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Robust PID Controller Design for Continuous-time Systems via Reduced Routh Parameters

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

Igor Artemtšuk ______________________
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Pidevaja süsteemide robustse PID kontrolleri süntees taandatud Routh parameetrite kaudu

IGOR ARTEMTŠUK

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Contents

List of Publications

Publication I: Ü. Nurges, I. Artemchuk, and J. Belikov. Generation of stable polytopes of Hurwitz polynomials via Routh parameters. In *Conference on Decision and Control*, pages 2390–2395, Los Angeles, CA, USA, Dec. 2014

Publication II: I. Artemchuk, Ü. Nurges, J. Belikov, and V. Kaparin. Stable cones of polynomials via Routh rays. In *The 20th International Conference on Process Control*, pages 255–260, Štrbské Pleso, High Tatras, Slovak Republic, 2015

Publication III: I. Artemchuk, Ü. Nurges, and J. Belikov. Robust pole assignment via Routh rays of polynomials. In *The 55th American Control Conference*, pages 7031–7036, Boston, MA, USA, July 2016

Publication IV: Ü. Nurges, I. Artemchuk, and J. Belikov. On stable cones of polynomials via reduced Routh parameters. *Kybernetika*, 52(3):461–477, 2016

Author's Contributions to the Publications

I In Publication I, the basic algorithm for generation of a stable polytope around a given polynomial of a continuous-time system is proposed. Implementation of the respective mathematical formulas and expressions in the Matlab software based package was performed by the author. Verification of the algorithm was carried out by means of extensive numeric simulations.

II In Publication II, the method from Publication I is further developed by the author providing new constructive formulas for efficient software implementation. The results were extended introducing new definitions and proofs. Corresponding changes have led to modifications and extension of software functions. Verification of introduced changes was performed using numeric simulations.

III In Publication III, a procedure for robust output controller design for continuous-time linear systems is presented. Approach is based on the reduced Routh parameters that are used to derive stable Routh rays and corresponding Routh cones of polynomials. The obtained region is used to designed a fixed-order controller. Procedures of pole placement and robust controller synthesis are developed and formulated by the author in the form of a step-by-step algorithm. The generic software package was extended with the respective functionality. Verification of the proposed methods was carried out by the author based on the software simulations and experiments with a laboratory prototype of a DC motor servo system.

IV In Publication IV, the method for generating stable cones in the polynomial coefficient space based on a construction of the so-called Routh stable line segments (half-lines) starting from a given stable point (originally introduced in Publication I and Publication II) is further extended. The author provided rigorous mathematical proofs and complemented the overall framework with additional theoretical results.

In all publications the author of the thesis is the main contributor. Results of the above papers were obtained in cooperation with Dr. Ü. Nurges and Prof. J. Belikov under the guidance of Prof. E. Petlenkov. Development of the main results, writing papers, and performing experiments was done by the author of the thesis.

Introduction

Control is essential and integral part in various engineering applications, e.g., space-vehicle, robotic systems and manufacturing systems, and any industrial operations involving control of temperature, pressure, humidity, flow, etc. [63, 75]. Controller design problem is known for a long time. Tracing back to the 18th century, three major sources of power: wind, water, and steam already required certain level of control. During the late 1920s improved management, the introduction of new machinery, and production processes resulted in a rapid growth in output and productivity in manufacturing industries. Engineers started to use technology more actively to ensure meeting the necessary requirements. They preferred continuous to batch processing and, whenever possible, used remote, semiautomatic, and in some cases, automatic control. The body of knowledge developed during the years 1930 to 1955 acquired the name of *classical or conventional control theory* [77] in the early 1960s, and since then it is known as a *modern control theory* [21]. Starting from simple deterministic mathematical models, research shifted to the study of more complex control systems [13, 29]. This complexity resulted into complication of stability related problems. In such systems uncertainties started to play more prominent role increasing the relevance of development of efficient stability methods.

State of the art

For linear systems the most intuitively understandable and inherently simple stability test is based on the location of roots of a characteristic polynomial. Other alternatives include Hurwitz, Routh, and Hermite-Bieler tests [20, 82] or frequency domain based techniques [105]. However, once the system contains uncertainties, these techniques cannot be directly applied. This resulted in the development of the so-called parametric approach [15], which links the study of relationships between roots of a polynomial and its coefficients [80]. The main problem appearing with the parametric approach is that, in general, the stability domain is nonconvex in the coefficient space. Over the last decades many efforts have been put towards the development of various techniques for convex approximation [17], including balls, ellipsoids and multi-ellipsoids, boxes, zonotopes, hyper-rectangles and polytopes, as summarized in Table 1.

Table 1. Stability domain approximation: Summary of existing methods.

Methods	**Approximation type**	
	Inner	**Outer**
Balls	[33, 37]	[33]
Ellipsoids	[27, 48, 58]	[17–19, 22, 28, 36, 43, 48, 49, 58, 61, 87, 103]
Boxes	[12]	[11, 39]
Zonotopes	[2, 16, 41, 65]	[16]
Hyper-rectangles	[26]	[26, 39, 53, 57]
Polytopic	[67, 68]	[67, 68]

For example, work [37] presents convex approximation of the stability domain by means of balls technique. The problem of localization of sensor nodes in a wireless network is addressed to approximate the uncertainty in true sensor's position. In this case a feasible set of intersections cannot be generally described by few parameters and outer-approximation by a simple shape of a ball is used. Later the algorithm was extended by means of ellipsoids that can generally capture more complex convex set due to the additional degree of freedom [36]. This method is further extended via intersection of multiple ellipsoids. However, task of finding the tightest ellipsoidal outer-approximation of the intersection of multiple ellipsoids is an NP-hard and the optimal algorithm still constitutes an open problem [17–19, 22, 28, 43, 49, 61, 87, 103]. Alternatively, approximation can be performed using approaches based on zonotopes [2,41] and hyper-rectangles [39,53,57], which grant better accuracy. However, the complexity of constructing hyper-rectangular approximations is $O(n)$, i.e., it grows linearly in the dimension n, as opposed to ellipsoidal and zonotopic approximations, which exhibit quadratic complexity. Another class of methods for convex approximation of the stability region is based on the class polytopic type techniques [47, 68]. In [67] both inner and outer approximations of the stability region are presented for discrete polynomials using their coefficients and reflection coefficients [30, 76]. This approach was further extended in [60] to solve the problem of pole placement, and eventually developed into Nurges-Schur method for discrete-time robust controller design in [9, 72, 73]. The subclasses of polytopic methods known as box type approximation and convex directions are used to solve convex approximation problems in [11,39] and [84], respectively. To get benefit of using different combination of methods, a new hybrid approach called successive convex approximation was developed. The principal idea is to iteratively approximate the problem with lower complexity subproblems. Unfortunately, due to the sequential approximation, the global optimality of the achieved solution is generally cannot be guaranteed. For example, this approach was used for the problem of weighted sum rate maximization with user specific quality-of-service constraints [50]. Same type of approximation has been considered in various forms by different authors, see [24, 62, 86, 88] and the references therein.

It is widely accepted that there is no universal method which can be effectively applicable in any situation. Each of the proposed methods has specific advantages and drawbacks. The existing parametric methods can be roughly divided into two categories: quadratic and multilinear criterions based techniques as summarized in Table 2. Ellipsoid type methods utilize nonlinear quadratic optimization criteria, contrary to the polynomial type methods that are based on the linear criteria, which is simple, except for cases with complex optimization areas. More extensive overview can be found in [68, 69, 72]. A simple comparison of controller performance can be found in [9] and stability area volumes are presented in [8] for different methods. Results show that volumes for ellipsoid and reflection methods are comparable. Though the comparison is presented for discrete-time case, many conclusions are valid for continuous case as well.

Unlike the stability problems that mainly refer to ideal case of mathematical models, the real applications have to deal with system uncertainties, and the control is expected to keep required performance within the specified uncertainty domain. This challenge has lead to the development of the robust control theory that has been thoroughly studied and different related topics were addressed during past decades [97]. Most of the recent studies are focused on the robust model reference

Table 2. Stability domain approximation: Comparison of existing methods.

Methods	**Accuracy**	**Computational complexity**	**Generality** (e.g., different uncertainty types)
Balls	low	medium	low
Ellipsoids	high	high	high
Boxes	low	low	low
Zonotopes	high	high	medium
Hyper-rectangles	high	high	high
Polytopic	high	medium	medium-high
Reduced Routh	high	medium	high

adaptive control techniques, which can be applied to different types of systems such as uncertain nonlinear systems with time delays [99], nonlinear uncertain dynamical systems [104], multi-agent systems with parametric uncertainties and external disturbances [59, 79], networked single-input single-output nonlinear systems with time delays [52], switched linear parameter-varying systems with parametric uncertainties [100], piecewise affine systems with input disturbances [14], and many others. On the other hand, several recent studies focus on the robust model reference nonadaptive control techniques. These techniques are applied to different types of systems, including polytopic uncertain systems [38], multivariable linear systems subject to model uncertainties [32, 92], decentralized linear systems with interactions treated as disturbances [45], descriptor linear systems subject to parametric uncertainties [31], Markovian jump linear systems with unknown transition probabilities [107], and uncertain network-based control systems [35].

Many of these robust model reference control techniques have been applied to practical systems. In case of the permanent magnet motor the position of motor is easily disturbed by external force, disturbance, and variation in parameters of a plant. To solve these problems and improve the system a number of solutions were proposed. For example, in [25] an internal model reference control algorithm is proposed to reduce the response time on the disturbance and static friction and eliminate the effect of parameter uncertainties. Improvements in robustness against load and system parameters' variations are done with sliding mode observer-based model reference adaptive algorithm [85]. For precise motion control of the piezoelectric actuation micropositioning systems, a model reference adaptive control with perturbation estimation is presented in [101]. This approach grants better stability of the closed-loop system and allows to predefine the size of tracking error. An optimized virtual model reference control synthesis method is proposed for semiactive suspension systems [23]. Designed robust adaptive controller allows to achieve the H_∞ performance of ride comfort and vehicle handling against the influence of parameter uncertainties and external disturbances of the system. Another practical applications are addressing robotic systems such as in [83], where physical human-robot interaction (pHRI) problem was addressed and an inner-loop robot-specific controller was developed, which enables the user to interact with the robotic system so that it behaves like a prescribed robot admittance model and allows to take into account the human dynamics and adapt the prescribed robot admittance model for different users. Similarly, the work [89] presents a nonlinear Model Reference Adaptive Impedance Controllers for the control of the robot impedance with uncertainties in model parameters. Work [95] introduces an obstacle-avoidance control algorithm for controller that is able to incorporate human operator's commands for a general-type two-wheeled human-operated mobile robot with several distance sensors to detect obstacles. Another study of robotic systems was done towards model reference adaptive control of robotic mechanisms in [106]. Robust control techniques can be used for geostationary satellite networks as in [81], where a multi-model reference control approach for queue-based bandwidth-on-demand procedures is presented and the problem of guaranteeing a high exploitation of the valuable satellite bandwidth while offering acceptable end-to-end delays to the traffic accessing the network is discussed. Or another application area touches chaotic systems, as in [94], where a model-reference control is successfully developed for stabilizing chaotic system so that system follows the desired model within a desired finite time.

The industry in most cases decides in the favor of low-order controllers due to their simplicity, low cost, and high reliability [40]. The majority of them (approx. 95%, according to [74]) are of PI/PID type, and are mainly (about 80%) poorly tuned. In brief, the PID controller uses the present (P component—proportional to the error output), the past (I component—proportional to the integral of the error), and the future (D component—proportional to the derivative of the error) of the error to adjust the control signal. To apply PID controllers, engineers must first decide which element(s) to keep in action and then adjust the parameters so that their control problems are tuned appropriately. To this end, they need to know the characteristics of the process. As the basis for this design procedure, they must have certain criteria to evaluate the performance of the control system [5]. The literature has a great variety of different (including robust) approaches and methods for PID controller design and tuning with application in different areas. In what follows, we briefly recall several recent studies on robust PID control design. PID controller design for unmanned aerial vehicle is presented in [102] using a reverse multi-variable root contour method. Work [42] presents an adaptive PID controller for wind turbines based on Lyapunov direct method. Simple internal model control, integer- and fractional-order PID controllers for motor-generator system are tuned in [3, 4, 98] using statistical analysis. Nelder Mead optimization technique is applied to tune both IOPID and FOPID for coupled tank system [64]. FOPID controller is tuned using multiobjective differential evolution method for the flight control system as shown in [54–56]. The frequency-domain performance criteria is used to tune FOPID controller as shown in [91].

Motivation and problem statement

The above overview and state of the art constitute motivation for the work presented in this thesis. In what follows we briefly indicate only the most important aspects.

The wide-spread of digitalization has resulted in the necessity to develop simple and efficient control algorithms. Recall that majority of processes cannot be modeled accurately enough and uncertainty is a common situation in real applications. These challenges together have resulted in an extensive research interest in the development of different discrete-time methods for robust controller design, see [67, 69] and the references therein. Meantime, various industrial processes are of continuous nature and controlled by devices operating with high sampling rate mimicking the continuous behavior. This motivates a further research especially with respect to continuous-time systems.

This work aims to fill the existing gaps and provide a method for the inner convex approximation of the stability domain and design of a robust output fixed-order controller for continuous-time systems with parametric uncertainties. The proposed approach partly employs ideas originally presented in [67] for the discrete-time systems. However, the continuous-time case is technically different and requires new mathematical definitions and tools to be developed. The overall problem can be roughly divided into two stages: (*i*) to construct convex approximation of the stability domain; (*ii*) to design robust output PID controller with the maximum possible stability measure. In other words, the goal is to find a point inside approximated stability area such that it is equally located from boundaries of the stability domain, and identify parameters of the controller based on the obtained approximation and selected structure. Furthermore, the developed procedure has to be efficient, yet simple enough to be suitable for software implementation and practical needs. In this work we extend the polytopic approach (see Table 2) by relying on the so-called reduced Routh parameters and inheriting all the benefits of the polytopic approach. We adopt the basics of the polytopic approach and first extend it to the continuous-time case. Next, we present a more general approach by including a cone-type uncertainty in addition to the polytopic one. Cone uncertainty that is addressed in this work is rather unexplored topic that might give great benefits in the future.

Author's contributions

The main contribution is the development of a simple and efficient procedure to design a robust output PID controller for continuous-time linear systems with uncertainties. The method is based on a new multilinear stability criterion for Hurwitz polynomials. This contribution comprises three main parts:

- *Convex approximation*: Development of methods for the solution of inner convex stability domain approximation problem of continuous-time linear systems relying on a multilinear stability criterion for Hurwitz polynomials. The approach is based on the so-called reduced Routh parameters that are used to construct stable Routh rays starting from a given stable polynomial. These lines may be used to construct a polytope or polyhedral cone inside the stability domain. A step-by-step algorithm is proposed for construction of a stable polytope around given starting point.

- *Controller design*: Based on the above algorithm for the convex approximation of a stability domain, a method for robust output PID controller design is proposed. The procedure starts from a stable simplex (or polytope) of the closed-loop characteristic polynomial. Then, we define a set of possible plant parameters as a convex polytope (polytopic plant model). Note that the number of vertices of the polytope determines complexity of the algorithm. Finally, we design a robust output controller for polytopic plant model using quadratic programming approach. The designed controller is able to operate within the limits of uncertainty presented in the plant.

- *Software implementation*: All the theoretical results are implemented in the Matlab software-based package. The package includes functions related to stability area approximation and controller design, as schematically depicted in Fig. 1.

The proposed theory and algorithms are illustrated by numeric examples and laboratory prototype setup of a DC motor servo system.

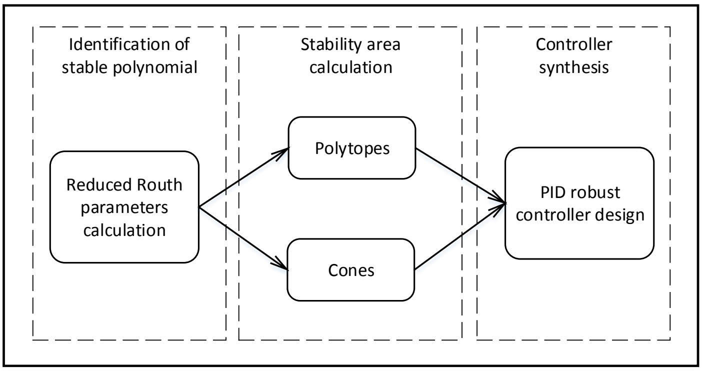

Figure 1. Schematic organization of the software package.

Thesis outline

Each section begins with a brief overview of the research problems and material discussed therein. The thesis opens with a brief overview of the stable (Hurwitz) polynomials, and is followed by theoretical results on continuous-time system convex approximation and respective robust PID controller design. Each section is concluded by a short discussion on key aspects. The concluding section comprises general comments as well as open directions for the possible future research. Throughout the thesis a number of illustrative numeric examples complements the obtained results.

Section 1

In this section, the reader is introduced to the main notions and concepts of Hurwitz stability, Hurwitz region, and stable polynomials. A method for generation convex sets of stable polynomials for continuous-time systems is explained. This method is based on the novel multilinear stability criterion using so-called reflection coefficients of Hurwitz polynomials. It allows to generate stable line segments in the directions of the Routh rays starting from an arbitrary Hurwitz polynomial. This leads to the inner approximation of a stability region based on the derived line segments. The developed results are summarized in the form of an algorithm for continuous-time system convex approximation. The section is accompanied by several illustrative examples.

Section 2

In this section, an algorithm for robust output PID controller design for continuous-time plant with uncertainties is presented. The solution is based on results from the previous section and continues with controller design based on the quadratic programming approach. The software implementation of the developed theory is discussed and complemented by description of the main functions and visualization of their mutual relations. Illustrative examples for the application of above introduced algorithms for Routh controller design are also provided.

Section 3

This section is devoted to numerical examples illustrating the main aspects of the developed theory. In particular, we consider 3 different cases. The first academic example is devoted to the design of a robust PI controller for the fourth-order system. Simulations are performed for the nominal plant and the case with minimal and maximum possible uncertainty. In the second example, a stabilizing PI controller for the second-order unstable system is designed. Laboratory experimental platform is analyzed in the third example. First, two types (*slow* and *fast*) of PI controllers are designed and tested for nominal plant and case with external friction. It is followed by comparison of PD and PI controllers designed using both polytope and cone techniques.

1 Identification of stable polynomials

The chapter opens with a brief overview of the Hurwitz stability, Hurwitz region, and notion of a stable polynomial, which are further used as a basis for construction of stable polytopes. Then, a method for generation of convex sets of stable polynomials (so-called polyhedral Routh cones and polytopes) for continuous-time systems is presented. The generation of stable polynomials is carefully studied in [90], where Lovinsone-Durbin parametrization is proposed as an efficient and numerically stable method. Here, this approach is generalized to generate stable line segments in the space of polynomial coefficients. The proposed method is based on a novel multilinear stability criterion for Hurwitz polynomials. In addition, for an arbitrary Hurwitz polynomial of the order n a method for generation of n stable line segments in the directions of the so-called Routh rays is proposed. This means that instead of single points bunches of stable half-lines in the polynomial coefficient space are constructed. Next, on the basis of the derived line segments an inner approximation of stability region is obtained.

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

with real coefficients $a_i \in \mathbb{R}$, for $i = 0, \ldots, n$, is said to be continuous-time stable in the Hurwitz sense, if all its roots λ_i, for $i = 1, \ldots, n$, are in the open left-half plane of $\mathbb{C}$, i.e., $\Re(\lambda_i) < 0$. Since polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^\mathsf{T}$ of its coefficients, i.e.,

$$a := a(s) = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^\mathsf{T}. \tag{2}$$

The Hurwitz *region* $\mathscr{H}_n$ is defined as the set

$$\mathscr{H}_n = \left\{ a \in \mathbb{R}^{n+1} \mid (1) \text{ is Hurwitz} \right\}. \tag{3}$$

1.1 Reduced Routh parameters of polynomials

In this section the reduced Routh parameters are introduced using Hurwitz and Routh stability notions. These parameters will be later used for construction of the stable line segments.

Definition 1.1 ([70])**.** The *reduced* Routh parameters w_j for normed $a_0 = 1$ polynomials

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1 \tag{4}$$

are defined as follows

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2, \\ w_1 &= a_1^2, \\ w_0 &= 1. \end{aligned} \tag{5}$$

Note that in (5) parameters a_j^j and a_{j-1}^j can be found explicitly via reduced Routh parameters w_k, $k = n, \ldots, 1$ as

$$\begin{aligned} a_{k-i-1}^{k-1} &= a_{k-i-1}^k, & i &= 0, \ldots, 2\lfloor (k-2)/2 \rfloor, \\ a_{k-i-2}^{k-1} &= a_{k-i-2}^k - w_k a_{k-i-3}^k, & i &= 0, \ldots, 2\lfloor (k-3)/2 \rfloor \end{aligned} \tag{6}$$

with $a_0 = 1$ or in the matrix form as $a^{k-1} = \overline{W}_k a^k$, where $\overline{W}_k$ is the $k \times k$ matrix

$$\overline{W}_k = I_k - w_k \begin{bmatrix} 0 & \bar{J}_{k-1} \\ \vdots & \vdots \\ 0 & 0^\mathsf{T} \end{bmatrix}, \tag{7}$$

and $\bar{J}_k$ is the $k \times k$ diagonal matrix, i.e., $\bar{J}_k = \mathrm{diag}\{0, 1, 0, 1, \ldots\}$.

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [34].

Proposition 1.1 ([71]). *A normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable if and only if $w_k > 0$, $k = 1, \ldots, n$.*

First, lets define Routh parameters h_k $k = 1, \ldots, n$ for polynomial (4) using reduced Routh parameters w_k as:

$$\begin{aligned} h_0 &= 1, \\ h_1 &= w_1, \\ h_2 &= w_2, \\ h_j &= w_j h_{j-1}, \quad j = 3, \ldots, n. \end{aligned} \tag{8}$$

Assume that a normed polynomial $a(s)$ of order n is stable in the Hurwitz sense. Then, according to Routh-Hurwitz stability criterion all first elements of stability table [34] have the same sign. Routh parameters h_k are equivalent to first elements of Routh table and method for constructing Hurwitz polynomials can be derived as follows [90]. Start with arbitrary Hurwitz polynomial of degree 2. Since positivity of the coefficients is equivalent to stability of the second-order polynomials, generate arbitrary positive numbers h_0, h_1, h_2 and compose the polynomial

$$a(s) = h_2 s^2 + h_1 s + h_0 \tag{9}$$

or

$$a = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} h_2 & h_1 & h_0 \end{bmatrix}^\mathsf{T}. \tag{10}$$

At the kth step, having a Hurwitz polynomial of degree k, i.e.,

$$a(s) = \begin{bmatrix} a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^\mathsf{T}, \tag{11}$$

consider two polynomials of degree $k+1$, e.g.

$$p(s) = \begin{bmatrix} 0 & a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^\mathsf{T} \tag{12}$$

and

$$q(s) = \begin{bmatrix} a_k & 0 & a_{k-2} & 0 & a_{k-4} & 0 & \cdots \end{bmatrix}^\mathsf{T}. \tag{13}$$

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{14}$$

which is Hurwitz polynomial of degree $k+1$, according to the Routh rule. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see

[93,96]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k=0,\dots,n$ recursively by increasing k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction [90]. Observe that, if $w_k > 0$, for $k = 1,\dots,n$, then all the Routh parameters of the polynomial $a(s)$ are positive $h_k > 0$, $k = 0,\dots,n$. Hence, it follows that the polynomial $a(s)$ is Hurwitz stable.

Example 1.1. Let us illustrate calculation procedure of Routh table stability criterions for normalized $n = 5$ polynomial via w_k and h_k. Denote degree of a polynomial by superscript.

$$a^5 = \begin{bmatrix} a_5^5 & a_4^5 & a_3^5 & a_3^5 & a_2^5 & a_1^5 & a_0^5 = 1 \end{bmatrix} \tag{15}$$

First elements of Routh-Hurwitz table are:

$$\begin{aligned}
&\text{row 1: } a_5^5,\\
&\text{row 2: } a_4^5,\\
&\text{row 3: } b_1 = \frac{a_4^5 a_3^5 - a_5^5 a_2^5}{a_4^5},\\
&\text{row 4: } c_1 = \frac{b_1 a_2^5 - b_2 a_4^5}{b_1},\\
&\text{row 5: } d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}.
\end{aligned} \tag{16}$$

Now, calculate w_k using (5) and (6) from w_5:

$$\begin{aligned}
w_5 &= \frac{a_5^5}{a_4^5} = \frac{h_5}{h_4},\\
w_4 &= \frac{a_4^4}{a_3^4} = \frac{a_4^5}{a_3^5 - w_5 a_2^5} = \frac{a_4^5}{b_1} = \frac{h_4}{h_3},\\
w_3 &= \frac{a_3^3}{a_2^3} = \frac{a_3^4}{a_2^4 - w_4 a_1^4} = \frac{b_1}{a_2^5 - \frac{a_4^5}{b_1} b_2} = \frac{b_1}{c_1} = \frac{h_3}{h_2},\\
w_2 &= a_2^2 = a_2^3 = c_1 = h_2,\\
w_1 &= a_1^2 = a_1^3 - w_3 a_0^3 = a_1^4 - \frac{b_1}{c_1} a_0^5 = a_1^5 - w_5 a_0^5 - \frac{b_1}{c_1} a_0^5\\
&= a_1^5 - \frac{a_5^5}{a_4^5} a_0^5 - \frac{b_1}{c_1} a_0^5 = b_2 - \frac{b_1}{c_1} a_0^5 = \frac{c_1 b_2 - b_1 c_2}{c_1} = d_1 = h_1,\\
w_0 &= 1 = h_0.
\end{aligned} \tag{17}$$

The results show that h_5 corresponds to first element of Routh table, h_4 corresponds to second element, and so on until h_1.

From (14) and (8) the relations for recursive generation of normed Hurwitz polynomials of order $k+1$, for $k > 2$, can be obtained as

$$a(s) = p(s) + w_{k+1} q(s). \tag{18}$$

Denote the degree of a polynomial by superscript to obtain

$$a^{k+1} = \begin{bmatrix} w_k a_k^k & a_k^k & a_{k-1}^k + w_k a_{k-2}^k & a_{k-2}^k a_{k-3}^k + w_k a_{k-4}^k & \cdots & 1 \end{bmatrix}^{\mathsf{T}}, \tag{19}$$

where

$$a^k = \begin{bmatrix} a_k^k & a_{k-1}^k & \cdots & 1 \end{bmatrix}^\mathsf{T}. \tag{20}$$

Using matrix notation, equation (19) can be rewritten as

$$a^{k+1} = W_k a^k, \tag{21}$$

where W_k is a $(k+1) \times k$ matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^\mathsf{T} \end{bmatrix} + \begin{bmatrix} 0^\mathsf{T} \\ \vdots \\ I_k \end{bmatrix} \tag{22}$$

with I_k being the $k \times k$ unit matrix and J_k being the $k \times k$ diagonal matrix $J_k = \operatorname{diag}\{1,0,1,0,\ldots\}$, i.e.,

$$W_k = \begin{bmatrix} w_k & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & w_k & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{23}$$

Next, using recursive relation (21), yields $a^n = W_k^n a^k$, where

$$W_k^n = W_n W_{n-1} \cdots W_k, \quad k = n, \ldots, 3 \tag{24}$$

or

$$a^n = W_3^n a^2 = W_n W_{n-1} \cdots W_3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}. \tag{25}$$

Example 1.2. Let us illustrate equation (25) on the basis of low order polynomials. In case $n = 3$, one gets

$$a^3 = W_2^3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix}, \tag{26}$$

and $n = 4$ yields

$$\begin{aligned} a^4 = W_2^4 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} &= \begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} w_2 w_3 w_4 \\ w_2 w_3 \\ w_2 + w_1 w_4 + w_3 w_4 \\ w_1 + w_3 \\ 1 \end{bmatrix}. \end{aligned} \tag{27}$$

Example 1.3. For numerical example let us consider the polynomial given as $a(s) = 8s^3 + 2s^2 + 6s + 1$ or

$$a^3 = \begin{bmatrix} a_3^3 & a_2^3 & a_1^3 & a_0^3 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^\mathsf{T}. \tag{28}$$

Then, the reduced Routh parameters can be computed as

$$\begin{aligned} w_0 &= a_0^3 = 1, \\ w_1 &= a_1^3 - \frac{a_3^3}{a_2^3} = 6 - 4 = 2, \\ w_2 &= a_2^3 = 2, \\ w_3 &= \frac{a_3^3}{a_2^3} = \frac{8}{2} = 4, \end{aligned} \tag{29}$$

and therefore $w(a) = \begin{bmatrix} 4 & 2 & 2 & 1 \end{bmatrix}^\mathsf{T}$.

Lemma 1.1 ([7])**.** *The elements in* (25)*, can be calculated using the direct formula*

$$a_l^n = \sum_{i_0=1}^{n} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-l}=1}^{i_{n-l-1}} \prod_{j=0}^{n-l} \overline{w}_{i_j} \mod(i_j + n - l - j, 2), \tag{30}$$

where $l = 1, \ldots, n$ is the index number of the corresponding row in (25)*, $n > 2$, and* $\mod(\alpha, 2)$ *is the usual modulus operation that returns either 1 or 0 depending on whether the number α is odd or even, respectively [7]. Elements $\overline{w}_{i_j}$ in* (30) *correspond to elements of the matrix W_k as*

$$\overline{w}_{i_j} := \begin{cases} w_2/\overline{w}_1 & \textit{for } i_j = 2, \\ w_{i_j} & \textit{otherwise.} \end{cases} \tag{31}$$

Let $n = 4$ and

$$w = \begin{bmatrix} w_4 & w_3 & w_2 & w_1 & 1 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 2 & 3 & 5 & 4 & 1 \end{bmatrix}^\mathsf{T}. \tag{32}$$

Next, calculate recursively the coefficients of polynomials. According to (21) and using the results from Example 1.2, for $k = 2, 3$, yields

$$a^2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \quad a^3 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix} \tag{33}$$

and for $k = 4$

$$a^4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 19 \\ 7 \\ 1 \end{bmatrix}. \tag{34}$$

1.2 Multilinear stability criterion and Routh rays of polynomials

In this section definition for Routh rays and Routh sources are given and multilinear stability criteria are presented. Henceforth, to simplify notation sometimes the power index of a polynomial is omitted, i.e., $a^n = a$.

Theorem 1.1 ([70]). *Through an arbitrary Hurwitz stable point*

$$a = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^\top \tag{35}$$

with reduced Routh parameters $w_k > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathscr{R}_k(a) \subset \mathscr{H}_n$ such that

$$\mathscr{R}_k(a) = \left\{ a \mid w_k \in (0, \infty), w_j = \text{const}, \, j \neq k; k, j \in \{1, \ldots, n\} \right\}. \tag{36}$$

Proof. Observe that all points of the line $\mathscr{R}_k(a)$ are Hurwitz stable, since

1. $n-1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are assumed to be fixed and positive $w_j > 0$;

2. the kth reduced Routh parameters $w_k > 0$, according to $w_k \in (0, \infty)$.

Next, we have to prove that $\mathscr{R}_k(a)$ is a line segment (half-line). It is easy to see that mapping (21) is multilinear. If $n-1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are fixed, then mapping (21) turns out to be linear with respect to the kth reduced Routh parameter w_k. The latter means that for each $k = 1, \ldots, n$ there is a half-line $\mathscr{R}_k(a)$, and altogether n half-lines $\mathscr{R}_k(a) \subset \mathscr{H}_n$. □

Definition 1.2 ([70]). The half-lines $\mathscr{R}_k(a)$, $k = 1, \ldots, n$ defined by (36) are called *Routh rays* of the polynomial $a(s)$. Moreover, their endpoints $v_k(a)$ such as

$$v_k(a) = a(w_k = 0) \tag{37}$$

are supposed to be the *Routh sources* of the polynomial $a(s)$.

Example 1.4 (Example 1.3 cont.). Let $n = 3$. Start from the polynomial

$$a = \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^\top \tag{38}$$

with reduced Routh parameters

$$w(a) = \begin{bmatrix} 4 & 2 & 2 & 1 \end{bmatrix}^\top. \tag{39}$$

By (21) we can easily calculate the Routh sources as follows

$$\begin{aligned} v_1 &= \begin{bmatrix} 8 & 2 & 4 & 1 \end{bmatrix}^\top, \\ v_2 &= \begin{bmatrix} 0 & 0 & 6 & 1 \end{bmatrix}^\top, \\ v_3 &= \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^\top \end{aligned} \tag{40}$$

and find the Routh rays $\mathscr{R}_1(a)$, $\mathscr{R}_2(a)$, $\mathscr{R}_3(a)$ through the corresponding Routh source v_k, $k = 1, 2, 3$, and the initial point a as

$$\begin{aligned} \mathscr{R}_1 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^\top + (1-\alpha) \begin{bmatrix} 8 & 2 & 4 & 1 \end{bmatrix}^\top, \\ \mathscr{R}_2 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^\top + (1-\alpha) \begin{bmatrix} 0 & 0 & 6 & 1 \end{bmatrix}^\top, \\ \mathscr{R}_3 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^\top + (1-\alpha) \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^\top, \end{aligned} \tag{41}$$

where $\alpha \in [0,\infty)$. Next, we calculate the roots

$$\lambda(v_k) = \begin{bmatrix} \lambda_1(v_k) & \lambda_2(v_k) & \lambda_3(v_k) \end{bmatrix}^\mathsf{T} \tag{42}$$

of the Routh sources as

$$\begin{aligned} \lambda(v_1) &= \begin{bmatrix} -0.25 \\ \pm 0.7071i \end{bmatrix}, \qquad \lambda(v_2) = \begin{bmatrix} 0 \\ 0 \\ -0.1667 \end{bmatrix}, \\ \lambda(v_3) &= \begin{bmatrix} 0 \\ -0.5 \pm 0.5i \end{bmatrix}. \end{aligned} \tag{43}$$

Indeed, all the Routh sources have at least one root on the imaginary axis, e.g., v_1 has a pair of imaginary roots, v_2 has two roots in the origin and v_3 has a root in the origin.

Theorem 1.2 (Multilinear stability criterion, [71]). *If a is a Hurwitz stable polynomial with reduced Routh parameters $w_k(a)$, $k = 1,\dots,n$, then all the Routh rays $\mathscr{R}_k(a)$ are Hurwitz stable.*

Proof. The proof follows directly from Theorem 1.1. □

According to Proposition 1.1, all Routh sources $v_k(a)$ of Hurwitz (stable) polynomials $a(s)$ are placed on the stability boundary. This means that some of the roots $\lambda_j(v_k)$, $j = 1,\dots,n$, $k = 1,\dots,n$ are placed on the imaginary axis. Using mapping (25) the following theorem can be formulated, regarding roots of Routh sources.

Theorem 1.3 ([70]). *All the Routh sources $v_j(a)$, $j = 2,\dots,n-1$ of a Hurwitz polynomial $a(s)$ of the order n have at least two roots at the origin*

$$\lambda_1(v_j) = \lambda_2(v_j) = 0, \quad j = 2,\dots,n-1 \tag{44}$$

and the last Routh source $v_n(a)$ has at least one root at the origin

$$\lambda_1(v_n) = 0. \tag{45}$$

Proof. To prove the theorem, the direct formula (30) from Lemma 1.1 is used. Indeed, take in (30) for $l = 1$ and $l = 2$ indices as

$$i_0 = n, \quad i_1 = n-1, \quad \dots, \quad i_{n-2} = 2, \quad i_{n-1} = 1 \tag{46}$$

and

$$i_0 = n-1, \quad i_1 = n-2, \quad \dots, \quad i_{n-3} = 2, \quad i_{n-2} = 1, \tag{47}$$

respectively. This yields the first two elements a_1^n, a_2^n of (25) given as

$$\begin{aligned} a_1^n &= \overline{w}_n \overline{w}_{n-1} \cdots \overline{w}_2 \overline{w}_1, \\ a_2^n &= \overline{w}_{n-2} \overline{w}_{n-3} \cdots \overline{w}_2 \overline{w}_1 \end{aligned} \tag{48}$$

or, using (31), in the simplified form as

$$\begin{aligned} a_1^n &= w_n w_{n-1} \cdots w_2, \\ a_2^n &= w_{n-2} w_{n-3} \cdots w_2. \end{aligned} \tag{49}$$

Hence, according to Definition 1.2, from the previous equations it follows

$$\begin{aligned}
&\lambda_1(v_j) = \lambda_2(v_j) = 0, \quad \text{for } j = 2, \ldots, n-1, \\
&\lambda_1(v_n) = 0.
\end{aligned} \tag{50}$$

□

Example 1.5 (Example 1.2 cont.)**.** Using (37) we can calculate the Routh sources $v_k(a)$ of some low-order polynomials $a(s)$. In case $n = 3$ from (26) we obtain

$$\begin{aligned}
v_1(a) &= \begin{bmatrix} w_2w_3 \\ w_2 \\ w_3 \\ 1 \end{bmatrix}, \quad v_2(a) = \begin{bmatrix} 0 \\ 0 \\ w_1 + w_3 \\ 1 \end{bmatrix}, \\
v_3(a) &= \begin{bmatrix} 0 \\ w_2 \\ w_1 \\ 1 \end{bmatrix}.
\end{aligned} \tag{51}$$

Indeed, as stated in Theorem 1.3, $\lambda_1(v_2) = \lambda_2(v_2) = \lambda_1(v_3) = 0$ for arbitrary positive reduced Routh parameters w_1, w_2 and w_3.

Now, in case $n = 4$ from (27) we obtain

$$\begin{aligned}
v_1(a) &= \begin{bmatrix} w_2w_3w_4 \\ w_2w_3 \\ w_2 + w_3w_4 \\ w_3 \\ 1 \end{bmatrix}, \quad v_2(a) = \begin{bmatrix} 0 \\ 0 \\ w_1w_4 + w_3w_4 \\ w_1 + w_3 \\ 1 \end{bmatrix}, \\
v_3(a) &= \begin{bmatrix} 0 \\ 0 \\ w_2 + w_1w_4 \\ w_1 \\ 1 \end{bmatrix}, \quad v_4(a) = \begin{bmatrix} 0 \\ w_2w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix}.
\end{aligned} \tag{52}$$

Hence, it follows that $\lambda_1(v_2) = \lambda_2(v_2) = \lambda_1(v_3) = \lambda_2(v_3) = \lambda_1(v_4) = 0$ for arbitrary positive reduced Routh parameters w_1, w_2, w_3, and w_4 as proposed in Theorem 1.3.

1.3 Stable (Hurwitz) polytopes of polynomials via Routh segments

In this section the algorithm for construction of Hurwitz stable polytopes is introduced. We now explain the generation procedure of stable polytopes of Hurwitz polynomials starting from a single Hurwitz polynomial a.

According to Theorem 1.1, the set of n Routh rays $\mathscr{R}_k(a)$, $k = 1, \ldots, n$ is Hurwitz stable. However, in general, the linear cover of the Routh rays $\mathscr{R}_k(a)$, $k = 1, \ldots, n$ is not Hurwitz stable. This results in a complex problem of finding a stable polytope $P(a)$ around the initial point a such that all the vertices are placed on the Routh rays $\mathscr{R}_k(a)$, $k = 1, \ldots, n$. Moreover, in many cases it is necessary to find the stable polytope $P_{\max}(a)$ with maximal possible volume

$$V(P_{\max}(a)) = \max_P V(P(a)). \tag{53}$$

A step-by-step algorithm to solve the problem of generating stable polytope via bunches of Routh segments is described next. Note that $+$ and $-$ signs stand to positive and negative directions with respect to the starting point, respectively.
Algorithm 1:

Step 1. Start from a given nth-order stable polynomial $a(s)$, or

$$a_n = \begin{bmatrix} a_n^n & a_{n-1}^n & \cdots & a_1^n & 1 \end{bmatrix}^\mathsf{T}. \tag{54}$$

Step 2. Using (5), calculate the reduced Routh parameters w_k for $k = n, \ldots, 1$.

Step 3. Calculate by (37) the Routh sources $v_k(a)$ for $k = 1, \ldots, n$.

Step 4. Using (36), find the Routh rays $\mathscr{R}_k(a)$ for $k = 1, \ldots, n$.

Step 5. Find the stable polytope of sources $P_0(a)$ of the polynomial $a(s)$ as follows:

- Start from the polytope $P_0^-(a)$ defined as the linear cover of the initial polynomial a and all of its sources $v_k(a)$, $k = 1, \ldots, n$, i.e., $P_0^-(a) = \mathrm{conv}\{a, v_1, \ldots, v_n\}$.
- Check the stability of single edges of $P_0^-(a)$ by Hurwitz Segment Lemma, see [15, p. 81]. Next, check the stability of the polytope $P_0^-(a)$ using Edge Theorem, see [15, p. 271].
- If thus obtained polytope $P_0^-(a)$ is not stable, then generate recursively using interval halving method (between a and $v_k(a)$, $k = 1, \ldots, n$) the new candidates for the polytope of sources $P_l^-(a)$, $l = 1, 2, \ldots$.
- If a stable polytope of sources $P_{\max}^-(a)$ with maximal volume $V(P^-(a)) = \max$ is found, then stop. The volume of polytopes $P^-(a)$ can be found by Triangulation method, see [8] for technical details.

Step 6. Similarly, find the stable polytope of rays $P^+(a)$ of the polynomial $a(s)$ starting from endpoints of the Routh rays $e_k(w_k = \gamma) \in \mathscr{R}_k(a)$ with γ being a big-enough number. If a stable polytope of rays $P_{\max}^+(a)$ with maximal volume $V(P^+(a)) = \max$ is found, then stop. The volume of polytopes $P^+(a)$ can be found by Triangulation method.

Step 7. Starting from the vertices of the polytopes $P^-(a)$ and $P^+(a)$, find using interval halving method the stable polytope of Routh (rays) segments $P(a)$ with vertices $\mathscr{R}_k^- \in \mathscr{R}_k(a)$ and $\mathscr{R}_k^+ \in \mathscr{R}_k(a)$ with maximal volume.

Algorithm 1 for generating stable polytope is visualized in Fig. 2.

Example 1.6. Consider the insulin model for a specific patient described by the state equations as [66]

$$\begin{aligned} \dot{x}_1 &= -0.435x_1 + 0.209x_2 + 0.02x_3 + u \\ \dot{x}_2 &= 0.268x_1 - 0.394x_2 \\ \dot{x}_3 &= 0.227x_1 - 0.02x_3 \\ y &= 0.0003x_1, \end{aligned} \tag{55}$$

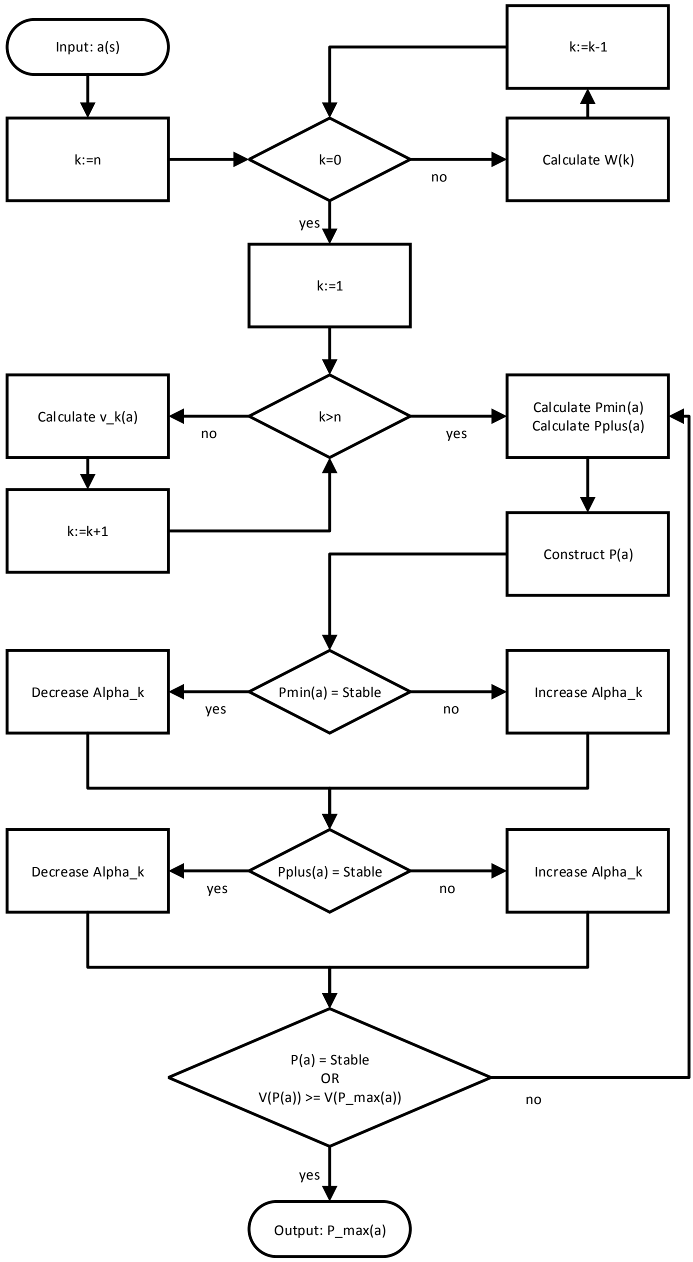

Figure 2. Schematic representation of Algorithm 1 for calculation of a stable polytope.

where x_1, x_2, x_3 denote the amount of insulin in plasma, liver, and interstitial compartment, respectively; u is the external insulin flow, and y is the plasma insulin concentration. The model (55) can be represented by the following transfer function

$$H(s) = \frac{0.0003s^2 + 0.1242 \times 10^{-3}s + 2.364 \times 10^{-6}}{s^3 + 0.849s^2 + 0.1274s + 0.5188 \times 10^{-3}}. \tag{56}$$

One can easily verify that the nominal system $H(s)$ is stable, since the poles $\lambda_1 = -0.656$, $\lambda_2 = -0.189$, and $\lambda_3 = -0.004$ have negative real parts. Our aim is to find the stable polytope (with maximal volume) in the coefficient space around the nominal characteristic polynomial

$$a(s) = s^3 + 0.849s^2 + 0.1274s + 0.5188 \times 10^{-3}. \tag{57}$$

Normalize polynomial $a(s)$ with respect to the free term to get

$$a(s) = 1927.53s^3 + 1636.47s^2 + 245.567s + 1. \tag{58}$$

Next, according to Algorithm 1, collect coefficients as

$$a = \begin{bmatrix} 1927.53 & 1636.47 & 245.567 & 1 \end{bmatrix}^{\mathsf{T}} \tag{59}$$

for which the reduced Routh parameters can be found as

$$w_k(a) = \begin{bmatrix} 1.17786 & 1636.47 & 244.389 & 1 \end{bmatrix}^{\mathsf{T}}. \tag{60}$$

Application of (21) yields the Routh sources given as

$$\begin{aligned} v_1 &= \begin{bmatrix} 1927.53 & 1636.47 & 1.17786 & 1 \end{bmatrix}^{\mathsf{T}}, \\ v_2 &= \begin{bmatrix} 0 & 0 & 245.567 & 1 \end{bmatrix}^{\mathsf{T}}, \\ v_3 &= \begin{bmatrix} 0 & 1636.47 & 244.389 & 1 \end{bmatrix}^{\mathsf{T}}. \end{aligned} \tag{61}$$

Now, using (36), one can find the Routh rays $\mathscr{R}_1(a)$, $\mathscr{R}_2(a)$, $\mathscr{R}_3(a)$ through the corresponding Routh source v_k, $k = 1,2,3$, and the initial point a as

$$\begin{aligned} \mathscr{R}_1 &= \alpha_1 \begin{bmatrix} 1927.53 \\ 1636.47 \\ 245.567 \\ 1 \end{bmatrix} + (1-\alpha_1) \begin{bmatrix} 1927.53 \\ 1636.47 \\ 1.17786 \\ 1 \end{bmatrix}, \\ \mathscr{R}_2 &= \alpha_2 \begin{bmatrix} 1927.53 \\ 1636.47 \\ 245.567 \\ 1 \end{bmatrix} + (1-\alpha_2) \begin{bmatrix} 0 \\ 0 \\ 245.567 \\ 1 \end{bmatrix}, \\ \mathscr{R}_3 &= \alpha_3 \begin{bmatrix} 1927.53 \\ 1636.47 \\ 245.567 \\ 1 \end{bmatrix} + (1-\alpha_3) \begin{bmatrix} 0 \\ 1636.47 \\ 244.389 \\ 1 \end{bmatrix}, \end{aligned} \tag{62}$$

where $\alpha \in [0, \infty)$. After several iterations algorithm terminates with the polytope

$P_{\max}(a)$, whose vertices have the following coordinates

$$\begin{aligned}
p_1^- &= (1927.53, 1636.47, 87.6285), \\
p_2^- &= (681.847, 578.888, 245.567), \\
p_3^- &= (681.847, 1636.47, 244.805), \\
p_1^+ &= (1.92752\times 10^3, 1.63647\times 10^3, 167.627\times 10^3), \\
p_2^+ &= (1.32208\times 10^6, 1.12245\times 10^6, 2.45567\times 10^2), \\
p_3^+ &= (1.32208\times 10^6, 1.63647\times 10^3, 1.05228\times 10^3),
\end{aligned} \tag{63}$$

where $+$ indicates that $\alpha_i > 1$, and $-$ corresponds to the case $1 > \alpha_i > 0$. Finally, the volume of the obtained polytope can be found as

$$V(P_{\max}(a)) = 4.1394\times 10^{16}. \tag{64}$$

1.4 Stable Routh cones of polynomials

In this section definitions and theorems for Routh cones and polynomials are presented, followed by several numeric examples.

We now study the stability of polynomials with conic uncertainty [44] by means of Routh rays. Define the so-called Routh cones[1] in the polynomial coefficient space $a \in \mathbb{R}^n$ starting from the reduced Routh parameter space $w \in \mathbb{R}^n$. Let $a^* \in \mathscr{H}_n$ be an arbitrary stable polynomial of the order n, and w^* be the respective vector consisting of the reduced Routh parameters.

Definition 1.3 ([7]).

1. A subset $\mathscr{K}_i(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a *Routh cone* of a polynomial $a^*(s)$ if it is closed under positive scalar multiplication of one of its reduced Routh parameters w_i^*, $i \in \{1,\ldots,n\}$, i.e., $a(w_i = \alpha w_i^*) \in \mathscr{K}_i$ when $a \in \mathscr{K}_i$ and $\alpha > 0$, where all the other reduced Routh parameters w_j, $j \neq i$, $j \in \{1,\ldots,n\}$ are fixed $w_j = w_j^*$.

2. If P is a subset of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$, then
$$\mathscr{K}_i(P) = \{a(w_i = \alpha w_i);\ a \in P, \alpha > 0, i \in \{1,\ldots,n\}\} \tag{65}$$
is called the Routh cone *generated* by P.

3. A convex cone $\mathscr{K}(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a *polyhedral* Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that
$$\mathscr{K}(a^*) = \left\{ \sum_{i=1}^{n} \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, 0 < \beta_i < 1, \right.$$
$$\left. \sum_{i=1}^{n} \beta_i = 1, w_j = w_j^* = \mathrm{const}, j \neq i, i = 1,\ldots,n \right\}. \tag{66}$$

[1]Note that the notion cone is used in consonance with results in [44, 71]. In this work definition of the Routh cone coincides with that of the Routh ray.

4. A convex cone $\mathscr{K}_{i,j}(a^*)$ of normed polynomial $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh i,j-*subcone* of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\mathscr{K}_{i,j}(a^*) = \big\{\beta_i a(w_i = \alpha_i w_i^*, w_j = w_j^*) + \beta_j a(w_j = \alpha_j w_j^*, w_i = w_i^*);\\ \alpha_i, \alpha_j > 1, 0 < \beta_i, \beta_j < 1, \beta_i + \beta_j = 1,\\ w_k = w_k^* = \text{const}, k \neq i,j; i,j,k \in \{1,\dots,n\}\big\}. \quad (67)$$

5. A convex set $\overline{\mathscr{K}}^n_{j,k}(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a *truncated* polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\overline{\mathscr{K}}^n_{j,k}(a^*) = \left\{\sum_{i=1}^{n} \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, i \neq j,k; 1 < \alpha_j < \overline{\alpha_j},\right.\\ 1 < \alpha_k < \overline{\alpha_k};\ 0 < \beta_i < 1,\\ \left.\sum_{i=1}^{n} \beta_i = 1, w_h = w_h^* = \text{const}, h \neq i, i = 1,\dots,n\right\}. \quad (68)$$

Remark 1.1. According to Theorem 1.1, it is possible to draw n stable Routh rays $\mathscr{R}_i(a^*)$ through an arbitrary stable point a^*. In [70] it was shown that if the point is not placed on the boundary of stability domain, then there are *positive* and *negative* directions with respect to a^*. The positive part of a Routh ray corresponds to $\alpha_i \in (1,\infty)$ while the negative to $\alpha_i \in (0,1)$, and for $\alpha_i = 1$ rays intersect at the point a^*. In this paper notions of Routh rays and Routh cones $\mathscr{K}_i(a^*)$ coincide for positive direction. Therefore, the point a^* has to be understood as a vertex of the polyhedral Routh cone.

Theorem 1.4 ([71])**.** *An arbitrary subset P of normed polynomials $a(s)$ of degree n, $a(s) \in \mathbb{R}^n$ has n Routh cones $\mathscr{K}_i(P)$, $i = 1,\dots,n$ generated by P. If the subset P is stable, then all Routh cones $\mathscr{K}_i(P)$ generated by P are stable.*

Proof. According to Theorem 1.1, through an arbitrary point $a \in P \subset \mathbb{R}^n$ it is possible to draw half-lines $\mathscr{R}_i(a)$ such that $w_i \in (0,\infty)$, $i = 1,\dots,n$. If polynomials $a \in P$ are stable, then all half-lines $\mathscr{R}_i(a)$ are stable, i.e., Routh cone $\mathscr{K}_i(P)$ is stable. □

Theorem 1.5 ([71])**.** *The n-times Routh cone of the polynomial $a(s) = 1$, i.e., $a = \begin{bmatrix}0 & \dots & 0\end{bmatrix} \in \mathscr{R}^n$, generates the whole stability domain $\mathscr{A}$ in polynomial coefficient space, $\mathscr{A} \subset \mathbb{R}^n$.*

Proof. Starting from the origin $a = 0$ it is possible to find the Routh ray $\mathscr{R}_1(0)$ which is placed on the stability boundary, since all the points $a \in \mathscr{R}_1(0)$ have $w_j = 0$, $j = 2,\dots,n$. The Routh cone $\mathscr{K}_{1,2}(0) = \mathscr{K}_2(\mathscr{R}_1(0))$ is also placed on the stability boundary, since all the points $a \in \mathscr{K}_{1,2}(0)$ have $w_j = 0$, $j = 3,\dots,n$ and $w_i \in (0,\infty)$, $i = 1,2$. Similarly, for all the points $a \in \mathscr{K}_{1,\dots,n-1}(0)$ it follows that $w_j = 0$, $j = n$ and $w_i \in (0,\infty)$, $i = 1,\dots,n-1$. Finally, the Routh cone $K_{1,\dots,n}(0)$ contains points a with $w_i \in (0,\infty)$, $i = 1,\dots,n$, i.e., $\mathscr{K}_{1,\dots,n}(0) = \mathscr{A}$. □

Theorem 1.6 ([7])**.** *If all the polyhedral Routh subcones $\mathscr{K}_{i,j}(a^*)$, $i,j \in \{1,\dots,n\}$ of a stable polynomial $a^*(s)$ are stable, then the polyhedral Routh cone $\mathscr{K}(a^*)$ is stable.*

Proof. Indeed, if α_i and α_j, $1<\alpha_i,\alpha_j<\infty$ are fixed, then the polyhedral Routh cone $\mathscr{K}(a^*)$ is a polytope with $n+1$ vertices a^* and $a(w_k=\alpha_k w_k^*, w_j=w_j^*)$, $j\neq k$, $k=1,\dots,n$. The edges $\mathrm{conv}\{a^*,a(w_k)\}$ are stable as Routh rays of a stable point a^*. The edges $\mathrm{conv}\{a(w_k),a(w_j)\}$ are stable, since $\mathrm{conv}\{a(w_k),a(w_j)\}\subset\mathscr{K}_{k,j}(a^*)$ for arbitrary $1<\alpha_{k,j}<\infty$. Thus, it remains to note that by Edge Theorem the polytope is stable for $1<\alpha_i,\alpha_j<\infty$, since all edges of the polytope are stable [10]. □

Let $\Gamma=\{1,\dots,n\}$ be a set of integers. Rewrite it as $\Gamma=\gamma_1\cup\gamma_2$, where γ_1 and γ_2 are sets that contain indices corresponding to ordinary and truncated Routh subcones, respectively, with $\dim\gamma_1=m_1$ and $\dim\gamma_2=m_2$ such that $m_1+m_2=n$.

Theorem 1.7 ([7])**.** *A truncated polyhedral Routh cone $\overline{\mathscr{K}}^n_{i_j}(a^*)$ such that $i_j\in\gamma_2$ and $j=1,\dots,m_2$ of a stable polynomial $a^*(s)$ is stable if the following conditions hold:*

1. *the polyhedral Routh subcones $\mathscr{K}_{r,s}(a^*)$, $r,s\in\gamma_1$ are stable;*

2. *the line segments $S_{u,v}(\overline{\alpha}_u,\overline{\alpha}_v)$, $u,v\in\gamma_2$ are stable, where*

$$S_{u,v}(\overline{\alpha}_u,\overline{\alpha}_v)=\Big\{a(w_u=\overline{\alpha}_{u,\min}w_u^*),a(w_v=\overline{\alpha}_{v,\min}w_v^*),w_i=w_i^*,i\neq u,v\Big\} \tag{69}$$

and $\overline{\alpha}_{u,\min}=\min_u\overline{\alpha}_u$.

Proof. Indeed, if α_r and α_s, $1<\alpha_r,\alpha_s<\infty$ are fixed, then the truncated polyhedral Routh cone $\overline{\mathscr{K}}^n_{i_j}(a^*)$ is a polytope with $n+1$ vertices a^*, $a(w_u,\overline{\alpha}_u)$, $a(w_v,\overline{\alpha}_v)$ and $a(w_r=\alpha_r w_k^*, w_l=w_l^*)$, $l\neq r$, $l\in\{1,\dots,n\}$, $a(w_s=\alpha_s w_k^*, w_l=w_l^*)$, $l\neq s$, $l\in\{1,\dots,n\}$. The edges $\mathrm{conv}\{a^*,a(w_u,\overline{\alpha}_u)\}$, $\mathrm{conv}\{a^*,a(w_v,\overline{\alpha}_v)\}$, $\mathrm{conv}\{a^*,a(w_r)\}$, and $\mathrm{conv}\{a^*,a(w_s)\}$ are stable as the Routh rays of a stable point a^*. The edges $\mathrm{conv}\{a(w_r),a(w_s)\}$ are stable, since $\mathrm{conv}\{a(w_r),a(w_s)\}\subset\mathscr{K}_{r,s}(a^*)$ for arbitrary $1<\alpha_r,\alpha_s<\infty$. It follows from condition 2) that the edges $S_{u,v}(\overline{\alpha}_u,\overline{\alpha}_v)$ are stable. Hence, by Edge Theorem the polytope is stable for $1<\alpha_r,\alpha_s<\infty$, since all edges of the polytope are stable [10]. □

Theorem 1.8 ([7])**.** *For $n=3$ the polyhedral Routh cone $\mathscr{K}(a^*)$ of an arbitrary stable polynomial $a^*(s)$ is stable.*

Proof. Assume without loss of generality that $\alpha_1=\alpha_2=\alpha_3=\alpha$. Then, by (21) we obtain the Routh cones $\mathscr{K}_i(a^*)$, $i=1,2,3$ for the polynomial $a^*(s)$

$$\begin{aligned}
\mathscr{K}_1(a^*)&=\begin{bmatrix}w_2^*w_3^* & w_2^* & \alpha w_1^*+w_3^* & 1\end{bmatrix}^\mathsf{T},\\
\mathscr{K}_2(a^*)&=\begin{bmatrix}\alpha w_2^*w_3^* & \alpha w_2^* & w_1^*+w_3^* & 1\end{bmatrix}^\mathsf{T},\\
\mathscr{K}_3(a^*)&=\begin{bmatrix}\alpha w_2^*w_3^* & w_2^* & w_1^*+\alpha w_3^* & 1\end{bmatrix}^\mathsf{T},
\end{aligned} \tag{70}$$

where $\alpha>1$ and w_1^*, w_2^*, w_3^* are the reduced Routh parameters of the polynomial $a^*(s)$.

Let $a\in\mathscr{K}(a^*)$ be an inner point of the polyhedral Routh cone $\mathscr{K}(a^*)$. Then, the convex combination can be expressed as

$$a=\beta_1\mathscr{K}_1(a^*)+\beta_2\mathscr{K}_2(a^*)+\beta_3\mathscr{K}_3(a^*), \tag{71}$$

where $0 < \beta_i < 1$, $\sum_{i=1}^{3} \beta_i = 1$ or in the explicit form as

$$a = \begin{bmatrix} (\beta_1 + \beta_2\alpha + \beta_3\alpha)w_2^* w_3^* \\ (\beta_1 + \beta_2\alpha + \beta_3)w_2^* \\ (\beta_1\alpha + \beta_2 + \beta_3)w_1^* + (\beta_1 + \beta_2 + \beta_3\alpha)w_3^* \\ 1 \end{bmatrix}. \tag{72}$$

Note that, according to Proposition 1.1, polynomial $a(s)$ is stable if the reduced Routh parameters $w_i > 0$, $i = 1,2,3$. From (5) one obtains

$$w_3 = \frac{(\beta_1 + \beta_2\alpha + \beta_3\alpha)w_2^* w_3^*}{(\beta_1 + \beta_2\alpha + \beta_3)w_2^*}. \tag{73}$$

Observe that, according to Proposition 1.1, the reduced Routh parameters w_i^*, $i = 1,2,3$, of the stable polynomial $a^*(s)$ are positive. Moreover, $\alpha > 1$ and $\beta_i > 0$, yielding $w_3 > 0$. Similarly, from (5), one obtains

$$w_2 = (\beta_1 + \beta_2\alpha + \beta_3)w_2^* > 0 \tag{74}$$

and

$$w_1 = (\beta_1\alpha + \beta_2 + \beta_3)w_1^* + (\beta_1 + \beta_2 + \beta_3\alpha)w_3^* - \frac{(\beta_1 + \beta_2\alpha + \beta_3\alpha)w_2^* w_3^*}{(\beta_1 + \beta_2\alpha + \beta_3)w_2^*}. \tag{75}$$

The latter after simple algebraic manipulations yields

$$\begin{aligned} w_1 = ((\beta_1 + \beta_2\alpha + \beta_3)w_2^*)^{-1} \times \big((\beta_1\alpha + \beta_2 + \beta_3)\times \\ (\beta_1 + \beta_2\alpha + \beta_3)w_1^* w_2^* + (1-\alpha)^2\beta_2\beta_3 w_2^* w_3^*\big) > 0. \end{aligned} \tag{76}$$

□

Example 1.7. Consider an Unmanned Free-Swimming Submersible vehicle [66] for which the relation of pitch angle to elevator surface angle can be represented by the transfer function

$$H(s) = \frac{-0.125(s+0.435)}{(s+1.23)(s^2+0.226s+0.0169)}. \tag{77}$$

Since the poles

$$\begin{aligned} \lambda_1 &= -1.23, \\ \lambda_{2,3} &= -0.113 \pm 0.0643i \end{aligned} \tag{78}$$

have negative real parts, it immediately follows that the nominal system $H(s)$ is stable. The goal is to construct the stable polyhedral cone in the coefficient space starting from the nominal characteristic polynomial (the denominator of $H(s)$)

$$a^*(s) = s^3 + 1.456s^2 + 0.2949s + 0.028. \tag{79}$$

Normalize the polynomial $a^*(s)$ dividing it by free term 0.028 to get

$$a^*(s) = 35.7143s^3 + 52s^2 + 10.5321s + 1 \tag{80}$$

or

$$a^3 = \begin{bmatrix} a_3^3 & a_2^3 & a_1^3 & 1 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 35.7143 & \underbrace{52 \quad 10.5321 \quad 1}_{\overline{a}^3} \end{bmatrix}^\mathsf{T}. \tag{81}$$

The reduced Routh parameters can be found using recursive relation (5) as follows. Start from

$$w_3^* = \frac{a_3^3}{a_2^3} = \frac{35.7143}{52} = 0.6868. \tag{82}$$

Next, find the second-order polynomial

$$a^2 = \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \overline{W}_3\bar{a}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.6868 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52 \\ 10.5321 \\ 1 \end{bmatrix} = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}, \tag{83}$$

yielding

$$w^* = \begin{bmatrix} w_3^* & w_2^* & w_1^* & w_0^* \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 0.6868 & 52 & 9.8453 & 1 \end{bmatrix}^\mathsf{T}. \tag{84}$$

Then, according to Definition 1.3, Routh cones can be calculated as

$$\mathscr{K}_i = \underbrace{\begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3} \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}. \tag{85}$$

Cone $\mathscr{K}_1$: Take $w_1 = \alpha_1 w_1^*$, $w_2 = w_2^*$, $w_3 = w_3^*$, $1 < \alpha_1 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix}. \tag{86}$$

Then,

$$\mathscr{K}_1 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136 \\ 52 \\ 9.8453\alpha_1 + 0.6868 \\ 1 \end{bmatrix}. \tag{87}$$

Cone $\mathscr{K}_2$: Take $w_1 = w_1^*$, $w_2 = \alpha_2 w_2^*$, $w_3 = w_3^*$, $1 < \alpha_2 < \infty$, and

$$a^2 = \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix}. \tag{88}$$

Then,

$$\mathscr{K}_2 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136\alpha_2 \\ 52\alpha_2 \\ 10.5321 \\ 1 \end{bmatrix}. \tag{89}$$

Cone $\mathscr{K}_3$: Take $w_1 = w_1^*$, $w_2 = w_2^*$, $w_3 = \alpha_3 w_3^*$, $1 < \alpha_3 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}. \tag{90}$$

Then,

$$\begin{aligned}\mathscr{K}_3 &= \begin{bmatrix} 0.68682\alpha_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.68682\alpha_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 35.7136\alpha_3 \\ 52 \\ 0.6868\alpha_3 + 9.8453 \\ 1 \end{bmatrix}.\end{aligned} \tag{91}$$

Let $a \in \mathscr{K}(a^*)$ be an inner point of the polyhedral Routh cone $\mathscr{K}(a^*)$. Then, the convex combination can be expressed as

$$a = \beta_1 \mathscr{K}_1(a^*) + \beta_2 \mathscr{K}_2(a^*) + \beta_3 \mathscr{K}_3(a^*), \tag{92}$$

where $0 < \beta_i < 1$, $\sum_{i=1}^{3} \beta_i = 1$ or in the explicit form as

$$a = \begin{bmatrix} 35.7136(\beta_1 + \beta_2\alpha + \beta_3\alpha) \\ 52(\beta_1 + \beta_2\alpha + \beta_3) \\ 9.8453(\beta_1\alpha + \beta_2 + \beta_3) + 0.6868(\beta_1 + \beta_2 + \beta_3\alpha) \\ 1 \end{bmatrix}. \tag{93}$$

From (5) it follows

$$\begin{aligned} w_3 &= \frac{0.6868(\beta_1 + \beta_2\alpha + \beta_3\alpha)}{\beta_1 + \beta_2\alpha + \beta_3}, \\ w_2 &= 52(\beta_1 + \beta_2\alpha + \beta_3), \\ w_1 &= \frac{511.956(\beta_1\alpha + \beta_2 + \beta_3)(\beta_1 + \beta_2\alpha + \beta_3) + 35.7136(1-\alpha)^2\beta_2\beta_3}{52(\beta_1 + \beta_2\alpha + \beta_3)}. \end{aligned} \tag{94}$$

Observe that $a^*(s)$ is stable. Then, it follows from Proposition 1.1 that $w_i^* > 0$, $i = 1,2,3$. It remains to show that the reduced Routh parameters w_i, $i = 1,2,3$ are also positive. This trivially follows from the fact that $\alpha_i > 1$ and $0 < \beta_i < 1$ with $\sum_{i=1}^{3} \beta_i = 1$. Therefore, the constructed polyhedral Routh cone

$$\mathscr{K}(a^*) = \left\{ \beta_1 \mathscr{K}_1(a^*) + \beta_2 \mathscr{K}_2(a^*) + \beta_3 \mathscr{K}_3(a^*) \mid \alpha_i > 1, 0 < \beta_i < 1, \right. \\ \left. \sum_{i=1}^{3} \beta_i = 1, i = 1,2,3 \right\} \tag{95}$$

is stable.

Theorem 1.9 ([7])**.** *The polyhedral subcones $K_{i,j}(a^*)$, $i,j \in \{1,2,3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable.*

Proof. By (21) we obtain the following Routh cones $\mathscr{K}_i(a^*)$, $i = 1,2,3$ for the polynomial $a^*(s)$, $a \in \mathbb{R}^n$

$$\begin{aligned}
\mathscr{K}_1(a^*) &= W_4^n(a^*) \begin{bmatrix} w_2^* w_3^* \\ \alpha w_1^* + w_3^* \\ 1 \end{bmatrix}, \\
\mathscr{K}_2(a^*) &= W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ \alpha w_2^* \\ w_1^* + w_3^* \\ 1 \end{bmatrix}, \\
\mathscr{K}_3(a^*) &= W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ w_2^* \\ w_1^* + \alpha w_3^* \\ 1 \end{bmatrix},
\end{aligned} \tag{96}$$

where

$$W_4^n(a^*) := W_n(a^*) \cdots W_4(a^*) \tag{97}$$

and $\alpha > 1$.

For $a \in \mathscr{K}_{1,2}(a^*)$ there exist constants $\alpha > 1$ and $0 < \beta < 1$ such that for an arbitrary $a \in \mathscr{K}_{1,2}(a^*)$

$$a = \beta a(w_1 = \alpha w_1^*) + (1-\beta) a(w_2 = \alpha w_2^*), \tag{98}$$

where

$$\begin{aligned}
a(w_1 = \alpha w_1^*) &\in \mathscr{K}_1, \\
a(w_2 = \alpha w_2^*) &\in \mathscr{K}_2.
\end{aligned} \tag{99}$$

The above relation can be rewritten in the explicit form as

$$a = W_n(a^*) \cdots W_4(a^*) \begin{bmatrix} (\beta + (1-\beta)\alpha) w_2^* w_3^* \\ (\beta + (1-\beta)\alpha) w_2^* \\ (\beta\alpha + 1 - \beta) w_1^* + w_3^* \\ 1 \end{bmatrix}. \tag{100}$$

Observe that the reduced Routh parameters $w_n, \ldots, w_4$ of the polynomial $a(s)$ are determined by the product of matrix multiplication $W_n(a^*) \cdots W_4(a^*)$, i.e., $w_i = w_i^*$, $i = 4, \ldots, n$. For the reduced Routh parameters w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathscr{K}_{1,2}(a^*)$, using (5), it follows

$$\begin{aligned}
w_2 w_3 &= (\beta + (1-\beta)\alpha) w_2^* w_3^*, \\
w_2 &= (\beta + (1-\beta)\alpha) w_2^*, \\
w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + w_3^*
\end{aligned} \tag{101}$$

or

$$\begin{aligned}
w_1 &= (\beta\alpha + 1 - \beta) w_1^*, \\
w_2 &= (\beta + (1-\beta)\alpha) w_2^*, \\
w_3 &= w_3^*.
\end{aligned} \tag{102}$$

Note that $\alpha > 1$, $0 < \beta < 1$, and $w_i^* > 0$, $i = 1, \ldots, n$. Then, $w_i > 0$, $i = 1, \ldots, n$, i.e., $a \in \mathscr{K}_{1,2}(a^*)$ is stable.

In the similar manner we obtain for $a \in \mathscr{K}_{1,3}(a^*)$ the reduced Routh parameters $w_n, \ldots, w_4$, $w_i = w_i^*$, $i = 4, \ldots, n$. For w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathscr{K}_{1,3}(a^*)$

from (5) we get the following relations

$$\begin{aligned} w_2 w_3 &= (\beta + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned} \tag{103}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta) w_1^* > 0, \\ w_2 &= w_2^* > 0, \\ w_3 &= (\beta + (1-\beta)\alpha) w_3^* > 0. \end{aligned} \tag{104}$$

Finally, for $a \in \mathscr{K}_{2,3}(a^*)$ we obtain the reduced Routh parameters $w_i = w_i^*$, $i = 4, \ldots, n$ and for w_i, $i = 1, \ldots, 3$

$$\begin{aligned} w_2 w_3 &= (\beta\alpha + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta\alpha + (1-\beta)) w_2^*, \\ w_1 + w_3 &= w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned} \tag{105}$$

that yields

$$\begin{aligned} w_1 &= w_1^* + \frac{(\beta(1-\beta)(1-\alpha)^2) w_3^*}{\beta\alpha + (1-\beta)} > 0, \\ w_2 &= (\beta\alpha + 1 - \beta) w_2^* > 0, \\ w_3 &= \frac{\alpha w_3^*}{\beta\alpha + 1 - \beta} > 0. \end{aligned} \tag{106}$$

Hence, all polyhedral subcones $\mathscr{K}_{i,j}(a^*)$, $i, j \in \{1,2,3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable. $\square$

1.5 Discussion

There are several aspects that make the method based on *Routh polytopes* suitable in a certain situation.

- It is especially well-suited to design a controller for an object with polytopic uncertainty.
- The largest volume of the stable polytope can be obtained if a^* is located relatively far from the stability boundary. In this case, both P^- and P^+ are almost proportional, and the initial point a^* is placed near the center of the constructed polytope.

At the same time, the proposed method does not use the whole length of the Routh rays. Moreover, if a^* is located close to the stability boundary, then volume P^- may appear to be small.

Similarly, the *polyhedral Routh cones* based method has several advantages.

- It is suitable to design a controller for an object with conic or one dominant uncertainty.
- The whole length of the Routh half-lines (i.e., $\alpha > 1$) is used.
- The largest volume can be obtained if a^* is located relatively close to the stability boundary.

Unlike the polytopes based case, if a^* is located far from the stability boundary, then the respective part P^- is not used at all. Moreover, the point a^* becomes a vertex of a polyhedral cone, which complicates the design of a controller.

2 Routh controller design and software implementation

In this chapter, we present a simple and efficient algorithm to design a robust output PID controller for continuous-time plant with uncertainties. The method is based on a new stability criterion for Hurwitz polynomials presented in the above section. The overall procedure can be briefly summarized as follows: (*i*) Start from a stable simplex (or polytope) of a closed-loop characteristic polynomial, which is defined via Routh rays of a pre-selected Hurwitz stable polynomial. (*ii*) Define the set of possible plant parameters as a convex polytope (polytopic plant model). This allows to determine properties that are common to all elements in the set analyzing vertices of the polytope. Hence, the number of vertices determines complexity of computations defined by n linear inequalities. (*iii*) Design a robust output controller for polytopic plant model using, for example, quadratic programming approach. Further in this section we present and describe a number of core functions that are used to construct stable Routh cones or polytopes and design the respective controller. Several illustrative examples are presented in the end of the section.

2.1 Robust controller design based on Routh cones and polytopes

In this section, the problem of a fixed-order robust output control with a pre-selected simplex is stated and solved using quadratic programming approach.

Given a plant with parametric uncertainties. Our goal is to design a robust output controller of a fixed-order so that the closed-loop poles are assigned in a specific region approximated by the Routh cone as explained in Section 1.4. For simplicity, we consider the problem of PID-controller design for a single-input single-output (SISO) plant with fixed parameters. Let the transfer function $H(s)$ of the plant is given

$$H(s) = \frac{g(s)}{f(s)} = \frac{g_{m-1}s^{m-1} + \cdots + g_1 s + 1}{f_m s^m + \cdots + f_1 s + f_0}, \tag{107}$$

and we are looking for a PID-controller $C(s)$ of the order $l = 2$ with the transfer function

$$C(s) = K_P + K_I \frac{1}{s} + K_D s \tag{108}$$

or

$$C(s) = \frac{q(s)}{p(s)} = \frac{q_2 s^2 + q_1 s + 1}{p_1 s}. \tag{109}$$

The closed-loop characteristic polynomial of degree $n = m + l = m + 2$ is

$$a(s) = f(s)p(s) + g(s)q(s). \tag{110}$$

It is known from the literature [51] that for $l = m - 1$ the above problem has a solution for arbitrary $a(s)$, whenever the plant has no common pole-zero pairs. However, in general, for $l < m - 1$ the exact attainment of the desired polynomial is not possible. We propose the following approach. Let us relax the requirement of attaining the desired polynomial $a(s)$ exactly and enlarge the target to a simplex S in a polynomial coefficient space containing the point representing the desired closed-loop characteristic polynomial. Without loss of generality we can assume that $g_0 = q_0 = 1$ and consider further normed polynomials $a(s)$ with $a_0 = 1$.

Consider a stability measure ρ defined in accordance with the simplex S as

$$\rho = c^{\mathsf{T}} c, \tag{111}$$

where

$$c = S^{-1} a, \tag{112}$$

and S is the $(m+l+1) \times (m+l+1)$ matrix of vertices s_i of the target simplex

$$S = \begin{bmatrix} s_1 & \cdots & s_{n+1} \end{bmatrix}. \tag{113}$$

Observe that for the normed polynomials

$$a_0 = s_{i0} = 1, \quad i = 1, \ldots, n+1, \quad \sum_{i=1}^{n+1} c_i = 1, \tag{114}$$

where $n = m + l$. If all coefficients $c_i > 0$, $i = 1, \ldots, n+1$, then the point a is placed inside the simplex S. It is easy to see that the minimum of ρ is obtained when

$$c_1 = c_2 = \cdots = c_{n+1} = \frac{1}{n+1}. \tag{115}$$

Then, the point a is placed in the center of the simplex S.

Now, we are ready to state the following problem of controller design:

$$\boxed{\text{Find a controller } C(s) \text{ such that the stability measure } \rho \text{ is minimal.}}$$

In other words, we are looking for a controller which places the closed-loop characteristic polynomial $a(s)$ as close as possible to the center of the target simplex S. In the matrix form notation we have

$$a = Gx, \tag{116}$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} f_m & 0 & g_{m-1} & g_{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ f_2 & g_3 & g_2 & g_1 \\ f_1 & g_2 & g_1 & 1 \\ f_0 & g_1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{117}$$

of dimension $(m+2) \times 4$ and x is a vector of controller parameters

$$x = \begin{bmatrix} p_1 & 1 & q_1 & q_2 \end{bmatrix}^{\mathsf{T}}. \tag{118}$$

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$J = \min_x x^{\mathsf{T}} G^{\mathsf{T}} (SS^{\mathsf{T}})^{-1} Gx \tag{119}$$

is obtained subject to the linear constraints

$$S^{-1} Gx > 0. \tag{120}$$

Note that constraints (120) follow from positivity requirement on coefficients c_i, $i = 1, \ldots, n$. Next, we summarize the above results in the form of the algorithm.

Algorithm 2:

Step 1. Start from a given transfer function $H(s)$ for uncertain plant (107) and desired controller type (PI, PD, or PID) function $C(s)$.

Step 2. Construct the closed-loop characteristic polynomial (110) and Sylvester matrix (117).

Step 3. Select the initial closed-loop characteristic polynomial $a^*(s)$ and check the stability.

Step 4. According to (8), find the reduced Routh parameters w_k, $k = n, \ldots, 1$ of the polynomial $a^*(s)$.

Step 5. According to (36), compute Routh rays $\mathscr{R}_k(a)$, $k = 1, \ldots, n$ of the polynomial $a^*(s)$ and, using (113), construct stable target simplex S with vertices on the Routh rays.

Step 6. Start with nominal plant (i.e., with values of uncertainties placed in the center of region) and find controller gains p and q by solving convex quadratic programming task (119) with restrictions (120).

Step 7. Check the stability of closed-loop system with polytopic plant, i.e., all the vertices of the closed-loop polytope must be located inside the target simplex S. If some points of the rectangle are located outside of S, then select different initial closed-loop characteristic polynomial $a^*(s)$ and repeat all the previous steps.

It is important to mention that the algorithm is flexible and one can either choose Routh cone or polytope to construct the target simplex on Step 5. The algorithm is visualized in Fig. 3.

2.2 Overview of the developed software package

In this section, we briefly describe the main functions implemented based on the above theoretical results. All the calculations and simulations were performed using MATLAB environment.

NormalizeA()

```
function NormInputA = NormalizeA(RawInputA)
% NORMALIZEA normalizes coefficients for polynomial a(s)
%
% Usage:
% [NormInputA] = NORMALIZEA(RawInputA)
%
% where
% RawInputA - array of coefficients for polynomial a(s)
%
% Outputs:
% NormInputA - normalized array of coefficients for polynomial
   a(s)
```

The function converts coefficients of polynomial $a(s)$ to a normalized form.

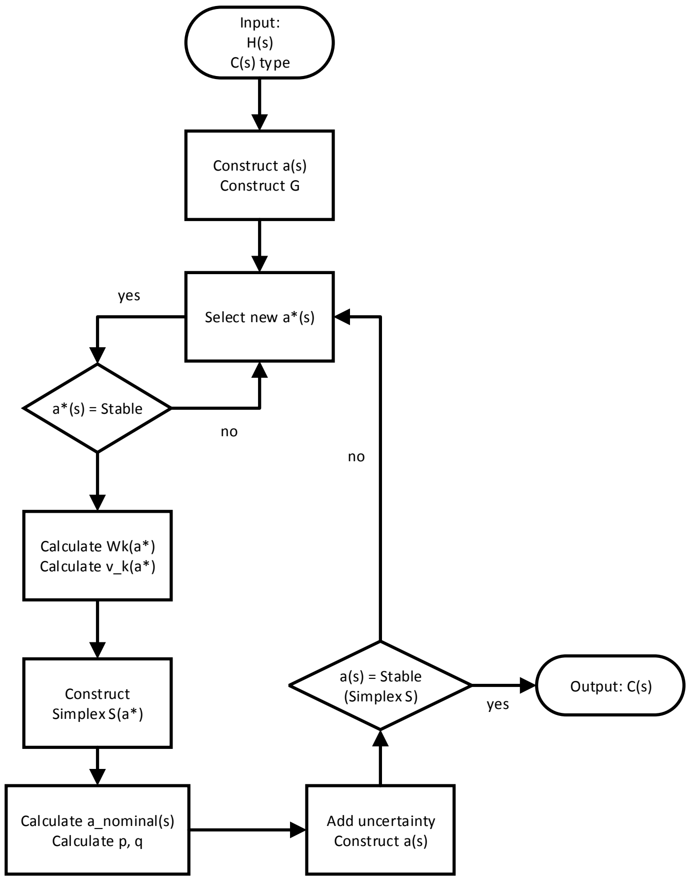

Figure 3. A schematic representation of Algorithm 2 robust controller design.

calcWfromA()

```
function OutputW = calcWfromA(InputA)
% CALCWFROMA calculates reduced Routh parameters for polynomial
    a(s) coefficients.
%
% Usage:
% [OutputW] = CALCWFROMA(InputA)
%
% where
% InputA - array of coefficients for polynomial a(s) (in a
   normalized format)
%
% Outputs:
% OutputW - reduced Routh parameters for polynomial a(s)
```

The function calculates Routh parameters from the polynomial $a(s)$.

calcAfromW()

```
function OutputA = calcAfromW(InputW)
% CALCAFROMW calculates polynomial a(s) coefficients based on
   reduced Routh parameters values.
%
% Usage:
% [OutputA] = CALCAFROMW(InputW)
%
% where
% InputW - array of reduced Routh parameters
%
% Outputs:
% OutputA - array of coefficients for polynomial a(s)
```

The function calculates polynomial $a(s)$ coefficients based on the reduced Routh parameters $w(s) = \{w_n, w_{n-1}, w_{n-2}, \dots, w_0\}$.

calcRouthSources()

```
function RouthSourceV = calcRouthSources(InputW)
% CALCROUTHSOURCES calculates of Routh sources from reduced
   Routh parameters.
%
% Usage:
% [RouthSourceV] = CALCROUTHSOURCES(InputW)
%
% where
% InputW - array of reduced Routh parameters
%
% Outputs:
% RouthSourceV - array of Routh sources
```

The function constructs stable half-lines (Routh rays) (36) from the reduced Routh parameters w_k, and calculates the Routh sources $v_k(a)$ (37) on the end-points of the respective Routh rays.

calcStablePmin()

```
function [OutPmin,minAlphaStable] = calcStablePmin(InputA,RouthSourceV,IterCnt,InitAlpha,IncrAlpha)
% CALCSTABLEPMIN calculates stable polytope P-(a) for "negative" direction.
%
% Usage:
% [OutPmin,minAlphaStable] = CALCSTABLEPMIN(InputA,RouthSourceV,IterCnt,InitAlpha,IncrAlpha)
%
% where
% InputA - array of coefficients for polynomial a(s) (in a normalized format)
% RouthSourceV - array of Routh sources
% IterCnt - maximum number of iterations in calculation
% InitAlpha - initial Alpha value in range between 0 and 1
% IncrAlpha - initial interval for Alpha change (should be less than InitAlpha)
%
% Outputs:
% OutPmin - matrix of vertex coordinates for found stable polytope (polytope for negative direction)
%               minAlphaStable - minimal stable Alpha value
```

The function calculates $P^{-}(a)$ stable polytope for the *negative* (α between 0 and 1 in (36)) direction, using the interval halving method.

calcStablePpluss()

```
function [OutPpluss,plussAlphaStable] = calcStablePpluss(InputA
    ,RouthSourceV,IterCnt,InitAlpha,IncrAlpha)
% CALCSTABLEPPLUSS calculates stable polytope P+(a) for "
    positive" direction.
%
% Usage:
% [OutPpluss,plussAlphaStable] = CALCSTABLEPPLUSS(InputA,
    RouthSourceV,IterCnt,InitAlpha,IncrAlpha)
%
% where
% InputA - array of coefficients for polynomial a(s) (in a
    normalized format)
% RouthSourceV - array of Routh sources
% IterCnt - maximum number of iterations in calculation
% InitAlpha - initial Alpha value in range between 1 and
    infinity
% IncrAlpha - initial interval for Alpha change
%
% Outputs:
% OutPpluss - matrix of vertex coordinates for found stable
    polytope (polytope for positive direction)
%                plussAlphaStable - maximal stable Alpha value
```

The function calculates of $P^{+}(a)$ stable polytope for *positive* (α between 1 and infinity in (36)) direction, using the interval halving method.

checkStablePAll()

```
function [OutPIsStable] = checkStablePAll(InPmin,InPpluss)
% CHECKSTABLEPALL check stability of polytope P(a) based on
    vertex coordinates of InPmin and InPpluss.
%
% Usage:
% [OutPIsStable] = CHECKSTABLEPALL(InPmin,InPpluss)
%
% where
% InPmin - vertex coordinates of "negative" direction polytope
% InPpluss - vertex coordinates of "positive" direction
    polytope
%
% Outputs:
% OutPIsStable - Flag for status of polytope stability: returns
     1 - for stable and 2 - for not stable polytope
```

This assistant function helps to verify whether a given polytope is stable or not.

calcStableP()

```
function [OutPminFin,OutPplussFin,minAlphaFin,plussAlphaFin] = calcStableP(InputA,RouthSourceV,minAlphaStable,CoefMin,plussAlphaStable,CoefPluss,IterCnt)
% CALCSTABLEP calculates final stable polytope P(a) starting from points of P-(a) and P+(a) cones.
%
% Usage:
% [OutPminFin,OutPplussFin,minAlphaFin,plussAlphaFin] = CALCSTABLEP(InputA,RouthSourceV,minAlphaStable,CoefMin,plussAlphaStable,CoefPluss,IterCnt)
%
% where
% InputA - array of coefficients for polynomial a(s) (in a normalized format)
% RouthSourceV - array of Routh sources
% minAlphaStable - initial minimal stable Alpha value
% CoefMin - value (in range between 0 and 1) that allows to change a speed at which alpha for Pmin will be adjusted; value = 0 will disable adjustment in "negative" direction
% plussAlphaStable - initial maximal stable Alpha value
% CoefPluss - value (in range between 0 and 1) that allows to change a speed at which alpha for Ppluss will be adjusted; value = 0 will disable adjustment in "positive" direction
% IterCnt - maximum number of iterations in calculation
%
% Outputs:
% OutPminFin - final matrix of vertex coordinates for "negative" direction of found stable polytope
%                 OutPplussFin - final matrix of vertex coordinates for "positeve" direction of found stable polytope
%                 minAlphaFin - final minimal stable Alpha value
%                 plussAlphaFin - final maximal stable Alpha value
```

The function returns the final $P(a)$ stable polytope. Calculations begin from end-points of stable $P^{-}(a)$ and $P^{+}(a)$ cones. Stable polytope $P(a)$ is found using interval halving method.

The following functions are designed for a robust controller design:

calcGainsSimplex()

```
function [OutputP_Q,OutputSimplex,OutputPlantNominal] =
    calcGainsSimplex(InputA,InputP)
% CALCGAINSSIMPLEX calculates controller gains p and q, and a
    nominal plant.
%
% Usage:
% [OutputP_Q,OutputSimplex,OutputPlantNominal] =
    CALCGAINSSIMPLEX(InputA,InputP)
%
% where
% InputA - array of coefficients for polynomial a(s) (in a
    normalized format)
% InputP - matrix of vertex coordinates for stable polytope
%
% Outputs:
% OutputP_Q - array of found gains p and q
%                 OutputSimplex - target Simplex matrix S
%                 OutputPlantNominal - matrix of coefficients
    representing nominal plant
```

The function is used to:

- Construct plant Sylvester matrix G (117) and target Simplex matrix S (113).
- Calculate controller $C(s)$ gains p and q using the optimization procedure (119).
- Calculate nominal plant with values of uncertainties placed in the center of region.

checkUncertRectangle()

```
function OutputPlantRectangle = checkUncertRectangle(InputP_Q,
    InputSimplex,FUncertValue)
% CHECKUNCERTRECTANGLE constructs Uncertainty rectangle and
    validates stability of the controller gains p and q.
%
% Usage:
% [OutputPlantRectangle] = CHECKUNCERTRECTANGLE(InputP_Q,
    InputSimplex,FUncertValue)
%
% where
% InputP_Q - array of controller gains p and q.
% InputSimplex - target Simplex matrix S.
% FUncertValue - array of uncertainty values for a plant
%
% Outputs:
% OutputPlantRectangle - Uncertainty rectangle end-points
```

The function constructs uncertainty rectangle and validates stability of the calculated controller gains p and q for the initial (with uncertainties) plant.

drawControllerN3()

```
function [] = drawControllerN3(InputA,InputP,
    InputPlantRectangle,InputPlantNominal)
% DRAWCONTROLLERN3 creates an 3D image of calculated controller
     stability area in correspondence to initial plant with
    uncertainty.
%
% Usage:
% DRAWCONTROLLERN3(InputA,InputP,InputPlantRectangle,
    InputPlantNominal)
%
% where
% InputA - initial closed-loop characteristic polynomial a*(s).
% InputP - Stable polytope P(s) end-points
% InputPlantRectangle - closed-loop polynomials for end-points
    of a plant
% InputPlantNominal - nominal plant closed-loop polynomial
%
% Outputs:
% MATLAB image of a controller stability area for initial plant
```

The function can be used to visualize the calculated controller $C(s)$ stability area with respect ot the initial plant. It is designed only for $n = 3$ systems.

Overall relations between the developed functions are presented in Fig. 4.

2.3 Discussion

The problem of designing a robust fixed-order controller for a continuous-time plant with uncertainties is addressed in this thesis. First, controller structure and order is selected based on the type of a controller (PI, PD or PID). Then, parameters of a selected controller are calculated solving linear quadratic problem for which the corresponding simplex is constructed using either polyhedral Routh cone or polytope. The simplex for both methods is calculated using reduced Routh parameters.

Controller is designed using inner approximation of the stability domain based on both Routh cones and polytopes. While, in general, the resulting area of polytope approach has larger volume, sometimes method based on polyhedral Routh cones may provide a better result especially when the starting stable point a^* is placed near the boundary of the stability domain. Thus, the selection of both controller type and algorithm should be done based on system prerequisites and requirements.

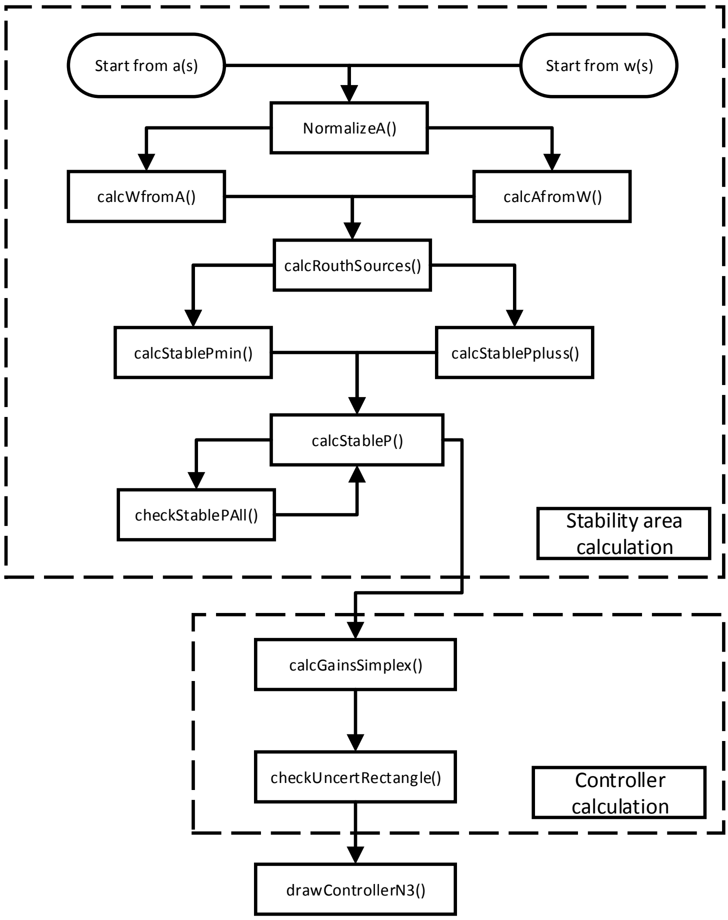

Figure 4. Overview of the developed package.

3 Numeric experiments

Example 3.1. Consider the normalized fourth-order system from [78]

$$H(s) = \frac{g_3 s^3 + g_2 s^2 + g_1 s + 1}{f_4 s^4 + f_3 s^3 + f_2 s^2 + f_1 s + f_0}, \tag{121}$$

where

$$\begin{aligned} &g_1 = 1, && g_2 = 0.29167, && g_3 = 0.04167, \\ &f_0 = 1, && f_1 = 2.083, && f_2 = 1.4583, \\ &f_3 = 0.4167, && f_4 = 0.04167. \end{aligned} \tag{122}$$

In order to illustrate the applicability of the algorithm proposed above, we introduce uncertainty to the plant as

$$\begin{aligned} f_0 &= 1 \pm 0.625, \\ f_1 &= 2.083 \pm 1.25. \end{aligned} \tag{123}$$

One may verify that the nominal plant (121) is stable. Our goal is to design a low-order robust controller. In particular, we consider the PI-type controller

$$C(s) = \frac{q_1 s + 1}{p_1 s}. \tag{124}$$

The characteristic polynomial $a(s)$ of the closed-loop system is given by

$$\begin{aligned} a(s) = p_1 s^5 &+ (0.4167 p_1 + 0.04167 q_1) s^4 \\ &+ (1.4583 p_1 + 0.29167 q_1 + 0.04167) s^3 \\ &+ [(2.083 \pm 1.25) p_1 + q_1 + 0.29167] s^2 \\ &+ [(1 \pm 0.625) p_1 + q_1 + 1] s + 1. \end{aligned} \tag{125}$$

Now, let us choose the initial stable closed-loop characteristic polynomial $a^*(s)$, whose poles are

$$r(a) = \{-3, -4, -5, -5, -7\}. \tag{126}$$

It means that the normed polynomial with $a_0^* = 1$

$$a^*(s) = 0.0005 s^5 + 0.0114 s^4 + 0.1076 s^3 + 0.4971 s^2 + 1.1262 s + 1 \tag{127}$$

has the reduced Routh parameters given as

$$w = \begin{bmatrix} 0.04167 & 0.1315 & 0.2451 & 0.3545 & 0.8394 & 1 \end{bmatrix}. \tag{128}$$

Take

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 4.4032 \tag{129}$$

and

$$a(w_i = \alpha_i w_i^*) \quad \text{for} \quad i = 1, \ldots, 5, \tag{130}$$

yielding the stable polynomials on the Routh rays of the polynomial $a^*(s)$

$$\begin{aligned} a_1^* &= \begin{bmatrix} 0.0005 & 0.0114 & 0.1233 & 0.8728 & 3.9828 & 1 \end{bmatrix}, \\ a_2^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.4536 & 1.7037 & 1.1262 & 1 \end{bmatrix}, \\ a_3^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.4079 & 0.6069 & 1.9604 & 1 \end{bmatrix}, \\ a_4^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.1278 & 0.9825 & 1.1262 & 1 \end{bmatrix}, \\ a_5^* &= \begin{bmatrix} 0.0021 & 0.0114 & 0.1781 & 0.4971 & 1.2680 & 1 \end{bmatrix}. \end{aligned} \tag{131}$$

Next, we solve the PI-controller design task for the nominal plant with $f_1 = 2.083$, $f_0 = 1$ via quadratic programming taking the target simplex of the closed-loop system by the above Routh rays as

$$S = \begin{bmatrix} a^* & a_1^* & a_2^* & a_3^* & a_4^* & a_5^* \end{bmatrix} = \begin{bmatrix} 0.0005 & 0.0005 & 0.0021 & 0.0021 & 0.0021 & 0.0021 \\ 0.0114 & 0.0114 & 0.0503 & 0.0503 & 0.0503 & 0.0114 \\ 0.1076 & 0.1233 & 0.4536 & 0.4079 & 0.1278 & 0.1781 \\ 0.4971 & 0.8728 & 1.7037 & 0.6069 & 0.9825 & 0.4971 \\ 1.1262 & 3.9828 & 1.1262 & 1.9604 & 1.1262 & 1.2680 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{132}$$

The optimization procedure returns parameters, resulting in the controller of the form

$$C(s) = \frac{0.4543s + 1}{0.0404s}. \tag{133}$$

The reference signal is chosen to be the step function. The simulation results for three variations (without, with maximum, and minimum possible uncertainties) of plant (121) are depicted in Fig. 5. It can be seen that outputs are capable of tracking reference signal for the same controller (133) with sufficient level of accuracy. Note that the overregulation depends on the choice of the initial stable polynomial $a^*(s)$, which itself is a different problem not covered here.

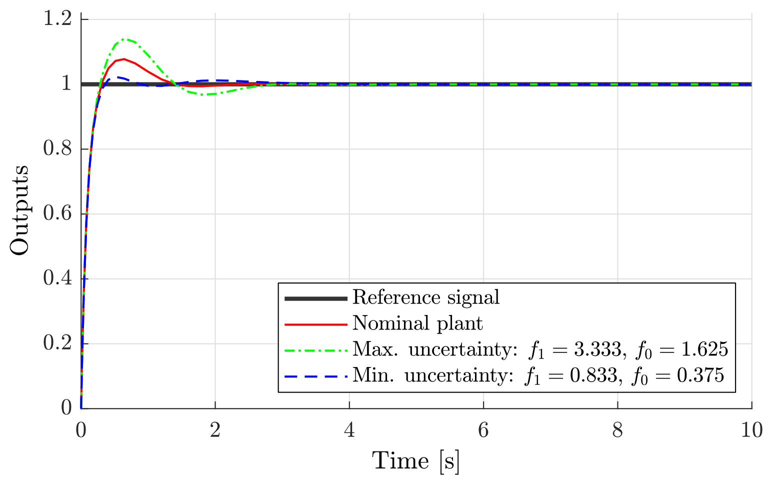

Figure 5. Simulation results for the designed PI controller operating for the nominal plant and plant with minimal and maximum possible uncertainty.

Example 3.2. Consider the second-order ($m = 2$) uncertain plant

$$H(s) = \frac{g(s)}{f(s)} = \frac{g_1 s + 1}{f_2 s^2 + f_1 s + f_0} \tag{134}$$

with

$$\begin{aligned} g_1 &= 0.5, \\ f_2 &= 1, \\ f_1 &= -1.2 \pm 0.8 \\ f_0 &= 0.52 \pm 1. \end{aligned} \tag{135}$$

One may verify that the nominal plant (i.e., without uncertainties) is unstable. Thus, our goal is to design a stabilizing robust PI-controller

$$C(s) = \frac{q_1 s + 1}{p_1 s}. \tag{136}$$

The characteristic polynomial $a(s)$ of the closed-loop system is given by

$$a(s) = p_1 s^3 - [(1.2 \pm 0.8)p_1 - 0.5q_1]s^2 + [(0.52 \pm 1)p_1 + 0.5 + q_1]s + 1. \tag{137}$$

Now, let us choose the initial stable closed-loop characteristic polynomial $a^*(s)$ with poles $r(a) = \{-4 \pm 0.5i, -0.5\}$. Take $\alpha_1 = \alpha_2 = \alpha_3 = 2$. Next, we solve the PI-controller design task for the nominal plant with $f_1 = -1.2$, $f_0 = 0.52$ via quadratic programming. The optimization procedure returns the optimal parameters, yielding

$$C(s) = \frac{2.7949s + 1}{0.1702s}. \tag{138}$$

Simulation results are presented in Fig. 6. The resulting pyramid corresponds to approximation of the stability domain by polyhedral Routh cone. The black (placed in the vertex) and blue dots are, respectively, defined by parameters of the initial stable polynomial $a^*(s)$ and coefficients of the characteristic polynomial $a(s)$ of the closed-loop system. The rectangular around blue dot determines bounds of uncertainties of (137). Note that it is inside the stability domain meaning that the designed controller is robust.

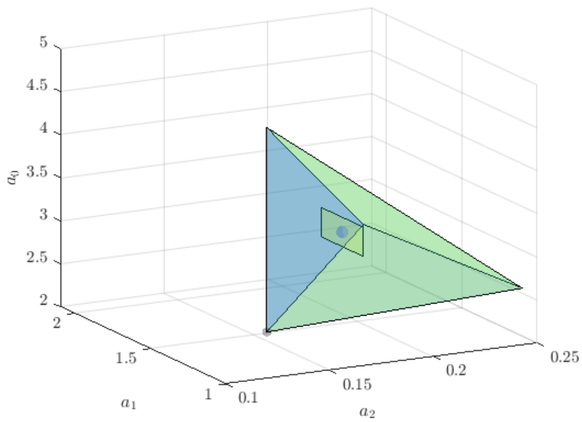

Figure 6. Approximation of the stability domain by polyhedral Routh cone.

Example 3.3. Consider the servo system provided by INTECO company [46] and available in [1], see Fig. 7.

This modular experimental platform consists of the following components: a tachogenerator, a 24V DC motor, an inertia load, a magnetic brake, an encoder, and a gearbox. The servo system may be interfaced with the MATLAB/Simulink environment through a specific PCI board, where data is collected from the encoder and tachogenerator, and is sent to the power drive box, which controls the DC motor. The data was collected from the plant and used for identification, yielding the following transfer function

$$H(s) = \frac{g(s)}{f(s)} = \frac{1}{0.0049s^2 + 0.0061s}. \tag{139}$$

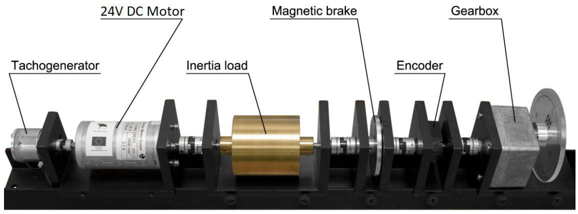

Figure 7. INTECO modular experimental platform.

We further add uncertainty to the identified model to verify the robustness of the designed controllers. Hence, the parameters are

$$\begin{aligned} &g_1 = 0, && g_0 = 1, \\ &f_2 = 0.0049, && f_1 = 0.0061 \pm 0.002, \quad f_0 = 0. \end{aligned} \tag{140}$$

Proceeding in the same manner as in the previous examples, and using two sets of poles

$$\begin{aligned} r_1(a) &= \{-1.5, -1, -0.5\}, \\ r_2(a) &= \{-7, -5, -3\} \end{aligned} \tag{141}$$

for the closed-loop characteristic polynomial, we get *slow* and *fast* PI controllers

$$\begin{aligned} C_s(s) &= \frac{97.1975s + 1}{13523s}, \\ C_f(s) &= \frac{72.2242s + 1}{22.0993s}. \end{aligned} \tag{142}$$

The results of laboratory experiments with controller $C_f(s)$ are presented in Fig. 8. Two types of scenarios are considered: (*i*) nominal plant, and (*ii*) plant with external friction between inertia load and the base. One can see that the controller is capable of tracking reference signal for both cases. Figure 9 shows the experimental results with controller $C_s(s)$ for varying set point. In addition, Fig. 10 depicts the comparison for PD and PI controllers designed using both polytope and cone techniques.

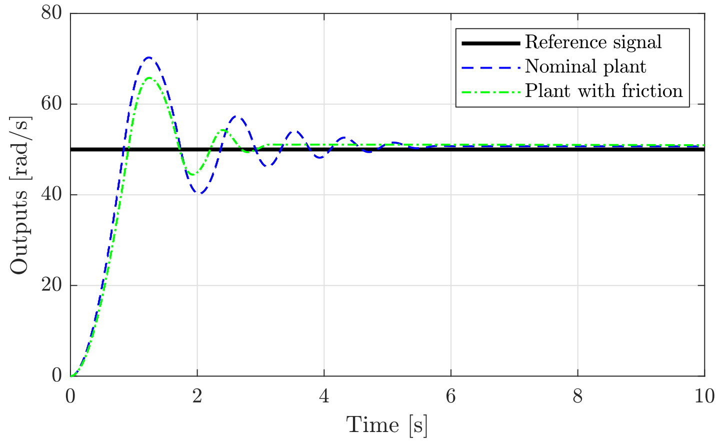

Figure 8. Laboratory experimental results for controller $C_f(s)$ operating for nominal plant and case with friction.

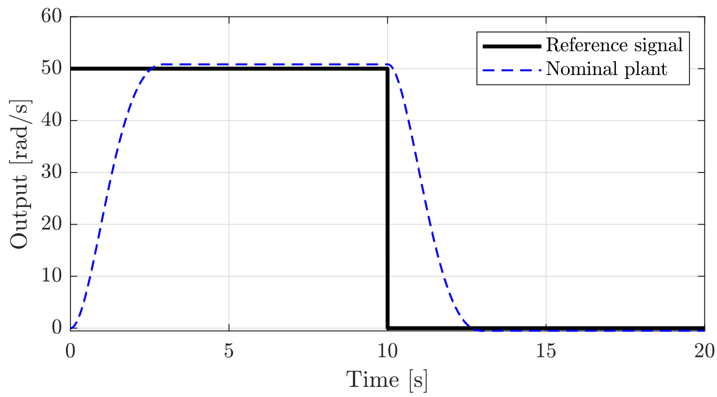

Figure 9. Laboratory experimental results for controller $C_s(s)$.

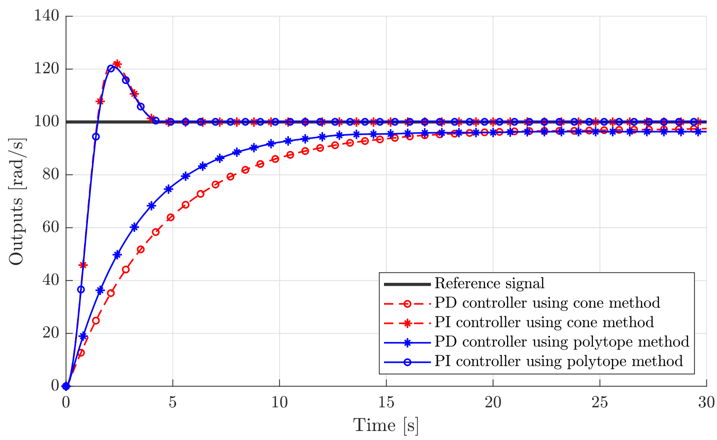

Figure 10. Laboratory experimental results for PD and PI controllers using cone and polytope based methods.

Conclusions and Future research

The problem of convex approximation is very challenging and is still in the focus of a community. The existing methods can tackle this problem under some specific settings. The most suitable method is selected based on a task and desired results with consideration of all advantages and disadvantages of selected approach.

In this thesis a simple and efficient method (with two algorithms) to generate stable line segments of Hurwitz polynomials is presented and explained. This method for convex approximation of a continuous-time system is based on the new multilinear stability condition for Hurwitz polynomials formulated via reduced Routh parameters. Starting from an arbitrary Hurwitz polynomial it is possible to generate stable line segments in the directions of the so-called Routh rays. This leads to the inner (convex) approximation of stability region based on the derived line segments. Two algorithms for generating stable polytopes and polyhedral cones with respect to a given polynomial are then developed. The functions and algorithms are presented in a format suitable for further software implementation. Each of the presented methods (based on Routh polytopes and cones) has a number of advantages and disadvantages. In the following we briefly discuss only the most important.

There are several key points that make the method based on Routh *polytopes* more preferable in a certain situation. First, it is well-suited to design a controller for an object with polytopic uncertainty. Second, the largest volume of the stable polytope can be obtained if a^* is located relatively far from the stability boundary. In this case, both P^- and P^+ are almost proportional. Finally, the initial point a^* is placed near the center of the constructed polytope, what significantly simplifies the procedure of robust controller design. On the other hand, this method does not use the whole length of the Routh rays. Moreover, if a^* is located close to the stability boundary, then the volume P^- is occurs to be small. Similarly, the polyhedral Routh *cones* based method has several strong points. First, it is suitable to design a controller for an object with conic or one dominant uncertainty. Second, the whole length of the Routh half-lines (i.e., $\alpha > 1$) is used. Finally, the largest volume can be obtained if a^* is located relatively close to the stability boundary. On the other hand, if a^* is located far from the stability boundary, then the respective part P^- is not used at all. Moreover, the point a^* is a vertex of a polyhedral cone, which complicates the design of a controller. In what follows we summarize the main procedure for the robust output controller design. The developed method has two branches in accordance with Routh polytopes and cones approaches. Proposed methods start from a stable simplex (or polytope) of the closed-loop characteristic polynomial, which is defined via Routh rays of a preselected Hurwitz stable polynomial. Then, the set of possible plant parameters as a convex polytope (polytopic plant model) is defined. As a final step for synthesis of robust output controller for polytopic plant model, a convex quadratic programming task is solved yielding a set of required parameters.

Mathematical equations and calculations were transformed into format that is suitable for software implementation, and were implemented in the form of a package. The simpleness of calculations and effectiveness of robust controllers designed using these methods is demonstrated on several illustrative systems, including system with uncertainties and laboratory prototype of a DC motor servo system.

One of the possible directions for the future research will be devoted to selection of the initial characteristic polynomial of the closed-loop system that is required for design of a suitable controller. According to the above discussion, the correct choice is not trivial and makes another open and challenging problem. There are different methods for points generation that might be applicable in this situation: randomly generated points, identification of the area of interest and selection points from it, application of already checked point as a basis for new points generation, etc.

Another direction for the future research lays in the detailed comparison of alternative methods for convex optimization and robust controller design (such as boxes, ellipsoids, hyper-rectangles, etc.) for continuous-time systems. The results should help to identify the most suitable types of problems for each of the existing methods.

One more possible extension of the research lays in the area of hybrid (combination of discrete and continuous) systems. The approach of convex approximation based on the reduced Routh parameters has proven itself for continuous (this thesis) and discrete-time [68] systems. Thus, the future research will be focused on the generalization of the developed results to the hybrid time scale.

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Abstract

Robust PID Controller Design for Continuous-time Systems via Reduced Routh Parameters

The present thesis is devoted to the research on convex approximation and robust output controller design for continuous-time linear systems. The proposed methods are based on the new multilinear stability condition for Hurwitz polynomials and formulated based on the so-called reduced Routh parameters. First part of the thesis presents two inner (convex) approximation algorithms based on the Routh polytopes and cones. The results of convex approximation are used for robust controller design in the second part of the thesis. The functions and algorithms are presented in a format suitable for further software implementation. The software-based package of functions for the practical implementation of both convex approximation and robust controller synthesis is presented in the last part of the thesis. The illustrative examples indicate the effectiveness of the proposed methods and ability of designed controllers to handle uncertainties in considered systems.

Kokkuvõte
Pidevaja süsteemide robustse PID kontrolleri süntees taandatud Routh parameetrite kaudu

Käesolev lõputöö on pühendatud kumerate aproksimatsioonide ja robustse väljundikontrolleri disaini uurimisele pidevaja lineaarsüsteemides. Väljapakutud meetodid tuginevad uuel multilineaarse stabiilsuse tingimusel Hurwitzi polonüümide jaoks ja on sõnastatud põhinedes nõndanimetatud vähendatud Routhi parameetritel. Lõputöö esimene osa tutvustab kahte sisemist (kumerat) aproksimatsiooni algoritmi, mis põhinevad Routh'i polütoopidel ja koonustel. Kumerate aproksimatsioonide tulemusi kasutatakse robustse kontrolleri disainimiseks lõputöö teises osas. Funktsioonid ja algoritmid on esitatud kujul, mis on sobilikud edasistes tarkvara rakendustes. Tarkvarapõhine funktsioonide pakett kumerate aproksimatsioonide ja robustse kontrolleri sünteesi praktiliseks rakendamiseks on esitatud lõputöö viimases osas. Illustreerivad näited toovad esile väljapakutud meetodite tõhususe ja disainitud kontrollerite suutlikkuse käsitleda määramatust vaadeldud süsteemides.

Appendix 1

Publication I

Ü. Nurges, I. Artemchuk, and J. Belikov. Generation of stable polytopes of Hurwitz polynomials via Routh parameters. In *Conference on Decision and Control*, pages 2390–2395, Los Angeles, CA, USA, Dec. 2014

Generation of stable polytopes of Hurwitz polynomials via Routh parameters

Ülo Nurges[1] and Igor Artemchuk[2] and Juri Belikov[1]

Abstract— **The paper addresses an important issue in the field of continuous-time linear control systems – convex stability domain approximation problem. The constructive procedure of generating a stable polytope is proposed. The main idea is based on constructing so-called Routh stable line segments (half-lines) starting from a given stable polynomial. It is summarized in the form of a step-by-step algorithm that results in a stable polytope around a given point. Several numerical examples are presented to demonstrate the covered concepts and the effectiveness of the proposed approach. Calculations are performed in a MATLAB environment.**

I. INTRODUCTION

In recent years, a growing interest is given to the use of randomized methods in system and control theory [1]. This new field of research demands efficient tools for generating random samples of entities encountered in the analysis of uncertain systems [2], [3]. The necessity of generating stable polynomials arises in such areas like fixed-order controller design [4], [5], [6], [7], static output feedback stabilization [7], and robust output controller design [8]. In general, the fixed-order output controller design is a hard problem, since it reduces to finding a stable polynomial in an affine family, which is known to be NP-hard, see [2]. The existence of stabilizing fixed-order controller for a given unstable plant is still an open problem.

Another practical issue is that of model uncertainty. If the model uncertainty is relatively small, then it is possible to use sensitivity-based methods. If the model uncertainty is large some robust formulation of the problem is needed, such as multimodel [9], polytopic model [10] or LMI approach [11].

The main hindrance of the parametric methods is the well-known fact that the stability domain in the space of polynomial coefficients is non-convex in general. That is why several convex approximations of the stability region such as ellipsoids [4], hyperrectangles [12] and polytopes [13], [10], [14] are well known and widely used in robust control.

The goal of this work is to provide a simple, fast enough and numerically stable algorithm for generation of sets of stable (Hurwitz) polynomials. The randomized generation of stable polynomials is carefully studied in [2], where the Levinsone-Durbin parametrization is suggested as an efficient and numerically stable method. We generalize this approach to generation of stable line segments in the space of polynomial coefficients. Our method is based on a new stability criterion for Hurwitz polynomials. Indeed, for an arbitrary Hurwitz polynomial of order n we introduce the method for generating n stable line segments in directions of the so-called Routh rays. The latter means that instead of single points we construct bunches of stable half-lines in the polynomial coefficient space. Next, on the basis of the derived line segments we obtain an inner approximation of the stability region. Note that the results of this paper can be understood as a step toward, for instance, robust output controller design.

The paper is organized as follows. Section II recalls the basic notions and definitions related to stability of polynomials in the continuous-time case. The reduced Routh parameters are introduced. It is followed by introducing stable half-lines (Routh rays) of polynomials. In Section IV a problem of generating stable polytopes via bunches of Routh segments is addressed, and a step-by-step algorithm is presented. The presented theory is illustrated by several numerical examples. Concluding remarks and possible directions for the future research are drawn in the last section.

II. REDUCED ROUTH PARAMETERS OF POLYNOMIALS

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \qquad (1)$$

with real coefficients $a_i \in \mathbb{R}$, for $i = 0, \ldots, n$, is said to be *continuous-time stable* in the Hurwitz sense, if all its roots λ_i, for $i = 1, \ldots, n$, are in the open left-half plane of $\mathbb{C}$, i.e. $\mathrm{Re}(\lambda_i) < 0$.

Since the polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = \begin{bmatrix} a_n & \ldots & a_0 \end{bmatrix}^{\mathrm{T}}$ of its coefficients, i.e. $a := a(s) = \begin{bmatrix} a_n & \ldots & a_0 \end{bmatrix}^{\mathrm{T}}$. Then, the Hurwitz *region* $\mathcal{H}_n$ is defined as the set

$$\mathcal{H}_n = \left\{ a \in \mathbb{R}^{n+1} \mid (1) \text{ is Hurwitz} \right\}.$$

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [15]. Based on this criterion, a method for constructing Hurwitz polynomials can be derived as follows [2]. Start with an arbitrary Hurwitz polynomial of degree 2. Since positivity of the coefficients is equivalent to stability for the second-order polynomials,

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generate arbitrary positive numbers h_0, h_1, h_2 and compose the Hurwitz polynomial

$$a(s) = h_2 s^2 + h_1 s + h_0$$

or

$$a = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} h_2 & h_1 & h_0 \end{bmatrix}^{\mathrm{T}}.$$

At the kth step, having a Hurwitz polynomial of degree k

$$a(s) = \begin{bmatrix} a_k & a_{k-1} & \ldots & a_0 \end{bmatrix}^{\mathrm{T}},$$

consider two polynomials of degree $k+1$

$$p(s) = \begin{bmatrix} 0 & a_k & a_{k-1} & \ldots & a_0 \end{bmatrix}^{\mathrm{T}},$$

and

$$q(s) = \begin{bmatrix} a_k & 0 & a_{k-2} & 0 & a_{k-4} & 0 & \ldots \end{bmatrix}^{\mathrm{T}}.$$

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{2}$$

which is, according to the Routh rule, Hurwitz polynomial of degree $k+1$. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see [16], [17]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k = 0, \ldots, n$ recursively by increasing degree k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction, i.e. the mapping from Routh parameters h to polynomial coefficients a is one-to-one [2]. Next, let us introduce the reduced Routh parameters that are used later in constructing stable line segments.

Definition 1: The *reduced* Routh parameters w_j for normed polynomials $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1$ are defined as follows

$$\begin{aligned} w_0 &= h_0 = 1, \\ w_1 &= h_1, \\ w_2 &= h_2, \\ w_j &= \frac{h_j}{h_{j-1}}, \quad j = 3, \ldots, n. \end{aligned} \tag{3}$$

From (2) and (3) we obtain the relations for recursive generation of normed Hurwitz polynomials of order $k+1$ for $k > 2$ as

$$a(s) = p(s) + w_{k+1} q(s).$$

Denote the degree of a polynomial by superscription to obtain

$$\begin{aligned} a^{k+1} = \big[w_k a_k^k \quad a_k^k \quad a_{k-1}^k + w_k a_{k-2}^k \quad a_{k-2}^k \\ a_{k-3}^k + w_k a_{k-4}^k \quad \ldots \quad 1 \big]^{\mathrm{T}}, \end{aligned} \tag{4}$$

where

$$a^k = \begin{bmatrix} a_k^k & a_{k-1}^k & \ldots & 1 \end{bmatrix}^{\mathrm{T}}.$$

Using matrix notation, we can rewrite equation (4) as

$$a^{k+1} = W_k a^k, \tag{5}$$

where W_k is a $(k+1) \times k$-dimensional matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0^{\mathrm{T}} \\ \vdots \\ I_k \end{bmatrix}$$

with I_k being the $k \times k$-dimensional unit matrix and J_k being the $k \times k$-dimensional diagonal matrix $J_k = \mathrm{diag}\{1, 0, 1, 0, \ldots\}$, i.e.

$$W_k = \begin{bmatrix} w_k & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & w_k & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The inverse mapping from polynomial coefficients a_k to the reduced Routh parameters w_k, $k = n, \ldots, 1$ can be recursively found starting from w_n via (5) as follows

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2 = a_2^3, \\ w_1 &= a_1^2 = a_1^3 - \frac{a_3^3}{a_2^3}. \end{aligned} \tag{6}$$

Note that in (6) parameters a_j^j can be found explicitly as

$$\begin{aligned} a_{k-i-1}^{k-1} &= a_{k-i-1}^k, & i &= 0, \ldots, 2\left\lfloor \frac{k-2}{2} \right\rfloor, \\ a_{k-i-2}^{k-1} &= a_{k-i-2}^k - w_k a_{k-i-3}^k, & i &= 0, \ldots, 2\left\lfloor \frac{k-3}{2} \right\rfloor \end{aligned}$$

with $a_0 := 1$.

Proposition 1: If the reduced Routh parameters $w_k > 0$, for $k = 1, \ldots, n$, then the normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable.

Proof: From (6) we obtain

$$\begin{aligned} h_0 &= 1, \\ h_1 &= w_1, \\ h_2 &= w_2, \\ h_j &= w_j h_{j-1}, \quad j = 3, \ldots, n. \end{aligned}$$

Now, if $w_k > 0$, for $k = 1, \ldots, n$, then all the Routh parameters of the polynomial $a(s)$ are positive $h_k > 0$, $k = 0, \ldots, n$. Therefore, the polynomial $a(s)$ is Hurwitz stable. ■

Proposition 2: The mapping (5) from the reduced Routh parameters w_k, $k = 1, \ldots, n$ to the normed polynomial coefficients a_k^n, $k = 1, \ldots, n$ with $a_0 = 1$ is a one-to-one mapping if $w_k > 0$, $k = 1, \ldots, n$.

Proof: The proof is straightforward, since mapping (3) between the reduced Routh parameters w_k, $k = 1, \ldots, n$ and the Routh parameters h_k, $k = 1, \ldots, n$ is one-to-one by $h_0 = 1$ as well as the the mapping between the Routh parameters h_k, $k = 1, \ldots, n$ and the polynomial coefficients a_k^n, $k = 1, \ldots, n$ by $a_0 = 1$, see [2] for technical details. ■

Example 1: Let $n = 4$ and

$$w = \begin{bmatrix} w_4^4 & w_3^4 & w_2^4 & w_1^4 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 & 2 & 2 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Next, we calculate recursively the coefficients of polynomials starting from $k = 2$. According to (6),

$$a^2 = \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

For $k = 3$ we get

$$a^3 = \begin{bmatrix} a_3^3 \\ a_2^3 \\ a_1^3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 6 \\ 1 \end{bmatrix}$$

and for $k = 4$

$$a^4 = \begin{bmatrix} a_4^4 \\ a_3^4 \\ a_2^4 \\ a_1^4 \\ 1 \end{bmatrix} = \begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 6 \\ 1 \end{bmatrix}.$$

III. STABLE ROUTH SEGMENTS OF POLYNOMIALS

In this section we introduce the stable line segments (half-lines) of polynomials that can be obtained starting from the reduced Routh parameters w_k, $k = 1, \ldots, n$ of a Hurwitz polynomial $a \in \mathcal{H}_n \subset \mathbb{R}^{n+1}$.

Theorem 1: Through an arbitrary Hurwitz stable point $a = \begin{bmatrix} a_n & a_{n-1} & \ldots & a_1 & 1 \end{bmatrix}^{\mathrm{T}}$ with reduced Routh parameters $w_k(a) > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$ such that

$$\begin{aligned} \mathcal{R}_k(a) = \big\{ a \mid\ & w_k(a) \in (0, \infty), \\ & w_j(a) = \text{const},\ j \neq k;\ k, j \in \{1, \ldots, n\} \big\}. \end{aligned} \tag{7}$$

Proof: Observe that all the points of the line $\mathcal{R}^k(a)$ are Hurwitz stable, since

1) $n-1$ reduced Routh parameters $w_j(a)$, $j \in \{1, \ldots, n\}$, $j \neq k$ are fixed and positive $w_j(a) > 0$ according to the first assumption;
2) the kth reduced Routh parameters $w_k(a) > 0$ according to assumption $w_k(a) \in (0, \infty)$.

Next, we have to prove that $\mathcal{R}_k(a)$ is a line segment (half-line). It is easy to see that the mapping (5) is multilinear. If $n-1$ reduced Routh parameters $w_j(a)$, $j \in \{1, \ldots, n\}$, $j \neq k$ are fixed, then the mapping (5) turns out to be linear with respect to the kth reduced Routh parameter $w_k(a)$. The latter means that for every k such that $k = 1, \ldots, n$ we obtain a half-line $\mathcal{R}_k(a)$ and altogether n half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$. ■

Definition 2: The half-lines $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ defined by (7) are called *Routh rays* of the polynomial $a(s)$. Furthermore, their endpoints $v_k(a)$ such as

$$v_k(a) = a(w_k = 0) \tag{8}$$

are supposed to be the *Routh sources* of the polynomial $a(s)$.

According to Proposition 1, all the Routh sources $v_k(a)$ of Hurwitz (stable) polynomials $a(s)$ are placed on the stability boundary. The latter means that some of the roots $\lambda_j(v)$, $j \in \{1, \ldots, n\}$ are placed on the imaginary axis.

Example 2: Let $n = 3$. Start from the polynomial $a = \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^{\mathrm{T}}$ with reduced Routh parameters $w(a) = \begin{bmatrix} 4 & 2 & 2 \end{bmatrix}^{\mathrm{T}}$. By (5) we can easily calculate the Routh sources as follows

$$\begin{aligned} v_1 &= \begin{bmatrix} 8 & 2 & 4 & 1 \end{bmatrix}^{\mathrm{T}}, \\ v_2 &= \begin{bmatrix} 0 & 0 & 6 & 1 \end{bmatrix}^{\mathrm{T}}, \\ v_3 &= \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

and find the Routh rays $\mathcal{R}_1(a)$, $\mathcal{R}_2(a)$, $\mathcal{R}_3(a)$ through the corresponding Routh source v_k, $k = 1, 2, 3$, and the initial point a as

$$\begin{aligned} \mathcal{R}_1 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^{\mathrm{T}} + (1 - \alpha) \begin{bmatrix} 8 & 2 & 4 & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{R}_2 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^{\mathrm{T}} + (1 - \alpha) \begin{bmatrix} 0 & 0 & 6 & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{R}_3 &= \alpha \begin{bmatrix} 8 & 2 & 6 & 1 \end{bmatrix}^{\mathrm{T}} + (1 - \alpha) \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

where $\alpha \in [0, \infty)$. Next, we calculate the roots $\lambda(v_k) = \begin{bmatrix} \lambda_1(v_k) & \lambda_2(v_k) & \lambda_3(v_k) \end{bmatrix}^{\mathrm{T}}$ of the Routh sources as

$$\lambda(v_1) = \begin{bmatrix} -0.25 \\ \pm 0.7071i \end{bmatrix}, \qquad \lambda(v_2) = \begin{bmatrix} 0 \\ 0 \\ -0.1667 \end{bmatrix},$$
$$\lambda(v_3) = \begin{bmatrix} 0 \\ -0.5 \pm 0.5i \end{bmatrix}.$$

Indeed, all the Routh sources have at least one root on the imaginary axis, e.g. v_1 has a pair of imaginary roots, v_2 has two roots in the origin and v_3 has a root in the origin.

Next, using the recursive relationship (5) we obtain

$$a^n = W_k^n a^k,$$

where

$$W_k^n = W_n W_{n-1} \cdots W_k$$

or

$$a^n = W_2^n a^2 = W_2^n \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}, \tag{9}$$

with

$$W_2^n = W_n W_{n-1} \cdots W_2. \tag{10}$$

From the mapping (9) and (10), we can easily obtain the following theorem regarding the roots of Routh sources.

Theorem 2: All the Routh sources $v_j(a)$, $j = 2, \ldots, n-1$ of a Hurwitz polynomial $a(s)$ of the order n have at least two roots at the origin

$$\lambda_1(v_j) = \lambda_2(v_j) = 0, \quad j = 2, \ldots, n-1$$

and the last Routh source $v_n(a)$ has at least one root at the origin

$$\lambda_1(v_n) = 0.$$

Proof: The proof is a straightforward conclusion from equations (9) and (10). Indeed, due to the structure of the matrix W_k the explicit calculation of the first two elements of $a^n = W_k^n a^k$ yields

$$a_n^n = w_2 \cdots w_n,$$
$$a_{n-1}^n = w_2 \cdots w_{n-1}.$$

Hence, according to Definition 2, from the previous equations one immediately gets $\lambda_1(v_j) = \lambda_2(v_j) = 0$, for $j = 2, \ldots, n-1$, and $\lambda_1(v_n) = 0$. ■

Example 3: Calculate by (9) and (10) the Routh sources $v_k(a)$ of some low order polynomials $a(s)$. In case $n = 3$ we obtain

$$a^3 = W_2^3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix}$$

and

$$v_1(a) = \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_3 \\ 1 \end{bmatrix}, \quad v_2(a) = \begin{bmatrix} 0 \\ 0 \\ w_1 + w_3 \\ 1 \end{bmatrix},$$
$$v_3(a) = \begin{bmatrix} 0 \\ w_2 \\ w_1 \\ 1 \end{bmatrix}.$$

Indeed, $\lambda_1(v_2) = \lambda_2(v_2) = \lambda_1(v_3) = 0$ for arbitrary positive reduced Routh parameters w_1, w_2 and w_3.

Now, in case $n = 4$ we obtain

$$\begin{aligned} a^4 &= W_2^4 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} w_2 w_3 w_4 \\ w_2 w_3 \\ w_2 + w_1 w_4 + w_3 w_4 \\ w_1 + w_3 \\ 1 \end{bmatrix} \end{aligned}$$

and

$$v_1(a) = \begin{bmatrix} w_2 w_3 w_4 \\ w_2 w_3 \\ w_2 + w_3 w_4 \\ w_3 \\ 1 \end{bmatrix}, \quad v_2(a) = \begin{bmatrix} 0 \\ 0 \\ w_1 w_4 + w_3 w_4 \\ w_1 + w_3 \\ 1 \end{bmatrix},$$
$$v_3(a) = \begin{bmatrix} 0 \\ 0 \\ w_2 + w_1 w_4 \\ w_1 \\ 1 \end{bmatrix}, \quad v_4(a) = \begin{bmatrix} 0 \\ w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix}.$$

Hence, it follows that $\lambda_1(v_2) = \lambda_2(v_2) = \lambda_1(v_3) = \lambda_2(v_3) = \lambda_1(v_4) = 0$ for arbitrary positive reduced Routh parameters w_1, w_2, w_3 and w_4.

IV. STABLE POLYTOPES VIA BUNCHES OF ROUTH SEGMENTS

In this section we generate stable polytopes of Hurwitz polynomials starting from a single Hurwitz polynomial a. According to Theorem 1, the set of n Routh rays $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ is Hurwitz stable. However, in general, the linear cover of the Routh rays $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ is not Hurwitz stable. Thus, one may ask the question: how to find a stable polytope $P(a)$ around the initial stable point a such that all of its vertices are placed on the Routh rays $\mathcal{R}_k(a)$, $k = 1, \ldots, n$. Moreover, it would be interesting to find the stable polytope $P_{\max}(a)$ with maximal possible volume $V(P(a))$, $V(P_{\max}(a)) = \max_P V(P(a))$.

Next, we formulate a step-by-step algorithm to solve the problem of generating stable polytope via bunches of Routh segments. Note that $+$ and $-$ signs stand to positive and negative directions with respect to the starting point, respectively.

Algorithm:

Step 1. Start from a given nth-order stable polynomial $a(s)$, or $a_n = \begin{bmatrix} a_n^n & a_{n-1}^n & \ldots & a_1^n & 1 \end{bmatrix}^{\mathrm{T}}$.

Step 2. Using (6), calculate the reduced Routh parameters w_k for $k = n, \ldots, 1$.

Step 3. Calculate by (8) the Routh sources $v_k(a)$ for $k = 1, \ldots, n$.

Step 4. Using (7), find the Routh rays $\mathcal{R}_k(a)$ for $k = 1, \ldots, n$.

Step 5. Find the stable polytope of sources $P_0(a)$ of the polynomial $a(s)$ as follows:

- Start from the polytope $P_0^-(a)$ defined as the linear cover of the initial polynomial a and all of its sources $v_k(a)$, $k = 1, \ldots, n$, i.e. $P_0^-(a) = \operatorname{conv}\{a \ v_1 \ \ldots \ v_n\}$.
- Check the stability of single edges of $P_0^-(a)$ by Hurwitz Segment Lemma [18, p. 81]. Next, check the stability of the polytope $P_0^-(a)$ using Edge Theorem [18, p. 271].
- If thus obtained polytope $P_0^-(a)$ is not stable, then generate recursively using interval halving method (between a and $v_k(a)$, $k = 1, \ldots, n$) the new candidates for the polytope of sources $P_l^-(a)$, $l = 1, 2, \ldots$.
- If a stable polytope of sources $P_{\max}^-(a)$ with maximal volume $V(P^-(a)) = \max$ is found, then stop. The volume of polytopes $P^-(a)$ can be found by Triangulation method, see [19] for technical details.

Step 6. Similarly, find the stable polytope of rays $P^+(a)$ of the polynomial $a(s)$ starting from endpoints of the Routh rays $e_k(w_k = \gamma) \in \mathcal{R}_k(a)$ with γ being a big-enough number. If a stable polytope of rays $P_{\max}^+(a)$ with maximal volume $V(P^+(a)) = \max$ is found, then stop. The volume of polytopes $P^+(a)$ can be found by Triangulation method.

Step 7. Starting from the vertices of the polytopes $P^-(a)$

and $P^+(a)$, find using interval halving method the stable polytope of Routh (rays) segments $P(a)$ with vertices $\mathcal{R}_k^- \in \mathcal{R}_k(a)$ and $\mathcal{R}_k^+ \in \mathcal{R}_k(a)$ with maximal volume.

Step 8. End of the Algorithm.

According to Theorem 1, the Routh rays $\mathcal{R}_k(a)$ are partly contained in the stable region. In addition, some of the edges connecting the points (vertices) e_k and $v_k(a)$ may fall into an unstable area. To avoid this, we can move along the stable segments on the Routh rays, changing the overall volume of the polytope, either increasing or decreasing the distance from or to the corresponding vertex. The simplest way to perform this action is due to the so-called interval halving method [20]. Note that proceeding this way after a finite number of steps the algorithm will converge to a polytope having maximal possible volume with respect to the starting point.

Example 4: Consider an Unmanned Free-Swimming Submersible vehicle [21] for which the relation of pitch angle to elevator surface angle can be represented by the following transfer function

$$H(s) = \frac{-0.125(s+0.435)}{(s+1.23)(s^2+0.226s+0.0169)}. \tag{11}$$

One can easily verify that the nominal system $H(s)$ is stable, since the poles $\lambda_1 = -1.23$, $\lambda_{2,3} = -0.113 \pm 0.0643i$ have negative real parts. Our aim is to find the stable polytope (with maximal volume) in the coefficient space around the nominal characteristic polynomial (denominator of $H(s)$)

$$a(s) = s^3 + 1.456s^2 + 0.2949s + 0.028.$$

First, we need to normalize polynomial $a(s)$ with respect to the free term (i.e. dividing polynomial by 0.028) as

$$a(s) = 35.7143s^3 + 52s^2 + 10.5321s + 1.$$

Next, according to the provided algorithm above, we collect the coefficients as

$$a = \begin{bmatrix} 35.7143 & 52 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}}$$

for which the reduced Routh parameters can be calculated as

$$w_k(a) = \begin{bmatrix} 0.6868 & 52 & 9.8453 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Application of (5) yields the Routh sources as

$$v_1 = \begin{bmatrix} 35.7143 & 52 & 0.6868 & 1 \end{bmatrix}^{\mathrm{T}},$$
$$v_2 = \begin{bmatrix} 0 & 0 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}},$$
$$v_3 = \begin{bmatrix} 0 & 53 & 9.845286 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Next, from (7) we find the Routh rays $\mathcal{R}_1(a)$, $\mathcal{R}_2(a)$, $\mathcal{R}_3(a)$ through the corresponding Routh source v_k, $k = 1, 2, 3$, and the initial point a as

$$\mathcal{R}_1 = \alpha \begin{bmatrix} 35.7143 & 52 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}} + (1-\alpha) \begin{bmatrix} 35.7143 & 52 & 0.6868 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{R}_2 = \alpha \begin{bmatrix} 35.7143 & 52 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}} + (1-\alpha) \begin{bmatrix} 0 & 0 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{R}_3 = \alpha \begin{bmatrix} 35.7143 & 52 & 10.5321 & 1 \end{bmatrix}^{\mathrm{T}} + (1-\alpha) \begin{bmatrix} 0 & 53 & 9.845286 & 1 \end{bmatrix}^{\mathrm{T}},$$

where $\alpha \in [0, \infty)$. After that we calculate the roots $\lambda(v_k) = \begin{bmatrix} \lambda_1(v_k) & \lambda_2(v_k) & \lambda_3(v_k) \end{bmatrix}^{\mathrm{T}}$ of the Routh sources as

$$\lambda(v_1) = \begin{bmatrix} -1.4560 \\ \pm 0.1387i \end{bmatrix}, \qquad \lambda(v_2) = \begin{bmatrix} 0 \\ 0 \\ -0.0949 \end{bmatrix},$$
$$\lambda(v_3) = \begin{bmatrix} 0 \\ -0.0947 \pm 0.1013i \end{bmatrix}.$$

Next, using calculated parameters, initial stable polytope was constructed, see Fig. 1. In fact, vertices of the positive direction have to be depicted a bit farther, but we decided to change coordinates (shifting them towards point a) for illustrative purposes. In Fig. 1 the big (blue) dot in the middle is a starting point a, the dark triangles are lower and upper bases, the light polygons are side faces, and the dashed lines are Routh rays. One can see that the dashed lines go beyond the border of polytope. This is due to the fact that the stable polytope with maximal volume, constructed on the Routh rays, is to be found.

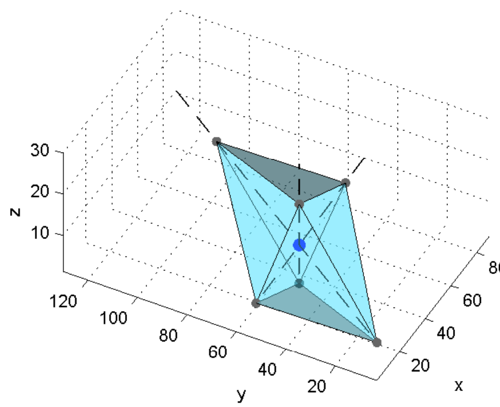

Fig. 1. Initial stable polytope

Repeating Steps 5-7 several times, we arrive at the polytope $P_{\max}(a)$ with vertices having the following coordinates

$$
\begin{aligned}
p_1^- &= (35.7143, 52, 8.9831),\\
p_2^- &= (30.0952, 43.8185, 10.5321),\\
p_3^- &= (30.0952, 52, 10.424),\\
p_1^+ &= (35.7143, 52, 1.9672 \cdot 10^4),\\
p_2^+ &= (7.1359 \cdot 10^4, 1.03898 \cdot 10^5, 10.5321),\\
p_3^+ &= (7.1359 \cdot 10^4, 52, 1.382 \cdot 10^3),
\end{aligned}
$$

where $+$ indicates positive direction (i.e. vertices placed above the point a) and $-$ indicates vertices placed below point a. Finally, the volume of the obtained polytope is $V(P_{\max}(a)) = 2.4277 \cdot 10^{13}$. Note that the volume of the initial polytope is $V(P_0(a)) = 2.2164 \cdot 10^4$.

V. CONCLUSIONS

A simple and efficient method is given for generation stable line segments of Hurwitz polynomials. It is based on the new stability condition for Hurwitz polynomials via reduced Routh parameters. The algorithm for generating stable polytopes around a given polynomial is developed.

The results of this paper can be extended in several ways. Note that application of the algorithm from Section IV results in a stable polytope (being a convex approximation of the stability domain) constructed with respect to a given point. However, the overall stability region is, in general, bigger. Therefore, the idea based on random generation of stable initial points and uniting resulting polytopes in one approximation with larger volume can be studied. In addition, the method of generating stable line segments and stable polytopes can be used for design of a stabilizing fixed-order controller as well as for robust fixed-order output controller synthesis.

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Appendix 2

Publication II

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Stable cones of polynomials via Routh rays

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***Abstract*—A problem of inner convex approximation of a stability domain is addressed in the paper for continuous-time linear control systems. A constructive procedure for generating stable cones of polynomials is provided. The main idea is based on the construction of so-called Routh rays (stable half-lines) starting from a given stable point. These lines serve as edges for the corresponding polyhedral Routh cones inside the stability domain. Several numerical examples are presented to illustrate introduced concepts and effectiveness of the proposed approach.**

I. Introduction

Progress in modern technologies stimulates deeper research in the field of control systems such as robust controller design and robust stabilization. Such controller should be able to provide required behavior of the controlled system and at the same time it should ensure desired stability and performance level for plants with uncertain parameters [1].

Industry in majority of cases decides in favor of the low-order controllers because of their simplicity, low cost, high reliability and low maintenance [2]. Those are mostly PI- and PID-controllers having two and three free parameters, respectively. Indeed, such controllers might be easy to adjust, however, the amount of control parameters may be insufficient to obtain desired performance and a stabilizing control law for unstable higher-order plants.

The difficulty of designing fixed-order output controllers lies in the fact that the set of all stabilizing fixed-order controllers is non-convex in the space of controller parameters. And the fixed-order output controller design task reduces to finding a stable polynomial in an affine family, which is known to be NP-hard [3]. It is well-known that, in general, the stability domain is non-convex. Over the years, a lot of techniques in robust control were developed relying on convex approximations of the stability region such as ellipsoids [4], hyperrectangles [5] and polytopes [6], [7], [8].

The purpose of this work is to provide simple yet efficient enough algorithm for generation of convex sets of stable polynomials (so called polyhedral Routh cones). The randomized generation method of stable polynomials is thoroughly studied in [3], where the Levinsone-Durbin parametrization is suggested as an efficient and numerically stable method. We generalize this approach to generation of stable line segments (Routh rays) in the space of polynomial coefficients. It means that instead of a single stable polynomial of nth order, we generate n-bunches of stable line segments of polynomials. The method is based on a new stability criterion for Hurwitz polynomials introduced in [9]. The application of this method has shown good results in robust fixed-order controller design for discrete-time single-input single-output plants [10]. This paper employs the idea from [10]; however, addresses the stability problem for continuous-time systems. Furthermore, the results presented in [9], where the approximation of the inner stability domain was made by polytopes, are extended by complementary results, definitions and proofs.

The paper is organized as follows. Section II recalls basic definitions related to stability of polynomials in the continuous-time case. The notion of the reduced Routh parameters is introduced. It is followed by the description of stable half-lines (Routh rays) of polynomials. The main results related to the approximation of the stability domain by the polyhedral Routh cones are presented in Section III. The presented theory is illustrated by numerical examples. Concluding remarks and possible directions for the future research are drawn in the final section.

II. Reduced Routh parameters and stable Routh rays of polynomials

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

with coefficients $a_i \in \mathbb{R}$, for $i = 0, \dots, n$, is said to be continuous-time stable in the Hurwitz sense, if all its roots λ_i, for $i = 1, \dots, n$, are in the open left-half plane of $\mathbb{C}$, i.e., $\mathrm{Re}(\lambda_i) < 0$.

Since polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = [a_n \quad \cdots \quad a_0]^{\mathrm{T}}$ of its coefficients, i.e., $a := a(s) = [a_n \quad \cdots \quad a_0]^{\mathrm{T}}$. Then, the Hurwitz *region* $\mathcal{H}_n$ is defined as the set

$$\mathcal{H}_n = \left\{ a \in \mathbb{R}^{n+1} \mid a(s) \text{ is Hurwitz} \right\}.$$

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [11]. Based on this criterion, a method for constructing Hurwitz polynomials can be derived as follows [3]. Start with arbitrary Hurwitz polynomial of degree 2. Since positivity of the coefficients is equivalent to stability for the second-order polynomials, generate arbitrary positive numbers h_0, h_1, h_2 and compose the polynomial $a(s) = h_2 s^2 + h_1 s + h_0$ or $a = [a_2 \quad a_1 \quad a_0]^{\mathrm{T}} =$

$[h_2 \quad h_1 \quad h_0]^{\mathrm{T}}$. At the kth step, having a Hurwitz polynomial of degree k, i.e., $a(s) = [a_k \quad a_{k-1} \quad \cdots \quad a_0]^{\mathrm{T}}$, consider two polynomials of degree $k+1$

$$p(s) = [0 \quad a_k \quad a_{k-1} \quad \cdots \quad a_0]^{\mathrm{T}}$$

and

$$q(s) = [a_k \quad 0 \quad a_{k-2} \quad 0 \quad a_{k-4} \quad 0 \quad \cdots]^{\mathrm{T}}.$$

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{2}$$

which is Hurwitz polynomial of degree $k+1$, according to the Routh rule. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see [12], [13]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k = 0, \ldots, n$ recursively by increasing k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction [3].

Definition 1. *The reduced Routh parameters w_j for normed polynomials $a(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$ are defined as follows*

$$\begin{aligned} &w_0 = h_0 = 1, \quad w_1 = h_1, \quad w_2 = h_2, \\ &w_j = \frac{h_j}{h_{j-1}}, \quad j = 3, \ldots, n. \end{aligned} \tag{3}$$

From (2) and (3) we obtain relations for recursive generation of normed Hurwitz polynomials of order $k+1$, for $k > 2$, as $a(s) = p(s) + w_{k+1}q(s)$. Denote the degree of a polynomial by superscript to obtain

$$\begin{aligned} a^{k+1} = \big[w_k a_k^k \quad a_k^k \quad a_{k-1}^k + w_k a_{k-2}^k \quad a_{k-2}^k \\ a_{k-3}^k + w_k a_{k-4}^k \quad \cdots \quad 1\big]^{\mathrm{T}}, \end{aligned} \tag{4}$$

where $a^k = \big[a_k^k \quad a_{k-1}^k \quad \cdots \quad 1\big]^{\mathrm{T}}$. Using matrix notation, we can rewrite equation (4) as

$$a^{k+1} = W_k a^k, \tag{5}$$

where W_k is a $(k+1) \times k$ matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0^{\mathrm{T}} \\ \vdots \\ I_k \end{bmatrix}$$

with I_k being the $k \times k$ unit matrix and J_k being the $k \times k$ diagonal matrix $J_k = \mathrm{diag}\{1, 0, 1, 0, \ldots\}$. Next, using recursive relation (5), we obtain $a^n = W_k^n a^k$, where $W_k^n = W_n W_{n-1} \cdots W_k$ or

$$a^n = W_2^n a^2 = W_n W_{n-1} \cdots W_2 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}. \tag{6}$$

Example 1: Let us illustrate equation (6) on the basis of low order polynomials. In case $n = 3$, one gets

$$a^3 = W_2^3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix}$$

and $n = 4$ yields

$$a^4 = W_2^4 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_2 w_3 \\ w_2 \\ w_1 + w_3 \\ 1 \end{bmatrix} = \begin{bmatrix} w_2 w_3 w_4 \\ w_2 w_3 \\ w_2 + w_1 w_4 + w_3 w_4 \\ w_1 + w_3 \\ 1 \end{bmatrix}.$$

Observe that formula (6) is iterative. This means that in order to calculate elements of the resulting matrix, one needs to multiply n matrices, whose dimensions increase by 1 with each iteration. To simplify calculations we derived the direct formula that can be represented as

$$a_l^n = \sum_{i_0=1}^{n} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-l}=1}^{i_{n-l-1}} \prod_{j=0}^{n-l} \overline{w}_{i_j} \mod (i_j + n - l - j, 2), \tag{7}$$

where $l = 1, \ldots, n$ is the index number of the corresponding row in (6), $n > 2$, and $\mod(\alpha, 2)$ is the usual modulus operation that returns either 1 or 0 depending on whether the number α is odd or even, respectively. Elements $\overline{w}_{i_j}$ in (7) correspond to elements of the matrix W_k as

$$\overline{w}_{i_j} := \begin{cases} w_2/\overline{w}_1 & \text{for } i_j = 2, \\ w_{i_j} & \text{otherwise.} \end{cases}$$

Example 2: Let $n = 4$ and

$$w = [w_4 \quad w_3 \quad w_2 \quad w_1 \quad 1]^{\mathrm{T}} = [2 \quad 3 \quad 5 \quad 4 \quad 1]^{\mathrm{T}}.$$

Next, we calculate recursively the coefficients of polynomials. According to (5) and using the results from Example 1, for $k = 2, 3$ we get

$$a^2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \ a^3 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix}$$

and for $k = 4$

$$a^4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 19 \\ 7 \\ 1 \end{bmatrix}.$$

Note that the same result can be achieved using the direct formula (7). One can observe that for $l = 1$ the only combination of indices $i_0 = 4, i_1 = 3, i_2 = 2, i_3 = 1$ gives a nonzero addend, yielding $a_1^4 = \overline{\omega}_4 \overline{\omega}_3 \overline{\omega}_2 \overline{\omega}_1 = \omega_4 \omega_3 \omega_2 = 30$. Next, for $l = 2$ again the sum has a single nonzero addend with the combination of indices $i_0 = 3, i_1 = 2, i_2 = 1$, resulting in $a_2^4 = \overline{\omega}_3 \overline{\omega}_2 \overline{\omega}_1 = \omega_3 \omega_2 = 15$. In the case of $l = 3$ three combinations of indices, i.e., $i_0 = 4, i_1 = 3; i_0 = 4, i_1 = 1; i_0 = 2, i_1 = 1$, leads to nonzero addends. Thus, $a_3^4 = \overline{\omega}_4 \overline{\omega}_3 + \overline{\omega}_4 \overline{\omega}_1 + \overline{\omega}_2 \overline{\omega}_1 = \omega_4 \omega_3 + \omega_4 \omega_1 + \omega_2 = 19$. Finally, for $l = 4$ two values of index $i_0 = 3$ and $i_0 = 1$ correspond to nonzero addends, which yields $a_4^4 = \overline{\omega}_3 + \overline{\omega}_1 = \omega_3 + \omega_1 = 7$.

The inverse mapping from polynomial coefficients a_k to the reduced Routh parameters w_k, $k = n, \ldots, 1$ can be recursively found starting from w_n via (5) as follows

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2 = a_2^3, \\ w_1 &= a_1^2 = a_1^3 - \frac{a_3^3}{a_2^3}. \end{aligned} \tag{8}$$

Note that in (8) parameters a_j^j and a_{j-1}^j can be found explicitly as $a_{k-i-1}^{k-1} = a_{k-i-1}^k, i = 0, \ldots, 2\lfloor (k-2)/2 \rfloor$ and $a_{k-i-2}^{k-1} = a_{k-i-2}^k - w_k a_{k-i-3}^k, i = 0, \ldots, 2\lfloor (k-3)/2 \rfloor$ with $a_0 = 1$ or in the matrix form as $a^{k-1} = \overline{W}_k a^k$, where $\overline{W}_k$ is a $k \times k$ matrix

$$\overline{W}_k = I_k - w_k \begin{bmatrix} 0 & \overline{J}_{k-1} \\ \vdots & \vdots \\ 0 & 0^{\mathrm{T}} \end{bmatrix}$$

and $\overline{J}_k$ is a $k \times k$ diagonal matrix $\overline{J}_k = \operatorname{diag}\{0, 1, 0, 1, \ldots\}$.

Proposition 1 ([9]). *A normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable if and only if $w_k > 0$, $k = 1, \ldots, n$.*

Proposition 2 ([9]). *The mapping* (5) *from the reduced Routh parameters w_k, $k = 1, \ldots, n$ to the normed polynomial coefficients a_k^n, $k = 1, \ldots, n$ with $a_0 = 1$ is a one-to-one mapping if $w_k > 0$, $k = 1, \ldots, n$.*

Next, we recall from [9] the notion of stable line segments (half-lines) of polynomials that can be obtained starting from the reduced Routh parameters w_k, $k = 1, \ldots, n$ of a Hurwitz polynomial $a \in \mathcal{H}_n \subset \mathbb{R}^{n+1}$.

Theorem 1 ([9]). *Through an arbitrary Hurwitz stable point $a = [a_n \quad a_{n-1} \quad \cdots \quad a_1 \quad 1]^{\mathrm{T}}$ with reduced Routh parameters $w_k > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$ such that*

$$\mathcal{R}_k(a) = \{a \mid w_k \in (0, \infty), w_j = \text{const}, \; j \neq k; \, k, j \in \{1, \ldots, n\}\}. \tag{9}$$

Definition 2. *The half-lines $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ defined by* (9) *are called* Routh rays *of the polynomial $a(s)$. Moreover, their endpoints $v_k(a)$ such as $v_k(a) = a(w_k = 0)$ are supposed to be the Routh sources of the polynomial $a(s)$.*

III. Stable Routh cones of polynomials

Next, we study the stability of polynomials with conic uncertainty [14] by the use of Routh rays. We define so-called Routh cones[1] in the polynomial coefficient space $a \in \mathbb{R}^n$ starting from the reduced Routh parameter space $w \in \mathbb{R}^n$. Let $a^* \in \mathcal{H}_n$ be an arbitrary stable polynomial of order n and w^* its reduced Routh parameters.

Definition 3. 1) *A subset $\mathcal{K}_i(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a* Routh cone *of a polynomial $a^*(s)$ if it is closed under positive scalar multiplication of one of its reduced Routh parameters w_i^*, $i \in \{1, \ldots, n\}$, i.e., $a(w_i = \alpha w_i^*) \in \mathcal{K}_i$ when $a \in \mathcal{K}_i$ and $\alpha > 0$, where all the other reduced Routh parameters w_j, $j \neq i$, $j \in \{1, \ldots, n\}$ are fixed $w_j = w_j^*$.*

2) *If P is a subset of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$, then*

$$\mathcal{K}_i(P) = \{a(w_i = \alpha w_i); \; a \in P, \alpha > 0, i \in \{1, \ldots, n\}\}$$

is called the Routh cone generated *by P.*

3) *A convex cone $\mathcal{K}(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a* polyhedral Routh cone *of a polynomial $a^*(s)$, if there exist α_i, β_i, such that*

$$\mathcal{K}(a^*) = \left\{ \sum_{i=1}^n \beta_i a(\alpha_i w_i^*); \; \alpha_i > 1, 0 < \beta_i < 1, \right. \left. \sum_{i=1}^n \beta_i = 1, w_j = w_j^* = \text{const}, j \neq i, i = 1, \ldots, n \right\}.$$

4) *A convex cone $\mathcal{K}_{i,j}(a^*)$ of normed polynomial $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a* polyhedral Routh i, j-subcone *of a polynomial $a^*(s)$, if there exist α_i, β_i, such that*

$$\begin{aligned} \mathcal{K}_{i,j}(a^*) = \big\{ &\beta_i a(w_i = \alpha_i w_i^*, w_j = w_j^*) \\ &+ \beta_j a(w_j = \alpha_j w_j^*, w_i = w_i^*); \\ &\alpha_i, \alpha_j > 1, 0 < \beta_i, \beta_j < 1, \beta_i + \beta_j = 1, \\ &w_k = w_k^* = \text{const}, k \neq i, j; \, i, j, k \in \{1, \ldots, n\} \big\}. \end{aligned}$$

5) *A convex set $\overline{\mathcal{K}}_{j,k}^n(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a* truncated polyhedral Routh cone *of a polynomial $a^*(s)$, if there exist α_i, β_i, such that*

$$\begin{aligned} \overline{\mathcal{K}}_{j,k}^n(a^*) = \Bigg\{ &\sum_{i=1}^n \beta_i a(\alpha_i w_i^*); \; \alpha_i > 1, i \neq j, k; \\ &1 < \alpha_j < \overline{\alpha_j}, 1 < \alpha_k < \overline{\alpha_k}; \; 0 < \beta_i < 1, \sum_{i=1}^n \beta_i = 1, \\ &w_h = w_h^* = \text{const}, h \neq i, i = 1, \ldots, n \Bigg\}. \end{aligned}$$

Proposition 3. *An arbitrary subset P of normed polynomials $a(s)$ of degree n, $a(s) \in \mathbb{R}^n$ has n Routh cones $\mathcal{K}_i(P)$, $i = 1, \ldots, n$ generated by P. If the subset P is stable, then all Routh cones $\mathcal{K}_i(P)$ generated by P are stable.*

Proposition 4. *The n-times Routh cone of the polynomial $a(s)$ with $a_i \to 0$, $i = 1, ..., n$, generates the whole stability domain $\mathcal{A}$ in polynomial coefficient space, $\mathcal{A} \subset \mathbb{R}^n$.*

Theorem 2. *If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, \ldots, n\}$ of a stable polynomial $a^*(s)$ are stable, then the polyhedral Routh cone $\mathcal{K}(a^*)$ is stable.*

Proof: Indeed, if α_i and α_j, $1 < \alpha_i, \alpha_j < \infty$ are fixed, then the polyhedral Routh cone $\mathcal{K}(a^*)$ is a polytope with $n + 1$ vertices a^* and $a(w_k = \alpha_k w_k^*, w_j = w_j^*)$, $j \neq k$, $k = 1, \ldots, n$. The edges $\operatorname{conv}\{a^*, a(w_k)\}$ are stable as Routh

[1] Note that the notion cone is used in consonance with results in [14]. In our paper definition of Routh cone coincides with that of the Routh ray.

rays of a stable point a^*. The edges $\mathrm{conv}\{a(w_k), a(w_j)\}$ are stable, since $\mathrm{conv}\{a(w_k), a(w_j)\} \subset \mathcal{K}_{k,j}(a^*)$ for arbitrary $1 < \alpha_{k,j} < \infty$. Thus, it remains to note that by Edge Theorem the polytope is stable for $1 < \alpha_i, \alpha_j < \infty$, since all the edges of the polytope are stable [15]. ∎

Let $\Gamma = \{1, \ldots, n\}$ be a set of integers. Rewrite it as $\Gamma = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are sets that contain indices corresponding to ordinary and truncated Routh subcones, respectively, with $\dim \gamma_1 = m_1$ and $\dim \gamma_2 = m_2$ such that $m_1 + m_2 = n$.

Theorem 3. *A truncated polyhedral Routh cone $\overline{\mathcal{K}}^n_{i_j}(a^*)$, $i_j \in \gamma_2$, $j = 1, \ldots, m_2$ of a stable polynomial $a^*(s)$ is stable if the following conditions hold:*

1) *the polyhedral Routh subcones $\mathcal{K}_{r,s}(a^*)$, $r, s \in \gamma_1$ are stable;*
2) *the line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$, $u, v \in \gamma_2$ are stable, where*

$$\begin{aligned} S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v) = \mathrm{conv}\{&a(w_u = \overline{\alpha}_{u,\min} w_u^*, \\ &a(w_v = \overline{\alpha}_{v,\min} w_v^*), w_i = w_i^*, i \neq u, v\} \end{aligned}$$

and $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$.

Proof: Indeed, if α_r and α_s, $1 < \alpha_r, \alpha_s \le \infty$ are fixed, then the truncated polyhedral Routh cone $\overline{\mathcal{K}}^n_{i_j}(a^*)$ is a polytope with $n+1$ vertices a^*, $a(w_u, \overline{\alpha}_u)$, $a(w_v, \overline{\alpha}_v)$ and $a(w_r = \alpha_r w_k^*, w_l = w_l^*)$, $l \neq r$, $l \in \{1, \ldots, n\}$, $a(w_s = \alpha_s w_k^*, w_l = w_l^*)$, $l \neq s$, $l \in \{1, \ldots, n\}$. The edges $\mathrm{conv}\{a^*, a(w_u, \overline{\alpha}_u)\}$, $\mathrm{conv}\{a^*, a(w_v, \overline{\alpha}_v)\}$, $\mathrm{conv}\{a^*, a(w_r)\}$, and $\mathrm{conv}\{a^*, a(w_s)\}$ are stable as the Routh rays of a stable point a^*. The edges $\mathrm{conv}\{a(w_r), a(w_s)\}$ are stable, since $\mathrm{conv}\{a(w_r), a(w_s)\} \subset \mathcal{K}_{r,s}(a^*)$ for arbitrary $1 < \alpha_r, \alpha_s < \infty$. It follows from condition 2) that the edges $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$ are stable. Hence, by Edge Theorem the polytope is stable for $1 < \alpha_r, \alpha_s < \infty$, since all edges of the polytope are stable [15]. ∎

Proposition 5. *For $n = 3$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of an arbitrary stable polynomial $a^*(s)$ is stable.*

Proof: Assume without loss of generality that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. Then, by (5) we obtain the Routh cones $\mathcal{K}_i(a^*)$, $i = 1, 2, 3$ for the polynomial $a^*(s)$

$$\begin{aligned} \mathcal{K}_1(a^*) &= \begin{bmatrix} w_2^* w_3^* & w_2^* & \alpha w_1^* + w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{K}_2(a^*) &= \begin{bmatrix} \alpha w_2^* w_3^* & \alpha w_2^* & w_1^* + w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{K}_3(a^*) &= \begin{bmatrix} \alpha w_2^* w_3^* & w_2^* & w_1^* + \alpha w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

where $\alpha > 1$ and w_1^*, w_2^*, w_3^* are the reduced Routh parameters of the polynomial $a^*(s)$.

Let $a \in \mathcal{K}(a^*)$ be an inner point of the polyhedral Routh cone $\mathcal{K}(a^*)$. Then, the convex combination can be expressed as

$$a = \beta_1 \mathcal{K}_1(a^*) + \beta_2 \mathcal{K}_2(a^*) + \beta_3 \mathcal{K}_3(a^*),$$

where $0 < \beta_i < 1, \sum_{i=1}^3 \beta_i = 1$ or in the explicit form as

$$a = \begin{bmatrix} (\beta_1 + \beta_2 \alpha + \beta_3 \alpha) w_2^* w_3^* \\ (\beta_1 + \beta_2 \alpha + \beta_3) w_2^* \\ (\beta_1 \alpha + \beta_2 + \beta_3) w_1^* + (\beta_1 + \beta_2 + \beta_3 \alpha) w_3^* \\ 1 \end{bmatrix}.$$

Note that, according to Proposition 1, polynomial $a(s)$ is stable if the reduced Routh parameters $w_i > 0$, $i = 1, 2, 3$. From (8) one obtains

$$w_3 = \frac{(\beta_1 + \beta_2 \alpha + \beta_3 \alpha) w_2^* w_3^*}{(\beta_1 + \beta_2 \alpha + \beta_3) w_2^*}.$$

Observe that, according to Proposition 1, the reduced Routh parameters w_i^*, $i = 1, 2, 3$, of the stable polynomial $a^*(s)$ are positive. Moreover, $\alpha > 1$ and $\beta_i > 0$, yielding $w_3 > 0$. Similarly, from (8), one obtains

$$w_2 = (\beta_1 + \beta_2 \alpha + \beta_3) w_2^* > 0$$

and

$$\begin{aligned} w_1 &= (\beta_1 \alpha + \beta_2 + \beta_3) w_1^* \\ &\quad + (\beta_1 + \beta_2 + \beta_3 \alpha) w_3^* - \frac{(\beta_1 + \beta_2 \alpha + \beta_3 \alpha) w_2^* w_3^*}{(\beta_1 + \beta_2 \alpha + \beta_3) w_2^*}. \end{aligned}$$

The latter after simple algebraic manipulations yields

$$\begin{aligned} w_1 = \big(&(\beta_1 + \beta_2 \alpha + \beta_3) w_2^*\big)^{-1} \cdot \\ &\big((\beta_1 \alpha + \beta_2 + \beta_3)(\beta_1 + \beta_2 \alpha + \beta_3) w_1^* w_2^* \\ &+ (1 - \alpha)^2 \beta_2 \beta_3 w_2^* w_3^*\big) > 0. \end{aligned}$$

∎

Proposition 6. *The polyhedral subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable.*

Proof: By (5) we obtain the following Routh cones $\mathcal{K}_i(a^*)$, $i = 1, 2, 3$ for the polynomial $a^*(s)$, $a \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{K}_1(a^*) &= W_4^n(a^*) \begin{bmatrix} w_2^* w_3^* & w_2^* & \alpha w_1^* + w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{K}_2(a^*) &= W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* & \alpha w_2^* & w_1^* + w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \\ \mathcal{K}_3(a^*) &= W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* & w_2^* & w_1^* + \alpha w_3^* & 1 \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

where $W_4^n(a^*) := W_n(a^*) \cdots W_4(a^*)$ and $\alpha > 1$.

Let $a \in \mathcal{K}_{1,2}(a^*)$ be an inner point of the polyhedral Routh subcone $\mathcal{K}_{1,2}(a^*)$. Then, the convex combination can be expressed as

$$a = \beta \mathcal{K}_1(a^*) + (1 - \beta) \mathcal{K}_2(a^*),$$

where $0 < \beta < 1$, or explicitly

$$a = W_n(a^*) \cdots W_4(a^*) \begin{bmatrix} (\beta + (1 - \beta)\alpha) w_2^* w_3^* \\ (\beta + (1 - \beta)\alpha) w_2^* \\ (\beta \alpha + 1 - \beta) w_1^* + w_3^* \\ 1 \end{bmatrix}.$$

Observe that the reduced Routh parameters $w_n, \ldots, w_4$ of a polynomial $a(s)$ are determined by the product of matrix multiplication $W_n(a^*) \cdots W_4(a^*)$, i.e., $w_i = w_i^*$, $i = 4, \ldots, n$. For the reduced Routh parameters w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathcal{K}_{1,2}(a^*)$, using (8), we obtain the following relations

$$\begin{aligned} w_2 w_3 &= (\beta + (1 - \beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta + (1 - \beta)\alpha) w_2^*, \\ w_1 + w_3 &= (\beta \alpha + 1 - \beta) w_1^* + w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta)w_1^*, \\ w_2 &= (\beta + (1-\beta)\alpha)w_2^*, \\ w_3 &= w_3^*. \end{aligned}$$

Observe next that $\alpha > 1$, $0 < \beta < 1$ and $w_i^* > 0$, $i = 1, \ldots, n$. Then $w_i > 0$, $i = 1, \ldots, n$, i.e., $a \in \mathcal{K}_{1,2}(a^*)$ are stable.

In the similar manner we obtain for $a \in \mathcal{K}_{1,3}(a^*)$ the reduced Routh parameters $w_n, \ldots, w_4$, $w_i = w_i^*$, $i = 4, \ldots, n$. For w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathcal{K}_{1,3}(a^*)$ we obtain by (8) the following relations

$$\begin{aligned} w_2 w_3 &= (\beta + (1-\beta)\alpha)w_2^* w_3^*, \\ w_2 &= w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta)w_1^* + (\beta + (1-\beta)\alpha)w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta)w_1^* > 0, \\ w_2 &= w_2^* > 0, \\ w_3 &= (\beta + (1-\beta)\alpha)w_3^* > 0. \end{aligned}$$

Finally, for $a \in \mathcal{K}_{2,3}(a^*)$ we obtain the reduced Routh parameters $w_i = w_i^*$, $i = 4, \ldots, n$ and for w_i, $i = 1, \ldots, 3$

$$\begin{aligned} w_2 w_3 &= (\beta\alpha + (1-\beta)\alpha)w_2^* w_3^*, \\ w_2 &= (\beta\alpha + (1-\beta))w_2^*, \\ w_1 + w_3 &= w_1^* + (\beta + (1-\beta)\alpha)w_3^* \end{aligned}$$

that after simple algebraic manipulations yield

$$\begin{aligned} w_1 &= w_1^* + \frac{(\beta(1-\beta)(1-\alpha)^2)w_3^*}{\beta\alpha + (1-\beta)} > 0, \\ w_2 &= (\beta\alpha + 1 - \beta)w_2^* > 0, \\ w_3 &= \frac{\alpha w_3^*}{\beta\alpha + 1 - \beta} > 0. \end{aligned}$$

Hence, all polyhedral subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable. ■

We suggest the following algorithm for solving the problem of generating stable truncated polyhedral Routh cones.

Algorithm:

Step 1. Start from a given n degree stable polynomial $a(s)$, or $a_n = [a_n^n \quad a_{n-1}^n \quad \cdots \quad a_1^n \quad 1]$.

Step 2. Find by (8) the reduced Routh parameters w_k, $k = n, \ldots, 1$ of the polynomial $a(s)$.

Step 3. Find by (9) the Routh rays $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ of the polynomial $a(s)$.

Step 4. Check the stability of all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \ldots, n\}$ of the polynomial $a(s)$ by Hurwitz Segment Lemma [1, p.81]. By Proposition 6 the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, 2, 3\}$ are stable. If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \ldots, n\}$ are stable, then by Theorem 2 the polyhedral Routh cone $\mathcal{K}(a)$ is stable.

Step 5. If some of the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \ldots, n\}$ are not stable, then find the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$ according to Theorem 3 with appropriate values of $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$ and $\overline{\alpha}_{v,\min} = \min_v \overline{\alpha}_v$.

Step 6. According to Theorem 3 the stable truncated polyhedral Routh cone $\overline{\mathcal{K}}^n(a)$ of the polynomial $a(s)$ is determined by the stable polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, \ldots, n\}$ and the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$.

Next, we consider an example that is designed to illustrate the approximation of stability domain by polyhedral Routh cone. Note that we deal with four dimensional coefficient space, and therefore, the presented figure (Fig. 1) has to be understood as a schematic illustration of the overall procedure.

Example 3: Consider the polynomial from Example 2

$$a^* = [30 \quad 15 \quad 19 \quad 7 \quad 1]^{\mathrm{T}},$$

whose reduced Routh parameters are

$$w^* = [w_4^* \quad w_3^* \quad w_2^* \quad w_1^* \quad 1]^{\mathrm{T}} = [2 \quad 3 \quad 5 \quad 4 \quad 1]^{\mathrm{T}}.$$

Then, according to 1) and 2) from Definition 3, Routh cones can be calculated as follows.

Cone $\boldsymbol{\mathcal{K}_1}$: $w_1 = \alpha_1 w_1^*$, $1 < \alpha_1 < \infty$, and

$$a^2 = \begin{bmatrix} 5 \\ 4\alpha_1 \\ 1 \end{bmatrix}, \quad a^3 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4\alpha_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 4\alpha_1 + 3 \\ 1 \end{bmatrix},$$

yielding

$$\mathcal{K}_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 5 \\ 4\alpha_1 + 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 8\alpha_1 + 11 \\ 4\alpha_1 + 3 \\ 1 \end{bmatrix}.$$

Cone $\boldsymbol{\mathcal{K}_2}$: $w_2 = \alpha_2 w_2^*$, $1 < \alpha_2 < \infty$, and

$$a^2 = \begin{bmatrix} 5\alpha_2 \\ 4 \\ 1 \end{bmatrix}, \quad a^3 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5\alpha_2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15\alpha_2 \\ 5\alpha_2 \\ 7 \\ 1 \end{bmatrix},$$

yielding

$$\mathcal{K}_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 15\alpha_2 \\ 5\alpha_2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 30\alpha_2 \\ 15\alpha_2 \\ 5\alpha_2 + 14 \\ 7 \\ 1 \end{bmatrix}.$$

Cone $\boldsymbol{\mathcal{K}_3}$: $w_3 = \alpha_3 w_3^*$, $1 < \alpha_3 < \infty$, and

$$a^2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \quad a^3 = \begin{bmatrix} 3\alpha_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3\alpha_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15\alpha_3 \\ 5 \\ 4 + 3\alpha_3 \\ 1 \end{bmatrix},$$

yielding

$$\mathcal{K}_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 15\alpha_3 \\ 5 \\ 4 + 3\alpha_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 30\alpha_3 \\ 15\alpha_3 \\ 6\alpha_3 + 13 \\ 3\alpha_3 + 4 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_4$: $w_4 = \alpha_4 w_4^*$, $1 < \alpha_4 < \infty$, and

$$a^2 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \quad a^3 = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix},$$

yielding

$$\mathcal{K}_2 = \begin{bmatrix} 2\alpha_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2\alpha_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 15 \\ 5 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 30\alpha_4 \\ 5 \\ 14\alpha_4 + 5 \\ 7 \\ 1 \end{bmatrix}.$$

Next, between obtained Routh cones we can draw six polyhedral Routh subcones. According to Proposition 6, the polyhedral Routh subcones $\mathcal{K}_{1,2}(a^*)$, $\mathcal{K}_{1,3}(a^*)$, and $\mathcal{K}_{2,3}(a^*)$ are stable. In addition, one may easily check that, according to Edge Theorem, $\mathcal{K}_{2,4}(a^*)$ and $\mathcal{K}_{3,4}(a^*)$ are stable as well. Note that the remaining subcone $\mathcal{K}_{1,4}(a^*)$ is not stable, whereas the truncated polyhedral Routh subcone $\overline{\mathcal{K}}_{1,4}(a^*)$ is stable for $\overline{\alpha}_1 = 2$ and $\overline{\alpha}_4 = 128$. Next, using obtained information, we can represent the inner approximation of the stability domain schematically by a truncated polyhedral Routh cone (Fig. 1).

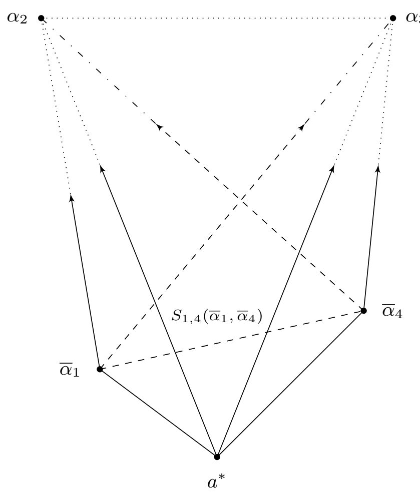

Fig. 1. Inner approximation of a stability domain by the truncated polyhedral Routh cone $\overline{\mathcal{K}}_{1,4}(a^*)$. Dotted and/or dashed style indicate that rays go to infinity and/or hidden, respectively.

IV. Conclusion

In this paper, the problem of convex approximation of the stability domain by a polyhedral Routh cone $\mathcal{K}(a^*)$ is considered. The novel approach based on the introduced notion of the reduced Routh parameters is presented. The main idea can be summarized as follows. First, one has to calculate the reduced Routh parameters. Then, through the initial stable point one may draw exactly n Routh rays (cones), which correspond to edges of polyhedral Routh subcones. If the constructed polyhedral Routh subcones are stable, then the whole polyhedral Routh cone is stable. If this is not the case, then it was explained how to calculate the stable truncated polyhedral Routh cone.

To conclude, this paper provides complementary results to those presented in [9]. Sometimes when the starting stable point a^* is placed too close to the boundary of the stability domain, the approximation method based on polyhedral Routh cones may provide a better result than the polytope generated by Routh rays. This is due to the fact that some edges of a polytope may fail to be stable or the volume of the stable polytope may appear to be small. The comparison of two techniques will make the subject for the future research.

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Appendix 3

Publication III

I. Artemchuk, Ü. Nurges, and J. Belikov. Robust pole assignment via Routh rays of polynomials. In *The 55th American Control Conference*, pages 7031–7036, Boston, MA, USA, July 2016

Robust pole assignment via Routh rays of polynomials

Igor Artemchuk[2] and Ülo Nurges[1] and Juri Belikov[1]

Abstract— **The paper presents a constructive procedure for robust output controller design for continuous-time linear systems. The approach is based on the so-called reduced Routh parameters that are used to derive stable Routh rays and corresponding Routh cones of polynomials (polyhedral Routh cones), which approximate stability domain. The obtained region is used to designed a fixed-order controller. The procedures of pole placement and robust controller synthesis are described and summarized in the form of step-by-step algorithm. Theoretical results are illustrated by two academic example and laboratory prototype of a DC motor servo system.**

I. INTRODUCTION

One of the common approaches in control theory to design a closed-loop controller for a continuous-time linear system may be seen in using pole placement or modal control methods. In case of a state feedback it is always possible to predefine arbitrary poles whenever the system is controllable. On contrary in case of the output feedback, in general, it is not possible to solve arbitrary pole assignment task relying on a fixed-order controller. Thus, alternative approaches based on the idea of placing poles of the closed-loop system in a suitable region of the complex plane were studied by different researchers [1].

It is well known that majority of practical systems are exposed to uncertainties. For example, most of sensors perform measurements with a certain level of precision. The problem of suppressing uncertainties can be solved by sensitivity-based methods that are however applicable in case of relatively small deviations only. For the models with large uncertainty there is a clear need for some robust formulations, such as multimodel [2], polytopic model [3], [4] or LMI approach [5]. In fact, the task of assigning poles of the closed-loop system can be replaced by an equivalent problem of assigning coefficients of characteristic polynomial, since coefficients are simply related to controller and plant parameters. However, the stability domain in the space of controller parameters is nonconvex, in general. This is the reason why, over the years, a lot of techniques in robust control, relying on convex inner approximations of the stability domain, were developed and taken into use, such as ellipsoids [6], [7], hyperrectangles [8] and polytopes [3], [4].

In this paper, we present a simple and efficient algorithm to design a robust output controller for continuous-time plant with uncertainties. Our method is based on a new stability criterion for Hurwitz polynomials. We move from arbitrary Hurwitz polynomial of order n to the construction of bunches of stable half-lines in the polynomial coefficient space (so-called polyhedral Routh cones) and obtain an inner approximation of the stability region. Similar approach for robust controller design using reflection coefficients was used in [9] for discrete-time systems. We start from a stable simplex (or polytope) of closed-loop characteristic polynomials, which is defined via Routh rays of a preselected Hurwitz stable polynomial. Next, we define the set of possible plant parameters as a convex polytope (polytopic plant model). Proceeding this way, we can determine properties that are common to all elements in the set from the analysis of its vertices only. Thus, the number of vertices of the polytope determines the complexity of computations. Then, we solve robust output controller for polytopic plant model design task by quadratic programming approach.

The paper is organized as follows. In Section II the main notions and definitions regarding Routh reduced parameters are presented. The results related to the approximation of the stability domain by the polyhedral Routh cones are presented in Section III. The problem of fixed-order robust output control with a preselected simplex is stated and solved by quadratic programming approach in the next section. Then, the presented theory is illustrated by examples. Concluding remarks and possible directions for the future research are drawn in the final section.

II. REDUCED ROUTH PARAMETERS AND STABLE ROUTH RAYS OF POLYNOMIALS

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

with coefficients $a_i \in \mathbb{R}$, for $i = 0, \ldots, n$, is said to be continuous-time stable in the Hurwitz sense, if all its roots λ_i, for $i = 1, \ldots, n$, are in the open left-half plane of $\mathbb{C}$, i.e., $\mathrm{Re}(\lambda_i) < 0$. Since polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$ of its coefficients, i.e., $a := a(s) = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$. Then, the Hurwitz *region* $\mathcal{H}_n$ is defined as the set $\mathcal{H}_n = \left\{ a \in \mathbb{R}^{n+1} \mid \text{(1) is Hurwitz} \right\}$.

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [10]. Based on this criterion, a method for constructing Hurwitz polynomials (1) can be derived as follows [11]. Start with arbitrary Hurwitz polynomial of degree 2. Since positivity of

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the coefficients is equivalent to stability of the second-order polynomials, generate arbitrary positive numbers h_0, h_1, h_2 and compose the polynomial $a(s) = h_2 s^2 + h_1 s + h_0$ or $a = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} h_2 & h_1 & h_0 \end{bmatrix}^{\mathrm{T}}$. At the kth step, having a Hurwitz polynomial of degree k, i.e., $a(s) = \begin{bmatrix} a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$, consider two polynomials of degree $k+1$, e.g. $p(s) = \begin{bmatrix} 0 & a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$ and $q(s) = \begin{bmatrix} a_k & 0 & a_{k-2} & 0 & a_{k-4} & 0 & \cdots \end{bmatrix}^{\mathrm{T}}$.

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{2}$$

which is a Hurwitz polynomial of degree $k+1$, according to the Routh rule. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see [12]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k = 0, \ldots, n$ recursively by increasing k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction [11]. Next, let us introduce the reduced Routh parameters that are used later in construction of the stable line segments.

Definition 1: The *reduced* Routh parameters w_j for a normed polynomial $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1$ are defined as follows

$$w_0 = h_0 = 1,\ w_1 = h_1,\ w_2 = h_2,$$
$$w_j = \frac{h_j}{h_{j-1}}, \quad j = 3, \ldots, n.$$

The relation for recursive generation of normed Hurwitz polynomials of order $k+1$, for $k > 2$ is given as $a(s) = p(s) + w_{k+1} q(s)$. Denote the degree of a polynomial by superscript to obtain

$$a^{k+1} = \begin{bmatrix} w_k a_k^k & a_k^k & a_{k-1}^k + w_k a_{k-2}^k & a_{k-2}^k \\ & & a_{k-3}^k + w_k a_{k-4}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}, \tag{3}$$

where $a^k = \begin{bmatrix} a_k^k & a_{k-1}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$. Using matrix notation, equation (3) can be rewritten as

$$a^{k+1} = W_k a^k, \tag{4}$$

where W_k is a $(k+1) \times k$ matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0^{\mathrm{T}} \\ \vdots \\ I_k \end{bmatrix}$$

with I_k being the $k \times k$ unit matrix and J_k being the $k \times k$ diagonal matrix $J_k = \mathrm{diag}\{1, 0, 1, 0, \ldots\}$. Next, using recursive relation (4), we obtain $a^n = W_k^n a^k$, where $W_k^n = W_n W_{n-1} \cdots W_k$ or

$$a^n = W_2^n a^2 = W_n W_{n-1} \cdots W_2 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}.$$

The inverse mapping from polynomial coefficients a_k to the reduced Routh parameters w_k, $k = n, \ldots, 1$ can be recursively found starting from w_n via (4) as follows

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2 = a_2^3, \\ w_1 &= a_1^2 = a_1^3 - \frac{a_3^3}{a_2^3}. \end{aligned} \tag{5}$$

Note that in (5) parameters a_j^j and a_{j-1}^j can be found explicitly as $a_{k-i-1}^{k-1} = a_{k-i-1}^k$, $i = 0, \ldots, 2\lfloor (k-2)/2 \rfloor$ and $a_{k-i-2}^{k-1} = a_{k-i-2}^k - w_k a_{k-i-3}^k$, $i = 0, \ldots, 2\lfloor (k-3)/2 \rfloor$ with $a_0 = 1$ or in the matrix form as $a^{k-1} = \overline{W}_k a^k$, where $\overline{W}_k$ is a $k \times k$ matrix

$$\overline{W}_k = I_k - w_k \begin{bmatrix} 0 & \overline{J}_{k-1} \\ \vdots & \vdots \\ 0 & 0^{\mathrm{T}} \end{bmatrix}$$

and $\overline{J}_k$ is a $k \times k$ diagonal matrix $\overline{J}_k = \mathrm{diag}\{0, 1, 0, 1, \ldots\}$.

Proposition 1 ([13]): A normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable if and only if $w_k > 0$, $k = 1, \ldots, n$.

Proposition 2 ([13]): The mapping (4) from the reduced Routh parameters w_k, $k = 1, \ldots, n$ to the normed polynomial coefficients a_k^n, $k = 1, \ldots, n$ with $a_0 = 1$ is a one-to-one mapping if $w_k > 0$, $k = 1, \ldots, n$.

Next, we recall from [13] the notion of stable line segments (half-lines) of polynomials that can be obtained starting from the reduced Routh parameters w_k, $k = 1, \ldots, n$ of a Hurwitz polynomial $a \in \mathcal{H}_n \subset \mathbb{R}^{n+1}$.

Theorem 1 ([13]): Through an arbitrary Hurwitz stable point $a = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{\mathrm{T}}$ with reduced Routh parameters $w_k > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$ such that

$$\begin{aligned} \mathcal{R}_k(a) = \{a \mid w_k \in (0, \infty), w_j = \text{const}, \\ j \neq k;\ k, j \in \{1, \ldots, n\}\}. \end{aligned} \tag{6}$$

Definition 2: The half-lines $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ defined by (6) are called Routh rays of the polynomial $a(s)$. Moreover, their endpoints $v_k(a)$ such as $v_k(a) = a(w_k = 0)$ are supposed to be the *Routh sources* of the polynomial $a(s)$.

III. Stable Routh cones of polynomials

Next, we study the stability of polynomials with conic uncertainty [14] by the use of Routh rays. We define so-called Routh cones in the polynomial coefficient space $a \in \mathbb{R}^n$ starting from the reduced Routh parameter space $w \in \mathbb{R}^n$. Let $a^* \in \mathcal{H}_n$ be an arbitrary stable polynomial of order n and w^* its reduced Routh parameters.

Definition 3: 1) A subset $\mathcal{K}_i(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a Routh cone of a polynomial $a^*(s)$ if it is closed under positive scalar multiplication of one of its reduced Routh parameters w_i^*, $i \in \{1, \ldots, n\}$, i.e., $a(w_i = \alpha w_i^*) \in \mathcal{K}_i$ when $a \in \mathcal{K}_i$ and $\alpha > 0$, where all the other reduced Routh parameters w_j, $j \neq i$, $j \in \{1, \ldots, n\}$ are fixed $w_j = w_j^*$.

2) If P is a subset of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$, then

$$\mathcal{K}_i(P) = \{a(w_i = \alpha w_i);\ a \in P, \alpha > 0, i \in \{1, \ldots, n\}\}$$

is called the Routh cone generated by P.

3) A convex cone $\mathcal{K}(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\mathcal{K}(a^*) = \left\{ \sum_{i=1}^{n} \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, 0 < \beta_i < 1, \sum_{i=1}^{n} \beta_i = 1, w_j = w_j^* = \text{const}, j \neq i, i = 1, \ldots, n \right\}.$$

4) A convex cone $\mathcal{K}_{i,j}(a^*)$ of normed polynomial $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh i,j-subcone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\begin{aligned} \mathcal{K}_{i,j}(a^*) = \{&\beta_i a(w_i = \alpha_i w_i^*, w_j = w_j^*) \\ &+ \beta_j a(w_j = \alpha_j w_j^*, w_i = w_i^*); \\ &\alpha_i, \alpha_j > 1, 0 < \beta_i, \beta_j < 1, \beta_i + \beta_j = 1, \\ &w_k = w_k^* = \text{const}, k \neq i, j;\ i, j, k \in \{1, \ldots, n\}\}. \end{aligned}$$

5) A convex set $\overline{\mathcal{K}}^n_{j,k}(a^*)$ of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$ is said to be a truncated polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\overline{\mathcal{K}}^n_{j,k}(a^*) = \left\{ \sum_{i=1}^{n} \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, i \neq j, k; 1 < \alpha_j < \overline{\alpha_j}, 1 < \alpha_k < \overline{\alpha_k};\ 0 < \beta_i < 1, \sum_{i=1}^{n} \beta_i = 1, w_h = w_h^* = \text{const}, h \neq i, i = 1, \ldots, n \right\}.$$

Remark 1: According to Theorem 1, it is possible to draw n stable Routh rays $\mathcal{R}_i(a^*)$ through an arbitrary stable point a^*. In [13] it was shown that if the point is not placed on the boundary of stability domain, then there are *positive* and *negative* directions with respect to a^*. The positive part of a Routh ray corresponds to $\alpha_i \in (1, \infty)$ while the negative to $\alpha_i \in (0, 1)$, and for $\alpha_i = 1$ rays intersect at the point a^*. In this paper notions of Routh rays and Routh cones $\mathcal{K}_i(a^*)$ coincide for positive direction. Therefore, the point a^* should be understood as a vertex of the polyhedral Routh cone.

Proposition 3: An arbitrary subset P of normed polynomials $a(s)$ of degree n, $a(s) \in \mathbb{R}^n$ has n Routh cones $\mathcal{K}_i(P)$, $i = 1, \ldots, n$ generated by P. If the subset P is stable, then all Routh cones $\mathcal{K}_i(P)$ generated by P are stable.

Theorem 2: If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, \ldots, n\}$ of a stable polynomial $a^*(s)$ are stable, then the polyhedral Routh cone $\mathcal{K}(a^*)$ is stable.

Let $\Gamma = \{1, \ldots, n\}$ be a set of integers. Rewrite it as $\Gamma = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are sets that contain indices corresponding to ordinary and truncated Routh subcones, respectively, with $\dim \gamma_1 = m_1$ and $\dim \gamma_2 = m_2$ such that $m_1 + m_2 = n$.

Theorem 3: A truncated polyhedral Routh cone $\overline{\mathcal{K}}^n_{i_j}(a^*)$, $i_j \in \gamma_2$, $j = 1, \ldots, m_2$ of a stable polynomial $a^*(s)$ is stable if the following conditions hold:

1) the polyhedral Routh subcones $\mathcal{K}_{r,s}(a^*)$, $r, s \in \gamma_1$ are stable;
2) the line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$, $u, v \in \gamma_2$ are stable, where

$$\begin{aligned} S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v) = \text{conv}\{&a(w_u = \overline{\alpha}_{u,\min} w_u^*, \\ &a(w_v = \overline{\alpha}_{v,\min} w_v^*), w_i = w_i^*, i \neq u, v\} \end{aligned}$$

and $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$.

Proposition 4: For $n = 3$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of an arbitrary stable polynomial $a^*(s)$ is stable.

Proposition 5: The polyhedral subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable.

IV. Fixed-order pole assignment

Assume that a plant with parametric uncertainties is given. Our goal is to design an output controller of a fixed-order so that the closed-loop poles are robustly assigned in a specific region approximated by the Routh cone, explained in the previous section. For simplicity, let us first consider the problem of PID-controller design for a SISO plant with fixed parameters. Let the plant transfer function $H(s)$ of order m be given

$$H(s) = \frac{g(s)}{f(s)} = \frac{g_{m-1}s^{m-1} + \cdots + g_1 s + 1}{f_m s^m + \cdots + f_1 s + f_0} \qquad (7)$$

and we are looking for a PID-controller $C(s)$ of an order $l = 2$ with the transfer function

$$C(s) = K_P + K_I \frac{1}{s} + K_D s$$

or

$$C(s) = \frac{q(s)}{p(s)} = \frac{q_2 s^2 + q_1 s + 1}{p_1 s}.$$

It means that the closed-loop characteristic polynomial

$$a(s) = f(s)p(s) + g(s)q(s) \qquad (8)$$

is of degree $n = m + l = m + 2$.

It is known in the literature [15] that when $l = m - 1$ the above problem has a solution for arbitrary $a(s)$ whenever the plant has no common pole-zero pairs. In general, for $l < m - 1$ exact attainment of the desired polynomial is impossible. Here we suggest the following approach. Let us relax the requirement of attaining the desired polynomial $a(s)$ exactly and enlarge the target to a simplex S in polynomial coefficient space containing the point representing the desired closed-loop characteristic polynomial. Without any restrictions we can assume that $g_0 = q_0 = 1$ and consider further normed polynomials $a(s)$ with $a_0 = 1$.

Let us now introduce a stability measure ρ in accordance with the simplex S as $\rho = c^{\mathrm{T}}c$, where $c = S^{-1}a$ and S is the $(m+l+1) \times (m+l+1)$ matrix of vertices s_i of the target simplex

$$S = \begin{bmatrix} s_1 & \cdots & s_{n+1} \end{bmatrix}. \tag{9}$$

Observe that for normed polynomials $a_0 = s_{i0} = 1$, $i = 1, \dots, n+1$, $\sum_{i=1}^{n+1} c_i = 1$, where $n = m + l$. If all coefficients $c_i > 0$, $i = 1, \dots, n+1$, then the point a is placed inside the simplex S. It is easy to see that the minimum of ρ is obtained by

$$c_1 = c_2 = \cdots = c_{n+1} = \frac{1}{n+1}.$$

Then, the point a is placed in the center of the simplex S.

Now we are ready to state the following problem of controller design: find a controller $C(s)$ such that the stability measure ρ is minimal. In other words, we are looking for a controller which places the closed-loop characteristic polynomial $a(s)$ as close as possible to the center of the target simplex S. In the matrix form we have $a = Gx$, where G is the plant Sylvester matrix

$$G = \begin{bmatrix} f_m & 0 & g_{m-1} & g_{m-2} \\ \vdots & \vdots & \vdots & \vdots \\ f_2 & g_3 & g_2 & g_1 \\ f_1 & g_2 & g_1 & 1 \\ f_0 & g_1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

of dimension $(m+2) \times 4$ and x is a vector of controller parameters $x = \begin{bmatrix} p_1 & 1 & q_1 & q_2 \end{bmatrix}^{\mathrm{T}}$.

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$J = \min_x x^{\mathrm{T}} G^{\mathrm{T}} (SS^{\mathrm{T}})^{-1} Gx \tag{11}$$

is obtained subject to the linear restrictions

$$S^{-1}Gx > 0. \tag{12}$$

Note that restrictions (12) follow from the positivity requirement of coefficients c_i, $i = 1, \dots, n$. Next, we summarize the above theory in the form of the algorithm.

Algorithm:

Step 1. Start from a given transfer function $H(s)$ for uncertain plant (7) and desired controller type (PI or PID) function $C(s)$.

Step 2. Construct the closed-loop characteristic polynomial (8) and plant Sylvester matrix (10).

Step 3. Choose the initial closed-loop characteristic polynomial $a^*(s)$ and check the stability.

Step 4. Find reduced Routh parameters w_k, $k = n, \dots, 1$ of the polynomial $a^*(s)$.

Step 5. According to (6), find Routh rays $\mathcal{R}_k(a)$, $k = 1, \dots, n$ of the polynomial $a^*(s)$ and, using (9), construct stable target simplex S with vertices on the Routh rays.

Step 6. Start with nominal plant (i.e., with values of uncertainties placed in the center of region) and find controller gains p and q by solving convex quadratic programming task (11) with restriction (12).

Step 7. Check the stability of closed-loop system with polytopic plant, i.e., all the vertices of the closed-loop polytope must be located inside the target simplex S. If some points of the rectangle are located outside of S, then select different initial closed-loop characteristic polynomial $a^*(s)$ and repeat all the previous steps.

Example 1: Consider the normalized fourth-order system from [16]

$$H(s) = \frac{g_3 s^3 + g_2 s^2 + g_1 s + 1}{f_4 s^4 + f_3 s^3 + f_2 s^2 + f_1 s + f_0}, \tag{13}$$

where $g_1 = 1, g_2 = 0.29167, g_3 = 0.04167$ and $f_0 = 1, f_1 = 2.083, f_2 = 1.4583, f_3 = 0.4167, f_4 = 0.04167$. In order to illustrate the applicability of the algorithm proposed above, we introduce uncertainty to the plant as $f_0 = 1 \pm 0.625$, $f_1 = 2.083 \pm 1.25$. One may easily check that the nominal plant (13) is stable. Our goal then is to design a low-order robust controller. In particular, we consider PI-controller

$$C(s) = \frac{q_1 s + 1}{p_1 s}.$$

The characteristic polynomial $a(s)$ of the closed-loop system is given by

$$\begin{aligned} a(s) = {} & p_1 s^5 + (0.4167 p_1 + 0.04167 q_1) s^4 \\ & + (1.4583 p_1 + 0.29167 q_1 + 0.04167) s^3 \\ & + [(2.083 \pm 1.25) p1 + q_1 + 0.29167] s^2 \\ & + [(1 \pm 0.625) p_1 + q_1 + 1] s + 1. \end{aligned}$$

Now, let us choose the initial stable closed-loop characteristic polynomial $a^*(s)$, whose poles are $r(a) = \{-3, -4, -5, -5, -7\}$. It means that the normed polynomial with $a_0^* = 1$

$$a^*(s) = 0.0005 s^5 + 0.0114 s^4 + 0.1076 s^3 + \\ 0.4971 s^2 + 1.1262 s + 1$$

has the reduced Routh parameters

$$w = \begin{bmatrix} 0.04167 & 0.1315 & 0.2451 & 0.3545 & 0.8394 & 1 \end{bmatrix}.$$

Take $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 4.4032$ and $a(w_i = \alpha_i w_i^*)$ for $i = 1, \dots, 5$, yielding the following stable polynomials on the Routh rays of the polynomial $a^*(s)$

$$\begin{aligned} a_1^* &= \begin{bmatrix} 0.0005 & 0.0114 & 0.1233 & 0.8728 & 3.9828 & 1 \end{bmatrix}, \\ a_2^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.4536 & 1.7037 & 1.1262 & 1 \end{bmatrix}, \\ a_3^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.4079 & 0.6069 & 1.9604 & 1 \end{bmatrix}, \\ a_4^* &= \begin{bmatrix} 0.0021 & 0.0503 & 0.1278 & 0.9825 & 1.1262 & 1 \end{bmatrix}, \\ a_5^* &= \begin{bmatrix} 0.0021 & 0.0114 & 0.1781 & 0.4971 & 1.2680 & 1 \end{bmatrix}. \end{aligned}$$

Next, we solve the PI-controller design task for the nominal plant with $f_1 = 2.083, f_0 = 1$ via quadratic programming taking the target simplex of the closed-loop system by the above Routh rays as

$$S = \begin{bmatrix} a^* & a_1^* & a_2^* & a_3^* & a_4^* & a_5^* \end{bmatrix} = \begin{bmatrix} 0.0005 & 0.0005 & 0.0021 & 0.0021 & 0.0021 & 0.0021 \\ 0.0114 & 0.0114 & 0.0503 & 0.0503 & 0.0503 & 0.0114 \\ 0.1076 & 0.1233 & 0.4536 & 0.4079 & 0.1278 & 0.1781 \\ 0.4971 & 0.8728 & 1.7037 & 0.6069 & 0.9825 & 0.4971 \\ 1.1262 & 3.9828 & 1.1262 & 1.9604 & 1.1262 & 1.2680 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The optimization procedure returns parameters, yielding the following controller

$$C(s) = \frac{0.4543s + 1}{0.0404s}. \quad (14)$$

The reference signal is chosen to be the step function. The experimental results for three variations (without, with maximum and minimum possible uncertainties) of plant (13) are depicted in Fig. 1. It can be seen that outputs are capable of tracking reference signal for the same controller (14) with acceptable level of accuracy. Note that the overregulation depends on the choice of the initial stable polynomial $a^*(s)$, which itself is a separate problem.

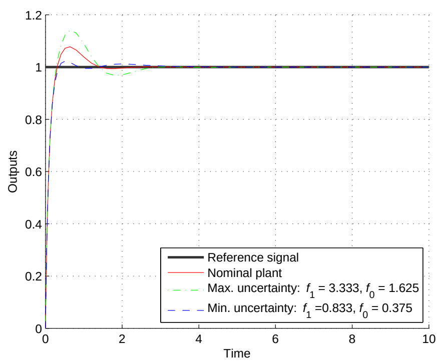

Fig. 1. Simulation results for the closed-loop system

Example 2: Consider the second-order ($m = 2$) uncertain plant

$$H(s) = \frac{g(s)}{f(s)} = \frac{g_1 s + 1}{f_2 s^2 + f_1 s + f_0}$$

with $g_1 = 0.5, f_2 = 1, f_1 = -1.2 \pm 0.8, f_0 = 0.52 \pm 1$. One may easily check that the nominal plant (i.e., without uncertainties) is unstable. Thus, our goal is to design a stabilizing robust PI-controller $C(s) = \frac{q_1 s+1}{p_1 s}$. The characteristic polynomial $a(s)$ of the closed-loop system is given by

$$a(s) = p_1 s^3 - [(1.2 \pm 0.8)p_1 - 0.5q_1]s^2 + [(0.52 \pm 1)p_1 + 0.5 + q_1]s + 1. \quad (15)$$

Now, let us choose the initial stable closed-loop characteristic polynomial $a^*(s)$ with poles $r(a) = \{-4 \pm 0.5i, -0.5\}$. Take $\alpha_1 = \alpha_2 = \alpha_3 = 2$. Next, we solve the PI-controller design task for the nominal plant with $f_1 = -1.2, f_0 = 0.52$ via quadratic programming. The optimization procedure returns the optimal parameters, yielding the following controller

$$C(s) = \frac{2.7949s + 1}{0.1702s}.$$

The result of the application of the above algorithm can be seen in Fig. 2. The resulting *pyramid* is the approximation of the stability domain by polyhedral Routh cone. The black (placed in the vertex) and blue dots are, respectively, defined by parameters of the initial stable polynomial $a^*(s)$ and coefficients of the characteristic polynomial $a(s)$ of the closed-loop system. The rectangular around blue dot determines bounds of uncertainties of (15). Note that it is placed inside the stability domain indicating that the designed controller is robust.

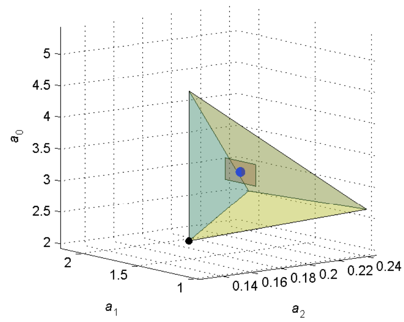

Fig. 2. Approximation of the stability domain by polyhedral Routh cone

Example 3: Consider configuration of the servo system provided by INTECO company [17]. The objective is to control a servo position. This modular experimental platform consists of the following components: a tachogenerator, a 24V DC motor, an inertia load, a magnetic brake, an encoder, and a gearbox. The servo system may be interfaced with the MATLAB/Simulink environment through a specific PCI board, where data is collected from the encoder and tachogenerator, and is sent to the power drive box, which controls the DC motor. The data was collected from the plant and used for identification, yielding the following transfer function

$$H(s) = \frac{g(s)}{f(s)} = \frac{1}{0.0049s^2 + 0.0061s}.$$

We may add uncertainty to the identified model to verify the robustness of the designed controllers. Hence, the parameters are $g_1 = 0, g_0 = 1, f_2 = 0.0049, f_1 = 0.0061 \pm 0.002, f_0 = 0$. Proceeding in the same manner as in the previous examples, and using two sets of poles $r_1(a) = \{-1.5, -1, -0.5\}$

and $r_2(a) = \{-7, -5, -3\}$ for the closed-loop characteristic polynomial, we get *slow* and *fast* controllers

$$C_s(s) = \frac{97.1975s + 1}{13523s}, \quad C_f(s) = \frac{72.2242s + 1}{22.0993s}.$$

Experiments from the real prototype for controller $C_f(s)$ are presented in Fig. 3. Two types of scenarios were considered: nominal plant and plant with external friction between inertia load and base. One may see that controller is capable of tracking reference signal for both cases.

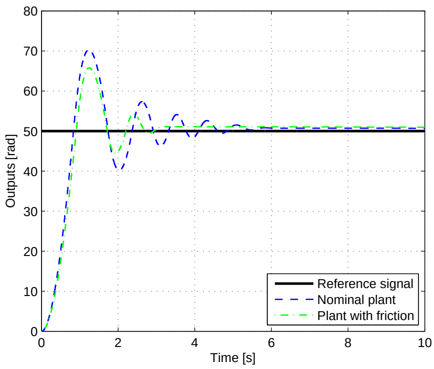

Fig. 3. Laboratory experimental results for controller $C_f(s)$

Fig. 4 shows the experimental results for controller $C_s(s)$ with varying set point.

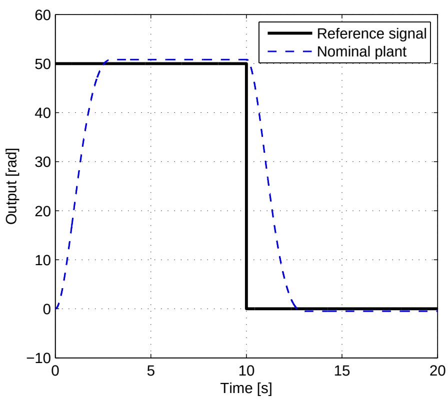

Fig. 4. Laboratory experimental results for controller $C_s(s)$

V. CONCLUSIONS

In this paper, the problem of designing a robust fixed-order controller for a continuous-time plant with uncertainties is addressed. Parameters of the controller are found using linear quadratic problem for which the corresponding simplex is constructed using so-called polyhedral Routh cone. The latter is calculated on the basis of the reduced Routh parameters.

In [13] a similar approach for the inner approximation of the stability domain was developed using idea based on polytopes. Though, in general, the resulting area has larger volume, sometimes method based on polyhedral Routh cones may provide a better result especially when the starting stable point a^* is placed near the boundary of the stability domain. Thus, the application of the design technique from this paper to the method from [13] as well as detailed comparison of the respective results will be subjects for the future research.

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Appendix 4

Publication IV
Ü. Nurges, I. Artemchuk, and J. Belikov. On stable cones of polynomials via reduced Routh parameters. *Kybernetika*, 52(3):461–477, 2016

ON STABLE CONES OF POLYNOMIALS VIA REDUCED ROUTH PARAMETERS

Ülo Nurges, Juri Belikov and Igor Artemchuk

A problem of inner convex approximation of a stability domain for continuous-time linear systems is addressed in the paper. A constructive procedure for generating stable cones in the polynomial coefficient space is explained. The main idea is based on a construction of so-called Routh stable line segments (half-lines) starting from a given stable point. These lines (Routh rays) represent edges of the corresponding Routh subcones that form (possibly after truncation) a polyhedral (truncated) Routh cone. An algorithm for approximating a stability domain by the Routh cone is presented.

Keywords: linear systems, Hurwitz stability, convex approximation

Classification: 93C05, 93D09

1. INTRODUCTION

The stability is one of the most important properties in the field of control systems. It arises in various applications and has to be taken into account while studying a system or designing an appropriate controller. The stability property can be analyzed in several ways. In case of linear systems the most intuitively understandable and inherently simple test is based on the location of roots of a characteristic polynomial. Other alternatives include Hurwitz, Routh, and Hermite–Bieler tests [6, 18] or frequency domain based techniques [23].

However, once a system contains uncertainties, these techniques cannot be directly applied. This resulted in the development of a parametric approach [4], which links the study of relationships between roots of a polynomial and its coefficients. The main problem appearing with the parametric approach is that, in general, the stability domain is nonconvex in the coefficient space. This challenge has led to the development of techniques for convex approximation of the stability domain such as based on ellipsoids [5, 9], polytopes [11, 14], hyperrectangles [8, 12], and convex directions [19]. The type of convex approximation of the stability domain depends on the type of system parameters uncertainty, for example, rectangular approximation is suitable for interval parameters, and polytopic approximation is applicable for polytopic uncertainties. This paper deals with conic approximation which may be useful for systems with one dominant uncertainty or with several conic type uncertainties.

In this paper, we provide a simple and efficient algorithm for the convex approximation of the stability domain by polyhedral Routh cones. The method is based on a new multilinear stability criterion for Hurwitz polynomials relying on the so-called reduced Routh parameters. For discrete-time systems the multilinear stability condition is introduced via reflection coefficients of polynomials [14] and the idea of random generation of stable line segments for stabilizing robust controller design is efficiently used [15, 20]. Here, we have proved that for continuous-time systems a similar approach can be used via reduced Routh parameters. The results of this paper can be understood as extension of those presented in [3] and [16]. For [3, 16], and this paper the multilinear stability condition is the main conception. In [16] the method for polytopic approximation of the stability domain is addressed. In the conference paper [3] the main idea of conic approximation of the stability domain is considered. However, the majority of facts are used without detailed proofs. In this paper, we provide the theoretical justification by giving complete proofs. Furthermore, the relevant additional material is added emphasizing relations between papers [3] and [16].

The paper is organized as follows. Section 2 recalls necessary definitions related to stability of polynomials in the continuous-time case. The notion of the reduced Routh parameters is introduced. The next section is devoted to the description of stable half-lines (Routh rays) of polynomials. The main results, related to the approximation of stability domain by polyhedral Routh cone, are addressed in Section 4. The presented material is illustrated by several numerical examples. Concluding remarks and possible directions for the future research are drawn in Section 5. Supplementary material is collected in the Appendix.

2. REDUCED ROUTH PARAMETERS OF POLYNOMIALS

A polynomial of degree n

$$a(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

with real coefficients $a_i \in \mathbb{R}$, for $i = 0, \dots, n$, is said to be continuous-time stable in the Hurwitz sense, if all its roots λ_i, for $i = 1, \dots, n$, are in the open left-half plane of $\mathbb{C}$, i. e., $\Re(\lambda_i) < 0$. Since polynomial (1) is uniquely defined by its coefficients, for simplicity, sometimes, we use a to denote both the polynomial $a(s)$ and the vector $a = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$ of its coefficients, i. e., $a := a(s) = \begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$. Then, the Hurwitz *region* $\mathcal{H}_n$ is defined as $\mathcal{H}_n = \left\{ a \in \mathbb{R}^{n+1} \mid (1) \text{ is Hurwitz} \right\}$.

A *stability boundary* is either the boundary of the stability domain in the coefficient space or the boundary of the root location domain (imaginary axis). The stability of polynomials $a(s)$ can be tested by Routh table, see [7]. Based on this criterion, a method for constructing Hurwitz polynomials can be derived as follows [20]. Start with arbitrary Hurwitz polynomial of degree 2. Since positivity of the coefficients is equivalent to stability of the second-order polynomials, generate arbitrary positive numbers h_0, h_1, h_2 and compose the polynomial $a(s) = h_2 s^2 + h_1 s + h_0$ or $a = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} h_2 & h_1 & h_0 \end{bmatrix}^{\mathrm{T}}$. At the kth step, having a Hurwitz polynomial of degree k, i. e., $a(s) = \begin{bmatrix} a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$, consider two polynomials of degree $k+1$

$$p(s) = \begin{bmatrix} 0 & a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}^{\mathrm{T}}$$

and

$$q(s) = \begin{bmatrix} a_k & 0 & a_{k-2} & 0 & a_{k-4} & 0 & \cdots \end{bmatrix}^{\mathrm{T}}.$$

Generate a positive random number h_{k+1} and compose

$$a(s) = p(s) + \frac{h_{k+1}}{a_k} q(s), \tag{2}$$

which is Hurwitz polynomial of degree $k+1$, according to the Routh rule. Proceeding in this manner up to $k = n$, we obtain a Hurwitz polynomial of degree n, see [21, 22]. Thus, the coefficients a_k of the nth-order polynomial are obtained from the Routh *parameters* h_k, $k = 0, \ldots, n$ recursively by increasing k. Furthermore, all Hurwitz polynomials of degree n can be obtained using this construction [20]. Next, we introduce the reduced Routh parameters that are used later in construction of stable line segments.

Definition 2.1. The *reduced* Routh parameters w_j for normed polynomials $a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + 1$ are defined as follows

$$\begin{aligned} w_0 &= h_0 = 1, \\ w_1 &= h_1, \\ w_2 &= h_2, \\ w_j &= \frac{h_j}{h_{j-1}}, \quad j = 3, \ldots, n. \end{aligned} \tag{3}$$

From (2) and (3) relations for recursive generation of normed Hurwitz polynomials of order $k+1$, for $k > 2$, can be obtained as

$$a(s) = p(s) + w_{k+1} q(s).$$

Denote the degree of a polynomial by superscript to get

$$a^{k+1} = \begin{bmatrix} w_k a_k^k & a_k^k & a_{k-1}^k + w_k a_{k-2}^k & a_{k-2}^k a_{k-3}^k + w_k a_{k-4}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}, \tag{4}$$

where $a^k = \begin{bmatrix} a_k^k & a_{k-1}^k & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$. Using matrix notation, equation (4) becomes

$$a^{k+1} = W_k a^k, \tag{5}$$

where W_k is a $(k+1) \times k$ matrix of the form

$$W_k = w_k \begin{bmatrix} J_k \\ \vdots \\ 0^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0^{\mathrm{T}} \\ \vdots \\ I_k \end{bmatrix}$$

with I_k being the $k \times k$ unit matrix and J_k being the $k \times k$ diagonal matrix $J_k = \operatorname{diag}\{1, 0, 1, 0, \ldots\}$, i. e.,

$$W_k = \begin{bmatrix} w_k & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & w_k & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

From recursive relation (5) it follows that $a^n = W_k^n a^k$, where $W_k^n = W_n W_{n-1} \cdots W_k$, $k = n, \ldots, 3$ or

$$a^n = W_3^n a^2 = W_n W_{n-1} \cdots W_3 \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}. \tag{6}$$

Lemma 2.2. The elements in (6), can be calculated using the direct formula

$$a_l^n = \sum_{i_0=1}^{n} \sum_{i_1=1}^{i_0} \cdots \sum_{i_{n-l}=1}^{i_{n-l-1}} \prod_{j=0}^{n-l} \overline{w}_{i_j} \bmod (i_j + n - l - j, 2), \tag{7}$$

where $l = 1, \ldots, n$ is the index number of the corresponding row in (6), $n > 2$, and $\bmod(\alpha, 2)$ is the usual modulus operation that returns either 1 or 0 depending on whether the number α is odd or even, respectively. Elements $\overline{w}_{i_j}$ in (7) correspond to entries of the matrix W_k as

$$\overline{w}_{i_j} := \begin{cases} w_2/\overline{w}_1 & \text{for } i_j = 2, \\ w_{i_j} & \text{otherwise.} \end{cases} \tag{8}$$

Proof. See the detailed explanation in [3] for the proof. □

The inverse mapping from polynomial coefficients a_k to the reduced Routh parameters w_k, $k = n, \ldots, 1$ can be recursively found starting from w_n via (5) as

$$\begin{aligned} w_j &= \frac{a_j^j}{a_{j-1}^j}, \quad j = n, \ldots, 3, \\ w_2 &= a_2^2, \\ w_1 &= a_1^2. \end{aligned} \tag{9}$$

Note that in (9) parameters a_j^j and a_{j-1}^j can be found explicitly as

$$\begin{aligned} a_{k-i-1}^{k-1} &= a_{k-i-1}^k, && i = 0, \ldots, 2\lfloor (k-2)/2 \rfloor, \\ a_{k-i-2}^{k-1} &= a_{k-i-2}^k - w_k a_{k-i-3}^k, && i = 0, \ldots, 2\lfloor (k-3)/2 \rfloor \end{aligned}$$

with $a_0 = 1$ or in the matrix form as $a^{k-1} = \overline{W}_k a^k$, where $\overline{W}_k$ is a $k \times k$ matrix

$$\overline{W}_k = I_k - w_k \begin{bmatrix} 0 & \overline{J}_{k-1} \\ \vdots & \vdots \\ 0 & 0^{\mathrm{T}} \end{bmatrix}$$

and $\overline{J}_k$ is a $k \times k$ diagonal matrix, i. e., $\overline{J}_k = \mathrm{diag}\{0, 1, 0, 1, \ldots\}$.

Proposition 2.3. A normed polynomial $a(s)$ with $a_0 = 1$ is Hurwitz stable if and only if $w_k > 0$, $k = 1, \ldots, n$.

Proof. *Necessity*: Assume that a normed polynomial $a(s)$ of order n is stable in the Hurwitz sense. Then, according to Routh stability criterion, all the Routh parameters h_k of stable polynomial $a(s)$ must be positive real numbers $h_k > 0$, $k = 1, \ldots, n$. Thus, (3) yields $w_k > 0$, $k = 1, \ldots, n$.

Sufficiency: From (3) it follows

$$\begin{aligned} h_0 &= 1, \\ h_1 &= w_1, \\ h_2 &= w_2, \\ h_j &= w_j h_{j-1}, \quad j = 3, \ldots, n. \end{aligned}$$

Observe that, if $w_k > 0$, for $k = 1, \ldots, n$, then all Routh parameters of the polynomial $a(s)$ are positive $h_k > 0$, $k = 0, \ldots, n$. Hence, the polynomial $a(s)$ is Hurwitz stable. □

Proposition 2.4. The mapping (5) from the reduced Routh parameters w_k, for $k = 1, \ldots, n$ to the normed polynomial coefficients a_k^n, $k = 1, \ldots, n$ with $a_0 = 1$ is a one-to-one mapping if $w_k > 0$, $k = 1, \ldots, n$.

Proof. According to the construction procedure, defined by (2), the mapping between the Routh parameters h_k, $k = 1, \ldots, n$ and the polynomial coefficients a_k^n, $k = 1, \ldots, n$ for $a_0 = 1$ is one-to-one, see [20]. Observe that mapping (3) between the reduced Routh parameters w_k, $k = 1, \ldots, n$ and the Routh parameters h_k, $k = 1, \ldots, n$ is one-to-one by $h_0 = 1$ as well. Hence, it remains to note that the composition of two injective functions is injective, and conclusion follows. □

3. STABLE ROUTH RAYS OF POLYNOMIALS

In this section we introduce the stable line segments (half-lines) of polynomials that can be obtained starting from the reduced Routh parameters w_k, $k = 1, \ldots, n$ of a Hurwitz polynomial $a \in \mathcal{H}_n \subset \mathbb{R}^{n+1}$.

Theorem 3.1. Through an arbitrary Hurwitz stable point

$$a = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{\mathrm{T}}$$

with reduced Routh parameters $w_k > 0$, $k = 1, \ldots, n$ one can draw n stable half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$ such that

$$\mathcal{R}_k(a) = \{a \mid w_k \in (0, \infty), w_j = \text{const},\ j \neq k;\ k, j \in \{1, \ldots, n\}\}. \tag{10}$$

P r o o f. Observe that all points of the line $\mathcal{R}_k(a)$ are Hurwitz stable, since

1. $n - 1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are assumed to be fixed and positive $w_j > 0$;

2. the kth reduced Routh parameters $w_k > 0$, according to assumption $w_k \in (0, \infty)$.

Next, we have to prove that $\mathcal{R}_k(a)$ is a line segment (half-line). It is easy to see that mapping (5) is multilinear. If $n - 1$ reduced Routh parameters w_j, $j \in \{1, \ldots, n\}$, $j \neq k$ are fixed, then mapping (5) turns out to be linear with respect to the kth reduced Routh parameter w_k. The latter means that for each $k = 1, \ldots, n$ there is a half-line $\mathcal{R}_k(a)$, and altogether n half-lines $\mathcal{R}_k(a) \subset \mathcal{H}_n$. □

Definition 3.2. The half-lines $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ defined by (10) are called *Routh rays* of the polynomial $a(s)$. Moreover, their endpoints $v_k(a)$ such as

$$v_k(a) = a(w_k = 0)$$

are supposed to be the *Routh sources* of the polynomial $a(s)$.

Proposition 3.3. (Multilinear stability criterion) If a is a Hurwitz stable polynomial with reduced Routh parameters $w_k(a)$, $k = 1, \ldots, n$, then all the Routh rays $\mathcal{R}_k(a)$ are Hurwitz stable.

P r o o f. The proof follows directly from Theorem 3.1. □

According to Proposition 2.3, all Routh sources $v_k(a)$ of Hurwitz (stable) polynomials $a(s)$ are placed on the stability boundary. This means that some of the roots $\lambda_j(v_k)$, $j = 1, \ldots, n$, $k = 1, \ldots, n$ are placed on the imaginary axis. Using mapping (6) the following theorem can be formulated, regarding roots of Routh sources.

Theorem 3.4. All the Routh sources $v_j(a)$, $j = 2, \ldots, n-1$ of a Hurwitz polynomial $a(s)$ of the order n have at least two roots at the origin

$$\lambda_1(v_j) = \lambda_2(v_j) = 0, \quad j = 2, \ldots, n-1$$

and the last Routh source $v_n(a)$ has at least one root at the origin

$$\lambda_1(v_n) = 0.$$

P r o o f . To prove the theorem, the direct formula (7) from Lemma 2.2 is used. Indeed, take in (7) for $l = 1$ and $l = 2$ indices as $i_0 = n, i_1 = n-1, \ldots, i_{n-2} = 2, i_{n-1} = 1$ and $i_0 = n-1, i_1 = n-2, \ldots, i_{n-3} = 2, i_{n-2} = 1$, respectively. This yields the first two elements a_1^n, a_2^n of (6) given as

$$\begin{aligned} a_1^n &= \overline{w}_n \overline{w}_{n-1} \cdots \overline{w}_2 \overline{w}_1, \\ a_2^n &= \overline{w}_{n-2} \overline{w}_{n-3} \cdots \overline{w}_2 \overline{w}_1 \end{aligned}$$

or, using (8), in the simplified form as

$$\begin{aligned} a_1^n &= w_n w_{n-1} \cdots w_2, \\ a_2^n &= w_{n-2} w_{n-3} \cdots w_2. \end{aligned}$$

Hence, according to Definition 3.2, from the previous equations it follows $\lambda_1(v_j) = \lambda_2(v_j) = 0$, for $j = 2, \ldots, n-1$, and $\lambda_1(v_n) = 0$. □

4. STABLE ROUTH CONES OF POLYNOMIALS

Next, we study the stability of polynomials with conic uncertainty [10] by means of Routh rays. We define so-called Routh cones[1] in the polynomial coefficient space $a \in \mathbb{R}^n$ starting from the reduced Routh parameter space $w \in \mathbb{R}^n$. Let $a^* \in \mathcal{H}_n$ be arbitrary stable polynomial of the order n and w^* its reduced Routh parameters.

Definition 4.1.

1. A subset $\mathcal{K}_i(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a Routh cone of a polynomial $a^*(s)$ if it is closed under positive scalar multiplication of one of its reduced Routh parameters w_i^*, $i \in \{1, \ldots, n\}$, i. e., $a(w_i = \alpha w_i^*) \in \mathcal{K}_i$ when $a \in \mathcal{K}_i$ and $\alpha > 0$, where all the other reduced Routh parameters w_j, $j \neq i$, $j \in \{1, \ldots, n\}$ are fixed $w_j = w_j^*$.

2. If P is a subset of normed polynomials $a(s)$ of degree n with coefficients $a \in \mathbb{R}^n$, then
$$\mathcal{K}_i(P) = \{a(w_i = \alpha w_i);\ a \in P, \alpha > 0, i \in \{1, \ldots, n\}\}$$
is called the Routh cone generated by P.

3. A convex cone $\mathcal{K}(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\mathcal{K}(a^*) = \left\{ \sum_{i=1}^{n} \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, 0 < \beta_i < 1, \right. \\ \left. \sum_{i=1}^{n} \beta_i = 1, w_j = w_j^* = \text{const}, j \neq i, i = 1, \ldots, n \right\}.$$

[1]Note that the notion cone is used in consonance with results in [10]. In our paper definition of the Routh cone coincides with that of the Routh ray.

4. A convex cone $\mathcal{K}_{i,j}(a^*)$ of normed polynomial $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a polyhedral Routh i,j-subcone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\begin{aligned}\mathcal{K}_{i,j}(a^*) = \big\{\beta_i a(w_i = \alpha_i w_i^*, w_j = w_j^*) + \beta_j a(w_j = \alpha_j w_j^*, w_i = w_i^*);\\ \alpha_i, \alpha_j > 1, 0 < \beta_i, \beta_j < 1, \beta_i + \beta_j = 1,\\ w_k = w_k^* = \text{const}, k \neq i, j;\ i, j, k \in \{1, \dots, n\}\big\}.\end{aligned}$$

5. A convex set $\overline{\mathcal{K}}_{j,k}^n(a^*)$ of normed polynomials $a(s)$ of the degree n with coefficients $a \in \mathbb{R}^n$ is said to be a truncated polyhedral Routh cone of a polynomial $a^*(s)$, if there exist α_i, β_i, such that

$$\begin{aligned}\overline{\mathcal{K}}_{j,k}^n(a^*) = \Bigg\{\sum_{i=1}^n \beta_i a(\alpha_i w_i^*);\ \alpha_i > 1, i \neq j, k;\\ 1 < \alpha_j < \overline{\alpha_j}, 1 < \alpha_k < \overline{\alpha_k};\ 0 < \beta_i < 1, \sum_{i=1}^n \beta_i = 1,\\ w_h = w_h^* = \text{const}, h \neq i, i = 1, \dots, n\Bigg\}.\end{aligned}$$

Remark 4.2. According to Theorem 3.1, it is possible to draw n stable Routh rays $\mathcal{R}_i(a^*)$ through an arbitrary stable point a^*. In [16] it was shown that if the point is not placed on the boundary of stability domain, then there are *positive* and *negative* directions with respect to a^*. The positive part of a Routh ray corresponds to $\alpha_i \in (1, \infty)$ while the negative to $\alpha_i \in (0, 1)$, and for $\alpha_i = 1$ rays intersect at the point a^*. In this paper notions of Routh rays and Routh cones $\mathcal{K}_i(a^*)$ coincide for positive direction. Therefore, the point a^* should be understood as a vertex of the polyhedral Routh cone.

Proposition 4.3. An arbitrary subset P of normed polynomials $a(s)$ of the degree n, $a(s) \in \mathbb{R}^n$ has n Routh cones $\mathcal{K}_i(P)$, $i = 1, \dots, n$ generated by P. If the subset P is stable, then all Routh cones $\mathcal{K}_i(P)$ generated by P are stable.

P r o o f. According to Theorem 3.1, through an arbitrary point $a \in P \subset \mathbb{R}^n$ it is possible to draw half-lines $\mathcal{R}_i(a)$ such that $w_i \in (0, \infty)$, $i = 1, \dots, n$. If polynomials $a \in P$ are stable, then all half-lines $\mathcal{R}_i(a)$ are stable, i. e., Routh cone $\mathcal{K}_i(P)$ is stable. □

Proposition 4.4. The n-times Routh cone of the polynomial $a(s) = 1$, i. e., $a = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \in \mathcal{R}^n$, generates the whole stability domain $\mathcal{A}$ in polynomial coefficient space, $\mathcal{A} \subset \mathbb{R}^n$.

P r o o f. Starting from the origin $a = 0$ it is possible to find the Routh ray $\mathcal{R}_1(0)$ which is placed on the stability boundary, since all the points $a \in \mathcal{R}_1(0)$ have $w_j = 0$,

$j = 2, \dots, n$. The Routh cone $\mathcal{K}_{1,2}(0) = \mathcal{K}_2(\mathcal{R}_1(0))$ is also placed on the stability boundary, since all the points $a \in \mathcal{K}_{1,2}(0)$ have $w_j = 0$, $j = 3, \dots, n$ and $w_i \in (0, \infty)$, $i = 1, 2$. Similarly, for all the points $a \in \mathcal{K}_{1,\dots,n-1}(0)$ it follows that $w_j = 0$, $j = n$ and $w_i \in (0, \infty)$, $i = 1, \dots, n-1$. Finally, the Routh cone $K_{1,\dots,n}(0)$ contains points a with $w_i \in (0, \infty)$, $i = 1, \dots, n$, i.e., $\mathcal{K}_{1,\dots,n}(0) = \mathcal{A}$. □

Theorem 4.5. (Artemchuk et al. [3]) If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, \dots, n\}$ of a stable polynomial $a^*(s)$ are stable, then the polyhedral Routh cone $\mathcal{K}(a^*)$ is stable.

Let $\Gamma = \{1, \dots, n\}$ be a set of integers. Rewrite it as $\Gamma = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are sets that contain indices corresponding to ordinary and truncated Routh subcones, respectively, with $\dim \gamma_1 = m_1$ and $\dim \gamma_2 = m_2$ such that $m_1 + m_2 = n$.

Theorem 4.6. (Artemchuk et al. [3]) A truncated polyhedral Routh cone $\overline{\mathcal{K}}^n_{i_j}(a^*)$ such that $i_j \in \gamma_2$ and $j = 1, \dots, m_2$ of a stable polynomial $a^*(s)$ is stable if the following conditions hold:

1. the polyhedral Routh subcones $\mathcal{K}_{r,s}(a^*)$, $r, s \in \gamma_1$ are stable;
2. the line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$, $u, v \in \gamma_2$ are stable, where

$$S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v) = \operatorname{conv}\left\{a(w_u = \overline{\alpha}_{u,\min} w_u^*), a(w_v = \overline{\alpha}_{v,\min} w_v^*), w_i = w_i^*, i \neq u, v\right\}$$

   and $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$.

Proposition 4.7. (Artemchuk et al. [3]) For $n = 3$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of an arbitrary stable polynomial $a^*(s)$ is stable.

Example 4.8. Consider an Unmanned Free-Swimming Submersible vehicle [13] for which the relation of pitch angle to elevator surface angle can be represented by the transfer function

$$H(s) = \frac{-0.125(s + 0.435)}{(s + 1.23)(s^2 + 0.226s + 0.0169)}.$$

Since the poles $\lambda_1 = -1.23$, $\lambda_{2,3} = -0.113 \pm 0.0643i$ have negative real parts, it immediately follows that the nominal system $H(s)$ is stable. The goal is to construct the stable polyhedral cone in the coefficient space starting from the nominal characteristic polynomial (denominator of $H(s)$)

$$a^*(s) = s^3 + 1.456s^2 + 0.2949s + 0.028.$$

Normalize the polynomial $a^*(s)$ dividing it by free term 0.028 to get

$$a^*(s) = 35.7143s^3 + 52s^2 + 10.5321s + 1$$

or

$$a^3 = \begin{bmatrix} a_3^3 & a_2^3 & a_1^3 & 1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 35.7143 & \underbrace{52 \quad 10.5321 \quad 1}_{\overline{a}^3} \end{bmatrix}^{\mathrm{T}}.$$

The reduced Routh parameters can be found using recursive relation (9) as follows. Start from

$$w_3^* = \frac{a_3^3}{a_2^3} = \frac{35.7143}{52} = 0.6868.$$

Next, find the second-order polynomial

$$a^2 = \begin{bmatrix} a_2^2 \\ a_1^2 \\ 1 \end{bmatrix} = \overline{W}_3 \overline{a}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.6868 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 52 \\ 10.5321 \\ 1 \end{bmatrix} = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}$$

and, therefore,

$$w^* = \begin{bmatrix} w_3^* & w_2^* & w_1^* & w_0^* \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.6868 & 52 & 9.8453 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Then, according to Definition 4.1, Routh cones can be calculated as

$$\mathcal{K}_i = \underbrace{\begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_1$: Take $w_1 = \alpha_1 w_1^*, w_2 = w_2^*, w_3 = w_3^*, 1 < \alpha_1 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_1 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52 \\ 9.8453\alpha_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136 \\ 52 \\ 9.8453\alpha_1 + 0.6868 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_2$: Take $w_1 = w_1^*, w_2 = \alpha_2 w_2^*, w_3 = w_3^*, 1 < \alpha_2 < \infty$, and

$$a^2 = \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_2 = \begin{bmatrix} 0.6868 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.6868 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52\alpha_2 \\ 9.8453 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136\alpha_2 \\ 52\alpha_2 \\ 10.5321 \\ 1 \end{bmatrix}.$$

Cone $\mathcal{K}_3$: Take $w_1 = w_1^*, w_2 = w_2^*, w_3 = \alpha_3 w_3^*,\ 1 < \alpha_3 < \infty$, and

$$a^2 = \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix}.$$

Then,

$$\mathcal{K}_3 = \begin{bmatrix} 0.68682\alpha_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0.68682\alpha_3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 52 \\ 9.8453 \\ 1 \end{bmatrix} = \begin{bmatrix} 35.7136\alpha_3 \\ 52 \\ 0.6868\alpha_3 + 9.8453 \\ 1 \end{bmatrix}.$$

Let $a \in \mathcal{K}(a^*)$ be an inner point of the polyhedral Routh cone $\mathcal{K}(a^*)$. Then, the convex combination can be expressed as

$$a = \beta_1\mathcal{K}_1(a^*) + \beta_2\mathcal{K}_2(a^*) + \beta_3\mathcal{K}_3(a^*),$$

where $0 < \beta_i < 1, \sum_{i=1}^{3} \beta_i = 1$ or in the explicit form as

$$a = \begin{bmatrix} 35.7136(\beta_1 + \beta_2\alpha + \beta_3\alpha) \\ 52(\beta_1 + \beta_2\alpha + \beta_3) \\ 9.8453(\beta_1\alpha + \beta_2 + \beta_3) + 0.6868(\beta_1 + \beta_2 + \beta_3\alpha) \\ 1 \end{bmatrix}.$$

From (9) it follows

$$\begin{aligned} w_3 &= \frac{0.6868(\beta_1 + \beta_2\alpha + \beta_3\alpha)}{\beta_1 + \beta_2\alpha + \beta_3}, \\ w_2 &= 52(\beta_1 + \beta_2\alpha + \beta_3), \\ w_1 &= \frac{511.956(\beta_1\alpha + \beta_2 + \beta_3)(\beta_1 + \beta_2\alpha + \beta_3) + 35.7136(1-\alpha)^2\beta_2\beta_3}{52(\beta_1 + \beta_2\alpha + \beta_3)}. \end{aligned}$$

Observe that $a^*(s)$ is stable. Then, it follows from Proposition 2.3 that $w_i^* > 0$, $i = 1, 2, 3$. It remains to show that the reduced Routh parameters w_i, $i = 1, 2, 3$ are also positive. This trivially follows from the fact that $\alpha_i > 1$ and $0 < \beta_i < 1$ with $\sum_{i=1}^{3} \beta_i = 1$. Therefore, the constructed polyhedral Routh cone

$$\mathcal{K}(a^*) = \Big\{ \beta_1\mathcal{K}_1(a^*) + \beta_2\mathcal{K}_2(a^*) + \beta_3\mathcal{K}_3(a^*) \mid \alpha_i > 1, 0 < \beta_i < 1, \sum_{i=1}^{3} \beta_i = 1, i = 1, 2, 3 \Big\}$$

is stable.

Proposition 4.9. The polyhedral subcones $K_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable.

P r o o f. See Appendix. □

The following algorithm allows to generate stable truncated polyhedral Routh cones for a given initial polynomial.

Algorithm:

Step 1. Start from a given n degree stable polynomial $a(s)$, or

$$a_n = \begin{bmatrix} a_n^n & a_{n-1}^n & \cdots & a_1^n & 1 \end{bmatrix}.$$

Step 2. Find the reduced Routh parameters w_k, $k = n, \ldots, 1$ of the polynomial $a(s)$ by solving (9).

Step 3. Find by (10) the Routh rays $\mathcal{R}_k(a)$, $k = 1, \ldots, n$ of the polynomial $a(s)$.

Step 4. Check the stability of all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$ with $i, j \in \{4, \ldots, n\}$ of the polynomial $a(s)$ by Hurwitz Segment Lemma [1, p.81]. By Proposition 4.9 the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, 2, 3\}$ are stable. If all the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \ldots, n\}$ are stable, then by Theorem 4.5 the polyhedral Routh cone $\mathcal{K}(a)$ is stable.

Step 5. If some of the polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{4, \ldots, n\}$ are not stable, then find the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$ using Theorem 4.6 with appropriate values of $\overline{\alpha}_{u,\min} = \min_u \overline{\alpha}_u$ and $\overline{\alpha}_{v,\min} = \min_v \overline{\alpha}_v$.

Step 6. According to Theorem 4.6 the stable truncated polyhedral Routh cone $\overline{\mathcal{K}}^n(a)$ of the polynomial $a(s)$ is determined by the stable polyhedral Routh subcones $\mathcal{K}_{i,j}(a)$, $i, j \in \{1, \ldots, n\}$ and the stable line segments $S_{u,v}(\overline{\alpha}_u, \overline{\alpha}_v)$.

Example 4.10. Consider the fourth-order system [17]

$$H(s) = \frac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24}.$$

The nominal system $H(s)$ is stable, since the poles are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$, $\lambda_4 = -4$. Our goal is to construct the stable polyhedral Routh cone around the nominal characteristic polynomial

$$a^*(s) = s^4 + 10s^3 + 35s^2 + 50s + 24.$$

Proceed in the same manner as in Example 4.8. Thus, first normalize the polynomial $a^*(s)$ dividing it by the free term 24 and then calculate the reduced Routh parameters as

$$w^* = \begin{bmatrix} w_4^* & w_3^* & w_2^* & w_1^* & w_0^* \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0.1 & 0.33 & 1.25 & 1.75 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Then, according to Definition 4.1, Routh cones can be calculated as

$$\mathcal{K}_i = \underbrace{\begin{bmatrix} w_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & w_4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{W_4} \cdot \underbrace{\begin{bmatrix} w_3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 0 & 1 \end{bmatrix}}_{W_3} \cdot \begin{bmatrix} w_2 \\ w_1 \\ 1 \end{bmatrix},$$

yielding

$$\begin{aligned}
\mathcal{K}_1 &= \begin{bmatrix}0.0417 & 0.4167 & 1.283 + 0.175\alpha_1 & 0.333 + 1.75\alpha_1 & 1\end{bmatrix}^{\mathrm{T}},\\
\mathcal{K}_2 &= \begin{bmatrix}0.0417\alpha_2 & 0.4167\alpha_2 & 0.2083 + 1.25\alpha_2 & 2.0833 & 1\end{bmatrix}^{\mathrm{T}},\\
\mathcal{K}_3 &= \begin{bmatrix}0.0417\alpha_3 & 0.4167\alpha_3 & 1.425 + 0.033\alpha_3 & 1.75 + 0.333\alpha_3 & 1\end{bmatrix}^{\mathrm{T}},\\
\mathcal{K}_4 &= \begin{bmatrix}0.0417\alpha_4 & 0.4167 & 1.25 + 0.2083\alpha_4 & 2.0833 & 1\end{bmatrix}^{\mathrm{T}}
\end{aligned}$$

with $1 < \alpha_i < \infty$, $i = 1, \ldots, 4$.

Next, between obtained Routh cones it is possible to draw six polyhedral Routh subcones. According to Proposition 4.9, the polyhedral Routh subcones $\mathcal{K}_{1,2}(a^*)$, $\mathcal{K}_{1,3}(a^*)$, and $\mathcal{K}_{2,3}(a^*)$ are stable. In addition, according to the Edge Theorem, $\mathcal{K}_{2,4}(a^*)$ and $\mathcal{K}_{3,4}(a^*)$ are stable as well. The remaining subcone $\mathcal{K}_{1,4}(a^*)$ is not stable, whereas the truncated polyhedral Routh subcone $\overline{\mathcal{K}}_{1,4}(a^*)$ is stable, for example, for $\overline{\alpha}_1 = \overline{\alpha}_4 = 6.2$.

5. DISCUSSION

This paper proposes the method for convex approximation of stability domain by the polyhedral Routh cone $\mathcal{K}(a^*)$. The main idea is based on the new multilinear stability criterion for Hurwitz polynomials relying on the reduced Routh parameters. The results presented in the paper extend those from [3] by giving rigorous mathematical proofs and providing additional theoretical material. Furthermore, Section 3 and Remark 4.2, in particular, explain how the results from [3] and [16] are related via Routh rays.

It was shown in Proposition 4.7 that for the particular case of the third-order system, the Routh cone of an arbitrary polynomial a^* is always stable. However, for higher order systems is remains an open challenge. Therefore, we state the following hypotheses that require theoretical proofs.

Conjecture 5.1. For $n = 4$ the polyhedral Routh cone $\mathcal{K}(a^*)$ of a stable polynomial $a^*(s)$ is stable if the polyhedral Routh subcone $\mathcal{K}_{1,4}(a^*)$ is stable.

Conjecture 5.2. The polyhedral Routh cone $\mathcal{K}(a^*)$ of a stable polynomial $a^*(s)$ of order n is stable if the polyhedral Routh subcones $K_{1,j}(a^*)$, $j = 4, ..., n$ are stable.

The convex inner approximation of the stability region and the multilinear stability conditions can be used, for example, to design an output controller of a fixed-order via quadratic programming approach so that the closed-loop poles are robustly assigned in the approximated region [2, 15]. This will make another direction for the future research.

APPENDIX

Proof of Proposition 4.9

Proof. By (5) we obtain the following Routh cones $\mathcal{K}_i(a^*)$, $i = 1, 2, 3$ for the polynomial $a^*(s)$, $a \in \mathbb{R}^n$

$$\mathcal{K}_1(a^*) = W_4^n(a^*) \begin{bmatrix} w_2^* w_3^* \\ \alpha w_1^* + w_3^* \\ 1 \end{bmatrix}, \quad \mathcal{K}_2(a^*) = W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ \alpha w_2^* \\ w_1^* + w_3^* \\ 1 \end{bmatrix},$$

$$\mathcal{K}_3(a^*) = W_4^n(a^*) \begin{bmatrix} \alpha w_2^* w_3^* \\ w_2^* \\ w_1^* + \alpha w_3^* \\ 1 \end{bmatrix},$$

where $W_4^n(a^*) := W_n(a^*) \cdots W_4(a^*)$ and $\alpha > 1$.

For $a \in \mathcal{K}_{1,2}(a^*)$ there exist constants $\alpha > 1$ and $0 < \beta < 1$ such that for an arbitrary $a \in \mathcal{K}_{1,2}(a^*)$

$$a = \beta a(w_1 = \alpha w_1^*) + (1 - \beta) a(w_2 = \alpha w_2^*),$$

where $a(w_1 = \alpha w_1^*) \in \mathcal{K}_1$ and $a(w_2 = \alpha w_2^*) \in \mathcal{K}_2$. The above relation can be rewritten in the explicit way as

$$a = W_n(a^*) \cdots W_4(a^*) \begin{bmatrix} (\beta + (1 - \beta)\alpha) w_2^* w_3^* \\ (\beta + (1 - \beta)\alpha) w_2^* \\ (\beta\alpha + 1 - \beta) w_1^* + w_3^* \\ 1 \end{bmatrix}.$$

Observe that the reduced Routh parameters $w_n, \dots, w_4$ of a polynomial $a(s)$ are determined by the product of matrix multiplication $W_n(a^*) \cdots W_4(a^*)$, i.e., $w_i = w_i^*$, $i = 4, \dots, n$. For the reduced Routh parameters w_i, $i = 1, \dots, 3$ of the polynomial $a \in \mathcal{K}_{1,2}(a^*)$, using (9), it follows

$$\begin{aligned} w_2 w_3 &= (\beta + (1 - \beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta + (1 - \beta)\alpha) w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta) w_1^*, \\ w_2 &= (\beta + (1 - \beta)\alpha) w_2^*, \\ w_3 &= w_3^*. \end{aligned}$$

Note that $\alpha > 1$, $0 < \beta < 1$, and $w_i^* > 0$, $i = 1, \dots, n$. Then, $w_i > 0$, $i = 1, \dots, n$, i.e., $a \in \mathcal{K}_{1,2}(a^*)$ is stable.

In the similar manner we obtain for $a \in \mathcal{K}_{1,3}(a^*)$ the reduced Routh parameters $w_n, \ldots, w_4$, $w_i = w_i^*$, $i = 4, \ldots, n$. For w_i, $i = 1, \ldots, 3$ of the polynomial $a \in \mathcal{K}_{1,3}(a^*)$ we obtain by (9) the following relations

$$\begin{aligned} w_2 w_3 &= (\beta + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= w_2^*, \\ w_1 + w_3 &= (\beta\alpha + 1 - \beta) w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned}$$

or

$$\begin{aligned} w_1 &= (\beta\alpha + 1 - \beta) w_1^* > 0, \\ w_2 &= w_2^* > 0, \\ w_3 &= (\beta + (1-\beta)\alpha) w_3^* > 0. \end{aligned}$$

Finally, for $a \in \mathcal{K}_{2,3}(a^*)$ we obtain the reduced Routh parameters $w_i = w_i^*$, $i = 4, \ldots, n$ and for w_i, $i = 1, \ldots, 3$

$$\begin{aligned} w_2 w_3 &= (\beta\alpha + (1-\beta)\alpha) w_2^* w_3^*, \\ w_2 &= (\beta\alpha + (1-\beta)) w_2^*, \\ w_1 + w_3 &= w_1^* + (\beta + (1-\beta)\alpha) w_3^* \end{aligned}$$

that yield

$$\begin{aligned} w_1 &= w_1^* + \frac{(\beta(1-\beta)(1-\alpha)^2) w_3^*}{\beta\alpha + (1-\beta)} > 0, \\ w_2 &= (\beta\alpha + 1 - \beta) w_2^* > 0, \\ w_3 &= \frac{\alpha w_3^*}{\beta\alpha + 1 - \beta} > 0. \end{aligned}$$

Hence, all polyhedral subcones $\mathcal{K}_{i,j}(a^*)$, $i, j \in \{1, 2, 3\}$ of an arbitrary stable polynomial $a^*(s)$ of order n are stable. $\square$

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