1) \cdot W(\delta) = 0;$$

The condition $\delta \notin [\, \xi_1, \xi_2 \,]$ must hold true.

It should be noted that, in the worst-case scenario δ, the wealth redistributed equals $W(\delta)$ — the average of expenses for funding the relief payments equal $B(\delta)$ — whereby the proposal δ depends on the endpoints of the bargaining interval $[\, \xi_1, \xi_2 \,]$. This dependence, provided that the Equation (6) can be solved for δ, serves as the basis for validation of the *pre-equity condition of breakdown,* as discussed in Section 7.

Observation 4. *In the alternating-offers game* $\Gamma(q)$ *with the risk* $0 < q << 1$ *of negotiations collapsing prematurely into the disagreement point* $\langle d_1, d_2 \rangle$, *the functions* $(u(\xi) - d_1)^{\alpha}$ *and* $(g(\xi) - d_2)^{1-\alpha}$ *imply bargaining payoffs of left- and right-wing politicians, respectively.*

Thus, (without proof) *for variables* λ_1, λ_2 *solving the equations* $(1-q)\cdot\left(u(\lambda_1)-d_1\right)^{\alpha} = \left(u(\lambda_2)-d_1\right)^{\alpha}$ *and* $(1-q)\cdot\left(g(\lambda_2)-d_2\right)^{1-\alpha} = \left(g(\lambda_1)-d_2\right)^{1-\alpha}$, *the solution* λ *of the well-defined bargaining problem* $\left\langle \boldsymbol{S}_b, d\right\rangle$ *is close to the pair* (λ_1, λ_2), $\lambda_1 \le \lambda \le \lambda_2$.

As explained by Osborn and Rubinstein, the outcome in our experiment of bargaining game $\Gamma(q)$ encapsulates the power indicators $(\alpha, 1-\alpha)$ of the left- and right-wing politicians. In the next section, we consider the design of political power indicators $(\alpha, 1-\alpha)$ using the solution λ that minimizes the tax burden with respect to an appropriately settled bargaining problem $\left\langle \boldsymbol{S}_b, d\right\rangle$.

6. ANALYSIS OF VOTING AND POLITICAL POWER DESIGN, *continued from Section 3.4*

Here, we will elaborate on power indicators $(\alpha, 1-\alpha)$ specifically, referring to the original bargaining scenario of \$1 division, based on the previously discussed axiomatic approach — α signifies LWP's political power, and $1-\alpha$ the political power of RWP, $0<\alpha<1$. Considering

$$\left(x^{\circ}, y^{\circ}\right) = \arg\max_{0 \le x+y \le 1} f(x, y, \alpha) =$$
$$= \left(u(x)-d_1\right)^{\alpha} \cdot \left(g(y)-d_2\right)^{1-\alpha}$$

the following questions emerge: What type of \$1 division will assist a moderator designing the power indicator α of the 1st actor? What will ensure that, during the negotiations, the 1st actor will obtain a desired or any other share x° of \$1? To answer these questions, let us assume that the 2nd actor might only accept or reject the 1st actor's proposals. We can thus start redesigning the power indicators $(\alpha, 1-\alpha)$ by replacing $y=1-x$, and taking the derivative of the resulting $f(x, 1-x, \alpha)$ with respect to the variable x by evaluating $f'_x(x, 1-x, \alpha)$. For a moment suppose, finally, that x° share of \$1 is a desirable solution.

Given $x = x^\circ$, the equation $f'_x(x^\circ, 1 - x^\circ, \alpha) = 0$ can be solved for $\alpha = \alpha^\circ$. In general, one might find comfort in the following egalitarian judgment:

> *To count on* x° *share of \$1 is a realistic attitude toward the 1st actor's position of negotiations. Indeed, even if the 2nd actor might have a stronger negotiating power than the 1st actor,* $\alpha^\circ < 1 - \alpha^\circ$*, the 1st actor, sooner rather than later, might predict the 2nd actor's preferences and thus force a concession.*

When, for example, the voters' representatives attempt to redesign political power indicators to $(\alpha, 1 - \alpha)$, we assume that politicians will try to share the wealth-pie in the manner in which \$1 was divided above. In doing so, we suppose that both politicians are ready to proceed with tax concessions. Reflecting just illustrated axiomatic bargaining toward allegedly desirable \$1 share x°, we proceed with our discussion.

In accordance with our analytical solution without constraints, the contract curve $\mathcal{S}_b = u(g)$ corresponds to a curve $\langle u(\xi), g(\xi) \rangle$. Moving along the curve while taking into account the scope of negotiations $[\xi_1, \xi_2]$, the expectations $\tau(\xi)$ of voters' majority lead to detection of $\tau_{min} \leftarrow \tau(\xi)$:

$$\min_{\xi \in [\xi_1, \xi_2]} \tau(\xi) \,\Big|\, \tau(\xi) = \frac{1}{v(\xi)}.$$

With the proviso that $\tau(\xi)$ is concave and sufficiently smooth, the detection point of τ_{min} is the root λ of the equation $\dot{\tau}(\xi) = 0$. Consequently, akin to the egalitarian judgment given above, the root λ might help in redesigning of the rules and norms of the wealth redistribution. This can be done by adjusting the α in a way that the political power α of the left-wing politicians will be sufficient to persuade the right-wing politicians to agree upon the poverty line residue $u(\lambda)$.

Indeed, in the left- and right- political bargaining, the old *standard* (discussed above) of how to share the \$1 can now be a new *Standard* pertaining to how to plan the wealth redistribution rules and norms. Under this premise, we can set $f(\xi,\alpha)=(u(\xi)-d_1)^{\alpha}\cdot(g(\xi)-d_2)^{1-\alpha}$, where α facilitates the political power of the LWP. Instead of $x=x^{\circ}$, planning the rules, we suppose that $\xi=\lambda$ is an allegedly desirable solution. Hence, we first take the derivative of $f(\xi,\alpha)$, with respect to ξ, evaluating $f'_{\xi}(\xi,\alpha)$, which allows us to solve the equation $f'_{\xi}(\xi\big|_{\xi=\lambda},\alpha)=0$ for α. As a result, the root α° will correspond to the redesigned political power of the left-wing politicians. This is the result as it appears.

Summary. To control the left- and right-wing political agreement on shares (x,y) of the wealth-pie, akin to the new *Standard* above, the majority of citizens can accept or reject a premature agreement archived at the a particular point during the negotiations, thereby voting for or against the division. As previously noted, the majority will favor the policy λ that minimizes the tax burden. This restriction allows us to rebalance the welfare institutions or finance resources by appropriate design of power indicators $(\alpha,1-\alpha)$ of the left- and right-wing politicians, ensuring that the most favorable shares (x°,y°) of the wealth-pie would incorporate the Nash axiomatic — the minimum tax — solution λ into the bargain portfolio as the most optimal outcome. This is our *case study* of tax policy in which only a minority would object to a proposal that corresponds to the tax rate minimum at the contract curve. In doing so, the implicit pressure of citizens will be lower. To be implemented in favor of majority, the minimum appears to be a desirable consensus.

Observation 5. *Given that politicians can reach a preliminary agreement on tax rate* $\tau=\tau(\xi)$, *condition* $\lambda=\arg\min_{\xi\in[\xi_1,\xi_2]}\tau(\xi)$ *is necessary to put forward a poverty proposal* λ *before voters by appropriately designing the power indicators* $(\alpha,1-\alpha)$ *in advance. At the contract curve* $\boldsymbol{S}_b$, *the proposal* λ *outlines a unique outcome*

$$\phi,\xi\Rightarrow z,x,\alpha,\tau(\lambda),\langle u(\lambda),g(\lambda)\rangle\in\boldsymbol{S}_b\ .$$

7. Discussion

The true essence of the economic reality behind the left- and right-wing political bargaining could be revealed by determining whether it is true that funding relief payments of the needy and maintaining the budget in balance will be difficult to sustain when the tax burden for all citizens is decreasing. On the surface, it seems that, at some point, fairness and equity might no longer be the main requirement because of the "risks becoming a Downton Abbey economy" (2014). Economists, including Kittel and Obinger (2003), have analyzed the poverty gap issue. In the face of these controversies, it is not possible to estimate the extent of potential fallout that might result from such outcomes of tax burden cut.

The citizens are those who decide what needs to be done and what should ultimately bring order to socially plan, or how to redesign the wealth redistribution rules and norms. Taking advantage of this opportunity, it is instructive to perform an exercise related to the most appropriate choice of welfare policy, as shown in the *"minimizing wealth-tax"* column of Table 1. [5] We illustrated that, despite minimizing the tax burden for all citizens, the minimum is, in fact, fiscally safe, while also ensuring just and fair redistribution of wealth for all citizens.

Due to the assumptions made during the analysis, the following discussion perhaps offers some guidance on doing the exercise. Before commenting on those, it is worth noting that the experiment presented here should be understood as purely normative — namely, "what ought to be" in economic or political matters, as opposed to "what is." Despite the fact that, in the preceding analysis, no actual situation was presented, our theoretical results rest on the assumptions delineated below.

First, our work is based on the premise that politicians would only make promises that can be fulfilled — fiscally safe proposals. Fiscal safety, when taken separately, even when attempted in accordance with the rules and norms in force, could lead to unjust and unfair solutions. Taken at will, fiscal safety might be a profoundly mistaken idea of justice. In Table 1, we presented the percentage of citizens below the poverty line, thus establishing the poverty rate. [6] Driven at will, the official poverty rate,

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[5] Table 1 was created by numerical simulation carried out upon imaginary distribution of citizens' incomes.

[6] Poverty rate determines the percent of anyone who lives with income below the official poverty line. The poverty line separates the rich (those with an income above the poverty line), from the less fortunate (having income below the line).

in accordance with the “disagreement” column of Table 1, could cause the poverty rate to decline below 0.41%, which wrongly appears to be the most just and the fairest.

Second, we postulated that the wealth redistribution compensates for the inequalities in the income of citizens that were below the poverty line. Usually, similar parameters are in the national government competence. While taking into account increases in the cost of living, the official number of individuals living in poverty should be adjusted annually according to government guidelines. Although our key assumption was that the right-wing politicians inherited no more than an advisory authority, the rules and norms that govern the poverty line determination have been solely under the mandate of the left-wing politicians. This decision was made because, in the analysis, we deliberately emphasized the distinctions between stereotypical motivations of left- and right-wing politicians. In our view, welfare protection that is most likely to be just as fair should be addressed as an independent institute, or better yet, as an assembly of independent institutes or legal charity foundations. We believe that, in our experiment of organizational independence, welfare protection could be expected to yield efficient welfare policies. Thus, in determining an efficient policy on poverty, we concluded that left-wing politicians should be in a privileged position that allows them to prescribe the poverty line independently. Only when these guidelines of independence are applied, the value judgment based upon the data presented in Table 1 makes sense. Still, it should be noted that the characterization of whether setting up such a privilege was a positive or negative restriction requires further investigation.

Next, we focused on the political power indicators $(\alpha, 1-\alpha)$, which highlight the amount of resources, skills and competence of left- and right-wing politicians. The fundamental factor in our analysis was the welfare protection of the society as a whole to justify and maintain welfare duties under the principle of how the state ought to act when attempting to fulfill its welfare mission. When the decision made by the politicians is not in line with the objectives of special interest groups, as previously pointed out, welfare protection could be a recurrent theme in political debates and election campaigns, and a source of significant political competition. A controversy with respect to political interests might lead to violent upsets, providing the opportunity to develop policy in favor of these groups. According to the foregoing account, which requires considerable administrative efforts and fiscally unrealistic expenses — and previous observations pertaining to the independence of the welfare services — we believe that having sophisticated left-wing institutions is unnecessary. Recognizing the vital role of the right-wing politicians, due to their central position in deciding who will be purchasing and delivering public goods, in the interpretation of the parameter α, we believed that it was beneficial to impose a lower α to the left-wing politicians, with a corresponding higher share $1-\alpha$ assigned to the right-wing politicians, i.e., $\alpha < 1-\alpha$, $0 < \alpha < 1$.

Thus, it was reasonable to assume that left-wing politicians, with almost no extra effort, would demonstrate an ample degree of readiness to make efficient decisions. Herewith, in planning and regulating the size of the wealth-pie to suit a fiscally realistic welfare policy to settle and assist the state welfare mission, we attempted to redesign the balance of political powers between the left- and right-wing politicians by adjusting the power indicators α and $1-\alpha$, imposed on the on the left- and right-wing politicians, respectively. With the goal in our view, to benefit all citizens in society, this enabled us to adjust the state rules and norms of the wealth redistribution, aligning them closer to the legal responsibilities and moral obligations of the citizens. We referred to the process of adjusting the power indicators $(\alpha, 1-\alpha)$ as a political power design. Such a politically designed outcome, as we supposed, justified the time and effort invested, even if the vision was a utopia.

The design of political power indicators $(\alpha, 1-\alpha)$ is a difficult and extremely time-consuming process. Indeed, prolonged political efforts might not be in the interest of anyone — citizens might not pursue such endeavor, even if the balance of political power can be ultimately reached. In particular, we supposed that electoral maneuvering of voters might put prolonged political efforts at risk of a premature collapse. It was deemed acceptable to assume presence of an implicit risk of voters defecting to the other side, which could interrupt negotiations ahead of the schedule. Thus, we brought the problem of likelihood of negotiations collapsing into focus. In our experiment, the failure of negotiations was deemed extremely undesirable for both politicians, as we hoped that this would be an incentive to move toward a solution faster. Alternatively, the actors would be more motivated to agree on terms of a contract, where both sides approach each other by making considerable concessions. In the view of receipt of relief payments, a policy of higher tax rates might be the most favorable and just solution for minority. From the majority perspective, however, the minimum tax rate is always preferable. For the citizens who finance the relief payments, as we assumed in the analysis, the minimum tax rate provides a more just and fair redistribution of wealth. In our experiment, the minimum rate also provided an outcome λ in which the designed political power indicators $(\alpha, 1-\alpha)$ visualize the society′s common denominator. Assuming, as we previously did, in accordance with the rules of the game, that outcome λ, minimizing taxes, could be politically designed — it provides insight into what policy should entail.

Table 1, presenting all four assumptions, suggests several proposals for citizens to vote on. Note that, when voting for policy of equal left- and right-wing political power, the policy $\eta = 79.23$ is less just and less fair than the outcome $\lambda = 45.50$, where the minimum *26.52%* of marginal tax rate is reached. Thus, only the policy/outcome λ on the poverty line (Figure 4) can be the desirable political consent. Indeed, in the variety of rules in the game the left- and right-wing politicians play, when engaged in an interaction aimed at implementing equal/egalitarian policy η, the equal political power $\alpha = 0.5$ of the LWP was stronger than *0.21*. Consumers' goal, however, can still be achieved by applying the weaker policy $\lambda = 45.50$ for the tax rate *26.52% < 28.21%*, although the outcome of the weakened political power indicator $\alpha = 0.21$ is yet to be confirmed. Through a reduction of citizens' obligations — even with LWP's weakened political position — the LWP will be able to come to a desirable agreement with the RWP, maintaining the most just and fair poverty line of wealth for all citizens.

In closing the discussion, we would like to point to a decision δ that corresponds to the political breakdown of negotiations. Utopian society, planned according to the event of a breakdown, as shown in Table 1, seemingly ignores welfare protection because practically all citizens are considered rich by default, i.e., poverty does not exist. Given this utopian society, financing expenses almost entirely with respect to vital public/non-basic goods, the breakdown policy δ, under the equity condition, requires *–2.49* public debt per capita. This, in turn, will require borrowing or money printing, promoting public spending, *e.g.,* through natural assets for refunding the debt. We admit that, based on the lowest tax burden of *26.52%*, a self-financing tax system has a better chance of being implemented.

8. CONCLUDING REMARKS

Given the ideological controversies of the left- and right-wing politicians, and the need to resolve the welfare policy dilemma, both actors should be willing to make concessions. In most cases, the root of the controversy is that, the left-wing politicians struggle — in response to public aspirations — in pursuing their own political causes for the increase of basic goods, whereas the right-wing politicians advocate for meeting the needs for non-basic goods. In our experiment, left-wing politicians gave credit to the tax system to guarantee a reasonably high living standard for benefit claimants. Whatever public spending voters preferred, both politicians were aware of voters' electoral maneuvering, which could put the negotiations at risk of a premature collapse. In our work, this threat was the only driving force in reaching the consensus. We argued that political arguments demanding higher taxes were weak, since overly costly welfare proposals lead to an excessive number of relief payments claimants, which, in spite of the tax increase, could diminish the quality of the welfare services. In turn, the excessive number of claims could generate further requests for the additional financial support through tax channels. In order to satisfy those who bear additional costs, and who could only approve the requests on the terms of fiscally safe welfare policies, we reduced the scope of negotiations to the fiscally realistic domain of voters' expectations.

In view of the above, a pretext for the analysis of the domain and the extent of bargain portfolio of two visionary politicians, denoted as LWP and RWP, were established. The portfolio was supposed to account for politicians having non-conforming expectations. Instead of the wealth-pie division, such an account allowed for including a guide on how the eventual consensus ought to be analyzed and interpreted within the scope of negotiations $\left[\xi_1, \xi_2\right]$ at the contract curve. In this context, the left- and right-wing political power indicators, specified by the bargaining problem solution, were supposed to be politically designed in advance and subsequently tailored in accordance with the citizens' visions and ambitions.

It was initially deemed that, due to the uncertainty in the selection of the breakdown policy, we could only treat the left- and right-wing political power indicators as given exogenously. While this is true at least in the valuable examples we provided, we found a condition where we can encode the indicators endogenously, to which we referred as *the pre-equity of political breakdown.*

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APPENDICES

A1. Example and results

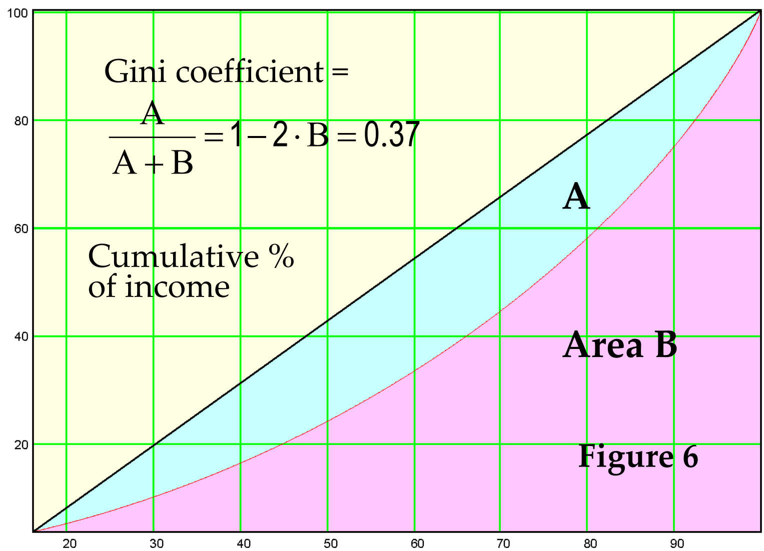
We proceed with a specific allocation of the welfare policy, encapsulating samples of income density distribution, parameterized by poverty line ξ, similar to an exponential function:

$$P(\sigma, \theta + h \cdot \xi) =$$
$$= \frac{1}{(\theta + h \cdot \xi) \cdot \Gamma(m)} \left(\frac{\sigma}{\theta + h \cdot \xi} \right)^{m-1} \cdot \exp\left(-\frac{\sigma}{\theta + h \cdot \xi} \right),$$

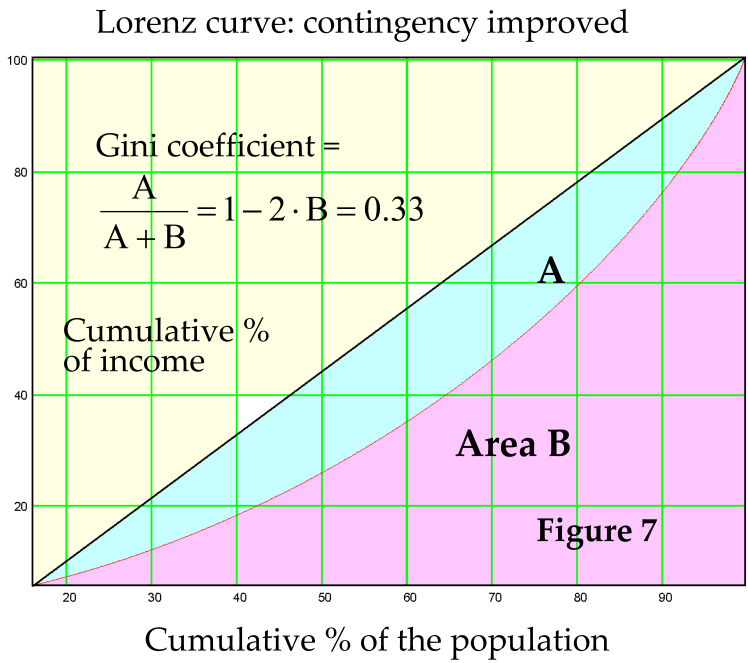
where $\theta = 61.9$, $m = 2.07$, and $h = -0.18$ are additional ex-ante parameters. More specifically, θ controls the wealth of citizens — a *horizontal shift* of samples; m controls inequality — a *vertical shift*; h is a hazard parameter; and $\Gamma(m)$ is an extension of $(m-1)!$ to real numbers. The sample $\xi = ½\mu$ (median income = μ) can be presented as **Lorenz Curve**, where citizens below an income ***95.1***, i.e., ***49.92%*** of the population, have <u>***24.13%***</u> of a total cumulative income, while the remaining ***50.08%***, with incomes at or above ***95.1***, have <u>***75.87%***</u>, Figure 6. Gini Coefficient equals ***0.37*** and is impervious to the horizontal shifts only. Relief payments, delivered to the population in line with Friedman personal exception rule in force equal to ½μ applied upon the income distribution sample $\xi = ½\mu$ diminished the Gini coefficient to ***0.33***. Indeed, on Figure 7 citizens below an income ***95.1***, i.e., ***49.83%*** of the population has slightly increased to ***25.83%*** of a total cumulative income, while the remaining ***50.17%***, with incomes at or above ***95.1***, have slightly decreased to ***74.17%***.

The density function $P(\sigma, \theta + h \cdot \xi)$, depending on ξ, reflects the initial wealth redistribution through tax channels. Political decision $\xi' > \xi$ shifts the density distribution $P(\sigma, \theta + h \cdot \xi)$ of incomes horizontally toward the allocation $P(\sigma, \theta + h \cdot \xi')$ that favors less wealthy. When shifted, the distribution $P(\sigma, \theta)$ masks the h-factor, $h = 0$, of the benefit claimants. The rate of change $Hz(\xi) = h \cdot \dot{a}(\theta + h \cdot \xi) < 0$ of the policy ξ quantifies a fiscally tolerable hazard ($h < 0$).

Lorenz curve without contingency



Cumulative % of the population



A2. Simulation foundation and illustration

In order to perform simulations, the expressions for average $B(\xi)$ of expenses on the relief payments and average taxable income — the wealth amount $W(\xi)$ — can incorporate income density distribution $P(\sigma, \theta + h \cdot \xi)$ in a more realistic but general form:

$B(\xi) = r \cdot \int_0^{\xi} (\xi - \sigma) \cdot P(\sigma, \theta + h \cdot \xi)\, d\sigma$; $r \cdot (\xi - \sigma)$ is the LI-relief payment, $0 < r < 1$;

$$W(\xi) = \int_0^{\xi} (\sigma + r \cdot (\xi - \sigma) - \phi) \cdot P(\sigma, \theta + h \cdot \xi)\, d\sigma +$$

$$+ \int_{\xi}^{\infty} (\sigma - \phi) \cdot P(\sigma, \theta + h \cdot \xi) \ .$$

In the left- and right-wing political bargaining, the choice of ξ, in general, is also determined by the ability to maintain the average income $a(\theta + h \cdot \xi)$, in order to uphold $a(\theta + h \cdot \xi) > W(\xi)$ within the "striking" distance from $W(\xi)$, which can be ensured through proper choice of the personal allowance constant $\phi > 0$, where ϕ identifies a flat tax bracket $[\phi, \infty)$. The average $a(\theta + h \cdot \xi)$ of income σ over the density sample $P(\sigma, \theta + h \cdot \xi)$ equals $\int_0^\infty \sigma \cdot P(\sigma, \theta + h \cdot \xi) d\sigma$.

The taxation of the total income $\sigma + r \cdot (\xi - \sigma)$ of the needy complies with the rules and norms in force, while the h-factor reveals the inverse working incentives, namely the feedback of the welfare recipients.

At this point, it is useful to verify that a disagreement policy δ under the primacy of equity principle of breakdown might be an outcome of the game. There is no reason to assume that the equation

$$(\delta - \phi) \cdot (B(\delta) + d_2) - (\delta - d_1) \cdot W(\delta) = 0,$$

in accordance with Observation 3, should have a solution in general. However, for the income density $P(\sigma, \theta + h \cdot \xi)$ (see above), a solution can be found. Given payoffs $\langle u, g \rangle$ at the endpoints $\langle u_1 = 6.44, g_1 = 47.18 \rangle$, $\langle u_2 = 89.26, g_2 = -2.49 \rangle$ of the scope of negotiations — within the interval $[\xi_1 = 8.00, \xi_2 = 144.54]$ — it can be shown that the pair $d = \langle d_1 = u_1, d_2 = g_2 \rangle = \langle 6.44, \ -2.49 \rangle$, $u_1 < u_2$, $g_1 > g_2$ consolidates an equity for breakdown policy $\delta = 6.39 \notin [\xi_1, \xi_2]$; wealth $W^* = 120.46$ and tax $\tau^* = -2.06\%$.

It should not be surprising that the amounts of public goods and tax rates may be negative. Ensuring this game outcome, the interpretation suggests that the simulated breakdown demonstrates a specific payoff deficit on public goods when it is impossible to cover all the costs through taxes. In such a scenario, as we have pointed out earlier, when discussing negotiations breakdown, it is necessary to resort to an external loan, money printing, or use of natural resources, if the latter are available.

The magnitude and dimension of poverty proposals to be debated or implemented, as *outcomes of the left- and right-wing political bargaining,* are given in Table 1.

Recall already known proposals for incomes η, λ_1, λ, λ_2, δ, whereby δ is outside of the scope of negotiations, $\delta \notin [\xi_1, \xi_2]$ and the poverty proposal $\frac{1}{2}\mu$, with their definitions given as follows:

η	the policy on poverty with equals left- and right-wing political power; the left- and right-wing political organizations are in symmetrical positions or in equal roles;
λ_1	the outcome of the alternating-offers game — representing what the right-wing politicians accept;
λ	the policy on poverty minimizing wealth-tax;
$\frac{1}{2}\mu$	½ of the median income, indicating that half of the population earns income above μ, while the income of the remaining half is below μ;
λ_2	the outcome of the alternating-offers game — representing what the left-wing politicians accept;
δ	the least desirable outcome, resulting in the policy breakdown or disagreement, which naturalizes the risk of negotiations' premature collapse, caused, for instance, by mutual traps.

A3. Verification

Proof of observation 1. Let us now assume an inverse scenario, whereby $u > u' = \pi(\xi, \tau(\xi, x))$. Here, the left-wing politicians — LWP — aim *to improve* the poverty line residue u', i.e., an after-tax residue of a marginal citizen $\sigma = \xi$ with income equal to the poverty line ξ. By initiating a new rule for policy $\xi' > \xi$, the LWP attempt to implement $u > u'$. Because of the inequalities $u \geq \pi(\sigma, \tau(\xi, x)) > u'$, for some highly pragmatic benefit claimants σ, it becomes apparent that they can be *better off* by *claiming relief payments*. Consequently, actions of these claimants will *increase* the expenditure $B(\xi') > B(\xi)$ on the relief payments and shift the balance of books $B(\xi) = x \cdot \tau(\xi, x) \cdot W(\xi)$ toward *deficit* $B(\xi') > x \cdot \tau(\xi, x) \cdot W(\xi)$. The balance was valid in the past, when

$\tau(\xi, x) \equiv \dfrac{B(\xi)}{x \cdot W(\xi)}$. Thus, the only option that would ensure that the balance in maintained, as the LWP must stay committed to x, is to adjust $\tau(\xi, x)$ to $\tau(\xi, \xi', x) = \dfrac{B(\xi')}{x \cdot W(\xi)} > \tau(\xi, x)$, as x was fixed by the agreement. Otherwise, keeping the old policy ξ intact, the LWP could — through a *decrease* in x — violate the commitment x. As LWP cannot directly change x, they resort to reducing the *deficit* via a tax increase. If $u > \pi(\xi', \tau(\xi, \xi', x))$, the LWP must continue with the tax adjustment policy by $\tau(\xi', \xi'', x) > \tau(\xi, \xi', x)$, now adjusting upon the welfare policy ξ' and proposing $\xi'' > \xi'$, whereby the new *deficit* becomes $B(\xi'') > x \cdot \tau(\xi, \xi', x) \cdot W(\xi')$. These *improvements* $u > u'' > u'$ initiate a sequence of poverty policies $(\ldots, \xi'' > \xi' > \xi, \ldots)$ and after-tax residues $(\ldots, u > u'' > u', \ldots)$ of marginal citizens. Thus, the conditions $u = u''$ and $\xi = \xi''$ can never be met, as this would contradict the assumption that the equation $u = \pi(\xi, \tau(\xi, x))$ cannot be solved for ξ. For this reason, the sequence $\ldots, \xi'' > \xi', \ldots$ is infinite. ■

The chain of reasoning regarding $u' > u$ is similar to that outlined above and is presented as a set of instructions. It should first be noted that, at low values $u' > u'' > u$, even when taxes are low, there would always be a surplus to finance the LI benefits and relief payments. The surplus masks a contradiction, since it is clear that, at low values of the after-tax residue parameter u, benefits financing can always be balanced.

Replace	*to implement an improved*	by	*to make a decline in*
–	*better off*	–	*worse off*
–	Improve *improvement*	–	Decline *deterioration*
–	*to claim for relief payments*	–	*that relief payments have been revoked*
–	*deficit*	–	*surplus*
–	$\geq, >$	–	$\leq, <$
Transpose:	*an increase*	with	*a decrease*

In what follows, we investigate the payoffs $\langle u, g \rangle \in \boldsymbol{S}_b$ of the left- and right-wing politicians. The consensus occurs at outcomes $\phi, \xi \Rightarrow z, x, \alpha, \tau, \langle u, g \rangle$ under the constraint that the variation in policy ξ does not improve the position of the left-wing politicians; rather, the policy emerges as the point on the contract curve $\boldsymbol{S}_b = u(g)$ as fiscally idempotent outcome.

For fiscally idempotent outcomes, the arguments of after-tax residue u, share x, policy ξ, and tax rate τ depend on each other. The share $x = x^{o}$, if settled as eventual agreement, redirects the residue $u = \pi(\xi, \tau(\xi, x^{o})$ to become a function $u = u(\xi, x^{o})$. Thus, the peak policy u with regard to the best welfare policy can be expressed as:

$$\xi^{o} = \operatorname{arg\ max}_{\xi} u(\xi, x^{o})$$

Lemma. *Let us assume that left-wing politicians do not shift from the share* $x = x^{o}$ *and that the volatility constraint* (4) *solves for two different policies* $\xi_1 < \xi_2$. *Let the tax sacrifice* $t(\xi, x^{o}) = \tau(\xi, x^{o}) \cdot (\xi - \phi)$ *be a differentiable function of* ξ *progressively increasing with* ξ *within the closed interval* $[\xi_1, \xi_2]$ *— namely, the following derivatives hold:*

$$\left.\frac{\partial}{\partial \xi} t(\xi, x^{o})\right|_{\xi=\xi_1} > 0, \left.\frac{\partial}{\partial \xi} t(\xi, x^{o})\right|_{\xi=\xi_2} < 0 \text{ and } \frac{\partial^2}{\partial \xi^2} t(\xi, x^{o}) > 0.$$

In such situation, the poverty line residue $u(\xi, x^{o}) = \xi - t(\xi, x^{o})$ *is a single* $\cap$*-peaked function of* ξ.

Corollary. *There exists a unique interior policy* ξ^{o} *maximizing* u *at*

$$\left.\frac{\partial}{\partial \xi} u(\xi, x^{o})\right|_{\xi=\xi^{o}} = 0.$$

Provided that the conditions of the lemma are fulfilled, the discussion that follows concerns the necessary and sufficient conditions for the fiscally idempotent policy ξ to occur at the contract curve.

Observation 2. *Let us assume that the volatility constraint* (4) *is differentiable from its arguments. The after-tax residue* $u = u(\xi, x^{o})$ *is differentiable and single peaked with respect to the policy* ξ *within some closed interval* $[\xi_1, \xi_2]$. *For a fiscally idempotent outcome* $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$ *to occur on the contract curve* $\boldsymbol{S}_{b} = u(g)$, *it is necessary and sufficient that the policy* ξ^{o} *solves the set of equations:*

$$\text{(i)} \qquad \left.\frac{\partial}{\partial \xi} L(\xi, x^{o}, u^{o})\right|_{\xi=\xi^{o}} = 0, \text{ where } u^{o} = u(\xi^{o}, x^{o})$$

provided that

$$\text{(ii)} \qquad \left.\frac{\partial}{\partial u} L(\xi^{o}, x^{o}, u)\right|_{u=u^{o}} \neq 0.$$

Proof

Necessity. Let the fiscally idempotent outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$ on the contract curve $\boldsymbol{S}_{b} = u(g)$ maximize (A1) at $u^{o} = u(\xi^{o}, \tau(\xi^{o}, x^{o}))$. Varying ξ in the vicinity of ξ^{o} of the outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$ and substituting $u = u(\xi, \tau(\xi, x^{o}))$ into the volatility constraint (4), we obtain an identity $L(\xi, x^{o}, \pi(\xi, \tau(\xi, x^{o}))) \equiv 0$. Within the proximity of (ξ^{o}, u^{o}), the following equation holds for arguments ξ, u:

$$\frac{\partial}{\partial \xi} L(\xi, x^{o}, u^{o}) + \frac{\partial}{\partial u} L(\xi^{o}, x^{o}, u) \cdot \frac{\partial}{\partial \xi} \pi(\xi, \tau(\xi, x^{o})) = 0, \text{(A2)}$$

from which we deduce the necessity statement for $\xi = \xi^{o}$ and $u = u^{o}$.

Sufficiency. Suppose the condition (**ii**) holds. Let (**i**) solve for ξ^{o} at the fiscally idempotent outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$. Combining (**i**) and (A2), we conclude that

$$\left. \frac{\partial}{\partial \xi} \pi(\xi, \tau(\xi, x^{o})) \right|_{\xi = \xi^{o}} = 0 .$$

The sufficiency clause (A1) holds, since $u = u(\xi, x^{o})$ is a convex function of ξ.■

Proof of Observation 3. The clause is correct, provided that there exists a fiscally idempotent policy δ for the implementation of the pair $\langle d_1, d_2 \rangle$.

In order to identify such a policy, we first replace the variable g with d_2 in the expression for the constraint (1). Next, we extract the expression for $\tau = \dfrac{B(\delta) + d_2}{W(\delta)}$ from (1) and substitute it into $(1 - \tau)\ldots$ of the constraint (3), where u should be replaced by d_1 in advance. By simplifying, we arrive at the statement of the observation.■

Sketch of the proof (Observation 5). Looking at the tax rate $\tau > \tau_{\min}$, for any outcome $\ldots, \tau, \langle u, g \rangle \in \boldsymbol{S}_b$, one may indeed prefer a counter outcome as a motion $\ldots, \tau, \langle u', g' \rangle$, which outlines $\ldots, \tau, \langle u' > u, g' < g \rangle$ or $\ldots, \tau, \langle u' < u, g' > g \rangle$. As the contract curve $\boldsymbol{S}_b = u(g)$ is a curve of efficient preferences $\langle u, g \rangle$ guaranteeing the poverty line residue $u(g)$, someone could put a motion $u' > u^{\circ}$ or $g' > g^{\circ}$ against an outcome $\ldots, \tau > \tau_{\min}, \langle u^{o}, g^{o} \rangle$. We argue that, in order to fulfill the expectations and requests of citizens' majority, it is necessary to pursue political consent via the proposal $\ldots, \tau_{\min} = \tau(\lambda), \langle u^{o} = u(\lambda), g^{o} = g(\lambda) \rangle$.■

$$\tau \cdot W(\xi) = B(\xi) + g$$

Delivery constraint: the size of the welfare pie, i.e., the average amount of tax returns is equal to the sum of the average monetary value per capita of primary goods and the average of non-primary goods g.

$$B(\xi) = x \cdot \tau \cdot W(\xi)$$

Budget constraint imposed on the relief payments finance in accordance with the share x of the wealth-pie — the tax-revenue.

$$u = (1-\tau)\cdot(\xi-\phi)+\phi$$

Stability constraint that determines fiscally idempotent policy ξ.

$$u = \xi - \tau\cdot(\xi-\phi)$$

After-tax residue constraint: an alternative form of stability constraint, where u is after-tax position of a marginal citizen with income $\sigma = \xi$, which concedes with the left-wing political aspirations.

A4. Mathematical derivation

Replacing $\tau = \dfrac{B(\xi)}{x \cdot W(\xi)}$ from the budget constraint into the stability constraint, we obtain the volatility constraint (4) as stated:

$$L(\xi, x, u) = (\xi-\phi)\cdot B(\xi) - x\cdot(\xi-u)\cdot W(\xi) = 0$$

that amalgamates budget constraint and after-tax residue. Contract curve (5) is thus given by:

$$D(\xi, x, u) = L'_{\xi}(\xi, x, u) =$$

$$= \left[(\xi-\phi)\cdot B(\xi) - x\cdot(\xi-u)\cdot W(\xi)\right]'_{\xi} = 0$$

$$L'_{\xi}(\xi, x, u) =$$

$$= B(\xi) + (\xi-\phi)\cdot \dot{B}(\xi) - x\cdot W(\xi) - x\cdot(\xi-u)\cdot \dot{W}(\xi) = 0^{\cdot}$$

The last expression may be rewritten as:

$$D(\xi, x, u) =$$
$$= B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi) - x \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \cdot$$

Extracting $x = \dfrac{(\xi - \phi) \cdot B(\xi)}{(\xi - u) \cdot W(\xi)}$ from the volatility constraint (4), we can substitute variable x into the rewritten expression for $D(\xi, x, u)$. The substitution results in the following expressions:

$$B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi) -$$
$$- \frac{(\xi - \phi) \cdot B(\xi)}{(\xi - u) \cdot W(\xi)} \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \text{, or}$$

$$\frac{\begin{array}{l}\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot (\xi - u) \cdot W(\xi) - \\ -(\xi - \phi) \cdot B(\xi) \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right)\end{array}}{(\xi - u) \cdot W(\xi)} = 0 .$$

Provided that $(\xi - u) > 0$ and $W(\xi) > 0$, we can conclude that the following is true:

$$\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot (\xi - u) \cdot W(\xi) -$$
$$-(\xi - \phi) \cdot B(\xi) \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \cdot$$

This allows writing the sub-expression $(\xi - u)$ in the form:

$$\begin{pmatrix} \left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - \\ -(\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi) \end{pmatrix} \cdot (\xi - u) -$$
$$-(\xi - \phi) \cdot B(\xi) \cdot W(\xi) = 0.$$

As a consequence of presenting the sub-expression $(\xi - u)$ in the form given above:

$$\xi - u = \frac{(\xi-\phi)\cdot B(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)}.$$

We observe that

$$u = \xi - \frac{(\xi-\phi)\cdot B(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)}.$$

We can now substitute the tax rate τ from the delivery constraint into the after-tax residue constraint. The result will be $u = \xi - \frac{B(\xi)+g}{W(\xi)}\cdot(\xi-\phi)$. After replacing the result into the observed u -expression, we obtain:

$$\xi - \frac{B(\xi)+g}{W(\xi)}\cdot(\xi-\phi) = $$
$$= \xi - \frac{(\xi-\phi)\cdot B(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)};$$

$$\frac{B(\xi)+g}{W(\xi)}\cdot(\xi-\phi) = $$
$$= \frac{(\xi-\phi)\cdot B(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)};$$

$$\left(B(\xi)+g\right)\cdot(\xi-\phi) = $$
$$= \frac{(\xi-\phi)\cdot B(\xi)\cdot W(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)};$$

$$B(\xi)+g = $$
$$= \frac{B(\xi)\cdot W(\xi)\cdot W(\xi)}{\left(B(\xi)+(\xi-\phi)\cdot \dot{B}(\xi)\right)\cdot W(\xi)-(\xi-\phi)\cdot B(\xi)\cdot \dot{W}(\xi)};$$

$$g = \frac{B(\xi) \cdot W(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)} -$$
$$- B(\xi)$$

.

We can now impose the denominator in the last expression for g on sub-expression for $(\xi - \phi)$, which can be written as:

$$\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi) =$$
$$= B(\xi) \cdot W(\xi) + (\xi - \phi) \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right).$$

Continuing with the expression for $g(\xi)$, we can replace the denominator transformed above:

$$g = \frac{B(\xi) \cdot W(\xi) \cdot W(\xi)}{B(\xi) \cdot W(\xi) + (\xi - \phi) \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right)} - ;$$
$$- B(\xi)$$

$$g = \frac{\begin{array}{l} B(\xi) \cdot W(\xi) \cdot W(\xi) - \\ - B(\xi) \cdot \left(\begin{array}{l} B(\xi) \cdot W(\xi) + (\xi - \phi) \cdot \\ \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right) \end{array} \right) \end{array}}{\begin{array}{l} B(\xi) \cdot W(\xi) + (\xi - \phi) \cdot \\ \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right) \end{array}}$$

Now, both the nominator and the dominator can be divided by $B(\xi) \cdot W(\xi)$, yielding:

$$g = \frac{W(\xi) - B(\xi) \cdot \dfrac{\begin{pmatrix} B(\xi) \cdot W(\xi) + (\xi - \varphi) \cdot \\ \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right) \end{pmatrix}}{B(\xi) \cdot W(\xi)}}{\begin{pmatrix} B(\xi) \cdot W(\xi) + (\xi - \varphi) \cdot \\ \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right) \end{pmatrix}}.$$

Let us define $v(\xi) = 1 + (\xi - \phi) \cdot \left(\dfrac{\dot{B}(\xi)}{B(\xi)} - \dfrac{\dot{W}(\xi)}{W(\xi)} \right)$, as this allows us to evaluate the expression for the right-wing political objective on public but vital goods as:

$$g(\xi) = \frac{W(\xi) - B(\xi) \cdot v(\xi)}{v(\xi)} = \frac{W(\xi)}{v(\xi)} - B(\xi).$$

In accordance with the delivery constraint, the size of the wealth-pie $\tau(\xi) \cdot W(\xi)$ equals $B(\xi) + g(\xi)$. Consequently, the tax rate is given by:

$$\tau(\xi) = \frac{B(\xi) + g(\xi)}{W(\xi)} = \frac{B(\xi) + \left(\dfrac{W(\xi)}{v(\xi)} - B(\xi) \right)}{W(\xi)} = \frac{1}{v(\xi)}.$$

Replacing the $\tau = \dfrac{1}{v(\xi)}$ in the after tax residue $u = \xi - \tau \cdot (\xi - \phi)$, we can finally evaluate the expression for the left-wing political wants on basic goods as: $u(\xi) = \xi - \dfrac{(\xi - \phi)}{v(\xi)}.$

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Name Restaurant

WINE TASTING

MENU
RESTAURANT

wine list

Rose Wines

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

White Wines

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Red Wines

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Your text here .... 100 g... 10.00
(your text here, your text here,
your text here, your text here)

Irrational Behavior when Buying and Selling Stocks:
Monotone Linkage Choice Model *

Abstract. We focused on the possibility of the irrationality of transactions when buying shares at a higher price. To show this phenomenon, we use an innovative procedure for statistical analysis of buying and selling stocks in the stock market, whose indicators with a higher dynamics of change compared to other indicators have an advantage in predictive ability.
Keywords: indicators; credentials; monotonic; system; kernel

1. INTRODUCTION

We start with a "visual" or "pedagogical exhibit". When accepting the order in a restaurant, the sommelier explains that at the moment some of the most expensive, and in other cases may be, the cheapest wines indicated by the guest on the list as possible favorite choices are temporarily absent. The absence on the list of the most expensive wines, for sure, will encourage guests to expand the list of cheap wines with those that at first glance have not yet been approved, or at least keep the list of those cheap wines that have already been approved. On the contrary, the lack of approved, at first glance, cheap wines may induce the sommelier to suggest more expensive wines in favor of others available for order, also cheaper, but quite good and better wines. More often than not, guests agree with such a proposal. This irrational behavior was our main motive for informing the reader about "these events." The phenomenon of irrationality is illustrated based on a probabilistic-statistical analysis of the numerical indicators of the exchange market.

In probabilistic-statistical analysis two opposing approaches can be distinguished: from subjective to objective knowledge and in the opposite direction — from objective to subjective. In the first approach, such specialists as a physician, biologist, astronomer, practitioner or market analyzer... those who have knowledge in their field, use data visualization for objective statements about the obtained estimates of experimental data,

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* Münich Personal RePEc Archive, https://mpra.ub.uni-muenchen.de/101591/ (Acessed 23.12.2021)

observations, etc. With this approach, from a subjective assessment to an objective assessment, probability research includes: Markov processes (Rogers and Williams, 2000), Lévy processes (Applebaum, 2004), Gaussian processes (Lifshits, 2012), random fields (Adler, 2010)… Statatistical analysis includes space state models (SSM, Koller and Friedman, 2009), parameter estimation (Walter and Pronzato, 1997), management and decision-making problems (Narula and Weistroffer, 1989), continuous modeling, multiple time series (Voelkl et al, 2012) and computational methods (Mirkin et al, 1995). In both areas the knowledge of the distribution of judgments about the object under study is necessary that is not always the case.

The objective to subjective assessment, both to statistical and probabilistic indicators, at first glance, seems to be contradictory. It seems that specialized knowledge is also required. Nevertheless, it is very possible to do without special knowledge, as well as knowledge about the distribution of numerical parameters — indicators.

The procedure of the objective to subjective approach considered below could be called the "blind glance of statistical scoring", which is what we need. The only thing the Data Explorer uses in blind scoring is that one number is greater/less than another. If common sense is achieved, then the well-known law of parsimony or **"Occam's razor"** will come into force. A procedure that requires fewer assumptions about reality can be considered the most reliable. Our parsimonious procedure is somewhat consistent with the postulates of bounded rationality of choice when buying and selling stocks on the stock market.

In particular, when studying the dynamics of time series, instead of absolute values, we are often interested in indicators over/under estimating these values relative to some given threshold u. Since the procedure described in the article can set the threshold of significance u of absolute values in a certain time interval, now the same procedure can be applied to the “induced” indicators $\pm\Delta$ of “overshoot” or “undershoot”. As a result, we will be able to calculate some interval $[-\Delta_1 + u, u + \Delta_2]$, where -$-\Delta_1$ is the underestimation undershoot, and $+\Delta_2$ is the overestimation overshoot threshold relative to the same absolute threshold u. Download an EXCEL spreadsheet (see below), where the Ctrl+s macro can be applied to both positive and negative values in columns, rows, or even tables.

2. Parsimonious approach

The wine list is ordered in descending order of price, and 1 multiplies the price of the most expensive wine, 2 multiplies the next local price, then 3 the next, and so on. We call these numbers as price credentials. The local maximum of credentials and the price of wine are selected when this peak location from the top of the ordered list — the maximum is reached. The guest decides to accept the price of the wine at the local credential maximum as an acceptable level of price significance when choosing wines with a higher or equal price level.

Before we get to the main part of the note, we will highlight some of the points of our short notice. We use extended quotes from well-known experts on the decision theory (bounded rationality) that was found online (Tomasz Strzalecki,TS, Accessed via Google: December 28, 2021, available to download from http://datalaundering.com/download/notes.pdf), since excessive or own repetition of the main provisions of the theory, it seems to us, cannot improve the explanation of the postulates of the theory.

3. Significance Indicatores

Here we will look at just a few details of our procedure for analyzing stock market data. Let's define a set of indicators $p_j \in W$, $|W| = n$, of n indicators, $j = \overline{1,n}$. In particular, let the sample, denoted as H, compare all indicators $p_j \in H$ as potential candidates for significance. We can further define a totality of sets $\{H\}$ of all 2^n samples $H \subseteq W$. Let credentials $\pi(p_j, H) = p_j \cdot |H|$ (in terms of Kempner et al., pp. 19-24, 1997, as monotone linkage functions) evaluate the level of significance.

The procedure for finding the most significant indicators is easy to set up. First, all the indicators p_j, are sorted in descending order, constituting (like wines order in wine list) the order $\langle p_j \rangle$, and then a sequence of credentials $\overline{\pi} = \langle \pi_j \rangle = \langle p_j \rangle \cdot j$, is constructed. The sequence $\overline{\pi}$ is called defining. The visualized indicators $\langle p_j \rangle$, in contrast to original indicators p_j, are necessary descending.

4. SIGNIFICANCE LEVEl

The credentials $\langle \pi_j \rangle$, $j = \overline{1,n}$, are single peaked, where the peak denotes the kernel H^* (Mullat, 1971-1995) of a monotone system. The set H^* constitutes the rational, i.e., the monotone linkage choice implemented in our findings. The local $\arg\max_{j=\overline{1,n}} \langle \pi_j \rangle$ denotes a peak indicator p_k^* at a location k^* from the top of the defining sequence, where the local maximum is reached. This value p_k^* will be called the level of significance of indicators p_j.

Proposition. *Among the totality of all samples* $H \subseteq W$*, i.e.„ among all the sets* $\{H\}$ *of all* 2^n *samples, the kernel* H^* *guarantees reaching the global maximum of the credential function* $F(H)$ *of samples* H *equal to* $\min_{p_j \in H} \pi(p_j, H)$*:* $H^* = \arg\max_{H \subseteq W} F(H)$.

The proposition confirms the postulate of independence from rejected alternatives, known in two-persons games since 1950 in the bargaining problem solution, Nash John F.

5. BOUNDED RATIONALITY POSTULATES

We emphasize it once again, as said, that the theory of bounded rationality choice (e.g., Arrow 1959) is better explained not by my own words, but by the words of specialists in this field. Below I shell slightly modify the nomenclature for my own purpose. I downloaded these notes from the public domain, licensed under a Creative Commons Attribution Non-Commercial-No Derivatives 4.0 license, Tomasz Strzalecki, TS, **available online** (Accessed: 9 July 2020):

"These notes are based on lectures I gave at Harvard in 2010{17 and also those I gave when visiting the Cowles Foundation at Yale in 2012…

…Models in Economics are micro founded, which means that at the bottom of every model there is an economic agent (often many of them) choosing an action from an opportunity set. This set, often called the budget set or a menu, represents the various actions that are available to the agent. But which action will the agent choose? By far the most popular theory is that the agent will choose the alternative with the highest utility. In the most basic version of the theory the utility function is

an as-if concept, which means that we don't claim it exists in any material sense nor is it necessarily related to the agent's well being, emotions, or biology. It is just a mathematical construct that helps the analyst make sense of observed choices: the agent behaves as if he maximizes a utility function. We don't care how the agent manages to maximize the utility (it may be a hard mathematical problem) because utility is just a language of description and it is not taken literally."...

...Let X *be a set of alternatives that the agent is choosing between. They can be either immediate "payoffs" (for example different candy bars of which your diet lets you eat only one) or more structured objects (such as retirement plans). There are three main "languages" that help us describe choice:"*

The following item is the most important for us.

"A choice function, TS. The analyst observes which element the agent chooses from every nonempty set $A \subseteq X$*. Let* $c(A)$ *denote that element."*

We use an uppercase letter for the selection function $C(A)$, which emphasizes that the user can observe for analysis not a single element but a set $C(A)$ to decide the set of best alternatives selected from A. In fact, the set A selected for choice action will be considered as an area in the EXCEL spreadsheets — as cells in a column, a row or tables that can be selected using the "pasted" option. The choice function $C(A)$, following the basic wine selection procedure described above (December 27, 2021), was programmed using the Cntrl-s macro, available online, http://datalaundering.com/download/TyskeAktier29062020%202.xls

Finally, we recall in a more formal form the postulates (cited by Aizerman and Malishevski, 1981, pp. 65-83, English version translated from Russian, p. 189) that we will note in connection with the procedure of supposedly rational choice of our guests in restaurant:

- *Independence with respect to dropping rejected alternatives (or, for brevity, elimination of options), Postulate 5 (Chernoff, 1954, pp. 422-443) or Axiom 2 (Jamison and Lau, 1973, pp. 901-912):*

  $C(X) \subseteq X' \subseteq X$ *that* $C(X') = C(X)$*;*

- *Succession, which is the same as Postulate 4 (Chernoff, 1954), or condition* α *(Sen, 1971, pp. 307-317) or the axiom C2 of Arrow-Uzawa (Arrow, 1959, pp. 121-127):*

  $C(X') \supseteq C(X) \cap X'$ [1]

[1] Recall that we spoke in the language of barmaid about this postulate in our "Game of Singles": *"The old love does not rust."*

- *Strict Succession or constant residual choice (it is the same as postulate 6 (Chernoff, 1954), and one of the forms of the "weak axiom of revealed preference" of Samuelson, , i.e., the axiom C4 (Arrow, 1959, pp. 121-127):*

$$C(X') = C(X) \cap X'.$$

6. Findings and Experiments

We tried to confirm these postulates predictive abilities in our experiments programmed in the EXCEL spreadsheet with share prices on the German stock market using the statistical approach of our parsimonious procedure. As one would expect, our experiments confirm the postulate of independence of rejected alternatives. However, succession or strict succession occurs only in relation to the most valuable stocks in the market. In contrast, the exclusion of certain cheap stocks from the sale presumably forces buyers in some situations to refuse to buy other cheap stocks that are still available for sale in favor of more expensive ones. Therefore, within the framework of the monotonic choice model, it must be admitted that transactions to buy more expensive shares are irrational.

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APPENDIX

Some comments are necessary in order to explain the implementation of the aforementioned “procedure” for the analysis of the stock market dynamics. The reliability of data on the sale and purchase of stocks on the market is guaranteed by the fact that the Nordet online, exchange was used, where all transaction data have been available to everyone. The spreadsheet was compiled using Nordnet public domain https://www.nordnet.dk/markedet/aktiekurser/ (Accessed: Monday, December 27, 2021).

The stock market monitoring spreadsheet consists of 620 Germany companies. Some columns show relative ±% rise and fall in stocks prices, last purchase, sale, etc. As you can see, some cells differ from others in certain patterns and frames. These highlighted patterns and frames are the result of using the macro — Cntrl-s. Cntrl-s means, as said, that an analysis of the significance levels of the negative/positive values of the stocks indicators dynamic has been conducted. Using the macro in columns, rows, or selected *("pasted")* areas of the spreadsheet in their entirety may consist of negative/positive numbers distributed throughout the areas without any special order for negative or positive numbers. However, the standard EXCEL data sorting options allow you to sort selected areas in ascending or descending order depending on the specified columns or rows. Thus, having, for example, negative values scattered across a spreadsheet in different cells, these cells can be redistributed together into "contiguous areas" of negative or positive values in the columns or row patterns to satisfy the necessary conditions. Such contiguous areas can help visualize the analysis results.

A note may be helpful. If macro Cntrl-s produces only a few selected cells or an unsatisfactory small number of special patterns and frames — in the case of very high positive or very low negative values — the same macro can be reused, now outside these sharp numerical jumps that were designated as unsatisfactory, i.e., when the macro detects too small areas. In doing so columns or rows must be first reordered in ascending descending order to bring the positive/negative indicators into contiguous areas. Now, reusing Cntrl-s macro in the gap, i.e., against not yet patterned and framed cells, to add additional meaningful cells in addition to previously selected cells, can help to improve the situation.

Negative significance level →	**-1.57%**	**€ 34.60**			
Positive significance level →	**1.36%**	**€ 0.60**	**€ 44.80**	**€ 43.01**	**€ 44.59**
Company in Germany	Today %	Today +/–	Last	Buy	Sell
1&1 DrillischAktiengesellschaft	**-1.65%**	€ 0.38	€ 22.67	€ 22.65	€ 22.70
11 88 0 Solutions AG	**-2.16%**	€ 0.03	€ 1.36	€ 1.36	€ 1.42
2G En. AG	**4.46%**	**€ 2.80**	**€ 65.60**	**€ 65.60**	**€ 66.40**
3M Co.	0.43%	€ 0.58	**€ 136.08**	**€ 134.86**	**€ 136.08**
3U Hold. AG	**2.56%**	€ 0.04	€ 1.60	€ 1.58	€ 1.64
4 SC AG	-0.60%	€ 0.01	€ 1.67	€ 1.63	€ 1.67
4basebio AG	**-1.97%**	€ 0.04	€ 1.99	€ 1.99	€ 2.00
....	....	....	....	....	....

Stock Market Table, shortened.

One can read the following about **Nordnet**

(https://nordnetab.com/about/nordnet-overview. Accessed: 23 August 2020):

> *"Our target group is Nordic savers and investors. We offer products and services to both experienced investors and beginners, no matter if they have knowledge or need guidance, wish to spend hours on your investments every day or simply review your savings a few minutes a week."*
>
> *Our vision is to become the Nordic private savers' first choice. To achieve this objective, we must always continue to challenge and innovate, keeping user-friendliness and savings benefit at the top of the agenda. Only then can we achieve the high level of customer satisfaction and brand strength required to become a leader in the Nordic region in terms of attracting new customers and producing loyal ambassadors for Nordnet.*
>
> *We help people and money grow. We are passionate about creating a world-class user experience. We simplify to make it easier for people to make smart investment decisions. We share our knowledge and inspiration without any hidden agendas.*
>
> *The overarching purpose of Nordnet's operations is to democratize savings and investments. By that, we mean giving private savers access to the same information and tools as professional investors. This purpose has driven us since we started in 1996 and remains our direction to this day. In the 1990s, the idea of democratization entailed offering easily accessible and inexpensive share trading via Internet, and building a fund supermarket with products from a number of different companies where savers could easily compare returns, risk and fees. During the journey, we have simplified matters and pressed down fees on, for example, pension savings, index funds and private banking services. In recent years, we have democratized the financial sector with, for example, the stock lending program. We are always on the savers' side, and pursue issues of, for example, the right to transfer pension savings free of charge and reasonable and predictable taxation of holdings of stocks and mutual funds.*
>
> *Our target group is Nordic savers and investors. We offer products and services to both experienced investors and beginners, no matter if they have knowledge or need guidance, wish to spend hours on your investments every day or simply review your savings a few minutes a week."*


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Financing Dilemma Supporting a Project *

Abstract. This article can be considered as an independent but at the same time complementary addendum to the previous article on bounded rationality in decision-making. With this in mind, the concept of rational decision-making core (the kernel) was re-visited to form coalitions in the game of interconnected players, characterized by monotonous contribution functions. We have focused on ad hoc coalitions that have an advantage over the rest due to the higher contribution of each individual member.
Keywords: coalition; game; contribution; donation; monotonic; project

1. Introduction

In multi-person games (Owen, 1971, 1982) a coalition is formed by a subset of participants. Among all coalitions, rational coalitions are of particular interest, as these allow all participants to gain individual benefits. It can further be stipulated that extraction of this benefit is ensured independently of the actions of players that are not coalition members. In this note, we will deal with one of the simplest cases of player-formed coalitions, all of which can be considered as "outstanding" in terms of bounded rationality. Bounded rationality is the idea that rational decision making of people is limited by people's irrational nature.

The class of games proposed is subjected to an additional monotonic condition, which has been studied in previous work of Mullat (1979). However, it should be noted that no prior knowledge of the subject matter discussed here is presupposed. Still, the formal theory of monotone systems adopted in this note is identical to that described earlier by Mullat (1971-1977); the only difference arises in interpretation, and pertains to the abstract indices of interconnection of the system elements, which are treated as donation intentions. The approach developed in this note enables us to establish, in one particular case, the possibility of finding rational coalitions in accordance with the principle of independence of rejected alternatives according to Nash (1950). However, for the purpose of simplicity, the following scenario might be informative.

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* Communicated with Е.Н. Кузнецов, Институт проблем управления им. В.А. Трапезникова РАН Россия, 117997, Москва, Профсоюзная ул., 65. Previous work in "Stable Coalitions in Monotonic Games", Avt.. i Tel., No. 10, pp. 84 – 94, October, 1979 (Russian version). Original article submitted October 3, 1978. Plenum Publishing Corporation, 227 West 17th Street, New York, 10011. We alert the readers' obligation with respect to copyrighted material; https://mpra.ub.uni-muenchen.de/96879/ (Accesed 24/01/2022).

2. Pedagogical scenario

Here we are dealing with participants who intend to fund a project being under development through donations. In principle, each participant $j=\overline{1,n}$ is willing to contribute a certain amount p_j supporting the project. In summary, each participant's donation amount p_j might be in accord with distribution defined by an exponential density function:

$$f(x,\beta)=\left|\begin{array}{l}\frac{1}{\beta}\cdot\exp(-x/\beta)\ \text{ for }\ x\geq 0,\\ 0\ \text{ for }\ x<0.\end{array}\right.$$

In favor of the project it is expected to collect a certain fund to finance the project. However, as a result of negotiations about the appropriateness of the planned project with like-minded participants, their preferences will be reoriented. It is assumed that a certain coalition game arises here in accordance with the monotonic game scheme, the solution of which is the concept of a kernel (Mullat 1979). Intricacies of financing interests of the participants are presented in the form of a solution called the kernel that will constitute a certain group of participants who agree to finance the project, but perhaps not to the extent to which they were originally intended, but still within reasonable limits. In fact, this reasonable limit is the one most reasonable of all possible options for financing the project in its final version. It should be noted here that a reasonable scenario is understood as a certain guaranteed payment, in which each project participant guarantees a contribution to the expected total amount.

We define the credential of participant $j\in H$ as $\pi(j,H)=|H|\cdot p_j$. Thus, it indicates that the total expected payments of all in H will not be less than $F(H)=\min_{j\in H}\pi(j,H)$. The kernel H^* in this scenario will be understood as participants $H^*=\arg\max_{X\subseteq W}F(X)$. The kernel H^* is remarkable in that it guarantees a contribution $F(H^*)$ to the project. Can more participants with lower individual p_j payments intentions fund the project to a greater extent? Such situation is possible, however, such payments cannot be guaranteed – this is the point. In what follows, we will focus only on payments guaranteed by project participants belonging to the kernel H^*.

The global maximum for the project funding by the kernel participants will form the basis of independence in accordance with the hypothesis of the so-called rejected alternatives, that is, regardless of the preferences of the participants not included in the kernel, if any are found, which nevertheless consider it appropriate to participate in the kernel. But we should not particularly believe them, as they will not be very reliable, and may seek to change their preferences not in favor of the project.

Therefore, we assume that non-kernel participants refusing to participate in the project will not affect those who belong to the kernel, i.e., the views and activities of the kernel members. Here we are dealing, as said, with the so-called principle of bounded rationality, that is, the principle of independence from rejected alternatives (cf. Nash, 1950). In essence, this principle in our particular case of project financing ensures that project participants are kept abreast of developments. The kernel participants will not change their decisions on financing regardless of what is happening or what change the conditions for participation in the project, despite the fact that some participants in the project refused to participate. If we give this last consideration a somewhat more formal character, then we can say that the stability property of decisions made by the kernel participants is nothing but the well-known so-called idempotent principle. After the decision is revised in the conditions when the commitments and priorities assumed remain unchanged, it will not require any new adjustments, and this decision will be made in the same form in which it was adopted earlier.

Example. Let we introduce in accord with exponential distribution the preferences p_j, of participants' $W = \{j = \overline{1,n}\}$. We can designate as X, all participants who prefer to participate in the project together with their like-minded people, while $\overline{X}$ prefer to reject the project or have other reasons for participating in the project.

Let we now try to determine the preferences π for the participants j in X, $j \in X$, supposing that their contributions in the project together with others in X be equal to $\pi(j, X) = |X| \cdot p_j$. Obviously, if some participant could not at all find a suitable partner for the project, the intention to contribute will be equal to $\pi(j, \{ j\}) = |\{j\}| \cdot p_j$, $|\{j\}| = 1$. Conversely, if all participants contribute to the project and all participants are in an adequate company W, the estimated contribution will be greater and equal to $\pi(j, W) = (|W| = n) \cdot p_j$. If now for any reason a participant $j \in X$ decides to spend the rest of the project development alone, the intention to contribute to all others remaining participants in X, including those to which some like-minded participants $X - \{j\}$ still join, will decrease: $\pi(i, X - \{j\}) \leq \pi(i, X)$ for $i \in X - \{j\}$. On the contrary, their intentions to contribute will increase if one $j \notin X$ of the previously single participants decides to join X and become a member of $X + \{j\}$: $\pi(i, X + \{j\}) \geq \pi(i, X)$ for $i \in X$.

The graph below shows the donations of the participants in% relative to the total amount of their initial intentions on the X-axis with the corresponding contributions in%, as well as to the same amount indicated on the Y-axis, where their donation preferences were reoriented. As the simulation shows, kernel members are almost always ready to finance approx. 50% of their original intentions.

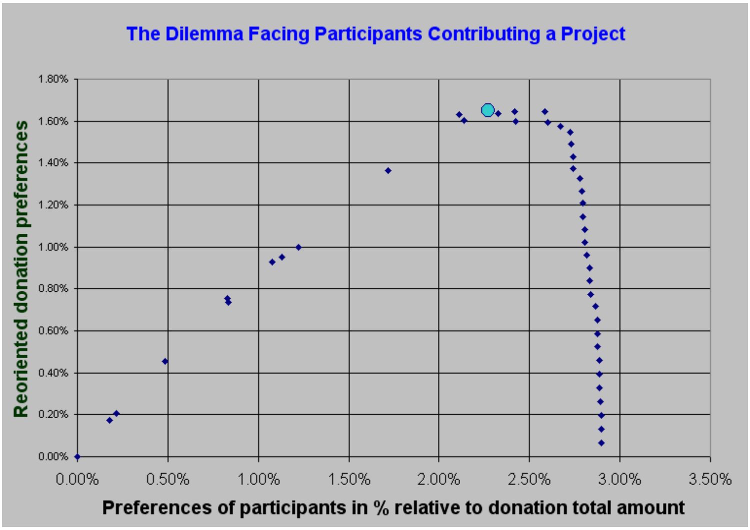

Figure 1. The kernel participants contribute at least 52.8% of their initial intentions to the project. The blue dot is the largest guaranteed contribution in which participants continue to agree to participate in the project.

To be more precise, in the initial state, the percentage of contribution to the total amount for financing the project, which reflects, as it was, the starting point of the participants' preferences on the X-axis — donation submission of participants.

The procedure for finding the kernel is very easy to set up. First, all the expected donation preferences $p_j, j=\overline{1,n}$, are sorted in descending order, constituting the order $\langle p_j \rangle$, the X-axis, and then a sequence $\overline{\pi}$ is constructed as $\overline{\pi}=\langle \pi_j \rangle = \langle p_j \rangle \cdot j$, by which we have denoted these reoriented $\langle \pi_j \rangle$ preferences, the Y-axis. The latter sequence is called defining. We then select the local maximum, i.e., the defining sequence. This is the kernel of Mullat's monotonic game, which is represented by a blue dot in Figure 1.

I. FINANSEERIMISE DILEMMA PROJEKTI TOETAMISEL

Kokkuvõtte. Tuuma mõistet külastati uuesti koalitsiooni moodustamiseks proekti financeerimise mängus, mida iseloomustavad monotoonsed panuse-funktsioonid. Keskendusime spetsiaalsetele koalitsioonidele, millel on eelis ülejäänud osas, kuna iga osalemine koalitsioonis annab suurema panuse.

Mitme-isiku mängudes (Owen 1971, 1982) moodustatakse koalitsioon osalejate alamrühmast. Kõigist koalitsioonidest pakuvad ratsionaalsed koalitsioonid eriti huvi, kuna need võimaldavad kõigil osalejatel saada individuaalseid eeliseid. Veel võib täpsustada, et selle hüvitise saamine tagatakse sõltumata mängijate tegevusest, kes ei ole koalitsiooni liikmed. Sõnumis käsitleme mängijate poolt moodustatud koalitsioonide ühte kõige lihtsamat juhtumit, mida võib pidada piiratud ratsionnaalsuse mõttes silmapaistvateks. Ratsionaalsus on piiratud sellega, et inimeste ratsionaalset otsustamist piirab inimeste irratsionaalne olemus.

Pakutud mängude klassile rakendatakse täiendavat monotoonset seisundit, mida on uuritud Mullati poolt (1979) monotoonses mängus ja varasemastes töödes. Tuleb märkida, et siin käsitletud teema eelteadmisi ei nõua. Kasutatud monotoonsete süsteemide teooria on identne sellega, mida Mullat (1971–1977) on varem kirjeldanud; ainus erinevus ilmneb tõlgendamises ja puudutab süsteemielementide abstraktseid sidumisnäitajaid, mida käsitletakse annetuste kavatsustena. Välja töötatud lähenemisviis võimaldab meil ühel konkreetsel juhul esiletuua lihtsa metoodika ratsionaalsete koalitsioonide leidmiseks, mis on kooskõlas (Nash, 1950) tagasilükatud alternatiivide sõltumatuse põhimõttega. Lihtsuse huvides järgmine pedagoogiline stsenaarium võib aga olla informatiivne.

II. PEDAGOGIKA

Siin on tegemist osalejatega, kes kavatsevad arendusjärgus olevat projekti rahastada annetuste kaudu. Põhimõtteliselt on iga osaleja $j=\overline{1,n}$ nõus projekti toetamiseks teatud summa p_j panustama. Kokkuvõtlikult võib iga osaleja annetussumma p_j olla kooskõlas jaotusega, mis on määratletud eksponentsiaalse tiheduse funktsiooniga:

$$f(x,\beta)=\left|\begin{array}{l}\frac{1}{\beta}\cdot\exp(-x/\beta)\ \text{ for }\ x\geq 0,\\ 0\ \text{ for }\ x<0.\end{array}\right.$$

Seetõttu loodetakse hankida projekti rahastamiseks teatud fond. Läbirääkimised mõttekaaslastega kavandatava projekti sobivuse üle viivad aga nende viimaste eelistused ümbersuunamiseks. Eeldatakse, et siin tekib vastavalt monotoonsele mänguskeemile teatud koalitsioonimäng, mille lahendab tuuma kontseptsioon (Mullat, 1979). Tuum on osalejate mõnevõrra tähelepanuväärne alamhulk.

Nagu juba ööldud on osalejate finantseerimishuvide keerukus esitatud lahenduse vormis, mida nimetatakse tuumaks, mis moodustab teatud osalejate rühma, kes nõustuvad projekti rahastama, kuid võib-olla mitte sellises mahus, nagu need algselt olid mõeldud, kuid siiski mõistlikkuse piires. Tegelikult on see mõistlik piir mis on parim tulemus projekti lõppfinantseerimisvõimaluste rahastamisel. Siinkohal tuleb märkida, et garanteeritud stsenaariumi all mõeldakse teatud garanteeritud makset, mille puhul iga projektis osaleja garanteerib oma panuse eeldatavasse kogusummasse.

Määratleme osaleja $j \in H$ mandaadi kui $\pi(j, H) = |H| \cdot p_j$. Seega näitab see, et kõigi sissemaksete eeldatav kogusumma ei ole väiksem kui $F(H) = \min_{j \in H} \pi(j, H)$. Selle stsenaariumi tuuma all mõistetakse osalejaid H^*. Tuum on tähelepanuväärne selle poolest, et see tagab projekti panuse $F(H^*)$. Kas väiksemate individuaalsete maksete kavatsustega p_j osalejad saavad projekti suuremal kui $F(H^*)$ määral rahastada? Selline olukord on võimalik, aga selliseid makseid garanteerida ei saa – see on asja mõte. Järgnevalt keskendume ainult nendele maksetele, mille tagavad tuuma H^* kuuluvad projektis osalejad.

Tuuma poolt projektile eraldatav globaalse maksimumi kogurahastus moodustab sõltumatuse aluse vastavalt juba nn tagasilükatud alternatiivide hüpoteesile, st sõltumata tuuma mittekuuluvate osalejate eelistustest, kui neid leidub, mis peavad tuumas osalemist siiski asjakohaseks. Kuid me ei tohiks eriti neid uskuda, kuna need ei ole väga usaldusväärsed ja võib-olla soovivad nad oma eelistusi projektis osalemise kohta muuta.

Seetõttu eeldame, et kui tuuma mittekuuluvad osalejad keelduvad projektis osalemast, siis ei mõjuta see neid kes kuuluvad tuuma, st tuumaliikmete vaateid ja nende tegevusi. Siin on tegemist nagu juba ööldud, nn piiratud ratsionaalsuse põhimõttega, see tähendab sõltumatuse põhimõttega tagasilükatud alternatiividest, vt Nash 1950. Sisuliselt tagab see põhimõte meie projekti rahastamise puhul, et projektis osalejad oleksid läbirääkimiste arengutega kursis. Tuuma osalejad ei muuda oma rahastamisotsuseid olenemata sellest, mis toimub või mis muudavad projektis osalemise tingimusi, hoolimata asjaolust, et mõned projektis osalejad keeldusid osalemast. Kui anname sellele viimasele kaalutlusele mõnevõrra formaalsema iseloomu, siis võime öelda, et tuumast osavõtjate tehtud otsuste stabiilsuse omadus pole midagi muud kui tuntud idempotentsuse põhimõte. Pärast otsuse läbivaatamist tingimustes, kus võetud kohustused ja prioriteedid jäävad muutumatuks, ei vaja see uusi muudatusi ning see otsus tehakse samal kujul, nagu see varem vastu võeti.

Näide. Tutvustame vastavalt eksponentsiaalsele jaotusele osalejate $W = \{j = \overline{1,n}\}$ eelistusi p_j, $j = \overline{1,n}$. Võime X-na tähistada kõiki osalejaid, kes eelistavad projektis osaleda, et koos oma mõttekaaslastega kokku leppida, samal ajal kui $\overline{X}$-s olevad osalejad eelistavad projekti tagasi lükata või on neil muud põhjused projektis osalemiseks.

Proovime nüüd määrata kindlaks X-s osalejate $j \in X$ eelistused, eeldades, et nende panus projekti koos teistega X-s on võrdne $\pi(j, X) = |X| \cdot p_j$. Ilmselt kui mõni osaleja ei suuda üldse projekti jaoks sobivat partnerit leida, on kaastöö tegemise kavatsus võrdne $\pi(j, \{j\}) = |\{j\}| = 1 \cdot p_j$-ga. Ja vastupidi, kui kõik osalejad panustavad projekti ja kõik osalejad on sobivas mõttekaaslaste seas W, on nende viimaste eeldatav panus suurem ja võrdne $\pi(j, W) = (|W| = n) \cdot p_j$-iga. Kui nüüd mõni osaleja $j \in X$ soovib või otsustab mingil põhjusel veeta ülejäänud projekti arenduse üksi, väheneb kavatsus panustama kõigile teistele X-is allesjäänud osalejatele, sealhulgas ka neile, kellega mõned mõttekaaslased X-ga endiselt liituvad: $i \in X - \{j\}$, $\pi(i, X - \{j\}) \le \pi(i, X)$. Vastupidi, nende panustamiskavatsused suurenevad, kui üks varem osalenud üksikliikmeline $j \notin X$ osaleja otsustab liituda X-iga ja saada $X + \{j\}$: $i \in X$ liikmeks: $\pi(i, X + \{j\}) \ge \pi(i, X)$.

Ülaloleval joonisel, Figure 1, on näidatud osalejate annetused protsentides, võrreldes nende esialgsete kavatsuste suhtes kogusumma panusena X-teljel koos vastava sissemaksega protsentides, samuti sama summa kohta, mis on näidatud Y-teljel, kus nende annetuseelistused olid ümber orienteeritud. Nagu simulatsioon näitab, on tuuma liikmed peaaegu alati valmis finantseerima umbes. 50% nende algsest kavatsusest. Kui täpsem olla, siis algseisundis on projekti finantseerimise kogusummast tehtud panuse protsent, mis peegeldab osalejate eelistuste lähtepunkti X-teljel — osalejate annetuste esitamine.

Tuuma H^* leidmise protseduuri on väga lihtne üles ehitada. Esiteks järjestatakse kõik arvud $p_j, j = \overline{1, n}$, langevas järjekorras, muutes järjestust p_j järjestuseks $\langle p_j \rangle$, ja seejärel konstrueeritakse järgmiste arvude jada, mida me nagu eelpool juba neid arvu tähistanud olime $\overline{\pi}$-ks: $\overline{\pi} = \langle \pi_j \rangle = \langle p_j \rangle \cdot j$ mis on Joonise 1 Y-teljel, nn osalejate panuste ümberorienteerimine. Seda jada nimetatakse määravaks jadaks. Seejärel valime selle viimase, järjestatud, st määratud jada põhjal, lokaalset maksimumi. See ongi monotoonse mängu Mullati tuum, mis on Joonisel 1 tähistatud sinise punktina.

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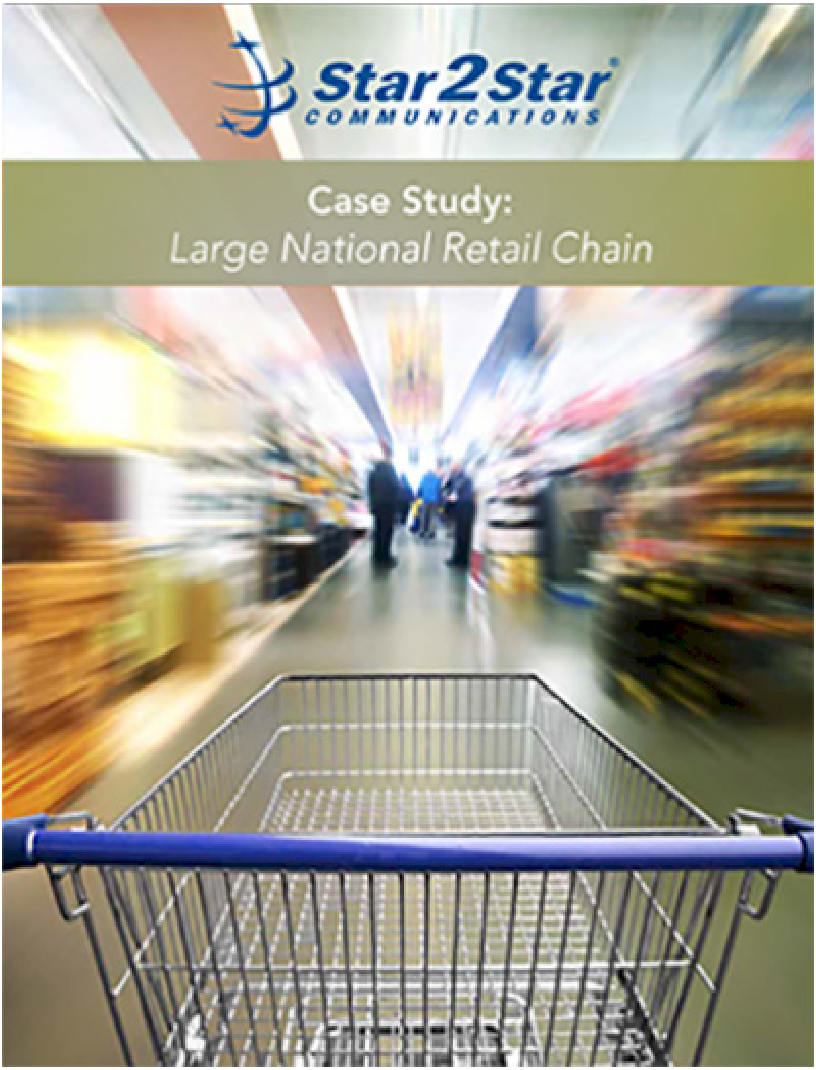
Star2Star
COMMUNICATIONS
Case Study:
Large National Retail Chain

Equilibrium in a Retail Chain with Transaction Costs: Rational Coalitions in Monotonic Games *

Abstract. In a given context, a situation is considered when a retail chain of suppliers, agents and distributors transforms while transaction costs increase. As costs increased, orders and deliveries between relevant chain's groups resulted in the most cost-resilient retail chain. The participants in such a resilient chain remain in equilibrium, provided that in any transaction, the profit from trading exceeds the cost of the transaction, including transportation costs. In making decisions about buying and selling, the participants in the chain had to follow the rules and regulations of what the author called a monotonous game. A formal scheme of coalition formation in this monotonic game of connected retail trade participants with monotonic utility functions is described. Special coalitions are studied that have an advantage for each of the participants over the rest in the sense of a greater ability to withstand the volatility of the supply market.
Keywords: suppliers; distributors; monotonic game; retail chain; coalition.

> *Businessmen in deciding on their ways of doing business and on what to produce have to take into account transaction costs. If the cost of making an exchange are greater than the gains which that exchange would bring, that exchange would not take place and the greater production that would flow from specialization would not be realized. In this way transaction costs affect not only contractual arrangements, but also what goods and services are produced.*
> Ronald H. Coase, "The Institutional Structure of Production," Ménard, C., and M. M. Shirley (eds.) (2005), *Handbook of New Institutional Economics*, Spriner: Dordrecht, Berlin, Heidelberg, New York. XIII. 884pp., p.35, ISBN 1-4020-2687-0.

1. INTRODUCTION

Everybody, probably knows that prices on commodity markets sometimes continue to rise unabated on the back of an anticipated shortage in the global raw materials availability and sharp volatility in the commodity future markets and terminal prices on fears of an immediate shortage of materials in the short term. Along with the significant increase in com-

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* A part of this article was translated from Avtomatica i Telemekhanika, 1980, 12, pp. 124 – 131. Original article submitted 1979. Automat. and Remote Control, Plenum Publishing Corporation, 1981, pp. 1724-1729. Russian version.

modity prices, on one hand, the transaction costs increase on inputs like petroleum, electricity, etc. On the other, while currency of exchange rates also moving adversely, the situation becomes uncertain. As an example, one may point at recent market price increase of coffee raw materials, which did not have immediate consequences for some known positions, while the distributors [1] of a retail chain, however, demonstrate readiness to make loosing transactions. With this in mind, distributors are trying to hold prices constant. However, it is also understandable that it would be impossible for the distributor to make frequent price changes again and again. Given the current context, they will have no other option but to seek price increase for distributed commodities with an immediate effect.

Uncertainties in market prices of commodities always lead to an increase of transaction costs. Transaction costs increase once again leads to additional uncertainties, and the distributors in the retail chain end up in a dead circle of price increase, which may result that the bilateral trade does not take place, and the market old supply and demand structure to be replaced with a new. In the environment of constant price increase, the orders and deliveries do not match any more for a given supply and demand structure. In such situations, individual participants in the retail chain are still assumed to act rationally finding a new ways of making business with the object of maximizing the profit by trying to restructure the chain. Worth to note that New Institutional Economics gives an explanation for transactions as mediated through the market in two directions: the vertical integration, Joskow (2005), where the market structure is mostly a vertical chain of semi-product components, and the horizontal chain of services and products outsourced by companies if needed to produce the end product.

This paper addresses the above situation in question by setting up a retail chain game of the participants in the chain grounding on supposition that orders and deliveries be met with uncertainty of transaction costs. In so doing, the paper attempts to develop a numerical description of the supply and demand structure for the deliveries of commodities in the retail chain. The allegedly rational behavior of a participant is not always such, because the participants on purpose may attempt to enter but irrationally into certain losing transactions in hope to offset the negative effect of the former. Given this irrational situation the prices will increase additionally upon already profitable transactions. Numerical analysis of irrational situations reveals, however, that in case the participants will try to avoid all

[1] A group of retail outlets owned by one firm and spread nationwide or worldwide.

losing transactions, their behavior is once again becoming rational and in such situations the participants of the retail chain will end up in the Nash equilibrium (1953).

To our knowledge (or lack of that), the retail chain formation, or in mundane terms the restructuring process of the retail chain is rather complicated mathematical problem, which do not have satisfactory solutions. However, in recent years it has become clear that a mathematical structure known as antimatroid is well suited for such type a retail chain formation process (cf. Algaba, et al, 2004). Antimatroid is a collection of potential interests groups — subsets of participants, i.e., those who make decisions to buy and sale in bilateral trade transactions. That is to say, within antimatroid one will always find a path of transactions connecting members of the retail chain — if the latter forms of course — with each other by mutual business interests inside groups/coalitions belonging to antimatroid and making the exchange as participants of a characteristic retail chain.

We step up beyond convention of the theory of coalition games that the solution mandatory has to be a core, and take the retail chain formation process in terms of so-called *defining sequence* of transactions (Mullat, 1979). The sequence facilitates the retail chain formation as a transformation process of nested sets of bilateral transactions, which ends at its last and highest costs' threshold — the most tolerant retail chain towards costs — a kernel. Hereby, the kernel operates as a retail chain of participants capable to cover the highest transaction costs in case of uncertainty. In our case, the *defining sequence* of transactions produces the elements of an antimatroid — some interest groups, cf. Levit and Kempner, (2001); see also (1991) Korte et al. The defining sequence on antimatroid, in particular, follows the Greedy heuristic procedure of Shapley's value, but in inverse order, cf. Rapoport (1985).

Bearing all this in mind, the suggested framework allows performing a series of computer simulations. First, to determine the possible response of the retail chain participants, to different supply and demand structures. Second, to identify the participants, where the executive efforts might be applied to prevent unpredictable actions that may misbalance the equilibrium in the retail chain. With this object, we used a model to assemble an "elasticity" measure for the choice of customers; this measure is represented by transaction costs' interval, for which the retail chain remains in equilibrium.

The rest of this paper is structured as follows. The next section sets up the basic concepts intending to bring at the surface the calculus of utilities of participants in the retail chain. It is a preliminary step necessary to move forward to the Section 3, where the general model of participants of the chain is described. In Section 4, which is main part of the paper, the retail chain game of customers addresses the process of the chain formation in details. Here the monotonic property of utilities plays its major role. In Sections 5-6, we construct different varieties of coalitions of retail network players that are "outstanding" in the sense of rationality, and indicate relations between such coalitions. Also, constructive processes described in Section 7 for discovering these *outstanding players*, described in additional Section 8. A summary of the results ends the study. The proofs of all theorems, etc., ... are given in the Appendix.

2. DESCRIPTION OF A RETAIL CHAIN: THE SIMPLE FORM

To consider the simplest case of commodities distribution in a retail chain might be instructive. This elementary model is used at current stage solely as a convenient means of simplifying the presentation.

The distribution of commodities in the retail chain is characterized by sales figures that may be expressed as one of the following three alternative numbers: a) a demand η which is disclosed to the particular participant either externally or by other participant in the chain; b) a capable supply ξ calculated at the cost of all commodities produced by the participant for delivery outside the chain or to the other participants; c) actual sales γ calculated at the prices actually paid by the customers for the delivered commodities.

An order is thus defined as a certain quantity of a particular commodity ordered by one of the participant's from another participant in the retail chain; a delivery is similarly defined as a certain quantity of a commodity delivered by one of the participant's to another participant in the chain. We assume that the chain includes suppliers who are only capable of making deliveries – the produces; participants, who both issue orders and make deliveries – the agents; and the distributors, who only order commodities from other participants. [2]

In what follows we consider the retail chain of orders and deliveries for the case like "pipeline" distribution without "closed circuits." Therefore, we can always identify a unique direction of "retail chain" of orders from the distributors to the produces via agents and a "retail chain" of deliveries in the reverse direction.

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[2] The distributors also act as suppliers to external customers.

Let us consider in more detail this particular retail chain of orders and deliveries of commodities. The direction of the chain of orders (deliveries) is defined by assigning serial numbers – the indexes $1,2$ and 3 – to the producer, to the agent, and to the distributor, respectively. The producer and the agent act as suppliers, the agent and the distributor act as customers. The agent thus has the dual role of a supplier and a customer, whereas the producer only acts as a supplier and the distributor only acts as a customer.

The chain of orders to the produces from the customers is characterized by two numbers η_{23} and η_{12}. The number η_{wj} $(w = 1,2; j = 2,3)$ is the demand η_{wj} disclosed by the customer j to the supplier w. We assume that sales are equal to deliveries. Two numbers ξ_{12} and ξ_{23}, which are interpreted as the corresponding capable sales similarly characterize the chain of deliveries to the distributor.

Suppose that the demand of the distributor to the external customers is fixed by d bank notes. The capable sales of the producer are s bank notes. In other words, d is the estimated amount of orders from the external customers and it plays the same role as the number η for the customers in the retail chain. Similarly, s is the intrastate amount of estimated deliveries by the producer, and it has the same role as ξ for the customers.

Let us now consider the exact situation in a chain. To make deliveries at a demand amount of d bank notes, the distributor have to place orders with the agent in the amount of $\eta_{23} = \nu_{23} \cdot d$ bank notes, where ν_{23} are the distributor's cost of commodities sold (the cost per 1 bank note of sales). The agent, having received an order from the distributor, will in turn place an order with the supplier in the amount $\nu_{12} \cdot \eta_{23}$, where ν_{12} is the agent's cost per one bank note of sales. On the other hand, the estimated sales of the producer are ξ_{12} bank notes, $\xi_{12} = s$. Assuming that all the transactions between the suppliers and the customers in the retail chain are materialized in amounts not less than those indicated in the purchase orders, the actual sales of the producer to the agent are given by $\gamma'_{12} = \min\{\xi_{12}, \eta_{12}\}$.

Now, since the agent paid the producer γ'_{12} for the commodities ordered, the agent's revenue is $\xi_{23} = \gamma'_{12} / \nu_{12}$, where clearly $\xi_{23} \geq \gamma'_{12}$. The difference between the revenue ξ_{23} and the costs γ'_{12} is defined as

$$\pi_{12} = \gamma'_{12} \cdot (1 - \nu_{12}) / \nu_{12} .$$

From the same considerations, $\gamma'_{23} = \min\{\xi_{23}, \eta_{23}\}$ [3]give the actual sales of the agent to the distributor. We similarly define the difference $\pi_{23} = \gamma'_{23} \cdot (1 - \nu_{23}) / \nu_{23}$. The numbers π_{12}, π_{23} represent the profit of the customers in the retail chain.

In conclusion of this section, let us consider the numbers π_{12}, π_{23} more closely. We see from the above discussion that the material costs are the only component of the costs of commodities sold for the customers in the retail chain; no other producing or transaction costs are considered. And yet in Section 4 the numbers π_{12}, π_{23} are used as the admissible bounds on transaction costs, which are assumed to be unknown. It is in this sense we construct a model of a monotonic game of customers (Mullat, 1979, p.6).

3. DESCRIPTION OF A RETAIL CHAIN: THE GENERAL FORM

Consider now a retail chain consisting of n participants indexed w, $j = 1,2,...,n$. The state of a supplier w is characterized by a ($m+1$)-component vector [4] $\langle d_w, y_w \rangle = \langle d_w, \eta_{wk+1,...,} \eta_{wn} \rangle$, ($n - k = m$); the state of a customer j by a ($v + 1$)-component vector $\langle s_j, x_j \rangle = \langle s_j, \gamma_{1j}, ..., \gamma_{vj} \rangle$. The components of the $\langle d_w, y_w \rangle$ and $\langle s_j, x_j \rangle$ vectors are interpreted as follows: d_w is the total orders

[3] In subsequent sections, γ'_{wj} is replaced by $\gamma_{wj} = \gamma'_{wj} / \nu_{wj}$. The numbers γ and γ' differ in the units of measurement of the commodities delivered to the user j. While γ' represents the sales at the cost, γ represents the same sales at actual selling prices.

[4] k is the number of produces, see below.

amount of the supplier w acting as a customer; s_j is the capable sales total amount of the customer j acting as a supplier; η_{wj} is the cost of orders placed by the customer j with the supplier w; γ_{wj} are actual sales (deliveries) to customer j from the supplier w. As indicated in the footnote, γ_{wj} represents the deliveries valued at the selling prices of the customer j acting as a supplier. The vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ are the order and the delivery vectors, respectively.

With each participant in the retail chain we associate certain domains in the nonnegative orthants $\Re^{m+1}$ of the ($m+1$) – and $\Re^{v+1}$ of the ($v+1$) – dimensional space. These domains $\Re^{m+1}$ and $\Re^{v+1}$ are the regions of feasible values of vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ in the ($m+v+2$) – dimensional space.

For some of the participants vectors with $\gamma_{wj} > 0$ are inadmissible, and for some participants vectors with $\eta_{wj} > 0$ are inadmissible. Participants having the former property will be called produces and those having the latter property will be called distributors; all other participants in the retail chain will be called agents. In what follows the numbers s_w ($w = 1,2,...,k$) characterize the k produces; the number s_w represents the capable sales controlled by the participant w. The numbers d_j ($j = v+1, v+2,...,n$) correspondingly characterize the r distributors: the number d_j represents the demand to the external customers ($n - v = r$).

Let us now impose certain constrains on the admissible vectors in this retail chain. The following constrains are strictly “local,” i.e., they apply to the individual participants in the retail chain.

The admissible retail chain states are constrained by balance conditions equating the actual sales from all the suppliers to a particular customer to capable sales of that customer acting as a supplier:

$$s_j = \sum_{w=1}^{v} \gamma_{wj} \; (j = k+1, k+2, ..., n). \tag{1}$$

We also require balance conditions between the cost of orders placed by all the customers with a particular supplier and the demand figure of that supplier acing as a customer:

$$d_w = \sum_{j=i+1}^{n} \eta_{wj} \; (w = 1,2,..., v). \tag{2}$$

As we have noted above, the retail chain considered in this article does not allow "closed-circuit motion" of orders or deliveries until a particular order reaches a producer or the delivery reaches a distributor. The indexes labeling the participants in such chains are ordered in a way [5] that if w is a supplier and j is a customer, then $w < j$ ($w = 1,2,..., v;$ $j = v+1, v+2,..., n$). We call such chains as of a retail-type, and their description requires certain additional assumptions.

Consider the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ satisfying the following constraints ($w < j; j = k+1,..., n$):

$$\sum_{j} \alpha_{wj} \leq 1 \; (j > w; w = 1,2,..., v), \; \sum_{w} \beta_{wj} \leq 1 \tag{3}$$

For the supplier w, the number α_{wj} is the fractional cost of orders made to the customer j. For customer j, the number $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$ is the fractional cost of the deliveries from supplier w, which are necessary for meeting the sales target.

Suppose that purchase of orders in the retail chain move from distributors through agents to suppliers. This chain is conducted at the wholesale prices. The deliveries, also conducted at the wholesale prices of the chain in the opposite direction. We express the effective wholesale prices by a set of constants ν_{wj} ($w = 1,2,..., v; j = k+1, k+2,..., n$), which represent the participant's cost per one bank note of sales for a customer acting as a supplier.

[5] The term topological sorting originates from Knuth (1972) to describe the ordering of indexes having this property.

The set of constants α_{wj}, β_{wj} and ν_{wj} make it possible to uniquely determine the amount of orders and deliveries in a given transaction. Indeed, the amount of orders to the supplier w from the customer j is given by $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$. The relation (see Section 2) determines the amount of deliveries $\gamma'_{wj} = \min\{\xi_{wj}, \eta_{wj}\}$, where $\xi_{wj} = s_w \cdot \alpha_{wj}$ are the capable sales values at cost prices. Considering the difference in revenue from sales of customer j acting as a supplier, we conclude that the deliveries from the supplier w to the customer j are given by $\gamma_{wj} = \gamma'_{wj} / \nu_{wj}$.

In conclusion, let us consider one computational aspect of order and delivery vectors in a retail-type distribution chain. [6] It is easily seen that the components d_j, s_w, η_{wj} and γ_{wj} $(w = 1,2,\ldots,v;$ $j = k+1, k+2,\ldots,n)$ as obtained from (1) and (2) are given by ($w < j; j = k+1,\ldots,n$)

$$d_w = \sum_j \beta_{wj} \cdot d_j \cdot \nu_{wj} \quad (j > w; w = 1,2,\ldots,v) \tag{4}$$

$$s_j = \sum_\omega \min\{s_w \cdot \alpha_{wj}; \beta_{wj} \cdot d_j \cdot \nu_{wj}\} / \nu_{wj} \tag{5}$$

The starting data in (4) is the demand of the distributors to external customers, i.e., the numbers $d_{v+1}, d_{v+2},\ldots, d_n$. The starting data in (5) are the capable sales amounts $s_1, s_2,\ldots, s_k$ of the produces, which together with the numbers $d_1, d_2,\ldots, d_v$ from (4) are used in (5) to compute the actual sales of the customers.

4. A MONOTONIC GAME OF CUSTOMERS IN THE RETAIL CHAIN

In the previous section we considered a retail-type distribution in the chain with participants indexed by $w = 1,2,\ldots,v;$ $j = k+1, k+2,\ldots,n$: the index j identifiers a customer, the index w identifiers a supplier.

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[6] Here we need only consider the principles of the computational procedure.

Let us interpret the activity of the retail chain as a monotonic game (Mullat, 1979), in which the customers need to decide from what supplier to order a particular commodity.

Suppose that in addition to the cost of materials, the customers bear uncertain transaction costs in their bilateral trade with suppliers. Because of the uncertainty of transaction costs, it is quite possible that in some transactions the costs will exceed the gross profit from sales. In this case, the potentially feasible transactions will not take place.

Let the set R_j represents all the potential transactions corresponding to the set of suppliers from which the customer j is to make his choice. The choice of the customer j ($j = k+1, k+2, ..., n$) is a subset A^j of the set R_j: $A^j \subseteq R_j$; the case $A^q = \varnothing$ is not excluded: it requires the customer's refusal to make a choice. The collection $\left\langle A^{k+1}, A^{k+2}, ..., A^n \right\rangle$ represents the customer's joint choice. It is readily seen that the sets R_j are finite and nonintersecting; their union corresponds to set $W = R_{k+1} \bigcup R_{k+1} \bigcup ... \bigcup R_n$.

In what follows, we focus on the criterion by which the customer j chooses his suppliers A^j while the lowest transaction costs, as a threshold u^o, increases. In contrast to the standard monotonic game (Mullat, 1979), which is based on a coalition formation, we will consider the strategy of individual customers whose objective is to maximize the profit from the actual sales revenues. We will thus essentially deal with m players' game, $m = n - k$.

Let us first introduce a measure of the utility of a transaction between customer j and supplier $w \in A^j$ ($j = k+1, k+2, ..., n$). The utility of a transaction between customer j and supplier w is expressed by the corresponding profit $\pi_{wj} = \gamma_{wj} \cdot (1 - \nu_{wj})$.

The utility of a transaction with a supplier $w \in A^j$ is a function $\pi_{wj}(X_{k+1}, X_{k+2}, ..., X_n)$ of many variables: the value of the variable X_j is the choice A^j of the customer j, the number of variables is

$m = n - k$. To establish this fact, it is sufficient to show how to compute the components of the order and delivery vectors from the joint choice $\langle X_{k+1}, X_{k+2}, \ldots, X_n \rangle$. Indeed, according to our description, a retail-type distribution in the chain requires defining the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ ($w = 1,2,\ldots, v; j = k+1,\ldots, n$) that satisfy the constraints (3). A pair of constants α_{wj} and β_{wj} can be assigned in a one-to-one correspondence to a supplier $w \in R_j$, rewriting (3) in the form

$$\sum_{w \in R_j} \alpha_{wj} \leq 1, (w = 1,2,\ldots, \nu), \sum_{w \in R_j} \beta_{wj} \leq 1, (j = k+1,\ldots, n) \qquad (6)$$

If the constrains (6) are satisfied, then the same constrains are of necessity satisfied on the subsets A^j of the set R_j. Thus, restricting (4) and (5) to the sets $X_j \subseteq R_j$, the numbers γ_{wj} can be uniquely calculated for every joint choice $\langle X_{k+1}, X_{k+2}, \ldots, X_n \rangle$. Finally, let us define the individual utility criterion of the customer j in the form:

$$\Pi_j = \sum_{w \in A^j} \left(\pi_{wj} - u_{wj} \right), \qquad (7)$$

where u_{wj} are the customer j transaction costs allocable to the supplier $w \in A^j$; we define $\Pi_j = 0$ if the customer j refused to make a choice — $A^j = \varnothing$. The function $\pi_{wj}(X_{k+1}, X_{k+2}, \ldots, X_n)$ has the obvious property of monotone utility, so that for every pair of joint choices of customers $\langle L^{k+1}, L^{k+2}, \ldots, L^n \rangle$ and $\langle G^{k+1}, G^{k+2}, \ldots, G^n \rangle$ such that $L^j \subseteq G^j$ ($j = k+1,\ldots, n$) we have

$$\pi_{wj}(L^{k+1}, L^{k+2}, \ldots, L^n) \leq \pi_{wj}(G^{k+1}, G^{k+2}, \ldots, G^n). \qquad (8)$$

The property of monotone utility leads to certain conclusions concerning the behavior of customers depending on the individual utility criterion. Under certain conditions, rational behavior of customer j (i.e., maximization of the profit Π_j) is equivalent to avoid profit-loosing transaction with all the suppliers $w \in A^j$. This aspect is not made explicit in Mullat (1979), although it is quite obvious. Thus, using the lemma, see the English version at p.1473, we can easily show that if the utilities $\pi_{wj}(\ldots, X_j, \ldots)$ are independent of the choice X_j, the customer j maximizes his profit Π_j by extending his choice to the set-theoretically largest choice. In what follows we will show that this result also applies under a weaker assumptions.

Below we first start with a few reservations about the proposed condition – see (9). This condition has a simple economic meaning: the customer j entering into loosing transactions cannot achieve a net increase in his utility of the losses. For example, if for fixed choices of all other customers in the retail chain, the utilities $\pi_{wj}(\ldots, X_j, \ldots)$ for $w \in X_j$ are independent of the choice X_j, the condition (9) hold as strict inequalities. These conditions are also reduces to strict inequalities when, for instance, the capable sales ξ_{wj} in each transaction between customer j and supplier $w \in A^j$ is not less than the demand η_{wj} so that every customer can receive the entire quantity ordered from his suppliers. In particular, by increasing the producers' supply $s_1, s_2, \ldots, s_k$ with unlimited manufacturing capacity, we can always increase the capable sales to such an extent that it exceeds the demand, so that the conditions (9) are satisfied.

We can now formulate the final conclusion: the following lemma suggests that each customer will make his choice so as to maximize the profit Π_j, providing all the other customers keep their choices fixed. [7]

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[7] The joint choice of users having this property is generally interpreted in the sense of Nash equilibrium (1953); see also Owen (1968).

Let the suppliers not entering the set A_j be assigned indexes $q = 1,2,\ldots$. Then the profit Π_j of customer j is represented by a many-variable function $\Pi_j(t_{1j}, t_{2j},\ldots)$ with variables t_{qj} varying on $[0, \beta_{qj}]$. [8] The value of the function $\Pi_j(t_{1j}, t_{2j},\ldots)$ is the customer's profit for the case when the customer j has extended the choice by placing orders in the amounts of $t_{qj} \cdot d_j \cdot \nu_{qj}$ with the suppliers $q = 1,2,\ldots$ outside the choice A_j. Thus, the customers j who expand their choice A_j, identify the suppliers $q = 1,2,\ldots$ by the set of variables t_{qj}. If all $t_{qj} = 0$, the choice A_j is not expanded and the profit $\Pi_j(0,0,\ldots)$ coincides with (7).

The profit function $\Pi_j(t_{1j}, t_{2j},\ldots)$ thus has to satisfy the following constraint: for every t_{qj} in $[0, \beta_{qj}]$ $q = 1,2,\ldots$

$$\Pi_j(t_{1j}, t_{2j},\ldots) \leq \Pi_j(0,0,\ldots). \tag{9}$$

Definition. *A joint choice $\langle A_o^{k+1},\ldots, A_o^n \rangle$ of the retail chain customers is said to be rational with the threshold u^o if, given an amount of transaction costs not less than $u^o > 0$, the utility measure $\pi_{wj} \geq u^o$ in every transaction of customer j with the supplier $w \in A_o^j$, $j = k+1,\ldots,n$.*

Lemma. *The set-theoretically largest choice $S^o = \langle A_o^{k+1},\ldots, A_o^n \rangle$ among all the joint choices rational with threshold $u^o > 0$ ensures that the retail-type distribution chain is in equilibrium relative to the individual profit criterion Π_j under the following conditions: a) the transaction costs u_{wj} for $w \in S^o$ do not exceed min π_{wj} over $w \in S^o \bigcap R_j$; b) inequality (9) holds.*

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[8] We recall that β_{qj} is the fractional cost of all the orders placed with supplier q.

Proof. Let S^o be a set-theoretically largest choice among all the joint choices rational with the threshold u^o, i.e., S^o is the largest choice H among all the choices such that $\pi_{wj}(H \bigcap R_{k+1},\ldots,H \bigcap R_n) \geq u^o$. Suppose that some customer *p* achieves a profit higher than Π_p by making the choice $A^p \subseteq R_p$, which is different from $S^o \bigcap R_p$; $\Pi_p' = \sum_{w \in A^p} \left(\pi_{wp}(\ldots, A^p, \ldots) - u_{wp}\right) > \Pi_p$, subject to $u^o \leq u_{wp} \leq \min_{w \in A^p} \pi_{wp}$. Clearly, the choice A^p is not a subset of S^o, since this would contradict the monotone property (8), so that $A^p \setminus S^o \neq \varnothing$. By the same monotone property, the customer making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ will achieve a profit not less than Π_p'. On the other hand, all transactions in $A^p \setminus S^o$ are losing transactions for this customer, since S^o is the set-theoretically largest set of non-losing bilateral trade agreements tolerant towards the transactions costs' threshold $u^o > 0$. For the customer p making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ the profit Π_p' does not decrease only if the total increase in utility due to the contribution π_{wp} of the transactions $w \in S^o \bigcap R_p$ exceeds the total negative utility due to the transactions in $A^p \setminus S^o$. Clearly, because of the constraint (9), the customer p has no such an opportunity. This contradiction establishes the truth of the lemma. ■

In conclusion, we would like to consider yet another point. With uncertain transaction costs, the refusal to enter into any transaction may lead to an undesirable "snowballing" of refusals by customers to choose their suppliers. It therefore seems that customers will attempt at least to conclude transactions with $\pi_{wj} \geq u^o$; even when there is some risk that the transaction costs will exceed the utility π_{wj}. Thus, without exaggeration, we may apparently state that the size of the interval $[u^o, \min \pi_{wj}]$ reflects the elasticity of the customer's choice: the number

$\min \pi_{wj} - u^{o}$ is thus a measure of a "risk" that the customer will get into non-equilibrium situation. Clearly, a customer with a small interval will have greater difficulties to maintain the equilibrium than a customer with a wide interval.

5. Rational Coalitions in Monotonic Games

In many-persons games (Owen, 1971) by a coalition we shall understand a subset of participants. Among all coalitions we usually single out rational coalitions — a participant in such coalition extracts from the interaction in the coalition a benefit, which satisfies him. In addition, sometimes it is further stipulated that extraction of this benefit is ensured independently of the actions of the players not entering into the coalition.

The class of games proposed in this paper is subjected to an additional monotonic condition, which has been studied earlier in Mullat (1976, 1977) (although knowledge of the latter is not presupposed). There is no difference between the formal scheme of the present paper and that of Mullat in essence; the difference involved in interpretation is in abstract indices of interconnection of elements of the system, which are understood as utility indices. The approach developed enables us to establish, in one particular case, the possibility of finding rational coalitions in the state of individual equilibrium according to Nash.

6. Formal Definitions and Concepts

We consider a set of n players denoted by I. Each player $j \in I$ $(j = \overline{1,n})$ is matched by a set R_j from which the player j can select elements. It is assumed that the sets R_j are finite and do not intersect. Their union forms a set $W = R_1 \cup R_2 \cup \ldots \cup R_n$. The elements selected by the player j from R_j compose a set $A^j \subseteq R_j$. The set A^j is called the choice of the player j, while the collection $\langle A^1, A^2, \ldots, A^n \rangle$ is called the joint choice. The case $A^k = \varnothing$ is not excluded and is called the refusal of k-th player from the choice.

We introduce the utility functions of elements $w \in A^j$. We assume that certain joint choice $\langle A^1, A^2, ..., A^n \rangle$ has been carried out. Let there be uniquely determined, with the respect to the result of the choice, a collection of numbers $\pi_w \geq 0$ that are assigned to the elements $w \in A^j, j = 1,2,...,n$; on the remaining elements of W the numbers are not determined. The numbers π_w are called utility indices, or simply utilities, and by definition, are in general case functions $\pi_w(X_1, X_2, ..., X_n)$ of n variables. The value of the variable X_j is the choice A^j of the player j. We shall single out utility functions possessing a special monotonic property.

Definition 1. *A set of utilities* π_w *is called monotonic, if for any pair of joint choices* $\langle L^1, L^2, ..., L^n \rangle$ *and* $\langle G^1, G^2, ..., G^n \rangle$ *such that* $L^j \subseteq G^j,\ j = 1,2,...,n$

$$\pi_w(L^1, L^2, ..., L^n) \leq \pi_w(G^1, G^2, ..., G^n) \tag{10}$$

is fulfilled for any $w \in L^j$[9].

We now turn to the problem of coalition formation. We shall call any nonempty subset of the set of players a coalition. Let there be given a coalition V, and let its participants have made their choices. We compose from the choices A^j of the participants of the coalition V a set-theoretic union H, which is called the choice of the coalition V: $H = \bigcup_{j \in V} A^j$.[10]

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[9] We note that fulfilment of (1) is not required for the element $w \notin L^j$. Furthermore, even the numbers π_w themselves may not be defined for $w \notin L^j$.

[10] A choice H without indication about the coalition V, which has affected it, is not considered, and if somewhere the symbol V is omitted, then under a coalition we understand a collection of players such and only such for which $H \cap R_j \neq \varnothing$.

To determine the degree of suitability of the selection of an element $w \in R_j$ for the player j, a participant of the coalition, we introduce an index of guaranteed utility. With this aim we turn our attention to the dependence of the utility indices on the choice of the players not entering into coalition. It is not difficult to note that as a consequence of the monotonic condition of the functions π_w the worst case for the participants of the coalition will be when all players outside the coalition V reject the choice: $A^k = \varnothing$, $k \notin V$, so that all elements outside H will not be chosen by any of the players who are capable of making their choices. In other words, the guaranteed (the least value) of utility π_w of an element *w* chosen by a player in the case of fixed choices $H \cap R_j$ of his partners in the coalition equals $\pi_w(H \cap R_1,\ldots,A^j,\ldots,H \cap R_n)$.

The quantity

$$g_j(H) = \min_{w \in A^j} \pi_w(H \cap R_1,\ldots,A^j,\ldots,H \cap R_n)$$

is called the guarantee of the participant j in the coalition V for the choice H.

We assume that according to the rules of the game, for each chosen element $w \in A^j$ a player $j \in V$ must make a payment u°. [i] It is obvious that under condition of the payment u° the selection of each element $w \in A^j$ is profitable or at least without loss to the player $j \in V$ if and only if $\pi_w \geq u^\circ$. In the calculation for the worst case this thus reduces to the criterion $g_j(H) \geq u^\circ$. In reality we shall be interested, in relation to the player $j \in V$, in all three possibilities: a) $g_j(H) > u^\circ$, b) $g_j(H) = u^\circ$ and c) $g_j(H) < u^\circ$. We shall say that a participant of the coalition V is above u°, on the level of u°, and below u°, if the conditions a), b), and c) are fulfilled respectively. The size of the payment is further considered as a parameter u of the game being described and is called the threshold. We shall say that a coalition V, having made a choice H, functions on the level $u[H] = \min_{j \in V} g_j(H)$.

Definition 2. *A coalition* V *is called rational with the respect to a threshold* $u^\circ = u[H]$ *if for a certain choice* H *all participants of the coalition are not below* u° *while someone in the coalition* $k \cup V$ *is below* u° *if any participant* $k \notin V$ *outside the coalition* V *makes a nonempty choice* $A^k \neq \varnothing$.

The set of numerical values being attained by the function $u[H]$ on rational coalitions will be called the spectrum. Each value of the function $u[H]$ will be called the spectral level (or simply the level). The entire construction described above will be called a monotonic parametric game on W.

Subsequently we will be interested in rational coalitions functioning on the highest possible spectral level. It is obvious that the spectrum of each monotonic game on a finite set W is bounded, and therefore there exists a maximum spectral level $u^\mu = \max_{H \subseteq W} u[H]$.

Definition 3. *A rational coalition* V^* *such that for a certain choice* H^* *the level* u^μ: $u[H] = u^\mu$ *is attained is called the kernel of the monotonic parametric game on* W.

Theorem 1. *If* V_1^* *and* V_2^* *are kernels of the monotonic game on* W, *then one can always find the minimum kernel (in set-theoretic sense)* V_c^* *such that* $V_c^* \supseteq V_1^* \cup V_2^*$. *The proof is presented in the appendix.*

Theorem 1 asserts that the set of kernels in the sense indicated by the binary operation of coalitions is closed. The closeness of a system of kernels allows as looking at the largest (in the set-theoretic sense) kernel, i.e., a kernel $K^\ominus$ such that all other kernels are included in it. From the Theorem 1 it follows the existence of the largest kernel in any finite monotonic parametric game.

The rest of the paper is devoted to the description of constructive methods of setting up coalitions that are rational with the respect to the threshold u°, including those rational with the respect to the threshold u^μ, i.e., the kernels coalitions. In particular, a method of constructing the largest kernel is suggested.

7. SEARCH OF RATIONAL COALITIONS

We consider a monotonic parametric game with n players. Below we bring together a system of concepts, which allows us constructively to discover rational coalitions with respect to an arbitrary threshold u° if they exist. In the monotonic game only a limited portion of subsets of the set W have to be searched in order to discover the largest rational coalition. With this aim in the following we study coalitions V whose participants do not refuse from a choice: for $j \in V$ the choice $A^{j} \neq \varnothing$. Such a coalition, which has affected a choice H, is denoted by $V[H]$. From here on, for the motive of simplicity of notation of guaranteed utility $\pi_{w}(H \cap R_{1},\ldots,A^{j},\ldots,H \cap R_{n})$, where H is a subset of the set W, we use $\pi(w;H)$.

Definition 4. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_{0},\alpha_{1},\ldots,\alpha_{m-1}\rangle$ *(*m *is the number of elements in* W*) from* W *is said to be in concord with respect to the threshold* u°*, if in a sequence of subsets of the set* W

$$\langle N_{0},N_{1},\ldots,N_{m-1},N_{m}\rangle,$$

where $N_{0}=W$, $N_{i+1}=N_{i}\setminus\alpha_{i}$, $N_{m}=\varnothing$, *there exists a subset* N_{p} *such that*:

a) *The utility* $\pi(\alpha_{i};N_{i})<u^{\circ}$ *for all* $i<p$;

b) *For each* $w \in N_{p}$ *the condition* $u^{\circ} \leq \pi(w;N_{p})$ *is fulfilled, or, this being equivalent, for each* $j \in V(N_{p})$ *the condition* $u^{\circ} \leq g_{j}(N_{p})$ [11] *is fulfilled.*

A sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, uniquely defines the set N_{p}. This fact is written in the form $N(\overline{\alpha})=N_{p}$.

[11] By definition $g_{j}(N_{p})=\min_{w\in N_{p}\cap R_{j}}\pi(w;N_{p})$.

Definition 5. *A set* $S^\circ \subseteq W$ *is said to be in concord with the respect to a threshold* u°, *if there exists a sequence* $\overline{\alpha}$ *of elements of* W, *in concord with respect to the threshold* u° *and such that* $S^\circ = N(\overline{\alpha})$, *while the coalition* $V(S^\circ)$ *is said to be in concord with respect to the threshold* u°.

The following two statements are derived directly from Definitions 4 and 5.

A. In the case where the set $S^\circ = W$ is in concord with the respect to the threshold u°, all players $j \in I$ are not below u°: $g_j(W) \geq u^\circ$.

B. If the set S°, in concord with the respect to the threshold u°, is empty, then there exists a chain of constructing sets

$$\langle N_0, N_1, \dots, N_{m-1}, N_m \rangle,$$

such that for each player $j \in I$, commencing with a certain N_t, in all those coalitions $V(N_i)$, $t \leq i$, where the player j enters, this player is below u°.

Theorem 2. *Let* S° *be a set that is in concord with respect to the threshold* u°. *Then any rational coalition* V *functioning on the level not less than* u° *makes a choice* H, *which is a subset of the set* S°: $H \subseteq S^\circ$. The proof is given in the appendix.

Corollary 1. *The set* S°, *in concord with respect to the threshold* u°, *is unique*. Indeed, if we assume that there exists a set S', in concord with the respect to the threshold u° and different from S°, then from theorem 2, $S' \subseteq S^\circ$. But analogously at the same time the inverse inclusion $S' \supseteq S^\circ$ must also be satisfied, which bring us to conclusion that $S' = S^\circ$.

Corollary 2. *As the spectral levels of functioning of coalitions in the monotonic parametric game grow, one can always find a chain of rational coalitions, included in one another and being in concord with respect to each increasing spectral level, as with respect to the growing threshold.*

Indeed, from the formulation of the theorem it follows that a rational coalition, in concord with the respect to a spectral level $\lambda < \mu$, satisfies the relation $V(S^{\lambda}) \supseteq V(S^{\mu})$, since in a set-theoretic sense $S^{\lambda} \supset S^{\mu}$.

Below we arrange a certain sequence $\overline{\alpha}$, which use up all elements of W. After the construction we formulate a theorem about the sequence $\overline{\alpha}$ thus constructed being in concord with respect to the threshold u°. The arrangement proves constructively the existence of a sequence of elements of W that is necessary in the formulation of the theorem.

Construction. Initial Step.

Stage 1. We consider a set of elements W. Among this set we search out elements γ_0 such that $\pi(\gamma_0; W) < u^{\circ}$, and order them in any arbitrary manner in the form of a sequence $\overline{\gamma}_0$. If there are no such elements, then all elements of W are ordered arbitrarily in the form of a sequence $\overline{\alpha}$, and the construction is completed. In this case W is assumed to be the set $N(\overline{\alpha})$.

Stage 2. Subsequently we examine the sequence $\overline{\gamma}_0$. When considering the t-th element $\gamma_0(t)$ of this sequence $\overline{\gamma}_0$, the sequence $\overline{\alpha}$ is supplemented by the element $\gamma_0(t)$, which is denoted by the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_0(t) \rangle$, while the set W is replaces by $W \setminus \overline{\alpha}$. After the last element of $\overline{\gamma}_0$ is examined we go over to the recursive step of the construction.

Recursive Step k.

Stage 1. Before constructions of the k-th step there is already composed a certain sequence $\overline{\alpha}$ of elements from W. Among the set $W \setminus \overline{\alpha}$ we seek out elements γ_k such that $\pi(\gamma_k; W \setminus \overline{\alpha}) < u^{\circ}$ and order them in any arbitrary manner in the form of a sequence $\overline{\gamma}_k$. Analogously to the initial step, if there happen to be no elements γ_k, the construction is ended. In this case in the role of the set $N(\overline{\alpha})$ we choose $W \setminus \overline{\alpha}$ while $\overline{\alpha}$ is completed in an arbitrary manner with all remaining elements from W.

Stage 2. Here we carry out constructions, which are analogous to stage 2 of the initial step. The entire sequence of elements $\overline{\gamma}_k$ is examined element by element. While examining the t-th element $\gamma_k(t)$ the sequence $\overline{\alpha}$ is complemented in accordance with the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_k(t) \rangle$. After examining the last element $\gamma_k(t)$ of the sequences $\overline{\gamma}_k$ we return to stage 1 of the recursive step.

On a certain step p, either initial or recursive, at stage 1 there are no elements γ, which are required by the inequalities (2) or (3), and the construction could not continue any more.

Theorem 3. *A sequence $\overline{\alpha}$ constructed according to the rules of the procedure is in concord with the respect to the threshold* u°. The proof is presented in the appendix.

In the current section, in view of the use, as an example, of the concepts just introduced, we consider a particular case of a monotonic parametric game in which the difference in the individual and cooperative behavior of the participants of the coalition is easily revealed. We assume that the utilities

$$\pi_w(A^1,\ldots,A^{j-1},X_j,A^{j+1},\ldots,A^n)$$

do not depend on X_j in the case that choices specified by the remaining players are fixed. In this case the j-s participant of the coalition V, under the condition that the remaining participants of it keep their choices, can limit his choice X_j to a single element $w' \in R_j$ on which the maximum guarantee $\max_{w' \in R_j} g_j(H)$ is attained. However, such a selection narrowing his choice down to a single-element, generally speaking, reduces the choice (in view of monotonicity of utility indices π_w) to the guarantee of the remaining participants of the coalition. Consequently, individual behavior of the participants of a coalition contradicts their cooperative behavior. In spite of this contradiction, in the general case, in the given case, using the concept of a rational coalition $V(S^{\circ})$ in concord with respect to the threshold u°, and having slightly modified the criteria of "individual interests" of the players, we can convince someone that there always exists a situation in which the individual interests do not contradict the coalition interests.

We define the winnings of the j-th participant of the coalition in the form of the sum of utilities after subtraction of all payments u°, i.e., as the number

$$f_j(H) = \sum_{w \in A_j} \left[\pi(w;H) - u^\circ\right]$$

(the winnings f_k for $k \notin V$ are not defined). Having represented H as a joint choice $\left\langle A^1, A^2, \ldots, A^{|V|} \right\rangle$, we can consider the behavior of each j-th participant as player in a certain non-cooperative game selecting a strategy A^j.

The situation of individual equilibrium in the sense of Nash (Owen, 1971) of the participants of the coalition V in the game with winnings f_j is defined as their joint choice $\bigcup_{j \in V} A_*^j = H^*$ such that for each $j \in V$

$$f_j(A_*^1, \ldots, A_*^{j-1}, A^j, A_*^{j+1}, \ldots, A_*^{|V|}) \le f_j(H^*)$$

for any $A^j \subseteq R_j$. In other word, the situation of equilibrium exists if none of the participants of the coalition has any sensible cause for altering his choice A_*^j under the condition that the rests keep to their choices.

Not every choice H of participants of the coalition V is an equilibrium situation. To see this it is sufficient to consider a choice H such that in the coalition V there are players having chosen elements $w \in A^j$ with utilities $\pi(w;H) < u^\circ$; for the selection of such an element the player pays more than this element brings in winnings $f_j(H)$ and, therefore, for the player, proceeding merely on the basis of individual interests, it would be advantageous to refrain from selection of such elements. Refraining from the selection of such elements of the set H is equivalent to non-equilibrium of H in the sense of Nash.

Lemma. *Let the utilities* $\pi(w;H)$ *be independent of* A^j. *Then a joint choice* S° *of the participants of the rational coalition* $V(S^\circ)$, *in concord with the respect to the threshold* u°, *is a situation of individual equilibrium.*

Indeed, according to Theorem 2, S° is the largest choice in the set-theoretic sense among all choices H of the rational coalition $V(S^\circ)$, where for any $w \in H$ the relation $\pi(w;H) \geq u^\circ$ is fulfilled. Let the choice of the participants of the coalition with an exception of that of the j-th participant be fixed. Since the utilities $\pi(w;S^\circ)$ do not depend on A^j, the j-th participant of $V(S^\circ)$ cannot secure an increase in the winnings $f_j(S^\circ)$ either by broadening or by narrowing his choice in comparison with $R_j \cap S^\circ$.

8. Coalitions functioning on the highest spectral level

We consider the problem of search of the largest kernel. First of all we present some facts, which are required for the solution of this problem.

From the definition of the guarantee $g_j(H)$ of the participant j effecting the choice H we see that the equality

$$g_j(H) = \min_{w \in A^j} \pi(w;H))$$

is fulfilled. Hence, according to the definition of the level $u[H]$ of functioning of the coalition $V(H)$ it follows that

$$u[H] = \min_{w \in H} \pi(w;H)$$

If we carry out a search of the subset H^* of the set W on which the value of the maximum of the function $u[H]$ is achieved, then thereby the search of a coalition functioning on the highest level $u^\mu = u[H]$ of the spectrum of a monotonic parametric game is affected. Without describing the search procedure, we give the definition of a sequence of elements W allowing us to discover the largest (in the set-theoretic sense) choice H^* of the largest coalition – a kernel K^*.

Definition 6. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_0, \alpha_1, ..., \alpha_{m-1} \rangle$ *(m is the number of elements in* W *) from* W *is called the defining sequence of the monotonic game, if in the sequence of sets* [12]

$$\langle N_0, N_1, ..., N_{m-1}, N_m \rangle$$

there exists a subsequence $\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \rangle$ *such that:*

a) *for any element* $\alpha_i \in \Gamma_k \setminus \Gamma_{k+1}$ *of the sequence* $\overline{\alpha}$ *the utility* $\pi(\alpha_i; N_i) < u[\Gamma_{k+1}]$ $(k = 0,1,..., p-1)$;

b) *in the rational coalition* $V(\Gamma_p)$ *no sub-coalition exists on a level above* $u[\Gamma_p]$.

From the Definition 6 one can see that the defining sequence in many ways is analogous to a sequence, which is in concord with the respect to the level u°. Since any rational coalition $V(\Gamma_k)$ functions on the level $u^k = u[\Gamma_k]$, it is not difficult to note that the defining sequence $\overline{\alpha}$ composes strictly increasing spectral levels $u[\Gamma_0] < u[\Gamma_1] < ... < u[\Gamma_p]$ of functioning of rational coalitions $V(\Gamma_k)$ in the monotonic parametric game. As a result, we require yet another formulation.

Definition 7. *A rational coalition* $V \subseteq I$ *is said to be determinable, if there exists a defining sequence* $\overline{\alpha}$ *of elements* W *such that among the choices of this coalition there is a choice* Γ_p *composed by* $\overline{\alpha}$ *according to Definition 6.*

Theorem 4. *For each monotonic parametric game a determinable coalition exists and is unique. Among the choices of the determinable coalition there is a choice on which the highest spectral level* u^μ *is attained.* The proof of the theorem is presented in the appendix.

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12 The given sequence is constructed exactly in the same way as the one in Definition 4.

Corollary to Theorem 4. *The concepts of a determinable coalition and the largest kernel are equivalent.*

Indeed, directly from the formulation of the Theorem 4 we see that a determinable coalition always is the largest kernel. Hence, since a determinable coalition always exists, while the largest kernel is unique, it follows that the largest kernel coincides with the determinable coalition.

Thus, the problem of search of the largest kernel is solved if we construct a defining sequence $\overline{\alpha}$ of elements W. The construction of $\overline{\alpha}$ can be effected by the procedure of discovering kernels (KFP) from Mullat. In conclusion we present yet another approach to the concept of "stability" of a coalition. [13]

Definition 8. *A coalition* $\hat{V}$ *is said to be a critical, if for a certain choice* $\hat{H}$ *of it no coalition* V *having a nonempty intersection with the coalition* $\hat{V}$ *functions on a level higher than* $u[\hat{H}]$. *The level* $\hat{u} = u[\hat{H}]$ *is called the critical level of the coalition* $\hat{V}$, *while the choice* $\hat{H}$ *is called its critical choice.*

From the Definition 8, in particular, it follows at once the uniqueness of the critical level of the coalition $\hat{V}$. Indeed, on the contrary, if were two different levels $\hat{u}'$ and $\hat{u}''$, $\hat{u}' < \hat{u}''$, then $\hat{u}'$ could not be a critical one according to the definition: it is sufficient to consider the coalition $V = \hat{V}$ itself with the choice $\hat{H}''$, which ensures $\hat{u}'' > \hat{u}'$.

It is obvious that kernels are critical coalitions. The inverse statement, generally speaking, is not true; a critical coalition is not necessarily a kernel.

We now consider the following hypothetical situation. Let $\hat{V}$ be a critical coalition and let $\hat{H}$ be its critical choice. We assume that this coalition is rational with respect to the threshold u°; i.e., $u^{\circ} \le u[\hat{H}]$ (see Definition 2). We assume that an increase of the threshold u° up to

[13] This approach is close to the concept of "M-stability" in cooperative n-person games, G. Owen.

the level $u^\circ > u[\hat{H}]$ took place and the critical coalition $\hat{V}$ with the critical choice $\hat{H}$ was transformed into unstable coalition with respect to the higher threshold u°. Let the participants of the coalition $\hat{V}$ preserving the stability of the coalition attempt to increase their guarantees. One of the possibilities for increasing the guarantee of a participant $j_0 \in \hat{V}$ is to refrain from the choice of an element $\alpha_0 \in A^{j_0}$ on which the value $g_{j_0}(H)$ - the minimum level of utility guaranteed for him, see (4), is attained. It is natural to assume that a participant with a level of guarantee $g_{j_0}(\hat{H}) = u[\hat{H}] < u^\circ$ will be among the participants attempting to increase their guarantees, and refrains from the selection of the element α_0 indicated above. It may happen that the refusal of α_0 gives rise, for another participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, to a decrease from his guarantee $g_{j_1}(\hat{H}) > u[\hat{H}]$ to the quantity $g_{j_1}(\hat{H} \setminus \alpha_0) \le u[\hat{H}]$. A participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, acting from the same considerations as j_0, refrains from the selection of an element α_1 on which $g_{j_1}(\hat{H} \setminus \alpha_0)$ is attained. Such a refusal of α_1 can give rise to subsequent refusals, and emerges hereby a chain of “refusing” participants $\langle j_0, j_1, \ldots \rangle$ of the coalition $\hat{V}$.

If a coalition V, rational with respect to the threshold u° in the sense of Definition 2, with the choice H became unstable as the threshold u° increases, then such a coalition, generally speaking, disintegrates; i.e., some of its participants may become participants of a new coalition which already is rational with the respect to the increased threshold u°. By definition of a critical coalition, transaction of its participants into new rational coalition, when the threshold u° increases is not possible, and it disintegrates completely. The theorem presented below and proved in the appendix reflects a possible character of complete disintegration of a critical coalition in terms of the hypothetical system described above.

Theorem 5. *Let there be given a critical coalition* $\hat{V}$ *having a non-empty intersection with a certain coalition* $V: \hat{V} \cap V \neq \varnothing$. *Let* H *be the choice of the coalition* V *and* $\hat{H}$ *the critical choice of the coalition* $\hat{V}$. *Then in the coalition* $\hat{V} \cap V$ *there exists a sequence of its participants* $\bar{j} = \langle j_0, j_1, \ldots, j_{r-1} \rangle$ *such that: a) in the sequence* $\bar{j}$ *there are represented all participants of the coalition* $\hat{V} \cap V$ *(the players* j_i *may be repeated,* r *is number of elements in* $\hat{H} \cup H$; *b) for the sequence* $\bar{j}$ *we can construct a chain of contracting coalitions*

$$\langle V(N_0), V(N_1), \ldots, V(N_{r-1}) \rangle,$$

where $N_0 = \hat{H} \cup H$, $N_{i+1} \subset N_i$, *so that for any* $j \in V$, *commencing from a certain* N_t, *in all those coalitions* $V(N_i)$, $t \leq i$, *into which the player* j *enters, this player is not above* $u[\hat{H}]$.

Appendix

Proof of Theorem 1. Let the level u^{μ} be attained for the coalitions V_1^* and V_2^*, which effect the choices H_1^* and H_2^* respectively; i.e., $u^{\mu} = u[H_1^*]$ and $u^{\mu} = u[H_2^*]$. For player $j \in I$ we consider two choices: $H_1^j = H_1^* \cap R_j$ and $H_2^j = H_2^* \cap R_j$ [14]. By the definition of guarantee $g_j(H_1^*)$ for the participant $j \in V_1^*$ of the coalition we have

$$\min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, \ldots, H_1^n) = g_j(H_1^*) \geq u^{\mu}; \qquad \text{(A1)}$$

for the participant $j \in V_2^*$ we respectively have

$$\min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, \ldots, H_2^n) = g_j(H_2^*) \geq u^{\mu}. \qquad \text{(A2)}$$

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[14] We note that, in the worst case, for player $k \notin V_1^*$ ($k \notin V_2^*$), $H_1^k = \varnothing$ ($H_2^k = \varnothing$).

We determine the choice of a participant $j \in V_1^* \cup V_2^*$ as $\Phi^j = H_1^j \cup H_2^j$. The monotonic property (1) allows us to conclude that the following inequalities are valid:

$$\begin{aligned}&\min_{w \in H_1^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq \\ &\triangleright \min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, ..., H_1^n)\end{aligned}; \tag{A3}$$

$$\begin{aligned}&\min_{w \in H_2^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq \\ &\geq \min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, ..., H_2^n)\end{aligned}. \tag{A4}$$

Combining (A1) – (A4), we obtain

$$\min_{w \in \Phi^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq u^\mu \tag{A5}$$

for any $j \in V_1^* \cup V_2^*$. If by Φ^* we denote the set $H_1^* \cup H_2^*$, then for the coalition $V_1^* \cup V_2^*$ affecting the choice Φ^* the inequality (A5) is rewritten in the form

$$g_j(\Phi^*) \geq u^\mu,\ j \in V_1^* \cup V_2^*. \tag{A6}$$

Due to the monotonic property (1) some elements $w \notin \Phi^*$ (if one can find such) may be added to Φ^* while the inequality (A6) is still true[15]. We will denote the enlarged set by Φ^c: $\Phi^c \supseteq \Phi^*$ and obviously for $V^c = V(\Phi^c)$ we have $V(\Phi^c) \supseteq V_1^* \cup V_2^*$. By the definition of a spectral level u^μ, for the participant $j' \in V^c$, on which $u[\Phi^c]$ is attained, we have

$$g_{j'}(\Phi^c) = u[\Phi^c] \leq u^\mu, \tag{A7}$$

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15 We suppose that such elements cannot be added to Φ^c.

since u^{μ} is the maximum spectral level of functioning of coalitions in the monotonic game. Applying (A7) and (A6) to the choice Φ^{c} for the participant $j = j'$, we see that $g_{j'}(\Phi^{c}) = u^{\mu}$, and the coalition $V^{c} \supseteq V_1^{*} \cup V_2^{*}$ functions on the spectral level u^{μ}. The theorem is proved. ■

Proof of Theorem 2. Let S° is a subset of the set W in concord with the respect to the threshold u°; i.e., there exists a sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, such that $S^{\circ} = N(\overline{\alpha})$. We assume that there exists a coalition V affecting a choice $H \subset S^{\circ}$ and functioning on the level $u[H] \geq u^{\circ}$, $H \setminus S^{\circ} \neq \varnothing$. Let $\alpha_1 \in H \setminus S^{\circ}$ and let α_t be an element, which is leftmost in the sequence $\overline{\alpha}$. Let p be the index of the set N_p in the sequence $\left\langle N_0, N_1, \ldots, N_{m-1}, N_m \right\rangle$. It is obvious that $t < p$ and, consequently,

$$\pi(\alpha_t; N_t) < u^{\circ} \tag{A8}$$

in accordance with a) of the Definition 4. Since the game being considered is monotonic, $\alpha_t \in H$ and $H \subseteq N_t$ there must hold

$$\pi(\alpha_t; H) \leq \pi(\alpha_t; N_t). \tag{A9}$$

From inequalities (A8) and (A9) it follows

$$\pi(\alpha_t; N_t) < u^{\circ} \leq u[H] \tag{A10}$$

(the latter $\leq$ by assumption). According to the inequality (A10) and by the definition of $u[H]$ we have

$$\pi(\alpha_t; H) < \min_{j \in V} g_j(H). \tag{A11}$$

Let the element α_t be chosen by a certain q-th player; i.e., $\alpha_t \in A^{q}$, $q \in V$. On the basis of (A11) we assume that

$$\pi(\alpha_t; H) < g_q(H) \tag{A12}$$

is valid. By definition $g_q(H) = \min_{w \in A^q} \pi(w;H)$ and following (A12), we note that $\pi(\alpha_t;H) < \min_{w \in A^q} \pi(w;H)$. The last inequality is contradictory, what proves the theorem. ■

Proof of Theorem 3. We assume that the construction of the sequence $\overline{\alpha}$ according to the rules of the procedure ended on a certain p-th step. This means that $\overline{\alpha}$ is made up of sequences $\overline{\gamma}_k$ $(k = \overline{0,p})$, and also of elements of the set N_p, found according to the rules of the procedure and being certainties for the sequences $\overline{\gamma}_k$. We consider any element α_i of the sequence thus constructed, being located on the left of the α-th element: $i < p$. The given element in the construction process falls into certain set $\overline{\gamma}_q$. By construction

$$\pi(\alpha_i; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} < u^{\circ}. \qquad \text{(A13)}$$

If to the sequence $\langle \overline{\gamma}_0, \overline{\gamma}_1, \ldots, \overline{\gamma}_{q-1} \rangle$ we add the elements $\overline{\gamma}_q$, which in $\overline{\alpha}$ are on the left of the α_i-th, then this set of elements together with the added part $\overline{\gamma}_q$ composes the complement $\overline{N}_i$ up to the set W (see Definition 4).

On the basis of the monotonic property (1) we conclude that $\pi(\alpha_i; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} \geq \pi(\alpha_i; W \setminus \overline{N}_i) = \pi(\alpha_i; N_i)$. The last relation in the combination with (A13) shows that $\pi(\alpha_i N_i) < u^{\circ}$. From the construction of the sequence $\overline{\alpha}$ it is also obvious that for any $j \in V(N_p)$ the guarantee $g_j(N_p) \geq u^{\circ}$. The theorem is proved. ■

Proof of the Theorem 4. Theorem can be proved as follows. First, a sequence $\overline{\alpha}$, in concord with respect to the highest spectral level u^{μ}, in the monotonic game exists, according to Theorem 3, and is, at the same time, a defining sequence; as the subsequence $\left\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \right\rangle$ in this case we have to choose the sequence $\left\langle W, S^{\mu} \right\rangle$, where S^{μ} is a set $S^{\mu} \subset W$ which is in concord with respect to the highest level u^{μ}. The determinable coalition is $V(S^{\mu})$. The uniqueness of the coalition $V(S^{\mu})$ is proved in Corollary 1 to the Theorem 1. Secondly, the choice S^{μ} of the coalition $V(S^{\mu})$, playing the part of the set Γ_p in the Definition 6, attains the maximum of the function $u[H]$, a fact which follows from Theorem 3 and b) of Definition 6; i.e., $u[S^{\mu}] = u^{\mu}$. Thirdly, the last statement of Theorem 4 is a particular case of the statement of Theorem 2, if we put $u^{\circ} = u^{\mu}$. The theorem is proved. ■

Proof of the Theorem 5. We consider a monotonic game of participants of a coalition $\hat{V} \cup V$ on the set $\hat{H} \cup H$, where $\hat{H}$ is the critical choice of the critical coalition $\hat{V}$, and H is some choice of the coalition V. Below the set $\hat{H} \cup H$ is denoted by Ω, while all concepts refer to a monotonic sub-game on Ω.

Let u° be the threshold of the parameter u of the game on Ω, and let $u^{\circ} > u[H]$. We construct a sequence $\overline{\alpha}$ of elements Ω, which is in concord with respect to the threshold u°. Two variants could be represented: 1) the set S°, in concord with the respect to the threshold u° is empty; 2) S° is not empty. We consider them one after the other. First, in the variant 1) from a sequence of elements $\overline{\alpha}$ of elements of Ω in concord with respect to the threshold u°, we uniquely determine a sequence of participants of the coalition $\hat{V} \cup V$ choosing elements α_i from sequence $\overline{\alpha}$ and composing a certain chain $\overline{j} = \left\langle j_0, j_1, \ldots, j_{r-1} \right\rangle$ (r is the number of elements Ω). Secondly, from the sequence $\overline{\alpha}$ we also uniquely determine the sequence of coalitions $\left\langle V(N_0), V(N_1), \ldots, V(N_{r-1}) \right\rangle$, where $N_0 = \Omega$, $N_{i+1} = N_i \setminus \alpha_i$, with $j_i \in V(N_i)$.

In the second variant none of the participants of the coalition $\overline{V}$ can be in a coalition, which is in concord with the respect to the threshold $u^{\circ} > u[H]$. This would contradict the definition of a critical coalition $\overline{V}$. Therefore in the chain $\bar{j}$ thus constructed of participants of the coalition $\hat{V} \cup V$ (by the same method as in the first variant) all participants of the coalition $\overline{V}$ are on the left of the j_p-th player; p is uniquely determined from the sequence $\overline{\alpha}$ (see Definition 4). By property a) of the Definition 4 and from the definition of the guarantee of a player $j_i \in V(N_i)$ we have

$$g_{j_i}(N_i) \leq \pi(\alpha_i; N_i) < u^{\circ}. \tag{A14}$$

Proceeding from the structure of the spectrum of a monotonic parametric game on Ω (see Corollary 2 to the Theorem 2) the value u° marginally close to $u[H]$ is satisfied successfully in the two variants considered. The first variant of the Theorem 5 forms the statement b) derived earlier from Definition 4 and 5 (see section 2). The second variant of the statement of the theorem is directly derived from the relation (A14). ■

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[i] In his book review of "Ménard, C. and M.M. Shirley. (eds., 2005) *Handbook of New Institutional Economics*, Springer: Dordrecht, Berlin, Heidelberg, New York. XIII. 884 pp., Rudolf Richer, University of Saarland, noticed that

> *North and Williamson stress, besides transaction costs, the role of bounded rationality, uncertainty, and imperfect rationality. Their objects of research differ: Northian NIE focuses on macro institutions that shape the functioning of markets, firms, and other modes of organizations such as the state (section II) and the legal system (section III). Williamsonian NIE concentrates on the micro institutions that govern firms (section IV), their contractual arrangements (section V), and issues of public regulation (section VI). Both the Northian and Williamsonian approaches to the NIE are used, i.e., in development and transformation economics: in efforts towards explaining the differences of exchange-supporting institutions (section VIII).*

It is worth to emphasize, in view of the above, that when the player $j \in V$ must make a payment u° for the element $w \in A^{j}$, the payment is well suited in the role of transaction cost, see below.

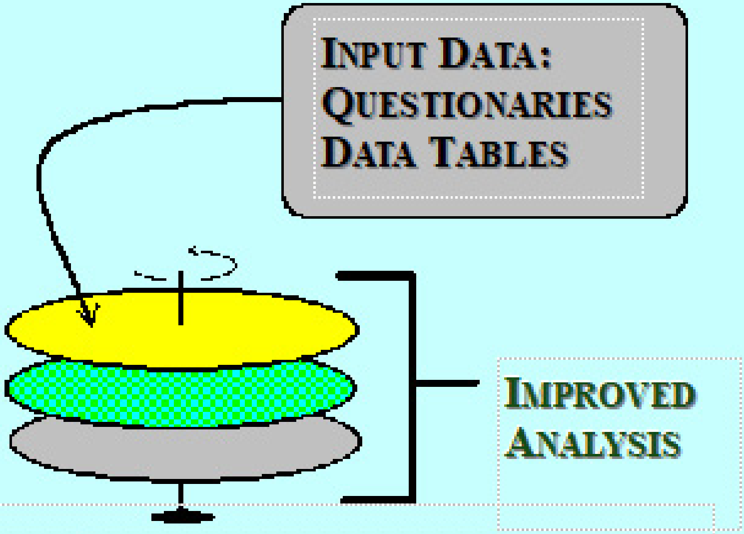
> *In economics and related disciplines, **a transaction cost** is a cost incurred in making an economic exchange. For example, most people, when buying or selling a stock, must pay a commission to their broker; that commission is a transaction cost of doing the stock deal. Or consider buying a banana from a store; to purchase the banana, your costs will be not only the price of the banana itself, but also the energy and effort it requires to find out which of the various banana products you prefer, where to get them and at what price, the cost of travelling from your house to the store and back, the time waiting in line, and the effort of the paying itself; the costs above and beyond the cost of the banana are the transaction costs. When rationally evaluating a potential transaction, it is important to consider transaction costs that might prove significant.*

FINAL REMARKS

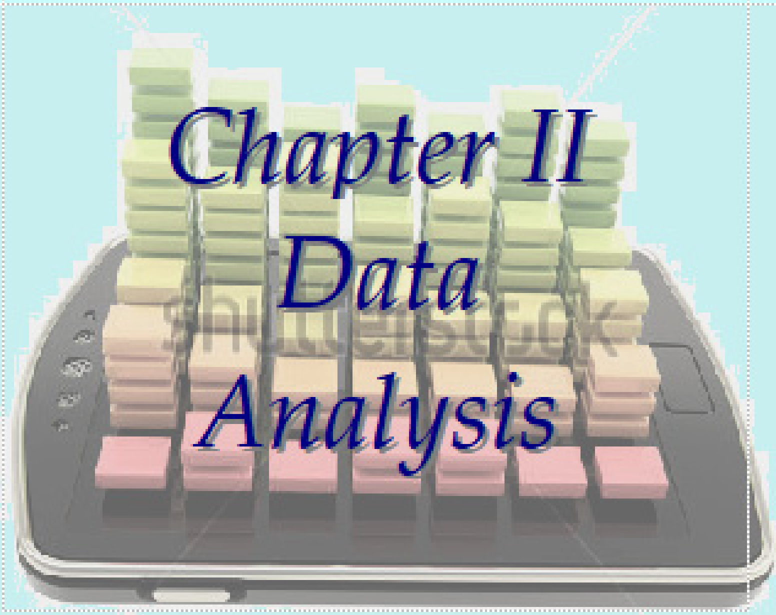
It ends where we started. The paper investigated a situation of distributing commodities in the retail chain with participants making "*to buy and sales*" decisions in a retail chain. One type of participants' produce and sale, others buy and sale, the third only buy for consumption. The price system was set up via some constants, which are nothing but percentages to perform calculus of how the sales price must depend and exceed the purchasing prices to archive a satisfactory results for participants maximizing their profits. The situation becomes complex as soon as to buy and sale decisions incorporated transaction costs. Transaction costs interact into the behavior of participants by transforming potentially profitable into loosing transactions. The paper investigated the situation, as global, depending on the transaction costs' threshold varying the threshold from low to high values until all transactions, allegedly profitable in bilateral trade agreements, became loosing and do not any more form a basis for an agreement between rational participants. The retail chain structure, while the transaction costs' threshold is increasing, changes like nested set of retail chains each of them on the higher threshold is capable to counteract higher transaction costs and still functioning in equilibrium. Condition for such a rational behavior was that all participants in the retail chain must avoid any loosing transaction. Beyond the goal of the retail chain formation to hold the retail chain in equilibrium, some elasticity intervals for transaction costs, where it still was realistic to buy and sale rationally, have been internally encoded into the scheme and calculated individually for all participants in the chain.

ACKNOWLEDGMENT

The Section 5 is a rewritten and expanded version of an article that appeared in "Automation and Remote Control", ISSN 005-1179 (Mullat, 1980, p.7), Russian periodical of MAIK Наука, Интерпериодика. In the expanded version, the introduction reflects the relationship with the new institutional economics, while final remarks point at elements of the theory of firm formation, as it has now become clear to the author. Nevertheless, significant amendments to the terminology have also been made in the main version of the article in English, which was originally distributed by Plenum Publishing Corporation.



Chapter II
Data Analysis



On The Maximum Principle for Some Set Functions [1]

Abstract. This article discusses the problem of finding extreme points for functions defined on all subsets of some large or general finite set. The construction method leads to the detection of extreme subsets. The main feature of the construction method is based on the assumption that on each subset, and for each of its elements, some numbers are given, i.e. credentials or weights, satisfying the monotonicity conditions p.1 and p.2.
Keywords: classification; graphs; convex functions; algorithm

1. INTRODUCTION [NB!]

In our study, we consider the problem of finding the global extremum of a function defined on all subsets of a given finite set. The described construction algorithm was used to solve some problems of object classification using the technique of homogeneous Markov chains. In general terms, the proposed construction allows one to solve some problems on graphs, for example, to single out, in a sense, "connected" subsets of the vertices of the graph. We formulate the theoretical foundations of our construction in terms of transparent rules for choosing subsets in a given finite set and some sequences of the same elements of a finite set. The result will be extracting the extreme subsets.

The types of problems of similar nature have a combinatorial character and do belong mostly to the discrete programming problems. Cherenin (1962), Cherenin and Hachaturov (1965) have successfully solved a pre-

[1] This idea at the moment, perhaps invisible from the first glance, is incorporated into "Left- and Right-Wing Political Power Design" as political parties bargaining game. Reg. "data analysis", see also, J. E. Mullat (1976-1977) Extremal Subsystems of Monotonic Systems, I,II,III, Automation and Remote Control, 37, pp. 758-766, 37, pp. 1286-1294; 38. pp. 89-96.

eminent class of similar problems on the finite sets. In the framework of these papers a functions have been considered satisfying condition, which can be formulated as follows. If ω_1 and ω_2 are two representatives for subsets of a given finite set then

$$f(\omega_1) + f(\omega_2) \le f(\omega_1 \cup \omega_2) + f(\omega_1 \cap \omega_2).$$

This condition with some reservation reflects the convexity of the function f.

The main property or requirement for the class of functions considered in the manuscript is the assumption of the existence of some numbers or weights that reveal for each element of a finite set the degree of its occurrence in the subset. The degree of occurrence must satisfy conditions (1) and (2), see below.

Concerning the current investigation it is worthwhile also to pay attention to Mirkin's (1970) work. In this work, a problem of optimal classification is reduced to finding special "painting" on a non-ordered graph. The optimal classification there is characterized by some maximum value of a function, corresponding in its form to the definition (1), however hereby we interpret (1) in a different sense. We do not consider in our function definition a decomposition of a given set into two non-intersecting subsets what was the main concern of Mirkin's work.

2. THE MODEL

Let $\{H\}$ is a set of subsets of some finite set W. Suppose that we introduce a π_H function for each set $H \subseteq W$ of its elements as arguments. Below by the collection $\{\pi_H\}$ we entitle a system of weights on the set H. The main supposition concerning the weight systems $\{\{\pi_H\}\}$ is as follows:

p.1 the credential $\pi_H(\alpha)$ of the element $\alpha \in H$ is a real number.

p.2 Following dependencies inhere between different credential, i.e., credential systems for different subsets of the set M: for each element $\alpha \in H$ and each $\beta \in H \setminus \{\alpha\}$ yields that
$\pi_{H\setminus\alpha}(\beta) \le \pi_H(\alpha)$.

In other words, following p.2, the requirement is that a removal of an arbitrary element α from a set H results in a new credential system $\left\{\pi_{H\backslash\alpha}\right\}$ and the effect of the removed element α on the credentials within the remaining part $H\backslash\{\alpha\}$ is only towards the direction of a decrease. We explain these two conditions by examples from the graph theory, although there are examples from other jurisdictions, however less convenient for a short discussion. Let consider non-oriented graphs, i.e., graphs with the property when a relation of a vertex x to y implies a reverse relation of vertex y to x.

Example 1. [2] [3]

Let W is a vertex set of a graph G. We define a credential system $\left\{\pi_H\right\}$ on each subset of vertexes H as a collection of numbers $\left\{\pi_H(\alpha)\right\}$, where the number $\pi_H(\alpha)$ is equal to the number of vertexes in H related to the vertex α. The truthfulness of the pp. 1 and 2 is easily checked, if one only remembers to recall that together with the removal of a vertex α all connected to it edges have to be removed concurrently.

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2 Kempner Y., Mirkin B. and I. Muchnik (1997) have given another example in Monotone Linkage Clustering and Quasi-Convex Set Functions, Appl. Math. Letters, v. 10, issue no. 4, pp. 19-24. Mirkin B. and I. Muchnik. (2002) Layered Clusters of Tightness Set Functions, Applied Mathematics Letters, v. 15, issue no. 2, pp. 147-151.

3 Yet another examples, Kuznetsov E.N. and I.B. Muchnik, Moscow (1982) Analysis of the Distribution Functions in an Organization, Automation and Remote Control, Plenum Publishing Corporation, pp. 1325-1332; Kuusik R. (1993) The Super-Fast Algorithm of Hierarchical Clustering and The Theory of Monotonic Systems, Data Processing, Problems of Programming, Transactions of Tallinn Technical University, No. 734, pp. 37-61; Mullat J.E., (1995) A Fast Algorithm for Finding Matching Responses in a Survey Data Table, Mathematical Social Sciences 30, pp. 195-205; Genkin A.V. and I. B. Muchnik (1993) Fixed Approach to Clustering, Journal of Classification, Springer, 10, pp. 219-240,.

Example 2.

Let W is a set of edges in a graph G or the set of pairs of vertexes related by the graph G. We define a credential system $\{\pi_H\}$ on arbitrary subset H of edges in the graph G as a collection of numbers $\{\pi_H(\alpha)\}$, where $\alpha \in H$ and $\pi_H(\alpha)$ is a number of triangles in the set of edges H, containing the edge α. The number $\pi_H(\alpha)$ is equal to the number of those vertexes on which the set H resides such, that if x is a pointed vertex and the edge $\alpha = [b,e]$, then it ensues that $[b,x] \in H$ and $[e,x] \in H$.

In the examples, we have exploited the fact, that a graph is a topological object from one side and a binary relation from the other side. Let now consider the following set function

$$f(H) = \min_{\alpha \in H} \pi_H(\alpha), \tag{1}$$

where $H \subseteq W$. We suggest below a principle, valid for the subset H, on which the global maximum of a type (1) function is reached. We formulate this principle in terms of some sequences of the set W elements and the sequences of the subsets of the same set W.

Let $\overline{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ is a sequence of elements of the set W and $k = |W|$. We define using the sequence $\overline{\alpha}$ a sequence of sets $\overline{H}(\overline{\alpha}) = \{H_0, H_1, \ldots, H_{k-1}\}$: as $H_0 = W$ and $H_{i+1} = H_i \setminus \{\alpha_i\}$.

Definition 1. We call a sequence of elements $\overline{\alpha}$ from the set W a defining sequence, if in the sequence of sets $\overline{H}(\overline{\alpha})$ there exists a sub sequence $\overline{\Gamma} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_p\}$ such that:

1°. The credential $\pi_{H_i}(\alpha_i)$ of an arbitrary element, belonging to Γ_j, but not belonging to Γ_{j+1}, is strictly less than $f(\Gamma_{j+1})$;

2° In Γ_p there do not exists such a strict subset L that $f(\Gamma_p) < F(L)$.

Definition 2. We call a subset H of the set W a definable, if there exists a defining sequence such that $H = \Gamma_p$.

Below, we simply refer to the notification $\{\pi_H\}$ as a credential system with respect to the set H.

Theorem. On the definable set H the function $f(H)$ reaches its global maximum. The definable set is unique. All sets, where the global maximum has been reached, lie within the definable set.

Proof. Let H is a definable set. Assume, that there exists L such that $f(H) \le f(L)$. Suppose that $L \setminus H \ne \varnothing$, [4] otherwise we have just to proof the uniqueness of H, what we will accomplish below. Let H_t is the smallest from the sets H_i $(i = 0,1,\dots,k-1)$, which include in it the set $L \setminus H$. From this fact one can conclude, that there exists an element $\ell \in L$ such, that $\ell \in H_t$, but $\ell \notin H_{t+1}$. Moreover, in combination with $L \setminus H \ne \varnothing$ the last conclusion ensues $t < p$. Inequality $t < p$ disposes to an existence of at least one a subset in the sequence of sets $\overline{\Gamma}$ such, that

$$\pi_{H_t}(\ell) < f(\Gamma_j) \qquad (2)$$

and $j \ge t+1$. Since $\ell \notin H_{t+1}$ and $\Gamma_j \subseteq H_{t+1}$ are true, it follows that $\ell \notin \Gamma_j$. Thus, the inequality

$$f(\Gamma_j) \le f(\Gamma_p) \qquad (3)$$

is valid as a consequence of the property 2° for the defining sequence.

Now, let $\ell \in L$ and the credential $\pi_L(\ell)$ is at the minimum in credential system with the respect to the set L. Inequalities (2) and (3) allow us to conclude, that $\pi_{H_t}(\ell) < \pi_L(\ell)$. Above we selected H_t on the condition that $L \subset H_t$. Hereby, recalling the main property p.2 of the credential system (the removal of elements), it is easily to establish that $\pi_L(\ell) \le \pi_{H_t}(\ell)$, i.e., in the credential system with the respect to the set

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[4] Here $\varnothing$ symbolizes an empty set.

L, there exists a credential, which is strictly less than the minimal. We came to a contradiction and by this, we have proved that on H the global maximum has been reached. Further, all such sets, different from H, where the global maximum is likewise reached, might really be located within H. It remains to be proved the uniqueness of the definable set. In connection of what we proved above, one might suppose that a definable set H' is located within H, however, proceeding with the line of reasoning towards H' similar to those we proposed above for L, we conclude, that $H \subset H'$. ■

Corollary. Let $\{R\}$ is a system of sets, where the function of type (1) reaches its global maximum. Hereby, if $H_1 \in \{R\}$ and $H_2 \in \{R\}$ are valid, then $H_1 \cup H_2 \in \{R\}$.

Proof. Following the p.2 (the main property) $f(H_1) \leq f(H_1 \cup H_2)$, but in addition $f(H_1 \cup H_2) \leq f(H_1)$, consequently $H_1 \cup H_2 \in \{R\}$. ■

Below we introduce an actual algorithm for constructing the defining sequences of elements of a set W. For the availability of the algorithm is exposed in the form of a block-scheme similar to some extent of a computer program.

3. ALGORITHM [5]

a.1. Let the set $R = W$ and sequences $\overline{\alpha}$ and $\overline{\beta}$ [6] be empty sets in the beginning, and let the index $i = 0$.

a.2. Find an element μ at the least credential with the respect to the set R, record the value $\lambda = \pi_R(\mu)$ and constitute $\overline{\alpha} = \overline{\alpha}, \overline{\beta}, \mu$ and thereafter $\overline{\beta} = \varnothing$.

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[5] Further developments, see Muchnik, I., and Shvartser, L. (1990) Maximization of generalized characteristics of functions of monotone systems, Automation and Remote Control, 51, pp. 1562-1572,

[6] Hereby $\overline{\beta} = \{ \beta_1, \beta_2, ..., \beta_i, ...\}$

a.3. Exclude the element μ from the set R and take into account the influence of the removed element $\mu \in R$ on remaining elements, i.e., recalculate all values $\pi_{R\backslash\mu}(\beta)$ for all $\beta \in R \setminus \{\mu\}$.

a.4. In case, among the remaining elements there exist such γ, that

$$\pi_{R\backslash\mu}(\gamma) \le \lambda \tag{4}$$

compose a sequence from those elements $\overline{\gamma} = \{\gamma_1, \gamma_2, \dots, \gamma_s\}$ and substitute $\overline{\beta} = \overline{\beta}, \overline{\gamma}$.

a.5. Substitute the set $R = R \setminus \{\mu\}$ and the element $\mu = \beta_{i+1}$. Return to the a.3 in case the element β_{i+1} is the element for the sequence $\overline{\beta}$ increasing in this moment the index i by one.

a.6. In case, when the sequence $\overline{\alpha}$ has utilized the whole set W, the construction is finished. Otherwise, return to a.2 initializing first $i = 0$.

Let us prove that the sequence $\overline{\alpha}$ just constructed by the proposed algorithm is defining. We consider the sequence $\overline{H}(\overline{\alpha})$ and let one selects in the role of the sequence $\overline{\Gamma}$ those sets, which start by the element μ found at the moment the algorithm is crossing the step a.2. The fact of crossing the a.2 of the algorithm guarantees, that the condition (4) is not valid before the cross was occurred, and the element β_{i+1} is not in the sequence β at this stage. The above guarantees as well the condition 1° fulfillment for the defining sequences. Suppose, that the condition 2° in the definition 1 do not hold, i.e., in the last set Γ_p in the sequence $\overline{\Gamma}$, there exists such a subset L, that $f(\Gamma_p) < f(L)$. Let us consider the sequence $\overline{\beta}$, which is generated at the last crossing through the a.2 of the above-described algorithm and let λ_p symbolize the highest value among all such λ. One has to conclude, that $\lambda_p = f(\Gamma_p)$, and, from the supposition of an existence of a set L, we come to the inequality $\lambda_p < f(L)$.

By the construction, the sequence $\overline{\alpha}$ and together with the sequence $\overline{\beta}$ (both of them), which is generated at last crossing though the a.2 of the algorithm has utilized all elements in W. Consequently, we can consider a set of elements K in the sequence $\overline{\beta}$, which start from the first confronted element $\ell \in L$, where $L \subset K$. On the basis justified above, we have $\pi_K(\ell) = \lambda_p$ and, recalling the main property of the credential system p.2 (the removal of elements), we conclude moreover that $\pi_L(\ell) \leq \lambda_p$. We reached to a contradiction and by that we have proved the property 2° of the definition 1 for the sequence $\overline{\alpha}$. On that account, the construction of defining sequences is possible by the pointed above algorithm.

We emphasize the necessity of concretizing the notion of credential system with the respect to a subset of a given finite set for solving some of the pattern recognition problems, what should be the subject for further investigation.

In conclusion, we will point out, that the construction of defining sequences has been realized in practice on a computer for one problem in graph theory, related to an extraction of "almost totally connected" subgraphs in a given graph. The number of edges in such graphs has been around 10^4.

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NB! In his work "Cores of Convex Games" Shapley investigated a class of n-person's games with special convex (supermodular) property, International Journal of Game Theory, Vol. 1, 1971, pp. 11-26. When writing current paper, in that time in the past, the author was not familiar with this work and could not predict the close connection between the basic monotonicity property pp.1-2, see above, and that of supermodular characteristics functions in convex games induce the same property upon marginal utilities. We are going to explain the connection. We will consequently do it in Shapley's own words to make the idea crystal clear.

The core of a n-person game is the set of feasible outcomes that cannot be improved upon by any coalition of players. A convex game is one that is based on a convex set function; intuitively this means that the incentives for joining a coalition increase as the coalition grows, so that one might expect a "snowballing" or "band-wagon" effect when the game is played cooperatively… In Shapley's paper a coalition game is a function v mapping a Ring of subsets from some set called a grand coalition $\mathcal{N}$ to the real numbers, satisfying $\mathrm{v}(\varnothing)=0$. *The function* v *is supperadditive if*

$\mathrm{v(S)+v(T)\le v(S\cup T)}$, i.e., all $\mathrm{S,T}\in\mathcal{N}$, with $\mathrm{S\cap T}=\varnothing$. It is convex if $\mathrm{v(S)+v(T)\le v(S\cup T)+v(S\cap T)}$ for all $\mathrm{S,T}\in\mathcal{N}$, p.12.

In the standard form in game theory, the elements of $\mathcal{N}$ are "players", the subsets of $\mathcal{N}$ are "coalitions"; $\mathrm{v(S)}$ is called the "characteristic function", which gives each coalition the best payoff that it can get without the help of other players.

Supper-additivity arises naturally in this interpretation, but convexity is another matter. For example, in voting situation S and T, but not $\mathrm{S\cap T}$, might be winning coalitions, causing "convexity" to fail. To see what convexity does entail, consider the function m:

$$\mathrm{m(S,T)=v(S\cup T)-v(S)-v(T)},$$

as defining the "incentive to merge" between disjoint coalitions S and T. Then it is a simple exercise to verify that convexity is equivalent to the assertion that $\mathrm{m(S,T)}$ is no decreasing in each variable – whence the "snowballing" or "band wagon" effect mentioned in the introduction.

Another condition that is equivalent to convexity (provided $\mathcal{N}$ is finite) is to require that

$$\mathrm{v(S\cup\{i\})-v(S)\le v(T\cup\{i\})-v(T)}$$

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for all individuals $i \in \mathcal{N}$ and all $S \subseteq T \subseteq \mathcal{N} \setminus \{i\}$. This expresses a sort of increasing marginal utility for coalition membership, and is analogous to *"increasing the returns to scale associated with convex production functions in economics.",* p.13

We return now back from the "expedition" into Shapley's work and make some comments. The latter condition, which is equivalent to convexity, is an exact, we repeat it once again, an exact utilization of our basic monotonicity property pp.1-2. Set functions of this type are also known in the literature as "suppermodular". As it turns out now the author knew such functions. To the knowledge of the author Cherenin was first who introduced functions of this type already in 1948. Nemhauser et al, also used $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ but an inverse property introduced in 1978 for computational optimization problems in "An Analysis of Approximation for Maximizing Submodular Set Functions", Mathematical Programming 14, 1978, 265-294. Shapley also notes the latter inverse property in connection with rank function of a matroid known as "submodular" or "lower semi-modular." Besides, in Nemhauser et al paper, the reader may find the proof of the conditions

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ and}$$
$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \text{ equivalency.}$$

However, the connection between the convex games and the monotonicity property pp.1-2 is invisible. Only recently Genkin and Muchnik pointed out (not in the connection with game theoretical models, but actually in connection with the problems of object classification, see "Submodular Set Functions and Monotone Systems in Aggregation Problems I,II," Translated from Automat. Telemekhanika No.5, pp.135-148, © 1987 0005-1179/87/4805-0679, Plenum Publishing Corporation), that the functions family $\pi_H(\alpha) = v(H) - v(H \setminus \{\alpha\})$ represent a derivatives of suppermodular set functions in the form just exhibited in Shapley's work.

SUMMARIZING

In convex games, following the theory developed in this work from 1971, one can always find a coalition, where it members will be awarded individually at least by some maximum payoff of guaranteed marginal utility, see the Theorem. We call this coalition the largest kernel (nuclei) or the definable set. A good example and its like, is the Example 1. Here, in economic terms, the marginal utility highlights the number of direct dealers with the player $i \in S$ (number of direct contacts, buyers, sellers, direct suppliers, etc.). On the contrary, the Example 2 is not its like and goes beyond the Shapley's Convex Game idea.

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О Принципе Максимума для некоторых Функций Множеств

Резюме. В статье рассматривается задача нахождения экстремальных точек функции, заданной на всех подмножествах конечного множества. Метод построения функции (1) приводит к выделению экстремальных множеств. Основная особенность метода построения основана на предположении, что для каждого элемента α существует набор чисел $\left\{\pi_H(\alpha)\right\}$, где H – подмножество конечного множества и $\alpha \in H$.

1. Введение

В нашем исследовании мы рассматриваем задачу нахождения глобального экстремума функции, заданной на всех подмножествах данного конечного множества. Описанный алгоритм построения применялся для решения некоторых задач классификации объектов с помощью метода однородных цепей Маркова. В общем виде предлагаемая конструкция позволяет решать некоторые задачи на графах, например, выделять в некотором смысле «связные» подмножества вершин графа. Теоретическая основа конструкции формулируется в терминах специальных правил отбора последовательностей подмножеств данного конечного множества и некоторых последовательностей его элементов, результатом которых является извлечение экстремальных подмножеств.

Задачи подобного типа имеют или носят комбинаторный характер и относятся скорее всего к задачам дискретного программирования. Определенный класс подобных задач на конечных множествах успешно решается в работах Черенина (1962), Черенина и Хачатурова (1965). В рамках этих работ рассматривались функции, удовлетворяющие условию, которое можно сформулировать следующим образом. Если ω_1 и ω_2 являются двумя представителями подмножеств данного конечного множества, то

$$f(\omega_1) + f(\omega_2) \le f(\omega_1 \cup \omega_2) + f(\omega_1 \cap \omega_2).$$

Это условие в некоторой степени отражает выпуклость функции f.

Главным свойством или требованием предъявляемым к рассматриваемому в рукописи класса функций является предположение о существовании некоторых чисел или весов (credentials, ed.), выявляющих для каждого элемента конечного множества степень его вхождения в подмножество. Степень вхождения должна удовлетворять условиям пп.1-2 (см. ниже).

Относительно настоящего исследования стоит также обратить внимание на работу Миркина (1970). В данной работе задача оптимальной классификации сводится к поиску специальной «раскраски» на неупорядоченном графе. Оптимальная классификация там характеризуется некоторым максимальным значением функции, соответствующим по своему виду определению (1), однако при этом мы интерпретируем (1) в ином смысле. Мы не рассматриваем в нашем определении функции разбиение заданного множества на два непересекающихся подмножества, что было основной задачей Миркина (cf., Võhandu & Frey, 1966, ed.).

2. Принцип максимума

Пусть $\{H\}$ множество подмножеств некоторого конечного множества W. Предположим, что мы вводим функцию π_H для каждого из элементов $H \subseteq W$ на совокупности подмножеств $\{H\}$ в качестве аргументов. Ниже под набором $\{\pi_H\}$ мы подразумеваем систему весов на множестве подмножеств $\{\{\pi_H\}\}$. Основное предположение относительно весовых систем следующее:

п.1 Весом $\pi_H(\alpha)$ элемента $\alpha \in H$ является действительное число;

п.2 Между различными системами весов $\{\{\pi_H\}\}$ для разных подмножеств $\{H\}$ набора $\{\pi_H\}$, существуют следующие зависимости: для каждого элемента $\alpha \in H$ и каждого $\beta \in H \setminus \{\alpha\}$ справедливо: $\pi_{H\setminus\alpha}(\beta) \leq \pi_H(\alpha)$.

Другими словами, согласно пункту 2, требование состоит в том, чтобы удаление произвольного элемента α из множества H приводило бы к новой системе весов $\{\pi_{H\setminus\alpha}\}$, а влияние удаленного элемента α на веса в оставшейся части $H \setminus \{\alpha\}$ было бы только в направлении уменьшения. Поясним эти два условия на примерах из теории графов, хотя есть и примеры из других областей познания, однако менее удобные для краткого обсуждения. Рассмотрим неориентированные графы, т.е. графы со свойством, когда отношение вершины x к y влечет обратное отношение вершины y к x.

Пример 1.

Пусть W – множество вершин графа G. Мы определяем систему весов $\{\pi_H\}$ на каждом подмножестве H вершин как набор чисел $\{\pi_H(\alpha)\}$, где число $\pi_H(\alpha)$ равно количеству вершин, в H связанных с вершиной α. Истинность пп. 1 и 2 легко проверяется, если только вспомнить, что вместе с удалением вершины α должны быть одновременно удалены все связанные с ней ребра.

Пример 2.

Пусть W это множество ребер в графе G или множество пар вершин, связанных графом G. Определим весовую систему $\{\pi_H\}$ на произвольном подмножестве H ребер в графе G как набор чисел $\{\pi_H(\alpha)\}$, где $\alpha \in H$ и $\pi_H(\alpha)$ – количество треугольников в множестве ребер H, содержащих ребро α. Число $\pi_H(\alpha)$ равно числу тех вершин, на которых находится множество H, такое, что если x вершина указывающая на ребро и ребро $\alpha = [b, e]$, то отсюда следует что $[b, x] \in H$ и $[e, x] \in H$.

В примерах мы использовали тот факт, что граф является топологическим объектом с одной стороны и бинарным отношением с другой стороны. Теперь рассмотрим следующую функцию множества

$$f(H) = \min_{\alpha \in H} \pi_H(\alpha), \tag{1}$$

где $H \subseteq W$. Ниже мы предлагаем принцип, справедливый для подмножества H, на котором достигается глобальный максимум функции типа (1). Сформулируем этот принцип в терминах некоторых последовательностей элементов множества W и последовательностей подмножеств того же множества W.

Пусть $\overline{\alpha} = \{\alpha_0, \alpha_1, ..., \alpha_{k-1}\}$ – последовательность элементов множества W и $k = |W|$. При помощи последовательности $\overline{\alpha}$ задана последовательность множеств $\overline{H}(\overline{\alpha}) = \{H_0, H_1, ..., H_{k-1}\}$, где $H_0 = W$ и $H_{i+1} = H_i \setminus \{\alpha_i\}$.

Определение 1. Назовем последовательность $\overline{\alpha}$ элементов из множества W определяющей, если в последовательности множеств $\overline{H}(\overline{\alpha})$ существует подпоследовательность $\overline{\Gamma} = \{\Gamma_0, \Gamma_1, ..., \Gamma_p\}$ такая, что:

1°. Вес $\pi_{H_i}(\alpha_i)$ произвольного элемента, принадлежащего Γ_j, но не принадлежащего Γ_{j+1}, строго меньше $f(G_{j+1})$;

2° В Γ_p не существует такого строгого подмножества L, что $f(G_p) < F(L)$.

Определение 2. Назовем подмножество H множества W определимым, если существует определяющая последовательность $\overline{\alpha}$ такая, что $H = \Gamma_p$.

Ниже мы вновь воспользуемся набором $\{\pi_H\}$ в виде системы весов по отношению к множеству H.

Теорема. На определимом множестве H функция $f(H)$ достигает своего глобального максимума. Определимое множество единственно. Все множества, в которых достигнут глобальный максимум, лежат в определяемом множестве.

Доказательство. Пусть H определимое множество. Предположим, что существует такое $L \subseteq W$, что $f(H) \leq f(L)$. Предположим, что $L \setminus H \neq \varnothing$; в противном случае нам остается только доказать единственность H, что мы и сделаем ниже. Пусть H_t есть наименьшее из множеств H_i $(i = 0,1,...,k-1)$, включающих в себя множество $L \setminus H$. Из этого факта можно заключить, что существует такой элемент $\ell \in L$, что $\ell \in H_t$, но $\ell \notin H_{t+1}$. Более того, в сочетании с последним $L \setminus H \neq \varnothing$ напрашивается вывод $t < p$. Неравенство $t < p$ располагает к существованию хотя бы одного такого подмножества в последовательности множеств $\overline{\Gamma}$, что

$$\pi_{H_t}(\ell) < f(\Gamma_j) \tag{2}$$

и $j \geq t+1$. Так как $\ell \notin H_{t+1}$ и $\Gamma_j \subseteq H_{t+1}$ верны, то следует, что $\ell \notin \Gamma_j$. Таким образом, неравенство

$$f(\Gamma_j) \leq f(\Gamma_p) \tag{3}$$

справедливо как следствие п. 2° определяющей последовательности.

Теперь пусть $\ell \in L$ и веса $\pi_L(\ell)$ минимальны в системе весов по отношению к множеству L. Неравенства (2) и (3) позволяют сделать вывод, что $\pi_{H_t}(\ell) < \pi_L(\ell)$. Выше мы выбрали H_t при условии, что $L \subset H_t$. При этом, вспоминая основное свойство п.2 системы весов (удаление элементов), нетрудно установить, что $\pi_L(\ell) \leq \pi_{H_t}(\ell)$, т. е. в системе весов по отношению к множеству существует вес, строго меньший, чем минимальный. Мы пришли к противоречию и тем самым доказали, что достигнут глобальный максимум. Далее, все такие множества H, отличные от L, где также достигается глобальный максимум, действительно могут находиться внутри H. Остается доказать лишь единственность определимого множества H. В связи с доказанным выше можно предположить, что некое определимое множество H' находится внутри H, однако, продолжая линию рассуждений, аналогичную предложенной нами выше для L, заключаем, что $H \subset H'$. ■

Следствие. Пусть $\{R\}$ – система множеств, в которой функция типа (1) достигает своего глобального максимума. Тогда, если $H_1 \in \{R\}$ и $H_2 \in \{R\}$, то $H_1 \cup H_2 \in \{R\}$.

Доказательство. Следуя пункту **2°** (основное свойство) $f(H_1) \le f(H_1 \cup H_2)$, а кроме того из $f(H_1 \cup H_2) \le f(H_1)$, следовательно $H_1 \cup H_2 \in \{R\}$. ■

Ниже мы приводим конкретный алгоритм построения определяющих последовательностей элементов множества W. Для доступности алгоритм представлен в виде блок-схемы, похожей в какой-то степени на компьютерную программу.

3. Алгоритм

a1. Пусть множество $R = W$ и последовательности $\overline{\alpha}$ и $\overline{\beta}$ вначале пусты, а индекс $i = 0$. Здесь $\overline{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots\}$, $\overline{\beta} = \{\beta_1, \beta_2, \ldots, \beta_i, \ldots\}$.

a2. Найдите элемент μ с наименьшим весом по отношению к множеству R, запоминаем значение $\lambda = \pi_R(\mu)$ и полагаем после этого $\overline{\alpha} = \overline{\alpha}, \overline{\beta}, \mu$, а затем $\overline{\beta} = \varnothing$.

a3. Исключаем элемент μ из множества R и учитываем влияние удаленного элемента на оставшиеся элементы $\mu \in R$, т.е. вычисляем все велечины $\pi_{R\setminus\mu}(\beta)$ для всех $\beta \in R \setminus \{\mu\}$.

a4. В случае, если среди остальных (оставшихся) элементов найдутся такие γ, что

$$\pi_{R\setminus\mu}(\gamma) \le \lambda \tag{4}$$

то образуем последовательность указанных элементов $\overline{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ и положим $\overline{\beta} = \overline{\beta}, \overline{\gamma}$.

а5. Положим множество $R = R \setminus \{\mu\}$ и элемент $\mu = \beta_{i+1}$ и . возвращаемся к пункту а3 в случае, если элемент β_{i+1} определен для последовательности элементов $\overline{\beta}$, увеличивая в этот момент индекс i на единицу.

а6. В случае, когда последовательность $\overline{\alpha}$ изчерпаала все множество W, построение закончено. В противном случае вернитесь к пункту а.2, полагая сначала индекс $i = 0$.

Докажем, что только что построенная по предложенному алгоритму последовательность $\overline{\alpha}$ является определяющей. Рассмотрим последовательность $\overline{H}(\overline{\alpha})$ и выделим в качестве последовательности $\overline{\Gamma}$ те множества, которые начинаются с элемента, найденного в момент перехода алгоритма через шаг а2. Факт пересечения а2 алгоритма гарантирует, что условие (4) не выполнялось до того, как произошло пересечение, и элемент β_{i+1} не находится в последовательности на данном этапе β. Сказанное выше гарантирует также выполнение условия 1° для определяющих последовательностей. Предположим, что условие 2° в определении 1 не выполнено, т.е. в последнем множестве Γ_p последовательности $\overline{\Gamma}$ существует такое подмножество L, что $f(\Gamma_p) < f(L)$. Рассмотрим последовательность $\overline{\beta}$, которая генерируется при последнем переходе через а2 вышеописанного алгоритма, и пусть λ_p символизирует наибольшее значение среди всех таких λ. Приходится заключить что из предположения о существовании множества L, и замечая что $\lambda_p = f(\Gamma_p)$ приходим к неравенству $\lambda_p < f(L)$. По построению последовательность $\overline{\alpha}$ и вместе с последовательностью $\overline{\beta}$ (обе они), которая генерируется при последнем переходе через а2 алгоритма, использовали все элементы W. Следовательно, мы можем рассматривать множество элементов

K в последовательности $\overline{\beta}$, которые начинаются с первого противостоящего элемента $\ell \in L$, где $L \subset K$. На основании обоснованного выше имеем $\pi_K(\ell) = \lambda_p$, и, вспоминая основное свойство учетной системы п.2 (удаление элементов), заключаем, кроме того, что $\pi_L(\ell) \leq \lambda_p$. Мы пришли к противоречию и тем самым доказали свойство 2° определения 1 для последовательности $\overline{\alpha}$. В связи с этим возможно построение определяющих последовательностей по указанному выше алгоритму.

Подчеркнем необходимость конкретизации понятия системы весов применительно к подмножеству заданного конечного множества для решения некоторых задач распознавания образов, что должно стать предметом дальнейшего исследования.

В заключение отметим, что построение определяющих последовательностей реализовано на практике на ЭВМ для одной задачи теории графов, связанной с выделением «почти полносвязных» подграфов в заданном графе. Количество ребер в таких графах составляет около 10^4.

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Data Structure Opening Method: Methodological Guide *

Abstract. This methodological study deals with hidden or rather unknown method of information processing, although this method has been actively used both in the past and in the last decades. The method has positively proven itself when opening a data structure in order to draw the necessary conclusions on issues related to people and economic activity, production processes, biology, demography, etc.
Keywords: data matrix; layering algorithm; graph; tournament

1. INTRODUCTION

If someone decides to collect data, the following questions must first be answered.

- What information is needed?
- Why is this information needed?
- To what extent are the reasons for gathering information?
- How can decisions be made based on the information gathered and thus influence the investigation process?

If answers are available, then the set of collected *"objects"*, those data, is also defined. For example, information may concern people living in a city, families in a given country, electronic equipment, factories made up of basic production units (objects in the terminology of the guide) etc.

Population information can be composed of a series of indicators that describe the population as a whole, such as the scales against which income is measured. In productive area, indicators determine the technical environment in which, e.g. equipment was manufactured and operated. Naturally, estimates based on the information collected differ from actual estimates. Thus, the researcher may draw incorrect conclusions if the error of the estimate is too large. This guide looks at one possible way to avoid the errors associated with the so-called stratification concept.

Let's give an example of the importance of this concept in information processing: in USA a presidential elections were held in 1932. Literary Digest sent postcards to voters with questions to predict Roosevelt's election to the presidency. Some 10 million postcards had been sent out. The results showed that the forecast made on the basis of the information collected was accurate within 1%. However, the prediction made using exactly the same technique in 1936, contained an error of almost 20%.

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* The original version in Estonian (Mullat, 1977), Protocol No. 9, approved by the TPI Council (Tallinn Polytechnic Institute) on March 15, 1977. TPI currently stands for Tallinn University of Technology – TalTech.

There is a general perception that the "postcard method" introduced a disproportion among voters who return postcards. It turned out that people with higher education and better conditions tended to return more postcards. People with a higher standard of living tended to prefer Roosevelt's competitor during the readiness period, and the forecast of results shifted away from the real thing.

This example shows that when the population is stratified (for example, only voters with higher education and better conditions are observed), a big mistake cannot be avoided. That is, in order to avoid such an error, the researcher must know in advance the subgroups of the population (classification), but usually the identification of subgroups is a complex and voluminous effort, which in turn is associated with the collection of information.

The guide looks at population stratification (classification) methods that currently exist in three types:

a) Methods that take into account the researcher's subjective opinion of the population. This means that classification with exact properties are known or simply assumed;
b) Methods to be used in the absence of any data or hypotheses about existing strategies and their attributes;
c) Methods, which are intended only to visualize a sample of the population in order for the researcher to be able to make a decision on the available strata.

Among methods a), b) and c), only the so-called monotonic layering (Mullat, 1971-1995) or known since then as the “monotonic linkage method” (Kempner et al, 1997) is considered. The last chapters are devoted to the theoretical study of these monotone systems and methods of monotone layering, in particular, on graphs. We do not discuss issues related to the use of standard statistical methods and algorithms. The additional tools and technologies needed for the monotone layering of data, the accompanying terminology and strict nomenclature are explained in the course of the narrative and defined where necessary.

The article consists of an introduction and a section that discusses the main concepts, a total of 8 sections. Section 3 discusses the different types of metric distances between objects to measure the difference between objects in classification problems. Section 4 describes the method itself at an informal level. Section 5 provides a more accurate construction at a precise mathematical level. In Sections 6-7, we consider the application of the method to the study of graphs, in particular, to determine the groups of strong players in tournaments as opposed to weak players. Concluding remarks are provided in Section 8.

2. KEY DEFINITIONS

First, we introduce the reader to the terminology and basic concepts used. The basic concept of data processing is a data matrix. The data X is a $n \times m$ matrix (n row and m columns), each row of which is called an object; one column of the matrix is called an attribute. This means that the data matrix is

$$X = \begin{Vmatrix} x_{1,1}, x_{1,2}, .., x_{1,m} \\ x_{2,1}, x_{2,2}, .., x_{2,m} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n,1}, x_{n,2}, .., x_{n,m} \end{Vmatrix}$$

and $x_{i,j}$ is the value of the j-th attribute of the i-th object. It is natural that the question immediately arises as to what the numerical values of the attribute in the data matrix reflect? There must be brands that the attributes may differ substantially. For example, the air temperature may be a characteristic when electric lamps are lit; the shoe number of the person; gender (male or female), etc. As the processing is formally based on mathematical apparatus, three types of attributes are distinguished in order to be able to interpret the final results and use them according to the purpose:

a) Attributes on a continuous scale (Interval scale), such as body credential, height, temperature (quantitative);
b) Attributes on a discrete ordinal scale, such as the grades a student receives in some subjects: unsatisfactory, satisfactory, good, and very good. At this point, the values of the attribute are considered ordered (in Points or ranked);
c) Attributes with discrete values that are not ranked (nominal scale or even qualitative attributes), For example, eye color, gender (male or female).

Quantitative attributes. The quantitative expression of an attribute is usually referred to as the value of the attribute can be compared. Questions about how many times the value of one attribute is greater than another can be answered. At first glance, the question does not seem to be very complicated, although a deeper examination in turn raises the question: "What is natural to compare?" Let's look at some more examples before answering this question.

Let us choose the cars that are described by the price tag. Undoubtedly, the attribute "price" is quantitative, the a car with the price of 1O.OOO€, is twice as expensive as the b car with the price of 5.000€. The characteristic "price" or "value" expressed by the function $f(a)$ can also be expressed by the function $\kappa \cdot f(a)$) (κ is a positive number). Every other type of conversion changes the price ratio of cars. The allowed transformations of the attribute “price” are multiplication by the constant κ. This property of the price makes it possible to determine how many times $f(a)$ is greater than $f(b)$ — the ratio $\kappa \cdot f(a) / \kappa \cdot f(b)$ does not depend on κ of the choice, and if κ is fixed, we can thus say how much is $f(a)$ greater than $f(b)$. This class of transformations allows for the universal presentation of concepts related to quantitative as well as other types of attributes. However, the determination of a unit of measurement requires only quantitative attributes

Definition. The permissible transformation of the value of an attribute $f(a)$ in the set of attributes $\mathcal{A}$ is called the function $\varphi(x)$ if the function $\varphi(f(a))$ $\left(a \in \mathcal{A}\right)$ shows the same attribute. If the values of the characteristic f are given together with the number of allowed conversions F, then we say that the measurements of the characteristic were performed on the F-type scale.

In the example of passenger cars $F_o \left\{\kappa \cdot x \,\middle|\, \kappa > 0\right\}$ and on the scale F_0 it is usually said that the measurements are made on a ratio scale. An interval scale is a scale where the number of transformations allowed is $F_x = \left\{\kappa \cdot x + o \,\middle|\, \kappa > 0\right\}$; the specific scale F_x is determined by the quantities κ and o, which give the unit of measurement and the starting point of the scale.

In most cases, the measurement results are presented in the form of a matrix, if after each allowed transformation the measurement results do not change. However, it should be noted that the results of matrix measurements do not allow them to be immediately used in arithmetic calculations. For example, the relationship $f(a) + f(b) > f(c)$ does not make sense in the scale with origin $o > 0$, since $\kappa \cdot \left[f(a) + f(b)\right] + 2 \cdot o$ is greater than $\kappa \cdot f(c) + o$ only for some κ and o values. Indeed, absolute zero is the natural and unambiguous presence of the zero point o that cannot be changed: °0-Kelvin is absolute zero on the scale, which characterizes the absence of the measured feature. However, °0-Celsius or °0-Fahrenheit are not. Two arbitrary physical phenomena are taken here: melting of ice, or an equal mixture of water, ice and salt at -21.1°C. Comparing the mean values of the interval scale is another matter.

Expression

$$\frac{1}{n} \cdot \sum_{i=1}^{n} f(a_i) > \frac{1}{m} \sum_{j=1}^{m} f(b_j) \qquad (1)$$

remains unchanged after using the allowed conversion. Namely

$$\frac{1}{n} \cdot \sum_{i=1}^{n} \kappa \cdot f(a_i) + o > \frac{1}{m} \sum_{j=1}^{m} \kappa \cdot f(b_j) + o$$

iff $$\frac{\kappa}{n} \cdot \sum_{i=1}^{n} f(a_i) + \frac{o \cdot n}{n} > \frac{\kappa}{m} \sum_{j=1}^{m} f(b_j) + \frac{o \cdot m}{m}$$

and the latter is equivalent to inequality (1).

It makes sense to compare the absolute differences in the values of the attributes, namely

$$\frac{|f(a) - f(b)|}{|f(c) - f(d)|} = \frac{|(\kappa \cdot f(a) + o) - (\kappa \cdot f(b) + o)|}{|(\kappa \cdot f(c) + o) - (\kappa \cdot f(d) + o)|}.$$

Now we ask the question what determines the number of allowed transformations $f(x)$? Usually the choice is related to other attributes with the possibility of forecasting. Formally expressed laws of science allow all these forecasting transformations not to change the law. For example, Clipperon's law $P \cdot V / T = \text{const}$ connects the scales of temperature T, volume V and pressure P of a given gas, allows transformation, leaving the law unchanged. Also in economics, in functional models, the price is determined fixed to within a multiplier.

Unknown patterns of relationships, characteristic of sociological or psychological research, allow transformations between objects in the form of empirical relationships, for example, by stratification methods. In these studies, however, interval or ratio scales are unacceptable.

Point or ordinal scales. Pupil assessment aims to test the degree of skill acquisition and achievement of primary education goals on a point scale: Fail (IN – Insuficiente); Pass (SU – Suficiente), Good (BI – Bien), Very Good (NT – Notable), Excellent (SB – Sobresaliente). Point scale gradations are limited by equal intervals of discrete numerical values. Expert judgments are often recorded as a sequence of natural numbers arranged symmetrically to the o point $(o, \pm 1, ...)$.

A distinction should be made between two types of point estimates. In the first case, the assessments reflect some well-known standards. The more opportunities you have to describe and characterize standards, the more accurately you can, for example, determine the deviation from the standard. Thus, the teacher depending on his work experience and personal experience forms the pedagogical level of high school students' performance. On the other hand, refining a benchmark helps predict attribute values; for example, a student who is very good at geometry usually also scores higher in algebra.

The second type of points occurs when there are no well-known standards or even the existence of an objective criterion is questionable, which may be reflected in subjective judgments, for example, the taste of culinary products. This type is also called an ordinal or ordered scale. The set of allowed transformations F consists of all monotonically increasing functions. The ordered values of the attributes are compared only on the basis of the relation "higher-lower". It is meaningless to compare the differences between the values of the attribute. For example, if $f(a) = 10, f(b) = 2, f(c) = 1, f(a) - f(b) = 8, f(b) - f(c) = 1$, $f(a) - f(b) = 8 > f(b) - f(c) = 1$, Then, using the monotonic transformation φ, where $\varphi(1) = 1$, $\varphi(2) = 20$, $\varphi(10) = 30$ gives a contradiction $10 = \varphi(f(a)) - \varphi(f(b)) > \varphi(f(b)) - \varphi(f(c)) = 19$.

It is, nonetheless, realistic to fix the values of original attributes using non-numerical terms. Eligible elements for each ordered set, such as alphabet, etc.

(c) The nominal scale. The scales of the above attributes — quantitative, point and ordinal scales — have general attributes. All scales define the binary relation B on the set of objects X. The relation is defined by the following rule: $(a, b) \in B$ then and only then when $f(a) > f(b)$. Quantitative and point measurements are informatively more voluminous than ordinary measurements, In practice, we can often only be interested in the information contained in the binary relation B. The researcher's conclusions about the functioning of the socio-economic system are usually qualitative (for example, stratification or ranking of objects in a sample).

It is natural to ask the question: is qualitative information not enough to draw conclusions? Qualitative information is easier to measure and more reliable. We do not have the means to accurately measure $f(a)$ and $f(b)$, while we can be sure that $f(a) > f(b)$.

On the other hand, the complex examination of data requires the transformation of the measurement results of individual assessments and objective indicators into a common type of data: quantitative or qualitative.

By limiting the number of transformations F allowed, complex data analysis is usually performed by quantifying all measurements. By limiting the number of transformations allowed sophisticated data analysis is usually performed by quantifying all measurements. Qualitative measurements can "suffer" in this way. When examining qualitative data, it is also possible to do the opposite: to transform quantitative measurements into qualitative ones. It is possible that even then the data will "suffer". However, if the results using quantitative methods are consistent with the results of qualitative data processing methods, the investigator is more likely to be sure of the conclusions reached,

Let the equivalence relation $\mathcal{J}$ be given for the cross product of objects $X \times X$. We assign to each object $x \in X$ the number of the i-th class of X, which contains the object X. Let's say that the measurements are made on a nominal scale, if the value of the attribute is the number of the i-th equivalent relation. Number of conversions allowed by F_n are unique functions. Thus the pair $(a, b) \in \mathcal{J}$ then and only then when attributes values $f(a) = f(b)$. Measurement on a nominal scale is the "weakest" measurement step, as it is only determined whether the equation $f(a) = f(b)$ truly applies.

3. METHODS FOR MEASURING DIFFERENCES BETWEEN OBJECTS

All of the methods that we will discuss in Sections 4-7 relate to some degree to the concept of distance or metric. This means that the task of stratification can be performed accurately only if the distance between objects is determined. Choosing a distance means also comparing distances that measure the similarity of two objects. The higher this number, the more the objects themselves differ, and vice versa.

The distance $\rho(x, y)$ between objects x and y is called a function that satisfies three conditions:

(a) for each x object $\rho(x, x) = 0$;
(b) for each pair (x, y) of objects $\rho(x, y) = \rho(y, x)$;
(c) there is a relationship for each of the three objects (x, y, z) that $\rho(x, y) + \rho(y, z) \geq \rho(y, z)$.

The following is a list of metrics or distances used. The notations are as follows: We denote the i-th, $i = \overline{1, n}$, object of the data matrix X as $x_i = \langle x_{i,1}, x_{i,2}, \ldots, x_{i,m} \rangle$, where $x_{i,j}$, $j = \overline{1, m}$, is the j-th attribute of the object i. The distance between two objects x_k and x_ℓ herein as said is nominated as $\rho(x_k, x_\ell)$.

Here are some of the most commonly used metrics.

Cubic distance:

$$\rho(x_k, x_\ell) = \max_{j=\overline{1,m}} \left| x_{k.j} - x_{\ell,j} \right|,$$

where $\left| \cdot \right|$ indicates an absolute value.

Octahedral distance:

$$\rho(x_k, x_\ell) = \sum_{j=1}^{m} \left| x_{k.j} - x_{\ell,j} \right|.$$

Euclidean distance,

$$\rho(x_k, x_\ell) = \sqrt{\sum_{j=1}^{m} \left(x_{k.j} - x_{\ell,j} \right)^2}$$

These three metrics are mostly useful for an interval scale. The following distance is useful when attributes are measured in points or on an ordinal scale: $\rho(x_k, x_\ell) = \sum_{j=1}^{m} \left| x_{k.j} - x_{\ell,j} \right| \Big/ \sum_{j=1}^{m} \max_{k,\ell} \left(x_{k,j}, x_{\ell,j} \right).$

There are distances that are valid when the attributes are binary. Binary is a sign of "marital" status, e.g. if there can be only two answers — "married-yes" or "married-no". These distances are valid even if the scale is nominal.

Hamming distance.

The notation is borrowed from set theory because objects can be interpreted as subsets of attributes. A value of 1 can be viewed as an indicator $x_{i,j}$ of whether the original attribute j belongs or does not belong to subset x_i. The object x_i is thus a Boolean vector $x_i = \left\langle x_{i,1}, \ldots, x_{i,m} \right\rangle$, where $x_{i,j}$ is the "1"-one or "0"-zero type, $j = \overline{1, m}$.

The absolute distance $\rho(x_k, x_\ell)$ is defined as follows: $\rho(x_k, x_\ell) = m - \left| x_k \cap x_\ell \right|$, which equals the number of missing matches in the objects x_k, x_ℓ. In this case, $\left| x_k \cap x_\ell \right|$ is the number of attributes matches in the data matrix, which takes into account 1-s in both objects x_k, x_ℓ, indicating the same attributes. The relative distance looks like $\rho(x_k, x_\ell) = 1 - \left| x_k \cap x_\ell \right| / \left| x_k \cup x_\ell \right|$, where $x_k \cup x_\ell$ is a set of only those attributes that are present in both x_k, x_ℓ objects, but do not necessarily indicate the same attributes.

The list of distances between objects can be continued, since the possibilities for determining the distances are not limited. It should only be noted that the choice of distances is a process that is difficult to formalize and is usually performed by a researcher based on his/her own experience. Measuring the differences or distances between attributes further complicates matters and differs from the above list. Inter-trait, or correlation coefficient between features/attributes is the most commonly used measure that shows the relative linearity of the change in a second identifier when the first identifier changes. The correlation coefficient c between attributes α, β can be determined using the following expression:

$$c_{\alpha,\beta} = \frac{\sum_{i=1}^{n} x_{i,\alpha} \cdot x_{i,\beta} - \left(\sum_{i=1}^{n} x_{i,\alpha} \cdot \sum_{i=1}^{n} x_{i,\beta} \right) \Big/ n}{\sqrt{\sum_{i=1}^{n} x_{i,\alpha}^2 - \left(\sum_{i=1}^{n} x_{i,\alpha}^2 \right)^2 \Big/ n} \cdot \sqrt{\sum_{i=1}^{n} x_{i,\beta}^2 - \left(\sum_{i=1}^{n} x_{i,\beta}^2 \right)^2 \Big/ n}}$$

In the case of the attributes "no", "yes", it is useful to apply a binary (Pirson's φ) correlation r between objects κ, ℓ in the form of:

$$r_{\kappa,\ell} = \frac{\left| x_k \cap x_\ell \right| \cdot \left| \overline{x}_k \cap \overline{x}_\ell \right| - \left| x_k \cap \overline{x}_\ell \right| \cdot \left| \overline{x}_k \cap x_\ell \right|}{\sqrt{\left| x_\kappa \right| \cdot \left| \overline{x}_\ell \right| \cdot \left| \overline{x}_\kappa \right| \cdot \left| x_\ell \right|}},$$

where $\overline{x}$ is a complement of x; $\left| x_\kappa \right| \cdot \left| \overline{x}_\ell \right| \cdot \left| \overline{x}_\kappa \right| \cdot \left| x_\ell \right| > 0$. Before selecting the distance/correlation between objects, one must perform a Class F independence check of the permitted transformations.

4. Data Layering Algorithm

The reader is probably aware that many models of automatic stratification or objective classification are given and described in the literature. We also know that quite a lot of algorithms of this type have been developed, but due to the lack of access to such knowledge, we independently developed and studied here only one, possibly new for many, method. This method is primarily intended for sociological data, but it can also be used to process the general data matrix X.

Let the information gathered be presented in a form that can depict a large graph. For example, some cities are divided into many quarters. The researcher collects information from the city's residents on movements from one quarter to another. Thus, quarters occur on top of a graph (graph) on the vertices of a graph. The arcs of Graph indicate where the local movements of the population are directed in the city. The task is to find out the movements global trends. So the task is basically in that not to stratify city quarters, but stratify possible directions of movement.

Let's match the number to each arrow (arc) in the graph indicating how many transit paths of length 2 the arrow around gives. Graphically, this means that the number of triangles attached to the arc of the graph has been enumerated, (Fig. 1).

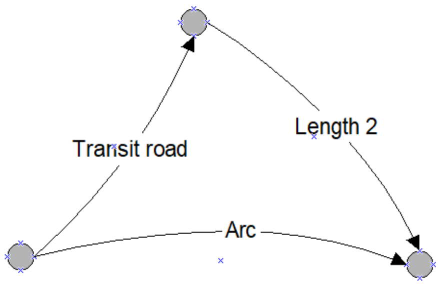


Figure 1

When this is done, the stratification of the arrows (arcs) is completed using the following algorithm. This algorithm was developed by Mullat (1971-1977). Everywhere, if necessary, we will call this algorithm using the abbreviation KSF — "Kernel Searching Routine".

1. **Zero step**

   Find the arc with the least number of triangles on the graph and set it to the value of the parameter u at the level u_0. The arc is removed from the graph. It may be that the removal operation at this point affects some other arcs in the graph and the number of triangles viewed on them changes, so that some other arcs with credentials become less than or equal to u_0. These arcs are also removed. This removal of arc or set of arcs shall be repeated until there are no more arcs whose credentials satisfy the condition: less than or equal to u_0,

2. **Recursive k-th step**

a) From the graph that developed in the previous $k-1$ steps when used, a new minimum credential arc, such as an arc with a minimum number of triangles but higher than previous u_{k-1} is found. The parameter u level u_k, $u_{k-1} < u_k$ of the credential of this arc remembers the level. The arc or arcs found is or are removed from the graph.

b) It may be that the removal operation in current step k affects some more arcs and that their credentials become less or equal to u_k. We repeat this "peeling" until there is no more arcs with credentials less or equal to u_k. All arcs are on some p-th step removed/reset from the graph. This terminates the algorithm.

As a result of the algorithm, all arcs of the graph are distributed into groups or layers, each of which is linked with the corresponding size (threshold) u_k, $k = (\overline{0, p})$. Observing these groups from the last, p-th group, the researcher can draw conclusions about the global or major movement directions on the graph.

Example Let this graph be in Fig. 2,

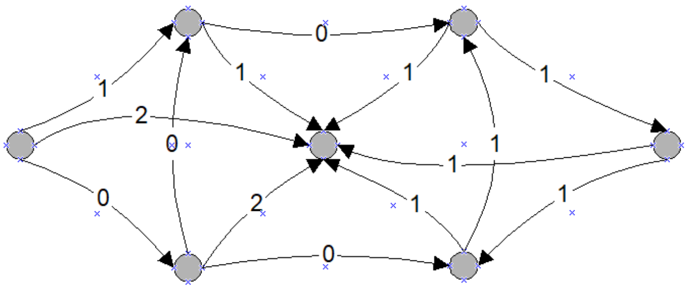


Figure 2.

This figure shows the transit number of routes defined above by Fig. 1 around with the arc in Fig. 2. According to the algorithm the performance of the zero step is the shape of the graph as shown in the Fig. 3.

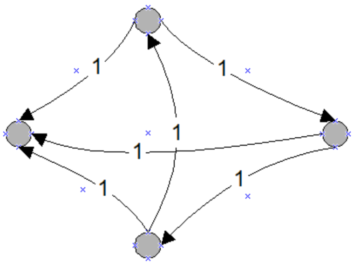


Figure 3

So, above in Fig. 2 it is determined that the given graph has three different 0-arcs. If it were a traffic intensity graph, then there should be two different u_0, u_1 values, or two different traffic layers: 0 and 1, in fact, meaning that the main traffic is possible only for the traffic shown in Fig. 3.

Another way to use the layering algorithm is more complex. An analogous algorithm can also be applied to the % processing (layering) of the data matrix. Only a few new concepts should be defined.

Based on data matrix X, we can create two frequency tables: the rows table and columns table, which will indicate the possible values of the attributes in a nominal scale. The maximum possible number **atr** of different attributes in the data matrix determines the nominal scale width or expansion.

By scanning the cells and at the same time summing the 1-s in the additional tables the two frequency tables $\boldsymbol{c}$ and $\boldsymbol{r}$ are progressively filled out. First, let's look at the corresponding cell of the κ-th object and its ℓ-th attribute in X. The $x_{\kappa,\ell}$ of this cell determines in which additional column $x_{\kappa,\ell}$ to the right of X, and in which additional row $x_{\kappa,\ell}$ at the bottom, in relation to X, the 1-s in cells of $r_{k,x_{k,\ell}}$ and 1-s in cells of $c_{x_{k,\ell},\ell}$ are summed up correspondingly. Namely, in relation to X, here $x_{\kappa,\ell}$ is the column No to the right, but also the row No at the bottom, in additional tables $\boldsymbol{r}$ and $\boldsymbol{c}$. We assume that table X (see example below) is filled with integer attributes or labels 1,2,1,3,… When filling out frequency tables, we initially look at the first object, then the next, and so on.

	Table 1	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**	*´1*	*´2*	*´3*		
	´1	**1**	**1**	**1**	**2**	**1**	**1**	**2**	**0**	5	2	0	7	
	´2	**1**	**1**	**1**	**3**	**1**	**1**	**3**	**3**	5	0	3	8	
	´3	**3**	**2**	**2**	**1**	**3**	**0**	**2**	**2**	1	4	2	7	$\Leftarrow \boldsymbol{r}$
	´4	**1**	**1**	**1**	**2**	**1**	**1**	**3**	**3**	5	1	2	8	
	´5	**1**	**1**	**1**	**0**	**1**	**1**	**2**	**1**	6	1	0	7	
	´1	4	4	4	1	4	4	0	1					
$\boldsymbol{c} \Rightarrow$	*´2*	0	1	1	2	0	0	3	1					
	´3	1	0	0	1	1	0	2	2					
		5	5	5	4	5	4	5	4					

In more compact form, the data cell (κ,ℓ) attribute determines the column No-$x_{k,\ell}$ of frequency $c_{x_{k,\ell},\ell}$ location in the table $\boldsymbol{c}=\left\|c_{t,\ell}\right\|$, $t=\overline{1,atr}$, while the cell (κ,ℓ) also determines the frequency $r_{k,x_{k,\ell}}$ location but in the row No-$x_{k,\ell}$ of table $\boldsymbol{r}=\left\|r_{\kappa,t}\right\|$; i.e. the cell (κ,ℓ) of the data matrix X, points at frequencies: $r_{k,x_{k,\ell}}$ and $c_{x_{k,\ell},\ell}$. Consider the following credentials: $\pi_{\kappa,\ell}=r_{k,x_{k,\ell}}+c_{x_{k,\ell},\ell}+\sum_{t=1}^{atr} r_{\kappa,t}+\sum_{t=1}^{atr} c_{t,\ell}$, where **atr** already has been determined as the nominal scale expansion or width.

Zero step. For all credentials $\pi_{\kappa,\ell}$ the minimum must be found and remembered using the auxiliary variable u_0. In the data matrix X the entry, where the minimum was found, — the κ-th row and ℓ-th column cell of the data table X is reset to zero or marked as processed. Thus, it usually happens that the corresponding cells to κ-th row and ℓ-th column in additional frequencies tables $\boldsymbol{c}$ and $\boldsymbol{r}$ change.

Recursive step. Thus, the reset operation may affect some of the other credentials $\pi_{\kappa,\ell}$ of the data matrix X cells, so that the credentials corresponding to those cells become less than or equal to the minor value u_k. Repeat the current step or steps for matrix X cells with this credential level u_k until no entries (cells) are found in the matrix X that satisfy the reset (zeroing) condition at the k-th step.

It is analogous to the zero step in the graph alignment algorithm. Examples of 5×8 matrix see the Table 1 above. The credential matrix corresponding to the data matrix is as follows:

Table 2	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**
´1	21	21	21	18	21	20	20	16
´2	22	22	22	19	22	21	21	17
´3	18	18	18	15	18	17	17	13
´4	22	22	22	19	22	21	21	17
´5	22	22	22	19	22	21	21	17

After the algorithm has been implemented against Table 2, it performs a transformation of the latter to Table. 3 (the reset cells are marked with the number 99):

Table 3	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**
´1	18	18	18	99	18	18	99	99
´2	18	18	18	99	18	18	99	99
´3	99	99	99	99	99	99	99	99
´4	18	18	18	99	18	18	99	99
´5	18	18	18	99	18	18	99	99

If the result needs to be interpreted essentially, the algorithm offers the researcher, after further investigation, the following interpretation: An area exists inside the data table X or block filled with 3-s labels, which consists of rows $1,2,4,5$ and columns $1,2,3,5,6$.

A similar algorithm can be used for the following two cases. Let's choose the credentials π as a cell value of the data matrix X in the κ-th row and ℓ-th column, which will be

$$\pi_{\kappa,\ell} = \sum_{t=1}^{atr} t \cdot r_{\kappa,t} + \sum_{t=1}^{atr} t \cdot c_{t,\ell} .$$

These types $\pi_{\kappa,\ell}$ of indicators in mechanics are called moments. The credential consists of row moment and column moment sum. We can act exactly according to the algorithm presented earlier.

Another example. The entropy of an object κ can be calculated by formula:

$$H(\kappa) = -\frac{1}{\sum_{t=1}^{atr} r_{\kappa,t}} \sum_{t=1}^{atr} r_{k,t} \cdot \log_{atr}\left(r_{k,t} \Big/ \sum_{t=1}^{atr} r_{k,t} \right),$$ as

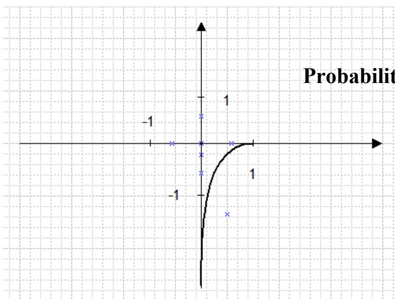
well as similar formula $H(\ell)$ for an attribute ℓ.

The quantities $H(\kappa)$ and $H(\ell)$ are the contributions of the κ-th object and ℓ-th attribute to the total entropies $\sum_{\kappa=1}^{n} H(\kappa)$ or $\sum_{\ell=1}^{m} H(\ell)$ of the data matrix X, which according to Shannon can be expressed as the sum of the entropies of individual objects or attributes respectively.

The maximum entropy in the frequency table is reached when the distribution of distribution in the data matrix X becomes uniform. To clarify the last statement, we draw a graph of the function:

$$-\log_{atr}\left(r_{\kappa,t} \Big/ \sum_{t=1}^{atr} r_{\kappa,t} \right):$$

$$-\log_{atr}\left(r_{\kappa,t} \Big/ \sum_{t=1}^{atr} r_{\kappa,t} \right)$$



$$r_{\kappa,t} \Big/ \sum_{t=1}^{atr} r_{\kappa,t}$$

.

The maximum entropy of the data matrix in the row direction is computed when the probabilities on the x-axis allocate a uniform frequency distribution, resulting in $\underset{max}{H}(\kappa) \approx 1$. Indeed, the value $-\log_{atr}\left(r_{\kappa,t} \Big/ \sum_{t=1}^{atr} r_{\kappa,t} \right)$ is at its maximum when $r_{\kappa,t} \Big/ \sum_{t=1}^{atr} r_{\kappa,t} \approx 1\!\!\Big/_{atr}$.

In case $r_{k,t} = 0$ then this zero value is not taken into account. Based on the maximum entropy, we get the actual information about the object κ equal to $1 - H(\kappa)$. Thus, the complete information contained in the data matrix X is calculated by the formula: $n - \sum_{\kappa=1}^{n} H(\kappa)$. The above layering algorithm can now be used.

For the credential of an individual object, we choose the entropy value $H(\kappa)$. Thus, the set of objects $x_1, x_2, ..., x_n$ is to be stratified. It is only necessary to keep in mind that after removing an object from the data matrix, changes occur in the frequency table (frequency bands). The changes consist in the fact that when using the values of the ℓ-th attribute $x_{\kappa,\ell}$ of the κ-th object, in the corresponding cells $r_{\kappa,x_{\kappa,\ell}}$ and $c_{x_{\kappa,\ell},\ell}$ of the frequency tables $\boldsymbol{\mathscr{r}}$ and $\boldsymbol{c}$, one is subtracted from the frequencies: $r_{\kappa,x_{\kappa,\ell}} = r_{k,x_{\kappa,\ell}} - 1$ and $c_{x_{\kappa,\ell},\ell} = c_{x_{\kappa,\ell},\ell} - 1$.

We will consider the properties of the stratification algorithm using the mentioned monotone systems in the next section, where the positive $\oplus$ and negative effects of elements are used. In graphs, the negative $\ominus$ effect on the arc was its removal. For data matrix, this is the reset of the ℓ-th attribute of the κ-th object or a series of $\ominus$ effects until the object will be completely removed by the entropy level u_k assessment.

5. MONOTONE SYSTEMS

We will continue our story about monotone systems now at a more precise level. A monotonous system manifests itself in the relationship between elements in the fact that if an element of the system is "positively influenced", then this effect is also positively reflected on its interrelated elements. It's the same with negative effects.

The monotonicity property as a central property allows us to formulate the concept of the system kernel or core in a general form. By the core, we mean a subset of the elements of "strongly attracting" or "strongly pushing" each other the elements of the system.

Consider any system W consisting of a finite set of elements, i.e., $|W| = n$ |. Quantities or credentials that indicate the level of "importance" of the element $\alpha \in W$ for the functioning of the system as a whole characterize the states of the elements of a system W.

It proves necessary to reflect the internal dependence of the elements of the system at the level of importance of the elements. In view of the fact that the elements of the system are interconnected, it is possible to take into account the effect of element α on other elements related to the change in the properties of element β. We assume that the level of importance of the element α itself also changes due to its effect. If elements α and β are in no way related in the system, it is natural to assume that the change caused by element α to the importance of element β is zero.

In the system W, we consider as an effect on the element α of two types of effects: $\oplus$ and $\ominus$ type effects ($\oplus$- and $\ominus$-effects). In the first case, the properties of element α are considered to improve as its importance to the system increases; in the second case, the properties of element α deteriorate as its level of importance in relation to the system decreases.

Now we can also provide a definition of a monotonic system. A monotonic system is a system in which the positive effect of $\oplus$ on any system element α causes the positive effect of $\oplus$ on all other elements of the system and the effect of the $\ominus$ type causes the effect of $\ominus$ type respectively.

System monotonicity conditions. The observed important concept — the effect on the element α of the system W and the accompanying effect on the other elements of the system — allows the set W to determine an infinite number of functions, since we have at least one actual function of the importance of the elements W of the system: $\pi : W \rightarrow \Re$, where $\Re$ is a set of real numbers.

If element α is affected, then it can be said that the function π is reflected in the function $\overset{+}{\pi}_{\alpha}$ for the effect of ⊕ and in the function $\overset{-}{\pi}_{\alpha}$ for the effect of ⊖ respectively. As a result of the effects ⊕ and ⊖ on the element implementation, the credentials of the system elements are redistributed from the function π to the functions $\overset{+}{\pi}_{\alpha}\overset{-}{\pi}_{\alpha}$ or the initial set of values $\{\pi \mid \pi(\delta \in W)\}$ is transferred to a new set $\{\pi \mid \overset{+}{\pi}_{\alpha}(\delta \in W)\}$ and $\{\pi \mid \overset{-}{\pi}_{\alpha}(\delta \in W)\}$ respectively. The functions π, $\overset{+}{\pi}_{\alpha}$, $\overset{-}{\pi}_{\alpha}$ are defined on the whole set W and thus are also defined $\overset{+}{\pi}_{\alpha}(\alpha)$ and $\overset{-}{\pi}_{\alpha}(\alpha)$. It is clear that if there is given a sequence α_1, α_2, α_3 …from the W set of elements (all repetitions and combinations of elements are allowed), and e.g. the a binary sequence ⊕, ⊖, ⊕,… then can be easily determined the combined effect in the form of a functional product of $\overset{+}{\pi}_{\alpha_1} \cdot \overset{-}{\pi}_{\alpha_2} \cdot \overset{+}{\pi}_{\alpha_3} \cdot \ldots$

The presented construction allows writing the monotonicity property of the systems as two main inequalities:

$$\overset{+}{\pi}_{\alpha}(\beta) \geq \pi(\beta) \geq \overset{-}{\pi}_{\alpha}(\beta)$$

for each element pair $\alpha, \beta \in W$, including pairs (α, α) and (β, β).

Identification of the system kernel. To determine the kernel of the system, consider the two subsets of W, namely H and $\overline{H}$, so that $H \cup \overline{H} = W$ and $H \cap \overline{H} = \varnothing$.

If only elements α_1, $\alpha_2, \ldots, \in H$ are positively affected then it determines for the set W a certain function $\overset{+}{\pi}_{\alpha_1} \cdot \overset{+}{\pi}_{\alpha_2} \cdot \ldots$, which can be considered determined only for the subset H. If we choose one of all possible sequences of a set H, namely $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$ where α_i does not repeat, then the function $\overset{+}{\pi}_{\alpha_1} \cdot \overset{+}{\pi}_{\alpha_2} \cdot, \ldots, \overset{+}{\pi}_{\alpha_{|\overline{H}|}}$ is denoted unambiguously on the set H function and call it a standard function. The function thus introduced is called the credential function on the set H and the individual value of the function on the element α is the credential. These credentials $\{\pi^{+}H(\alpha) \mid \alpha \in H\}$ we denote by $\Pi^{+}H$ and call this set of credentials specified for a given set H, i.e., for the set of credentials with respect to the set H.

Suppose that the set of credentials sets $\{\Pi^{+}H \mid H \subseteq W\}$ for all possible subsystems 2^{W} of system W — the number of all possible subsystems is $2^{|W|}$.

Instead of the plus effects of the standard function, we can look at the analogous ⊖ effects function $\pi^{-}_{\alpha_1} \cdot \pi^{-}_{\alpha_2} \cdot,\ldots,\pi^{-}_{\alpha_{|\overline{H}|}}$. Similarly to the function $\pi^{+}H(\alpha)$, we also determine, the set of credentials $\{\pi^{-}H(\alpha) \mid \alpha \in H\}$ and also the collections of sets of credentials $\{\Pi^{-}H \mid H \subseteq W\}$. In addition, to obtain a process of type ⊖ effects — an analogous process $\Pi^{-}H$ is performed. All elements of the set H are affected in sequence according to the ordered list $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$.

On the subsets or arrays $\{\Pi^{+}H \mid H \subseteq W\}$ and $\{\Pi^{-}H \mid H \subseteq W\}$ of credentials given on the sets $H \subseteq W$, the following two functions can be defined for each subset H:

$$F_{+}(H) = \min_{\alpha \in H} \pi^{+}H(\alpha),$$

$$F_{-}(H) = \max_{\alpha \in H} \pi^{-}H(\alpha).$$

By the kernels of W we call the global minimum of the function $F_{+}(H)$ and the global maximum of the function $F_{-}(H)$. The subsystem $H^{\oplus}$ that reaches the global minimum of the F_{+} function is called the system ⊕-kernel, and the subsystem $H^{\ominus}$ that reaches the global maximum of the F_{-} function is called the ⊖-kernel, respectively.

Definition. The defining set considered in monotone systems theory is the last set in the layer algorithm with level u_p (see the section 3 above), where the sequence $\overline{\alpha} = \left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$ of system elements by which such a defining set is found is called the defining sequence.

Theorem 1. The defining set $H^{\ominus}$ is the set where the F_{-} function reaches the global maximum. There is only one defining set $H^{\ominus}$ set. All other subsets if they exist where F_{-} reach the global maximum are within the defining set $H^{\ominus}$.

Theorem 2. For the definite set of $H^{\oplus}$, the function F_{+} reaches a global minimum. There is only one defining set $H^{\oplus}$. All sets that reach the global minimum are enclosed in the defining set $H^{\oplus}$.

The existence of defining sets $H^{\ominus}$ and $H^{\oplus}$ is ensured by a special constructive routine. The defining sets are kernels of Monotone Systems, because on these sets the functions F_{-} and F_{+} reach the global maximum (minimum) accordingly. Theorems 1 and 2 guarantee that all kernels are located in one "large" kernel — the defining set.

6. Monotone Systems Subsystems on Graphs

Let us have a "*big*" graph G and a "*small*" graph g. It is necessary to select a part of the "*big*" graph G (a set of arcs or edges) so that this set is the most *"saturated"* with "*small*" graphs g. For example, we can assume that one part of the graph is more saturated than the other if the first contains more small graphs g than the second.

With some complexity, saturation can also be approached as follows. Consider the arcs, edges or vertices of G that belong to the part we are interested in. We now count in integers: how many there are small graphs g, separately those g graphs that are located "near" each vertex, arc or edge. By this integers is meant the number of graphs g that contain a given vertex, arc or edge, and are thus expressed as an integer. By doing this, we get exactly such an integer or credential that characterizes the part of G we are interested in. Each such integer reflects a certain "*local*" saturation of the graph G with the graphs g.

Based on the obtained integers, several variants open to determine the saturation of the G part of the graph. The mean, variance, etc., of these numbers can be calculated. We consider the simplest credential magnitude, namely the entity of small graphs g, which are located in a separate part of a large graph G, i.e., the smallest value of the local parts. Figuratively speaking, this number of sub-graphs is in the most "empty" location of the graph G, which we should further on remove by $\ominus$ type actions.

Below we give an exact representation of the problem of determining the most saturated parts of the graph G with small graphs. We set the problem as follows: From all possible parts (or a large number of parts) of a graph G we find the one with the maximum value of the smallest number of local sets of small graphs g.

It is natural that in this method many small graphs g can be placed in a part in the usual way, because the number of small sub-graphs g on each vertex or arc is not less than on the vertex or arc on which it is minimal. At the same time, however, this minimum number in the extreme part is quite large, because we specifically chose the part where the local number of graphs condition reaching the global maximum of the minimum would be satisfied,

Similarly, we can set the task of finding the part of the graph G that is least saturated with small graphs g. The number of sub-graphs g at the vertex or arc where this number is maximal characterizes then each part of the graph. Instead of looking for the part of the graph where the minimum local number of graphs is the maximum, we look for the part where the maximum local number is the minimum. In this case, the number g of the sub-graphs of each vertex or arc is not greater than the "maximum" vertex or arc, and the latter has a default due to the global minimum condition.

The extreme parts of a graph are usually uniformly saturated or unsaturated with small graphs. In a saturated extreme part, no single vertex or arc can usually have very few graphs g, because without the arc of this vertex the part of the graph is probably more saturated at the top or arc with sub-graphs g in the more complex sense mentioned above.

7. GENERAL MODEL OF KERNEL EXTRACTION ON GRAPHS

If a graph G is given, then with $V(G)$ or by V we denote the set of vertices of the graph. We denote the set of arcs of an oriented graph G by $U(G)$ or U and the set of edges of an unoriented graph by $E(G)$ or E.

In graph theory, the concept of a sub-graph of a given graph G is used. A graph G' is a sub-graph of the graph $[V(G), U(G)]$ if $V(G') \subset V(G)$ and $U(G')$ is the set of arcs of all and only those that bind the pair from $V(G')$. Similarly, we can define a sub-graph of an undirected graph if the term edge is used instead of the arc.

Sometimes the term part G of a graph is also used. The graph G is called the part of the graph $G[V, U]$ if $V(G'') \subseteq V(G')$ and $U(G'') \subseteq U(G')$. In terms of the oriented graph, some arcs of the graph G are simply missing. Similarly, an undirected sub-graph is determined.

The design of concepts described in the previous two sections of this guide must begin with the identification of the elements of the system W. Two structural units can be separated from graphs — a vertex and an arc. Let us consider first the case where the vertex of the graph G is chosen as an element of the system. We now determine the effects of the ⊕- and ⊖-effects on the vertices, i.e., on the elements of the system W. Determining the effects of ⊕ and ⊖ requires the addition of a special significance function π to the vertices of the graph G. The action has already been mentioned in the previous two sections of the guide, that the credentials in the system must increase as a result of the ⊕ effect and decrease as a result of the ⊖ effects.

We need to define saturation indicators, or whatever we call them, credentials for the elements α of each subset of H from W. To get this, we need to set up an initial set of credentials for W, as well as a framework how to express ⊕ and ⊖ effects.

An initial set of credentials $\{\pi(\alpha) \mid \alpha \in W\}$ can be specified, for example, as follows. Let g be a small graph given a large graph G. We count the number of different sub-graphs of graph G that are isomorphic to graph g and whose vertices include vertex α. We set the just obtained number to the initial credential level $\pi(\alpha)$. To underline the introduced dependence of the level $\pi(\alpha)$ on the small graph g, we use the expression — the credential of the vertex α of the graph G with respect to g. Next, we consider two operations for obtaining new graphs from G, namely the ⊕ and ⊖ operations.

Let a graph G be given and an empty graph Λ (a graph that has no arcs but has $|V(G)|$ vertices). We assume that $V(\Lambda)$ is an exact copy of $V(G)$. And when we talk about the vertex α, we mean the vertex of a graph G, which appears in two forms — like the vertex of a graph G and like the vertex of a graph Λ.

A ⊖-type operation of a graph G with a vertex α is to carry out removing all the arcs or edges leading to that vertex. On an empty graph Λ, however, the ⊕-type operation is a recovery operation for all edges leading to that vertex α. It appears that if a ⊕-type operation is applied to a vertex, the credentials of all other vertices (relative to the small graph g) either decrease or, in some cases, remain the same. When performing a ⊕-type operation, a natural question arises: what should be considered the credential of the vertex after restoring the vertex?

The solution to this question lies in the following construction. Let us count the credentials of the vertices of the graph Λ (with respect to the small graph g) and add the credentials of the vertices of the graph G. We consider the obtained amounts as the total credentials of the vertices. In this case, the opposite effect can be observed: as a result of the ⊕-type operation, the total credentials increase or, like the ⊖-type credentials, remain at the same level. Generally speaking, the initial credential set $\{\pi(\alpha) \mid \alpha \in W\}$ (the credential set before any ⊕-type operation) of the vertices of graph G can be considered as a general credential set to be built since any part of graph G is initially empty. At this stage, minimizing the maximum credentials means some options for the vertices of graph G to be isolated. In this approach, the monotonicity condition is satisfied.

When constructing sets of credentials in system W, it must be demonstrated how the initial set of credentials $\{\pi(\alpha) \mid \alpha \in W\}$ found is redistributed due to ⊕ and ⊖ operations.

Let be given a certain sequence of vertices $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$, which forms a set of $\overline{H} \subseteq W$. We express the effect of ⊕ on the vertices of G according to their occurrence in the sequence. As a result, a sub-graph of G is formed on the graph $V(\Lambda)$. At the vertex of each resulting sub-graph we can count the number of isomorphic sub-graphs with a small graph g, so we get the credentials of a set of H (the complement of $\overline{H}$ to W) elements. Consistent with the above theory, we can state that the set H determines a new significance function in the form,

$$\pi^{+}_{\alpha_1} \cdot \pi^{+}_{\alpha_2} \cdot \ldots \tag{2}$$

obtained from the initial credential collection $\{\pi(\alpha) \mid \alpha \in W\}$.

Thus, if a sequence of vertices $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$ is given that promotes the set $\overline{H}$, then the set H forms a set of credentials determined by (2) or (3). We denote this set by $\Pi^{+}H$, and we call the set of credentials by the set of vertices induced on H. The sets of induced credentials form the set $\{\Pi^{+}H \mid H \subseteq W\}$. Sometimes it is appropriate to use the expression of ⊕-collection of sets with respect to the small graph g.

The collection or array $\{\Pi^{-}H \mid H \subseteq W\}$ of sets of credentials is determined analogously. The collection $\Pi^{-}H$ of the credentials is determined by the function

$$\pi^{-}_{\alpha_1} \cdot \pi^{-}_{\alpha_2} \cdot \ldots \tag{3}$$

given in part G of the graph, which remains after the application of the $\ominus$-activities to the sequence of vertices forming $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$. It only needs to be emphasized that each subset $H \subseteq W$ of the set of credentials is in fact the set of the remaining part, but not the total, i.e., not the part given by the set of graph Λ, which actually is an empty graph.

Next, let's take the arc as the system element. The system is defined as the set of interconnected arcs $U(G)$ of the graph G, determining the $\oplus$ and $\ominus$ effects again requires setting the values of the initial function π.

Let be given a small graph of g. We count the number of different sub-graphs of the graph G that are isomorphic to the graph g and whose arcs or edges include this arc or edge. The resulting integer is taken as the significance level of the arc α of the graph G. This is called the credential of the arc α with respect to the graph g.

Similarly to those described at the vertices of G, the concepts of $\oplus$ and $\ominus$ activities are also determined by the arcs or edges of the graph G. Arcs or edges are now removed or restored instead of vertices.

Let's look at the $\ominus$ operation first. It is obvious that as a result of removing the arc (edge), the initial set of credentials with respect to the small graph g may decrease or remain the same. A decrease in importance of credentials indicates that the $\ominus$ operation is equivalent to defining $\ominus$ activity for system elements.

Let $\langle \alpha_1, \alpha_2, \ldots \rangle$ be a sequence of different arcs on G, including arcs forming $\overline{H} \subseteq U(G)$. We perform $\ominus$-actions sequentially on the arcs of the graph G according to the given sequence. As a result, we get a certain part of the graph G, the elements of which are arcs (edges) belonging to the set $H \subseteq U(G)$. For each arc $\alpha \in H$, count the number of isomorphic graphs with the graph g, which is considered to be the credential or significance of the element α with respect to the set H.

According to the notations used, the method for determining the given credentials creates a function on the elements of the set H of arcs. Similarly to the case where the number of sets of credentials was assigned to the vertices of a given graph, arcs (edges) are created that belong to the set of credentials $\{\pi^{-}H(\alpha) \mid \alpha \in H\}$, which we denote again $\Pi^{-}H$. We proceed in a similar way to find the set of credentials $\{\Pi^{-}H \mid H \subseteq U(G)\}$. On an empty graph Λ, defining the ⊕-activity on the basis of the ⊕-operation requires a more detailed analysis.

Let again the sequence of arcs $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$ in the given graph G (said arcs form the set $\overline{H}$), we perform ⊕-operations on the set $\overline{H}$ arcs sequentially. As a result, the set of vertices $V(\Lambda)$ forms a part of a graph G whose list of arcs is equal to $\overline{H}$. For the vertex model, we calculated the total credential of each vertex $\alpha \in V(G)$. In this case, too, we try to do the same and find the total credential of the arcs forming H.

The arcs belonging to the set H are not present in the graph g and the question is how to count the number of sub-graphs isomorphic to the graph g and containing the arc α (which is not present in the graph Λ). Proceed as follows: we read that this arc α is fictitious only at the moment of counting the sub-graphs. In this case, the set of arcs H forms certain integers that depend on both the graph and the part of the graph formed on the empty graph g.

In the method described above, the function $\pi^{+}_{\alpha_1} \cdot \pi^{+}_{\alpha_2} \cdot \ldots$ is determined from the quantity H, which creates a set of ⊕-credentials $\{\pi^{+}H(\alpha) \mid \alpha \in H\}$.

In this case, even in the case of a ⊕-operation, the set of credentials of the ⊕-activities can be determined with respect to a small graph. The use of the term "⊕-activity" is perfectly legal here, as the total credentials of those elements that are not yet subject to ⊕-activity may increase or remain the same.

Illustrative Examples on Directed Graphs. A graph G of partial ordering is defined as a binary relation G with the following properties:

a) Reflexivity, i.e., if $\alpha \in V(G)$, then $\alpha G \alpha$. The graph G has a loop at the vertex α.

b) Transitivity, if there exists an arc (α, β) and (β, γ), then the graph G has an arc (α, γ), or from $\alpha \, G \, \beta$ and $\beta \, G \, \gamma$ it follows that $\alpha \, G \, \gamma$.

A complete order is defined as a graph of partial ordering in which any pair of vertices α and β is connected by an arc.

It is possible to formulate the following problem: in a given directed graph it is required to find the (in certain sense) most “saturated” regions that are “close” to a graph of partial ordering or to graphs of complete ordering. This problem will be solved by a method of organization (on a graph) of a monotonic system with subsequent determination of kernels.

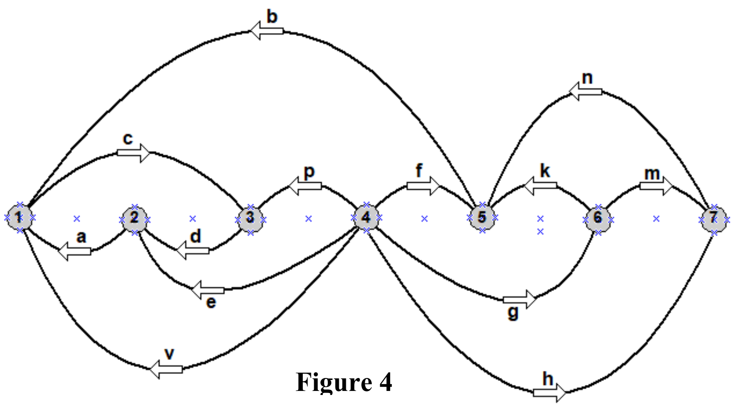


Figure 4

In accordance with the scheme of organization of a monotonic system on graphs described in the previous section, it is necessary to assign a small graph g. Suppose that this graph consists of three vertices x, y, z, and it is such that $U(\Gamma) = \{ (x, y), (y, z), (x, z) \}$. The graph has a total of three arcs (a transitive triple).

Now let us consider the assignment of collection of credentials arrays at the vertices of a graph shown in Fig. 4. The loops on this graph have been omitted.

According to the scheme of assignment of collections of credential arrays at the vertices of a graph, it is required to determine an initial array of credentials $\{\pi(\alpha)\}$, where $\alpha = 1,2,3,...,7$. According to the method of calculation of the values $\pi(\alpha)$ with respect to the graph g (a transitive triple), we obtain $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 7$, $\pi(5) = 4$, $\pi(6) = 3$, $\pi(7) = 3$. As an example, let us determine a credential array on a subset of vertices $H = \{1,2,3,4,5\}$. By successively performing ⊖ actions on the set $\overline{H} = \{6,7\}$, we obtain on the set H a new credential array $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 4$, $\pi(4) = 4,$, $\pi(5) = 1$.

The values of the function $\overset{+}{\pi}_6 \overset{+}{\pi}_7$ can be obtained in a similar way, but for this purpose it is necessary to use the assignment of collections of total ⊕ arrays with respect to a transitive triple. According to Fig. 5, the values of this function in their order at the vertices $\{1,2,3,4,5\}$ are as follows: $\pi(1) = 3,$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 8$, $\pi(5) = 4$. In exactly the same way we can determine on any subset H of vertices $V = \{1,2,3,4,5,6,7\}$ a proper credential array of ⊕ or □ actions with respect to a transitive triple.

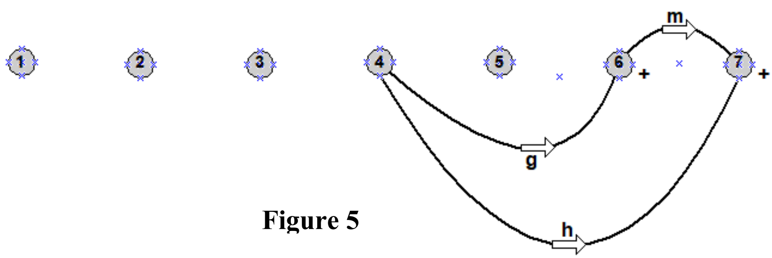


Figure 5

Now let us consider a construction that is assigned not on vertices, but on the arcs of the graph presented on Fig. 4. In this case the set of elements of the system W will be $U(G) = \{a, b, c,..., n, m\}$. As the small graph g we shall take the same graph as above, with a set $U(g) = \{(x, y),(y, z),(x, z)\}$.

By analogy with the foregoing, we realize the construction in the same succession. We determine an initial credential array $\left\{ \pi(\alpha) \middle| \alpha \in U \right\}$ on the arcs of the graph G in accordance with the general scheme.

We find that

$$\pi(a)=1, \pi(b)=1, \pi(c)=1, \pi(d)=1, \pi(e)=2, \pi(f)=3,$$
$$\pi(g)=2, \pi(h)=2, \pi(k)=2, \pi(n)=2,$$
$$\pi(m)=1, \pi(v)=3, \pi(p)=2$$

As an example, let us now perform ($\oplus$ and $\ominus$ actions on the arcs f,k and m, i.e., on the set $H=\left\{ f,k,m \right\}$. On the set H we hence obtain

$$\pi(a)=1, \pi(b)=0, \pi(c)=1, \pi(d)=1, \pi(e)=2,$$
$$\pi(g)=0, \pi(h)=0, \pi(n)=0, \pi(v)=2, \pi(p)=2.$$

In accordance with the adopted system of notations this array of numbers will be denoted by $\Pi^{-}H$. For obtaining a $\Pi^{+}H$ array, we must calculate the total credentials. The dashed lines in Fig. 6 represent the arcs of graph Λ that experience the effect of actions performed on the arcs f,k and m.

According to Fig. 6, the total credential array will be as follows:

$$\pi(a)=1, \pi(b)=1, \pi(c)=1, \pi(d)=1, \pi(e)=2,$$
$$\pi(g)=3, \pi(h)=2, \pi(n)=3, \pi(v)=2, \pi(p)=2.$$

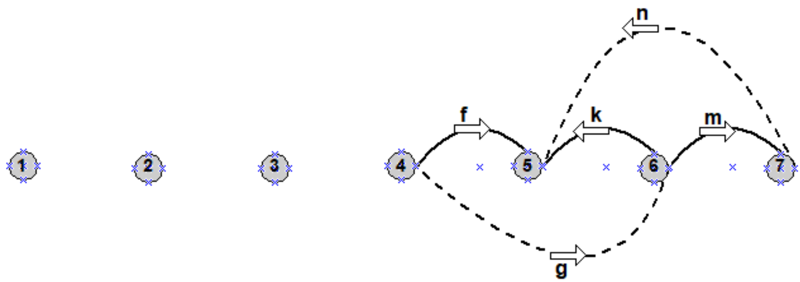


Figure 6

Thus on any subset H of arcs of the graph shown in Fig. 4 we can construct the credential arrays $\Pi^{-}H$ and $\Pi^{+}H$.

Next we describe the procedures of construction of determining sequences of $\oplus$ or $\ominus$ actions, at first for vertices, and then for arcs of the graph shown in Fig. 4. The construction is carried out for the purpose of illustrating the concepts of $\oplus$ or kernels of the monotonic system and also for ascertaining the effect of the duality theorem formulated by Mullat (1976-1977).

Let us consider an example in which $\ominus$ credential arrays are assigned at vertices with respect to a transitive triple. According to the scheme prescribed in Mullat's routine of construction of a determining $\oplus$ and $\ominus$ sequence of vertices of a graph on the basis of $\oplus$ and $\ominus$ actions. For the graph shown in Fig. 4, the Kernel-Searching Routine consists of two steps: the zero-th and the step one. It yields two subsets $\Gamma_0^{-}, \Gamma_1^{-} \subseteq V(G)$, where

$$\Gamma_0^{-} = V(G) = \{1,2,3,...,7\},\ \Gamma_1^{-} = \{4,5,6,7\},$$

and the thresholds $u_0 = 2$, $u_1 = 3$.

The determining sequence of vertices constructed with the aid of $\ominus$ actions is as follows: $\overline{\alpha}_{-} = \langle 3,2,1,4,5,6,7 \rangle$. Thus on the basis of: a) according to Theorems 1,3 (Mullat, 1971) and b) according to Theorem 1 (Mullat, 1976) about KSR, it can be argued that the set $\{4,5,6,7\}$ is the definable set of vertices of the graph shown in Fig. 4, and, therefore, this set is also the largest kernel $K^{\ominus}$.

Now let apply the KSR for constructing a $\oplus$-determining sequence. We find that $\overline{\alpha}_{+} = \{4,5,6,7,1,2,3\}$. The routine terminates at the third step, and it consists of four steps, namely the zero-th, the first, the second and the third. According to the construction of $\oplus$ sequences prescribed in the KSR, we produce the sets Γ_j^{+}: $\Gamma_0^{+} = \{4,5,6,7,1,2,3\}$, $\Gamma_1^{+} = \{5,6,7,1,2,3\}$, $\Gamma_2^{+} = \{6,7,1,2,3\}$, $\Gamma_3^{+} = \{2,3\}$ and a sequence of thresholds $u_0 = 7$, $u_1 = 4$, $u_2 = 3$, $u_3 = 2$. As in the case of a $\oplus$ sequence, we conclude on the basis of Theorems 2 and 3 of a) Mullat, and of Theorem 1 of b) Mullat, that $\{2,3\}$ is the largest $K^{\oplus}$ kernel of the system of vertices of the graph in Fig.1.

A careful analysis of Fig.1 shows that the $K^{\oplus}$ kernel is in fact completely ordered set, i.e., $\langle 4,5,6,7 \rangle$. On the other hand the $K^{\oplus}$ indicates from the point of view of the "structure" of a graph that the region, in which the vertices are least ordered, it is ordered itself as well. This is in agreement with the our formulation of the problem of finding kernels as representatives of "saturated" or "unsaturated" regions (parts of a graph) with small graphs g

Now let us use the KSR for constructing determining sequences of arcs of the graph in Fig.1. The graph has a total of 13 arcs. After applying the KSR, we obtain on the basis of $\ominus$ actions the following sequence:

$$\overline{\alpha}_{-} = \langle a, b, c, d, v, e, p, f, k, n, m, h, g \rangle .$$

The routine terminates at first step and it consists of two steps, namely the zero-th step and the first step. At the zero-th step we have $\Gamma_0^- = U(G)$, and at the first step we have $\Gamma_1^- = \{ f, k, n, m, h, g \}$, with the thresholds $u_0 = 1$ and $u_1 = 2$ respectively. Summing up, we can assert on the basis of the results of a), b) Mullat, that this is a definable set and at the same time the largest $K^{\ominus}$ kernel in the system of arcs.

From the point of view of the graph structure, the application of the KSR to arcs in the construction of a $\ominus$ determining sequence does not yield anything new compared to the application of the KSR to vertices. We obtain the same complete order $\langle 4,5,6,7 \rangle$ represented in the form of a string of arcs, and it also corroborates our assertions concerning the saturation of a $K^{\ominus}$ kernel by transitive triples. On the other hand the use of KSR for constructing $\oplus$ determining sequence of arcs yields a $K^{\oplus}$ kernel

$$\Gamma_1^+ = \{k, m, n, g, h, e, p, b, a, c, d\}$$

whose meaning with regard to "non-saturation" with transitive triples cannot be determined.

Below we shall illustrate the peculiar features of using the duality theorem from b) Mullat (1976) for finding $K^{\oplus}$ and $K^{\ominus}$ kernels of a monotonic system specified by vertices or arcs of a directed graph.

At first let us consider the monotonic system of vertices of the graph in Fig.1. The sequence of sets $\langle \Gamma_j^+ \rangle$ specified by the KSR on the basis of $\oplus$ actions uniquely determines the sets $V \setminus \Gamma_1^+ = \{4\}$, $V \setminus \Gamma_2^+ = \{4,5\}$, $V \setminus \Gamma_3^+ = \{1,4,5,6,7\}$. Above we have found that $F_+(\Gamma_2^+) = u_2 = 3$. From the construction of a determining sequence $\overline{\alpha}_-$ of vertices of a graph we know that $F_-\{4,5,6,7\} = 3$. Hence by virtue of Corollary 1 of Theorem 1 of b) Mullat, we can assert already after the second step of construction of an $\overline{\alpha}_+$ sequence that the set $\{1,4,5,6,7\}$ contains the largest $K^{\ominus}$ kernel. Thus we have shown that the sufficient conditions of the duality theorem of b) Mullat, are satisfied in the example of the graph represented in Fig. 1.

Now let us consider the set $V \setminus \Gamma_1^- = \{1,2,3\}$. As was shown above, inside this set there exists a set $\Gamma_3^+ = \{2,3\}$ such that $F_+(\Gamma_3^+) = 2$; $F_-(\Gamma_1^-) = 3$ on the other hand. By virtue of Corollary 4 of the duality theorem we can assert that set $\{1,2,3\}$ contains the largest $K^{\oplus}$ kernel of the system of vertices of the graph (Fig.1); this likewise confirms that existence of the conditions governing the theorem.

At last let us consider a collection of credential arrays on the arcs of the graph. The determining $\overline{\alpha}_+$ sequence of arcs specifies a set $\Gamma_1^+ = \{k,m,n,g,h,e,p,b,a,c,d\}$. It is easy to see that inside the set $U \setminus \Gamma_1^+$ there does not exist a set H as required by the conditions of Corollaries 1 and 2 of the duality theorem in Mullat (1976). This shows that in comparison to arrays on vertices, credential arrays on arcs do not satisfy the duality theorem.

Monotonic systems on special classes of graphs. In contrast to the previous section, we do not carry out here a detailed construction of collections of credential arrays and determining sequences and kernels on any illustrative example. Here we shall show how to select a small graph g and $\oplus$ and $\ominus$ actions so as to match the selection of these elements with the desired "saturation" of the investigated graph. The desired saturation of a graph can be understood as the saturation desirable for the investigator who usually has a working hypothesis with respect to the graph structure. In view of this, we shall consider the following classes of graphs: tournaments, a-cyclic (directed) graphs, and (directed or undirected) trees.

Let us recall the definitions of these classes of graphs. A tournament is a directed graph in which each pair of vertices (x, y) is connected by an arc, cf. Harari (1969). A none-cyclic graph is a graph without cycles (in case of an undirected graph), and a graph without circuits (in case of a directed graph). None-cyclic undirected graphs are trees, and we shall consider the most general class of trees, as well as the class of directed trees.

In tournaments it is appropriate to consider regions of vertices that are "saturated" with cyclic triples. A cyclic triple is a graph g such that $V(g) = \{ x, y, z \}$, $U(g) = \{ (x, y), (y, z), (x, z) \}$. It can be assumed that a tournament in which there exists such a region represents a structure of the participants of the tournament. This structure is non-uniform; i.e., there exists a central region (set) of participants who can win against the other players, but they are in neutral position with respect to one another.

For solving the above problem, we propose the following exact formulation in the language of monotonic systems. In Section 2 we have considered credential arrays on vertices and arcs of a graph. Now let us consider the above models on vertices or arcs in a certain order. In both models we take a cyclic triple as the small graph g with respect to which the π function is calculated. Suppose that the methods of assignment of collections of credential arrays on vertices are the same as in Section 2. It is possible to modify this scheme by taking as a $\ominus$-action on the vertex α the removal of all arcs of a tournament that originates at α, whereas $\oplus$-action is the restoration of all the arcs in the graph Λ that originate at α. In Section 2 we performed the opposite operation, i.e., the removal of incoming arcs and the restoration of these same incoming arcs.

The assignment of credential arrays on arcs of a tournament graph must be carried out in accordance with a scheme similar to that described in Section 2. Within the framework of the theory it is apparently impossible to decide whether the scheme of determination of kernels on arcs of a tournament is preferable to the scheme using vertices; therefore, it is necessary to carry out computer experiments. There exists only one heuristic consideration. If in a tournament there can exist several central regions saturated with cyclic triples, it will be preferable to use the scheme of determination of kernels on the arcs of tournament, since these regions can be found. The model based on vertices makes it possible to find a kernel that consists also of regions, but it does not permit finding an individual region. We do not possess a string of arcs representing these regions.

None-cyclic directed graphs are a convenient language for describing operation systems (Kendal, 1940). An operation system can be regarded as a system of modules and interpreted as a library of programs. Each working program is a path in a none-cyclic graph, or, in other words, the set of modules of a library needed at a given instant. The modules are called one after another if not all of them can be stored in the main memory. In case of a library of a large size, there naturally arises the question of fixing the modules on information carriers. Prior to solving this problem, it is appropriate to ascertain the "structure" of a none-cyclic graph of a library of modules.

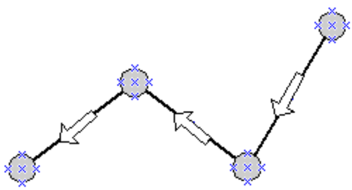


Figure 7

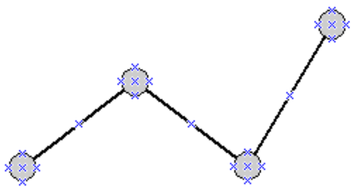


Figure 8

For ascertaining the structure of a graph and for just-mentioned task of fixing the modules, we have to find the principal (nodal) vertices or arcs. The nodes are the "bottlenecks" of graphs or, in other words, the modules that occur in many working programs.

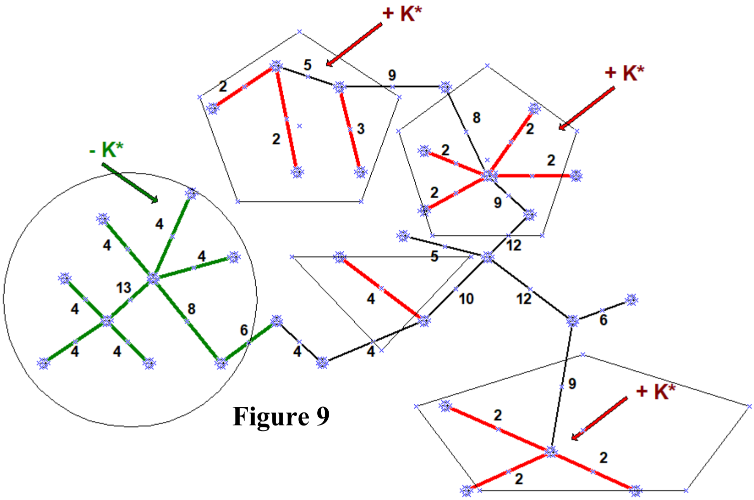


Figure 9

We shall now formally describe this problem with the aid of a model of organization of a monotonic system on a graph. As a small graph we shall take directed graph in Fig.7. The structure of this graph is in accordance with the above definition of bottlenecks of the none-cyclic graph under consideration. It is possible to construct a monotonic system also on the arcs of a none-cyclic graph of a library of modules. With the respect to the graph on Fig.7, the collection of credential arrays and ⊕ and ⊖ actions, in accordance with the general scheme of Section 2, must be defined. After this it is necessary to use the routine of finding vertex kernels or arc kernels, which in conjunction must indicate the bottlenecks in accordance with the above definition. As in case of tournaments, which a monotonic system is preferable of arcs or vertices requires experimental checking.

In comparison to the two previous examples, the last example does not have the aim of associating the application or description of any actual problem with tees. Our aim is to try and find in a tree a region, which in some sense is more similar to “cluster” than any other part of the tree.

At first let us consider undirected trees. We shall use a model of organization of a monotonic system on the branches of a tree. As a small graph g we shall take the graph shown on Fig. 8. As in the case of assignment of collections of ⊕ and ⊖ credential arrays on arcs, we assign the corresponding ⊕ and ⊖ arrays with respect to the graph shown in Fig.9. The $\ominus$ arrays appear as a result of $\ominus$ actions (removal of edges), whereas the $\oplus$ arrays result from $\oplus$ actions (restoration of edges on empty graph Λ by calculating the total credentials of the tree G and its copy on Λ. As an example we presented the $\oplus$ and $\ominus$ kernels in Fig.9 of this tree. Together with each edge we indicated the number of sub-graphs g that contain this edge and which are isomorphic to the graph shown in the Fig.8.

Now let us consider directed trees. If it is of interest to separate "clusters" in a directed tree, we shall proceed as follows. Let us consider the following small graphs: g_1, g_2 and g_3 (see Fig. 10).

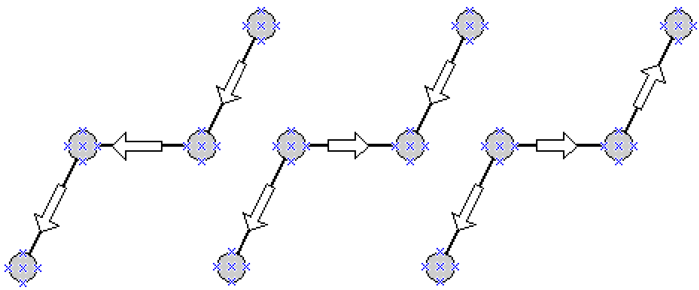


Figure 10

The credential function π on a directed tree can be calculated separately with respect to each small graph g_1, g_2 and g_3; then the values of all these three functions can be added up (a linear combination), thus yielding the overall function with respect to the graphs g_1, g_2 and g_3. In the same way we can assign a monotonic system on arcs of a tree if $\ominus$ action signifies the removal of an arc of a tree, $\oplus$ action the restoration of an arc on a copy of given tree on Λ. Thus we can pose on directed trees a similar problem of finding cluster kernels. Let us note that we use in the last example with trees a more general model of assignment of collections of credential functions with respect to a series of small graphs. The model in Section 2 has been presented for one graph g. A collection of credential arrays with respect to a series of graphs has also the property of monotonicity, and apparently such a model is more interesting in solving problems of determination of "saturated" parts of graphs.

Let us consider how the g, $\oplus$ and $\ominus$ activities of a small graph can be selected to coordinate the selection of these elements with the desired "saturation" of the graph under study. The desired saturation of a graph can be understood as desirable from the researcher's point of view, because the researcher has a certain working hypothesis about the structure of the graph.

For the small graph g for which the functions π were calculated, we choose a cyclic triangle. We use the method described in the previous subsections to create a set of credentials. The removal of all the arcs in the tournament $\langle x \text{ wins } y \rangle$ from the vertex x is the $\ominus$ action on the vertex x and the $\oplus$-action on the graph Λ is the restoration of all pairs where x wins y. The set of credentials on the graph tournament arcs must be created analogously to the previous sections.

The question of which is more preferable, whether the scheme is done on the arcs of the tournament (a game between two participants) or on the vertices of a graph, cannot be solved within the theory. It can only be said that if there are several central regions in the tournament that are saturated with cyclic triplets, the scheme of separating the kernel by arcs will be better, because these regions can be separated. A model that uses vertices separates the kernel that consists of these regions, but does not allow a single region to be found. We don't have a list of arcs that represent these areas.

Non-cyclic oriented graphs are a suitable tool for describing operating systems. The operating system can be thought of as a system of modules and interpreted as a library of programs. Each work program is a set of modules activated from a library, or in other words, in a non-cyclic graph of the path form. The modules call each other in sequence if they are not all in RAM or for some other reasons.

If the library is large, the natural idea is to place the modules on data carriers. Before solving this task, it is reasonable to explain the structure of the non-cyclic graph of the library of modules. The latter can be understood as the separation of the main sub-vertices or arrows. Vertices are very important places in the graph, in this case they are modules that are available in many work programs.

This task can be formally described in a graph by a monotonic system organization model. The question of the preference of monotonic systems formed by arrows or vertices again requires experimental control. Looking at the trees, we try to separate them from an area that is in some way more like a "bush" than the rest of the tree.

8. Discussions and summary

Usually, information is collected in to draw the necessary conclusions on issues related to human collectives, economic activity, production processes, demography, etc. If you are more interested in the verbal history itself, then the numerical experiments in Tables 1-3 can still be interesting of themselves. Indeed, with the help of these tables, the main feature of the analysis method is manifested, namely, the independence from any prior knowledge or specific information that is necessary for data analysis. This is especially true of the usual practice of personal and expensive interviews in sociological research. In this regard, the algorithm described in the manual for decomposing the data matrix into layers can be called "blind eye of statistical evaluation or scoring", which is what we need (Võhandu, 1979, 1989). This methodological guide looked at this information processing method that often has been used.

Although the main component of this methodological guide was prepared and presented for publication many years ago, as it seems to us everything that is given here is still relevant. It’s not a secret that with the development of information technologies, methods for analyzing data extracted from our environment not only become more complicated, but also their volume has grown to enormous sizes when you have to deal with databases whose size reaches many gigabytes in the amount of collected information. One thing is that all the information in such well-known applications as Facebook and the like are always reflected in some graphs of mutual relations between the participants, whether it is Linkedin or Twitter, etc. Many do not even suspect that our technology for analyzing relationships reflected in these applications are fully adapted to the analysis of such information. The problem here is that such information must be collected and presented either in tabular form or in the form of graphs. Graphs, however, must again be presented in tabular form, which, as we have already indicated, is the main form of data to be analyzed.

The algorithm for decomposing data into layers given in this tutorial turned out to be effective in many specific problems as we can apply here in the form of data viewing technology. Moreover, as already indicated throughout the book, the entire analysis process begins with the construction of the so-called defining sequence, whether it be elements of graphs or data tables, when it is required to find a local maximum at which the global maximum is reached when moving along the defining sequence from weak elements in the direction of strong ones. It turns out that a more effective method of searching for the core or kernel of a monotonic system is to move from top to bottom, from strong to weak elements. Such a search for the kernel is much more economical than the one that was proposed at that time in the original of this methodological manual.

On the other hand, the model of a monotonic system turned out to be a more complex than the author had assumed, who initiated the theoretical and practical use of monotonic systems. The fact is that on graphs when arcs of a graph or edges are taken as elements of the system, it is required to formulate very precisely what are $\oplus$ and $\ominus$ actions. If the $\ominus$ action is to remove or $\oplus$ is add both arcs and edges of the graph together with arcs and edges adjacent to an arc or edge, then monotone systems of a special type arise when the layering algorithm does not always lead to an optimal layer in the global sense. This white area has not yet been sufficiently studied, and here it is quite possible to discover some new features of monotonic systems of the indicated unusual type. We have already indicated this feature earlier in the article on how to organize a party in order to make the optimal combination of participants.

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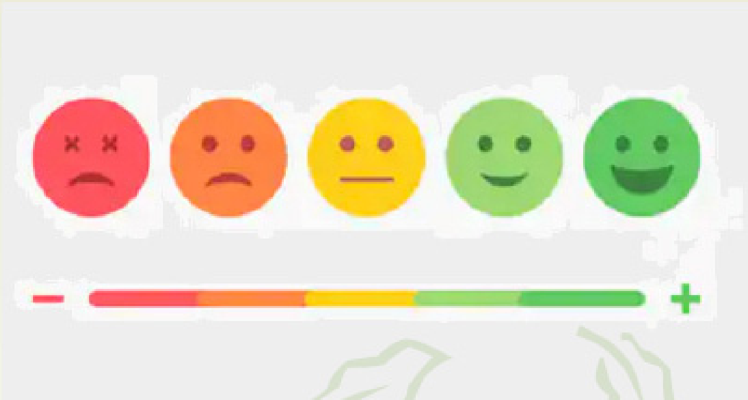

SURVEY DATA CLEANING

Survey Data Cleaning: Monotone Linkage*

Abstract. The note addresses a data cleaning principle. The principle implementation procedure presented here includes a recommendation that might be well suited for explicating and illustrating the results yielded by survey data analysis.
Keywords: data cleaning; dirty data; customer satisfaction

1. INTRODUCTION

Every day, in an endless stream, we are presented with various polls, studies, statistics, opinions, measurements, research results, etc. Enterprises, media experts, universities and other interested organizations try to present reality in a certain way or explain how it all works using information in the form of data collected during the interview. While we take this influx of data for granted, very few of us question whether this way of having reality served on a platter is actually helpful. Most people merely accept what the various analysts have presented and treat it as factual information. Thus, if more people in a survey have answered that they prefer rye bread to the white variety, does the same assertion apply to the world population? Should we infer from this finding that people in general eat more rye bread instead of white? Certainly not, reality is complex and consists of numerous choices, possibilities, behavioral patterns, preferences, etc. As a result, a typical survey based on which such 'facts' are reported can never cover all relevant data pertaining to any given subject and would without doubt lead to completely nonsensical conclusions. More accurate approximations of reality require a comprehensive statistical investigation. Therefore, as a rule, when aiming to interpret data gathered based on a sample drawn from a population of interest, one should seek input from a researcher or some other qualified person, so that the results can be interpreted and analyzed. Additionally, it is essential to take into consideration the researcher's knowledge and expertise on the subject, as well as carefully assess whether the questions discussed pertain to the aim of the survey. It is equally important to evaluate the respondents' credibility and ability to answer the questions posed, as this is one of the means to ensure the instrument reliability.

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* Presented at the 19th Nordic Conference on Mathematical Statistics, June 9-13, 2002, Stockholm, Sweden and at the "Symposium i Advent Statistik," January 23-26, 2006, København. Linkage term was used by Ylia Kempner (2008).

2. RELIABILITY

Reliability, as a generic concept, is difficult to define. In most cases, it is interpreted in a specific context. Nevertheless, it can be shows that adopting the "maximum principle" will not only help the researcher in his/her analytical endeavors, but will also "clean up" the investigation, filtering out the more "unreliable" answers and thus remove some "interference" or "outliers" — i.e., answers that are overly dissimilar from the rest or are incongruent with the most conceivable result. However, it must be emphasized that the method of analysis is still central to the success of the outcome. In other words, in spite of the aforementioned argument, the final estimation should still be based on the subjective perception of reality. After all, the primary difference between this method and the conventional statistical analysis employed to interpret survey results is that the former identifies both *unreliable* respondents and their *unreliable* answers. Consequently, we hereby obtain a much more comprehensive picture of reality simply by examining patterns that conform to the answers provided by the remaining group members. In order to describe the method, an example of a survey in progress, not having a serious purpose or value, will be used. It should be noted that what follows is significantly simplified, as the main objective is to outline the foundations of the method.

Food is a subject of public interest and related data is thus frequently under the analyst's scrutiny. Hence, in our hypothetical or frivolous example, the objective is to map people's taste preferences. To do so, the survey respondents are presented with five menus listed below and are asked to state their daily consumption of each of the given food groups.

The options they are given are as follows:

1. *Dairy produce: cheese and milk*
2. *Cereals: bread, potatoes, rise and pasta*
3. *Vegetables: vegetables, fruit, etc.*
4. *Fish: shrimp, frozen/fresh fish*
5. *Meat products: various meats, sandwich spreads and sausages*

The results pertaining to seven study participants are presented in Table 1, which will suffice for the upcoming food preferences investigation.

Table 1.

	Dairy	Cereal	Vegetables	Fish	Meat	Total
Respond. no. 1		X	X			2
Respond. no. 2	X	X		X	X	4
Respond. no. 3			X	X		2
Respond. no. 4	X	X		X	X	4
Respond. no. 5			X	X		2
Respond. no. 6	X	X	X	X	X	5
Respond. no. 7		X	X			2
Total	3	5	5	5	3	21

Considering the total score given at the bottom of the table, people's food choices seem healthy and nutritional. Moreover, it can be discerned that "cereals," "vegetables" and "fish" are most frequently consumed food groups, as five of seven respondents stated that they consume these foodstuffs daily. Can we conclude that, in general, people's lifestyle is healthy? Moreover, does this mean that 71% of population eats cereals, fish and vegetables every day? This conclusion could be clearly misleading. In addition, even conclusions pertaining to this small group require close examination of the individual respondents' answers, because some of them differ from those of the other respondents in certain ways. For example, respondents 1, 3, 5 and 7 have chosen only two food groups from the given list. Respondents no. 1 and 7 stated that they consume only "cereals" and "vegetable" products on a daily basis, while no. 3 and 5 eat only "vegetables" and "fish" every day. Assuming that this is an exhaustive list (again, note the simplifications in this example), it seems highly unlikely that someone would not eat any products from other food groups. This is a crucial point to consider, as we must believe that the answers respondents provide and factual in order to include them in the analysis. Thus, responses like those noted above are clearly unreliable reflections of reality. Let us therefore experimentally discard the unreliable respondents together with their answers to see whether we obtain a more credible result, which is a more accurate representation of reality.

3. Agreement Level – Tuning Parameter

Just as it is unusual to rely on only two food groups for sustenance, it is unlikely that an individual would eat, for example, only bread from the cereal menu, or solely shrimp from the fish menu. Thus, in "fine-tuning" the experiment, the aim is to identify all the respondents that have chosen only these two menus. The objective is, as was already emphasized above, to obtain a clearer picture of reality. Table 2 below represents the results of this data “cleaning,” based on the chosen “agreement level” or “tuning parameter”. In this case, the agreement level is set to 4, i.e., none of the totals in the last column is less than 4.

Table 2.

	Dairy	Cereal	Vegetables	Fish	Meat	Total
Respond. no. 2	X	X		X	X	4
Respond. no. 4	X	X		X	X	4
Respond. no. 6	X	X	X	X	X	5
Total	3	3	1	3	3	13

This seems to be a very useful instrument for the experiment. However, the tuning parameter will only be relevant when its value exceeds one. If, for example, we try to set the agreement level (tuning) to 1 in Table 1, this would render ALL respondents reliable, even though menus “Dairy” and “Meat” are associated with the lowest frequency number, namely three. What can we conclude from the outcome of adopting tuning parameter = 1? The conclusion is exactly the same as that yielded by the original analysis — “people’s lifestyle is healthy.” In contrast, setting the tuning parameter to 2, 3 or a higher value allows us to explore patterns in answers that would not be otherwise apparent. Table 2 shows the distribution of respondents based on the tuning parameter = 4.

Why should we use this particular value as a tuning parameter? Yes, indeed, in the following analysis we intend to adopt the maximum principle as a method for selecting reliable respondents. This will be done through “agreement level”, see “*totals*” of columns, pertaining to a single respondent. The value of the tuning parameter is not fixed, and can be changed depending on the purpose of analysis, and is typically set at the level that reveals the most adequate picture of reality. Roughly speaking, we can compare the situation to rotating a tuner on TV or Radio, when we attempt to receive a clear picture/sound by trying to select the right frequency. The tuner value here is 4, and we assume that the selected respondents are now reliable.

4. MAXIMUM PRINCIPLE

Finding the correct tuner position is not sufficient, as will be shown in the discussion that follows. For example, only one of the remaining, supposedly reliable, respondents chose the ”vegetable” menu. This would imply that only 33% of the sample is consuming vegetables daily. While this is likely for such a small group of respondents, it is important to reiterate that this example is a simplification of an actual, much larger survey, where such results would indeed be odd. Thus, the fine-tuning must proceed further, this time addressing the menu content. Fist, we can remove “vegetables” from the available options and see what effect this would have on the analysis.

The next step in our analysis is called “maximum principle” (Mullat, 1971a) and will be illustrated using an old merchant marketing example. If a merchant wants to make a compromise between the highest possible demand on some assortment of his/her commodities and to shorten the list of assortments as well, he would intuitively do so by removing from the assortment the commodity for which the demand is the lowest, assuming that it is identified from the purchasing patterns of reliable customers only. In the example considered in this study, the “vegetables” menu has the lowest demand. Moreover, its removal from the available options results in equal frequencies associated with the remaining menus. In general, removal of available options must be done with care, as it should not result in a simultaneous removal of reliable respondents. In some cases, however, it might be necessary to add further reliable respondents to the sample, complying with our tuning parameter once again, etc.

In general, the maximum principle can be formulated as follows: among all the reliable respondents, first remove options with the lowest agreement level, those with the lowest frequency (in our example, the menu “vegetables” in Table 2). As a result, the number of choices is reduced, but the remaining answers with the lowest frequency have a higher contingency compared to those that have been removed. In short, the aim is to remove available options in such a manner that ensures that those remaining have high representation and there are more matches in their answers. In other words, in the menu, where the matching is low, the low match becomes relatively high due to the removal, which would not be the case if the removed menus will still occupy a place in the table. In other words, the goal is not only to separate a group of menus from those that have higher matching responses, but also to find a group of respondents for whom the menu with the lowest level of matching is on a relative high level. This is the key for understanding the maximum principle. The respondents included in the analysis must be reliable, but the answers producing such reliability must also be more or less identical.

In accordance with this argument, the menu “vegetables” is removed, since the responses associated with it are not aligned with the general answer pattern based on the maximum principle. Note that here, the removal is not based on any qualitative tests, but is rather guided purely by a pattern disclosed by matching the answers!

<u>Table 3</u>.

	Dairy	Grain	Fish	Meat	Total
Respond. no. 2	X	X	X	X	4
Respond. no. 4	X	X	X	X	4
Respond. no. 6	X	X	X	X	4
Total	3	3	3	3	12

5. Conclusion

What can be concluded from the simplified survey scenario discussed above? Put it simply: it is evident that the final outcome is completely different from the results yielded by the initial analysis. According to Table 1, in general, people's food preferences are healthy and in accordance with current recommendations. On the other hand, Table 3 indicates that food habits are, in fact, less healthy. Implementing our analysis principle has reduced the panel of reliable respondents, and this has changed the outcome of our analysis.

Of course, it is natural to ask whether the proposed principle is more credible than other methods of analysis. It is true that a subjective consideration and personal choice have played in instrumental role in the analytical framework adopted to produce the final results. Some may argue that this approach is flawed, as analyst/researcher intuition was the only basis for tuning the parameters, i.e., adjusting the "agreement level." This personal consideration cannot be excluded because the method described here will sometimes coincide with what we might otherwise call common sense, where the most frequent answers reflect the actual reality. This should be the case when dealing with simple surveys in which the respondents are asked questions such as "Will you vote for so and so the coming election*?"* The value of this approach is really evident when surveys including hundreds or thousands of respondents and many hundreds of questions are conducted. They will inevitably generate diverse responses forming patterns that "common sense" will be impossible to wield, since unaided human intellect is incapable of grasping such complicated patterns. This is where our method can make a substantial difference, because it is a way of locating erroneous or misleading patterns, based on a comprehensive comparison within the full data set. This, however, does not undermine the analysts' role, as these experts will be responsible for making the relevant judgments/decisions as to why certain data is removed from the set. The goal is to identify and remove all "unreliable" respondents with the help of the "tuning parameter." The aim of this "cleansing procedure" is to retain only the most usable answers, in accordance with our maximum principle. Thus, the method presented here should be treated as an instrument, which has to be used correctly by the analyst to tune into the clearest picture of reality. The aim is to reduce the interference effect produced by unreliable respondents.

APPENDIX

A.1 Practical recommendations

The preliminary explanation above is a general introduction to our maximum principle, the background of which is found in a much more complex methodology and theory.[1] First, it is beneficial to demonstrate how the results can be used and presented for the analyst, making the use of the notion of positive/negative profile.

When designing a questionnaire, it is widely accepted that the available responses associated with the individual questions should be presented in the "same direction," i.e., from positive to negative values/opinions or vice versa. Using a more rigorous terminology, such ordering would be denoted numerically and represented on a nominal/ordinal scale. This nomenclature is used primarily because, when implementing our method in the form of computer software, the analyst must separate the answers by grouping them together into positive/negative scale ends — the $(+/-)$ pools. The next step will be to create profile groups within each $(+)$ or $(-)$ pool range. A profile group of answers is created following their subject-oriented field of interest. For example, one might be interested in participants' lifestyle, nutritional practices, exercising, etc. Thus, these profiles, distinguished by their placement in $(+/-)$ pools, are also either positive or negative.

Once the analyst has created the $(+/-)$ profiles, an automated process utilizing our maximum principle, which further organizes the data into what we call a series of profile components, conducts the subsequent analysis. Each profile component is a table, as above, located within particular profile limits. Clearly, a component is differentiated from the profile by the fact that, while a profile is a list of subject-specific questions and the corresponding options/answers composed by the analyst, the

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[1] Some theoretical aspects may be found in Appendix A.2

component is a table formed using the maximum principle. Therefore, the list of answers constituting a component (and the resulting set of table columns) is smaller, as only specific answers/columns from the full profile are included. Thus, once again the components will be separated into $(+/-)$ components $K_1^{\pm}, K_2^{\pm}, \ldots$, just as the profiles were separated into $(+/-)$ profiles. The $K_1^{\pm}, K_2^{\pm}, \ldots$ separation provides not only conceptual advantages, but also allows for more transparent illustration of the survey findings.

Analysis findings increase in value if they are presented in the format that can be easily comprehended. The simplest tool available for graphical presentation is a pie chart. Here, the pie can be divided into positive $K_1^{+}, K_2^{+}, \ldots$, and negative $K_1^{-}, K_2^{-}, \ldots$ components, represented in green and red color, respectively. However, to depict these components accurately, it is necessary to calculate some statistical parameters beforehand. For example, one can merge the $(+/-)$ components into single $(+/-)$ table and calculate the $(+/-)$ probabilities.[2] Hereby, statistical parameters based on the $(+/-)$ probabilities may be evaluated and illustrated by a pie chart divided into green and red area, effectively representing the $(+/-)$ elements.[3] There are many techniques and graphical tools at the analyst's disposal, and a creative analyst may proceed in this direction indefinitely. Still, it is plausible to wonder if the creation of the $(+/-)$ components is worthwhile. In other words, what is the advantage of using the "maximum principle" when interpreting the survey findings? The answer, see above, is that the blurred nature of the data may hinder clear interpretation of the reality underlying the data.

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[2] Certainly, some estimates only.

[3] Please, find below a typical pie chart pertinent to what we just discussed. The positive and negative profiles relate to 21 questions highlighting people's behaviour, responses, opinions, etc., regarding their daily work and habits. Answers to these questions can be presented using an ordinal scale 1, 2, ..., 5, where 1, 2, 3 are at the negative, and 3, 4, 5 at the positive end of the scale.

A.2 Some theoretical aspects

Suppose that respondents $N = \{1,\ldots,i,\ldots n\}$ participate in the survey. Let x, $x \in 2^N$, denote those who expressed their preferences towards certain questions $M = \{1,\ldots, j,\ldots, m\}$. We lose no generality in treating the list M as at a profile, whether negative or positive. Let a Boolean table $W = \|a_{i,j}\|_n^m$ reflect the survey results related to respondents' preferences, whereby $a_{i,j} = 1$ if respondent i prefers the answer j, $a_{i,j} = 0$ otherwise. In addition, all lists 2^M of answers $y \in 2^M$ within the profile M have been examined. Let an index $\delta^k_{i,j} = 0$, $i \in x, j \in y$ if $\sum_{j\in y} a_{i,j} < k$, otherwise $\delta^k_{i,j} = 1$, e.g., $\sum_{j\in y} a_{i,j} \geq k$, where k is our tuning parameter. We can calculate an indicator $F_k(H)$, using sub-table H formed by crossing entries of the rows x and columns y in the original table W. The number of 1-entries $\delta^k_{i,j} \cdot a_{i,j} = 1$ in each column within the range y determines the indicator $F_k(H)$ by further selection of a column with the minimum number $F_k(H)$ from the list y.

Identification of the component K seems to be a tautological issue, in the sense that following our maximum principle we have to solve the indicator maximization problem $K = \arg\max_{(x,y)} F_k(H)$. The task thus becomes an NP-hard problem, the solution of which includes operations that grow exponentially in number. Fortunately, we claim that our $K^{\pm}$ components might be found by polynomial $O(m \cdot n \cdot \log_2 n)$ algorithm, as shown in the cited literature. Finally, we can restructure the entire procedure by extracting a component $K_1^{\pm}$ first, before removing it from the original table W and repeating the extraction procedure on the remaining content, thus obtaining components $K_2^{\pm}$, $K_3^{\pm}$,... etc. From now on, statistical parameters and other table characteristics, which empower

$(+/-)$ share, arise from components $K_1^-, K_2^-, \ldots$ and $K_1^+, K_2^+, \ldots$ only, and are available to the analyst for illustration purposes, as depicted in the example below.

A.3 Illustration

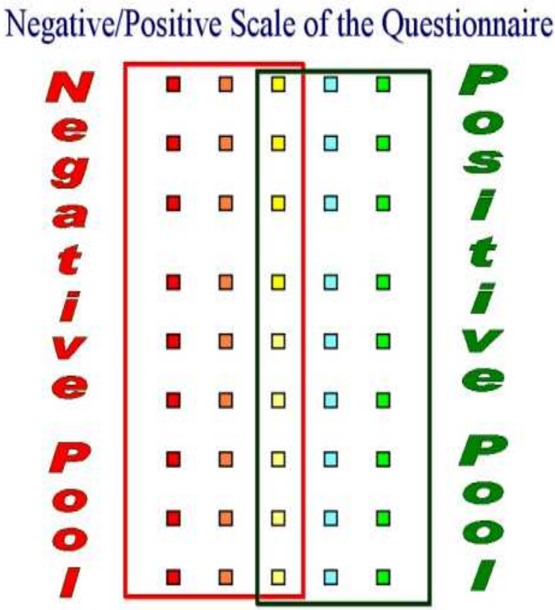
In the example, we use a sampling highlighting 383 people's attitudes towards 21 phenomenal questions. Each question requires a response on an ordinal scale, with $1 < 2, \ldots, < 5$, where $1 < 2 < 3$ are positive values at the left end, and $3 < 4 < 5$ are negative values at the right end.[4] Hence, our sampling, depicted as a Boolean table, has 383×105 dimensions. As the tuning parameter $k = 5$ was chosen, we also extracted a set of three positive K_1^+, K_2^+, K_3^+ and negative K_1^-, K_2^-, K_3^- components. The actual values in the title and those shares illustrate our positive (green) and negative (red) (+/–) components.

Some typical sampling questions are given below:

1. Is your behavior slow/quick? – eating, talking, gesticulating,...
   - 1.1 Absolutely slow
   - 1.2 Somewhat slow
   - 1.3 Sometimes slow and sometimes quick
   - 1.4 Somewhat quick
   - 1.5 Absolutely quick
2. Are you a person who prefers deadlines/postpones duties?
   - 2.1 Absolutely always prefer deadlines
   - 2.2. Often prefer deadlines
   - 2.3. Sometimes prefer deadlines or sometimes postpone my duties
   - 2.4. Often postpone my duties
   - 2.5. Absolutely always postpone my duties

. . . .

[4] Sampling owner (Scanlife Vitality ApS in Denmark) kindly provided us with a permission to use the data for analysis purposes. We are certainly very grateful for such help.



The figure shows more clearly the methodology of the positive/negative analysis of surveys data tables to identify hidden preferences of respondents. Whatever the analyst is doing to build a negative ordering of the left half of the questionnaire, our negative defining sequence is then compared with similar sequence of the right half of the questionnaire. As a result, two credential scales have been formed, which can then be visualized graphically in two-dimensional coordinate system on the plane.

At first glance that being said, our story may seem perhaps frivolous, but we say that it is much easier to suggest something new if the essence of the matter is presented in the form of an allegory, which can be interpreted in such a way as to reveal the hidden meaning of reality.

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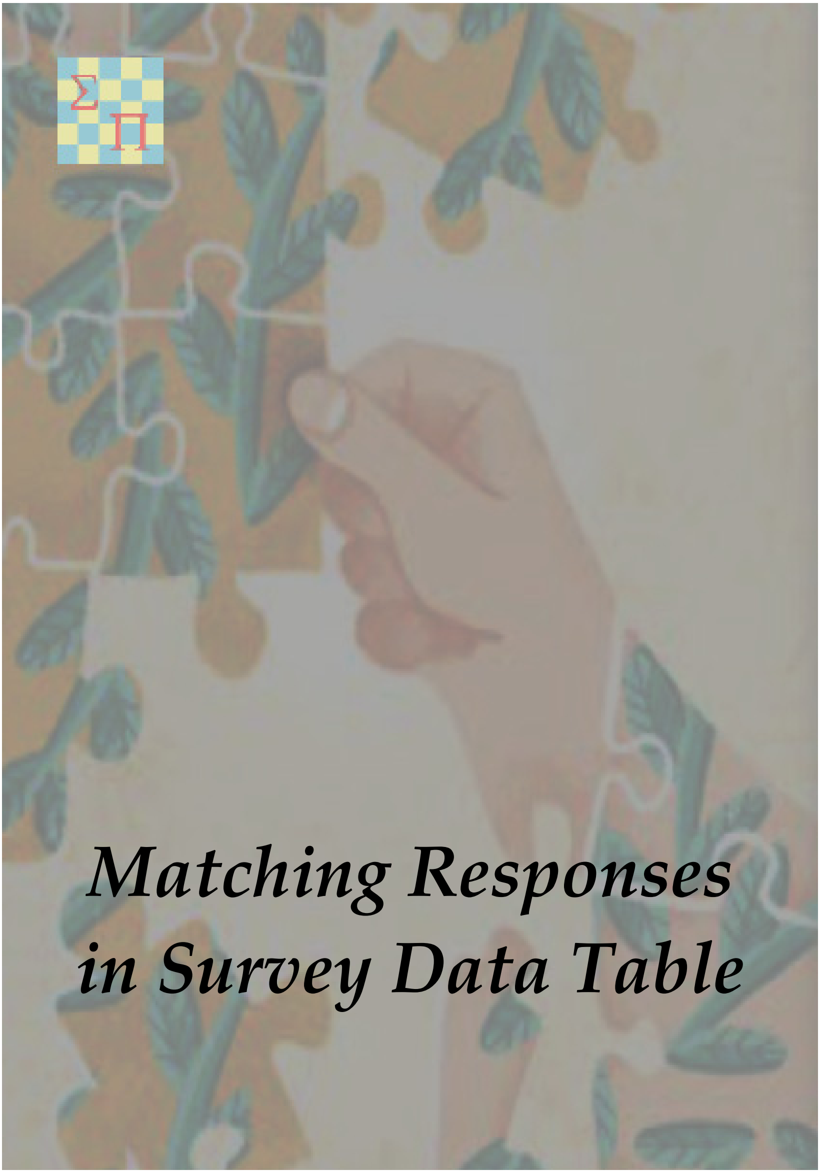

Matching Responses in Survey Data Table

A Fast Algorithm for Finding Matching Responses in Survey Data Table

Abstract. The paper addresses an algorithm to perform an analysis on survey data tables with some unreliable entries. The algorithm has almost linear complexity depending on the number of elements in the table. The proposed technique is based on a monotonicity property. An implementation procedure of the algorithm contains a recommendation that might be realistic for clarifying the analysis results.
Keywords: survey; boolean; data table; matrix.

1. INTRODUCTION

Situations in which customer responses being studied are measured by means of survey data arise in the market investigations. They present problems for producing long-term forecasts because the traditional methods based on counting the matching responses in the survey with a large customer population are hampered by unreliable human nature in the answering and recording process. Analysis institutes are making considerable and expensive efforts to overcome this uncertainty by using different questioning techniques, including private interviews, special arrangements, logical tests, "random" data collection, questionnaire scheme preparatory spot tests, etc. However, percentages of responses representing the statistical parameters rely on misleading human nature and not on a normal distribution. It appears thereby impossible to exploit the most simple null hypothesis technique because the distributions of similar answers are unknown. The solution developed in this paper to overcome the hesitation effect of the respondent, and sometimes unwillingness, rests on the idea of searching so-called "agreement lists" of different questions. In the agreement list, a significant number of respondents do not hesitate in choosing the identical answer options, thereby expressing their willingness to answer. These respondents and the agreement lists are classified into some two-dimensional lists – "highly reliable blocks".

For survey analysts with different levels of research experience, or for the people mostly interested in receiving results by their methods, or merely for those who are familiar with only one, "the best survey analysis technique", our approach has some advantages. Indeed, in the survey, data are collected in such a way that can be regarded as respondents answering a series of questions. A specific answer is an option such as displeased, satisfied, well contented, etc. Suppose that all respondents participating in the survey have been interviewed using the same questionnaire scheme. The resulting survey data can then be arranged in a table $X = \langle x_{iq} \rangle$, where x_{iq} is a Boolean vector of options available, while the respondent i is answering the question q. In this respect, the primary table X is a collection of Boolean columns where each column in the collection is filled with Boolean elements from only one particular answer option. Our algorithm will always try to detect some highly reliable blocks in the Table X bringing together similar columns, where only some trustworthy respondents are answering identically. Detecting these blocks, we can separate the survey data. Then, we can reconstruct the data back from those blocks into the primary survey data table $X' = \langle x'_{iq} \rangle$ format, where some "non-matching/ doubtful" answers are removed. Such a "data-switch" is not intended to replace the researchers' own methods, but may be complementary used as a "preliminary data filter" - separator. The analysts' conclusions will be more accurate after the data-switch has been done because each filtered data item is a representative for some "well known sub-tables".

Our algorithm in an ordinary form dates back to Mullat (1971). At first glance, the ordinary form seems similar to the greedy heuristic (Edmonds 1971), but this is not the case. The starting point for the ordinary version of the algorithm is the entire table from which the elements are removed. Instead, the greedy heuristic starts with the empty set, and the elements are added until some criterion for stopping is fulfilled. However, the algorithm developed in the present paper is quite different. The key to our paper is that the properties of the algorithm remain unchanged under the current construction. For matching responses in the Boolean table, it has a lower complexity.

The monotone property of the proposed technique - "monotone systems idea" - is a common basis for all theoretical results. It is exactly the same property (iii) of submodular functions brought up by Nemhauser et al (1978, p.269). Nevertheless, the similarity does not itself diminish the fact that we are studying an independent object, while the property (iii) of submodular set functions is necessary, but not sufficient.

From the very start, the theoretical apparatus called the "monotone system" has been devoted to the problem of finding some parts in a graph that are more "saturated" than any other part with "small" graphs of the same type (see Mullat, 1976). Later, a Markov chain replaced the graph presentation form where the rows-columns may be split implementing the proposed technique into some sequence of submatrices (see Mullat, 1979). There are numerous applications exploiting the monotone systems ideas; see Ojaveer et al (1975). Many authors have developed a thorough theoretical basis extending the original conception of the algorithm; see Libkin et al (1990) and Genkin and Muchnik (1993).

The rest of the paper is organized as follows. In Section 2, a reliability criterion will be defined for blocks in the Boolean table B. This criterion guarantees that the shape of the top set of our theoretical construction is a sub-matrix - a block; see the Proposition 1. However, the point of the whole monotone system idea is not limited by our specific criterion as described in Section 2. This idea addresses the question: How to synthesize an analysis model for data matrix using quite simple rules? In order to obtain a new analysis model, the researcher has only to find a family of π-functions suitable for the particular data. The shape of top sets for each particular choice of the family of π-functions might be different; see the note prior to our formal construction. For practical reasons, especially in order to help the process of interpretation of the analysis results, in Section 3 there are some recommendations on how to use the algorithm on the somewhat extended Boolean tables $B^{\pm}$. Section 4 is devoted to an exposition of the algorithm and its formal mathematical properties, which are not yet utilized widely by other authors.

2. RELIABILITY CRITERION

In this Section we deal with the criterion of reliability for blocks in the Boolean tables originating from the survey data. In our case we analyze the Boolean table $B = \langle b_{ij} \rangle$ representing all respondents $\langle 1,\ldots,i,\ldots,n \rangle$, but including only some columns $\langle 1,\ldots,j,\ldots m \rangle$ from the primary survey data table $X = \langle x_{iq} \rangle$; see above. The resulting data of each table B can be arranged in a $n \times m$ matrix. Those Boolean tables are then subjected to our algorithm separately, for which reason there is no difference between any sub-table in the primary survey data and a Boolean table. A typical example is respondent satisfaction with services offered, where $b_{ij} = 1$ if respondent i is satisfied with a particular service j level, and $b_{ij} = 0$ if he is unsatisfied. Thus, we analyze any Boolean table of the survey data independently.

Let us find a column j with the *most* significant frequency F of 1-elements among all columns and throughout all rows in table B. Such rows arrange a $g=1$ *one*-column sub-table pointing out only those respondents who prefer *one* specific *most* significant column j. We will treat, however, a more general criterion. We suggest looking at some significant number of respondents where at least F of them are granting at least g Boolean 1-elements in each single row within the range of a particular number of columns. Those columns arrange what we call an agreement list, $g=2,3,\dots$; g is an agreement level.

The problem of how to find such a significant number of respondents, where the F criterion reaches its global maximum, is solved in Section 4. An optimum table S^*, which represents the outcome of the search among all "subsets" H in the Boolean table B, is the solution; see Theorem I. The main result of the Theorem I ensures that there are at least F positive responses in each column in table S^*. No superior sub-table can be found where the number of positive responses in each column is greater F. Beyond that, the agreement level is at least equal to $g=2,3,\dots$ in each row belonging to the best sub-table S^*; g is the number of positive responses within the agreement list represented by columns in sub-table S^*. In case of an agreement level $g=1$, our algorithm in Section 4 will find out only *one* column j with the *most* significant positive frequency F among all columns in table B and throughout all respondents, see above. Needless to say that it is worthless to apply our algorithm in that particular case $g=1$, but the problem becomes fundamental as soon as $g=2,3,\dots$.

Let us look at the problem more closely. The typical attitude of the respondents towards the entire list of options — columns in table B — can be easily "accumulated" by the total number of respondent i positive hits selected:

$$r_i = \sum_{j=1,\dots,m} b_{ij}.$$

Similarly, each column - option can be measured by means of the entire Boolean table B as

$$c_j = \sum_{i=1,\dots,n} b_{ij} .$$

It might appear that it should be sufficient to choose the whole table B to solve our problem provided that $r_i \geq g, i = \overline{1,n}$. Nevertheless, let us look throughout the whole table and find the worse case where the number $c_j, j = \overline{1,m}$ reaches its minimum F. Strictly speaking, it does not mean that the whole table B is the best solution just because some "poor" columns (options with rare responses - hits) may be removed in order to raise the worst-case criterion F on the remaining columns. On the other hand, it is obvious that while removing "poor" columns, we are going to decrease some r_i numbers, and now it is not clear whether in each row there are at least $g = 2,3,\dots$ positive responses. Trying to proceed further and removing those "poor" rows, we must take into account that some of c_j numbers decrease and, consequently, the F criterion decreases as well. This leads to the problem of how to find the optimum sub-table S^*, where in the worst-case F criterion reaches its *global maximum*? The solution is in Section 4.

Finally, we argue that the intuitively well-adapted model of 100% matching 1-blocks is ruled out by any approach trying to qualify the real structure of the survey data. It is well known that the survey data matrices arising from questionnaires are fairly empty. Those matrices contain plenty of small 100% matching 1-blocks, whose individual selection makes no sense. We believe that the local worst-case criterion F top set, found by the algorithm, is a reasonable compromise. Instead of 100% matching 1-blocks, we detect somewhat blocks less than 100% filled with 1-elements, but larger in size.

3. RECOMMENDATIONS

We consider the interpretation of the survey analysis results as an essential part of the research. This Section is designed to give guidance on how to make the interpretation process easier. In each survey data it is possible to conditionally select two different types of questions: (1) The answer option is a fact, event, happening, issue, etc.; (2) The answer is an opinion, namely displeased, satisfied, well contented etc.; see above. It does not appear from the answer to options of type 1, which of them is positive or negative, whereas type 2 allows us to separate them. The goal behind this splitting of type 2 opinions is to extract from the primary survey data table two Boolean sub-tables: table B^{+}, which includes type 1 options mixed with the positive options from type 2 questions, and table B^{-} where type 1 options are mixed together with the negative type 2 options - opinions. It should be noticed that doing it this way, we are replacing the analysis of primary survey data by two Boolean tables where each option is represented by one column. Tables B^{+} and B^{-} are then subjected to the algorithm separately.

To initiate our procedure, we construct a sub-table K_1^{+} implementing the algorithm on table B^{+}. Then, we replace sub-table K_1^{+} in B^{+} by zeros, constructing a restriction of table $B^{\pm}$. Next, we implement the algorithm on this restriction and find a sub-table K_2^{+}, after which the process of restrictions and sub-tables sought by the algorithm may be continued. For practical purposes we suggest stopping the extraction with three sub-tables: K_1^{+}, K_2^{+} and K_3^{+}. We can use the same procedure on the table B^{-}, extracting sub-tables K_1^{-}, K_2^{-} and K_3^{-}.

The number of options-columns in the survey Boolean tables $B^{\pm}$ is quite significant. Even a simple questionnaire scheme might have hundreds of options - the total number of options in all questions. It is difficult, perhaps almost impossible, within a short time to observe those options among thousands of respondents. Unlike Boolean tables $B^{\pm}$, the sub-tables $K_{1,2,3}^{\pm}$ have reasonable dimensions. This leads to the following interpretation opportunity: the positive options in $K_{1,2,3}^{+}$ tables indicate some most successful phenomena in the research while the negative options in $K_{1,2,3}^{-}$ point in the opposite direction. Moreover, the positive and negative sub-tables $K_{1,2,3}^{\pm}$ enable the researcher in a short time to “catch” the “sense” in relations between the survey options of type 1 and positive/negative options of the type 2. For instance, to observe all Pearson's *r* correlations a calculator has to perform $O(n \cdot m^2)$ operations depending on the $n \times m$ table dimension, n-rows and m-columns. The reasonable dimensions of the sub-tables $K^{\pm}_{1,2,3}$ can reduce the amount of calculations drastically. Those sub-tables - blocks $K^{\pm}_{1,2,3}$, which we recommend to select in the next Section as index-function $F(H)$ top sets found via the algorithm, are not embedded and may not have intersections; see the Proposition 1. Concerning the interpretation, it is hoped that this simple approach can be of some use to researchers in elaborating their reports with regard to the analysis of results.

4. Definitions and Formal Mathematical Properties

In this Section, our basic approach is formalized to deal with the analysis of the Boolean $n \times m$ table B, n-rows and m-columns. Henceforth, the table B will be the Boolean table $B^{\pm}$ - see above - representing certain options-columns in the survey data table. Let us consider the problem of how to find a sub-table consisting of a subset S_{max} of the rows and columns in the original table B with the properties: (1) that $r_i = \sum_j b_{ij} \geq g$ and (2) the minimum over j of $c_j = \sum_i b_{ij}$ is as large as possible, precisely – the global maximum. The following algorithm solves the problem.

Algorithm.

Step I. Set up the initial values.

1i. Set minimum and maximum bounds a, b on threshold u for c_j values.

Step A. To find that the next step **B** produces a non-empty sub-table.

1a. Using step **B**, test u as $(a+b)/2$.

If it succeeds, replace a by u. If it fails replace b by u.

2a. Go to **1a**.

Step B. To test whether the minimum over j can be at least u.

1b. Delete all rows whose sums $r_i < g$.

This step **B** fails if all must be deleted; return to step **A**.

2b. Delete all columns whose sums $c_j \leq u$.

This step **B** fails if all must be deleted, return to step **A**.

3b. Perform step **T** if none deleted in **1b** and **2b**; otherwise go to **1b**.

Step T. Test that the global maximum is found.

1t. Among numbers c_j find the minimum.

With this new value as u test performing step **B**.
If it succeeds, return to step **A**, otherwise final stop.

Step **B** performed through the step **T** tests correctly whether a sub-matrix of B can have the rows sums at least g and the column sums at least u. Removing row i, we need to perform no more than m operations to recalculate c_j values; removing column j, we need no more than n-operations. We can proceed through **1b** no more than n-times and through **2b**, m-times. Thus, the total number of operations in step **B** is $O(nm)$. The step **A** tests the step **B** no more than $\log_2 n$ times. Thus, the total complexity of the algorithm is $O(\log_2 n \times nm)$ operations.

Note. It is important to keep in mind that the algorithm itself is a particular case of our theoretical construction. As one can see, we are deleting rows and columns including their elements all together, thereby ensuring that the outcome from the algorithm is a sub-matrix. But, in order to expose the properties of the algorithm, we look at the Boolean elements separately. However, in our particular case of π-functions it makes no difference. The difference will be evident if we utilize some other family of π-functions, for instance $\pi = c_j \max(r_i, c_j)$. We may detect top binary relations, which we call kernels, different from submatrices. It may happen that some kernel includes two blocks - one quite long in the vertical direction and the other - in the horizontal. All elements in the empty area between these blocks in some cases cannot be added to the kernel. In general, we cannot guarantee either the above low complexity of the algorithm for all families of π-functions, but the complexity still remains in reasonable limits.

We now consider the properties of the algorithm in a rigorous mathematical form. Below we use the notation $\mathrm{H} \subseteq \mathrm{B}$. The notation H contained in B will be understood in an ordinary set-theoretical vocabulary, where the Boolean table B is a set of its Boolean 1-elements. All 0-elements will be dismissed from the consideration. Thus, H, as a binary relation, is also a subset of a binary relation B. However, we shall soon see that the top binary relations - kernels from the theoretical point of view are also sub-matrices for our specific choice of π-functions. Below, we refer to an element we assume that it is a Boolean 1-element.

For an element $\alpha \in \mathrm{B}$ in the row i and column j we use the similarity index $\pi = \mathrm{c}_{\mathrm{j}}$ if $\mathrm{r}_{\mathrm{i}} \geq \mathrm{g}$ and $\pi = 0$ if $\mathrm{r}_{\mathrm{i}} < \mathrm{g}$, counting only on Boolean elements belonging to H. The value of π depends on each subset $\mathrm{H} \subseteq \mathrm{B}$ and we may thereby write $\pi \equiv \pi(\alpha, \mathrm{H})$: the set H is called the π-function parameter. The π-function values are the real numbers - the similarity indices. In Section 2 we have already introduced these indices on the entire table B. Similarity indices, as one can see, may only concurrently increase with the "expansion" and decrease with the "shrinking" of the parameter H. This leads us to the fundamental definition.

Definition 1. Basic monotone property. *By a monotone system will be understood a family* $\{\pi(\alpha, \mathrm{H}) : \mathrm{H} \subseteq \mathrm{B}\}$ *of* π*-functions, such that the set* H *is to be considered as a parameter with the following monotone property: for any two subsets* $\mathrm{L} \subset \mathrm{G}$ *representing two particular values of the parameter* H *the inequality* $\pi(\alpha, \mathrm{L}) \leq \pi(\alpha, \mathrm{G})$ *holds for all elements* $\alpha \in \mathrm{B}$.

We note that this definition indicates exactly that the fulfilment of the inequality is required for all elements $\alpha \in \mathrm{B}$. However, in order to prove the Theorems 1,2 and the Proposition 1, it is sufficient to demand the inequality fulfillment only for elements $\alpha \in \mathrm{L}$; even the numbers π themselves may not be defined for $\alpha \notin \mathrm{L}$. On the other hand, the fulfillment of the inequality is necessary to prove the argument of the Theorem 3 and the Proposition 2. It is obvious that similarity indices $\pi = \mathrm{c}_{\mathrm{j}}$ comply with the monotone system requirements.

Definition 2. *Let* $V(H)$ *for a non-empty subset* $H \subseteq B$ *by means of a given arbitrary threshold* u° *be the subset* $V(H) = \{\alpha \in B : \pi(\alpha, H) \geq u^{\circ}\}$. *The non-empty* H*-set indicated by* S° *is called a stable point with reference to the threshold* u° *if* $S^{\circ} = V(S^{\circ})$ *and there exists an element* $\xi \in S^{\circ}$, *where* $\pi(\xi, S^{\circ}) = u^{\circ}$. See Mullat (1981, p.991) for a similar concept.

Definition 3. *By monotone system kernel will be understood a stable set* S^* *with the maximum possible threshold value* $u^* = u_{\max}$.

We will prove later that the very last pass through the step **T** detects the largest kernel $\Gamma_p = S^*$. Below we are using the set function notation $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$.

Definition 4. *An ordered sequence* $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ *of distinct elements in the table B, which exhausts the whole table,* $d = \sum_{i,j} b_{ij}$, *is called a defining sequence if there exists a sequence of sets* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *such that:*

A. *Let the set* $H_k = \{\alpha_k, \alpha_{k+1}, \ldots, \alpha_{d-1}\}$. *The value* $\pi(\alpha_k, H_k)$ *of an arbitrary element* $\alpha_k \in \Gamma_j$, *but* $\alpha_k \notin \Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$, $j = 0,1,\ldots, p-1$.

B. *In the set* Γ_p *there does not exist a proper subset* L, *which satisfies the strict inequality* $F(\Gamma_p) < F(L)$.

Definition 5. *A subset* D^* *of the set* B *is called definable if there exists a defining sequence* $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ *such that* $\Gamma_p = D^*$.

Theorem 1. *For the subset* S^* *of* B *to be the largest kernel of the monotone system - to contain all other kernels - it is necessary and sufficient that this set is definable:* $S^* = D^*$. *The definable set* D^* *is unique.*

We note that the Theorem 3 will establish the existence of the largest kernel later.

Proof.

Necessity. If the set S^* is the largest kernel, let's look at the following sequence $B = \Gamma_0 \supset \Gamma_1 = S^*$ of only two sets. Suppose we have found elements $\alpha_0, \alpha_1, ...\alpha_k$ in $B \setminus S^*$ such that for each $i = \overline{1,k}$ the value $\pi(\alpha_i, B \setminus \{\alpha_0, ..., \alpha_{i-1}\})$ is less than $u^o = u_{max}$ and $\alpha_0, \alpha_1, ...\alpha_k$ does not exhaust $B \setminus S^*$. Then, in $(B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}$ some α_{k+1} exists such that $\pi(\alpha_{k+1}, (B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}) < u^*$. Otherwise, the set $(B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}$ is a larger kernel than with the same value u^*. Thus, the induction is complete.

This gives the ordering with the property (a). If the property (b) failed, then u^* would not be a maximum, contradicting the definition of the kernel. This proves the necessity.

Sufficiency. Note that every time the algorithm — see above — goes through step **T**, some stable point, a set S° is put in the form of a set $\Gamma_j = S^\circ$, $j = 0,1,..., p-1$, where $u_j = \min_{\alpha \in S^\circ} \pi(\alpha, S^\circ)$. Obviously, these stable "layering" points (stable sets) form an embedded chain of sets $B = \Gamma_0 \supset \Gamma_1 \supset ... \supset \Gamma_p = D^*$. Let the set $L \subseteq B$ be the largest core. Suppose that this L is a proper subset of D^*, then by property (b) $F(D^*) \geq F(L)$ and hence D^* is also a kernel. The set L as the largest kernel cannot be a proper subset of D^* and therefore must be equal to D^*.

Suppose now that L is not the subset of D^*. Let H_s be the smallest set $H_k = \{\alpha_k, \alpha_{k+1}, ..., \alpha_{d-1}\}$, which includes L. The value $\pi(\alpha_s, H_s)$ by our basic monotone property must be grater than, or at least equal to u^*, since α_s is an element of H_s and it is also an element of the kernel L and $L \subseteq H_s$. By property (a) this value is strictly less than $F(\Gamma_{j+1})$ for some $j = 0,1,\ldots,p-1$. But that contradicts the maximality of u^*. This proves the sufficiency. Moreover, it proves that any largest kernel equals D^* so that it is the unique largest kernel. This concludes the proof. ■

Proposition 1. *The largest kernel is a sub-matrix of the table* B.

Proof. Let S^* be the largest kernel. If we add to S^* any element lying in a row and a column where S^* has existing elements, then the threshold value u^* cannot decrease. So by maximality of the set S^* this element must already be in S^*. ■

Now, we need to focus on the individual properties of the sets $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$, which have a close relation to the case $u < u_{max}$ - a subject for a separate inquiry. Let us look at the step **T** of the algorithm originating the series of mapping initiating from the whole table B in form of $V(B), V(V(B),\ldots$ with some particular threshold u. We denote $V(V(B))$ by $V^2(B)$, etc.

Definition 6. *The chain of sets* $B, V(B), V^2(B),\ldots$ *with some particular threshold u is called the central series of monotone system; see Mullat (1981) for exactly the same notion.*

Theorem 2. *Each set* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *in the defining sequence* $\alpha_0, \alpha_1, \ldots, \alpha_{d-1}$ *is the central series convergence point* $\lim_{k=2,3,\ldots} V^k(B)$ *as well as the stable point for some particular thresholds values* $F(W) = u_0 < u_1 < \ldots < u_n = F(S^*)$. *Each* Γ_j *is the largest stable point - including all others for threshold values* $u \geq u_j = F(\Gamma_j)$.

It is not our intention to prove the statement of Theorem 2 since this proof is similar to that of Theorem 1. Theorem 1 is a particular case for Theorem 2 statement regarding threshold value $u = u_p$.

Next, let us look at the formal properties of all kernels and not only the largest one found by the algorithm. It can easily be proved that with respect to the threshold $u_{max} = u_p$ the subsystem of all kernels classifies a structure, which is known as an upper semilattice in lattice theory.

Theorem 3. *The set of all kernels - stable points - for* u_{max} *is a full semilattice.*

Proof. Let Ω be a set of kernels and let $K_1 \in \Omega$ and $K_2 \in \Omega$. Since the inequalities $\pi(\alpha, K_1) \geq u$, $\pi(\alpha, K_2) \geq u$ are true for all K_1 and K_2 elements on each K_1, K_2 separately, they are also true for the union set $K_1 \cup K_2$ due to the basic monotone property. Moreover, since $u = u_{max}$, we can always find an element $\xi \in K_1 \cup K_2$ where $\pi(\xi, K_1 \cup K_2) = u$. Otherwise, the set $K_1 \cup K_2$ is some H-set for some u' greater than u_{max}. Now, let us look at the sequence of sets $V^k(K_1 \cup K_2)$, $k = 2,3,\ldots$, which certainly converges to some non empty set - stable point K. If there exists any other kernel $K' \supset K_1 \cup K_2$, it is obvious, that applying the basic monotone property we get that $K' \supseteq K$. ■

With reference to the highest-ranking possible threshold value $u_p = u_{max}$, the statement of Theorem 3 guarantees the existence of the largest stable point and the largest kernel S^* (compare this with equivalent statement of Theorem 1).

Proposition 2. *Kernels of the monotone system are submatrices of the table* B.

Proof. The proof is similar to proposition 1. However, we intend to repeat it. In the monotone system all elements outside a particular kernel lying in a row and a column where the kernel has existing elements belong to the kernel. Otherwise, the kernel is not a stable point because these elements may be added to it without decreasing the threshold value u_{max}.

Note that Propositions 1,2 are valid for our specific choice of similarity indices $\pi = c_j$. The point of interest might be to verify what π-function properties guarantee that the shape of the kernels still is a submatrix. The defining sequence of table B elements constructed by the algorithm represents only some part $u_0 < u_1 < u_2 < ... < u_p$ of the threshold values existing for central series in the monotone system. On the other hand, the original algorithm, Mullat (1971), similar to the inverse Greedy Heuristic, produces the entire set of all possible threshold values u for all possible central series, what is sometimes unnecessary from a practical point of view. Therefore, the original algorithm always has the higher complexity.

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ЭКСТРЕМАЛЬНЫЕ ПОДСИСТЕМЫ МОНОТОННЫХ СИСТЕМ. I

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Рассматривается общая теоретическая модель, предназначенная для начального этапа анализа систем взаимосвязанных элементов. В рамках модели и исходя из специально постулированного свойства монотонности систем гарантируется существование особых под-систем — ядер. Устанавливается ряд экстремальных свойств и структура ядер в монотонных системах. Детализируется язык описания монотонных систем взаимосвязанных элементов на общем теоретико-множественном уровне, и на его основе вырабатывается конструктивная система понятий в случае систем с конечным числом элементов. Изучается ряд свойств особых конечных последовательностей элементов системы, с помощью которых осуществимо выделение ядер в монотонных системах.

1. Введение

При изучении поведения сложной системы часто приходится сталкиваться с задачей анализа конкретных числовых данных о функционировании системы. На основе подобных данных иногда требуется выяснить, существуют ли в системе особые элементы или подсистемы элементов, реагирующих однотипно на какие-либо «воздействия», а также «отношения» между однотипными под-системами. Сведения о существовании указанных особенностей или о «структуре» изучаемой системы необходимы, например, до проведения обширных или дорогостоящих статистических исследований.

В связи с широким применением вычислительной техники в настоящее время на начальном этапе выявления структуры системы намечается подход, основанный на различного рода эвристических моделях [1—4]. При построении моделей многие авторы исходят из содержательных постановок задач, а также из формы представления исходной информации [5, 6].

Естественной формой представления информации для целей изучения сложных систем является форма графа [7]. Распространенным носителем информации служит также матрица, например матрица данных [8]. Матрицы и графы легко допускают выделение двух минимальных структурных единиц системы: «элементов» и «связей» между элементами*. В данной работе понятия «связь» и «элемент» трактуются достаточно широко. Так, инргда желательно рассматривать связи в виде элементов системы; в этом случае можно обнаружить более «тонкие» зависимости в исходной системе. Представление системы в виде единого объекта — элементы и связи между элементами — позволяет придать более четкий смысл задаче выявления структуры системы. Структура системы — это такая организация элементов системы в подсистемы, которая складывается в виде множестваотношений между подсистемами. Структурой системы, например, может быть естественно сложившийся способ объединения подсистем в единую систему, который определяется на основе «сильных» и «слабых» связей между элементами системы. Подобный подход к анализу систем описан, например, в [9], где рассматривается вопрос агрегирования систем взаимосвязанных элементов. Агрегирование оказывается удобным макроязыком для вскрытия структуры системы.

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* В литературе подобные системы называются системами взаимосвязанных элементов.

Extremal Subsystems of Monotonic Systems, I [i]

Abstract. A general theoretical method is described which is intended for the initial analysis of systems of interrelated elements. Within the framework of the model, a specially postulated monotonicity property for systems guarantees the existence of a special kind of subsystems called kernels. A number of extremal properties and the structure of the kernels are found. The language of description of monotonic systems of interrelated elements is described in general set-theoretic terms and leads to a constructive system of notions in the case of systems with finite number of elements. A series of properties of special finite sequences of elements are studied whereby kernels in monotonic systems are classified.
Keywords: monotonic; system; matrix; graph; cluster

1. INTRODUCTION

For the study of a complex system, it is often necessary to encounter the problem of analyzing concrete numerical data about the system functioning. Sometimes based on similar data it is required to show whether in the system there exist special elements or subsystems, reacting in one way to some “actions” as well as “relations” between one-type subsystems. Information on the existence of the indicated peculiarities or on the “structure” of the system under study is necessary, for example, before carrying out extensive or expensive statistical investigation.

Concerning wide application of computational techniques, at the present time, to initial detection of the structure of a system an approach based on various kind of heuristic models is planned (Braverman et al, 1974; McCormik, 1972; Deutch, 1971; Zahn, 1971). For constructing models, many authors start with intuitive formulations of the problem and also with the form of presentation of the initial data (Võhandu, 1964; Terent’ev, 1959).

A natural form of presentation the data for the purpose of studying complex systems is that of a graph (Muchnik, 1974). A matrix, for example, a data matrix (Hartigan, 1972) also serves as a widely spread carrier of information. Matrices and graphs easily admit isolation of two minimal structural units of the system: “elements” and “connections” between elements.[1] In this paper the notions “connections” and “elements” are interrelated in a sufficiently broad fashion. Thus, sometimes it is desirable to consider connections in the form of elements of a system; in this case, it is possible to find more “subtle” relations in the original system.

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[1] Analogous systems are called systems of interrelated elements in the literature.

Representation of the system in the form of a unique object — elements and connections between elements — makes it possible to give a more precise meaning to the problem of revealing the structure of the system. The structure of a system is the organization of system elements into subsystems, which are composed as a set of relationships between subsystems. The structure can, for example, be a natural way of combining subsystems into a single system, which is determined on the basis of "strong" and "weak" links between the elements of the system. A similar approach to systems analysis is described (for example, Braverman et al, 1971), where the issue of assembling systems from interconnected elements is considered. Assembly turns out to be a convenient macro language for expressing the structure of a system.

In the theory of systems, usually direct connections between elements are considered. Situation, however, sometimes requires considering indirect connections as well. This requirement is distinguished thus: that indirect connections are dynamic relations in the sense that “degree” of dependence is determined by a subsystem, in which this or that connection is considered. Below we describe and study a certain subclass of similar “dynamic” systems called monotonic systems.

The monotonicity property for systems allows us to formulate in a general form the concept of a kernel of a system as a subsystem, which in the originally indicated sense reflects the structure of the whole system in the large. A kernel represents a subsystem whose elements are “sensitive” in the highest degree to one of two types of actions (positive or negative), since “sensibility” to actions is determined by the intrinsic structure of the system. The definition of positive and negative actions reduces to the existence of two types of kernels – positive and negative kernels.

Existence of kernels (special subsystems) is guaranteed by the mathematical model described in this paper and the problem of “isolating” kernels is typical problem in the description of a “large” system in the language of a “small” system – kernel. In this sense, figuratively speaking, a kernel of a system is a subsystem whose removal inflicts “cardinal” changes the properties of that system: The system "gives up" the existing structure.

For exposition of the material terminology and symbolism, the theory of sets is used which requires no special knowledge. One should turn attention to the special notation introduced, since the apparatus developed in this paper is new.

2. Examples of Monotonic Systems [2]

1. In the n-dimensional vector space let there be given N vectors. For each pair of vectors x and y one can define in many ways a distance $\rho(x, y)$ between these vectors (i.e., to scale the space). Let us assume that the set of given vectors forms an unknown system W.

For every vector in an arbitrary subsystem of W we calculate the sum of distances to all vectors situated inside the selected subsystem. Thus, with the respect to each subsystem of W and each vector situated inside that subsystem, a characteristic sum of distances is defined, which can be different for different subsystems.

It is not difficult to establish the following property of the set of sums of distances. Because of removing a vector from the subsystem the sums computed for the remaining vectors decrease while because of adding a vector to the subsystem they increase. A similar property of sums for every subsystem of system W is called in this paper the monotonicity property and a system W having such a property is called a monotonic system.

2. For studying schools, directions in various branches of science, the so-called graphs of cited publications (Nalimov and Mul'chenko, 1969) are used. These are directed a-cyclic graphs, since each author can cite only those authors whose papers are already published. It is entirely reasonable to assume that the set of publications W forms a certain system, where the system elements (published papers) exchange with each other information by special way, namely, by the help of citation. If we consider a subset from an available survey of the set of publications W, then the number of bibliographical tittles can characterize each publication, taken only over the subset – subsystem – considered. It is clear that "removal" of publication from the subsystem only decreases the quantitative evalua-

[2] Kempner, Y., Mirkin, B., and Muchnik, I. B., "Monotone linkage clustering and quasi-concave set functions," Applied Mathematics Letters, 1997, 4, 19; B. Mirkin and I. Muchnik, "Layered Clusters of Tightness Set Functions," Applied Mathematics Letters, **2002**, v. 15, issue no. 2, pp. 147-151.; see also, A. V. Genkin (Moscow), I. B. Muchnik (Boston), "Fixed Approach to Clustering, Journal of Classification," Springer, 1993, 10, pp. 219-240,; and latest connection, Kempner, Y., Levit V. E., "Correspondence between two antimatroid algorithmic characterizations," Dept. of Computer Science, Holon Academic Institute of Technology, July, 2003, Israel,.

tion thus introduced for the degree of exchange of information in the subsystem while the "addition" of a publication in the subsystem only increases that evaluation for all publications in the subsystem. Thus, we have here a monotonic citation system given in the form of a graph.

In connection with the above example, it is interesting to note (Trybulets, 1970), where the author involuntarily considers an example of a monotonic system in the form of a directed graph.

3. Let us assume that there is a set W of telephone exchanges or points of connection that are joined by lines of two-sided connections. Under the absence of any connection between points in a system with communications, it is possible to organize a transit connection. If a functioning of a similar system is observed for a long time, then the "quality" of connection" between each pair of points can be expressed, independently of whether there exists a two-sided connection or not, by the average number of "denials" in establishing a connection between them in a standard unit of time. Generally speaking, if it is desired to characterize each point of the system W in the sense of "unreliability" of establishing connections with other points, then this second characteristic can be taken to be the average number of denials in establishing connection with at least one point of the system in a unit time. It is clear that these same numerical qualities (quality of connection, unreliability characteristic) can be defined only inside every subsystem of the system with communications W.

The proposed model has the following obvious properties. A gap in any line of two-sided connection increases the average number of denials among all other points of connection; introduction of any new line, in contrast decreases the average number of denials. This is related with the fact that load on the realization of a transit connection in a telephone communication network increases (decreases). In the case of curtailment of activity at any point of connection inside the given subsystem the unreliability of all points of subsystem increases while in case of addition of a point of connection to the subsystem the unreliability decreases.

Thus, there is a complete similarity with the examples of monotonic systems considered above and one can state that the model described for telephone communications is a monotonic system.

In the present paper a monotonic system is defined, to be a system over whose elements one can perform “positive” and “negative” actions. In addition, positive actions increase certain quantitative indicators of the functioning of a system while the negative actions decrease those indicators. In the second example considered above the positive action is the addition of an element to a subsystem while the negative action is removing an element from the subsystem; in the third example the converse holds.

In the second and third examples above, the kernel must have an intuitive meaning. Thus, in the citation graphs, a negative kernel must turn out to be the set of publications citing each other in a considerable degree (by authors representing a single scientific school) while a positive kernel must consist of publications citing each other to a lesser degree (representing different schools).

In telephone communications networks the intuitive sense of a kernel must manifest itself in the following. If we take as elements of a communication network the lines of connection, then a negative kernel is a collection of lines that give on the average a “mutually agreed upon” large number of denials while a positive kernel has the opposite sense – a collection of lines that give on the average less denials. In case the system elements are taken to be the connection points of a telephone communication network, a negative kernel is a set of mutually unreliable points while a positive kernel is a set of more reliable points.

The intuitive meaning given to kernels of citation graphs and communication network is not based on a sufficient number of experimental facts. The indicated properties are noted in analogy with available intuitive interpretation of kernels obtained for solutions of automatic-classification problems (Mullat, 1975).

3. Description of a Monotonic System

One considers some system W consisting of a finite number of elements,[3] i.e., $|W| = N$, where each element α of the system W plays a well-defined role. It is supposed that the states of elements α of W are described by definite numerical quantities characterizing the “significance” level of elements α for the operation of the system as a whole and that from each element of the system one can construct some discrete actions.

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[3] If W is a finite set, then $|W|$ denotes the number of its elements.

We reflect the intrinsic dependence of system elements on the significance levels of individual elements. The intrinsic dependence of elements can be regarded in a natural way as the change, introducible in the significance levels of elements β, rendered by a discrete action produced upon element α.

We assume that the significance level of the same element varies as a result of this action. If the elements in a system are not related with each other in any way, then it is natural to suppose that the change introduced by element α on significance β (or the influence of α on β) equals zero.

We isolate a class of systems, for which global variations in the significance levels introduced by discrete actions on the system elements bears a monotonic character.

Definition. By a monotonic system, we understand a system, for which an action realized on an arbitrary element α involves either only decrease or only increase in the significance levels of all other elements.

In accordance with this definition of a monotonic system two types of actions are distinguished: type $\oplus$ and type $\ominus$. An action of type $\oplus$ involves increase in the significance levels while $\ominus$ involves decrease.

The formal concept of a discrete action on an element α of the system W and the change in significance levels of elements arising in connection with it allows us to define on the set of remaining elements of W an uncountable set of functions whenever we have at least one real significance function $\pi : W \rightarrow D$ (D being the set of real numbers).

Indeed, if an action is rendered on element α, the starting from the proposed scheme one can say that function π is mapped into π_α^+ or π_α^- according as a the action $\oplus$ or $\ominus$. Significance of system elements is redistributed as action on element α changes from function π to π_α^+ $\left(\pi_\alpha^-\right)$ or, otherwise, the initial collection of significance levels $\left\{ \pi(\partial) \,\middle|\, \partial \in W \right\}$ changes into a new collection $\left\{ \pi_\alpha^+(\partial) \,\middle|\, \partial \in W \right\}$.[4]

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[4] Functions π, π_α^+ and π_α^- are defined on the whole set W and, consequently, $\pi_\alpha^+(\partial)$ and $\pi_\alpha^-(\partial)$ are defined.

Clearly, if we are given some sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ of elements of W (arbitrary repetitions and combinations of elements being permitted) and the binary sequence $+,-,+,\ldots$, then by the usual means one can define the functional product of functions $\pi^{+}_{\alpha_1}, \pi^{-}_{\alpha_2}, \pi^{+}_{\alpha_3}$ in the form $\pi^{+}_{\alpha_1}\pi^{-}_{\alpha_2}\pi^{+}_{\alpha_3}$.

The construction presented allows us to write the property of monotonic systems in the form of the following basic inequalities:

$$\pi^{+}_{\alpha}(\partial) \geq \pi(\partial) \geq \pi^{-}_{\alpha}(\partial) \qquad (1)$$

for every pair of elements $\alpha, \partial \in W$, including the pairs α, α or ∂, ∂.

Let there be given a partition of set W into two subsets, i.e., $H \cup \overline{H} = W$ and $H \cap \overline{H} = \varnothing$. If we subject the elements $\alpha_1, \alpha_2, \alpha_3, \ldots \in \overline{H}$ to positive actions only, then by the same token on set W there is defined some function $\pi^{+}_{\alpha_1}\pi^{-}_{\alpha_2}\pi^{+}_{\alpha_3}\ldots$, which can be regarded as defined only on the subset H of W. [5]

If from all possible sequences of elements of set $\overline{H}$ we select a sequence $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{H}|} \right\rangle$, [6] where α_i are not repeated, then on the set H the function $\pi^{+}_{\alpha_1}\pi^{+}_{\alpha_2}\ldots$ is induced ambivalently.

We denote this function $\pi^{+}H$ and call it a standard function. We shall also refer to the function thus introduced as a credential function and to its value on an element as an α credential.

In accordance with this terminology the set $\left\{ \pi^{+}H(\alpha) \,\middle|\, \alpha \in H \right\}$, which is denoted by $\Pi^{+}H$ is called a credential collection given on the set H or a credential collection relative to set H. Let us assume that we

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[5] We are not interested in significance levels obtained as a result of operations on elements of $\overline{H}$ onto the same set $\overline{H}$.

[6] Here symbols $\langle , \rangle$ are used to stress the ordered character of a sequence of $\overline{H}$.

are given a set of credential collections $\left\{ \Pi^{+}\mathrm{H} \,\middle|\, \mathrm{H} \subseteq \mathrm{W} \right\}$ on the set of all possible subsystems $\mathrm{P(W)}$ of system W. The number of all possible subsystems is $\left|\mathrm{P(W)}\right| = 2^{|\mathrm{W}|}$.

Instead of considering a standard function for positive actions $\pi^{+}_{\alpha_1}\pi^{+}_{\alpha_2}\ldots$ one can consider a similar function for negative actions $\pi^{-}\mathrm{H}$. Thus, by exact analogy one defines single credential collection $\Pi^{-}\mathrm{H} = \left\{ \pi^{-}\mathrm{H}(\alpha) \,\middle|\, \alpha \in \mathrm{H} \right\}$ and the aggregate of credential collections $\left\{ \Pi^{-}\mathrm{H} \,\middle|\, \mathrm{H} \subseteq \mathrm{W} \right\}$.

Let us briefly summarize the above construction. Starting with some real function π defined on a finite set W and using the notion of positive and negative actions on elements of system W, one can construct two types of aggregate collections $\Pi^{+}\mathrm{H}$ and $\Pi^{-}\mathrm{H}$ defined on each of the H of subsets of W. Each function from the aggregate (credential collection) is constructed by means of the complement to H, equaling $\mathrm{W} \setminus \mathrm{H}$, and a sequence $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{\mathrm{H}}|} \right\rangle$ of distinct elements of the set $\overline{\mathrm{H}}$. For this actions of types ⊕ and ⊖ are applied to all elements of set $\overline{\mathrm{H}}$ in correspondence with the ordered sequence $\left\langle \alpha_1, \alpha_2, \ldots, \alpha_{|\overline{\mathrm{H}}|} \right\rangle$ in order to obtain $\Pi^{+}\mathrm{H}$ and $\Pi^{-}\mathrm{H}$ respectively.

Credential collections/arrays concept of $\Pi^{+}\mathrm{H}$ and $\Pi^{-}\mathrm{H}$ needs refinement. The definition given above does not taken into account the character of dependence of function $\pi\mathrm{H}$ on the sequence of actions realized on the elements of set $\overline{\mathrm{H}}$.[7] Generally speaking, credential collection $\Pi^{+}\mathrm{H}(\Pi^{-}\mathrm{H})$ is not defined uniquely, since it can happen that for different orderings of set $\overline{\mathrm{H}}$ we obtain different function $\pi\mathrm{H}$.

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[7] In the sequel, if sign "–" or "+" is omitted from our notation, then it is understood to be either "–" or "+"

In order that credential collection $\Pi^{+}H$ $(\Pi^{-}H)$ be uniquely defined by subset H of the set W it is necessary to introduce the notion of commutability of actions.

Definition. An action of type ⊕ or ⊖ is called commutative for system W if for every pair of elements $\alpha, \beta \in W$ we have

$$\pi^{+}_{\alpha}\pi^{+}_{\beta} = \pi^{+}_{\beta}\pi^{+}_{\alpha}, \ \pi^{-}_{\alpha}\pi^{-}_{\beta} = \pi^{-}_{\beta}\pi^{-}_{\alpha}$$

In this case it is easy to show that the values of function πH on the set H do not depend on any order defined for the elements of the set $\overline{H}$ by sequence $\langle \alpha_1, \alpha_2, \ldots \rangle$. The proof can be conducted by induction and is omitted.

Thus, for commutative actions the function $\pi^{+}H$ $(\pi^{-}H)$ is uniquely determined by a subset of W.

In concluding this section, we make one important remark of an intuitive character. As is obvious from the above-mentioned definition of aggregates of credentials collection of type ⊕ and ⊖, the initial credential collection serves as the basic constructive element in their construction. The initial credential collection is a significance function defined on the set of system elements before the actions are derived from the elements. In other words, it is the initial state of the system fixed by credential collection ΠW. It is natural to consider only those aggregates of credential collections that are constructed from an initial ⊕ collection, which is the same as the initial ⊖ collection. The dependence indicated between ⊕ and ⊖ credential collections is used considerably for the proof of the duality theorem in the second part of this paper.

4. EXTREMAL THEOREMS. STRUCTURE OF EXTREMAL SETS [8]

Let us consider the question of selecting a subset from system W whose elements have significance levels that are stipulated only by the internal "organization" of the subsystem and are numerically large or, conversely, numerically small. Since this problem consists of selecting from the whole set of subsystems $P(W)$ a subsystem having desired properties, therefore it is necessary to define more precisely how to prefer one subsystem over another.

Let there be given aggregates of credential collections $\left\{\Pi^{+}H \,\middle|\, H \subseteq W\right\}$ and $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$. On each subset there are defined the following two functions:

$$F_{+}(H) = \max_{\pi \in H} \pi^{+}H(\alpha),$$

$$F_{-}(H) = \min_{\pi \in H} \pi^{-}H(\alpha).$$

Definition of Kernels. By kernels of set W we call the points of global minimum of function F_{+} and of global maximum of function F_{-}.

A subsystem, on which F_{+} reaches a global minimum is called a ⊕ kernel of the system W, while a subsystem on which F_{-} reaches a global maximum, is called ⊖ kernel. Thus, in every monotonic system the problem of determining ⊕ and ⊖ kernels is raised.

With the purpose of intuitive interpretation as well as with the purpose of explaining the usefulness of the notion of kernels introduced above we turn once again to the examples of citation graphs and telephone commutation networks.

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8 See also, Muchnik, I., and Shvartser, L., 1990, "Maximization of generalized characteristics of functions of monotone systems," Automation and Remote Control, 51, 1562-1572,.

The definition of the kernel can be formulated using the levels of significance of the elements of the system, that is: the ⊕ kernel is a subsystem of a monotonic system, for which the maximum level among the levels of significance is determined only by the internal organization of the system is the minimum, and the ⊖ kernel is the subsystem for which the minimum level among the same significance levels is the maximum.

The definition of a kernel accords with the intuitive interpretation of a kernel in citation graphs and telephone commutation networks. Thus, in citation graphs a ⊕ kernel is a subset (subsystem) of publications, in which the longest list of bibliographical titles is at the same time very short; though not inside the subset, but among all possible subsets of the selected set of publications (among the very long lists). If in our subset of publications a very short list of bibliographical titles is at the same time very long among the very short ones relative to all the subsets, then it is a ⊖ kernel of the citation graph. It is clear that a ⊖ kernel publications cite one another often enough, since for each publication the list of bibliographical titles is at any rate not less than a very short one while a very short list is nevertheless long enough. In a ⊕ kernel the same reason explains why in this subset one must find representatives of various scientific schools.

In telephone commutation networks, one can consider two types of system elements – lines of connections and points of connections. In a system consisting of lines, a ⊖ kernel turns out to be a subset of lines, for which the lines with the least number of denials in that subset are at the same time the lines with the greatest number of denials among all possible sets of lines. This means that at least the number of denials stipulates only by the internal organization of a sub-network of lines of a ⊖ kernel is not less than the number of denials for lines with the smallest number of denials and, besides, this number is large enough. Hence one can expect that the number of denials for lines of a ⊖ kernel is sufficiently large. Similarly one should expect a small number of denials for lines of a ⊕ kernel. Formulation for ⊕ and ⊖ kernels for points of connection is exactly the same as for the lines and is omitted here.

Before stating the theorems, we need to introduce some new definitions and notations.

Let $\overline{\alpha} = \langle \alpha_0, \alpha_1, \ldots, \alpha_{k-1} \rangle$ be an ordered sequence of distinct elements of set W, which exhausts the whole of this set, i.e., $k = |W|$. From sequence $\overline{\alpha}$ we construct an ordered sequence of subsets of W in the form $\Delta_{\overline{\alpha}} = \langle H_0, H_1, \ldots, H_{k-1} \rangle$ with the help of the following recurrent rule $H_0 = W$, $H_{i+1} = H_i \setminus \{\alpha_i\}$; $i = 0, 1, \ldots, k-2$ [9]

Definition. Sequence $\overline{\alpha}$ of elements of W is called a defining sequence relative to the aggregate of credentials collections $\{\Pi^- H \mid H \subseteq W\}$ if there exists in sequence $\Delta_{\overline{\alpha}}$, a subsequence of sets $\Gamma_{\overline{\alpha}} = \langle \Gamma_0^-, \Gamma_1^-, \ldots, \Gamma_p^- \rangle$, such that:

a) credential $\pi^- H_i(\alpha_i)$ of an arbitrary element α_i in sequence $\overline{\alpha}$, belonging to set Γ_j^- but not belonging to set Γ_{j+1}^- is strictly less than values of $F_-(\Gamma_{j+1})$; [10]

b) in set Γ_p^- there does not exist a proper subset L, which satisfies the strict inequality $F_-(\Gamma_p) < F_-(L)$.

A sequence $\overline{\alpha}$ with properties a) and b) is denoted by $\overline{\alpha}_-$. One similarly defines a sequence $\overline{\alpha}_+$.

c) arbitrary element α_i in sequence $\overline{\alpha}$, belonging to set Γ_j^+ but not belonging to set Γ_{j+1}^+ is strictly greater than values of $F_+(\Gamma_{j+1})$;

d) in set Γ_q^+ there does not exist a proper subset L, which satisfies the strict inequality $F_+(\Gamma_q) > F_+(L)$.

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[9] Sign $\setminus$ denotes the subtraction operation for sets.

[10] Here and everywhere, for simplification of expression, where it is required, the sign "–" or "+" is not used twice in notations. We should have written $F_-(\Gamma_{j+1}^-)$ or $F_+(\Gamma_{j+1}^+)$.

Definition. Subset H_{+}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{+}$ such that $H_{+}^{*}=\Gamma_{q}^{+}$.

Definition. Subset H_{-}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{-}$ such that $H_{-}^{*}=\Gamma_{p}^{-}$.

Below we formulate, but do not prove, a theorem concerning properties of points of global maximum of function F_{-}. The proof is adduced in Appendix 1. A similar theorem holds for function F_{+}. In Appendix 1 the parallel proof for function F_{+} is not reproduced. The corresponding passage from the proof for F_{-} to that of F_{+} can be effected by simple interchange of verbal relations "greater than" and "less than", inequality signs "≥" and "≤", ">", "<" as well as by interchange of signs "+" and "–". The passage from definable set H_{+}^{*} to H_{-}^{*} and from definition of sequence $\overline{\alpha}_{+}$ and $\overline{\alpha}_{-}$, is affected by what has just been said.

Theorem 1. On a definable set H_{-}^{*} function F_{-} reaches a global maximum. There is a unique definable set H_{-}^{*}. All sets, on which a global maximum is reached, lie inside the definable set H_{-}^{*}.

Theorem 2. On a definable set H_{+}^{*} function F_{+} reaches a global minimum. There is a unique definable set H_{+}^{*}. All sets, on which a global minimum is reached, lie inside the definable set H_{+}^{*}.

In the proof of Theorem 1 (Appendix 1) it is supposed that definable set H_{-}^{*} exists. It is natural that this assumption, in turn, needs proof. The existence of H_{-}^{*} is secured by a special constructive procedure. [11]

The proof of Theorem 2 is completely analogous to the proof of Theorem 1 and is not adduced in Appendix 1.

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[11] This procedure will be presented in the second part of the article, since here only the extremal properties of kernels and the structure of the set of kernels are established.

We present a theorem, which reflects a more refined structure of kernels of W as elements of the set $P(W)$ of all possible subsets (subsystems) of set W.

Theorem 3. The system of all sets in $P(W)$, on which function F_- (F_+) reaches maximum (minimum), is closed with the respect to the binary operation of taking union of sets.

The proof of this theorem is given in Appendix 2 and only for the function F_-. The assertion of the theorem for F_+ is established similarly.

Thus, it is established that the set of all ⊕ kernels (⊖ kernels) forms a closed system of sets with respect to the binary operation of taking the unions. The union of all kernels is itself a large kernel and, by the statements of Theorems 1 and 2, is a definable set.

APPENDIX 1

Proof of Theorem 1. We suppose that a definable set H^*_- exists.

(Conducting the proof by contradiction) let us assume that there exists a set $L \subseteq W$, which satisfies the inequality

$$F_-(H^*) \leq F_-(L). \qquad \text{(A.1)}$$

Thus two sets H^*_- and L are considered. One of the following statements holds:

1) Either $L / H^*_- \neq \varnothing$, which signifies the existence of elements in L, not belonging to H^*_-;

2) or $L \subseteq H^*_-$.

We first consider 2). By a property of definable set H^*_- there exists a defining sequence $\overline{\alpha}_-$ of elements of set W with the property b) (cf. the definition of $\overline{\alpha}_-$) such that the strict inequality $F_-(H^*) < F_-(L)$ does not hold and, consequently, only the equality holds in (A.1). In this case, the first and the third statements of the theorem are proved. It remains only to prove the uniqueness of H^*_-, whish is done after considering 1).

Thus, let $L/H_{-}^{*} \neq \varnothing$ and let us consider set H_t – the smallest of those H_i $(i = 0,1,...,k-1)$ from the defining sequence $\overline{\alpha}_{-}$ that include the set L/H_{-}^{*}. Then the fact that H_t is the smallest of the indicated sets implies the following: there exists element $\lambda \in L$, such that $\lambda \in H_t$, but $\lambda \notin H_{t+1}$.

Below, we denote by $i(\Omega)$ the smallest of the indices of elements of defining sequence $\overline{\alpha}_{-}$ that belong to the set $\Omega \subseteq W$.

Let Γ_p^- be the last in the sequence of sets $\langle \Gamma_j^- \rangle$, whose existence is guaranteed by the sequence $\overline{\alpha}_{-}$. For indices t and $i(\Gamma_p^-)$ we have the inequality $t < i(\Gamma_p^-)$.

The last inequality means that in sequence of sets $\langle \Gamma_j^- \rangle$ there exists at least one set Γ_s^-, which satisfies

$$i(\Gamma_{s+1}^-) \geq t + 1. \tag{A.2}$$

Without decreasing generality, one can assume that Γ_s^- is the largest among such sets.

It has been established above that $\lambda \in H_t$, but $\lambda \notin H_{t+1}$. Inequality (A.2) shows that $\Gamma_s^- \subset H_{t+1}$, since the opposite assumption $\Gamma_s^- \supseteq H_{t+1}$ leads to the conclusion that $i(\Gamma_s^-) \geq t+1$ and, consequently Γ_s^- is not the largest of the sets, for which (A.2) holds.

Thus, it is established that $\Gamma_{s-1}^- \supset H_t$. Indeed, if $\Gamma_{s-1}^- \subseteq H_t$, then for indices $i(\Gamma_{s-1}^-)$ and *t* we have $i(\Gamma_{s-1}^-) \geq t$.

Hence $i(\Gamma_{s-1}^-) + 1 \geq t + 1$ and the inequality $i(\Gamma_s^-) \geq i(\Gamma_{s-1}^-) + 1$ implies $i(\Gamma_s^-) \geq t + 1$. The last inequality once again contradicts the choice of set Γ_s^- as the largest set, which satisfies inequality (A.2).

Thus, $\lambda \notin \Gamma_s^-$, but $\lambda \in \Gamma_{s-1}^-$, since $\lambda \in H_t$, $H_t \subseteq \Gamma_{s-1}^-$. On the basis of property a) of the defining sequence $\overline{\alpha}_-$, we can conclude that

$$\pi^- H_t(\lambda) < F_-(\Gamma_s), \qquad \text{(A.3)}$$

where $0 \le s \le p$.

Let us consider an arbitrary set Γ_j^- $(j = 0,1,\dots,p-1)$ and an element $\tau \in \Gamma_j^-$, which has the smallest index in the sequence $\overline{\alpha}_-$. In other words, set Γ_j^- starts from the element τ in sequence $\overline{\alpha}_-$. In this case, set Γ_j^- is a certain set H_i in the sequence of imbedded sets $\langle H_i \rangle$. The definition of $F_-(H)$ and the property a) of defining sequence $\overline{\alpha}_-$ implies that

$$F_-(\Gamma_j) \le \pi^- \Gamma_j(\tau) < F_-(\Gamma_{j+1}).$$

Hence

$$F_-(\Gamma_0) < F_-(\Gamma_1) < \dots < F_-(\Gamma_p)$$

and as a corollary we have for $j = 0,1,\dots,p$

$$F_-(\Gamma_j) \le F_-(\Gamma_p) = F_-(H^*), \qquad \text{(A.4)}$$

since $\Gamma_p^- = H_-^*$.

Let $\mu \in L$ and let credential $\pi^- L(\mu)$ be minimal in the collection of credentials relative to set L. On the basis of inequalities (A.1), (A.3), and (A.4) we deduce that

$$\pi^- H_t(\lambda) < \pi^- L(\mu) = F_-(L).$$

Above, H_t was chosen so that $L \subseteq H_t$. Recalling the fundamental monotonicity property (1) for collection of credentials (the influence of elements on each other), it easy to establish that

$$\pi^- L(\lambda) \le \pi^- H_t(\lambda). \qquad \text{(A.6)}$$

Inequalities (A.5) and (A.6) imply the inequality

$$\pi^- L(\lambda) < \pi^- L(\mu),$$

i.e., there exists in the collection of credentials relative to set L a credential, which is strictly less than the minimal credential.

A contradiction is obtained and it is proved that set L can only be a subset of H_-^* and that all sets, distinct from H_-^*, on which the global maximum is also reached, lie inside H_-^*.

It remains to prove that if a definable set H_-^* exists, then it is unique. Indeed, in consequence of what has been proved above we can only suppose that some definable set $H_-^{'}$, distinct from H_-^*, is included in H_-^*.

It is now enough to adduce arguments for definable set $H_-^{'}$ similar to those adduced above for L, considering it as definable set $H_-^{'}$; this implies that $H_-^* \subseteq H_-^{'}$. The theorem is proved.

APPENDIX 2

Proof of Theorem 3. Let Ω be the system of set in $P(W)$, on which function F_- reaches a global maximum, and let $K_1 \in \Omega$ and $K_2 \in \Omega$.

Since on K_1 and K_2 the function F_- reaches a global maximum, therefore we might establish the inequalities

$$F_-(K_1 \cup K_2) \leq F_-(K_1),\ F_-(K_1 \cup K_2) \leq F_-(K_2). \quad \text{(A.7)}$$

We consider element $\mu \in K_1 \cup K_2$, on which the value of function F_- on set $K_1 \cup K_2$, is reached, i.e.,

$$\pi^- K_1 \cup K_2(\mu) = \min_{\alpha \in K_1 \cup K_2} \pi^- K_1 \cup K_2(\alpha).$$

If $\mu \in K_1$, then by rendering $\ominus$ actions on all those elements of set $K_1 \cup K_2$, that do not belong to K_1, we deduce from the fundamental monotonicity property of collections of credentials (1) the validity of the inequality

$$\pi^- K_1(\mu) \leq \pi^- K_1 \cup K_2(\mu).$$

Since the definition of F_- implies that $F_-(K_1) \le \pi^- K_1(\mu)$ and by the choice of element μ we have $\pi^- K_1 \cup K_2(\mu) = F_-(K_1 \cup K_2)$, therefore we deduce the inequality

$$F_-(K_1) \le F_-(K_1 \cup K_2).$$

Now from the inequality (A.7) it follows that

$$F_-(K_1) = F_-(K_1 \cup K_2).$$

If, however, it is supposed that $\mu \in K_2$, then $\ominus$ actions are rendered on elements of $K_1 \cup K_2$, not belonging to K_2; in an analogous way we obtain the equality

$$F_-(K_2) = F_-(K_1 \cup K_2),$$

which was to be proved.

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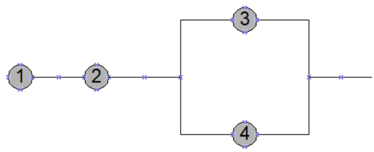
*i The name "Monotonic System" at that moment in the past was the best match for our scheme. However, this name "Monotone System" was already occupied in "Reliability Theory" unknown to the author. Below we reproduce a fragment of a "monotone system" concept different from ours in lines of Sheldon M. Ross "Introduction to Probability Models", Fourth Ed., Academic Press, Inc., pp. 406-407.

Example (A four-Component Structure):



Consider a system consisting of four components, and suppose that the system functions if and only if components 1 and 2 both function and at least one of components 3 and 4 function. Its structure function is given by

$$\phi(x) = x_1 \cdot x_2 \cdot \max(x_3, x_4).$$

Pictorially, the system is shown in Figure. A useful identity, easily checked, is that for binary variables, (a binary variable is one which assumes either the value 0 or 1) x_i, $i = 1,\dots,n$,

$$\max(x_1,\dots,x_n) = 1 - \prod_{i=1}(1 - x_i).$$ When $n = 2$, this yields

$$\max(x_1, x_2) = 1 - (1 - x_1)\cdot(1 - x_2) = x_1 + x_2 - x_1 \cdot x_2.$$

Hence, the structure function in the above example may be written as

$$\phi(x) = x_1 \cdot x_2 \cdot (x_3 + x_4 - x_3 \cdot x_4)$$

It is natural to assume that replacing a failed component by a functioning one never lead to a deterioration of the system. In other words, it is natural to assume that the structure function $\phi(x)$ is an increasing function of x, that is, if $x_i \le y_i$, $i = 1,\dots,n$, then $\phi(x) \le \phi(y)$. Such an assumption shall be made in this chapter and the system will be called *monotone*.

ЭКСТРЕМАЛЬНЫЕ ПОДСИСТЕМЫ МОНОТОННЫХ СИСТЕМ. II

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Предлагаетсяконструктивнаяпроцедурапостроения особых определяющих последовательностей элементов монотонных систем, рассмотренных в [1]. Изучаются взаимные свойства двух определяющих последовательностей $\overline{\alpha}_{-}$ и $\overline{\alpha}_{+}$, и полученный результат форму-лируется в виде теоремы двойственности. На основе теоремы двойственности описан способ сужения области поиска экстремальных подсистем — ядер монотонной системы и приведена соответствующая схема поиска.

1. Введение

В [1] разработан основной аппарат выделения в монотонных системах особых подсистем — ядер, обладающих экстремальными свойствами. Основным понятием развитого аппарата является определимое множество [2]. В принятой терминологии определимое множество оказывается наибольшим ядром монотонной системы взаимосвязанных элементов. Понятие определимого множества в [1] вводилось с помощью предположенияо существовании особых подпоследовательностей элементов изучаемой системы, названных определяющими ($\overline{\alpha}_{-}$ и $\overline{\alpha}_{+}$) -последовательностями.

В данной работе вопрос существования определяющих последоательностей решается конструктивно в виде процедур — алгоритмов. Основные свойства определяющей последовательности, построенной по правилам процедуры и исчерпывающей все множе-ство элементов системы ***W*,** гарантируется теоремой.

Рассматривая также вопрос о том, какая существует связь между определяющими последовательностями $\overline{\alpha}_{-}$ и $\overline{\alpha}_{+}$. Можно предположить, что если построена определяющая последовательность $\overline{\alpha}_{-}$, то стоит взять эту последовательность в обратном порядке, как получится $\overline{\alpha}_{+}$ последовательность. В общем случае это не так. Тем не менее имеет место более слабое утверждение. На основе опреeленных в [1] понятий дискретных действий типа ⊕ и ⊖ и на элеенты системы ***W*** данное утверждение формулируется здесь в виде теоремы двойственности. В случае выполнения условий еоремы двойственности изложенные алгоритмы построения определяющих последовательностей используются для значительного сужения облати поиска ⊕ и ⊖ ядер системы ***W***. Алгоритм сужения области поиска изложен также в виде процедуры — конструктивно.

Extremal Subsystems of Monotonic Systems, II

Abstract. A constructive routine is considered for obtaining singular-determining sequence of elements of monotonic systems studied by Mullat (1976). The relationship between two determining sequences $\overline{\alpha}_-$ and $\overline{\alpha}_+$ is also examined, and the obtained result is formulated as a duality theorem. This theorem is used for describing a routine of restricting the domain of search for extremal subsystems (or kernels of a monotonic system); the corresponding search scheme is also presented.
Keywords: monotonic; system; matrix; graph; cluster

1. Introduction

In Mullat (1976) we have developed the basic method of selection (from monotonic systems) of singular subsystem, i.e., the kernels possessing extremal properties. The main concept of this method is that of a definable set Mullat (1971). In the terminology adopted by us, a definable set is the largest kernel of a monotonic system of interrelated elements. In 1971 we introduced the concept of a definable set with the aid of the system under consideration called determining $\overline{\alpha}_- (\overline{\alpha}_+)$ sequences.

In this paper the problem of existing of determining sequences is solved constructively in the form of routines (algorithms). The principal properties of determining sequences sequence constructed according to the rules of a routine and that exhausts the entire set of elements of the system W are specified by a theorem.

We shall also examine the relationship between two determining sequences $\overline{\alpha}_-$ and $\overline{\alpha}_+$. It can be assumed that after constructing a determining sequence $\overline{\alpha}_-$, we could take this sequence in inverse order, thus obtaining an $\overline{\alpha}_+$ sequence. But in the general case this is not so. Nevertheless we can make a weaker assertion. On the basis of the concepts (defined in Mullat (1976) of discrete operations of type $\oplus$ and $\ominus$ on the elements of a system W, this assertion will be formulated below as a duality theorem. Under the conditions of the duality theorem, the algorithms of construction of determining sequences described here will be used foe considerably restricting the domain of search for $\oplus$ and $\ominus$ kernels of the system W. The algorithm of restriction of the domain of search is presented in the form of a constructive routine.

2. ROUTINE OF FINDING THE KERNELS

Below we describe a routine of construction of an ordered sequence $\overline{\alpha}$ of all the elements of W. In abbreviated form, this routine is called KSR (kernel-searching routine).

This routine consists of rules of generation and scanning of an ordered series of ordered sets $\langle\overline{\beta}_j\rangle$ (sequences); here j varies from zero to a value p, which is automatically determined by the rules of the routine, whereas the elements of each sequence $\overline{\beta}_j$ are selected from the set W[1].

This series $\langle\overline{\beta}_j\rangle$ constructed by this rule forms a numerical sequence of thresholds $\langle u_j\rangle$ and a sequence of sets $\langle\Gamma_j\rangle$. On the other hand the sequence of thresholds governs the transactions from $\overline{\beta}_{j-1}$ to $\overline{\beta}_j$ in the chain $\langle\overline{\beta}_j\rangle$, and the sequence $\langle\Gamma_j\rangle$ terminates with a set, which is definable.

In the description of a rule we use the operation of extending a sequence $\overline{\beta}_j$ by adjoining to it another sequence $\overline{\gamma}$. This operation is symbolically expressed by $\overline{\beta} \leftarrow \langle\overline{\beta},\overline{\gamma}\rangle$.

This rule of construction of the sequence $\overline{\alpha}$ of all elements of the set W can be described stages: by step **Z** and **R**.

Z. In the set W we find an element μ_0 such that

$\pi^{-}W(\mu_0) = \min_{\delta\in W}\pi^{-}W(\delta) = F_{-}(W)$ we are constructing a determining sequence $\overline{\alpha}_{-}$. The construction of $\overline{\alpha}_{+}$ is entirely similar and therefore not presented here. We shall only indicate where it is necessary to invert the sign of inequalities, and where the search for an element with the minimal credential must be replaced by search for an element with maximal credential, so as to be able to construct $\overline{\alpha}_{+}$.

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1 Let us recall that in a) the brackets $\langle,\rangle$ denoted an ordered set; in the case under consideration they denote an ordered set of ordered sets $\overline{\beta}_j$.

Thus the construction here of $\overline{\alpha}_+$, the element μ_0 is obtained from the condition $\pi^+W(\mu_0) = \max_{\delta\in W}\pi^+W(\delta) = F_+(W)$. We shall write $u_0 = \pi^-W(\mu_0)$, $\overline{\alpha} = \langle\mu_0\rangle$ and the set $\Gamma_0 = W$. We select a subset of elements γ from W such that $\pi^-W \setminus \overline{\alpha}(\gamma) \le u_0$. The construction of $\overline{\alpha}_+$ requires the selection of such γ that $\pi^+W \setminus \overline{\alpha}(\gamma) \ge u_0, u_0 = \pi^+W(\mu_0)$.
After that we order the elements in a certain manner (which can be arbitrary selected). The thus-obtained ordered set is denoted by $\overline{\gamma}$. Let us write $\overline{\beta}_0 = \overline{\gamma}$.

R. We construct a recursive routine for extending the sequences $\overline{\alpha}$ and $\overline{\beta}_0$. Here we denote by $\beta_0(i)$ the i-th element of the sequence $\overline{\beta}_0$. We specify one after another the elements of the sequence $\overline{\beta}_0$. At each instant of specification we extend the sequence $\overline{\alpha}$ by the elements from $\overline{\beta}_0$ of the sequence fixed at this instant. In accordance with the symbolic notation of the operation of extension of a sequence $\overline{\alpha}$, we perform at each instant t of specification the operation $\overline{\alpha} \leftarrow \langle\overline{\alpha}, \beta_0(t)\rangle$. Suppose that all the elements of $\overline{\beta}_0$ up to $\beta_0(i-1)$ inclusive have been fixed. Then the sequence $\overline{\alpha}$ will have the form $\langle\mu_0, \beta_0(1), \beta_0(2), \ldots, \beta_0(i-1)\rangle$, which corresponds to the symbolic notation of the operation of extension of the sequences $\overline{\alpha} \leftarrow \langle\overline{\alpha}, \beta_0(1), \beta_0(2), \ldots, \beta_0(i-1)\rangle$ in the case that $\overline{\alpha}$ inside the brackets consists of one element μ_0. Let us consider an element $\beta_0(i-1)$ of the sequence $\overline{\beta}_0$. At the instant of specification of the element $\beta_0(i-1)$ we decide during the above-mentioned operation of extension of $\overline{\alpha}$ also about any further extension or about stopping the extension of the sequence $\overline{\beta}_0$. We must check the following two conditions:

a) In the set $W \setminus \overline{\alpha}$ there exist elements such that $\pi^{-}W \setminus \overline{\alpha}(\gamma) \le u_0$ In constructing $\overline{\alpha}_{+}$, this condition is replaced by $\pi^{+}W \setminus \overline{\alpha}(\gamma) \ge u_0$;

b) the element $\beta_0(i)$ is defined for the sequence $\overline{\beta}_0$. By assumption an element $\beta_0(i)$ to be defined for a sequence $\overline{\beta}_0$ if the sequence $\overline{\beta}_0$ has an element with an ordinal number i. Otherwise the element $\beta_0(i)$ is not defined. There can be four cases of fulfillment or no fulfillment of these conditions. In two cases, when the first condition is satisfied, irrespective of whether or not the second condition holds, the sequence $\overline{\beta}_0$ will be extended. This means that the set of elements γ in $W \setminus \overline{\alpha}$ specified by the first condition is ordered in the form of sequence $\overline{\gamma}$. The sequence $\overline{\beta}_0$ is extended in accordance with the formula $\overline{\beta}_0 \leftarrow \left\langle \overline{\beta}_0, \overline{\gamma} \right\rangle$. In case when the first condition is not satisfied, whereas the second condition is satisfied, we shall fix the element $\beta_0(i)$ and at the same time extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha} \leftarrow \left\langle \overline{\alpha}, \beta_0(i) \right\rangle$, and proceed to new recursion stage. In case neither the first nor the second condition holds, the sequence $\overline{\beta}_0$ will not be extended nor the last fixed element in the sequence $\overline{\beta}_0$ will be the element $\beta_0(i-1)$. Suppose that we have fixed all the elements of the sequence $\overline{\beta}_j$. By that time we have constructed a sequence $\overline{\alpha}$. Let us consider the set $W \setminus \overline{\alpha}$ and the credential system $\Pi^{-}W \setminus \overline{\alpha}$. We shall find an element in $\Pi^{-}W \setminus \overline{\alpha}$ on which the minimum is reached in the credential system $\Pi^{-}W \setminus \overline{\alpha}$. The obtained element is denoted by μ_{j+1}. We obtain $\overline{\alpha}_{+}$ the element μ_{j+1} from the condition:

$$\pi^{+}W \setminus \overline{\alpha}(\mu_{j+1}) = \max_{\delta \in W \setminus \overline{\alpha}} \pi^{+}W \setminus \overline{\alpha}(\delta) = F_{+}(W \setminus \overline{\alpha}).$$

Thus, $\pi^{-}W\setminus\overline{\alpha}(\mu_{j+1})=F_{-}(W\setminus\overline{\alpha})$. Let us write $u_{j+1}=\pi^{-}W\setminus\overline{\alpha}(\mu_{j+1})$, and for the set $\Gamma_{j+1}=W\setminus\overline{\alpha}$; then we supplement the sequence $\overline{\alpha}$ by the element μ_{j+1}, i.e., $\overline{\alpha}\leftarrow\left\langle\overline{\alpha},\mu_{j+1}\right\rangle$. In the same way as during the zero step, we select a subset of elements γ from $W\setminus\overline{\alpha}$ such that $\pi^{-}W\setminus\overline{\alpha}(\gamma)\le u_{j+1}$. Here we select for $\overline{\alpha}_{+}$ a set of elements γ such that $\pi^{+}W\setminus\overline{\alpha}(\gamma)\ge u_{j+1}$. The selected set can be ordered in any manner. The ordered set is denoted by $\overline{\gamma}$. The set $\overline{\beta}_{j+1}$ is assumed to be equal to $\overline{\gamma}$.

c) By analogy with previous b) the recursion step will be described as a recursion routine. At this stage we also use the rule of extension of the sequences $\overline{\alpha}$ and $\overline{\beta}_{j+1}$. Suppose that we have fixed all elements of $\overline{\beta}_{j+1}$ up to $\beta_j(i-1)$ inclusive. Then the sequence $\overline{\alpha}$ will have the form $\overline{\alpha}=\left\langle\overline{\alpha},\mu_{j+1},\beta_j(1),\ldots,\beta_j(i-1)\right\rangle$, where $\overline{\alpha}$ denotes the sequence $\overline{\alpha}$ obtained at the instant of fixing all the elements of $\overline{\beta}_j$, or, to rephrase, the sequence $\overline{\alpha}$ prior to the $(j+1)$-th step. The last equation corresponds to the symbolic operation of extension of the sequence $\overline{\alpha}=\left\langle\overline{\alpha},\mu_{j+1},\beta_j(1),\ldots,\beta_j(i-1)\right\rangle$ in the case that $\overline{\alpha}$ inside the brackets denotes the sequence $\left\langle\overline{\alpha},\mu_{j+1}\right\rangle$. Let us consider an element $\beta_{j+1}(i-1)$ of the sequence $\overline{\beta}_{j+1}$. At the instant of fixing the element $\beta_{j+1}(i-1)$ we decide about a further extension or about stopping the extension of the sequence $\overline{\beta}_{j+1}$. For this purpose we consider the credential system $\Pi^{-}W\setminus\overline{\alpha}$ and we check two conditions:

1) The set $W \setminus \overline{\alpha}$ contains elements γ such that $\pi^{-} W \setminus \overline{\alpha}(\gamma) \leq u_{j+1}$ For constructing $\overline{\alpha}_{+}$ we must take elements γ such that $\pi^{+} W \setminus \overline{\alpha}(\gamma) \geq u_{j+1}$;
2) the element $\beta_{j+1}(i)$ is defined for the sequence $\overline{\beta}_{j+1}$.

By analogy with the step **Z**, we find that the sequence $\overline{\beta}_{j+1}$ is extended in two cases in which the first condition is satisfied irrespective of whether or not the second condition holds. The set of elements γ in $W \setminus \overline{\alpha}$ specified by the first condition is ordered in the form of a sequence $\overline{\gamma}$. The sequence $\overline{\beta}_{j+1}$ is extended in accordance with the formula $\overline{\beta}_{j+1} \leftarrow \left\langle \overline{\beta}_{j+1}, \overline{\gamma} \right\rangle$. In the case that the first condition does not hold, whereas the second condition is satisfied, the element $\beta_{j+1}(i)$ will be fixed and at the same time we extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha} \leftarrow \left\langle \overline{\alpha}, \beta_{j+1}(i) \right\rangle$, and after that we proceed again in accordance with the rules of Stage 2 of the recursion routine of extension of the sequence $\overline{\beta}_{j+1}$. In the case that neither the first, nor the second condition holds, the sequence $\overline{\beta}_{j+1}$ will not be extended, and the last fixed element of the sequence $\overline{\beta}_{j+1}$ will be the element $\beta_{j+1}(i-1)$. At some step p the sequence $\overline{\alpha}$ will exhaust the entire set of elements W.

Theorem 1. A sequence $\overline{\alpha}$ constructed on the basis of a collection of credential system $\left\{ \Pi^{-} H \,\middle|\, H \subseteq W \right\}$ is a determining sequence $\overline{\alpha}_{-}$, whereas a sequence $\overline{\alpha}$ constructed on the basis of $\left\{ \Pi^{+} H \,\middle|\, H \subseteq W \right\}$ is a determining sequence $\overline{\alpha}_{+}$.

The first part of the theorem (for $\overline{\alpha}_{-}$) is proved in Appendix 1. The second part (for $\overline{\alpha}_{+}$) can be proved in the same way.

NB1. Let us note that a sequence $\overline{\alpha}$ constructed by KSR rules has somewhat stronger properties than required in obtaining a determining sequence. More precisely, there does not exist a proper subset L for $j = 0,1,...,p-1$ such that $\Gamma_j \supset L \supset \Gamma_{j+1}$ and $F_-(\Gamma_j) < F_-(L)$. This is not required for obtaining a determining sequence $\overline{\alpha}_-$ ($\overline{\alpha}_+$). The corresponding proof is not given here.

NB2. Let us note another circumstance. With the aid of the kernel-searching routine it is possible to effectively find (without scanning) the largest kernel, i.e., a definable set. It is not possible to find an individual kernel strictly included in a definable set (if the latter exists) by constructing a determining sequence.

3. DUALITY THEOREM

Let us establish a relationship between the determining sequences $\overline{\alpha}_-$ and $\overline{\alpha}_+$ of a system W.

Theorem 2. Let $\overline{\alpha}_-$ and $\overline{\alpha}_+$ be determining sequences of the set W with respect to the collection of credential system $\left\{\Pi^- H \,\middle|\, H \subseteq W\right\}$, $\left\{\Pi^+ H \,\middle|\, H \subseteq W\right\}$ respectively. Let $\left\langle \Gamma_j^- \right\rangle$ be the subsequence of the sequence $\Delta_{\overline{\alpha}_-}$ $(j = 0,1,...,p)$ needed in the determination of $\overline{\alpha}_-$, and let $\left\langle \Gamma_j^+ \right\rangle$ be the corresponding subsequence of the sequence $\Delta_{\overline{\alpha}_+}$ $(j = 0,1,...,q)$.

Hence if for an m and a n we have

$$F_+(\Gamma_n^+) = F_-(\Gamma_m^-), \tag{1}$$

then $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$, $\Gamma_n^+ \subseteq W \setminus \Gamma_{m+1}^-$. If

$$F_+(\Gamma_n^+) < F_-(\Gamma_m^-)^2, \tag{2}$$

then $\Gamma_m^- \subseteq W \setminus \Gamma_n^+$, $\Gamma_n^+ \subseteq W \setminus \Gamma_m^-$.

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[2] In the following, the + and – sign will not be used twice in notation. This rule applies also to Appendices 1 and 2

This theorem is important from two points of view. Firstly, under the conditions (1) and (2) there exists a relationship between an $\overline{\alpha}_-$ sequence and $\overline{\alpha}_+$. This relationship consists in the fact that elements of $\overline{\alpha}_+$ which are at the "beginning" and form either the set $W \setminus \Gamma^+_{n+1}$ or the set $W \setminus \Gamma^+_n$ will include all the elements of the set Γ^-_m that are at the "end" of $\overline{\alpha}_-$. The same applies also to sets $W \setminus \Gamma^-_{m+1}$ or $W \setminus \Gamma^-_m$ which are at the beginning of $\overline{\alpha}_-$, since they include in a similar way the set Γ^+_n. In other words, the theorem states that the sequence $\overline{\alpha}_+$ does not differ "very much" (under certain conditions) from the sequence, which is the inverse to $\overline{\alpha}_-$.

Let us note that the conditions (1) and (2) are sufficient conditions, and it can happen that actual monotonic systems satisfying these conditions do not exist. Nevertheless, in the third part of this article, we shall describe actual examples of such systems.

4. Kernel Search Routine Based on Duality Theorem

We just noted that a determining sequence $\overline{\alpha}_+$ differs "slightly" from the inverse sequence of $\overline{\alpha}_-$. For elucidating the possibility of a search for kernels on the basis of the duality theorem, let us rephrase the latter. This assertion can be formulated as follows: at the beginning of the sequence $\overline{\alpha}_+$ we often encounter elements of the sequence $\overline{\alpha}_-$, which are at the end of the latter.

Such an interpretation of the duality theorem yields an efficient routine of dual search for $\oplus$ and $\ominus$ kernels of the system W. This is due to the fact if the elements are often encountered, there exists a higher possibility of finding a $\oplus$ kernel at the beginning of the sequence $\overline{\alpha}_+$ as compared to finding it at the end of $\overline{\alpha}_-$; the same applies also to a $\ominus$ kernel in the sequence $\overline{\alpha}_-$.

The routine under construction is based on Corollaries I-IV of the duality theorem presented in Appendix II, where we also prove this theorem.

The routine of dual search for kernels described below is an application of two constructive routines, i.e., a KSR for constructing $\overline{\alpha}_{+}$ and a KSR for constructing $\overline{\alpha}_{-}$. The routine is stepwise, with two constructing stages realized at each step, i.e., a stage in which the KSR is used for constructing $\overline{\alpha}_{+}$ with $\oplus$ operations, and a stage in which the same routine is used for constructing $\overline{\alpha}_{-}$ with the aid of $\ominus$ operations on the elements of the system.

Z. At first we store two numbers: $u_0^{+} = F_{+}(W)$ and $u_0^{-} = 0$. After that we perform precisely Stage 1 and 2 of the zero step of the KSR used for constructing the determining sequence $\overline{\alpha}_{+}$. This signifies that the set W contains an element μ_0 such that $\pi^{+}W(\mu_0) = \max_{\delta\in W}\pi^{+}W(\delta) = F_{+}(W)$. The threshold u_0^{+} is equal to $\pi^{+}W(\mu_0)$, etc. By using the constructions of the zero step of KSR at the previous stage of the dual routine under construction, we obtained a set $\Gamma_1^{+} \subset W$. Then we examine the set $W \setminus \Gamma_1^{+}$ and the credential system $\Pi^{+}W \setminus \Gamma_1^{+}$. On the set $W \setminus \Gamma_1^{+}$ with the credential system $\Pi^{+}W \setminus \Gamma_1^{+}$ we perform a complete kernel-searching routine for the purpose of constructing a determining sequence of $\oplus$ operations only for the set $W \setminus \Gamma_1^{+}$. As a result, we obtain in the set $W \setminus \Gamma_1^{+}$ a subset K^{1} on which the function F_{-} reaches a global maximum among all the subsets of the set $W \setminus \Gamma_1^{+}$.

R. By applying the previous $(j-1)$ steps to the j-th step, we obtained a sequence of sets $\Gamma_0^{+}, \Gamma_1^{+}, \ldots, \Gamma_j^{+}$, and according to the construction of a determining sequence we have $\Gamma_0^{+} \supset \Gamma_1^{+} \supset \ldots \supset \Gamma_j^{+}$ and

$\Gamma_0^+ = W$. At first we store two numbers: $u_j^+ = F_+(\Gamma_j^+)$ and $u_j^- = F_-(H^j)$. By analogy, we perform the same construction consisting of two stages of a KSR recursion step for constructing $\overline{\alpha}_+$ with the aid of $\oplus$ operations. At a given instant of such dual construction we obtained a set $\Gamma_{j+1}^+ \subset \Gamma_j^+$. Then we consider the set $W \setminus \Gamma_{j+1}^+$ and the credential system $\Pi^- W \setminus \Gamma_{j+1}^+$. In the same way as at the zero step, we perform on the set $W \setminus \Gamma_{j+1}^+$ a complete kernel-searching routine with the purpose of constructing a sequence $\overline{\alpha}_-$ only on the set $W \setminus \Gamma_{j+1}^+$. As a result we obtain in the set $W \setminus \Gamma_{j+1}^+$ a subset H^{j+1} on which the function F_- reaches a global maximum among all subsets of the set $W \setminus \Gamma_{j+1}^+$.

Rule of Termination of Construction Routine. Before starting the construction of the j-th step of the routine under construction, we check the condition

$$u_j^+ \le u_j^- . \qquad (3)$$

If (3) is satisfied as a strict inequality, the construction will terminate before the j-th step. If (3) is an equality, the construction will terminate after the j-th step.

5. Definable Sets of Dual Kernel-Search Routine

At the end of the construction process, the above routine yields a set H^j or a set H^{j+1}. It can be asserted that one of the sets is definable set or the largest kernel of the system W with respect to a collection of credential system $\left\{\Pi^- H \mid H \subseteq W\right\}$.

The assertion is based on the following. Firstly, by applying the KSR we obtained the second stage of the j-th step of a dual routine the maximal set H^{j+1} among all the subsets of the set $W \setminus \Gamma_{j+1}^+$ on which the function $F_{\_}$ reaches a global maximum in the system of sets of all the subsets of the set $W \setminus \Gamma^{+}_{j+1}$. Secondly, by virtue of Corollary 1 of the Theorem 2 (the duality theorem), it follows that, prior to the j-th step and provided that (3) is a strict inequality, the largest kernel (a definable set) will be contained in the set $W \setminus \Gamma^{+}_{j}$, or it follows from the Corollary 2 of the Theorem 2, if (3) is a equality, that the largest kernel is included in the set $W \setminus \Gamma^{+}_{j+1}$.

Thus by comparing these two remarks we can see that either H^{j} or H^{j+1} is a definable set.

By virtue of Corollaries 3 and 4 of the duality theorem, it is possible to find by similar dual routine also the largest kernel $K^{\oplus}$- definable set. This assertion can be proved in the same way as the assertion about H^{j} and H^{j+1}; therefore this proof is not given here.

APPENDIX 1

Proof of Theorem 1. We shell prove that a sequence $\overline{\alpha}$ constructed by the KSR rules is a determining sequence for a collection of credential systems

$$\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}.$$

First of all let us recall the definition of a determining sequence of elements of the system W. We shall use the notation $\Delta_{\overline{\alpha}} = \left\langle H_0, H_1, \ldots, H_{k-1} \right\rangle$, where $H_0 = W$, $H_{i+1} = H_i \setminus \alpha_i$ $(i = 0,1,\ldots,k-2)$. A sequence of elements of a set W is said to be determining with respect to a coalition of credential system $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$ if the sequence $\Delta_{\overline{\alpha}}$ has a subsequence of sets $\Gamma_{\overline{\alpha}} = \left\langle \Gamma_0, \Gamma_1, \ldots, \Gamma_p \right\rangle$, such that

a) The credential $\pi^{-}H_i(\alpha_i)$ of any element α_i of the sequence $\overline{\alpha}$ that belongs to the set Γ_j, but does not belong to the set Γ_{j+1}, is strictly smaller than the credential of an element with minimal credential with respect to the set Γ_{j+1}, i.e., $\pi^{-}H_i(\alpha_i) < F_{-}(\Gamma_{j+1})$, $j = 0,1,...,p-1$[3];

b) the set Γ_p does not have a proper subset L such that the strict inequality $F_{-}(\Gamma_p) < F_{-}(L)$ is satisfied (the "–" symbol has been omitted; see previous footnote).

We shall consider a sequence of sets $\Delta_{\overline{\alpha}}$ and take the subsequence $\Gamma_{\overline{\alpha}}$ in the form of the sets Γ_j $(j = 0,1,...,p)$ constructed by the KSR rules. We have to prove that sets Γ_j have the required properties of a determining sequence. Assuming the contrary carries out the proof.

Let us assume that Mullat property (1971) of a determining sequence is not satisfied. This means that for any set Γ_j there exists in the sequence of elements

$$\overline{\beta}_j = \langle \beta_j(1), \beta_j(2), ..., \rangle$$

an element $\beta_j(r)$ such that

$$\pi^{-}H_{v+r}(\beta_j(r)) \geq F_{-}(\Gamma_{j+1}) = u_{j+1} \tag{A.1}$$

Here v is the index number of the element μ_j selected in Stage 1 of the recursion step of the constructive routine of determination of $\overline{\alpha}$; in the vocabulary of notation used in Mullat (1976) we have $v = i(\Gamma_j)$.

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[3] In the definition of $\overline{\alpha}_+$ sequence it is required that the following strict inequality be satisfied: $\pi^{+}H_i(\alpha_i) > F_{+}(\Gamma_{j+1})$, $j = 0,1,...,q-1$

According to the method of construction, the sequence $\overline{\beta}_j$ consists of sequences γ formed at the second stage of the j-th step of the constructive routine. Let M be a set in a sequence of sets $\Delta_{\overline{\alpha}}$ such that the first element $\alpha_{i(M)}$ of the set M in the constructed sequence $\overline{\alpha}$ is used at the second stage of the j-th step for constructing the sequence γ to which the element $\beta_j(r)$ belongs. This definition of M shows that $H_{v+r} \subseteq M$.

From the construction of the second stage of the j-th step and the principal property of monotonicity of ⊖ operations in the system we obtain the inequalities

$$\pi^{-}H_{v+r}(\beta_j(r)) \leq \pi^{-}M(\beta_j(r)) \leq \pi^{-}\Gamma_j(\mu_j) = u_j \qquad \text{(A.2)}$$

By virtue of the above method of selection of the set Γ_{j+1} from the sequence of sets $\langle \Gamma_j \rangle$ and of the properties of a fixed sequence $\overline{\beta}_j$, we obtain at the j-th step

$$u_j = \pi^{-}\Gamma_j(\mu_{j+1}) < \pi^{-}\Gamma_{j+1}(\mu_{j+1}) = u_{j+1}, \qquad \text{(A.3)}$$

where $j = 0,1,\ldots,p-1$.

According to the rule of constructing of the sequence $\overline{\alpha}$, the function F_{-} reaches its value on the elements μ_j and μ_{j+1}. The elements μ_j and μ_{j+1} belong to the sets Γ_j and Γ_{j+1} respectively; therefore the inequalities (A.1) – (A.3) are contradictory.

Thus our assumption is not true and Mullat Property of the determining sequence $\overline{\alpha}$ constructed by KSR rules has been proved.

Let as assume that Property b) does not hold, i.e., the last Γ_p of the sequence $\langle \Gamma_j \rangle$ contains a proper subset L such that

$$F_{-}(\Gamma_p) < F_{-}(L). \tag{A.4}$$

Let the element $\lambda \in L$, and suppose that it is the element with minimal ordinal number in $\overline{\alpha}$ belonging to L; moreover, let t denotes this number, i.e., $t = i(L)$, $\alpha_t = \lambda$. From the definition of t it follows that $L \subseteq H_t$.

Our analysis carried out above for the set H_{v+r} we repeat below for the set H_t. By analogy with the definition of the set M we define a set M' with the aid of the element λ and the sequence $\overline{\alpha}$.

The set M' is equated with the set of the sequence of sets $\Delta_{\overline{\alpha}}$ that begins with an element used in the formation of a set $\overline{\gamma}$ at the p-th step of the constructive routine such that $\lambda \in \overline{\gamma}$.

By analogy with derivative of (A.2) we obtain

$$\pi^{-}H_t(\lambda) \leq \pi^{-}M'(\lambda) \geq \pi^{-}\Gamma_p(\mu_p) = u_p. \tag{A.5}$$

Since $F_{-}(L) \leq \pi^{-}L(\lambda)$, it follows from (A.4) and (A.5) that $\pi^{-}H_t(\lambda) < \pi^{-}L(\lambda)$.

We noted above that $L \subseteq H_t$, by virtue of the monotonicity of $\ominus$ operations, it hence follows that

$$\pi^{-}L(\lambda) \leq \pi^{-}H_t(\lambda).$$

The last two inequalities are contradictory, and hence Property b) of the determining sequence is satisfied.

Thus we have proved that the sequence $\overline{\alpha}$ constructed by the KSR rules is a determining sequence with respect to a collection of credential systems $\{\Pi^{-}H \mid H \subseteq W\}$, and hence it can be denoted by $\overline{\alpha}_{-}$, whereas the sequence $\langle \Gamma_j \rangle$ obtained by a constructive routine can be denoted by $\Gamma^{-}_{\overline{\alpha}_{-}}$.

APPENDIX 2

Proof of Duality Theorem. Below we shall show that $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$, if $F_+(\Gamma_n^+) = F_-(\Gamma_m^-)$ (we omit a twice notation of + and – symbols; a promised above the + and – sign will not be used twice in notation. This rule have been applied also to Appendices 1 and 2.

Let us assume that there exists an element $\xi \in \Gamma_m^-$ and that $\xi \in \Gamma_{m+1}^-$, i.e., $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$. Hence follows that we have defined a credential $\pi\Gamma_{n+1}^+(\xi)$. According to the definition of the function F_+ we have the inequality $\pi\Gamma_{n+1}^+(\xi) \le F(\Gamma_{n+1}^+)$.

For a determining sequence $\overline{\alpha}_+$ and for any $j = 0,1,...,q-1$ we have inequalities $F(\Gamma_{n+1}^+) < F(\Gamma_n^+)$.

Let us consider an element $g \in \Gamma_n^+$ with the smallest index number in $\overline{\alpha}_+$. It follows from the definition of $\overline{\alpha}_+$ that

$$\pi\Gamma_n^+(g) > F(\Gamma_{n+1}^+). \tag{A.7}$$

The choice of element g is convenient because it permits the use of Mullat Property of a determining sequence (see Appendix 1), i.e., in this case the set Γ_n^+ is in the form of $H_t = \Gamma_n^+$. Since $F(\Gamma_n^+) \ge \pi\Gamma_n^+(g)$, we have proved (A.6).

Since $\xi \in \Gamma_m^-$, it follows that we have defined a credential $\pi\Gamma_m^-(\xi)$. We have the following chain of inequalities:

$$F(\Gamma_m^-) \le \pi\Gamma_m^-(\xi) \le \pi^- W(\xi) = \pi^+ W(\xi) \le \pi\Gamma_n^+(\xi).$$

Let us recall that for any element δ of the system W under consideration, we have in a) the relation $\pi^- W(\delta) = \pi^+ W(\delta)$. The first inequality follows from the definition of the function F_-, and the second inequality from the monotonicity of $\ominus$ operations. The equality follows from the definition of the functions π^- and π^+, whereas the last inequality follows from the monotonicity of $\ominus$ operations.

By virtue of (A.6) and of the conditions of the theorem, we have also the following chain of inequalities:

$$\pi\Gamma_{n+1}^{+}(\xi) \leq F(\Gamma_{n+1}^{+}) < F(\Gamma_{n}^{+}) = F(\Gamma_{m}^{-}).$$

By supplementing this chain by the previous chain of inequalities, we hence obtain $\pi\Gamma_{n+1}^{+}(\xi) < \pi\Gamma_{n}^{+}(\xi)$. Since $\Gamma_{n+1}^{+} \subset \Gamma_{n}^{+}$, it follows from the monotonicity of $\oplus$ operations that $\pi\Gamma_{n+1}^{+}(\xi) < \pi\Gamma_{n+1}^{+}(\xi)$. The logical step used for obtaining the last inequality is valid, and therefore the assumption that $\Gamma_{m}^{-} \subseteq W \setminus \Gamma_{n+1}^{+}$ is untrue.

In the same way we can prove the inclusion $\Gamma_{n}^{+} \subseteq W \setminus \Gamma_{m+1}^{-}$. For this purpose it suffices to change the signs of the inequalities and (whenever necessary) to replace the set Γ_{n+1}^{+} by Γ_{n+1}^{-}, and Γ_{m}^{-} by Γ_{n}^{+}.

If condition (2) of the theorem holds, it is not necessary to use (A.6). In this case the proof will be similar, being based on the following chain of inequalities (The proof is based on assuming the contrary, so that $\Gamma_{m}^{-} \not\subset W \setminus \Gamma_{n}^{+}$, i.e., there exists, as it were, an element $\xi \in \Gamma_{m}^{-}$ and $\xi \in \Gamma_{n}^{+}$.):

$$\begin{aligned}&\pi\Gamma_{n}^{+}(\xi) \leq F(\Gamma_{n}^{+}) < F(\Gamma_{m}^{-}) \leq \\ &\leq \pi\Gamma_{m}^{-}(\xi) \leq \pi^{-}W(\xi) \leq \pi\Gamma_{n}^{+}(\xi)\end{aligned}.$$

The first inequality follows from the definition of $F(\Gamma_{n}^{+})$, the second follows from Condition (2) of the theorem, and the third from the definition of $F(\Gamma_{m}^{-})$. The last two relations express the properties of monotonic systems. Hence in this case we have under Condition (2) also

$$\pi\Gamma_{n}^{+}(\xi) < \pi\Gamma_{n}^{+}(\xi).$$

This completes the proof of the theorem. ■ Now follows several corollaries of Theorem 2.

Corollary 1. If for $n=\overline{0,q}$ the determining sequence is $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) > F(\Gamma_n^+)$. Thus kernel $K^{\oplus}$ will belong to the set $W \setminus \Gamma_n^+$. Indeed, since a definable set is also kernel, it follows that $F_-(H) \le F(\Gamma_p^-)$, $m=0,1,...,p$, and hence (in any case) if $m=p$, and n is selected on the basis of the condition of the corollary, then $F(\Gamma_n^+) < F(\Gamma_p^-)$. By virtue of the theorem, we therefore obtain the assertion of the corollary.

Corollary 2. If for $n=0,1,...,q-1$ of a determining sequence $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) = F(\Gamma_n^+)$, then the kernel $K^{\oplus}$ will belong to the set $W \setminus \Gamma_{n+1}^+$.

The proof follows directly from Corollary 1, by virtue of (A.6).

Corollary 3. If for $m=0,1,...,p$ of a determining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) < F(\Gamma_m^-)$ then the kernel $K^{\ominus}$ will belong to the set $W \setminus \Gamma_m^-$. The proof of Corollary 3 is entirely similar to that of Corollary 1. It is only necessary to change the signs of the inequalities and replace the set Γ_n^+ by Γ_m^-.

Corollary 4. If for $m=0,1,...,p-1$ of a determining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) = F(\Gamma_m^-)$, then the kernel $K^{\ominus}$ will belong to the set $W \setminus \Gamma_{m+1}^-$.

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КОНТРМОНОТОННЫЕ СИСТЕМЫ В АНАЛИЗЕ СТРУКТУРЫ МНОГОМЕРНЫХ РАСПРЕДЕЛЕНИЙ

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Ставится задача выделения сгущений в многомерном пространстве измерений на основе векторного критерия качества. Для поиска решений используется специальная параметризация функций, при которой с увеличением значений параметров значение функций во всей области определения уменьшается.

1. Введение

Анализ структуры распределения плотности измерений в n-мерном пространстве — традиционная тематика исследований в таких прикладных областях, как планирование эксперимента [1], анализ изображений [2], анализ принятия решений [3], распознавание образов [4] и т. д.

На содержательном уровне структура распределения обычно представляется совокупностью сгущений, которые иногда называются также модами [5]. Анализ подобной структуры, если не явно, то косвенно, почти всегда сводится к вариационной задаче оптимизации — максимизации какого-либо скалярного критерия качества, оценивающего выделяемые сгущения. Вместо скалярного в данной работе используется векторный критерий, а в основу понятия оптимальности положено так называемое равновесное состояние в смысле Нэша [6].

Правомерность подхода с позиции состояния равновесия к анализу структуры распределения плотности измерений в n-мерном пространстве объясняется тем, что здесь по существу происходит замена одной многомерной многими «почти одномерными» задачами в проекциях на оси координат. На каждой оси сгущение выделяется так, что оси «увязываются» между собой строго определенным образом: сгущение на данной оси нельзя «сдвинуть в сторону» без какого-либо ухудшения сгущения на других осях в смысле рассматриваемого критерия при условии, что эти другие уже фиксированы.

Преимущество предложенного подхода не исчерпывается указанной «технической подробностью» замены одного многомерного пространства одномерными проекциями. Дело в том, что состояние равновесия, выделяемое при помощи используемого векторного критерия, параметризируется так называемыми порогами, которые задают уровни плотности сгущений. По крайней мере в некоторых частных случаях состояние равновесия как решение системы уравнений можно аналитически выразить в форме функций порогов и тем самым полностью обозреть выделяемые сгущения в спектре возможных уровней плотности.

Counter Monotonic Systems in the Analysis of the Structure of multivariate Distributions

Abstract. The problem of distinguishing condensations in multivariate space of measurements based on a qualitative vector criterion is presented. We find solutions by a special parameterization of functions, the values of which decrease in all regions of the definition in inverse proportion to the values of the parameters.
Keywords: monotonic; distributions; equilibrium; cluster

1. INTRODUCTION

The analysis of the structure of the probability density function of measurements in an n-dimensional space is a traditional topic of investigation in such applied fields as experimental design (Finney, 1964), image analysis (Rosenfeld, 1969), the analysis of decision making (Fishburn, 1970), pattern recognition (Aizerman et al, 1970), etc…

At a conceptual level, a distribution structure is usually represented by a set of data clusters, sometimes called modes (Zagoruiko and Zaslavskaya, 1968). The analysis of such a structure is indirectly, if not explicitly, usually reduced to the problem of variational optimization. That is, maximizing some scalar performance metrics that characterize the identified clusters. Instead of a scalar performance index, in this article we use a vector index and base the concept of optimality on the so-called Nash equilibrium state (Owen, 1968).

Approaching the analysis of the structure of a measurement density function in n-dimensional space, our standpoint is the equilibrium state concept. It is justified by the fact that, essentially, what happens, is the replacement here of a single multidimensional problem by many “almost one-dimensional” problems in projections onto the coordinate axes. On each axis a cluster is delineated in such a way as to “bind” the axes together in a rigorously defined way. So, exposed to such a “bind” the cluster on a given axis cannot be “nudged” without in some measure deteriorating itself on the other axes in the sense of investigated performance index, subject to the condition that these others are fixed.

The superiority of the proposed approach is not restricted to the indicated “technical detail” of replacing one multidimensional space by one-dimensional projections. Indeed, an equilibrium state identified by means of the given vector index is parameterized by so-called thresholds, which satisfy the density levels of the clusters. In certain special cases, at any rate, an equilibrium state as the solution of a system of equations can be expressed analytically in the form of threshold functions, whereupon the identified clusters can be fully scanned in the spectrum of possible density levels.

The proposed theory for the identification of clusters of the probability density of measurements in n-dimensional space is set forth in two parts. In the first part (sec.2) the theory is not taken beyond the scope of customary multivariate functions and it concludes with a system equations, namely the system whose solution in the form of threshold functions makes it possible to scan the identified clusters. In the second part (Sec.3) the theory now rests on a more abundant class of measurable functions specified by the class of sets represented on the coordinate axes by at most countable set of unions or intersections of segments. Overall the construction described in this part is so-called counter-monotonic system; actually, the first part on multi-parameter counter-monotonic systems is also discussed in these terms (special case).

The fundamental result of the second part does not differ, in any way, from the form of the system of equations in the first part; the essential difference is in the space of admissible solutions. Whereas in the system of equations of the first part the solution is a numerical vector, in the second part it is a set of measurable sets containing the sought-after measurable density clusters. As the solution of the system of equations, the set of measurable sets serves as a fixed point of special kind mapping of subsets of multidimensional space. This particular feature is utilized in an iterative solving procedure.

2. COUNTER-MONOTONIC SYSTEMS OVER A FAMILY OF PARAMETERS

Here a monotonic system represents first a one-parameter and then a multi-parameter family of functions defined on real axis. This type of representation is a special case of a more general monotonic system described in the next section.

We consider a one-parameter family of functions $\pi(x;h)$ defined on the real axis, where h is a parameter. For definiteness, we assume that an individual copy π of the indicated family is a function integrable with respect to x and differentiable with respect to h. The family of functions π is said to be counter-monotonic if it obeys the following condition: for any pair of quantities ℓ and g such that $\ell \leq g$ the inequality

$$\pi(x;\ell) \geq \pi(x;g) \text{ holds for any } x.$$

The specification of a multi-parameter family of functions π is reducible to the following scheme. We replace the one function π by a vector function $\pi = \left\langle \pi_1, \pi_2, \ldots, \pi_n \right\rangle$, each j-th component of which is a copy of the function depending now on n parameters $h_1, h_2, \ldots, h_n$, i.e., $\pi_j = \pi_j(x; h_1, h_2, \ldots, h_n)$. We wrote down the counter-monotonicity condition for any pair of vectors $\ell = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle$ and $g = \langle g_1, g_2, \ldots, g_n \rangle$ such that $\ell_k \leq g_k$, $k = (1, 2, \ldots, n)$ in the form of n inequalities $\pi_j(x; \ell_1, \ell_2, \ldots, \ell_n) \geq \pi_j(x; g_1, g_2, \ldots g_n)$. We also note that this condition rigorously associates with family of vector functions a component-wise partial ordering of vector parameters.

We give special attention to the case of a so-called de-coupled multi-parameter family of functions π. The family π arrange de-coupled functions if the j-th component of the vector function π does not depend on the j-th component of the vector of parameters h, i.e., on h_j. Therefore, the function π of a de-coupled multi-parameter family is written in the form $\pi_j(x, h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_n)$ $(j = 1, \ldots, n)$.

We now return to the original problem of analyzing a multi-modal empirical distribution in multidimensional space. We first investigate the case of one axis probability distribution of only one random variable (univariate distribution).

Let $p(x)$ be the probability density function of points in the *x*-axis. For the counter-monotonic family π we can choose, for example, the functions $\pi(x; h) = p(x)^h$. It is easy verified that the counter-monotonicity condition is satisfied.

We consider the following variational problem. With respect to an externally specified threshold u^o $(0 \leq u^o \leq 1)$ let it be necessary to maximize the functional

$$\Pi(h) = \int_{-h}^{+h} [\pi(x; h) - u^o] \cdot dx.$$

It is clear that for small h the quantity $\Pi(h)$ will be small because of the narrow interval of integration, while for the large h it will be small by the counter-monotonicity condition. Consequently, the value of $\max_h \Pi(h)$ will necessarily be attained for certain finite nonzero h^o.

It is easy to see that if $p(x)$ is a unique function of the density of modes with zero mathematical expectation, then maximizing the functional $\Pi(h)$ implies identifying the interval on the axis corresponding to the density $p(x)$ concentration. But if $p(x)$ has a more complex form, then the maximum $\Pi(h)$ determines the interval in which the "essential part" in a certain sense of the density function $p(x)$ is concentrated.

Directly from the form of the function $\Pi(h)$ we derive necessary condition for the local maximum (the zero equation of the derivative with respect to h : $\frac{\partial}{\partial h}\Pi(h)=0$: or, in expanded form, the equation

$$\pi(-h;h)+\pi(h;h)+\int_{-h}^{+h}\tfrac{\partial}{\partial y}\pi(x;y)\,|_{y=h}\,dx=2u^{o}. \qquad (1)$$

The root of the given equation will necessarily contain one at which $\Pi(h)$ attains a global maximum. We have thus done with the problem: we found the central cluster points of the density function on one axis in terms of a counter-monotonic family of functions.

To find the central clusters of a multivariate distribution in n-dimensional space we invoke the notion of a multi-parameter counter-monotonic family of functions π. Let the family of functions π in vector form be written, say, in the form $\pi_j(x;h_1,...,h_n)=p_j(x)^h$, where $h=\sum_{k=1}^{n}h_k$, and $p_j(x)$ is a projection of the multivariate distribution on the axis j-th axis. In the stated sense the goodness of the delineated central cluster is evaluated by the multivariate (vector) performance index $\Pi=\langle\Pi_1,...,\Pi_n\rangle$, where

$$\Pi_j(h_1,h_2,...,h_n)=\int_{-h_j}^{h_j}[\pi_j(x;h_1,...,h_n)-u_j]\cdot dx \qquad (2)$$

and u_j is the component of the corresponding externally specified multi-dimensional threshold vector u: $u=\langle u_1,u_2,...,u_n\rangle$. As in the one-dimensional case, of course, it is meaningful to use the given functional only distributions $p_j(x)$ with zero expectation.

Once the goodness of a delineated cluster has been evaluated by the vector index, it must be decided, based on standard (Becker and McClintock, 1967) vector optimization principles, what is an acceptable cluster. In this connection it is desirable to indicate simultaneously a procedure for finding an extremal point in the space of parameters. It turns out that for so-called Nash-optimal Equilibrium State there is a simple technique for finding solutions at least in de-coupled family of counter-monotonic functions π.

En equilibrium situation (Nash point) in the parameter space $h = \langle h_1, \ldots, h_n \rangle$ with indices Π_j is defined as a point $h^* = \langle h_1^*, h_2^*, \ldots, h_n^* \rangle$ such that for every j the inequality

$$\Pi_j(h_1^*, \ldots, h_{j-1}^*, h_j, h_{j+1}^*, \ldots, h_n^*) \le \Pi_j(h_1^*, \ldots, h_j^*, \ldots, h_n^*)$$

holds for any value of h_j. In other words, if there are no sensible bases in the sense of index Π_j on the one (j-th) axis, then the equilibrium situation is shifted with respect to the parameter h_j, subject to the condition that the quantities h_k^*, $k \ne j$, are fixed on all other axes.

Clearly, a necessary condition at a Nash point in the parameter space (as in the one-dimensional case) is that the partial derivatives tend to zero, i.e., the n equalities $\partial / \partial h_j \Pi_j(h_1^*, \ldots, h_n^*) = 0$ must hold. The sufficient condition comprises the n inequalities $\partial^2 / \partial h_j^2 \Pi_j(h_1^*, \ldots, h_n^*) \le 0$.

An essential issue here, however, is the fact that the necessary condition (equalities) acquires a simpler form for de-coupled family of counter-monotonic functions than in the general case. Thus, by the decoupling of the family π the partial derivative $\partial \Pi_j / \partial h_j$ is identically zero, and the system of equations, see (1) by analogy, with respect to the sought-after point h^* is reducible to the form

$$\begin{aligned} &\pi_j(-h_j; h_1, \ldots, h_{j-1}, h_{j+1}, \ldots h_n) + \\ &\quad + \pi_j(h_j; h_1, \ldots, h_{j-1}, h_{j+1}, \ldots h_n) = 2u_j \end{aligned} \tag{3}$$

Now the sufficient condition is satisfied automatically for any solution h^* of Eqs. (3).

In conclusion we write out the system of equations for two special cases of a de-coupled family of counter-monotonic functions π.

1. Let $\pi_j(x;h_1,...,h_{j-1},h_{j+1},...,h_n) = p_j(x)^{\sigma-h_j}$, where $\sigma = h_1 + h_2 + ... + h_n$. The system of equations (3) is reducible to the form $p_j(-h_j)^{\sigma-h_j} + p_j(h_j)^{\sigma-h_j} = 2u_j$, $j = \overline{1,n}$.
2. Let the role of $\pi_j(x;h_1,...,h_{j-1},h_{j+1},...,h_n)$ be taken by the $p_1(x)^{h_1}...p_{j-1}(x)^{h_{j-1}}p_{j+1}(x)^{h_{j+1}}...p_n(x)^{h_n}$ function.

The system of equations (3) for finding a solution, i.e., an equilibrium situation (Nash point) h^*, is written

$$p(-h_j)/p_j(-h_j)^{h_j} + p(h_j)/p_j(h_j)^{h_j} = 2u_j \quad (j = \overline{1,n}),$$

where $p(x) = p_1(x)^{h_1}p_2(x)^{h_2}...p_n(x)^{h_n}$ is the product of univariate density functions.

We conclude this section with an important observation affecting the vector of thresholds $u =< u_1, u_2, ..., u_n >$. By straightforward reasoning we infer that each component h_j^* of the equilibrium situation h^* is a function of thresholds and h^* can be represented by a vector function of thresholds in the form $h_j^* = h_j^*(u_1, u_2, ..., u_n)$. If the solution of the system of equations (3) can be expressed analytically, then prolific possibilities are afforded for scanning the equilibrium situations in the parameter space and, accordingly, selecting an "acceptable" cluster in the spectrum of existing densities of measurements in a multidimensional space of thresholds. A similar approach can be used when solutions of Eqs. (3) are sought by numerical methods.

3. COUNTER-MONOTONIC SYSTEMS OVER A FAMILY OF SEGMENTS

A multi-parameter family of counter-monotonic functions used for the analysis of multivariate distributions, unfortunately, has one substantial drawback. Generally speaking, there is no way to guarantee the identification of homogeneous distribution clusters in projection onto the j-th axis, because the segment $[-h_j, h_j]$ can contain several distinct modes. On the other hand, it is sometimes desirable to identify modes by merely indicating a family of segments containing each mode separately. The construction proposed below enlarges the possibilities for the solution of such a problem by augmenting the counter-monotonic systems of the proceeding section in natural way.

Thus, on real axis we consider subsets represented by at most countable set of operations of union, intersection, and difference of segments. The class of all such subsets is denoted by B, and each representative subset by $H \in B$ (which we call a B set) is distinguished from like sets by length μ (by measure zero). A set L is congruent with G $(G = L)$ if the measure of the symmetric difference $G\Delta L$ is equal to zero $(\mu G\Delta L = 0)$; a set L is contained in G $(L \subseteq G)$ with respect to measure μ if $\mu G \setminus L = 0$. A measure on the real axis, being an additive function of sets (the length), is determined by taking to the limit the length of the sets in the set of unions, intersections, and differences of segments forming the B set. Then set-theoretic operations over B sets will be understood to mean up to measure zero. By convention, all B sets of measure zero are indistinguishable.

We associate with every B set H a nonnegative function $\pi(x; H)$, which is Borel measurable (or simply measurable) and whose domain of definition is on the real axis.[1] In other words, in contrast with the one-parameter family of counter-monotonic functions of the preceding section, the parameter h is now generalized, namely, it is extended to the B set H. As before, we say that a family of measurable functions π is counter-monotonic if it obeys the following condition: for any pair of sets L and G such that $L \subseteq G$ the inequality

$$\pi(x; L) \geq \pi(x; G)$$

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[1] A function $\pi(x; H)$ is Borel measurable if for any numerical threshold u^o the set of all *x* of the real scale for which $\pi(x; H) > u^o$ is measurable: $\{x : \pi(x; H) > u^o\}$ is B set.

holds for any x.

The scheme of specification of a multi-parameter family of functions is analogous to the previous situation. In place of a scalar function π we now specify a vector function $\pi = \langle \pi_1, \pi_2, \ldots, \pi_n \rangle$, each j-th component of which is a copy of a function depending at the outset on n parameters $\langle H_1, H_2, \ldots, H_n \rangle$, i.e., $\pi_j = \pi_j(x; H_1, H_2, \ldots, H_n)$ (B sets). Again, the counter-monotonicity condition is reducible to the statement that for any pair of vectors (ordered sets of B sets) of the form $L = \langle L_1, \ldots, L_n \rangle$ and $G = \langle G_1, \ldots, G_n \rangle$ such that $L_k \subseteq G_k$ $(k = 1, 2, \ldots, n)$, the following n inequalities are satisfied:[2]

$$\pi_j(x; L_1, \ldots, L_n) \geq \pi_j(x; G_1, \ldots, G_n).$$

These inequalities associate a partial ordering of sets of B sets with a family of vector functions π in a rigorously defined way.

In the case of a de-coupled family of counter-monotonic functions, where the j-th component of a copy of the vector function π does not depend on the parameter H_j, or B set on the j-th axis of definition of the function π_j, this component π_j of the vector function π is written $\pi_j = \pi_j(x; H_1, H_2, \ldots, H_n)$.

Following again the order of discussion of Sec.2, we now consider the original problem of analyzing the structure of a multi-modal empirical distribution in a multidimensional space. We first investigate the case of a one-dimensional (univariate) distribution.

Let $p(x)$ be the density function of points on the *x*-axis. In the role of the counter-monotonic family of functions π, we adopt functions of the form $\pi(x; H) = p(x)^{F(H)}$, where $F(H) = \int_H p(x)dx$ is the probability of a random variable occurring in a B set under the probability

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[2] Here x is a point on the j-th axis. This is tacitly understood everywhere.

density function $p(x)$. It is clear that the counter-monotonicity condition is satisfied.

We consider the following variational problem. Given the externally specified threshold u^o $(0 \le u^o \le 1)$, maximize the functional

$$\Pi(H) = \int_H [\pi(x; H) - u^o] d\mu .$$

The integral here is understood in the Lebegue sense with respect to measure μ, where μ, as mentioned before, is the length of the B set on the x axis.

Clearly, the quantity $\Pi(H)$ as a function of the length μ (measure of set H) increases first and then, as $\mu H \to \infty$, reverts to zero by the counter-monotonicity condition on the family of functions π. Therefore, the value of $\max_H \Pi(H)$ will necessary is attained on a certain B set of finite measure μ (see the analogous assertion in Sec.2).

It is impossible in the same simple way to deduce directly from the form of the functional $\Pi(H)$ any maximum condition comparable with the like condition of the preceding section (Eq.1). To do so would require elaborating the notation of a "virtual translation" from a B set H to a set $\widetilde{H}$ similar to it in some sense, in such a way as to establish the necessary maximum condition. These circumstances exclude the case of a univariate distribution from further consideration. Nonetheless, as will be shown presently, for multivariate distribution there are means for finding a B set that will maximize the function $\Pi(H)$ at least in the case of a de-coupled family of counter-monotonic functions.

As in the preceding section, we evaluate the goodness of an identified central cluster by the multivariate (vector) performance index

$$\Pi = \langle \Pi_1, \Pi_2, ..., \Pi_n \rangle :$$
$$\Pi_j(H_1, H_2, ..., H_n) = \int_{H_j} [\pi(x; H_1, ..., H_n) - u_j] d\mu ,$$

where u_j is the coordinate of the corresponding multidimensional vector of thresholds u, specified externally: $u = \langle u_1, u_2, \ldots, u_n \rangle$.

At this point we call attention to the fact that, in contrast with the analogous multivariate index of Sec.2, the given functional now has significance for an arbitrary distribution, rather than only for the centered condition of "*zero-valued-ness*" of the expectation. We again look for the required cluster in multidimensional space as an equilibrium situation according to the vector index $\Pi = \langle \Pi_1, \Pi_2, \ldots, \Pi_n \rangle$. We regard a cluster as a set of B sets $H^* = \langle H_1^*, H_2^*, \ldots, H_n^* \rangle$ such that the following inequalities holds for every j:

$$\begin{aligned} &\Pi_j(H_1^*, \ldots, H_{j-1}^*, H_j, H_{j+1}^*, \ldots, H_n^*) \le \\ &\le \Pi_j(H_1^*, \ldots, H_j^*, \ldots, H_n^*) \end{aligned} \qquad (j = \overline{1, n}).$$

In a de-coupled family of counter-monotonic functions it is feasible, as in the multi-parameter case, see Eq. (3), to find an equilibrium situation. Equilibrium situations are sought to be a special technique of mappings of B sets onto real axes.

We define the following type of mappings of B sets onto real axes:

$$V_j(H_j) = \{x : \pi_j(x; H_j) > u_j\},$$

where u_j is the threshold involved in the expression for the functional Π_j $(j = \overline{1, n})$. Thus defined, n such mappings are uniquely expressible in the vector form

$$V(H) = \{x : \pi(x; H) > u\}.$$

Here $H = H_1 \times H_2 \times \ldots \times H_n$ denotes the direct product of sets H_j. We define a fixed point of the mapping $V(H)$ as a set H^* for which the equality $H^* = V(H^*)$ holds.

Theorem 1. *For a de-coupled family of counter-monotonic functions* π, *a fixed point of the mapping* $V(H)$ *generates an equilibrium situation according to the vector index* $\Pi = \langle \Pi_1, \Pi_2, ..., \Pi_n \rangle$.

The proof of the theorem is simple. Thus, because π_j is independent of the parameter H_j, the form of the function $\pi_j(x; H_1^*, ..., H_{j-1}^*, H_{j+1}^*, ..., H_n^*)$ does not depend on H_j. Also, the set $H^* = H_1^* \times H_2^* \times ... \times H_n^*$ in projection onto the j-th axis intersects the set H_j^* consisting exclusively of all points *x* for which $\pi_j(x; H_j^*) > u_j$: $H_j^* = \{x : \pi_j(x; H_j^*) > u_j\}$. It is immediately apparent that any H_j distinct from H_j^* the value of the functional $\Pi_j(H_1^*, ..., H_{j-1}^*, H_j, H_{j+1}^*, ..., H_n^*)$ for immovable sets H_k^* $(k \neq j)$ cannot be anything but smaller than the quantity $\Pi_j(H_1^*, ..., H_{j-1}^*, H_j^*, H_{j+1}^*, ..., H_n^*)$.

It is important, therefore, to find the fixed points of the constructed mapping of B sets.

4. METHODS OF FINDING EQUILIBRIUM STATE FOR DE-COUPLED FAMILIES OF COUNTER-MONOTONIC FUNCTIONS

The ensuing discussion rests heavily on the counter-monotonicity property of a function π. To facilitate comprehension of the formulations and propositions we use the language of diagrams reflecting the structure of the relations involved in the constructed mappings of B sets, in particular the symbol $\rightarrow$ denoting the relation "set X_1 is nested in set X_2 $(X_1 \subseteq X_2)$: $X_1 \rightarrow X_2$.

All diagrams of the relations between B sets are based on the following proposition: the relation $X_1 \rightarrow X_2$ (as a consequence of the counter-monotonicity condition on π) implies that $V(X_1) \leftarrow V(X_2)$.

Now let the mapping V be applied to the original space W of axes on which the functions π_j $(j = \overline{1,n})$ are defined. After the image $V(W)$ has been obtained, we again apply the mapping V with the B set $V(W)$ as its inverse image, i.e., we consider the image $V^2(W)$, and so on. In this way we construct a chain of B sets W, $V(W)$, $V^2(W)$,..., which we call the central series of the counter-monotonic system.

The following diagram of nestling of B sets of the central series is inferred directly from the above stated proposition:

$$W \leftarrow V(W) \rightarrow V^2(W) \leftarrow V^3(W) \rightarrow V^4(W) \leftarrow V^5(W)$$

It is evident from the diagram that there exist in the central series two monotonic chains of B sets: one shrinking and one growing. The monotonically shrinking chain of B sets comprises the sequence $V^2(W) \leftarrow V^4(W) \leftarrow \ldots$ with even powers of the mapping V. The monotonically growing chain is the sequence $V(W) \rightarrow V^3(W) \rightarrow V^5(W) \rightarrow \ldots$ with odd powers of V.

It is well known (Shilov and Gurevich, 1967) that monotonically decreasing (increasing) chains in the class of B sets always converge in the limit of sets of the same class. For example, the limit of the sets $V^{2k}(W)$ with even powers is the intersection $L = \bigcap_{k=1}^{\infty} V^{2k}(W)$, and the limit of sets $V^{2k-1}(W)$ with odd powers is the union $G = \bigcup_{k=1}^{\infty} V^{2k-1}(W)$.

Theorem 2. *For the central series of a counter-monotonic system the nesting* $L \subseteq G$ *of the limiting* B set L *of even powers of the mapping* $V(X)$ *in the limiting* B *set* G *of odd powers of the same mapping is always true.*

The theorem follows at once from the diagram of nestlings of the central series.

We now resume our at the moment interrupted discussion of the problem of finding a fixed point of a mapping of B sets, such point generating an equilibrium situation according to the vector index Π (Theorem 1). In counter-monotonic systems, as a rule, the strict nesting $L \subset G$ of limiting B sets holds in the statement of Theorem 2. The equality $L = G$ would imply convergence of the central series in the limit to a single set, namely a fixed pint. In view of the exceptional status of the equality $L = G$, we give a "more refined" procedure, which automatically in the number of cases of practical importance yields the desired result, a solution of the equation $X = V(X)$.

Procedure for Solving the Equation $X = V(X)$. A chain of B sets $H_0, H_1, \ldots,$ is generated recursively according to the following rule. Let the set H_k (where H_0 is any B set of finite measure) be already generated in the chain. We use the mapping $V(X)$ to transform the following B sets:

$$V\{V^2(H_k) \bigcup V(H_k)\}, \qquad V\{V(H_k) \bigcap H_k\},$$
$$V\{V(H_k) \bigcup H_k\}, \qquad V\{V^2(H_k) \bigcap V(H_k)\}.$$

We denote these sets by L_k^2, G_k, L_k, G_k^2 accordingly. By the counter-monotonicity of the family of functions π it turns out that L_k^2 is a subset of G_k and that L_k is a subset of G_k^2. Picking any A_k based on the condition $L_k^2 \subset A_k \subset G_k$, and then B_k from the analogous condition $L_k \subset B_k \subset G_k^2$, we put the set H_{k+1} following H_k in the constructed series of B sets equal to $A_k \bigcup B_k$: $H_k = A_k \bigcup B_k$. The sets A_k

and B_k can be chosen, for example, according to mapping rules in the class of B sets, namely,

$$A_k = \{x : ½[\pi(x;L_k^2) + \pi(x;G_k)] > u\},$$
$$B_k = \{x : ½[\pi(x;L_k) + \pi(x;G_k^2)] > u\}.$$

The conditions imposed on A_k and B_k are satisfied in this case.

Theorem 3. For the series of sets $V(H_k)$ to contain the limiting set $V(H^*)$ as $k \to \infty$, which would be a solution of the equation $X = V(X)$, the following two conditions are sufficient:

a) $\lim_{k\to\infty} \mu G_k \setminus L_k^2 = 0$,

b) $\lim_{k\to\infty} \mu G_k^2 \setminus L_k = 0$.

The plan of the proof is quickly grasped in the following nesting diagrams, which are consequences of the counter-monotonicity property of the functions π, i.e.,

I. $V^2(H_k) \leftarrow L_k^2 \to G_k \leftarrow V(H_k)$,

II. $V(H_k) \leftarrow L_k \to G_k^2 \leftarrow V^2(H_k)$.

Diagrams **I** and **II** imply the validity of the two chains:

1) $V^2(H_k) \setminus V(H_k) \subseteq V^2(H_k) \setminus G_k \subseteq L_k^2 \setminus G_k$,

2) $V(H_k) \setminus V^2(H_k) \subseteq V(H_k) \setminus G_k^2 \subseteq L_k \setminus G_k^2$.

The first chain implies that for the limiting set H^* of the series $H_0, H_1, \ldots,$ the equality $\mu V^2(H_k) \setminus V(H^*) = 0$ holds, i.e., $V(H^*) \subset V^2(H^*)$; the second chain implies the opposite relation:

$V^2(H^*) \subseteq V(H^*)$. Consequently, $V(H^*)$ is the solution of the equation $X = V(X)$: $V(H^*) = V(V(H^*))$. Of course, the conditions of the theorem are sufficient for the existence of a solution of the equation $X = V(X)$, and their absence does not in any way negate some other solving technique, provided that solutions exist in general. The possibility that solution H^* of the equation $X = V(X)$ do not exist should certainly not be dismissed.

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Application of Monotone Systems to the Study of the Structure of Markov Chains

Abstract. The method of analysis of "Markov chain" is described. The method is based on the transformation of the Markov chain into a monotonic system and on the separation of kernels from the transformed chain.
Keywords: Markov chain; communication line; network; transition matrix; kernel

1. INTRODUCTION

In the work presented here, the theory of monotonic systems developed in an earlier publication (Mullat, a) 1976) is applied to the Markov chains. In the study of Markov chains the interest stems from the fact that it is convenient to interpret a special class of absorbing chains as monotonic systems. On the other hand, it also provides a meaningful way of illustrating the main properties of monotonic systems, as shown here using an example based on communication networks.[1] In order to disclose on conceptual level the technology developed for extracting the extreme subsystems in Markov chains discussed in the current paper, we employ the communication network as an example of monotonic system, albeit in a slightly modified form relative to that originally proposed in the context of telephone network. This will enable us to elucidate the manner in which a Markov chain may be associated with the monotonic system and what principal operations may be performed on it towards utilization of monotonic systems theoretical apparatus described in the Mullat original work.

In the earlier paper on which this Mullat work is based, an example of a communication network has been considered, whereby a set W comprising of communication lines/channels between some nodes — communicating units — was introduced.[2] Here, we will assume that each line has certain built-in redundancy mechanisms, such as the main and the reserved channels.[3] Thus, if a direct line is not available between nodes, analo-

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[1] Translated from Russian, Mullat, c) 1979. In the original paper, the term used was "telephone switch net," which was not adopted here, as it is outdated. Still, the concept underpinning the work remains highly relevant, as forms of "switches" are still used in redirecting TCP/IP packages, in a manner comparable to the telephone net.

[2] Switch is a device of such type and can learn where to address the communication packages.

[3] In practice, network redundancy may be guaranteed by some additional channels/lines activated only in urgent situations when the net usage exceeds some predefined threshold.

gously to what was described in Mullat's work 1976, the traffic might be organized through pass-around channels. In addition to this mechanism, in the present case, the possibility of employing pass-around communication is not excluded even if a direct channel is available.

In the example presented in the original paper (Mullat, 1976), an average number of "denials" before establishing the contact characterizing each pair of nodes was utilized. The number of denials usually characterizes the communication lines in the communication network.[4] In the model described below, and for the purpose of current investigation, it is more convenient to use a value inverse to the number of denials, as this will characterize the communication line throughput.

Let us assume that each communication line (comprising of both the main and the reserved channels) is characterized by the throughput $c_{i,j}$ or, in other words, by the maximum allowed bandwidth usage, expressed in kilobytes for example. The value $c_{i\,j}$ thus denotes the throughput of main and reserved channels. We then explicate the communication center s by the maximum permissible usage

$$c_i = \sum_{j=1}^{n} c_{i,j} .$$

The traffic redirected through the node s along the main communication channel, as well as the reserved channel, between nodes s and j specifies thereupon a share of maximum permitted usage c_s. In an actual communication network, the usage share must be lower than the maximum allowed share $p_{s\,j} = {c_{s,j}}/{c_s}$. Moreover, the usage share $p_{s,j}$ of the communication channel can be interpreted as a probability of establishing contact between the nodes s and j. Assuming that the main and the reserved channels are treated as equitable, the quantity must satisfy an inequality

$$2 \cdot \sum_{j=1}^{n} p_{i,j} < 1 \qquad (1)$$

for all s without exception,

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4 Network protocol analyzers can collect such types of statistical data.

Let a communication network, characterized by the aforementioned pass-around traffic feasibility, function during a long period of time by originating its main channels. We can characterize the traffic along each main channel (more precisely, the nodes i and j) by the average number of hits $\overline{p}_{i,j}$ that occur in the process of establishing either direct or indirect (pass-around) contact. It is apparent that $\overline{p}_{i,j}$ is slightly greater than the corresponding $p_{i,j}$.

If a malfunction occurs somewhere along the channel, the change[5] in the communication network will be reflected in a decrease in $\overline{p}_{i,j}$. In such a scenario, activating a reserved channel can accommodate higher network usage. It is obvious that, in this case, all $\overline{p}_{i,j}$ values will increase accordingly. Organized in this manner, the communication network represents a monotonic system.

However, a problem arises with respect to identifying the type of change malfunctioning/activating of a main/reserved channel that would influence the $\overline{p}_{i,j}$ values. In order to find an appropriate solution, it is necessary to explain the problem in Markov chains nomenclature.

Consider a set W of communication channels described by a square matrix $\left\| p_{i,j} \right\|_n^n$, when no channels exist, $p_{i,j} = 0$. According to the theory of Markov chains (Chung 1960). Such matrices may be associated with a set of returning states for some absorbing Markov chain. In the nomenclature pertaining to chains of this type, the value $\overline{p}_{i\,j}$ can be interpreted as an average number of hits from node i into node j along the Markov chain. Similarly, a malfunction in the main channel, resulting in the activation of the reserved channels, can be described through recalculating the average hit values $\overline{p}_{i,j}$. The above can be denoted as an action of type $\ominus$, whereas in the nomenclature of monotonic systems, an action of type $\oplus$ pertains to activating the reserved channel due to the malfunctioning in the main channel.

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5 For example, the OSPF (Open Short Path First) protocol will automatically redirect the traffic.

From the above discussion, it is evident that adopting this special class of absorbing Markov chains allows approaching the problem from the perspective of how to differentiate the Extremal Subsystem of Monotonic System — the kernels. Along with the KSR — Kernel Search Routine elaborated for this purpose in (Mullat, 1976), this approach can actually accomplish the kernel search task.

In Section II below, the problem of kernel extraction on Markov chains is described in more detail. In Section III, we show that the results of performing the $\oplus$ and $\ominus$ actions upon Markov chain entries in a transition matrix lead to Sherman-Morrison (Dinkelbach, 1969) expressions for recalculating the numbers of average hits (see Appendix).

2. THE PROBLEM OF KERNEL EXTRACTION ON MARKOV CHAINS

Consider a stationary Markov chain with a finite number of states and discrete time. We denote the set of states by V. Stationary Markov chain can be characterized by the property that the pass probability from the state i to the state j at a certain point in time $t+1$ does not depend upon the state s $(s = 1,2,...,n)$ the considered chain arrived in i in the preceding moment t. We denote by $p(i, j, k)$ $(p(i, j, 1) = p_{i,j})$ the conditional probability of this pass from i to j within k units of time.

Below, we consider only a special class of Markov chains that, for arbitrary states i and j within some subset in V, is constrained by

$$\lim_{k\to\infty} p(i, j, k) = 0 .$$

According to the theory of Markov chains, this limit equals zero when the state j is returning, implying that there must be some reversible states in such Markov chains. Without diminishing the generality of this consideration, we will further examine chains with only one reversible state, which must simultaneously be an absorbing state.

The absorbing chains utilized below satisfy the following properties:

1. There exist only one absorbing state $\theta \in V$
2. All remaining states are returning, and the probability of a pass between the states in one step corresponds to an entry in the square matrix $\left\| p_{i,j} \right\|_n^n$.
3. The probability of a pass into an absorbing state θ from some returning state i in one step, in accordance with 1 and 2, is equal to

$$p_{i\theta} = 1 - \sum_{\theta=1}^{n} p_{i,\theta} .$$

The monotonic system mandates a definition of some positive and negative (⊕, ⊖) actions upon system elements. For this purpose, we make use of the average number of hits $\overline{p}_{i,j}$ from the state i into the state j along the chain (Chung 1960). It is known that the value of $\overline{p}_{i\,j}$ is specified by the series

$$\overline{p}_{i,j} = \sum_{k=1}^{\infty} p(i,j,k) . \tag{2}$$

The sufficient condition for series (2) to converge is established if the sum of entries in each row of the matrix $\left\| p_{i,j} \right\|_n^n$ is less than one. We consider that elements elsewhere in the chains fulfill the conditions 1-3.

Let W be the set of all nonzero entries in the matrix $\left\| p_{i,j} \right\|$. On the transition W set of the Markov chain described above, we define the following actions.

Definition. The action type ⊖ on the element of the system W (nonzero element of the matrix $\left\| p_{i,j} \right\|$) denotes a decrease in its value by some Δp of its probability to pass in one step.

By analogy, we define the action ⊕. In this case, the probability of a pass in one step, which corresponds to the entry value $p_{i,j}$, is increased by Δp. In case of some nonzero increment in the matrix $\left\| p_{i,j} \right\|$ element (based on straightforward probability considerations), all average numbers of hits $\overline{p}_{i\,j}$ must also increase accordingly. On the other hand, a Δp decrement would result in a decrease in the corresponding $\overline{p}_{i\,j}$ values. In sum, introduced actions upon system W elements fully meet the monotonic condition (Mullat, 1976), and system W transforms into a monotonic system.

At this juncture, it is important to emphasize that the Δp changes in values of probabilities in one step within W are not specified in the definition of $\oplus$ and $\ominus$ actions upon the entries in the matrix $\overline{p}_{i,j}$. Relatively rich possibilities exist for the change definition. For example, it can denote an increase (decrease) in each probability on a certain constant, or the same change, but this time depending upon the probability value itself, etc. When providing the definitions of $\oplus$ and $\ominus$ actions on an absorbing Markov chain, it is desirable to utilize authentic considerations. Below, using an example of communication network, we describe one of such considerations.

Let W be the set of all possible transitions in one step among all returning states of an absorbing chain. These transitions in the set W retain the correspondence with nonzero elements of the matrix $\left\|p_{i,j}\right\|$. Let T be a certain subset of the set W, relating to the nonzero elements noted above. Denote by $p(T,i,j,k)$ the probability that the chain passes from the state i into the state j within k time units, constrained by the condition that, during this period, all passes in one step upon the set T have been changed by either $\oplus$ or $\ominus$ actions. This condition corresponds to the assertion that the passes along the set $W \setminus T \equiv \overline{T}$ proceed in accordance with the “old” probabilities, while those along T are in governed by the “new” Probabilities. We do not exclude the case when no $\oplus$ or $\ominus$ actions have been implicated — the set $T = \varnothing$. In this case, we simply omit the T symbol notation in the corresponding probabilities.[6]

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6 We suppose that actions do not violate the convergence of probability series, see condition (1).

The average number of hits from i into j, subject to the constraint that some passes in the set T have been changed by actions, is specified by a series

$$\overline{p}(T,i,j)=\sum_{m=1}^{\infty}p(T,i,j,m). \tag{3}$$

Let us now focus on the collections of credentials specified by a monotonic system W. We define a collection $\Pi^{+}H$ on the subset $H \in W$ as a collection of real numbers $\left\{\overline{p}(\overline{H},i,j)\,\middle|\,(i,j)\in H\right\}$ in case that the positive $\oplus$ actions occur on $\overline{H}=W\setminus H$, while $\Pi^{-}H=\left\{\overline{p}(\overline{H},i,j)\,\middle|\,(i,j)\in H\right\}$ collection corresponds to the case of the negative $\ominus$ actions taking place.

In the original paper (Mullat, 1976), we have proved that, in a monotonic system, two kinds of subsystems always exist — the $\oplus$ and $\ominus$ kernels. The definitions introduced above, pertaining to the average number of hits $\overline{p}(\overline{H},i,j)$, allow us to formulate the notion of $\oplus$ and $\ominus$ kernels in the Markov chain.

Definition. By the Extremal Subsystem of passes on absorbing Markov chain — the $\oplus$ and $\ominus$ kernels — we call a system $H^{\oplus}\subseteq W$, on which the functional

$$\max_{(i,j)\in H}\overline{p}(\overline{H},i,j) \tag{4}$$

reaches its global minimum on 2^{W}, whereby $\ominus$ kernels will be a subsystem $H^{\ominus}\subseteq W$ where the functional

$$\min_{(i,j)\in H}\overline{p}(\overline{H},i,j) \tag{5}$$

reaches its global maximum as well.

We will now turn the focus toward the notions of ⊕ and ⊖ kernels introduced above, using an example on communication network described earlier.

The probabilities of hits p_{ij} (without any passes, i.e., in a single step) between nodes i and j $\left(i, j = \overline{1, n}\right)$ allow us to construct for the communication network an absorbing chain satisfying the conditions 1-3 above. In fact, as we already noted, only one condition is mandatory to satisfy the inequality (1), which is a natural condition for any communication network. Conditions 2 and 3, on the other hand, can be guaranteed by the Markov chain design. In this case, numbers $p_{i,j}$ may be interpreted as probabilities of a pass in one step, whereby $\overline{p}_{i,j}$ denotes an average number of hits from i into j, whether directly, or via an indirect pass-around along other lines in the chain.

The search for the ⊕ and ⊖ kernels on an actual Markov chain, reconstructed from a communication network, mandates a precise definition of ⊕ and ⊖ actions. In the beginning of the discussion, we observed that ⊖ action might represent a malfunctioning in the main channel, whereas ⊕ action might pertain to the activation of a reserved channel. On the Markov chain, the malfunctioning is denoted as null, reducing the corresponding probability, while the activating of a reserved channel is reflected in the doubling of its initial probability value.[7] The condition (1) guarantees that, in any circumstance that would necessitate such ⊕ and ⊖ actions, the convergence of series (2) and (3) will not be violated.

We suggest a suitable interpretation of ⊕ and ⊖ kernels in Markov chain below, starting from the Markov chain characteristics, introduced here in terms of communication network.

In Extreme Subsystem $H^{\ominus}$, none of the communication lines/channels are subject to changes, whereas in all lines outside $H^{\ominus}$, they're reserved channels have been activated. The extreme value of the functional (4) shows that the average number of hits within channels belonging to $H^{\ominus}$,

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[7] We stress once again that ⊕ and ⊖ actions are subjective evaluations of an actual situation.

including the indirect pass-around hits (by definition, an indirect hit requires at least two steps to reach the destination), is relatively low. This assertion implies that the lines within the $H^{\ominus}$ kernel are "immune" with respect to package delivery malfunctions, i.e., most of the transported packages pass along direct lines. The set of lines in $H^{\oplus}$ kernel is characterized by a reverse property. Thus, the main channels in $H^{\ominus}$ kernel are the most "appropriate" for organizing "high-quality" indirect communications, but are also a sensible choice for mitigating the malfunctions that may result in a "snowballing" or "bandwagon" effects. Conversely, along $H^{\oplus}$, the indirect communication is typically hampered for some reason.

3. MONOTONE SYSTEM CREDENTIAL FUNCTIONS ON MARKOV CHAINS

In Section II, we defined some ⊕ and ⊖ actions upon the transition matrix entries in one step corresponding to returning states. In this section, we will develop an apparatus that allows us to incorporate the changes induced by these two types of actions into the average numbers of hits from one returning state i into the other state j. We describe here and deduce some tangible credential functions intended for use alongside our formal monotonic system description, following the conventions presented in the previous work (Mullat, 1976). Let us first recollect the notion of credential function before providing an account of the main section contents.

Suppose that, in the system W, which in the case of Markov chain is characterized as a collection of entries in matrix $\left\|p_{i,j}\right\|_n^n$ corresponding to passes among returning states, a subset H has been extracted. As a result, the set H consists of one-step transitions. Owing to the successive actions of type ⊖, by accounting for all individual sequential steps in the process (see Section II) taken upon the elements in $\overline{H}$ (a complementary of H to W), it is possible to establish the average number of hits within the transition set H — the credential system $\Pi^{-}H$. By analogy, on the set $\overline{H}$, a succession of ⊕ actions establishes the credential system $\Pi^{+}H$. The average number of hits in the nomenclature given in Section II may be represented as $\overline{p}(\overline{H},i,j)$ — i.e., the limit values for series (2) on non-zero elements for the transition matrix P corresponding to the entries/lines within the set H. Further, we will refer to the numbers $\overline{p}(\overline{H},i,j)$ as the credential functions.

Let us now establish the general form of the credential functions on Markov chains as a matrix series. This can explain the mechanism of actions the defined in Section II, performed upon the elements of a monotonic system — the Markov chain.

The credential function on Markov chain may be found using the series (2), where the single element (i,j) in the series presents the probability of the chain pass from i into j, constrained by the condition that actions have been performed upon the set $\overline{H}$.

The general matrix form of such transition probabilities described in Section II is given below: θ

$$\left\|\begin{matrix} 1 & 0 & . & . & . & 0 \\ p_{1,\theta} & & & & & \\ . & & P & & & \\ p_{n,\theta} & & & & & \end{matrix}\right\|, \text{ where} \tag{6}$$

θ – absorbing state of the chain;

$p_{i,\theta}$ – the probability of a pass from the i's returning state into the absorbing state θ;

P – the transition matrix of probabilities between the returning states within one step, where the matrix dimension is $n \times n$.

Using Chapman-Kolmogorov equations (Chung 1960), the element $p(T,i,j,m)$ in series (3) may be found as the m-s power of the matrix (6), whereby it occupies an entry in the matrix P^m.

In summary, the collection of series (3) may be written as the following matrix series

$$\overline{P}_T = I + P_T + P_T^2 + \ldots,^8 \qquad (7)$$

P_T – the matrix, where type ⊕ and ⊖ actions have been performed upon all nonzero elements within the set. Recall that, in the definition of a monotonic system, the credential function on the set $H \subseteq W$ takes advantage of a complementary set $\overline{H}$ to the set H only. The set $\overline{H}$ is actually the set of performed actions. Given that the elements of the set W are also presented as matrix entries $\overline{P}_{\overline{H}} = \left\| I - P_{\overline{H}} \right\|^{-1}$, the matrix is the credential functions collection on the Markov chain, identical to the matrix limit of (7).

In the nomenclature of fundamental matrices, the actions upon the monotonic system elements are transformations, taking place in succession, from the matrix $\left\| I - P_T \right\|^{-1}$ to the matrix $\left\| I - P_{T\cup\alpha} \right\|^{-1}$. Calculus of such a transformation is, however, a very "hard operation." In order to organize the search of ⊕ and ⊖ kernels on the basis of constructive procedures (KSR) described previously (Mullat, 1976), the utilization of matrix form is inappropriate. To extract the extreme subsystems on Markov chains successfully and take full advantage of the developed theory of monotonic systems, a more effective technology is needed, which leads us to Sherman-Morrison relationships (Dinkelbach, 1969).

The solution that can account for the changes emerging as a result of the ⊕ and ⊖ actions upon the transition matrix elements within one step in the fundamental matrix of Markov chain may be archived in the following manner. Suppose that, instead of the old probability p_o denoting a pass in between the returning states i and j, an updated (new) probability

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[8] We suppose that $p(T, i, j, 0) = \delta_{i,j}$, which is what the unity matrix in Section I highlights. In the nomenclature of the Markov chains (Kemeny et al, 1976) theory, matrices of type P_T are referred to as the fundamental matrices.

$p_n = p_o + \Delta p$ is utilized, where the action $(\pm \Delta p)$ results in either an increment or a decrement. In case of $(+ \Delta p)$, the ⊕ action has occurred, whereas $(- \Delta p)$ implies the ⊖ action. The change induced by one of these actions may be treated as two successive effects. First, the probability p_o is replaced by 0 and the replacement is recalculated. Second, the transition probability is subsequently reestablished with the new value p_n and the change in the fundamental matrix is recalculated immediately after the first recalculation.

The relationships accounting for the changes in the fundamental matrix $\overline{P}_T$ as a result of the element α having a null value and affecting the matrix P_T, as well as the relationships accounting for the changes in $\overline{P}_T$, also in the reverse case of ⊕ actions, may be found in Appendix I.

In sum, for the search of extreme subsystems following the theory of constructing the defining sequences on system W elements with the aid of KSR routines introduced in the previous work (Mullat, 1976), it is necessary to obtain some well-organized and distinct recurrent expressions, which can account for the changes in the matrix $\overline{P}_T$ whereby it is transformed to the matrix $\overline{P}_{T \cup \alpha}$. The formulas for specified Δp, which allow us to transform from $\overline{P}_T$ in order to find the matrix $\overline{P}_{T \cup \alpha}$ are given in Appendix II on the basis of the expressions II 1.3 and II 1.4.

With the aid of these recurrent expressions, in Appendix II, it is possible to obtain on each set $H \subseteq W$ the collection of credentials $\Pi^{+}H$ or $\Pi^{-}H$ by performing the successive implementation of expressions II 2.5 to all elements upon the set $\overline{H}$. These expressions mirror the transformation of system element credentials π into π_α in view of the theoretical apparatus of monotonic systems (Mullat, 1976). Indeed, we construct the collection $\Pi^{+}H$ in the case of $\Delta p > 0$, whereas the collection $\Pi^{-}H$ is constructed if $\Delta p < 0$.

4. On homogeneous Markov chains

In this section, we consider homogeneous Markov chains with a finite number n of states and a discrete time. A chain is called homogeneous if and only if the transition probabilities $p_{i,j}$ are independent of time t.

Our goal is to establish the relations between the elements of fundamental matrix denoting an absorbing chain (Chung, 1960), p. 66), see the definition below on the condition that certain transitions per time unit have been declared as prohibited. These relations are used in adjusting the corresponding elements without imposing this restriction. It should be noted that similar relations are encountered in compositions pertaining to the first and the last occurrence of some Markov chain states (see (Chung, 1960), p. 75). However, in spite of this obvious resemblance, such relations have not yet been considered in the literature.

Given without proof, the relations given in the form of theorems I-IV allow making a case for implementation of a general principle of maximum for some functions, defined on finite sets (Mullat, 1971). The foundation for the construction scheme (1971), in particular, is contingent upon requirements applied to the functions in the form of inequalities given as a result of this research.

In developing an efficient algorithm at the computer center of the Tallinn University of Technology, the theorems I-IV served as a foundation for finding solutions for some notable pattern recognition classification problems. Application of the algorithm improved the solution quality and speed with which problems were solved computationally, in comparison with those achieved by currently used algorithms.

Usually, homogenous chain can be represented as a directed graph whose vertices correspond to the state of the chain, whereby the arcs denote possible unit transitions from one state to another at any point in time. In addition, when the transition probability $p_{i,j}$ is zero, the arc $u = (i, j)$ is not depicted on the graph. On the other hand, any graph Γ can be represented in the form of a homogeneous chain attributing the arcs of the chain by satisfying the relation of the conditional probabilities. These chains are referred to as chains associated with the graph Γ.

Let $U(G)$ be the set of arcs of the graph G, and $V(G)$ the set of vertices. Adding to the set of vertices $V(G)$ a vertex θ, which is in turn connected to any vertex in $V(G)$ by an arc leading into θ, can hence reproduce a graph Γ

Consider the following homogeneous Markov chain associated with the graph G:

1) There exists a unique absorbing state $\theta \notin V(G)$;
2) The probability of transition from i to j, $i, j \in V(G)$, $p_{i,j} = p_j$, if the arc $(i, j) \in U(G)$, and $p_{i,j} = 0$ otherwise;
3) The probability of transition from the state $i \in V(G)$ to the absorbing state θ is given by $p_{i,\theta} = 1 - \sum_{i=1}^{n} p_{i,j}$.

It can easily be verified that all states of the chain, identified by the vertices of the graph G, are irrevocable, whereby the designated Markov chain belongs to a class of absorbing chains (see (Chung, 1960), p. 55).

Here, some of the tuning indicators v_j refer to the parameters of the Markov chain associated with the graph G. Further, we assume that for any $v_j = \sum_i^n p_{i,j} < 1$. For all vertices of the graph G, it can be demonstrated that for any graph G, one can find a tuning parameter v for which a given constraint $0 < v < 1/k$ is satisfied. Indeed, let k represent the largest number of nonzero elements in the rows of the fundamental matrix corresponding to the vertices of the graph G.

Moreover, let H denote an arbitrary subset of arcs of the graph G, i.e., $H \subset U(G)$. Here, $p(H, i, j, k)$ designates the probability of transition from the state i to the state j in k units of time, on the condition that the transitions along the arcs of the subset H are prohibited during this period. Owing to this restriction, the subset H denotes a prohibited set of arcs, all of which are thus prohibited as well.

Let $p(H, i, j, 0) = \delta_{i,j}$ (where $\delta_{i,j}$ represents the Kronecker's symbol) and

$$\bar{p}(H, i, j) = \sum_{n=0}^{\infty} p(H, i, j, n).$$

Due to the existence of a Markov chain associated with the graph Γ of an absorbing state θ, the entire set $V(G)$ is irrevocable, see Chung, 1960, p. 45, and the series (1) converges.

We use the Greek letters $\alpha, \beta, \ldots$ to denote prohibited arcs of the graph G, whereby α^+ refers to the vertex (state) from which the arc emerges, and α^- is the vertex toward which the arc is pointing.

Theorem I. We denote by $H + \alpha$ a set-theoretic operation. $H \cup \alpha$.

$$\bar{p}(H + \alpha, i, j) = \bar{p}(H, i, j) -$$
$$- v \cdot \frac{\bar{p}(H, i, \alpha^+) \cdot \bar{p}(H, \alpha^-, j)}{1 + p_{\alpha^-} \cdot \bar{p}(H, \alpha^-, \alpha^+)} \cdot$$

This expression might be interpreted as a consequence of malfunctions in the communication line α.

Theorem II. This can be interpreted as an increase in traffic efficiency after repairs on the line.

$$\bar{p}(H, i, j) = \bar{p}(H + \alpha, i, j) +$$
$$+ v \cdot \frac{\bar{p}(H + \alpha, i, \alpha^+) \cdot \bar{p}(H + \alpha, \alpha^-, j)}{1 - p_{\alpha^-} \cdot \bar{p}(H, \alpha^-, \alpha^+)}$$

Corollary.

From the form of the dependence in the formulations of Theorems I-II it immediately follows that the following inequalities are valid for the case of directed and undirected graphs, respectively

$$\bar{p}(H + \alpha, i, j) \leq \bar{p}(H, i, j),\ i, j = \overline{1, n}).$$

These inequalities guarantee the fulfillment of the monotonicity condition for the realization of a monotonic system on homogeneous Markov chains.

APPENDIX I

Consider the value $\overline{p}(T, i, j)$ produced by the series (3). Each component of this series may be treated as the measure of all passes in m time steps (time units) commencing in i and terminating in j. This assemblage of transitions is a union of two nonintersecting collections. The first set pertains to the passes from i to j with a mandatory transition, at least once, along $\alpha \in W$. On the other hand, the second relates to the set of passes from i to j avoiding this transition α. Each passage from the first set consists of two passes: a pass avoiding α being in t steps long, and a pass in $m - t - 1$ steps (time units), passing along α. In other words, the passages in t steps avoid the pass along α, whereas passages in $m - t - 1$ steps make use of this pass α.

We introduce the following notation: $\overline{p}(T^0, i, j, k)$ represents the average number of hits from i into j with the transition matrix P_T, where the nonzero element α is null, and $p(T^0, i, j, k)$ denotes the probability of transition without making use of α. Implementation of the introduced notification results in:

$$p(T,i,j,m) = p(T^0,i,j,m) + \\ p_\alpha \cdot \sum_{t=0}^{m-1} p(T^0,i,\alpha_b,t) \cdot p(T,\alpha_e,j,m-t-1);$$

$$p(T,i,j,m) = p(T^0,i,j,m) + \\ p_\alpha \cdot \sum_{t=0}^{m-1} p(T,i,\alpha_b,t) \cdot p(T^0,\alpha_e,j,m-t-1),$$

where α_b – the state from which a one-step pass begins, ending in α_e; p_α – the pass along α in one step, corresponding to the element α of the matrix P_T.

The first component in II 1.1 and II 1.2 introduces the value of $p(T,i,j,m)$, denoting the measure of transitions avoiding the pass along α. In addition, the components included in the summation represent the probability that the states α_b (for the relationship II 1.1) and α_e (for the relationship II 1.2) have been reached by the first and the last pass along α in the moments t and $t+1$, respectively.

Let us calculate the $\overline{p}(T,i,j)$ values using the relationship II 1.1. We conclude, after performing the summation of each of the equations II 1.1 from 1 to M and thereafter changing the order of sums in the double summation, that

$$\sum_{m=1}^{M} p(T,i,j,m) = \sum_{m=1}^{M} p(T^0,i,j,m)$$

$$p_\alpha \cdot \sum_{t=0}^{M-1} p(T^0,i,\alpha_b,t) \cdot \sum_{s=1}^{M-t} p(T,\alpha_e,j,s-1)$$

Dividing both parts of the latter equation yields $\sum_{t=0}^{M-1} p(T^0,i,\alpha_b,t)$.

Thus, based on the theorem of Norlund averages (Chung 1960) considering the sequence $a_t = p(T^0,i,\alpha_b,t)$ and $b_{m-t} = \sum_{s=1}^{M-t} p(T,\alpha_e,j,s-1)$, while increasing $M \to \infty$ for the sequences a_n and b_n, it can be concluded that the following relations are valid:

$$\overline{p}(T,i,j) = \overline{p}(T^0,i,j) + p_\alpha \cdot \overline{p}(T^0,i,\alpha_b) \cdot \overline{p}(T,\alpha_e,j). \quad \text{II 1.3}$$

Analogous relationship can be deduced by exploiting the composition II 1.2, namely:

$$\overline{p}(T,i,j) = \overline{p}(T^0,i,j) + p_\alpha \cdot \overline{p}(T,i,\alpha_b) \cdot \overline{p}(T^0,\alpha_e,j). \quad \text{II 1.4}$$

APPENDIX II

We introduce the following notifications. Let $\overline{p}(T_o, i, j)$ represent the matrix $\overline{P}_T$ element, and $\overline{p}(T_n, i, j)$ denote the matrix $\overline{P}_{T\cup\alpha}$ element. Let us also rewrite II 1.3 and II 1.4 with respect to these notifications, which results in:

$$\overline{p}(T_n, i, j) = \overline{p}(T^0, i, j) + p_n \cdot \overline{p}(T^0, i, \alpha_b) \cdot \overline{p}(T_n, \alpha_e, j); \qquad \text{II 2.1}$$

$$\overline{p}(T_o, i, j) = \overline{p}(T^0, i, j) + p_o \cdot \overline{p}(T_o, i, \alpha_b) \cdot \overline{p}(T^0, \alpha_e, j). \qquad \text{II 2.2}$$

From the relationships II 2.1 and II 2.2, it follows that the new value for the average hits from i into j is equal to

$$\begin{aligned}\overline{p}(T_n, i, j) = \overline{p}(T_o, i, j) + p_n \cdot \overline{p}(T^0, i, \alpha_b) \cdot \overline{p}(T_n, \alpha_e, j) - \\ - p_o \cdot \overline{p}(T_o, i, \alpha_b) \cdot \overline{p}(T^0, \alpha_e, j)\end{aligned}. \qquad \text{II 2.3}$$

Substituting in II 2.1 the state $i = \alpha_e$, we obtain

$$\overline{p}(T_n, \alpha_e, j) = \overline{p}(T^0, \alpha_e, j) / \left(1 - p_n \cdot \overline{p}(T^0, \alpha_e, \alpha_b)\right),$$

and from II 2.2, with the same $i = \alpha_e$, we get

$$\overline{p}(T^0, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right).$$

Replacing the latter expression into the preceding one, and taking into account that

$$\overline{p}(T^0, \alpha_e, \alpha_b) = \overline{p}(T_o, \alpha_e, \alpha_b) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right),$$

we finally arrive at

$$\overline{p}(T_n, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 - \Delta p \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right). \qquad \text{II 2.4}$$

The expression II 2.1 is valid if we replace T_n by T_o and p_n by p_o, and if in the expression II 2.2 we make a reverse replacement. Substituting $j = \alpha_n$ in the expression II 2.2, first regrouping it by this reverse replacement, results in

$$\overline{p}(T^0, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right).$$

Finally, we deduce the expression that can account for the changes in the fundamental matrix $\overline{P}_T$ by simplifying the last two equalities and the expression II 2.4, after collecting sub-expressions and making rearrangements to transform $\overline{P}_T$ into the matrix $\overline{P}_{T\cup\alpha}$. Adopting the standard nomenclature given in Section III, the ultimate form of the expression is given as follows:

$$\overline{p}(T \cup \alpha, i, j) = \overline{p}(T, i, j) + \Delta p \cdot \frac{\overline{p}(T, i, \alpha_b) \cdot \overline{p}(T, \alpha_k, j)}{1 - \Delta p \cdot \overline{p}(T, \alpha_e, \alpha_b)} . \quad \text{II 2.5}$$

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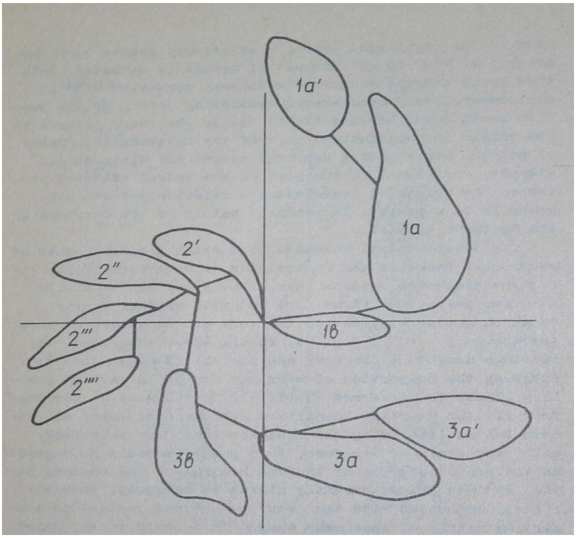
A Study of Infraspecific Groups of the Baltic East Coast Autumn Herring by new Method based on Cluster Analysis



Positions of the autumn herring subgroups differentiated by the method described.

Figure 1

E. Ojaveer, Estonian Laboratory of Marine Ichthyology (1975)

"In the Baltic Sea the autumn spawning herring forms a smaller number of groups than the spring herring does. This is probably connected with the different location of their spawning grounds. Spawning grounds of the spring herring are concentrated in favorable sites near the coast (in gulf, estuaries, etc.) while between such spawning centers gaps occur usually. Contrary to it, in most parts of the Baltic spawning places of the autumn herring form a continuous chain situated in the open sea. Therefore, differences in environment conditions between the autumn spawning grounds of neighboring areas are small and in large districts the characters of the autumn herring do not reveal essential differences. For instance, there is no significant difference between the autumns herrings caught on various grounds off the Polish coasts. The autumn herring of the Swedish Baltic coasts can be divided into four groups (that of the Gulf of Bothnia, that of the Bothnia Sea, the herring of the Swedish east coast and that of the Swedish south coast), between which a gradual transition occurs."

Appendix 1, J. Mullat (1975), Tallinn Technical University

While cluster is a concept in common usage, there is currently no consensus on its exact definition. There are many intuitive, often contradicting, ideas on the meaning of cluster. Consequently, it is difficult to develop exact mathematical formulation of the cluster separation task. Yet, several authors are of view that clustering techniques are already well established, suggesting that the focus should be on increasing the accuracy of data analysis. The available examples of data clustering tend to be rather badly structured, whereas application of the formal techniques on such data fails to yield results when the classification is known *a priori*. These issues are indicative of the fundamental deficiencies inherent in many numerical taxonomy techniques.

Following the standard nomenclature, a vector of measurements can describe every object $\langle x_1, x_2, ..., x_k \rangle$. Thus, for every pair of objects E_i and E_j a distance d_{ij} between those objects can be defined as

$$d_{ij} = \sqrt{(x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + ... + (x_{ik} - x_{jk})^2} \qquad (1)$$

However, it should be noted that all measurements are usually standardized beforehand.

Applying Eq. (1) on N objects yields a full matrix of distances

$$D = \begin{Vmatrix} 0 & d_{12} & d_{13} & . & . & d_{1k} \\ d_{21} & 0 & d_{23} & . & . & d_{2k} \\ . & . & . & . & . & . \\ d_{k1} & d_{k2} & . & . & . & d_{kk} \end{Vmatrix} \qquad (2)$$

Authors of many empirical studies employ methods utilizing the full matrix of distances as a means of identifying clusters on the set $\{E_1, ..., E_i, ..., E_k\}$.

In this section, we describe a new and highly effective clustering method, underpinned by some ideas offered by the graph theory. As the first step in our novel approach, we emphasize that, for elucidating the structure of the system of objects, knowledge of all elements of the matrix of distances given above is rarely needed. We further posit that, for every object, it is sufficient to consider no more than M of its nearest neighbors.

To explicate this strategy, let us consider a system of 9 objects (Fig. 2) with their interconnections — edges. The matrix of nearest neighbors for such a graph is given by:

$$
MND = \left\| \begin{array}{cccccc}
5(1) & 6(1) & 3(2) & 0 & 0 & 0 \\
4(1) & 3(2) & 7(3) & 0 & 0 & 0 \\
4(1) & 5(1) & 1(2) & 2(2) & 0 & 0 \\
2(1) & 3(1) & 5(1) & 7(3) & 0 & 0 \\
1(1) & 3(1) & 4(1) & 6(1) & 7(3) & 0 \\
1(1) & 5(1) & 7(3) & 0 & 0 & 0 \\
2(3) & 4(3) & 5(3) & 6(3) & 8(3) & 9(3) \\
7(3) & 9(3) & 0 & 0 & 0 & 0 \\
7(3) & 8(3) & 0 & 0 & 0 & 0
\end{array} \right\|
$$

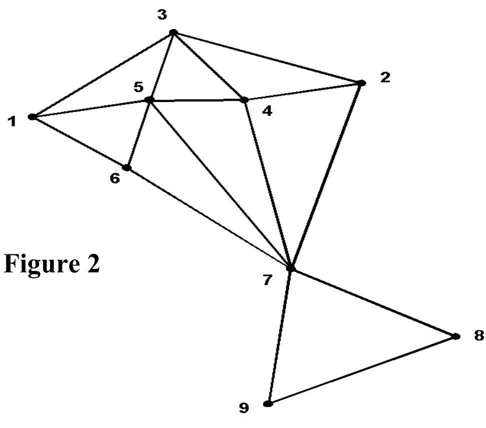


Figure 2

It can be easily verified that each row i of that matrix contains a list of objects j directly connected with a given object $\mathrm{E_i}$, with the distances $\mathrm{d_{ij}}$ given in parentheses. Based on this argument, henceforth, we will denote the matrix of nearest neighbor distances by *MND*.

In most cases, having data pertaining to about 8-10 nearest neighbors is sufficient. This is highly important for computation, where the goal is to minimize the required memory space. By applying this method on, e.g., the case of 1,000 objects, only 10,000 memory locations would be needed, which is a significant saving relative to the 500,000 required when the full matrix is processed.

We will use the *MND* defined above as a starting point to create some useful mathematical constructs.

Let W be the list of edges (pairs of objects) in the *MND*. For every edge $e = [a, b]$, a subset W_b^a of the list W can be defined as follows.

Definition 1. Subset W_b^a of W represents a proximity space of edge $[a, b]$ if

a) for every pair of objects x and y, which are connected with at least one edge in W_b^a, there exists a path joining x and y, and

b) every edge that is a member of that path belongs to the subset W_b^a.

According to the graph theory postulates, proximity space is a sub-graph connected with the edge $[a, b]$.

Example. Let us consider the edge $[4{,}5]$ shown in Fig. 1. According to the aforementioned rules, its proximity space, denoted as W_5^4, is the sub-graph $W_5^4 = \{[3{,}4],[3{,}5],[4{,}7],[5{,}7],[2{,}4],[1{,}5],[5{,}6],[4{,}5]\}$.

Definition 2. The system of proximity spaces is referred to, as the proximity structure if for each edge $w = [a, b]$ there exists a nonempty proximity space W_b^a in the system.

Sometimes it is useful to exclude the edge $[a, b]$ from the proximity space W_b^a. In line with the Venn diagram annotation, this exclusion is denoted as $W_b^a \setminus [a, b]$, whereby the resulting subset can be referred to as a reduced proximity space.

In the preceding discussion, for every edge $[a, b]$, only the value of the distance $d[a, b]$ between $[a, b]$ was taken into account. In what follows, it is useful to introduce a new notation. For example, it is beneficial to assign a real number (credential π), which is different from the distance, to every edge on the graph. For example, let us define the credential of every edge in the diagram shown in Fig. 1 as

$$\pi[x, y] = d[x, y] + r[x, y].$$

For example, $\pi[4,7] = 3 + 2$, $\pi[7,8] = 3 + 1$ on the edge $[x, y]$, where $d[x, y]$ is the Euclidean distance (1) between x, y and $r[x, y]$; $r[x, y]$ is the number of triangles that can be built around $[x, y]$.

Let us further assume that a proximity structure $\mathcal{L}$ of a graph W is known and that $f(x)$ is a real function.

Definition 3. The function $f_b^a(\pi)$ defined for all credentials of the edges in W_b^a is called the influence function of the proximity structure $\mathcal{L}$ if the following holds $f_a^b(\pi[x, y]) \leq \pi[x, y]$ for each $[x, y] \in W_b^a \setminus [a, b]$, where $\pi[x, y]$ is the credential of the edge $[x, y]$.

In other words, for every edge $[x, y]$, we can find a new credential in the reduced proximity space $W_b^a \setminus [a, b]$

$$\pi'[x, y] = f_b^a(\pi[x, y]).$$

To demonstrate the benefit of introducing the influence function, let us again consider the diagram depicted in Fig. 1. Graphically, the influence function represents the value of the number of triangles after the elimination of the edge $[a, b] \in W_b^a$ from the list W_b^a. Using the set W_5^4 as an example, this corresponds to

$$f_5^4(\pi[3,4]) = f_5^4((d_{34} + r_{34}) = (1 + 1)) = (d_{34} + r_{34}) = (1 + 0) = 1;$$

$$f_5^4(\pi[3,4]) = f_5^4((d_{56} + r_{56}) = (1 + 0)) = (d_{34} + r_{34}) = (1 + 0) = 1;$$

$$f_5^4(\pi[3,4]) = f_5^4((d_{47} + r_{47}) = (3 + 1)) = (d_{34} + r_{34}) = (3 + 0) = 3.$$

$$
MNW = \begin{Vmatrix}
5(3) & 6(2) & 3(3) & 0 & 0 & 0 \\
4(3) & 3(3) & 7(4) & 0 & 0 & 0 \\
4(3) & 5(3) & 1(3) & 2(3) & 0 & 0 \\
2(3) & 3(3) & 5(3) & 7(5) & 0 & 0 \\
1(3) & 3(3) & 4(3) & 6(3) & 7(5) & 0 \\
1(2) & 5(3) & 7(4) & 0 & 0 & 0 \\
2(4) & 4(5) & 5(5) & 6(4) & 8(4) & 9(4) \\
7(4) & 9(4) & 0 & 0 & 0 & 0 \\
7(4) & 8(4) & 0 & 0 & 0 & 0
\end{Vmatrix}
$$

It is evident that knowledge of the influence function of an edge allows us to easily find the set of new credentials for an entire subset $H \in W$. Let us consider the set $\overline{H} = W \setminus H$ and arrange its edges in some order $\langle e_1, e_2, \ldots \rangle$. Applying the steps shown above, we can find the proximity spaces of the edges in $\langle e_1, e_2, \ldots \rangle$ and apply Eq. (3) recursively.

Using the information delineated thus far, we can now introduce our algorithm, the aim of which is to identify the data structure.

At this point, we can assume that steps pertaining to the selection of the proximity structure and the influence function have been completed. Thus, we can proceed through the algorithm as follows:

A1. Find the edge with the minimum credential and store its value.

A2. Eliminate the edge from the list of all edges and compute the credentials for proximity spaces of the minimal edge using the recursive procedure (3).

A3. Traverse through the list of edges and identify the first edge with the credential less or equal to the stored credential. Return to A2 to eliminate that edge. If no such edge exists, proceed to A4.

A4. Check whether there are any further edges in W. If yes, return to A1, otherwise terminate the calculations.

Performance of the algorithm will be demonstrated by applying the aforementioned steps to the graph shown in Fig. 1.

First, the credentials for all edges should be defined using the following expression:

$$\pi[x, y] = d[x, y] + r[x, y].$$

To do so, we must compute the matrix of credentials using the matrix of distances (2).

We will demonstrate all steps of the algorithm described above.

A1. Minimal edge is $[1,6]$ and the associated credential is $\pi[1,6]=2$. To store its value, let $u=2$.

A2. We eliminate the edge $[1,6]$ from the list W and therefore have to change the credentials of $\pi'[6,7]=4$:

$$W_6^1 \setminus [1,6]:\ \pi'[1,3]=3;\ \pi'[1,5]=2;\ \pi'[5,6]=2.$$

A3. Proceeding through the list, we encounter the edge $[1,5]$ as the first edge with the credential less or equal to u. Now, we return to step A2. After 9 steps with $u=2$, we have the following sequence of edges:

$$\langle [1,6],[1,5],[1,3],[3,5],[3,4],[2,4],[2,3],[4,5],[5,6] \rangle.$$

Now, we consider the case $u=3$, and after applying the preceding steps, we obtain $\langle [2,7],[4,7],[5,7],[6,7] \rangle$. Finally, using $u=4$ yields $\langle [7,8],[7,9],[8,9] \rangle$.

It can be easily verified that those ordered lists of edges provide accurate representation of our graph’s structure.

For graphical output, we can utilize the ordered edges to construct a connected tree (a tree is a graph without circles).

For the example given above, we can construct the tree using the ordered lists of edges, while excluding all edges $[a,b]$ if both their end points, a and b, are already members of the list. This approach results in the sequence

$$\langle [1,6],[1,5],[1,3],[3,4],[2,4],[2,7],[7,8],[7,9] \rangle$$

based on which the tree in Fig. 3 can be constructed.

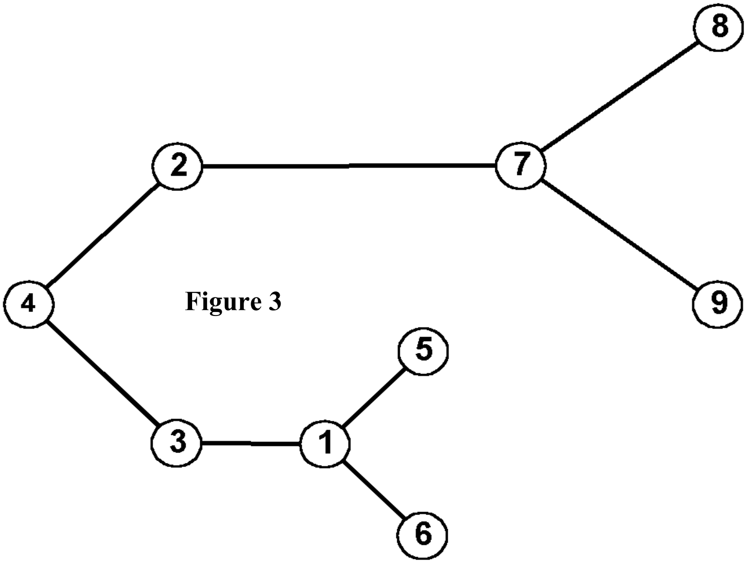


Figure 3

Using this simplified diagram, relative position of any object in the tree can be established by considering the number $S(x, y)$ of steps needed to reach the point y from the point x on the tree (e.g., $S(1,2) = 3$, $S(1,8) = 5$). Hence, for every object x, we can identify another object from which the maximum number of steps is required to reach x. For example, to identify the object at the top of the tree, we will take the object for which that maximum is minimum. Using real data, and applying these rules, we obtain the tree shown in Fig. 1.

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Prospects

POSTSCRIPT, ACKNOWLEDGEMENT AND PROSPECTS

Incidentally, the phenomena occurring in nature and in everyday life were referred to in this book as Monotonic systems, without the knowledge that this term has already been used in a different context. This coincidence, however, does not prevent us from discussing the contribution of our efforts presented here.

In the discussions, we investigated Greedy type algorithms, which allowed us to arrive at some ordering, as they facilitated arranging what we called the defining sequence. According to the prerequisites of the defining sequence, the credentials increase or decrease according to the partial order of some sub-lists of elements belonging to the main ordering such as: price credentials for wines, nodes on graphs, records of overview tables, radiotrancievers in cellular networks, routes along communications lines, agents in retail network, transfer payments, tax relief, etc. The list of indicators suitable for presentation in our defining sequence was indeed unlimited. Our aim, when using a defining sequence to arrange the order of elements, was two-fold. First, the credentials increase to some peak point, after which their value decreases to zero. Alternatively, the work-around scheme could be applied when the picture is reversed. We could easily perform some actions $\oplus$ and $\ominus$ with elements in the sub-lists among all possible sub-lists — Totality of sets, where the General Ordering was a representative of the Totality. Actions $\oplus$ improved phenomena, and $\ominus$ actions were believed to have adverse effects on the same phenomena.

The sub-lists in our Totality, which remained intact after $\oplus$, $\ominus$ actions, were investigated. We also introduced a notion of stable/steady sets, or fixed points, which cannot be improved by $\oplus$ or worsened by $\ominus$ actions when applied upon subsets. In other words, we established that a fixed point couldn't be destabilized by some predefined mappings. However, the ultimate aim was to find an optimal solution using the Greedy type algorithms in the form of defining sequence of ordering. We have proved that the defining sequence guaranteed the optimal ordering, as well as ensured discovery of optimal stable subsets — the kernels. In general, as a side effect, any defining sequence formation complied with the Fibonacci rule.

Other researchers have also investigated the Monotone System approach, the root of which is important to discuss. Some different types of more light Monotone Systems were established, allowing more effective implementation of Greedy type algorithms due to their simplified architecture. Such a convenient architecture of Monotone Systems was found when the standard order of credentials in the direction of increase or decrease on the Grand Ordering of elements did not change while the defining sequence was under formation. As was shown, any subset of credentials in such a Totality of subsets remained in harmony with the initial Grand ordering of credentials. In particular, the Totality of wine menu credentials of wine prices satisfies the harmony or light property.

Light Monotone Systems provided the opportunity to present the Grand Ordering in either increased or decreased order using standard ordering procedures — any procedure is adequate for this purpose. As a result, formation of the defining sequence would require operations the extent of which is proportional to the logarithmic scale of complexity, in contrast to the hard general scheme. It is important to note here some postulates related to the theory of bounded rationality, Arrow, Rubinstein, Sen, Uzawa,…, both for general systems and for light monotone systems, i.e., the postulate of independence from rejected alternatives, as well as the postulate of succession, for example in a Singles Party game. The latter in the language of a barmaid reads: "Old love does not rust."

It is important to remind once again that light monotonic systems supposedly allow algorithms like Greedy to find an optimal solution with much less computational effort than those required to solve complex NP-problems. Indeed, the optimality that is claimed to be guaranteed for a function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$, as noted by other researchers, [1] the $F(X)$ must obey the quasi-convex property as a whole when the function $F(X)$ is optimized among subsets $X \subseteq W$ in Grand Ordering W. Quasi-convexity on W means that for any pair $[X, Y]$ of subsets X and Y the inequality $F(X \cup Y) \geq \min[F(X), F(Y)]$ must hold. Exactly this inequality is guaranteed, allegedly, that the NP hard problem could be substituted by polynomial complexity procedures, allowing the Greedy type algorithms to perform in reasonable time.

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[1] a) Yulia Kempner, Vadim E. Levit and Ilya Muchnik. (2008) Quasi-Concave Functions and Greedy Algorithms, Advances in Greedy Algorithms, Book edited by: Witold Bednorz, ISBN 978-953-7619-27-5, p. 586, November I-Tech, Vienna, Austria; b) Yulia Kempner and Ilya Muchnik. (2008) Quasi-concave functions on meet-semilattices, Discrete Applied Mathematics 156, pp. 492-499.

We found, however, through relatively simple examples, such as our single game scheme, that quasi-convex property was not always satisfied for some Monotone Systems. This means that Monotone Systems in general are richer or more complex objects than was postulated in the beginning. Disappointingly, the techniques based on the defining sequence of ordering will fail for such systems, as they cannot be applied to search for optimal solution when the goal is to find kernels. However, it is possible to find the optimal solution by other means. Branch and Bound algorithms may be suitable for this purpose. Despite the need for applying the twisted rules of Branch and Bound algorithms, the complexity of which is much higher than Greedy type used in case of quasi-convex set functions, the Branch and Bound algorithms work effectively, when investigating the conflict situations. They are particularly useful for describing, e.g., the phenomenon of bilateral agreements, where the data set is usually of reasonable size.

Acknowledgement. In conclusion, it would be, perhaps, interesting for the reader to learn about the history of the Monotone Systems as it appears to the author of these lines. Indeed, the author had the opportunity to attend the Institute for Management Problems in Moscow, a laboratory under the guidance of late prof. Aizerman. Since the mid-50s of the last century, methods for automatic classification of objects have been investigated in the laboratory. One of the working hypotheses on the basis of which these methods were supposed to work was that objects in a multidimensional space related to similar phenomena, such as analysis of data, visual objects, sequences of letters and words, etc., are usually located closer to each other than the objects responsible for different phenomena. Most of the statistical data are always represented in this way and, thus, the hypothesis of the so-called compactness of similar objects was expressed which should be distant to dissimilar objects.

Based on the compactness hypothesis, it was possible to develop numerous classification algorithms (Braverman et al, 1975 [2]; Mirkin et al [3], the list goes on). It is important here that all these methods were based on the fact that it was necessary to classify the objects in such a way that

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[2] Braverman É.M., Litvakov B.M., Muchnik I.B. and S.G. Novikov. (1975) Stratified sampling in the organization of empirical data collection, Autom. Remote Control, 36:10, pp. 1629–1641

[3] Mirkin B.G. and L.B. Cherny. (1970) On a distance measure between partitions of a finite set, Automation and remote Control, 31, 5, pp. 786-792.

within classes the objects would be located close to each other in the sense of some metric, and objects from different classes would be far from each other in the same sense the metric itself. In connection to this task, it is noticeable to note the work of Professor in biometric of Leningrad State University P.V. Terentyev, who developed the method of correlation Pleiades, which allowed him to successfully solve the problem of choosing from among a mass of signs the most stable, “independent” ones. Terentyev 1959 [4], applied his own method of his Pleiades in order to build a classification of biological objects, which, as it seems, has in his time served and as well as now still going serving on as the basis of a whole group of methods of the so-called nearest neighbor linkage.

One of the simplest cases here is the problem of classifying objects into two classes. Indeed, Võhandu and Frey 1966 [5] published a similar method in the Biological Series of the Estonian Academy of Sciences in order to enlighten biologists in the new achievements of statistics.

The fact is that being a post graduate student of Tallinn University of Technology in 1969-1971, whose supervisor was L.K. Võhandu (LV), and thanks to LV, he was familiar with similar methods what was the topic of communication especially fruitfully with the late Prof. E.M. Braverman from the Institute of Control Problems in Moscow. As far as the author remembers, when presenting his views on the problem of classification in terms of monotone systems, Braverman noted that this was something new. Indeed, in contrast to the nearest-neighbor method, a formal mathematical construction of a purely combinatorial nature was proposed at the same time with the possibility of constructing algorithms for the effective search for so called kernels of Monotonic Systems. The author interpreted in his own words and invented a general and new procedure of data analysis thanks to a “blind glance” or specific data scoring ideology of LV. Within the framework of this ideology, the author developed a theory that is now known in the literature as "Monotonic Linkage Functions" (Maximum Margin Separations in Finite Closure Systems, Florian Seiffarth et al, 2021) [6], although this model was origin ally named by the author as "Monotone / Monotonic System".

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[4] Терентьев П.В. (1959) Метод Корреляционных Плеяд, Вестник ЛГУ№9.

[5] Frey T. and L. Võhandu. (1966) Uus Meetod Klassifikatsiooniühikute Püstitamiseks, Eesti NSV Teaduste Akadeemia Toimetised, XV Köide, Bioloogiline Seeris, Nr.4. Известия Академии Наук Эстонской ССР, Том XV, Серия Биологическая, №4б.

[6] Available online, https://link.springer.com/chapter/10.1007/978-3-030-67658-2_1 (Accessed 23.12.2021).

As a result, the essence of this method was an article published by the author in 1971 in the Proceedings of Tallinn Technical University, where the method was presented formally in the language of set theory and the Totality of partially ordered subsets in standard language used in mathematics. Since then the theory was further developed and published in Russian periodical of “Автоматика и Телемеханика” in 1976, where the author proposed to call the scheme by a monotone system In English the Plenum Publishing Corporation originally distributed the idea in “Automation and Remote Control” publications. We hope that these lines will probably explain to those who doubt what exactly is called the Monotone System.

Prospects. We also hope that the Monotone Systems scheme will be subject to more extensive research, as this will contribute to the theoretical understanding, as well as assist in developing more affective algorithms aimed at finding the best solutions. The most promising avenue to pursue going forward, in our view, is the approach of steady states, or stable sets, which have been demonstrated in the collection of papers presented here. In order to discover some important phenomena hiding in plain sight, we have offered various perspectives on different subjects, in atomic or continuous form. Our motive was to demonstrate the opportunities for those enthusiasts that wish to open their minds and devote their time to promotion and advancement of science.

The concept or category of monotone (monotonic) system, independent and distinct from all that is usually referred to in the relevant literature as dynamic systems, is applied to computer science and communications, social sciences, social and network economics. It will appeal to specialists in specific areas of game theory and data analysis.