THESIS ON NATURAL AND EXACT SCIENCES B86

ON LAGRANGE FORMALISM FOR LIE THEORY AND OPERADIC HARMONIC OSCILLATOR IN LOW DIMENSIONS

JÜRI VIRKEPU

TUT Press

TALLINN UNIVERSITY OF TECHNOLOGY
Faculty of Science
Department of Mathematics

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Supervisor: Prof Eugen Paal, Department of Mathematics, Tallinn University of Technology

Opponents: DSc, Prof Rein-Karl Loide, Tallinn University of Technology, Estonia

PhD, Prof Alexander Stolin, Göteborg University and Chalmers University of Technology, Sweden

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for any academic degree.

/Jüri Virkepu/

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LAGRANGE'I FORMALISMIST LIE TEOORIALE JA HARMOONILINE OPERAADOSTSILLAATOR MADALATES DIMENSIOONIDES

JÜRI VIRKEPU

TTÜ Kirjastus

TALLINNA TEHNIKAÜLIKOOL
Matemaatika-loodusteaduskond
Matemaatikainstituut

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Juhendaja: Prof Eugen Paal, Matemaatikainstituut, Tallinna Tehnikaülikool

Opponendid: DSc, Prof Rein-Karl Loide, Tallinna Tehnikaülikool, Eesti

PhD, Prof Alexander Stolin, Göteborgi Ülikool ja Chalmersi Tehnikaülikool, Rootsi

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Contents

Introduction

Motives and aims of the thesis

The theory of quantum groups is a relatively new, but powerful and quickly developing branch of mathematics, that has many applications in modern physics. These objects are conventionally constructed via deformations (e.g [3]). But it is also interesting to consider other quantization methods of the algebraic systems, in particular e.g the canonical and path integral quantizations of the Lie groups. Then one has to construct the Lagrangian and Hamiltonian of a Lie group under consideration. The crucial idea of such an approach is that the Lie equations of the Lie (transformation) group are represented as the Euler-Lagrange and the Hamilton canonical equations. In operad theory, for introducing dynamics in algebraic systems one can consider the operadic Lax equation.

The main task of the present thesis is to present these two alternative novel ways of introducing dynamics in algebraic systems. Concisely speaking, these can be realized through the Lagrange and Hamiltonian formalisms in the Lie theory and constructing the operadic Lax representations for the harmonic oscillator.

Outline of the thesis

This thesis consists of two parts. The first part (Chapters 1 to 3) deals with a Lie transformation group in terms known from the classical mechanics. It contains the short description of the basic topics of the Lie theory (Chapter 1), that covers the notions of a Lie group, Lie algebra, Lie transformation group, the Lie and Maurer-Cartan equations. Then the group $SO(2)$ is taken as the main model. It turns out that $SO(2)$ is a constrained mechanical system in the sense of P. Dirac. The Lagrangian and Hamiltonian are defined for $SO(2)$ both in real and complex representation. The Lagrange and Hamiltonian equations turn out to be Lie equations for $SO(2)$. The canonical formalism is developed and the physical interpretation of the Lagrangian and Hamiltonian are given (Chapter 2). It is shown that the constraints satisfy the canonical commutation relations. The consistency of the constraints is checked.

In Chapter 3, a general method for constructing Lagrangians for the Lie transformation groups is presented. It turns out that one has to use vector Lagrangians. Two examples are provided.

Motivated by the results of the first part of the thesis, in the second part a similar task is formulated by using operads. In classical mechanics the dynamics of a system with the Hamiltonian $H(q^i, p_i)$ can be given either by the Hamiltonian system

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

or by the equivalent Lax equation

$$\frac{dL}{dt} = [M, L] := ML - LM$$

Thus, from the algebraic point of view, mechanical systems can be described by linear operators, i.e by linear maps $V \to V$ of a vector space V. As a generalization of this the following question can be posed: how can the time evolution of the linear operations (multiplications) $V^{\otimes n} \to V$ be described?

The second part (Chapters 4 to 6) starts with the overview of the Gerstenhaber theory (Chapter 4) and explaining basic concepts of the operadic dynamics. The main idea of the operadic dynamics is as follows. If $L : V^{\otimes n} \to V$ is an n-ary operation, then we can modify the Lax equation by replacing the commutator bracketing on the r.h.s. of it by the Gerstenhaber brackets. Using the Gerstenhaber brackets is natural, because these brackets satisfy the graded Jacobi identity and if $n = 1$, then the Gerstenhaber brackets coincide with the ordinary commutator bracketing. Thus, the time evolution of the operadic variables may be given by the operadic Lax equation. The concept of the operadic (Lax representation for) harmonic oscillator is explained as well.

As examples, in Chapter 5, the low-dimensional ($\dim V = 2, 3$) operadic Lax representations for the harmonic oscillator are constructed.

In Chapter 6, by using the operadic Lax representations for the harmonic oscillator, the dynamical deformations of the 3-dimensional real Lie algebras in the Bianchi classification are constructed. Then the Jacobi identities of these algebras are studied. Finally, quantum counterparts of 3-dimensional real Lie algebras are defined and studied.

There are three appendices in the thesis. Appendix A includes detailed proofs of Theorems 6.12-6.13. Appendix B has a discussion on operadic quantization over the harmonic oscillator, containing the conjecture about the corresponding quantum conditions. Appendix C covers some additional topics on dynamical deformations of 2-dimensional binary real algebras.

In Chapters 5–6 and Appendices A–C, the mathematical computer program MAPLE™ 13 was used to check most of calculation.

Main novelties of the thesis

1. The Lagrange and canonical formalisms for $SO(2)$ are developed and their physical interpretation is given.
2. A general method for constructing Lagrangians for the Lie transformation groups is presented. The method is illustrated with two examples.
3. Examples of the low-dimensional ($\dim V = 2, 3$) operadic Lax representations for the harmonic oscillator are constructed.
4. The dynamical deformations of the 3-dimensional real Lie algebras in the Bianchi classification over the harmonic oscillator are constructed. It is shown that the energy conservation of the harmonic oscillator is related to the Jacobi and associativity identities of the dynamically deformed algebras. Based on this observation, it is proved that the dynamical deformations of 3-dimensional real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras.
5. Quantum counterparts of 2- and 3-dimensional real Lie algebras are defined and their Jacobi operators are calculated. It is discussed how the operadic dynamics in 3-dimensional real Lie algebras over the harmonic oscillator leads to quantization of a 3-dimensional space.

List of preprints and other publications

The results of the thesis have, for the most part, been published in the papers given in List of Publications on page 78. The other part is presented as the following preprint and publication:

1. E. Paal and J. Virkepu. Operadic quantization of VII_a, $\mathrm{III}_{a=1}$, $\mathrm{VI}_{a\neq 1}$ over harmonic oscillator. Preprint ArXiv: `0903.3702` (2009).
2. J. Virkepu. On Lie theory. Annual Book 2005, Estonian Mathematical Society, 2006, 30-51 (in Estonian).

The research of the author has been an essential part of the above.

Conference reports

The results of the thesis have been presented on the following conferences and seminars:

1. The 5th Baltic-Nordic Workshop on Algebra, Geometry, and Mathematical Physics, Bedlewo (Poland), October 12-16, 2009. "*Dynamical deformations and quantum counterparts of three-dimensional real Lie algebras over harmonic oscillator.*"
2. The 4th Baltic-Nordic Workshop on Algebra, Geometry, and Mathematical Physics, Tartu (Estonia), October 9-11, 2008. "*Operadic harmonic oscillator in low dimensions.*"
3. Noncommutative Structures in Mathematics and Physics (Satellite Conference to the 5th European Congress of Mathematics), Brussels (Belgium), July 22-26, 2008. "*Operadic harmonic oscillator.*"
4. Seminar dedicated to memory of the Estonian academician Arnold Humala, Tallinn (Estonia), March 10, 2008. "*Some results on operadic harmonic oscillator.*"
5. The 3rd Baltic-Nordic Workshop on Algebra, Geometry, and Mathematical Physics, Göteborg (Sweden), October 11-13, 2007. "*Operadic harmonic oscillator.*"
6. The 2nd Baltic-Nordic Workshop on Algebra, Geometry, and Mathematical Physics, Lund (Sweden), October 12-14, 2006. "*How to construct Lagrangian?*"
7. The 1st Baltic-Nordic Workshop on Algebra, Geometry, and Mathematical Physics, Tallinn (Estonia), October 8, 2005. "*The group SO(2) and Hamilton-Dirac mechanics.*"
8. Workshop on Algebra and its Applications, Nelijärve (Estonia), May 7-8, 2005. "*Lie theory and Hamilton-Dirac mechanics.*"
9. Workshop on Algebra and its Applications, Kääriku (Estonia), May 8-9, 2004. "*On Lie theory.*"

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Symbols

α	angle (unless used as an index)
S	action of a local Lie group G on a set $\mathcal{X}$
x, y, z	algebra elements
$[\cdot,\cdot]$	anti-commutative product, Gerstenhaber brackets
$[\cdot,\cdot]_t, [\cdot,\cdot]_\hbar$	anti-commutative product with parameters $t, \hbar$
$\mathcal{H}(\mathcal{C}), H^n(\mathcal{C})$	associated cohomology with homogeneous components $H^n(\mathcal{C})$
$A(x;y;z)$	associator of algebra elements x, y, z
$\hbar$	atomic Planck constant
$G_\pm^{\omega/2}, G_\pm^{3\omega/2}$	auxiliary functions for operadic Lax equations
$u_j^i, S_j^\mu; \psi_{jk}^i$	auxiliary functions for the Lie; Euler-Lagrange equations
Γ	auxiliary matrix for operadic Lax equations
$\langle\cdot,\cdot,\cdot\rangle$	auxiliary operation for the Getzler identity
b_i, e_i	basis elements
$\mathcal{A}$	binary (real) algebra
$\partial, \partial_\mu, \mathrm{ad}_\mu^{\mathrm{right}}$	(pre-)coboundary operator
$\bar{z}$	complex conjugation of a complex number z
f, g, q	canonical coordinates
p, s	canonical momenta
$\bar{\mathcal{E}}_R^n, \mathcal{C}oEnd_R^n$	coendomorphism operad
$\mathrm{Com}^-\mathcal{C}$	commutator algebra of an algebra $\mathrm{Com}\,\mathcal{C}$
$\mathrm{Com}\,\mathcal{C}$	composition algebra of an operad $\mathcal{C}$
$\circ, \circ_i, \bullet$	compositions: ordinary, partial, total
$\varphi_i, \varphi_i^\alpha$	constraints
U	coordinate neighbourhood
$g^i, h^j, \dots$	coordinates of a Lie group or algebra elements $g, h, \dots$
$\mathrm{Cup}\,\mathcal{C}$	cup-algebra of an operad $\mathcal{C}$
$\smile$	cup-multiplication
$\deg f; \deg\partial$	degrees: of a homogeneous element $f \in C^n$; of an operator ∂
$\mathrm{dev}_\bullet\,\partial$	derivation deviation of ∂ over $\bullet$
$\dot{f}, \dot{g}$	derivative of f, g (in case of one variable)

det	determinant
r	dimension of a group
$\times$	direct product
$\mathrm{I}^t, \mathrm{II}^t, \ldots$	dynamical deformations of the algebras I, II,...
X, X'	elements of a set $\mathcal{X}$
$\mathcal{E}_V^n, \mathcal{E}nd_V^n$	endomorphism operad of a unital K-module V
$L_{k\alpha} = 0$	Euler-Lagrange equations
μ^2	formal associator
f, g, h, v	group or K-module elements
H, H'	Hamiltonian
id	identity mapping
$\operatorname{Im} \partial$	image of an operator ∂
ξ, η	infinitesimal coefficients
S_x	infinitesimal operator of an action S
L_x, L_y, L_z	infinitesimal translations
p_0	initial value of p
g^{-1}	inverse element of g
$\hat{J}_\hbar^i$	Jacobi operator coordinates
J_t^i	Jacobiator coordinates
$\operatorname{Ker} \partial$	kernel of an operator ∂
T	kinetic energy
l	kinetic momentum
δ_j^i	Kronecker delta
$\lambda_j, \lambda_{j\alpha}^i$	Lagrange multipliers
$\mathcal{L}$	(scalar) Lagrangian
$\mathbf{L} = (\ldots, L_i, \ldots)$	(vector) Lagrangian with components L_i
$(L, M), (\mu, M)$	(operadic) Lax pair
$\mathcal{C}$	linear operad
φ	local coordinate mapping
(U, φ)	local coordinate system
G	local Lie group
$\overset{\circ}{\mu}$	(undeformed) multiplication
$\cdot$, μ	multiplication (unless used as an index)
μ^i	multiplication functions
$\hat{\mu}$	multiplication of a quantum algebra
$A, B, F, \varepsilon, \theta_i$	observables
$\xi_\pm$	observables, used for energy conservation law
$B_\pm, D_\pm, K_\pm$	observables, used for Lax representations
$A_\pm$	observables, quasi-canonical coordinates of h. oscillator
$\hat{q}, \hat{p}, \hat{A}_\pm, \hat{H}, \hat{\varepsilon}$	operators, acting on a Hilbert space of quantum states
$\hat{\xi}_\pm$	operators, quantum analogue of $\xi_\pm$

$\partial_x f$	partial derivative of f with respect to x
$\{\cdot,\cdot\}$	Poisson brackets
$\mu(g,h), gh$	product of group or algebra elements g and h
$F^0_\hbar, \mathcal{A}^\hbar$, $\mathrm{I}^\hbar$, $\mathrm{II}^\hbar, \dots$	quantum algebras
$\mathbb{R}^2$	real two-plane
$C, \omega, \zeta_1, \zeta_2$	real-valued constants
$a, b, C_i, \alpha_i, \beta_i, \gamma, \tau$	real-valued parameters
$\|f\|$; $\|a\|$	reduced degree of $f \in C^n$; absolute value (of a number a)
$(\cdot,\cdot,\cdot)$	scalar triple product
$\mathcal{X}$	set
$\mathbb{C}$	set of all complex numbers
$\mathrm{Hom}(V, V^{\otimes n})$	set of all homomorphisms $V \to V^{\otimes n}$
$\mathbb{N}$	set of all natural numbers
$\mathbb{R}$	set of all real numbers
R	K-space
c^i_{jk}	structure constants of a Lie group or Lie algebra
$\overset{\circ}{\mu}{}^i_{jk}$	structure constants of an algebra
μ^i_{jk}	structure functions, dynamical deformations of $\overset{\circ}{\mu}$
$\hat{\mu}^i_{jk}$	structure functions of a quantum algebra
$T_e(G)$	tangent space of a Lie group G at e
$\otimes$ ($\otimes_K$)	tensor product (over K)
$\{\cdot,\cdot,\cdot,\cdot\}$	tetrabraces
$\mathfrak{h}_1$	three-dimensional (real) Heisenberg algebra
E	total energy
$\mathfrak{T}(\mathcal{X})$	transformation group of a set $\mathcal{X}$
S_g, S_h	(G-)transformations of a set $\mathcal{X}$
$\{\cdot,\cdot,\cdot\}$	(Gerstenhaber) tribraces
$e \in G$, $I \in C^1$; i	unit; imaginary unit (unless used as an index)
K	unital associative commutative ring
C^n, V	unital K-module
$\approx$	weak equality

Other symbols

$\varphi : \mathcal{X}_1 \to \mathcal{X}_2$	"A mapping φ maps from a set $\mathcal{X}_1$ to a set $\mathcal{X}_2$"
$x \mapsto y$	"An element y is assigned to an element x"
$\mathcal{X}_1 \subset \mathcal{X}_2$	"A set $\mathcal{X}_1$ is a subset of a set $\mathcal{X}_2$"
$x \in \mathcal{X}$, or $\mathcal{X} \ni x$	"An element x belongs to a set $\mathcal{X}$"
$\forall$; $\exists$; $\vee$	"for all"; "there exist(s)"; "or"
$\Rightarrow$; $\Leftrightarrow$	"implies"; "is equivalent to"
$:=$	introduces a new notation
$\square$	indicates the end of proof

Part I

Lie theory and Lagrange formalism

CHAPTER

1

Topics on Lie theory

For the convenience of the reader, in this chapter some basic notions and topics of the Lie theory are concisely presented. There are several presentations of the Lie theory. Here we follow L. Pontryagin in his classical book on continuous groups [32]. The differential calculus of functions of several variables and some basic results of theory of differential equations are used.

1.1 Introduction

Sophus Lie (1842-1899) was an outstanding mathematician of the 19th century. His works on continuous transformation groups influenced the whole development of mathematics. Methods of algebra, geometry and mathematical analysis are simultaneously used in the Lie theory. The Lie theory is widely used in contemporary mathematics, theoretical and mathematical physics.

1.2 Lie group

The main subject of the *Lie theory* is the correspondence between the local Lie groups and Lie algebras. A *Lie group* is a group G having the structure of an analytical manifold such that the mapping $\mu : (g, h) \mapsto gh^{-1}$ of the direct product $G \times G$ into G is analytic. In other words, a Lie group is a set which has compatible structures of a group and an analytical manifold. The dimension of G will be denoted by r. By considering a neighbourhood of the unit e in G, one can formalize the notion of a *local* Lie group (see e.g [32] for the detailed definition).

Let G be a local Lie group. Since G is a differentiable manifold, one can fix a *local coordinate system* (U, φ) with the coordinate neighbourhood $U \subset G$ of the unit element e of G and a homeomorphism

$$\varphi : U \to \mathbb{R}^r, \quad \varphi(e) = 0$$

also called the *local coordinate mapping.* Denote the local coordinates of the point $g \in U$ by $g^1, g^2, \ldots, g^r$. Let $g, h \in U$ be such that $gh \in U$. For such elements g, h in U their product $gh := \mu(g, h)$ can be presented in the coordinate form by

$$(gh)^i = \mu^i(g, h) := \mu^i(g^1, \ldots, g^r, h^1, \ldots, h^r), \quad i = 1, \ldots, r \tag{1.1}$$

The functions μ^i are called the *multiplication functions.* From $ge = g = eg$ one gets

$$\mu^i(g^1, \ldots, g^r, 0, \ldots, 0) = g^i = \mu^i(0, \ldots, 0, g^1, \ldots, g^r) \tag{1.2}$$

The Lie theory explores the multiplication functions (1.1).

Expand the functions (1.1) into the Taylor series in a neighbourhood of the point $g^i = h^i = 0$ $(i = 1, \ldots, r)$:

$$\begin{aligned}(gh)^i &= g^i + u^i_j(g)h^j + \cdots \\ &= g^i + h^i + a^i_{jk} g^j h^k + \cdots \end{aligned} \tag{1.3}$$

Here, one has to note that $\det u^i_j(g) \neq 0$ and $u^i_j(e) = \delta^i_j$. The coefficients

$$c^i_{jk} := a^i_{jk} - a^i_{kj}$$

are called the *structure constants of* G. In the Lie theory, it is shown how the structure constants of G are related to the multiplication functions (1.2).

1.3 Lie and Maurer-Cartan equations. Lie algebra

The Lie theory can be seen as a differential-integral calculus on groups. Via the *Lie theorems* one can assign a *tangent Lie algebra* to a local Lie group and study relation between the tangent algebra and the local Lie group. It turns out that a local Lie group is determined by its structure constants.

Let G be a local Lie group with the coordinate system given in Section 1.2, e be the unit element of G. Then the multiplication functions (1.1) satisfy the *Lie equations* (*the first Lie theorem*)

$$u^k_j(gh)\frac{\partial (gh)^i}{\partial g^k} = u^i_j(g) \quad i, j = 1, \ldots, r \tag{1.4}$$

with initial conditions (1.2). To integrate (1.4) one has to know the auxiliary functions u^i_j. By using the integrability conditions of (1.4), i.e

$$\frac{\partial (gh)^i}{\partial g^j \partial g^k} = \frac{\partial (gh)^i}{\partial g^k \partial g^j}, \quad i, j, k = 1, \ldots, r$$

one can prove that the auxiliary functions satisfy the *Maurer-Cartan equations* (*the second Lie theorem*)

$$u_k^s(g)\frac{\partial u_j^i(g)}{\partial g^s} - u_j^s(g)\frac{\partial u_k^i(g)}{\partial g^s} = c_{jk}^p u_p^i(g) \tag{1.5}$$

with the initial conditions $u_j^i(e) = \delta_j^i$. The integration of (1.5) is explained in [32]. In $T_e(G)$, let us use the basis

$$b_i := \frac{\partial}{\partial g^i}\bigg|_{g=e} \quad \in T_e(G)$$

Thus e.g $T_e(G) \ni x = x^i b_i$. Let x and y be the tangent vectors from the tangent space $T_e(G)$ of G at e. Their product $[x,y] \in T_e(G)$ can be defined in the component form by

$$[x,y]^i := c_{jk}^i x^j y^k = -[y,x]^i, \quad i,j,k = 1,\dots,r$$

The tangent space $T_e(G)$ equipped with the anti-commutative multiplication $[\cdot,\cdot] : T_e(G) \times T_e(G) \to T_e(G)$ is called the *tangent algebra* of G.

For x in $T_e(G)$ define the *infinitisemal translations*

$$L_x := x^j u_j^k(g)\frac{\partial}{\partial g^k} \quad \in T_g(G)$$

One can see that $L_x = 0$ implies that $x = 0$. The Maurer-Cartan equations (1.5) can be rewritten as

$$[L_x, L_y] = -L_{[x,y]}, \quad x,y \in T_g(G)$$

It follows from the *Jacobi identity*

$$[[L_x,L_y],L_z] + [[L_y,L_z],L_x] + [[L_z,L_x],L_y] = 0$$

that the Jacobi identity in the tangent algebra holds as well:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \quad \forall\, x,y,z \in T_e(G) \tag{1.6}$$

Nowadays the anti-commutative algebras that satisfy the Jacobi identity (1.6) are called the *Lie algebras*. Thus, the *tangent algebra* $\{T_e(G),[\cdot,\cdot]\}$ of a local Lie group is a Lie algebra (the *third Lie theorem*).

It turns out that by using the structure constants of a real finite-dimensional Lie algebra one can (locally) find the multiplication gh of a local Lie group (the *third inverse Lie theorem*).

The inverse Lie theorems give an algorithm for constructing local Lie groups by its structure constants: first integrate the Maurer-Cartan equations to find the auxiliary functions and then integrate the Lie equations to find the multiplication μ. This is the essence of the *inverse Lie theorems*.

1.4 Lie transformation group

Let $\mathcal{X}$ be a set and let $\mathfrak{T}(\mathcal{X})$ denote the transformation group of $\mathcal{X}$, i.e. the group of bijective maps $\mathcal{X} \to \mathcal{X}$. Elements of $\mathfrak{T}(\mathcal{X})$ are called the *transformations* of $\mathcal{X}$. Multiplication in $\mathfrak{T}(\mathcal{X})$ is defined as the composition of transformations, and the unit element of $\mathfrak{T}(\mathcal{X})$ coincides with the identity transformation id of $\mathcal{X}$.

A map

$$S : G \to \mathfrak{T}(\mathcal{X}), \quad g \mapsto S_g$$

of a Lie group G into the group $\mathfrak{T}(\mathcal{X})$ is said to be an *action* of G on $\mathcal{X}$ if

$$S_e = \mathrm{id}, \quad S_g S_h = S_{gh}$$

The map S is also called a *representation* of G in $\mathfrak{T}(\mathcal{X})$. The transformations $S_g \in \mathfrak{T}(\mathcal{X})$ $(g \in G)$ are called *G-transformations* of $\mathcal{X}$. One can easily see that

$$S_g^{-1} = S_{g^{-1}}, \quad \forall\, g \in G$$

It seems quite natural to make G-transformations continuous as well. So, let G be a Lie group and let $\mathcal{X}$ denote a real, analytic manifold. The dimensions of G and $\mathcal{X}$ will be denoted as r and n, respectively.

The action S of G on $\mathcal{X}$ is said to be differentiable if the local coordinates of the point $S_g X$ are differentiable functions of the points $g \in G$ and $X \in \mathcal{X}$. In this case, the representation is said to be differentiable as well. The group $\mathfrak{T}(\mathcal{X})$ is said to be a *Lie transformation group* if G-transformations are continuous. In what follows, we shall consider continuous transformations only locally, and by "continuity" we mean differentiability as many times as needed. The action of g (from the vicinity of the unit $e \in G$) on $X \in \mathcal{X}$ we can write in local coordinates as

$$(S_g X)^\mu = S^\mu(X^1, \ldots, X^n; g^1, \ldots, g^r) := S^\mu(X; g)$$

As in the case of the Lie group (see (1.3)), the Taylor expansion

$$(S_g X)^\mu = X^\mu + S_j^\mu(X) g^j + O(g^2)$$

can be used to introduce the auxiliary functions S_j^μ of S. The functions $(S_g X)^\mu$ satisfy the Lie equations (the first Lie theorem, see (1.4))

$$u_j^k(g) \frac{\partial (S_g X)^\mu}{\partial g^k} = S_j^\mu(S_g X) \tag{1.7}$$

The integrability conditions of (1.7) read

$$\frac{\partial (S_g X)^\mu}{\partial g^j \partial g^k} = \frac{\partial (S_g X)^\mu}{\partial g^k \partial g^j}, \quad \mu = 1, \ldots, n, \quad j, k = 1, \ldots, r$$

and imply the *Lie-Cartan equations* (the second Lie theorem, see (1.5))

$$S_k^\nu(X)\frac{\partial S_j^\mu(X)}{\partial X^\nu} - S_j^\nu(X)\frac{\partial S_k^\mu(X)}{\partial X^\nu} = c_{jk}^p S_p^\mu(X) \tag{1.8}$$

By introducing the *infinitisemal operators* of S as

$$S_x := x^j S_j = x^j S_j^\mu(X)\frac{\partial}{\partial X^\mu}, \quad x \in T_e(G)$$

the Lie-Cartan equations (1.8) read

$$[S_x, S_y] = -S_{[x,y]}, \quad x, y \in T_e(G)$$

Thus, the vector space spanned by all infinitisemal operators of the Lie transformation group is a Lie algebra as well. It turns out that the infinitisemal operators locally determine the Lie transformation group (the inverse Lie theorem for Lie transformation groups) [32]. In a sense, the infinitisemal operators of S represent the tangent Lie algebra of G.

CHAPTER

2

Group $SO(2)$ and Hamilton-Dirac mechanics

In this chapter, the canonical formalism for the group $SO(2)$ is developed. The Lagrangian and Hamiltonian are constructed and their physical interpretation is given. It is shown that the constraints satisfy the canonical commutation relations and their consistency holds. The material of this chapter is based on [2, 30].

2.1 Introduction

The Lie group multiplication can be locally given as an integral of the first order partial differential equations called the Lie equations. One may ask for such a Lagrangian recapitulation of the Lie theory that the Euler-Lagrange equations coincide with the Lie equations. Based on the Lagrangian one can try to elaborate the corresponding canonical formalism for a Lie group.

In this chapter, the canonical formalism for real plane rotations is developed. It is shown that the one-parametric real plane rotation group $SO(2)$ can be seen as a toy model of the *Hamilton-Dirac mechanics* with constraints [4]. The Lagrangian and Hamiltonian are explicitly constructed. The Euler-Lagrange and the Hamilton equations coincide with the Lie equations. Consistency of the constraints is checked. It is also shown that the constraints satisfy the canonical commutation relations (CCR).

2.2 Real representation

The material presented in this section has been published in [30].

2.2.1 Lie equations and Lagrangian

Let $SO(2)$ be the rotation group of the real two-plane $\mathbb{R}^2$. Rotation of the plane $\mathbb{R}^2$ by an angle $\alpha \in \mathbb{R}$ is given by the transformation

$$\begin{cases} x' = f(x, y, \alpha) := x \cos\alpha - y \sin\alpha \\ y' = g(x, y, \alpha) := x \sin\alpha + y \cos\alpha \end{cases}$$

We consider the rotation angle α as a dynamical variable and the functions f and g as *field variables* for $SO(2)$. Denote

$$\dot{f} := \partial_\alpha f, \quad \dot{g} := \partial_\alpha g$$

The *infinitesimal coefficients* of the transformation are

$$\begin{cases} \xi(x, y) := \dot{f}(x, y, 0) = -y \\ \eta(x, y) := \dot{g}(x, y, 0) = x \end{cases}$$

and the *Lie equations* read

$$\begin{cases} \dot{f} = \xi(f, g) = -g \\ \dot{g} = \eta(f, g) = f \end{cases}$$

Our first aim is to find such a Lagrangian $\mathcal{L}(f, g, \dot{f}, \dot{g})$ that the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{f}} = 0, \quad \frac{\partial \mathcal{L}}{\partial g} - \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{g}} = 0$$

correspondingly coincide with the Lie equations.

Definition 2.1 (Lagrangian)**.** The Lagrangian $\mathcal{L}$ for $SO(2)$ can be defined by

$$\mathcal{L}(f, g, \dot{f}, \dot{g}) := \frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)$$

Theorem 2.2. *The Euler-Lagrange equations of $SO(2)$ coincide with its Lie equations.*

Proof. Calculate

$$\frac{\partial \mathcal{L}}{\partial f} = \frac{\partial}{\partial f}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = \frac{1}{2}\dot{g} - f$$
$$\frac{\partial \mathcal{L}}{\partial \dot{f}} = \frac{\partial}{\partial \dot{f}}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = -\frac{1}{2}g \quad \Longrightarrow \quad \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{f}} = -\frac{1}{2}\dot{g}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{f}} = 0 \quad \Longleftrightarrow \quad \frac{1}{2}\dot{g} - f + \frac{1}{2}\dot{g} = 0 \quad \Longleftrightarrow \quad \dot{g} = f$$

Analogously calculate

$$\frac{\partial \mathcal{L}}{\partial g} = \frac{\partial}{\partial g}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = -\frac{1}{2}\dot{f} - g$$
$$\frac{\partial \mathcal{L}}{\partial \dot{g}} = \frac{\partial}{\partial \dot{g}}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = \frac{1}{2}f \quad \Longrightarrow \quad \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{g}} = \frac{1}{2}\dot{f}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial g} - \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{g}} = 0 \quad \Longleftrightarrow \quad -\frac{1}{2}\dot{f} - g - \frac{1}{2}\dot{f} = 0 \quad \Longleftrightarrow \quad \dot{f} = -g \qquad \square$$

2.2.2 Physical interpretation

It follows from the Lie equations that

$$\ddot{f} + f = 0 = \ddot{g} + g$$

The Lagrangian of the latter is

$$\mathcal{L}(f, g, \dot{f}, \dot{g}) := \frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right) - \frac{1}{2}\left(f^2 + g^2\right)$$

The quantity

$$T := \frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right)$$

is the *kinetic energy* of a point $(f, g) \in \mathbb{R}^2$, meanwhile

$$l := f\dot{g} - g\dot{f}$$

is its *kinetic momentum* with respect to origin $(0, 0) \in \mathbb{R}^2$. By using the Lie equations one can easily check that

$$\dot{f}^2 + \dot{g}^2 = f\dot{g} - g\dot{f}$$

This relation has a simple explanation in the kinematics of a rigid body [10]. The kinetic energy of a point can be represented via its kinetic momentum as follows:

$$\frac{1}{2}\left(\dot{f}^2 + \dot{g}^2\right) = T = \frac{l}{2} = \frac{1}{2}(f\dot{g} - g\dot{f})$$

This relation explains the equivalence of the Lagrangians. Both Lagrangians give rise to the same extremals. Thus we can conclude, that for the given Lie equations (that is, on the extremals) of $SO(2)$ the Lagrangian $\mathcal{L}$ gives rise to a Lagrangian of the 2-dimensional harmonic oscillator.

2.2.3 Hamiltonian and Hamilton equations

Our aim is to develop canonical formalism for $SO(2)$ with the Lagrangian given by Definition 2.1. According to canonical formalism, define the *canonical momenta* as

$$p := \frac{\partial \mathcal{L}}{\partial \dot{f}} = \frac{\partial}{\partial \dot{f}}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = -\frac{g}{2}$$
$$s := \frac{\partial \mathcal{L}}{\partial \dot{g}} = \frac{\partial}{\partial \dot{g}}\left[\frac{1}{2}(f\dot{g} - \dot{f}g) - \frac{1}{2}\left(f^2 + g^2\right)\right] = +\frac{f}{2}$$

Note that the canonical momenta do not depend on velocities and so we are confronted with a *constrained system* [4] with two *constraints*

$$\varphi_1(f, g, p, s) := p + \frac{g}{2} = 0, \quad \varphi_2(f, g, p, s) := s - \frac{f}{2} = 0$$

Definition 2.3 (Hamiltonian)**.** According to Dirac [4], the *Hamiltonian* H for $SO(2)$ can be defined by

$$\begin{aligned} H &:= \overbrace{p\dot{f} + s\dot{g} - \mathcal{L}}^{H'} + \lambda_1\varphi_1(f, g, p, s) + \lambda_2\varphi_2(f, g, p, s) \\ &= p\dot{f} + s\dot{g} - \mathcal{L} + \lambda_1\left(p + \frac{g}{2}\right) + \lambda_2\left(s - \frac{f}{2}\right) \end{aligned}$$

where λ_1 and λ_2 are the *Lagrange multipliers*.

Lemma 2.4. *The Hamiltonian of $SO(2)$ can be presented as*

$$H = \frac{1}{2}\left(f^2 + g^2\right) + \lambda_1\left(p + \frac{g}{2}\right) + \lambda_2\left(s - \frac{f}{2}\right)$$

Proof. It is sufficient to calculate

$$\begin{aligned} H' &:= p\dot{f} + s\dot{g} - \mathcal{L} \\ &= p\dot{f} + s\dot{g} - \frac{1}{2}(f\dot{g} - \dot{f}g) + \frac{1}{2}\left(f^2 + g^2\right) \\ &= \dot{f}\left(p + \frac{g}{2}\right) + \dot{g}\left(s - \frac{f}{2}\right) + \frac{1}{2}\left(f^2 + g^2\right) \\ &= \frac{1}{2}\left(f^2 + g^2\right) \end{aligned}$$

□

Theorem 2.5 (Hamilton equations)**.** *If the Lagrange multipliers*

$$\lambda_1 = -g, \quad \lambda_2 = f$$

then the Hamilton equations

$$\dot{f} = \frac{\partial H}{\partial p}, \quad \dot{g} = \frac{\partial H}{\partial s}, \quad \dot{p} = -\frac{\partial H}{\partial f}, \quad \dot{s} = -\frac{\partial H}{\partial g}$$

coincide with the Lie equations of $SO(2)$*.*

Proof. Really, first calculate

$$\begin{aligned}
\dot{f} &= \frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left[\frac{1}{2}\left(f^2+g^2\right) - g\left(p+\frac{g}{2}\right) + f\left(s-\frac{f}{2}\right)\right] = -g \\
\dot{g} &= \frac{\partial H}{\partial s} = \frac{\partial}{\partial s}\left[\frac{1}{2}\left(f^2+g^2\right) - g\left(p+\frac{g}{2}\right) + f\left(s-\frac{f}{2}\right)\right] = f
\end{aligned}$$

Similarly calculate

$$\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial f} = -\frac{\partial}{\partial f}\left[\frac{1}{2}\left(f^2+g^2\right) - g\left(p+\frac{g}{2}\right) + f\left(s-\frac{f}{2}\right)\right] \\
&= -f - s + f = -s \\
\dot{s} &= -\frac{\partial H}{\partial g} = -\frac{\partial}{\partial g}\left[\frac{1}{2}\left(f^2+g^2\right) - g\left(p+\frac{g}{2}\right) + f\left(s-\frac{f}{2}\right)\right] \\
&= -g + p + g = p
\end{aligned}$$

Now use here the constraints $p = -g/2$ and $s = f/2$ to obtain

$$\begin{cases} \dot{p} = -s \\ \dot{s} = p \end{cases} \implies \begin{cases} -\frac{1}{2}\dot{g} = -\frac{1}{2}f \\ +\frac{1}{2}\dot{f} = -\frac{1}{2}g \end{cases} \implies \begin{cases} \dot{g} = f \\ \dot{f} = -g \end{cases}$$ □

Remark 2.6. One must remember that on the constraints must be applied **after** the calculations of the partial derivatives of H.

Corollary 2.7. *The Hamiltonian of* $SO(2)$ *can be presented in the form*

$$H = fs - gp$$

Then the Hamilton equations coincide with the Lie equations of $SO(2)$*.*

Remark 2.8. Note that our Hamiltonian H is the *angular momentum* of the point $(f, g) \in \mathbb{R}^2$. This is natural, because we consider plane rotations and the angular momentum is the generator of the rotations. The Hamiltonian obtained from the conventional Lagrangian (see Subsection 2.2.2) will be the *total energy*

$$\begin{aligned}
E &:= \frac{1}{2}\left(p^2+s^2\right) + \frac{1}{2}\left(f^2+g^2\right) \\
&= \frac{1}{2}(fs - gp) + \frac{1}{2}\left(f^2+g^2\right)
\end{aligned}$$

2.2.4 Poisson brackets and constraint algebra

Definition 2.9 (observables and Poisson brackets)**.** Sufficiently smooth functions of the canonical variables are called *observables*. The *Poisson brackets* of the observables A and B are defined by

$$\{A, B\} := \frac{\partial A}{\partial z}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial z} + \frac{\partial A}{\partial \overline{z}}\frac{\partial B}{\partial \overline{p}} - \frac{\partial A}{\partial \overline{p}}\frac{\partial B}{\partial \overline{z}}$$

Example 2.10. In particular, one can easily check that

$$\{f, p\} = 1 = \{g, s\}$$

and all other Poisson brackets between canonical variables vanish.

Example 2.11. In particular,

$$\{\varphi_1, H'\} = \left\{p + \frac{g}{2}, H'\right\} = -\frac{\partial H'}{\partial f} + \frac{1}{2}\frac{\partial H'}{\partial s} = -\frac{1}{2}\frac{\partial}{\partial f}\left(f^2 + g^2\right) = -f$$

and similarly

$$\{\varphi_2, H'\} = \left\{s - \frac{f}{2}, H'\right\} = -\frac{1}{2}\frac{\partial H'}{\partial p} - \frac{\partial H'}{\partial g} = -\frac{1}{2}\frac{\partial}{\partial g}\left(f^2 + g^2\right) = -g$$

Definition 2.12 (weak equality)**.** The observables A and B are called *weakly equal*, if

$$(A - B)\Big|_{\varphi_1 = 0 = \varphi_2} = 0$$

In this case we write $A \approx B$.

Theorem 2.13. *The Lie equations read*

$$\dot{f} \approx \frac{\partial H}{\partial p}, \quad \dot{g} \approx \frac{\partial H}{\partial s}, \quad \dot{p} \approx -\frac{\partial H}{\partial f}, \quad \dot{s} \approx -\frac{\partial H}{\partial g}$$

Theorem 2.14. *The Lie equations of $SO(2)$ can be presented in the Poisson-Hamilton form*

$$\dot{f} \approx \{f, H\}, \quad \dot{g} \approx \{g, H\}, \quad \dot{p} \approx \{p, H\}, \quad \dot{s} \approx \{s, H\}$$

Proof. As an example, check the third equation. We have

$$\{p, H\} := \frac{\partial p}{\partial f}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial f} + \frac{\partial p}{\partial g}\frac{\partial H}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial H}{\partial g} = -\frac{\partial H}{\partial f} \approx \dot{p} \qquad \square$$

Theorem 2.15. *The equation of motion of an observable F reads*

$$\dot{F} \approx \{F, H\}$$

Proof. By using the Hamilton equations, calculate

$$\begin{aligned}\dot{F} &= \frac{\partial F}{\partial f}\dot{f} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial g}\dot{g} + \frac{\partial F}{\partial s}\dot{s} \\ &\approx \frac{\partial F}{\partial f}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial f} + \frac{\partial F}{\partial g}\frac{\partial H}{\partial s} - \frac{\partial F}{\partial s}\frac{\partial H}{\partial g} \\ &:= \{F, H\}\end{aligned}$$

□

Theorem 2.16 (constraint algebra)**.** *The constraints of $SO(2)$ satisfy the canonical commutation relations (CCR)*

$$\{\varphi_1, \varphi_1\} = 0 = \{\varphi_2, \varphi_2\}, \quad \{\varphi_1, \varphi_2\} = 1$$

Proof. First two relations are evident. To check the third one, calculate

$$\begin{aligned}4\{\varphi_1, \varphi_2\} &= \{2p + g, 2s - f\} \\ &:= \frac{\partial(2p+g)}{\partial f}\frac{\partial(2s-f)}{\partial p} - \frac{\partial(2p+g)}{\partial p}\frac{\partial(2s-f)}{\partial f} \\ &+ \frac{\partial(2p+g)}{\partial g}\frac{\partial(2s-f)}{\partial s} - \frac{\partial(2p+g)}{\partial s}\frac{\partial(2s-f)}{\partial g} \\ &= -2\frac{\partial(2s-f)}{\partial f} + \frac{\partial(2s-f)}{\partial s} \\ &= 2 + 2 = 4\end{aligned}$$

□

2.2.5 Consistency

Now consider the dynamical behaviour of the constraints. Note that

$$\varphi_1 = 0 = \varphi_2 \quad \Longrightarrow \quad \dot{\varphi}_1 = 0 = \dot{\varphi}_2$$

To be consistent with equations of motion we must prove

Theorem 2.17 (consistency)**.** *The constraints of $SO(2)$ satisfy equations*

$$\{\varphi_1, H\} \approx \dot{\varphi}_1 = 0, \quad \{\varphi_2, H\} \approx \dot{\varphi}_2 = 0$$

Proof. Really, first calculate

$$\begin{aligned}\{\varphi_1, H\} &:= \{\varphi_1, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\ &\approx \{\varphi_1, H'\} + \lambda_1\{\varphi_1, \varphi_1\} + \lambda_2\{\varphi_1, \varphi_2\} \\ &= -f + \lambda_1 \cdot 0 + \lambda_2 \cdot 1 \\ &= -f + f \\ &= 0\end{aligned}$$

$$= \dot{\varphi}_1$$

Similarly,

$$\begin{aligned}\{\varphi_2, H\} &:= \{\varphi_2, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\ &\approx \{\varphi_2, H'\} + \lambda_1\{\varphi_2, \varphi_1\} + \lambda_2\{\varphi_2, \varphi_2\} \\ &= -g - \lambda_1 \cdot 1 + \lambda_2 \cdot 0 \\ &= -g + g \\ &= 0 \\ &= \dot{\varphi}_2\end{aligned}$$

□

Concluding remark 2.18. Once the canonical structure of $SO(2)$ established, one can perform the canonical quantization of $SO(2)$ as well. Physically this actually means the quantization of the angular momentum.

2.3 Complex representation

The material presented in this section has been published in [2].

2.3.1 Lie equations and Lagrangian

Consider the rotation group $SO(2)$ of the real two-plane $\mathbb{R}^2$. Rotation of $\mathbb{R}^2$ by an angle $\alpha \in \mathbb{R}$ is given by the transformation

$$\begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \end{cases}$$

In matrix notations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

By denoting $i = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, a generic element $z \in SO(2)$ reads as a complex number

$$z = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos\alpha + i\sin\alpha = e^{i\alpha}$$

We consider the rotation angle α as a dynamical variable and z as a field variable for $SO(2)$. The Lie equations read

$$\dot{z} := \partial_\alpha z = iz, \quad \dot{\overline{z}} := \partial_\alpha \overline{z} = -i\overline{z}$$

Definition 2.19 (Lagrangian)**.** The Lagrangian $\mathcal{L}$ for $SO(2)$ can be defined by

$$\mathcal{L}(z, \dot{z}, \overline{z}, \dot{\overline{z}}) := \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}$$

Remark 2.20. The Lie equations can be seen as an analogue of the Dirac equations. The corresponding Lagrangian is then a Dirac Lagrangian in 1-dimensional case.

Theorem 2.21. *The Euler-Lagrange equations of $SO(2)$ coincide with its Lie equations.*

Proof. Calculate

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \overline{z}} &= \frac{\partial}{\partial \overline{z}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = \frac{1}{2i}\dot{z} - z \\
\frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}} &= \frac{\partial}{\partial \dot{\overline{z}}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = -\frac{1}{2i}z \quad \Longrightarrow \quad \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}} = -\frac{1}{2i}\dot{z}
\end{aligned}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial \overline{z}} - \frac{\partial}{\partial \alpha}\frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}} = 0 \quad \Longleftrightarrow \quad \frac{1}{2i}\dot{z} - z + \frac{1}{2i}\dot{z} = 0 \quad \Longleftrightarrow \quad \dot{z} = iz \qquad \square$$

2.3.2 Hamiltonian and Hamilton equations

Our aim is to develop canonical formalism for $SO(2)$ in complex representation. We have already constructed such a Lagrangian $\mathcal{L}$ that the Euler-Lagrange equations coincides with the Lie equations. According to canonical prescription, define the *canonical momenta* as

$$\begin{aligned}
p &:= \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = +\frac{\overline{z}}{2i}, \\
s &:= \frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}} = \frac{\partial}{\partial \dot{\overline{z}}}\left[\frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) - z\overline{z}\right] = -\frac{z}{2i} = \overline{p}
\end{aligned}$$

Note that the canonical momenta do not depend on velocities and so we are confronted with a *constrained system* with two *constraints*

$$\varphi_1(z, \overline{z}, p, \overline{p}) := p - \frac{\overline{z}}{2i} = 0, \quad \varphi_2(z, \overline{z}, p, \overline{p}) := \overline{p} + \frac{z}{2i} = 0$$

Definition 2.22 (Hamiltonian)**.** According to Dirac theory [4] of constrained systems, the *Hamiltonian* H for $SO(2)$ can be defined by

$$H := \overbrace{p\dot{z} + \overline{p}\,\dot{\overline{z}} - \mathcal{L}}^{H'} + \lambda_1\varphi_1(z, \overline{z}, p, \overline{p}) + \lambda_2\varphi_2(z, \overline{z}, p, \overline{p})$$

$$= p\dot{z} + \overline{p}\,\dot{\overline{z}} - \mathcal{L} + \lambda_1\left(p - \frac{\overline{z}}{2i}\right) + \lambda_2\left(\overline{p} + \frac{z}{2i}\right)$$

where λ_1 and λ_2 are the *Lagrange multipliers*.

Lemma 2.23. *The Hamiltonian of $SO(2)$ can be presented as*

$$H = z\overline{z} + \lambda_1\left(p - \frac{\overline{z}}{2i}\right) + \lambda_2\left(\overline{p} + \frac{z}{2i}\right)$$

Proof. It is sufficient to calculate

$$\begin{aligned} H' &:= p\dot{z} + \overline{p}\,\dot{\overline{z}} - \mathcal{L} \\ &= p\dot{z} + \overline{p}\,\dot{\overline{z}} - \frac{1}{2i}(\dot{z}\overline{z} - z\dot{\overline{z}}) + z\overline{z} \\ &= \dot{z}\left(p - \frac{\overline{z}}{2i}\right) + \dot{\overline{z}}\left(\overline{p} + \frac{z}{2i}\right) + z\overline{z} \\ &= z\overline{z} \end{aligned}$$

□

Theorem 2.24 (Hamilton equations)**.** *If the Lagrange multipliers*

$$\lambda_1 = iz, \quad \lambda_2 = -i\overline{z} = \overline{\lambda_1},$$

then the Hamilton equations

$$\dot{z} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial z}$$

coincide with the Lie equations of $SO(2)$.

Proof. Really, calculate

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left[z\overline{z} + iz\left(p - \frac{\overline{z}}{2i}\right) - i\overline{z}\left(\overline{p} + \frac{z}{2i}\right)\right] = iz = \dot{z}, \\ \frac{\partial H}{\partial z} &= \frac{\partial}{\partial z}\left[z\overline{z} + iz\left(p - \frac{\overline{z}}{2i}\right) - i\overline{z}\left(\overline{p} + \frac{z}{2i}\right)\right] \\ &= \overline{z} + i\left(p - \frac{\overline{z}}{2i}\right) - i\overline{z}\frac{1}{2i}\frac{\overline{z}}{2} + ip - \frac{\overline{z}}{2} = ip \\ &- \left(\frac{\overline{z}}{2i}\right)^{\cdot} = -\dot{p} \end{aligned}$$

□

Remark 2.25. One must remember that the constraints must be applied **after** the calculations of the partial derivatives of H.

Corollary 2.26. *The Hamiltonian of $SO(2)$ can be presented in the form*

$$H = i(zp - \overline{z}\,\overline{p})$$

Then the Hamilton equations coincide with the Lie equations of $SO(2)$.

2.3.3 Poisson brackets and constraint algebra

By analogy with Definiton 2.9, propose

Definition 2.27 (Poisson brackets)**.** The *Poisson brackets* of the observables A and B are defined by

$$\{A, B\} := \frac{\partial A}{\partial z}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial z} + \frac{\partial A}{\partial \overline{z}}\frac{\partial B}{\partial \overline{p}} - \frac{\partial A}{\partial \overline{p}}\frac{\partial B}{\partial \overline{z}}$$

Example 2.28 (well known)**.** In particular, one can easily check that

$$\{z, p\} = 1 = \{\overline{z}, \overline{p}\}$$

and all other Poisson brackets between canonical variables identically vanish.

Example 2.29. In particular,

$$\{\varphi_1, H'\} = \left\{p + \frac{\overline{z}}{2i}, H'\right\} = -\frac{\partial H'}{\partial z} + \frac{1}{2i}\frac{\partial H'}{\partial \overline{p}} = -\frac{\partial}{\partial z}(z\overline{z}) = -\overline{z}$$

and similarly

$$\{\varphi_2, H'\} = \left\{\overline{p} + \frac{z}{2i}, H'\right\} = \frac{1}{2i}\frac{\partial H'}{\partial p} - \frac{\partial H'}{\partial \overline{z}} = -\frac{\partial}{\partial \overline{z}}(z\overline{z}) = -z$$

Using the notion of a weak equality one can propose

Theorem 2.30. *The Lie equations of $SO(2)$ read*

$$\dot{z} \approx \frac{\partial H}{\partial p}, \quad \dot{p} \approx -\frac{\partial H}{\partial z}$$

Theorem 2.31. *Lie equations of $SO(2)$ can be presented in the Poisson-Hamilton form*

$$\dot{z} \approx \{z, H\}, \quad \dot{p} \approx \{p, H\}$$

Proof. As an example, check the second equation. We have

$$\{p, H\} = \frac{\partial p}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial z} + \frac{\partial p}{\partial \overline{z}}\frac{\partial H}{\partial \overline{p}} - \frac{\partial p}{\partial \overline{p}}\frac{\partial H}{\partial \overline{z}} = -\frac{\partial H}{\partial z} \approx \dot{p} \qquad \square$$

Theorem 2.32. *The equation of motion of an observable F reads*

$$\dot{F} \approx \{F, H\}$$

Proof. By using the Hamilton equations, calculate

$$\begin{aligned}\dot{F} &= \frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial p}\dot{p} + \frac{\partial F}{\partial \overline{z}}\dot{\overline{z}} + \frac{\partial F}{\partial \overline{p}}\dot{\overline{p}} \\ &\approx \frac{\partial F}{\partial z}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial z} + \frac{\partial F}{\partial \overline{z}}\frac{\partial H}{\partial \overline{p}} - \frac{\partial F}{\partial \overline{p}}\frac{\partial H}{\partial \overline{z}} \\ &:= \{F, H\}\end{aligned}$$

□

Theorem 2.33 (constraint algebra)**.** *Constraints of $SO(2)$ satisfy the commutation relations*

$$\{\varphi_1, \varphi_1\} = 0 = \{\varphi_2, \varphi_2\}, \quad \{\varphi_1, \varphi_2\} = i$$

Proof. First two relations are evident. To check the third one, calculate

$$\begin{aligned}4i^2\{\varphi_1, \varphi_2\} &= \{2ip - \overline{z}, 2i\overline{p} + z\} \\ &:= \frac{\partial(2ip - \overline{z})}{\partial z}\frac{\partial(2i\overline{p} + z)}{\partial p} - \frac{\partial(2ip - \overline{z})}{\partial p}\frac{\partial(2i\overline{p} + z)}{\partial z} \\ &+ \frac{\partial(2ip - \overline{z})}{\partial \overline{z}}\frac{\partial(2i\overline{p} + z)}{\partial \overline{p}} - \frac{\partial(2ip - \overline{z})}{\partial \overline{p}}\frac{\partial(2i\overline{p} + z)}{\partial \overline{z}} \\ &= -2i\frac{\partial(2i\overline{p} + z)}{\partial z} - \frac{\partial(2i\overline{p} + z)}{\partial \overline{p}} \\ &= -2i - 2i = -4i\end{aligned}$$

□

2.3.4 Consistency

Now consider the dynamical behaviour of the constraints. Note that

$$\varphi_1 = 0 = \varphi_2 \quad \Longrightarrow \quad \dot{\varphi}_1 = 0 = \dot{\varphi}_2$$

To be consistent with equations of motion we must prove

Theorem 2.34 (consistency)**.** *The constraints of $SO(2)$ satisfy equations*

$$\{\varphi_1, H\} \approx \dot{\varphi}_1 = 0, \quad \{\varphi_2, H\} \approx \dot{\varphi}_2 = 0$$

Proof. Really, first calculate

$$\begin{aligned}\{\varphi_1, H\} &:= \{\varphi_1, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\ &\approx \{\varphi_1, H'\} + \lambda_1\{\varphi_1, \varphi_1\} + \lambda_2\{\varphi_1, \varphi_2\} \\ &= -\overline{z} + \lambda_1 \cdot 0 + \lambda_2 \cdot i \\ &= -\overline{z} + \overline{z} \\ &= 0\end{aligned}$$

$$= \dot{\varphi}_1$$

Similarly,

$$\begin{aligned}
\{\varphi_2, H\} &:= \{\varphi_2, H' + \lambda_1\varphi_1 + \lambda_2\varphi_2\} \\
&\approx \{\varphi_2, H'\} + \lambda_1\{\varphi_2, \varphi_1\} + \lambda_2\{\varphi_2, \varphi_2\} \\
&= -z - \lambda_1 \cdot i + \lambda_2 \cdot 0 \\
&= -z + z \\
&= 0 \\
&= \dot{\varphi}_2
\end{aligned}$$ □

CHAPTER

3

Constructing a vector Lagrangian

A method for constructing Lagrangians for the Lie transformation groups is explained. As an example, the Lagrangians for real plane rotations and affine transformations for the real line are constructed. The material presented in this chapter has been published in [26].

3.1 Main idea

It is a well-known problem in physics and mechanics how to construct Lagrangians for mechanical systems via their equations of motion. This *inverse variational problem* has been investigated for some types of equations of motion in [18].

In Chapter 2, the plane rotation group $SO(2)$ was considered as a toy model of the Hamilton-Dirac mechanics with constraints. By introducing a Lagrangian in a particular form, canonical formalism for $SO(2)$ was developed. The crucial idea of this approach is that the Euler-Lagrange and the Hamilton canonical equations must in a sense coincide with the Lie equations of the Lie transformation group.

In this chapter, the method for constructing such a Lagrangian is proposed. It is shown, how it is possible to find a Lagrangian, based on the Lie equations of the Lie transformation group.

By composing a Lagrangian, it is possible to describe the given Lie transformation group as a mechanical system and to develop the corresponding Lagrange and Hamilton formalisms for the Lie transformation group.

3.2 General method for constructing Lagrangians

Let G be a real r-parametric Lie group with unit $e \in G$ and let g^i $(i = 1, \ldots, r)$ denote the local coordinates of an element $g \in G$ from the vicinity of e. Let $\mathcal{X}$ be a real analytic n-dimensional manifold and denote the local coordinates of $X \in \mathcal{X}$ by X^α, $\alpha = 1, \ldots, n$ (see Section 1.4). Consider a (left) differentiable action of G on $\mathcal{X}$ given by

$$X' = S_g X \quad \in \mathcal{X}$$

Let gh denote the multiplication of G. Then

$$S_g S_h = S_{gh}, \quad \forall\, g, h \in G$$

By introducing the *auxiliary functions* u^i_j and S^α_j by

$$\begin{aligned} (gh)^i \quad &:= \quad h^i + u^i_j(h) g^j + \ldots\ , \\ (S_g X)^\alpha \quad &:= \quad X^\alpha + S^\alpha_j(X) g^j + \ldots \end{aligned}$$

the Lie equations read

$$\varphi^\alpha_j(X; g) := u^s_j(g) \frac{\partial (S_g X)^\alpha}{\partial g^s} - S^\alpha_j(S_g X) = 0$$

Then we search for such a vector Lagrangian $\mathbf{L} := (L_1, \ldots, L_r)$ with components

$$L_k := \sum_{\alpha=1}^{n} \sum_{l=1}^{r} \lambda^l_{k\alpha} \varphi^\alpha_l, \quad k = 1, 2, \ldots, r$$

and such *Lagrange multipliers* $\lambda^l_{k\alpha}$ that the Euler-Lagrange equations in a sense coincide with the Lie equations.

The notion of a vector Lagrangian was introduced and developed in [6,36].

By analogy with Definition 2.12, we generalize the notion of weak equality to the case of $n \cdot r$ constraints:

Definition 3.1 (weak equality)**.** The functions A and B are called *weakly equal*, if

$$(A - B)\Big|_{\varphi^\alpha_j = 0} = 0, \quad j = 1, 2, \ldots, r, \quad \alpha = 1, 2, \ldots, n$$

In this case we write $A \approx B$.

By denoting

$$X'^\alpha_i := \frac{\partial X'^\alpha}{\partial g^i}$$

the conditions for the Lagrange multipliers read as the *weak Euler-Lagrange equations*

$$L_{k\alpha} := \frac{\partial L_k}{\partial X'^\alpha} - \sum_{i=1}^{r} \frac{\partial}{\partial g^i} \frac{\partial L_k}{\partial X'^\alpha_i} \approx 0$$

Finally, one must check by direct calculations that the Euler-Lagrange equations $L_{k\alpha} = 0$ imply the Lie equations of the Lie transformation group.

3.3 Lagrangian for the group $SO(2)$

Consider the 1-parameter Lie transformation group $SO(2)$, *the rotation group of the real two-plane* $\mathbb{R}^2$. In this case $n = 2$ and $r = 1$. Rotation of the plane $\mathbb{R}^2$ by an angle $g \in \mathbb{R}$ is given by the transformation

$$\begin{cases} (S_g X)^1 = X'^1 = X'^1(X^1, X^2, g) := X^1 \cos g - X^2 \sin g \\ (S_g X)^2 = X'^2 = X'^2(X^1, X^2, g) := X^1 \sin g + X^2 \cos g \end{cases}$$

We consider the rotation angle g as a dynamical variable and the functions X'^1 and X'^2 as *field variables* for the plane rotation group $SO(2)$.

Denote

$$\dot{X}'^\alpha := \frac{\partial X'^\alpha}{\partial g}$$

The *infinitesimal coefficients* of the transformation are

$$\begin{cases} S^1(X^1, X^2) := \dot{X}'^1(X^1, X^2, e) = -X^2 \\ S^2(X^1, X^2) := \dot{X}'^2(X^1, X^2, e) = X^1 \end{cases}$$

and the Lie equations read

$$\begin{cases} \dot{X}'^1 = S^1(X'^1, X'^2) = -X'^2 \\ \dot{X}'^2 = S^2(X'^1, X'^2) = X'^1 \end{cases}$$

Rewrite the Lie equations in implicit form as follows:

$$\begin{cases} \varphi_1^1 := \dot{X}'^1 + X'^2 = 0 \\ \varphi_1^2 := \dot{X}'^2 - X'^1 = 0 \end{cases}$$

We search a Lagrangian of $SO(2)$ in the form

$$L_1 = \sum_{\alpha=1}^{2} \sum_{l=1}^{1} \lambda_{1\alpha}^l \varphi_l^\alpha = \lambda_{11}^1 \varphi_1^1 + \lambda_{12}^1 \varphi_1^2$$

It is more convenient to rewrite it as follows:

$$\mathcal{L} := \lambda_1 \varphi^1 + \lambda_2 \varphi^2$$

where the Lagrange multipliers λ_1 and λ_2 are to be found from the weak Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial X'^1} - \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^1} \approx 0, \quad \frac{\partial \mathcal{L}}{\partial X'^2} - \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} \approx 0$$

Calculate

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X'^1} &= \frac{\partial}{\partial X'^1}\left[\lambda_1(\dot{X}'^1 + X'^2) + \lambda_2(\dot{X}'^2 - X'^1)\right] \\
&= \frac{\partial \lambda_1}{\partial X'^1}\varphi^1 + \frac{\partial \lambda_2}{\partial X'^1}\varphi^2 - \lambda_2 \approx -\lambda_2 \ , \\
\frac{\partial \mathcal{L}}{\partial \dot{X}'^1} &= \frac{\partial}{\partial \dot{X}'^1}\left[\lambda_1(\dot{X}'^1 + X'^2) + \lambda_2(\dot{X}'^2 - X'^1)\right] \approx \lambda_1 \ , \\
\frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^1} &= \frac{\partial \lambda_1}{\partial g} = \frac{\partial \lambda_1}{\partial X'^1}\dot{X}'^1 + \frac{\partial \lambda_1}{\partial X'^2}\dot{X}'^2 \approx -\frac{\partial \lambda_1}{\partial X'^1}X'^2 + \frac{\partial \lambda_1}{\partial X'^2}X'^1
\end{aligned}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial X'^1} - \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^1} \approx 0 \quad \Longleftrightarrow \quad -\lambda_2 + \frac{\partial \lambda_1}{\partial X'^1}X'^2 - \frac{\partial \lambda_1}{\partial X'^2}X'^1 \approx 0$$

Analogously calculate

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X'^2} &= \frac{\partial}{\partial X'^2}\left[\lambda_1(\dot{X}'^1 + X'^2) + \lambda_2(\dot{X}'^2 - X'^1)\right] \\
&= \frac{\partial \lambda_1}{\partial X'^2}\varphi^1 + \frac{\partial \lambda_2}{\partial X'^2}\varphi^2 + \lambda_1 \approx \lambda_1 \ , \\
\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} &= \frac{\partial}{\partial \dot{X}'^2}\left[\lambda_1(\dot{X}'^1 + X'^2) + \lambda_2(\dot{X}'^2 - X'^1)\right] \approx \lambda_2 \ , \\
\frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} &= \frac{\partial \lambda_2}{\partial g} = \frac{\partial \lambda_2}{\partial X'^1}\dot{X}'^1 + \frac{\partial \lambda_2}{\partial X'^2}\dot{X}'^2 \approx -\frac{\partial \lambda_2}{\partial X'^1}X'^2 + \frac{\partial \lambda_2}{\partial X'^2}X'^1
\end{aligned}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial X'^2} - \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} \approx 0 \quad \Longleftrightarrow \quad \lambda_1 + \frac{\partial \lambda_2}{\partial X'^1}X'^2 - \frac{\partial \lambda_2}{\partial X'^2}X'^1 \approx 0$$

So the calculations imply the following system of differential equations for the Lagrange multipliers:

$$\begin{cases} -\frac{\partial \lambda_1}{\partial X'^1}X'^2 + \frac{\partial \lambda_1}{\partial X'^2}X'^1 \approx -\lambda_2 \\ -\frac{\partial \lambda_2}{\partial X'^1}X'^2 + \frac{\partial \lambda_2}{\partial X'^2}X'^1 \approx \lambda_1 \end{cases}$$

We are not searching for the general solution for this system of partial differential equations, but the Lagrange multipliers are supposed to be a linear combination of the field variables X'^1 and X'^2,

$$\begin{cases} \lambda_1 := \alpha_1 X'^1 + \alpha_2 X'^2 \\ \lambda_2 := \beta_1 X'^1 + \beta_2 X'^2, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \end{cases}$$

By using these expressions, one has

$$\begin{cases} -\alpha_1 X'^2 + \alpha_2 X'^1 \approx -\beta_1 X'^1 - \beta_2 X'^2 \\ -\beta_1 X'^2 + \beta_2 X'^1 \approx \alpha_1 X'^1 + \alpha_2 X'^2 \end{cases} \iff \begin{cases} (\alpha_2 + \beta_1) X'^1 + (\beta_2 - \alpha_1) X'^2 \approx 0 \\ (\beta_2 - \alpha_1) X'^1 - (\alpha_2 + \beta_1) X'^2 \approx 0 \end{cases}$$

This is a homogeneous system of two linear equations of four unknowns $\alpha_1, \alpha_2, \beta_1, \beta_2$. The system is satisfied, if

$$\begin{cases} \alpha_2 + \beta_1 = 0 \\ \beta_2 - \alpha_1 = 0 \end{cases} \iff \begin{cases} \beta_1 = -\alpha_2 \\ \beta_2 = \alpha_1 \end{cases}$$

The parameters α_1, α_2 are free. Thus

$$\begin{cases} \lambda_1 = \alpha_1 X'^1 + \alpha_2 X'^2 \\ \lambda_2 = -\alpha_2 X'^1 + \alpha_1 X'^2 \end{cases}$$

and the desired Lagrangian for $SO(2)$ reads

$$\mathcal{L} = \alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left[X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right] \quad (3.1)$$

with free real parameters $\alpha_1, \alpha_2 \neq 0$. Now we can propose

Theorem 3.2. *Let $\alpha_2 \neq 0$. The Euler-Lagrange equations for the Lagrangian (3.1) coincide with the Lie equations of $SO(2)$.*

Proof. Calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X'^1} &= \frac{\partial}{\partial X'^1} \left[\alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left(X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right) \right] \\ &= \alpha_1 \dot{X}'^1 - \alpha_2 \dot{X}'^2 + 2\alpha_2 X'^1 , \\ \frac{\partial \mathcal{L}}{\partial \dot{X}'^1} &= \frac{\partial}{\partial \dot{X}'^1} \left[\alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left(X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right) \right] \\ &= \alpha_1 X'^1 + \alpha_2 X'^2 \quad \Longrightarrow \quad \frac{\partial}{\partial g} \frac{\partial \mathcal{L}}{\partial \dot{X}'^1} = \alpha_1 \dot{X}'^1 + \alpha_2 \dot{X}'^2 \end{aligned}$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial X'^1} - \frac{\partial}{\partial g} \frac{\partial \mathcal{L}}{\partial \dot{X}'^1} = 0 \iff 2\alpha_2 X'^1 - 2\alpha_2 \dot{X}'^2 = 0 \iff \dot{X}'^2 = X'^1$$

Analogously calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X'^2} &= \frac{\partial}{\partial X'^2} \left[\alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left(X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right) \right] \\ &= \alpha_2 \dot{X}'^1 + 2\alpha_2 X'^2 + \alpha_1 \dot{X}'^2 , \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} = \frac{\partial}{\partial \dot{X}'^2}\left[\alpha_1(X'^1\dot{X}'^1 + X'^2\dot{X}'^2) + \alpha_2\left(X'^2\dot{X}'^1 + (X'^2)^2 - X'^1\dot{X}'^2 + (X'^1)^2\right)\right]$$
$$= -\alpha_2 X'^1 + \alpha_1 X'^2 \quad \Longrightarrow \quad \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} = -\alpha_2\dot{X}'^1 + \alpha_1\dot{X}'^2$$

from which it follows

$$\frac{\partial \mathcal{L}}{\partial X'^2} - \frac{\partial}{\partial g}\frac{\partial \mathcal{L}}{\partial \dot{X}'^2} = 0 \quad \Longleftrightarrow \quad 2\alpha_2\dot{X}'^1 + 2\alpha_2 X'^2 = 0 \quad \Longleftrightarrow \quad \dot{X}'^1 = -X'^2 \ \square$$

Remark 3.3. While the Lagrangian $\mathcal{L}$ of $SO(2)$ contains two free parameters α_1, α_2, particular forms of it can be found taking into account physical considerations. In particular, if $\alpha_1 = 0$ and $\alpha_2 = -1/2$, then the Lagrangian of $SO(2)$ reads

$$\mathcal{L}(X'^1, X'^2, \dot{X}'^1, \dot{X}'^2) := \frac{1}{2}(X'^1\dot{X}'^2 - \dot{X}'^1 X'^2) - \frac{1}{2}\left[(X'^1)^2 + (X'^2)^2\right]$$

By using the Lie equations one can easily check that

$$X'^1\dot{X}'^2 - \dot{X}'^1 X'^2 = (\dot{X}'^1)^2 + (\dot{X}'^2)^2$$

The interpretation of this result was given in Subsection 2.2.2.

Remark 3.4. Thus, one has explained how it is possible to derive the Lagrangian postulated in Definition 2.1.

3.4 Lagrangian for the group of affine transformations

Now consider the affine transformations of the real line. The latter may be represented by

$$\begin{cases} X'^1 = X'^1(X^1, X^2, g^1, g^2) := g^1 X^1 + g^2 \\ X'^2 = X'^2(X^1, X^2, g^1, g^2) := 1, \qquad 0 \neq g^1, g^2 \in \mathbb{R} \end{cases}$$

Thus $r = 2$ and $n = 2$. Denote

$$e := (1, 0), \quad g^{-1} := \frac{1}{g^1}(1, -g^2)$$

First, find the multiplication rule

$$(X'')^1 := (X'^1)' = S_{gh}X^1 = S_g(S_h X^1) = S_g(h^1 X^1 + h^2)$$
$$= g^1(h^1 X^1 + h^2) + g^2 = (g^1 h^1)X^1 + (g^1 h^2 + g^2)$$

Calculate the infinitesimal coefficients

$$\begin{aligned} S_1^1(X^1,X^2) &:= X_1'^1\big|_{g=e} = X_1 \ , \\ S_2^1(X^1,X^2) &:= X_2'^1\big|_{g=e} = 1 \ , \\ S_1^2(X^1,X^2) &:= X_1'^2\big|_{g=e} = 0 \ , \\ S_2^2(X^1,X^2) &:= X_2'^2\big|_{g=e} = 0 \end{aligned}$$

and the auxiliary functions

$$\begin{aligned} u_1^1(g) &:= \left.\frac{\partial (S_{gh}X)^1}{\partial g^1}\right|_{h=g^{-1}} = \frac{1}{g^1} \ , \\ u_2^1(g) &:= \left.\frac{\partial (S_{gh}X)^1}{\partial g^2}\right|_{h=g^{-1}} = 0 \ , \\ u_1^2(g) &:= \left.\frac{\partial (S_{gh}X)^2}{\partial g^1}\right|_{h=g^{-1}} = -\frac{g^2}{g^1} \ , \\ u_2^2(g) &:= \left.\frac{\partial (S_{gh}X)^2}{\partial g^2}\right|_{h=g^{-1}} = 1 \end{aligned}$$

Next, write Lie equations and find constraints

$$\left\{ \begin{array}{l} X_1'^1 = \frac{1}{g^1}X'^1 - \frac{g^2}{g^1} \\ X_2'^1 = 1 \\ X_1'^2 = 0 \\ X_2'^2 = 0 \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \varphi_1^1 := X_1'^1 - \frac{1}{g^1}X'^1 - \frac{g^2}{g^1} \\ \varphi_2^1 := X_2'^1 - 1 \\ \varphi_1^2 := X_1'^2 \\ \varphi_2^2 := X_2'^2 \end{array} \right.$$

We search for a vector Lagrangian $\mathbf{L} = (L_1, L_2)$ as follows:

$$\begin{aligned} L_k &= \sum_{\alpha=1}^{2}\sum_{l=1}^{2} \lambda_{k\alpha}^l \varphi_l^\alpha = \lambda_{k1}^1\varphi_1^1 + \lambda_{k1}^2\varphi_2^1 + \lambda_{k2}^1\varphi_1^2 + \lambda_{k2}^2\varphi_2^2 \\ &= \lambda_{k1}^1\left(X_1'^1 - \frac{1}{g^1}X'^1 - \frac{g^2}{g^1}\right) + \lambda_{k1}^2\left(X_2'^1 - 1\right) + \lambda_{k2}^1 X_1'^2 + \lambda_{k2}^2 X_2'^2, \quad k = 1,2 \end{aligned}$$

By substituting the Lagrange multipliers $\lambda_{k\alpha}^s$ into the weak Euler-Lagrange equations

$$\frac{\partial L_k}{\partial X'^\alpha} - \sum_{i=1}^{2} \frac{\partial}{\partial g^i}\frac{\partial L_k}{\partial X_i'^\alpha} \approx 0$$

we get the following PDE system

$$\left\{ \begin{array}{l} (X'^1 - g^2)\frac{\partial \lambda_{k1}^1}{\partial X'^1} + g^1\frac{\partial \lambda_{k1}^2}{\partial X'^1} + \lambda_{k1}^1 \approx 0 \\ (X'^1 - g^2)\frac{\partial \lambda_{k2}^1}{\partial X'^1} + g^1\frac{\partial \lambda_{k2}^2}{\partial X'^1} \approx 0, \quad k = 1,2 \end{array} \right.$$

We find some particular solutions for this system. For example,

$$k=1:\begin{cases}\lambda_{11}^1 := 0\\ \lambda_{11}^2 := \psi_{11}^2(X'^2)\\ \lambda_{12}^1 := \psi_{12}^1(X'^2)\\ \lambda_{12}^2 := \psi_{12}^2(X'^2)\end{cases} \quad \text{and} \quad k=2:\begin{cases}\lambda_{21}^1 := \psi_{21}^1(X'^2)\\ \lambda_{21}^2 := -\frac{X'^1}{g^1}\psi_{21}^1(X'^2)\\ \lambda_{22}^1 := 0\\ \lambda_{11}^2 := 0\end{cases}$$

with $\psi_{21}^1(X'^2), \psi_{11}^2(X'^2), \psi_{12}^1(X'^2), \psi_{12}^2(X'^2)$ as arbitrary real-valued functions of X'^2.

Thus we can define the Lagrangian $\mathbf{L} = (L_1, L_2)$ with

$$\begin{cases} L_1 = \psi_{11}^2(X'^2)(X_2'^1 - 1) + \psi_{12}^1(X'^2)X_1'^2 + \psi_{12}^2(X'^2)X_2'^2 \\ L_2 = \psi_{21}^1(X'^2)\left(X_1'^1 - \frac{1}{g^1}X'^1 + \frac{g^2}{g^1}\right)\end{cases} \tag{3.2}$$

and propose

Theorem 3.5. *The Euler-Lagrange equations for the vector Lagrangian $\mathbf{L} = (L_1, L_2)$ with components (3.2) coincide with the Lie equations of the affine transformations of the real line.*

Proof. Calculate

$$\begin{aligned}
\frac{\partial L_1}{\partial X'^1} &= \frac{\partial}{\partial X'^1}\left[\psi_{11}^2(X'^2)(X_2'^1 - 1) + \psi_{12}^1(X'^2)X_1'^2 + \psi_{12}^2(X'^2)X_2'^2\right] = 0\ ,\\
\frac{\partial}{\partial g^1}\frac{\partial L_1}{\partial X_1'^1} &= \frac{\partial}{\partial g^1}0 = 0\ ,\\
\frac{\partial}{\partial g^2}\frac{\partial L_1}{\partial X_2'^1} &= \frac{\partial \psi_{11}^2(X'^2)}{\partial g^2} = \frac{\partial \psi_{11}^2(X'^2)}{\partial X'^2}X_2'^2
\end{aligned}$$

from which it follows

$$\frac{\partial L_1}{\partial X'^1} - \sum_{i=1}^{2}\frac{\partial}{\partial g^i}\frac{\partial L_1}{\partial X_i'^1} = 0 \quad\Longleftrightarrow\quad \frac{\partial \psi_{11}^2(X'^2)}{\partial X'^2}X_2'^2 = 0 \quad\Longrightarrow\quad X_2'^2 = 0$$

Analogously calculate

$$\begin{aligned}
\frac{\partial L_1}{\partial X'^2} &= \frac{\partial}{\partial X'^2}\left[\psi_{11}^2(X'^2)(X_2'^1 - 1) + \psi_{12}^1(X'^2)X_1'^2 + \psi_{12}^2(X'^2)X_2'^2\right]\\
&= \frac{\partial \psi_{11}^2(X'^2)}{\partial X'^2}(X_2'^1 - 1) + \frac{\partial \psi_{12}^1(X'^2)}{\partial X'^2}X_1'^2 + \frac{\partial \psi_{12}^2(X'^2)}{\partial X'^2}X_2'^2\ ,\\
\frac{\partial}{\partial g^1}\frac{\partial L_1}{\partial X_1'^2} &= \frac{\partial \psi_{12}^1(X'^2)}{\partial g^1} = \frac{\partial \psi_{12}^1(X'^2)}{\partial X'^2}X_1'^2\ ,\\
\frac{\partial}{\partial g^2}\frac{\partial L_1}{\partial X_2'^2} &= \frac{\partial \psi_{12}^2(X'^2)}{\partial g^2} = \frac{\partial \psi_{12}^2(X'^2)}{\partial X'^2}X_2'^2
\end{aligned}$$

from which it follows

$$\frac{\partial L_1}{\partial X'^2}-\sum_{i=1}^{2}\frac{\partial}{\partial g^i}\frac{\partial L_1}{\partial X_i'^2}=0 \quad\Longleftrightarrow\quad \frac{\partial \psi^2_{11}(X'^2)}{\partial X'^2}(X_2'^1-1)=0 \quad\Longrightarrow\quad X_2'^1-1=0$$

Now we differentiate the second component of the Lagrangian $\mathbf{L}$. Calculate

$$\begin{aligned}
\frac{\partial L_2}{\partial X'^1} &= \frac{\partial}{\partial X'^1}\left[\psi^1_{21}(X'^2)\left(X_1'^1-\frac{1}{g^1}X'^1+\frac{g^2}{g^1}\right)-\frac{1}{g^1}X'^1\psi^1_{21}(X'^2)(X_2'^1-1)\right]\\
&= -\frac{1}{g^1}\psi^1_{21}(X'^2)-\frac{1}{g^1}\psi^1_{21}(X'^2)X_2'^1+\frac{1}{g^1}\psi^1_{21}(X'^2)=-\frac{1}{g^1}\psi^1_{21}(X'^2)X_2'^1\ ,\\
\frac{\partial}{\partial g^1}\frac{\partial L_2}{\partial X_1'^1} &= \frac{\partial\psi^1_{21}(X'^2)}{\partial g^1}=\frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}X_1'^2\ ,\\
\frac{\partial}{\partial g^2}\frac{\partial L_2}{\partial X_2'^1} &= \frac{\partial}{\partial g^2}\left(-\frac{1}{g^1}X'^1\psi^1_{21}(X'^2)\right)=-\frac{1}{g^1}\left(\psi^1_{21}(X'^2)X_2'^1+X'^1\frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}X_2'^2\right)
\end{aligned}$$

from which it follows

$$\frac{\partial L_2}{\partial X'^1}-\sum_{i=1}^{2}\frac{\partial}{\partial g^i}\frac{\partial L_2}{\partial X_i'^1}=0 \quad\Longleftrightarrow\quad \frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}\left(X_1'^2-\frac{1}{g^1}X'^1X_2'^2\right)=0$$

$$\Longrightarrow\quad X_1'^2-\frac{1}{g^1}X'^1X_2'^2=0$$

Analogously calculate

$$\begin{aligned}
\frac{\partial L_2}{\partial X'^2} &= \frac{\partial}{\partial X'^2}\left[\psi^1_{21}(X'^2)\left(X_1'^1-\frac{1}{g^1}X'^1+\frac{g^2}{g^1}\right)-\frac{1}{g^1}X'^1\psi^1_{21}(X'^2)(X_2'^1-1)\right]\\
&= \frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}\left(X_1'^1-\frac{1}{g^1}X'^1+\frac{g^2}{g^1}\right)-\frac{1}{g^1}\frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}X'^1\left(X_2'^1-1\right)\ ,\\
\frac{\partial}{\partial g^1}\frac{\partial L_2}{\partial X_1'^2} &= \frac{\partial}{\partial g^1}0=0\ ,\\
\frac{\partial}{\partial g^2}\frac{\partial L_2}{\partial X_2'^2} &= \frac{\partial}{\partial g^2}0=0
\end{aligned}$$

from which it follows

$$\frac{\partial L_2}{\partial X'^2}-\sum_{i=1}^{2}\frac{\partial}{\partial g^i}\frac{\partial L_2}{\partial X_i'^2}=0 \quad\Longleftrightarrow\quad \frac{\partial\psi^1_{21}(X'^2)}{\partial X'^2}\left(X_1'^1-\frac{1}{g^1}X'^1X_2'^1+\frac{g^2}{g^1}\right)=0$$

$$\Longrightarrow\quad X_1'^1-\frac{1}{g^1}X'^1X_2'^1+\frac{g^2}{g^1}=0$$

Thus the Euler-Lagrange equations read

$$\begin{cases} X_2'^2 = 0 \\ X_2'^1 - 1 = 0 \\ X_1'^2 - \frac{1}{g^1} X'^1 X_2'^2 = 0 \\ X_1'^1 - \frac{1}{g^1} X'^1 X_2'^1 + \frac{g^2}{g^1} = 0 \end{cases}$$

It can be easily verified, that the latter is equivalent to the system of the Lie equations. □

Remark 3.6. In the proof it is assumed that

$$\frac{\partial \psi_{21}^1(X'^2)}{\partial X'^2} \neq 0$$

Remark 3.7. While the Lagrangian **L** contains four arbitrary functions, particular forms of it can be fixed by taking into account physical considerations.

Part II

Operadic dynamics and harmonic oscillator

CHAPTER

4

Operadic dynamics

Based on the Gerstenhaber theory, it is explained how the operadic dynamics may be introduced. Operadic observables satisfy the Gerstenhaber algebra identities and their time evolution is governed by operadic evolution equation. The notion of an operadic Lax pair is also introduced. As an example, an operadic (representation of) harmonic oscillator is proposed.

Sections 4.1–4.7 are mostly based on the material from [22]. The material of Sections 4.8–4.10 has been published in [28].

4.1 Introduction

In 1963, Gerstenhaber invented [7] an *operad calculus* in the Hochschild complex of an associative algebra; operads were introduced under the name of *pre-Lie systems*. In the same year, Stasheff constructed [35] (see also [33]) quite an original geometrical operad, which nowadays is called an *associahedra*. The notion of an operad was further formalised by May [21] as a tool for iterated loop spaces. Examples of operads are algebraic operations and co-operations, rooted trees, little squares and Feynman diagrams.

The main principles of the operad calculus (brace algebra) were presented by Gerstenhaber and Voronov [9,37]. Some quite remarkable research activity in the operad theory and its applications can be observed in the last decade (eg. [17,20,34]). It may be said that operads are also becoming an important tool for quantum field theory and deformation quantization [14].

Today, much attention is given to static operadic constructions. For dynamical operations one has to prescribe their time evolution. In this section, based on the Gerstenhaber theory, clarification is given on how operadic dynamics may be introduced.

We start from simple algebraic axioms. Basic algebraic constructions associated with linear operads are introduced. Their properties and the first derivation deviations for the coboundary operator are presented explicitly. Under certain conditions (a formal associativity constraint), the Gerstenhaber algebra structure appears in the associated cohomology of an operad.

The operadic dynamics may be introduced by simple and natural analogy with the Hamiltonian version. Operadic observables satisfy the Gerstenhaber algebra identities and their time evolution is governed by the operadic analogue of the Hamilton equations, the operadic evolution equation. The latter describes the time evolution of operations. In particular, the notion of an operadic Lax pair may be introduced as well.

4.2 Operad

Let K be a unital associative commutative ring, and let C^n ($n \in \mathbb{N}$) be unital K-modules. For $f \in C^n$, we refer to n as the *degree* of f and often write (when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f := (-1)^n$, $C^f := C^n$ and $\circ_f := \circ_n$. Also, it is convenient to use the *reduced* degree $|f| := n-1$. Throughout this paper, we assume that $\otimes := \otimes_K$.

Definition 4.1 (operad (e.g [7, 8])). A linear (non-symmetric) *operad* (i.e pre-operad) with coefficients in K is a sequence $\mathcal{C} := \{C^n\}_{n\in\mathbb{N}}$ of unital K-modules (an $\mathbb{N}$-graded K-module), such that the following conditions are held to be true:

(1) For $0 \le i \le m-1$ there exist *partial compositions*

$$\circ_i \in \operatorname{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0$$

(2) For all $h\otimes f\otimes g \in C^h\otimes C^f\otimes C^g$, the *composition (associativity) relations* hold,

$$(h \circ_i f) \circ_j g = \begin{cases} (-1)^{|f||g|}(h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \le j \le i-1, \\ h \circ_i (f \circ_{j-i} g) & \text{if } i \le j \le i+|f|, \\ (-1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i+f \le j \le |h|+|f|. \end{cases}$$

(3) There exists the unit $\mathrm{I} \in C^1$ such that

$$\mathrm{I} \circ_0 f = f = f \circ_i \mathrm{I}, \quad 0 \le i \le |f|$$

In the second item, the *first* and *third* parts of the defining relations turn out to be equivalent.

Example 4.2 (endomorphism operad [7])**.** Let V be a unital K-module and $\mathcal{E}_V^n := \mathcal{E}nd_V^n := \mathrm{Hom}(V^{\otimes n}, V)$. Define the partial compositions for $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$ as

$$f \circ_i g := (-1)^{i|g|} f \circ (\mathrm{id}_V^{\otimes i} \otimes g \otimes \mathrm{id}_V^{\otimes(|f|-i)}) \in \mathcal{E}_V^{m+n-1}, \quad 0 \leq i \leq |f|$$

The sequence $\mathcal{E}_V := \{\mathcal{E}_V^n\}_{n\in\mathbb{N}}$, equipped with the partial compositions $\circ_i$, is an operad (with the unit $\mathrm{id}_V \in \mathcal{E}_V^1$) called the *endomorphism operad* of V.

Therefore, algebraic operations can be seen as elements of the endomorphism operad.

Example 4.3 (coendomorphism operad)**.** Let R be a K-space and

$$\overline{\mathcal{E}}_R^n := \mathcal{C}o\mathcal{E}nd_R^n := \mathrm{Hom}(R, R^{\otimes n})$$

Define the partial compositions for $f \otimes g \in \overline{\mathcal{E}}_R^f \otimes \overline{\mathcal{E}}_R^g$ as

$$f \circ_i g := (-1)^{i|g|} (\mathrm{id}_R^{\otimes i} \otimes g \otimes \mathrm{id}_R^{\otimes(|f|-i)}) \circ f, \quad 0 \leq i \leq |f|$$

Then $\overline{\mathcal{E}}_R := \{\overline{\mathcal{E}}_R^n\}_{n\in\mathbb{N}}$ is an operad (with the unit $\mathrm{id}_R \in \overline{\mathcal{E}}_R^1$) called the *coendomorphism operad* of R. Thus, algebraic co-operations can be seen as elements of a coendomorphism operad.

Just as elements of a vector space are called *vectors*, it is natural to call elements of an abstract operad *operations*. The endomorphism operads can be seen as the most suitable objects for modelling operadic systems.

4.3 Cup and braces

Throughout this section, fix a binary operation $\mu \in C^2$ in an operad $\mathcal{C}$.

Definition 4.4. The *cup-multiplication* $\smile\colon C^f \otimes C^g \to C^{f+g}$ is defined by

$$f \smile g := (-1)^f (\mu \circ_0 f) \circ_f g \in C^{f+g}, \quad |\smile| = 1$$

The pair $\mathrm{Cup}\,\mathcal{C} := \{\mathcal{C}, \smile\}$ is called a $\smile$-algebra (cup-algebra) of $\mathcal{C}$.

Example 4.5. For the endomorphism operad (Example 4.2) $\mathcal{E}_R$ one has

$$f \smile g = (-1)^{fg} \mu \circ (f \otimes g), \quad \mu \otimes f \otimes g \in \mathcal{E}_R^2 \otimes \mathcal{E}_R^f \otimes \mathcal{E}_R^g$$

Definition 4.6. The *total composition* $\bullet : C^f \otimes C^g \to C^{f+|g|}$ is defined by

$$f \bullet g := \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0$$

The pair $\mathrm{Com}\,\mathcal{C} := \{\mathcal{C}, \bullet\}$ is called the *composition algebra* of $\mathcal{C}$.

Definition 4.7 (tribraces)**.** Define the Gerstenhaber *tribraces* $\{\cdot,\cdot,\cdot\}$ as a double sum

$$\{h,f,g\} := \sum_{i=0}^{|h|-1}\sum_{i+f}^{|f|+|h|}(h\circ_i f)\circ_j g \in C^{h+|f|+|g|}, \quad |\{\cdot,\cdot,\cdot\}| = 0$$

Definition 4.8 (tetrabraces)**.** The *tetrabraces* $\{\cdot,\cdot,\cdot,\cdot\}$ are defined by

$$\{h,f,g,v\} := \sum_{i=0}^{|h|-2}\sum_{j=i+f}^{|h|+|f|-1}\sum_{k=j+g}^{|h|+|f|+|g|}((h\circ_i f)\circ_j g)\circ_k v \in C^{h+|f|+|g|+|v|}$$

with $|\{\cdot,\cdot,\cdot,\cdot\}| = 0$.

It turns out that

$$f \smile g = (-1)^f\{\mu,f,g\}$$

In general, $\operatorname{Cup}\mathcal{C}$ is a *non-associative* algebra. By denoting $\mu^2 := \mu\bullet\mu$, it turns out that the associator in $\operatorname{Cup}\mathcal{C}$ reads

$$(f\smile g)\smile h - f\smile(g\smile h) = \{\mu^2, f, g, h\}$$

Therefore the *formal associator* (*micro-associator*) μ^2 is an obstruction to the associativity of $\operatorname{Cup}\mathcal{C}$. For an endomorphism operad $\mathcal{E}_R$, the ternary operation μ^2 also reads as an associator:

$$\mu^2 = \mu\circ(\mu\otimes\operatorname{id}_R - \operatorname{id}_R\otimes\mu), \quad \mu\in\mathcal{E}_R^2$$

4.4 Associated graded Lie algebra

In an operad $\mathcal{C}$, the Getzler identity

$$\langle h,f,g\rangle := (h\bullet f)\bullet g - h\bullet(f\bullet g) = \{h,f,g\} + (-1)^{|f||g|}\{h,g,f\}$$

holds, which easily implies the Gerstenhaber identity

$$\langle h,f,g\rangle = (-1)^{|f||g|}(h,g,f)$$

The *Gerstenhaber brackets* $[\cdot,\cdot]$ are defined in $\operatorname{Com}\mathcal{C}$ as a graded commutator by

$$[f,g] := f\bullet g - (-1)^{|f||g|}g\bullet f = -(-1)^{|f||g|}[g,f], \quad |[\cdot,\cdot]| = 0 \tag{G1}$$

The *commutator algebra* of $\operatorname{Com}\mathcal{C}$ is denoted as $\operatorname{Com}^-\mathcal{C} := \{\mathcal{C},[\cdot,\cdot]\}$. By using the Gerstenhaber identity, one can prove that $\operatorname{Com}^-\mathcal{C}$ is a *graded Lie algebra*. The Jacobi identity reads

$$(-1)^{|f||h|}[[f,g],h] + (-1)^{|g||f|}[[g,h],f] + (-1)^{|h||g|}[[h,f],g] = 0 \tag{G2}$$

4.5 Coboundary operator

In an operad $\mathcal{C}$, by using the Gerstenhaber brackets, a *(pre-)coboundary* operator $\partial := \partial_\mu$ may be defined by

$$\begin{aligned}\partial f := \mathrm{ad}_\mu^{\mathrm{right}} f := [f, \mu] := f \bullet \mu - (-1)^{|f|} \mu \bullet f \\ = f \smile \mathrm{I} + f \bullet \mu + (-1)^{|f|}\, \mathrm{I} \smile f, \quad \deg \partial = +1 = |\partial|\end{aligned}$$

It follows from the Jacobi identity in $\mathrm{Com}^- \mathcal{C}$ that ∂ is a (right) derivation of $\mathrm{Com}^- \mathcal{C}$,

$$\partial[f, g] = (-1)^{|g|}[\partial f, g] + [f, \partial g]$$

and one has the commutation relation

$$[\partial_f, \partial_g] := \partial_f \partial_g - (-1)^{|f||g|} \partial_g \partial_f = \partial_{[g,f]}$$

Therefore, since $|\mu| = +1$ is *odd*, then

$$\partial_\mu^2 = \frac{1}{2}[\partial_\mu, \partial_\mu] = \frac{1}{2}\partial_{[\mu,\mu]} = \partial_{\mu \bullet \mu} = \partial_{\mu^2}$$

Here we assumed that $2 \neq 0$, the proof for an arbitrary characteristic may be found from [13]. But ∂ need not be a derivation of $\mathrm{Cup}\, \mathcal{C}$, and μ^2 again appears as an obstruction:

$$\partial(f \smile g) - f \smile \partial g - (-1)^g \partial f \smile g = (-1)^g \{\mu^2, f, g\}$$

4.6 Derivation deviations

The *derivation deviation* of ∂ over $\bullet$ is defined by

$$(\mathrm{dev}_\bullet \partial)(f \otimes g) := \partial(f \bullet g) - f \bullet \partial g - (-1)^{|g|} \partial f \bullet g$$

Theorem 4.9. *In an operad $\mathcal{C}$, one has*

$$(-1)^g (\mathrm{dev}_\bullet \partial)(f \otimes g) = f \smile g - (-1)^{fg} g \smile f$$

Proof. The full proof is presented in [12]. □

The derivation deviation of ∂ over $\{\cdot, \cdot, \cdot\}$ is defined by

$$\begin{aligned}(\mathrm{dev}_{\{\cdot,\cdot,\cdot\}} \partial)(h \otimes f \otimes g) := \partial\{h, f, g\} - \{h, f, \partial g\} \\ - (-1)^{|g|}\{h, \partial f, g\} - (-1)^{|g|+|f|}\{\partial h, f, g\}\end{aligned}$$

Theorem 4.10. *In an operad $\mathcal{C}$, one has*

$$(-1)^g(\mathrm{dev}_{\{\cdot,\cdot,\cdot\}}\,\partial)(h\otimes f\otimes g) = (h\bullet f)\smile g + (-1)^{|h|f} f\smile (h\bullet g) - h\bullet(f\smile g)$$

Proof. The full proof is presented in [13]. $\square$

Therefore the *left* translations in $\mathrm{Com}\,\mathcal{C}$ are not derivations of $\mathrm{Cup}\,\mathcal{C}$, the corresponding deviations are related to $\mathrm{dev}_{\{\cdot,\cdot,\cdot\}}\,\partial$. It turns out that the *right* translations in $\mathrm{Com}\,\mathcal{C}$ are derivations of $\mathrm{Cup}\,\mathcal{C}$,

$$(f\smile g)\bullet h = f\smile (g\bullet h) + (-1)^{|h|g}(f\bullet h)\smile g$$

By combining this formula with the one from Theorem 4.10 we obtain

Theorem 4.11. *In an operad $\mathcal{C}$, one has*

$$(-1)^g(\mathrm{dev}_{\{\cdot,\cdot,\cdot\}}\,\partial)(h\otimes f\otimes g) = [h,f]\smile g + (-1)^{|h|f} f\smile [h,g] - [h,f\smile g]$$

4.7 Gerstenhaber theory

Now, clarification can be supplied to show how the Gerstenhaber algebra can be associated with a linear operad. If (formal associativity) $\mu^2 = 0$ holds, then $\partial^2 = 0$, which in turn implies $\mathrm{Im}\,\partial \subseteq \mathrm{Ker}\,\partial$. Then one can form an associated cohomology ($\mathbb{N}$-graded module) $\mathcal{H}(\mathcal{C}) := \mathrm{Ker}\,\partial/\,\mathrm{Im}\,\partial$ with homogeneous components

$$H^n(\mathcal{C}) := \mathrm{Ker}(C^n \xrightarrow{\partial} C^{n+1})/\,\mathrm{Im}(C^{n-1} \xrightarrow{\partial} C^n)$$

where, by convention, $\mathrm{Im}(C^{-1} \xrightarrow{\partial} C^0) := 0$. Also, in this ($\mu^2 = 0$) case, $\mathrm{Cup}\,\mathcal{C}$ is *associative*,

$$(f\smile g)\smile h = f\smile (g\smile h) \tag{G3}$$

and ∂ is a *derivation* of $\mathrm{Cup}\,\mathcal{C}$. Remember from previously that $\mathrm{Com}^-\mathcal{C}$ is a graded Lie algebra and ∂ is a derivation of $\mathrm{Com}^-\mathcal{C}$. Due to the derivation properties of ∂, the multiplications $[\cdot,\cdot]$ and $\smile$ induce corresponding (factor) multiplications on $\mathcal{H}(\mathcal{C})$, which we denote by the same symbols. Then $\{\mathcal{H}(\mathcal{C}),[\cdot,\cdot]\}$ is a *graded Lie algebra*. It follows from Theorem 4.9 that the induced $\smile$-multiplication on $\mathcal{H}(\mathcal{C})$ is *graded commutative*,

$$f\smile g = (-1)^{fg} g\smile f \tag{G4}$$

for all $f\otimes g \in H^f(\mathcal{C})\otimes H^g(\mathcal{C})$, hence $\{\mathcal{H}(\mathcal{C}),\smile\}$ is an *associative graded commutative* algebra. It follows from Theorem 4.11 that the *graded Leibniz rule* holds,

$$[h,f\smile g] = [h,f]\smile g + (-1)^{|h|f} f\smile [h,g] \tag{G5}$$

for all $h \otimes f \otimes g \in H^h(\mathcal{C}) \otimes H^f(\mathcal{C}) \otimes H^g(\mathcal{C})$. At last, it is also relevant to note that

$$0 = |[\cdot, \cdot]| \neq |\smile| = 1 \tag{G6}$$

In this way, the triple $\{\mathcal{H}(\mathcal{C}), \smile, [\cdot, \cdot]\}$ turns out to be a *Gerstenhaber algebra* [8]. The defining relations of a Gerstenhaber algebra are (G1)-(G6).

In the case of an endomorphism operad, the Gerstenhaber algebra structure appears on the Hochschild cohomology of an associative algebra [7]. This is the essence of the Gerstenhaber theory.

In particular, in the case of a coendomorphism operad, the Gerstenhaber algebra structure appears on the Cartier cohomology of a coassociative coalgebra.

Remark 4.12. The unique properties (G1)-(G6) show that the Gerstenhaber brackets can be seen as a graded analogue of the Poisson brackets in classical mechanics. Thus, these brackets can be used as a tool for defining a graded analogue of mechanics in algebra, called the operadic dynamics (see next section).

4.8 Operadic mechanics

In Hamiltonian formalism, a mechanical system is described by canonical variables q^i, p_i and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{4.1}$$

By a Lax representation [1, 16] of a mechanical system one means such a pair (L, M) of matrices (linear operators) L, M that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = [M, L] := ML - LM \tag{4.2}$$

Thus, from the algebraic point of view, mechanical systems can be described by linear operators, i.e by linear maps $V \to V$ of a vector space V. As a generalization of this one can pose the following question [22]: how can the time evolution of the linear operations (multiplications) $V^{\otimes n} \to V$ be described?

Assume that $K := \mathbb{R}$ or $K := \mathbb{C}$. It is known that the Poisson algebras can be seen as an algebraic abstraction of mechanics. Consider the following figurative commutative diagram:

$$\begin{array}{ccc}
\text{Poisson algebras} & \xleftarrow{\text{algebra}} & \text{mechanics} \\
\wr & & \wr \\
\text{Gerstenhaber algebras} & \xleftarrow{\text{algebra}} & \text{operadic mechanics}
\end{array}$$

Concisely speaking, *operadic observables* are elements of a Gerstenhaber algebra. If an operadic system depends on time, one can speak about *operadic dynamics*. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics by using the Gerstenhaber brackets instead of the commutator bracketing in the Lax equation (4.2).

The time evolution of an operadic observable f is then governed by the *operadic evolution equation*

$$\frac{df}{dt} = [H, f] := H \bullet f - (-1)^{|H||f|} f \bullet H$$

with the (model-dependent) *operadic Hamiltonian* H. The most simple assumption for its degree is

$$\left|\frac{d}{dt}\right| = |H| = 0 \quad \Longrightarrow \quad [H, f] := H \bullet f - f \bullet H$$

In particular,

$$|H| = |f| = 0 \quad \Longrightarrow \quad [H, f] = H \circ f - f \circ H$$

and in this case one finds the well-known evolution equation

$$\frac{df}{dt} = [H, f] := H \circ f - f \circ H$$

In this way one can describe the time evolution of operations as they can be seen as an example of the *operadic* variables [7]. In particular, one can propose

Definition 4.13 (operadic Lax pair)**.** Allow a classical dynamical system to be described by the Hamiltonian system (4.1) An *operadic Lax pair* is a pair (L, M) of operations $L, M \in \mathcal{E}_V$, such that the Hamiltonian system (4.1) may be represented as the *operadic Lax equation*

$$\frac{dL}{dt} = [M, L] := M \bullet L - (-1)^{|M||L|} L \bullet M$$

The pair (L, M) is also called an *operadic Lax representations* of/for the Hamiltonian system (4.1).

Evidently, the degree constraints $|M| = |L| = 0$ give rise to the ordinary Lax equation (4.2). If only one of the operations M, L turns out to be unary, i.e either $|M| \neq 0$ or $|L| \neq 0$, then the Gerstenhaber brackets do not coincide with the ordinary commutator.

Endomorphism and co-endomorphism operads are the most natural objects for modelling operadic dynamical systems.

Surprisingly, examples are at hand. By using the Lax pairs one may extend these to operadic area via the operadic Lax equation.

4.9 Evolution of binary algebras

Let $\mathcal{A} := \{V, \mu\}$ be a binary algebra with a (linear) operation $xy := \mu(x \otimes y)$. For simplicity assume that $|M| = 0$. We require that $\mu = \mu(q, p)$ so that (μ, M) is an operadic Lax pair, i.e the Hamiltonian system (4.1) of the harmonic oscillator may be written as the operadic Lax equation

$$\dot{\mu} = [M, \mu] := M \bullet \mu - \mu \bullet M, \quad |\mu| = 1, \quad |M| = 0$$

Note that under conditions $|\mu| = 1, |M| = 0$ the Gerstenhaber brackets of μ and M do not coincide with the ordinary commutator bracketing that is used in the case of the ordinary Lax representations.

Let $x, y \in V$. Assuming that $|M| = 0$ and $|\mu| = 1$, one has

$$\begin{aligned} M \bullet \mu &= \sum_{i=0}^{0} (-1)^{i|\mu|} M \circ_i \mu = M \circ_0 \mu = M \circ \mu \\ \mu \bullet M &= \sum_{i=0}^{1} (-1)^{i|M|} \mu \circ_i M = \mu \circ_0 M + \mu \circ_1 M \\ &= \mu \circ (M \otimes \mathrm{id}_V) + \mu \circ (\mathrm{id}_V \otimes M) \end{aligned}$$

Therefore,

$$\frac{d}{dt}(xy) = M(xy) - (Mx)y - x(My)$$

Let $\dim V = n$. In a basis $\{e_1, \ldots, e_n\}$ of V, the structure constants μ^i_{jk} of $\mathcal{A}$ are defined by

$$\mu(e_j \otimes e_k) := \mu^i_{jk} e_i, \quad j, k = 1, \ldots, n$$

In particular,

$$\frac{d}{dt}(e_j e_k) = M(e_j e_k) - (Me_j)e_k - e_j(Me_k)$$

By denoting $Me_i := M^s_i e_s$, it follows that

$$\dot{\mu}^i_{jk} = \mu^s_{jk} M^i_s - M^s_j \mu^i_{sk} - M^s_k \mu^i_{js}, \quad i, j, k = 1, \ldots, n$$

4.10 Operadic harmonic oscillator

Consider the Lax pair for the harmonic oscillator:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the Hamiltonian of the harmonic oscillator is

$$H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

it is easy to check that the Lax equation

$$\dot{L} = [M, L] := ML - LM$$

is equivalent to the Hamiltonian system of the harmonic oscillator

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q \tag{4.3}$$

If μ is a linear algebraic operation one can use the above Hamilton equations to obtain

$$\begin{aligned}\frac{d\mu}{dt} &= \frac{\partial \mu}{\partial q}\frac{dq}{dt} + \frac{\partial \mu}{\partial p}\frac{dp}{dt} = p\frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} \\ &= [M, \mu] = M \bullet \mu - \mu \bullet M\end{aligned}$$

Therefore, we get the following linear partial differential equation for $\mu(q,p)$:

$$p\frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu] = M \bullet \mu - \mu \bullet M \tag{4.4}$$

By integrating (4.4) one can get collections of operations called the *operadic (Lax representations for/of the) harmonic oscillator*. Since the general solution of a partial differential equation depends on arbitrary functions, these representations are not uniquely determined.

CHAPTER

5

Operadic Lax representations for harmonic oscillator in low dimensions

Operadic Lax representations for the harmonic oscillator in certain types of 2- and 3-dimensional binary real algebras are found. The material of this chapter is based on [23, 24, 27, 28].

5.1 Operadic Lax representations for harmonic oscillator in a 2d real Lie algebra

Operadic Lax representations for harmonic oscillator in a 2-dimensional real Lie algebra are found. The material of this section has been published in [28].

Lemma 5.1. *Let* $\dim V = 2$ *and* $M := (M^i_j) := \frac{\omega}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$*. Then the* 2*-dimensional binary operadic Lax equations read*

$$\begin{cases} \dot{\mu}^1_{11} = -\frac{\omega}{2}\left(\mu^2_{11} + \mu^1_{12} + \mu^1_{21}\right), & \dot{\mu}^2_{11} = \frac{\omega}{2}\left(\mu^1_{11} - \mu^2_{12} - \mu^2_{21}\right) \\ \dot{\mu}^1_{12} = -\frac{\omega}{2}\left(\mu^2_{12} - \mu^1_{11} + \mu^1_{22}\right), & \dot{\mu}^2_{12} = \frac{\omega}{2}\left(\mu^1_{12} + \mu^2_{11} - \mu^2_{22}\right) \\ \dot{\mu}^1_{21} = -\frac{\omega}{2}\left(\mu^2_{21} - \mu^1_{11} + \mu^1_{22}\right), & \dot{\mu}^2_{21} = \frac{\omega}{2}\left(\mu^1_{21} + \mu^2_{11} - \mu^2_{22}\right) \\ \dot{\mu}^1_{22} = -\frac{\omega}{2}\left(\mu^2_{22} - \mu^1_{12} - \mu^1_{21}\right), & \dot{\mu}^2_{22} = \frac{\omega}{2}\left(\mu^1_{22} + \mu^2_{12} + \mu^2_{21}\right) \end{cases}$$

In what follows, consider only anti-commutative algebras. Then one has

Corollary 5.2. *Let* $\mathcal{A}$ *be a* 2*-dimensional anti-commutative real algebra, i.e*

$$\mu^1_{11} = \mu^1_{22} = \mu^2_{11} = \mu^2_{22} = 0, \quad \mu^1_{12} = -\mu^1_{21}, \quad \mu^2_{12} = -\mu^2_{21}$$

Then the operadic Lax equations read

$$\begin{cases} \dot{\mu}^1_{12} = -\frac{\omega}{2}\mu^2_{12} \\ \dot{\mu}^2_{12} = \frac{\omega}{2}\mu^1_{12} \end{cases}$$

Thus, one has to specify μ^1_{12} and μ^2_{12} as functions of the canonical variables q and p. Define

$$\begin{cases} K_+ := \sqrt{\sqrt{2H}+p} \\ K_- := \sqrt{\sqrt{2H}-p} \end{cases}$$

and

$$\begin{cases} B_+ := K_+ + K_- = \sqrt{\sqrt{2H}+p} + \sqrt{\sqrt{2H}-p} \\ B_- := K_+ - K_- = \sqrt{\sqrt{2H}+p} - \sqrt{\sqrt{2H}-p} \end{cases}$$

Then one has

Theorem 5.3. *The formulae*

$$M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{cases} \mu^1_{11} = \mu^1_{22} = \mu^2_{11} = \mu^2_{22} = 0 \\ \mu^1_{12} = -\mu^1_{21} = B_- \\ \mu^2_{12} = -\mu^2_{21} = B_+ \end{cases}$$

represent a 2*-dimensional binary operadic Lax pair of the harmonic oscillator. The algebra given by the above structure functions μ^i_{jk} is a* 2*-dimensional real Lie algebra.*

Proof. The operadic Lax equations read

$$\begin{cases} \dot{B}_- = -\frac{\omega}{2} B_+ \\ \dot{B}_+ = \ \ \frac{\omega}{2} B_- \end{cases}$$

That is

$$\begin{cases} \left[\frac{1}{2K_+}\left(\frac{p}{\sqrt{2H}}+1\right) - \frac{1}{2K_-}\left(\frac{p}{\sqrt{2H}}-1\right)\right]\dot{p} + \left[\left(\frac{1}{2K_+} - \frac{1}{2K_-}\right)\frac{q\omega^2}{\sqrt{2H}}\right]\dot{q} = -\frac{\omega}{2}B_+ \\ \left[\frac{1}{2K_+}\left(\frac{p}{\sqrt{2H}}+1\right) + \frac{1}{2K_-}\left(\frac{p}{\sqrt{2H}}-1\right)\right]\dot{p} + \left[\left(\frac{1}{2K_+} + \frac{1}{2K_-}\right)\frac{q\omega^2}{\sqrt{2H}}\right]\dot{q} = \ \ \frac{\omega}{2}B_- \end{cases}$$

Multiplying both equations by $2K_+K_-$ one gets

$$\begin{cases} \left[K_-\left(\frac{p}{\sqrt{2H}}+1\right) - K_+\left(\frac{p}{\sqrt{2H}}-1\right)\right]\dot{p} - \frac{q\omega^2 B_-}{\sqrt{2H}}\dot{q} = -\omega B_+ K_+ K_- \\ \left[K_-\left(\frac{p}{\sqrt{2H}}+1\right) + K_+\left(\frac{p}{\sqrt{2H}}-1\right)\right]\dot{p} + \frac{q\omega^2 B_+}{\sqrt{2H}}\dot{q} = \ \ \omega B_- K_+ K_- \end{cases}$$

Now use the Cramer formulae. By using the relations

$$B_+^2 - B_-^2 = 4K_+K_-, \quad (K_+K_-)^2 = q^2\omega^2$$

first calculate the determinants

$$\Delta = \begin{vmatrix} K_-\left(\frac{p}{\sqrt{2H}}+1\right) - K_+\left(\frac{p}{\sqrt{2H}}-1\right) & -\frac{q\omega^2 B_-}{\sqrt{2H}} \\ K_-\left(\frac{p}{\sqrt{2H}}+1\right) + K_+\left(\frac{p}{\sqrt{2H}}-1\right) & \frac{q\omega^2 B_+}{\sqrt{2H}} \end{vmatrix} = \frac{4q^2\omega^3}{\sqrt{2H}}$$

$$\Delta_{\dot{p}} = \begin{vmatrix} -\omega B_+ K_+ K_- & -\frac{q\omega^2 B_-}{\sqrt{2H}} \\ \omega B_- K_+ K_- & \frac{q\omega^2 B_+}{\sqrt{2H}} \end{vmatrix} = -\frac{4q^3\omega^5}{\sqrt{2H}}$$

$$\Delta_{\dot{q}} = \begin{vmatrix} K_-\left(\frac{p}{\sqrt{2H}}+1\right) - K_+\left(\frac{p}{\sqrt{2H}}-1\right) & -\omega B_+ K_+ K_- \\ K_-\left(\frac{p}{\sqrt{2H}}+1\right) + K_+\left(\frac{p}{\sqrt{2H}}-1\right) & \omega B_- K_+ K_- \end{vmatrix} = \frac{4pq^2\omega^3}{\sqrt{2H}}$$

Thus one obtains the Hamiltonian system of the harmonic oscillator

$$\dot{q} = \frac{\Delta_{\dot{q}}}{\Delta} = p, \quad \dot{p} = \frac{\Delta_{\dot{p}}}{\Delta} = -q\omega^2$$

and the latter is equivalent to the above operadic Lax system of the harmonic oscillator.

The Jacobi identity for μ^i_{jk} can be checked by direct calculation. □

Remark 5.4. The real Lie algebra $A := \{V, \mu\}$ found in Theorem 5.3 is isomorphic to the real Lie algebra $\overline{A} := \{V, \overline{\mu}\}$ given by

$$\overline{\mu}^1_{12} = -\overline{\mu}^1_{21} = 1, \quad \overline{\mu}^2_{12} = -\overline{\mu}^2_{21} = 0$$

The isomorphism $\overline{A} \to A$ can be represented by the linear map

$$\begin{cases} e_1 = -B_-\overline{e}_1 - B_+\overline{e}_2 \\ e_2 = -B_+\overline{e}_1 + B_-\overline{e}_2 \end{cases}$$

The multiplication $\overline{\mu}$ does not depend on the canonical variables.

5.2 Operadic Lax representations for harmonic oscillator in a general 2d binary real algebra

The operadic Lax representations for the harmonic oscillator are constructed in a 2-dimensional real algebra. The material of this section has been partially published in [27]. For the proof of the main theorem see [23].

Definition 5.5 (Quasi-canonical coordinates)**.** For the harmonic oscillator, define its *quasi-canonical coordinates* $A_\pm$ by

$$A_+^2 - A_-^2 = 2p, \quad A_+A_- = \omega q \tag{5.1}$$

and the auxiliary functions $D_\pm$ by

$$D_\pm := \pm\frac{1}{2}A_\pm\left(A_\pm^2 - 3A_\mp^2\right)$$

Remark 5.6. Note that $A_{\pm}$ can not be simultaneously zero. The Hamiltonian of the harmonic oscillator $H = \frac{1}{2}(p^2 + \omega^2 q^2)$ together with (5.1) imply via the bi-quadratic equation the equality

$$A_+^2 + A_-^2 = 2\sqrt{2H} \tag{5.2}$$

By differentiating defining relations (5.1) of $A_{\pm}$ and (5.2) with respect to t one gets

$$\begin{cases} A_+\dot{A}_+ + A_-\dot{A}_- = \frac{1}{\sqrt{2H}}(p\dot{p} + \omega^2 q\dot{q}) \\ A_+\dot{A}_+ - A_-\dot{A}_- = \dot{p} \\ A_-\dot{A}_+ + A_+\dot{A}_- = \omega\dot{q} \end{cases} \tag{5.3}$$

Propose the following

Theorem 5.7. *Let $C_\nu \in \mathbb{R}$ ($\nu = 1, \ldots, 8$) be arbitrary real–valued parameters, not simultaneously zero, $M := \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and*

$$\begin{cases} \mu^1_{11}(q,p) = \quad C_5A_- + C_6A_+ + C_7D_- + C_8D_+ \\ \mu^1_{12}(q,p) = \quad C_1A_+ + C_2A_- - C_7D_+ + C_8D_- \\ \mu^1_{21}(q,p) = -C_1A_+ - C_2A_- - C_3A_+ - C_4A_- - C_5A_+ + C_6A_- - C_7D_+ + C_8D_- \\ \mu^1_{22}(q,p) = -C_3A_- + C_4A_+ - C_7D_- - C_8D_+ \\ \mu^2_{11}(q,p) = \quad C_3A_+ + C_4A_- - C_7D_+ + C_8D_- \\ \mu^2_{12}(q,p) = \quad C_1A_- - C_2A_+ + C_3A_- - C_4A_+ + C_5A_- + C_6A_+ - C_7D_- - C_8D_+ \\ \mu^2_{21}(q,p) = -C_1A_- + C_2A_+ - C_7D_- - C_8D_+ \\ \mu^2_{22}(q,p) = -C_5A_+ + C_6A_- + C_7D_+ - C_8D_- \end{cases}$$

be the structure constants of the multiplication $\mu : V \otimes V \to V$ in a 2-dimensional real vector space V. Then (μ, M) is a 2-dimensional binary operadic Lax pair of the harmonic oscillator.

Proof. Denote

$$\begin{cases} G_\pm^{\omega/2} \quad := \dot{A}_\pm \pm \frac{\omega}{2}A_\mp \\ G_\pm^{3\omega/2} \quad := \dot{D}_\pm \pm \frac{3\omega}{2}D_\mp \end{cases}$$

Define the matrix $\Gamma = (\Gamma^\beta_\alpha)$ by

$$\Gamma := \begin{pmatrix}
0 & G_+^{\omega/2} & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 \\
0 & G_-^{\omega/2} & -G_-^{\omega/2} & 0 & 0 & -G_+^{\omega/2} & G_+^{\omega/2} & 0 \\
0 & 0 & -G_+^{\omega/2} & -G_-^{\omega/2} & G_+^{\omega/2} & G_-^{\omega/2} & 0 & 0 \\
0 & 0 & -G_-^{\omega/2} & G_+^{\omega/2} & G_-^{\omega/2} & -G_+^{\omega/2} & 0 & 0 \\
G_-^{\omega/2} & 0 & -G_+^{\omega/2} & 0 & 0 & G_-^{\omega/2} & 0 & -G_+^{\omega/2} \\
G_+^{\omega/2} & 0 & G_-^{\omega/2} & 0 & 0 & G_+^{\omega/2} & 0 & G_-^{\omega/2} \\
G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2} & -G_-^{3\omega/2} & G_+^{3\omega/2} \\
G_+^{3\omega/2} & G_-^{3\omega/2} & G_-^{3\omega/2} & -G_+^{3\omega/2} & G_-^{3\omega/2} & -G_+^{3\omega/2} & -G_+^{3\omega/2} & -G_-^{3\omega/2}
\end{pmatrix}$$

Then, by using Lemma 5.1, it follows that the 2-dimensional binary operadic Lax equations read

$$C_\beta \Gamma^\beta_\alpha = 0, \quad \alpha = 1, \dots, 8$$

Since the parameters C_β are arbitrary, the latter constraints imply $\Gamma = 0$. Thus one has to consider the following differential equations

$$G_\pm^{\omega/2} = 0 = G_\pm^{3\omega/2}$$

We show that

$$\begin{cases} \dot{p} = -\omega^2 q \\ \dot{q} = p \end{cases} \quad \overset{(I)}{\Longleftrightarrow} \quad G_\pm^{\omega/2} = 0 \quad \overset{(II)}{\Longleftrightarrow} \quad G_\pm^{3\omega/2} = 0$$

First prove (I). $\Longrightarrow$: Assume that the Hamilton equations (4.3) for the harmonic oscillator hold. Then it follows from (5.3) that

$$\begin{cases} A_+\dot{A}_+ + A_-\dot{A}_- = 0 \\ A_+\dot{A}_+ - A_-\dot{A}_- = -\omega^2 q \\ A_-\dot{A}_+ + A_+\dot{A}_- = \omega p \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 2A_-\dot{A}_- = \ \ \omega^2 q \\ 2A_+\dot{A}_+ = -\omega^2 q \\ A_-\dot{A}_+ + A_+\dot{A}_- = \omega p \end{cases}$$

$$\Longleftrightarrow \quad \begin{cases} \dot{A}_- = \frac{\omega^2 q}{2A_-} = \frac{\omega^2 q A_+}{2A_-A_+} = \ \ \frac{\omega}{2}A_+ \\ \dot{A}_+ = \frac{-\omega^2 q}{2A_+} = \frac{-\omega^2 q A_-}{2A_+A_-} = -\frac{\omega}{2}A_- \\ A_+^2 - A_-^2 = 2p \end{cases}$$

$$\Longleftrightarrow \quad G_\pm^{\omega/2} = 0$$

and the latter system is the required system for $A_\pm$.
$\Longleftarrow$: Assume that the system of differential equations $G_\pm^{\omega/2} = 0$ holds. Then it follows from (5.3) that

$$\begin{cases} A_-A_+ - A_+A_- = \frac{2(p\dot{p} + \omega^2 q\dot{q})}{\omega\sqrt{2H}} \\ A_+A_- + A_-A_+ = -\frac{2}{\omega}\dot{p} \\ A_+^2 - A_-^2 = 2\dot{q} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} p\dot{p} + \omega^2 q\dot{q} = 0 \\ A_+A_- = -\frac{1}{\omega}\dot{p} \\ A_+^2 - A_-^2 = 2\dot{q} \end{cases}$$

$$\iff \begin{cases} p\dot{p} + \omega^2 q\dot{q} = 0 \\ \dot{p} = -\omega A_+ A_- = -\omega^2 q \\ \dot{q} = \frac{1}{2}(A_+^2 - A_-^2) = p \end{cases}$$

where the first relation easily follows from the Hamiltonian system (4.3).

Now prove (II). Differentiate the auxiliary functions $D_\pm$ to get

$$\begin{cases} \dot{D}_+ = \frac{1}{2}\dot{A}_+(A_+^2 - 3A_-^2) + A_+(A_+\dot{A}_+ - 3A_-\dot{A}_-) \\ \dot{D}_- = \frac{1}{2}\dot{A}_-(3A_+^2 - A_-^2) + A_-(3A_+\dot{A}_+ - A_-\dot{A}_-) \end{cases}$$

$\Longrightarrow$: Assume that the functions $A_\pm$ satisfy the system of differential equations $G_\pm^{\omega/2} = 0$. Then

$$\begin{cases} \dot{D}_+ = -\frac{\omega}{4}A_-(A_+^2 - 3A_-^2) - \frac{A_+\omega}{2}(A_+A_- + 3A_-A_+) \\ \dot{D}_- = \quad \frac{\omega}{4}A_+(3A_+^2 - A_-^2) - \frac{A_-\omega}{2}(3A_+A_- + A_-A_+) \end{cases}$$

and

$$\begin{cases} \dot{D}_+ = -\frac{3\omega}{2}\frac{A_-}{2}(3A_+^2 - A_-^2) = -\frac{3\omega}{2}D_- \\ \dot{D}_- = \quad \frac{3\omega}{2}\frac{A_+}{2}(A_+^2 - 3A_-^2) = \quad \frac{3\omega}{2}D_+ \end{cases} \iff G_\pm^{3\omega/2} = 0$$

$\Longleftarrow$: Assume that the functions $D_\pm$ satisfy the system of differential equations $G_\pm^{3\omega/2} = 0$. Then

$$\begin{cases} -\frac{3\omega}{2}D_- = \frac{\dot{A}_+}{2}(A_+^2 - 3A_-^2) + A_+(A_+\dot{A}_+ - 3A_-\dot{A}_-) \\ \quad \frac{3\omega}{2}D_+ = \frac{\dot{A}_-}{2}(3A_+^2 - A_-^2) + A_-(3A_+\dot{A}_+ - A_-\dot{A}_-) \end{cases}$$

$$\iff \begin{cases} \dot{A}_+(3A_+^2 - 3A_-^2) + \dot{A}_-(-6A_-A_+) = -3\omega D_- \\ \dot{A}_+(6A_+A_-) + \dot{A}_-(3A_+^2 - 3A_-^2) = 3\omega D_+ \end{cases}$$

$$\iff \begin{cases} \quad p\dot{A}_+ - \omega q\dot{A}_- = -\frac{\omega}{2}D_- \\ \omega q\dot{A}_+ + \quad p\dot{A}_- = \quad \frac{\omega}{2}D_+ \end{cases}$$

To use the Cramer formulae, calculate

$$\Delta = \begin{vmatrix} p & -\omega q \\ \omega q & p \end{vmatrix} = p^2 + \omega^2 q^2 = 2H$$

$$\Delta_{\dot{A}_+} = \begin{vmatrix} -\frac{\omega}{2}D_- & -\omega q \\ \frac{\omega}{2}D_+ & p \end{vmatrix} = -\frac{\omega}{2}(D_-p - D_+\omega q)$$

$$\Delta_{\dot{A}_-} = \begin{vmatrix} p & -\frac{\omega}{2}D_- \\ \omega q & \frac{\omega}{2}D_+ \end{vmatrix} = \frac{\omega}{2}(D_+p + D_-\omega q)$$

Note that

$$\begin{aligned}
D_-p - D_+\omega q &= \frac{A_-}{2}p(3A_+^2 - A_-^2) - \frac{A_+}{2}\omega q(A_+^2 - 3A_-^2) \\
&= \frac{A_-}{2}\frac{1}{2}(A_+^2 - A_-^2)(3A_+^2 - A_-^2) - \frac{A_+}{2}A_+A_-(A_+^2 - 3A_-^2) \\
&= \frac{A_-}{4}(A_+^2 + A_-^2)^2 = 2A_-H \\
D_+p + D_-\omega q &= \frac{A_+}{2}p(A_+^2 - 3A_-^2) + \frac{A_-}{2}\omega q(3A_+^2 - A_-^2) \\
&= \frac{A_+}{2}\frac{1}{2}(A_+^2 - A_-^2)(A_+^2 - 3A_-^2) + \frac{A_-}{2}A_+A_-(3A_+^2 - A_-^2) \\
&= \frac{A_+}{4}(A_+^2 + A_-^2)^2 = 2A_+H
\end{aligned}$$

Thus,

$$\begin{cases} \dot{A}_+ = \frac{\Delta_{\dot{A}_+}}{\Delta} = -\frac{\omega}{2}\frac{2HA_-}{2H} = -\frac{\omega}{2}A_- \\ \dot{A}_- = \frac{\Delta_{\dot{A}_-}}{\Delta} = \frac{\omega}{2}\frac{2HA_+}{2H} = \frac{\omega}{2}A_+ \end{cases} \iff G_{\pm}^{\omega/2} = 0 \qquad \square$$

Further study of 2-dimensional binary real algebras can be found in Appendix C.

5.3 Operadic Lax representations for harmonic oscillator in a 3d binary anti-commutative real algebra

The material of this section is based on [24].

Lemma 5.8. *Matrices*

$$L := \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \frac{\omega}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

give a 3-dimensional Lax representation for the harmonic oscillator.

Lemma 5.9. *Let* $\dim V = 3$ *and* M *be defined as in Lemma 5.8. Then the* 3*-dimensional binary operadic Lax equations read*

$$\begin{cases} \dot{\mu}^1_{11} = -\frac{\omega}{2}\left(\mu^2_{11} + \mu^1_{12} + \mu^1_{21}\right), & \dot{\mu}^1_{13} = -\frac{\omega}{2}\left(\mu^2_{13} + \mu^1_{23}\right), & \dot{\mu}^1_{33} = -\frac{\omega}{2}\mu^2_{33} \\ \dot{\mu}^1_{12} = -\frac{\omega}{2}\left(\mu^2_{12} - \mu^1_{11} + \mu^1_{22}\right), & \dot{\mu}^1_{23} = -\frac{\omega}{2}\left(\mu^2_{23} - \mu^1_{13}\right), & \dot{\mu}^2_{33} = \frac{\omega}{2}\mu^1_{33} \\ \dot{\mu}^1_{21} = -\frac{\omega}{2}\left(\mu^2_{21} - \mu^1_{11} + \mu^1_{22}\right), & \dot{\mu}^1_{31} = -\frac{\omega}{2}\left(\mu^2_{31} + \mu^1_{32}\right), & \dot{\mu}^3_{13} = -\frac{\omega}{2}\mu^3_{23} \\ \dot{\mu}^1_{22} = -\frac{\omega}{2}\left(\mu^2_{22} - \mu^1_{12} - \mu^1_{21}\right), & \dot{\mu}^1_{32} = -\frac{\omega}{2}\left(\mu^2_{32} - \mu^1_{31}\right), & \dot{\mu}^3_{23} = \frac{\omega}{2}\mu^3_{13} \\ \dot{\mu}^2_{11} = \frac{\omega}{2}\left(\mu^1_{11} - \mu^2_{12} - \mu^2_{21}\right), & \dot{\mu}^2_{13} = -\frac{\omega}{2}\left(\mu^2_{23} - \mu^1_{13}\right), & \dot{\mu}^3_{22} = \frac{\omega}{2}\left(\mu^3_{12} + \mu^3_{21}\right) \\ \dot{\mu}^2_{12} = \frac{\omega}{2}\left(\mu^1_{12} + \mu^2_{11} - \mu^2_{22}\right), & \dot{\mu}^2_{23} = \frac{\omega}{2}\left(\mu^1_{23} + \mu^2_{13}\right), & \dot{\mu}^3_{21} = \frac{\omega}{2}\left(\mu^3_{11} - \mu^3_{22}\right) \\ \dot{\mu}^2_{21} = \frac{\omega}{2}\left(\mu^1_{21} + \mu^2_{11} - \mu^2_{22}\right), & \dot{\mu}^2_{31} = -\frac{\omega}{2}\left(\mu^2_{32} - \mu^1_{31}\right), & \dot{\mu}^3_{11} = -\frac{\omega}{2}\left(\mu^3_{21} + \mu^3_{12}\right) \\ \dot{\mu}^2_{22} = \frac{\omega}{2}\left(\mu^1_{22} + \mu^2_{12} + \mu^2_{21}\right), & \dot{\mu}^2_{32} = \frac{\omega}{2}\left(\mu^1_{32} + \mu^2_{31}\right), & \dot{\mu}^3_{12} = \frac{\omega}{2}\left(\mu^3_{11} - \mu^3_{22}\right) \\ \dot{\mu}^3_{33} = 0, & \dot{\mu}^3_{32} = \frac{\omega}{2}\mu^3_{31}, & \dot{\mu}^3_{31} = -\frac{\omega}{2}\mu^3_{32} \end{cases}$$

In what follows, consider only anti-commutative real algebras. Then one has

Corollary 5.10. *Let A be a 3-dimensional anti-commutative real algebra, i.e*

$$\mu^i_{jk} = -\mu^i_{kj}, \quad i,j,k = 1,2,3$$

Then the operadic Lax equations for the harmonic oscillator read

$$\begin{cases} \dot{\mu}^1_{12} = -\frac{\omega}{2}\mu^2_{12}, & \dot{\mu}^2_{12} = \frac{\omega}{2}\mu^1_{12}, & \dot{\mu}^3_{12} = 0 \\ \dot{\mu}^1_{13} = -\frac{\omega}{2}\left(\mu^1_{23} + \mu^2_{13}\right), & \dot{\mu}^2_{13} = -\frac{\omega}{2}\left(\mu^2_{23} - \mu^1_{13}\right), & \dot{\mu}^3_{13} = -\frac{\omega}{2}\mu^3_{23} \\ \dot{\mu}^1_{23} = \frac{\omega}{2}\left(\mu^1_{13} - \mu^2_{23}\right), & \dot{\mu}^2_{23} = \frac{\omega}{2}\left(\mu^2_{13} + \mu^1_{23}\right), & \dot{\mu}^3_{23} = \frac{\omega}{2}\mu^3_{13} \end{cases}$$

Theorem 5.11. *Let $C_\nu \in \mathbb{R}$ ($\nu = 1, \ldots, 9$) be arbitrary real–valued parameters, such that*

$$C_2^2 + C_3^2 + C_5^2 + C_6^2 + C_7^2 + C_8^2 \neq 0 \tag{5.4}$$

Let M be defined as in Lemma 5.8, and

$$\begin{cases} \mu^1_{11} = \mu^1_{22} = \mu^1_{33} = \mu^2_{11} = \mu^2_{22} = \mu^2_{33} = \mu^3_{11} = \mu^3_{22} = \mu^3_{33} = 0 \\ \mu^1_{23} = -\mu^1_{32} = C_2 p - C_3\omega q - C_4 \\ \mu^2_{13} = -\mu^2_{31} = C_2 p - C_3\omega q + C_4 \\ \mu^1_{31} = -\mu^1_{13} = C_2\omega q + C_3 p - C_1 \\ \mu^2_{23} = -\mu^2_{32} = C_2\omega q + C_3 p + C_1 \\ \mu^1_{12} = -\mu^1_{21} = C_5 A_+ + C_6 A_- \\ \mu^2_{12} = -\mu^2_{21} = C_5 A_- - C_6 A_+ \\ \mu^3_{13} = -\mu^3_{31} = C_7 A_+ + C_8 A_- \\ \mu^3_{23} = -\mu^3_{32} = C_7 A_- - C_8 A_+ \\ \mu^3_{12} = -\mu^3_{21} = C_9 \end{cases} \tag{5.5}$$

be the structure constants of the multiplication $\mu : V \otimes V \to V$ in a 3-dimensional real vector space V. Then (μ, M) is a 3-dimensional anti-commutative binary operadic Lax pair for the harmonic oscillator.

Proof. Denote

$$\begin{cases} G_+^{\omega} := \ \dot{p} + \omega^2 q, & G_+^{\omega/2} := \dot{A}_+ + \frac{\omega}{2} A_- \\ G_-^{\omega} := \omega(\dot{q} - p), & G_-^{\omega/2} := \dot{A}_- - \frac{\omega}{2} A_+ \end{cases}$$

Define the matrix

$$\Gamma = (\Gamma_\alpha^\beta) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_+^{\omega} & G_+^{\omega} & G_-^{\omega} & G_-^{\omega} & 0 & 0 & 0 & 0 & 0 \\ -G_-^{\omega} & -G_-^{\omega} & G_+^{\omega} & G_+^{\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_+^{\omega/2} & G_-^{\omega/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G_-^{\omega/2} & -G_+^{\omega/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_+^{\omega/2} & G_-^{\omega/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_-^{\omega/2} & -G_+^{\omega/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then it follows from Corollary 5.10 that the 3-dimensional anti-commutative binary operadic Lax equations read

$$C_\beta \Gamma_\alpha^\beta = C_2 \Gamma_\alpha^2 + C_3 \Gamma_\alpha^3 + C_5 \Gamma_\alpha^5 + C_6 \Gamma_\alpha^6 + C_7 \Gamma_\alpha^7 + C_8 \Gamma_\alpha^8 = 0, \quad \alpha = 1, \ldots, 9$$

Since the parameters C_β $(\beta = 2, 3, 5, 6, 7, 8)$ are arbitrary, not simultaneously zero, the latter constraints imply $\Gamma = 0$.

Thus we have to consider the following differential equations

$$G_\pm^{\omega} = 0 = G_\pm^{\omega/2}$$

We show that

$$G_\pm^{\omega} = 0 \quad \overset{(I)}{\Longleftrightarrow} \quad \begin{cases} \dot{p} = -\omega^2 q \\ \dot{q} = p \end{cases} \quad \overset{(II)}{\Longleftrightarrow} \quad G_\pm^{\omega/2} = 0$$

First note that (I) immediately follows from the definition of $G_\pm^{\omega}$.

For the proof of (II) see Section 5.2 on page 61. □

CHAPTER

6

Dynamical deformations and quantum counterparts of 3d real Lie algebras in Bianchi classification over harmonic oscillator

In this chapter, operadic Lax representations for the harmonic oscillator are used to construct the dynamical deformations of 3-dimensional real Lie algebras in the Bianchi classification. It is shown how the energy conservation of the harmonic oscillator is related to the Jacobi identities of the dynamically deformed algebras. Based on this observation, it is proved that the dynamical deformations of 3-dimensional real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras. Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of 3-dimensional real Lie algebras in the Bianchi classification. The Jacobi operators of the quantum algebras are calculated. This material has been presented in [24, 25, 31].

6.1 Initial conditions and dynamical deformations

The material of this section has been published in [24, 25].

It seems attractive to specify the coefficients C_ν in Theorem 5.11 by the initial conditions

$$\mu|_{t=0} = \overset{\circ}{\mu}, \quad p|_{t=0} := p_0 \neq 0, \quad q|_{t=0} = 0$$

The latter together with (5.1) yield the initial conditions for $A_\pm$:

$$\begin{cases} \left(A_+^2 + A_-^2\right)\big|_{t=0} = 2\,|p_0| \\ \left(A_+^2 - A_-^2\right)\big|_{t=0} = 2p_0 \\ A_+A_-|_{t=0} = 0 \end{cases} \iff \begin{cases} p_0 > 0 \\ A_+|_{t=0} = \pm\sqrt{2p_0} \\ A_-|_{t=0} = 0 \end{cases} \vee \quad \begin{cases} p_0 < 0 \\ A_+|_{t=0} = 0 \\ A_-|_{t=0} = \pm\sqrt{-2p_0} \end{cases}$$

In what follows assume that $p_0 > 0$ and $A_+|_{t=0} > 0$. Other cases can be treated similarly. Note that $p_0 = \sqrt{2E}$, where $E > 0$ is the total energy of the harmonic oscillator, $H = H|_{t=0} = E$.

From (5.5) we get the following linear system:

$$\begin{cases} \overset{\circ}{\mu}{}^1_{23} = C_2p_0 - C_4, & \overset{\circ}{\mu}{}^1_{31} = C_3p_0 - C_1, & \overset{\circ}{\mu}{}^1_{12} = C_5\sqrt{2p_0} \\ \overset{\circ}{\mu}{}^2_{13} = C_2p_0 + C_4, & \overset{\circ}{\mu}{}^2_{12} = -C_6\sqrt{2p_0}, & \overset{\circ}{\mu}{}^2_{23} = C_3p_0 + C_1 \\ \overset{\circ}{\mu}{}^3_{13} = C_7\sqrt{2p_0}, & \overset{\circ}{\mu}{}^3_{23} = -C_8\sqrt{2p_0}, & \overset{\circ}{\mu}{}^3_{12} = C_9 \end{cases} \tag{6.1}$$

One can easily check that the unique solution of the latter system with respect to C_ν ($\nu = 1, \ldots, 9$) is

$$\begin{cases} C_1 = \frac{1}{2}\left(\overset{\circ}{\mu}{}^2_{23} - \overset{\circ}{\mu}{}^1_{31}\right), & C_2 = \frac{1}{2p_0}\left(\overset{\circ}{\mu}{}^2_{13} + \overset{\circ}{\mu}{}^1_{23}\right), & C_3 = \frac{1}{2p_0}\left(\overset{\circ}{\mu}{}^2_{23} + \overset{\circ}{\mu}{}^1_{31}\right) \\ C_4 = \frac{1}{2}\left(\overset{\circ}{\mu}{}^2_{13} - \overset{\circ}{\mu}{}^1_{23}\right), & C_5 = \frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^1_{12}, & C_6 = -\frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^2_{12} \\ C_7 = \frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^3_{13}, & C_8 = -\frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^3_{23}, & C_9 = \overset{\circ}{\mu}{}^3_{12} \end{cases}$$

Remark 6.1. Note that the parameters C_ν have to satisfy condition (5.4) to get the operadic Lax representations.

Definition 6.2. If $\mu \neq \overset{\circ}{\mu}$, then the multiplication μ is called a *dynamical deformation of* $\overset{\circ}{\mu}$ (over the harmonic oscillator). If $\mu = \overset{\circ}{\mu}$, then the multiplication $\overset{\circ}{\mu}$ is called *dynamically rigid*.

Example 6.3 ($\mathfrak{so}(3)$)**.** As an example consider the Lie algebra $\mathfrak{so}(3)$ with the structure equations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$

Thus, the nonzero structure constants are

$$\overset{\circ}{\mu}{}^1_{23} = \overset{\circ}{\mu}{}^2_{31} = \overset{\circ}{\mu}{}^3_{12} = -\overset{\circ}{\mu}{}^1_{32} = -\overset{\circ}{\mu}{}^2_{13} = -\overset{\circ}{\mu}{}^3_{21} = 1$$

Using the above initial conditions (6.1), we get

$$\begin{cases} \overset{\circ}{\mu}{}^1_{23} = C_2p_0 - C_4 = 1, & \overset{\circ}{\mu}{}^1_{31} = C_3p_0 - C_1 = 0, & \overset{\circ}{\mu}{}^1_{12} = C_5\sqrt{2p_0} = 0 \\ \overset{\circ}{\mu}{}^2_{13} = C_2p_0 + C_4 = -1, & \overset{\circ}{\mu}{}^2_{12} = -C_6\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^2_{23} = C_3p_0 + C_1 = 0 \\ \overset{\circ}{\mu}{}^3_{13} = C_7\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{23} = -C_8\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{12} = C_9 = 1 \end{cases}$$

From this linear system it is easy to see that the only nontrivial constants are $C_9 = -C_4 = 1$. Replacing these constants into (5.5) we get

$$\mu^i_{jk} = \overset{\circ}{\mu}{}^i_{jk}, \quad i, j, k = 1, \ldots, 9 \quad \Longrightarrow \quad \dot{\mu}|_{\mathfrak{so}(3)} = 0$$

Thus we can see that the present selection of the parameters C_ν ($\nu = 1, \dots 9$) via the structure constants of $\mathfrak{so}(3)$ does not give rise to the operadic Lax representation for the harmonic oscillator. In a sense, we can also say that the algebra $\mathfrak{so}(3)$ is dynamically rigid over the harmonic oscillator. This happens because condition (5.4) is not satisfied.

Example 6.4 (Heisenberg algebra)**.** As another example, consider the 3-dimensional Heisenberg algebra $\mathfrak{h}_1$ with the structure equations

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0$$

We can see that the only nonzero structure constant is $\overset{\circ}{\mu}{}^3_{12} = 1$. System (6.1) reads

$$\begin{cases} \overset{\circ}{\mu}{}^1_{23} = C_2 p_0 - C_4 = 0, & \overset{\circ}{\mu}{}^1_{31} = C_3 p_0 - C_1 = 0, & \overset{\circ}{\mu}{}^1_{12} = C_5\sqrt{2p_0} = 0 \\ \overset{\circ}{\mu}{}^2_{13} = C_2 p_0 + C_4 = 0, & \overset{\circ}{\mu}{}^2_{12} = -C_6\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^2_{23} = C_3 p_0 + C_1 = 0 \\ \overset{\circ}{\mu}{}^3_{13} = C_7\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{23} = -C_8\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{12} = C_9 = 1 \end{cases}$$

Thus, the only nontrivial constant is $C_9 = 1$. We conclude that

$$\mu^i_{jk} = \overset{\circ}{\mu}{}^i_{jk}, \quad i, j, k = 1, \dots, 9 \quad \Longrightarrow \quad \dot{\mu}|_{\mathfrak{h}_1} = 0$$

and the algebra $\mathfrak{h}_1$ turns out to be dynamically rigid over the harmonic oscillator as well. Again we can see that condition (5.4) is not satisfied.

Example 6.5 ($\mathfrak{sl}(2)$)**.** Finally consider the Lie algebra $\mathfrak{sl}(2)$ with the structure equations

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = 2e_1, \quad [e_2, e_3] = 2e_2$$

We can see that the nonzero structure constants are

$$\overset{\circ}{\mu}{}^1_{31} = \overset{\circ}{\mu}{}^2_{23} = 2\overset{\circ}{\mu}{}^3_{12} = 2$$

System (6.1) reads

$$\begin{cases} \overset{\circ}{\mu}{}^1_{23} = C_2 p_0 - C_4 = 0, & \overset{\circ}{\mu}{}^1_{31} = C_3 p_0 - C_1 = 2, & \overset{\circ}{\mu}{}^1_{12} = C_5\sqrt{2p_0} = 0 \\ \overset{\circ}{\mu}{}^2_{13} = C_2 p_0 + C_4 = 0, & \overset{\circ}{\mu}{}^2_{12} = -C_6\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^2_{23} = C_3 p_0 + C_1 = 2 \\ \overset{\circ}{\mu}{}^3_{13} = C_7\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{23} = -C_8\sqrt{2p_0} = 0, & \overset{\circ}{\mu}{}^3_{12} = C_9 = 1 \end{cases}$$

from which it follows that the only nontrivial constants are $C_3 = \frac{2}{p_0}$, $C_9 = 1$. From (5.5) we get the operadic Lax system

$$\begin{cases} \mu^1_{12} = \mu^2_{12} = \mu^3_{13} = \mu^3_{23} = \mu^3_{12} - 1 = 0 \\ \mu^1_{23} = \mu^2_{13} = -\frac{2\omega}{p_0} q, \quad \mu^1_{31} = \mu^2_{23} = \frac{2}{p_0} p \end{cases}$$

It turns out that the algebra $\{V,\mu\}$ is also a Lie algebra and isomorphic to $\mathfrak{sl}(2)=\{V,\overset{\circ}{\mu}\}$. The isomorphism

$$\mu^s_{jk}(q,p)A^i_s=\overset{\circ}{\mu}{}^i_{lm}A^l_jA^m_k$$

is realized by the matrix

$$A=(A^i_j):=\frac{1}{2p_0}\begin{pmatrix}\frac{2p_0}{\omega q}(p+\sqrt{2H}) & 2p_0 & 0\\ p-\sqrt{2H} & \omega q & 0\\ 0 & 0 & 2\sqrt{2H}\end{pmatrix}$$

6.2 Bianchi classification of 3d real Lie algebras

The material of Sections 6.2-6.4 has been published in [25].

We use the Bianchi classification of the 3-dimensional real Lie algebras given in [15]. The structure equations of the latter can be presented as follows:

$$[e_1,e_2]=-\alpha e_2+n^3e_3,\quad [e_2,e_3]=n^1e_1,\quad [e_3,e_1]=n^2e_2+\alpha e_3$$

The values of the parameters α, n^1, n^2, n^3 and the corresponding structure constants are presented in Table 6.1.

Type	α	(n^1,n^2,n^3)	$\overset{\circ}{\mu}{}^1_{12}$	$\overset{\circ}{\mu}{}^2_{12}$	$\overset{\circ}{\mu}{}^3_{12}$	$\overset{\circ}{\mu}{}^1_{23}$	$\overset{\circ}{\mu}{}^2_{23}$	$\overset{\circ}{\mu}{}^3_{23}$	$\overset{\circ}{\mu}{}^1_{31}$	$\overset{\circ}{\mu}{}^2_{31}$	$\overset{\circ}{\mu}{}^3_{31}$
I	0	$(0,0,0)$	0	0	0	0	0	0	0	0	0
II	0	$(1,0,0)$	0	0	0	1	0	0	0	0	0
VII	0	$(1,1,0)$	0	0	0	1	0	0	0	1	0
VI	0	$(1,-1,0)$	0	0	0	1	0	0	0	-1	0
IX	0	$(1,1,1)$	0	0	1	1	0	0	0	1	0
VIII	0	$(1,1,-1)$	0	0	-1	1	0	0	0	1	0
V	1	$(0,0,0)$	0	-1	0	0	0	0	0	0	1
IV	1	$(0,0,1)$	0	-1	1	0	0	0	0	0	1
VII_a	a	$(0,1,1)$	0	$-a$	1	0	0	0	0	1	a
$\mathrm{III}_{a=1}$	1	$(0,1,-1)$	0	-1	-1	0	0	0	0	1	1
$\mathrm{VI}_{a\neq 1}$	a	$(0,1,-1)$	0	$-a$	-1	0	0	0	0	1	a

Table 6.1: 3d real Lie algebras in Bianchi classification. Here $a>0$

The Bianchi classification is for instance used in cosmology to describe spatially homogeneous spacetimes of dimension 3+1. In particular, the Lie algebra VII_a is very interesting for cosmological applications, because it is related to the Friedmann-Robertson-Walker metric. One can find more details in [5, 11, 38].

6.3 Dynamical deformations of 3d real Lie algebras

By using the structure constants of the 3-dimensional real Lie algebras in the Bianchi classification, Theorem 5.11 and relations (6.1) one can propose that evolution of the 3-dimensional algebras real Lie algebras can be prescribed as given in Table 6.2.

Type	μ_{12}^1	μ_{12}^2	μ_{12}^3	μ_{23}^1	μ_{23}^2	μ_{23}^3	μ_{31}^1	μ_{31}^2	μ_{31}^3
I^t	0	0	0	0	0	0	0	0	0
II^t	0	0	0	$\frac{p+p_0}{2p_0}$	$\frac{\omega q}{2p_0}$	0	$\frac{\omega q}{2p_0}$	$\frac{p-p_0}{-2p_0}$	0
VII^t	0	0	0	1	0	0	0	1	0
VI^t	0	0	0	$\frac{p}{p_0}$	$\frac{\omega q}{p_0}$	0	$\frac{\omega q}{p_0}$	$-\frac{p}{p_0}$	0
IX^t	0	0	1	1	0	0	0	1	0
VIII^t	0	0	-1	1	0	0	0	1	0
V^t	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	0	0	0	$\frac{-A_-}{\sqrt{2p_0}}$	0	0	$\frac{A_+}{\sqrt{2p_0}}$
IV^t	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	1	0	0	$\frac{-A_-}{\sqrt{2p_0}}$	0	0	$\frac{A_+}{\sqrt{2p_0}}$
VII_a^t	$\frac{aA_-}{\sqrt{2p_0}}$	$\frac{-aA_+}{\sqrt{2p_0}}$	1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aA_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aA_+}{\sqrt{2p_0}}$
$\mathrm{III}_{a=1}^t$	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-A_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{A_+}{\sqrt{2p_0}}$
$\mathrm{VI}_{a\neq 1}^t$	$\frac{aA_-}{\sqrt{2p_0}}$	$\frac{-aA_+}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aA_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aA_+}{\sqrt{2p_0}}$

Table 6.2: Evolution of 3d real Lie algebras. Here $p_0 = \sqrt{2E}$

From this table one can see that the real Lie algebras I, VII, VIII, IX do not give rise to the operadic Lax representation for the harmonic oscillator, because condition (5.4) is not satisfied. In a sense, these Lie algebras are dynamically rigid over the harmonic oscillator in the Bianchi classification. However, for some other particular basis one can get (see Example 6.5) an operadic Lax representation for $\mathfrak{sl}(2)$ as well (type VIII in the Bianchi classification).

6.4 Jacobi identities and energy conservation

Theorem 6.6 (dynamically rigid algebras). *The algebras I, VII, VIII, and IX are dynamically rigid over the harmonic oscillator.*

Proof. This is evident from Tables 6.1 and 6.2. □

Denoting $\mu := [\cdot,\cdot]_t$, define the *Jacobiator* of the algebra elements x, y, z by

$$\begin{aligned} J_t(x;y;z) &:= [x,[y,z]_t]_t + [y,[z,x]_t]_t + [z,[x,y]_t]_t \\ &= J_t^1(x;y;z)e_1 + J_t^2(x;y;z)e_2 + J_t^3(x;y;z)e_3 \end{aligned}$$

Theorem 6.7 (dynamical Lie algebras). *The algebras II^t, IV^t, V^t, VI^t, $III^t_{a=1}$, $VI^t_{a\neq 1}$, and VII^t_a are Lie algebras.*

Proof. It follows from Theorems 6.11-6.12 in the classical case that the algebras II^t, IV^t, V^t, and VI^t are Lie algebras. From Theorem 6.13 in the classical case one gets that the Jacobiator coordinates for the algebras $\mathrm{VI}^t_{a\neq 1}$, and VII^t_a read

$$\begin{cases} J_t^1(x;y;z) = \frac{a(x,y,z)}{\sqrt{2p_0^3}}\left[A_+(p_0-p) - A_-\omega q\right] \\ J_t^2(x;y;z) = \frac{a(x,y,z)}{\sqrt{2p_0^3}}\left[A_-(p_0+p) - A_+\omega q\right] \\ J_t^3(x;y;z) = 0 \end{cases} \tag{6.2}$$

and for the algebra $\mathrm{III}^t_{a=1}$ one has the same formulae with $a = 1$. We use notation (x, y, z) for the scalar triple product of elements x, y, z (see page 88).

Now, by using relations (5.1) calculate:

$$\begin{aligned} A_+(p_0-p) - A_-\omega q &= A_+(p_0-p) - A_+A_-^2 \\ &= A_+(p_0 - p - A_-^2) \\ &= A_+\left(p_0 - \frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 - A_-^2\right) \\ &= A_+\left(p_0 - \frac{1}{2}A_+^2 - \frac{1}{2}A_-^2\right) \\ &= A_+(p_0 - \sqrt{2H}) \\ &= A_+(p_0 - \sqrt{2E}) \\ &= A_+0 \\ &= 0 \end{aligned}$$

Here we used the fact that the Hamiltonian H is a *conserved* observable, i.e

$$H = H|_{t=0} = E = \frac{p_0^2}{2} \tag{6.3}$$

Thus, we have proved that $J^1_t = 0$. In the same way one can check that $J^2_t = 0$. □

When proving Theorem 6.7 we observed how the conservation of energy $H = E$ implies the Jacobi identities $J^1_t = J^2_t = 0$ of the dynamically deformed algebras. Now let us expose *vice versa*, i.e

Theorem 6.8. *The Jacobi identities $J^1_t = J^2_t = 0$ imply conservation of energy $H = E$.*

Proof. By setting in (6.2) $J^1_t = J^2_t = 0$, we obtain the following system:

$$\begin{cases} A_-\omega q + A_+ p = A_+ p_0 \\ A_+\omega q - A_- p = A_- p_0 \end{cases}$$

Now use defining relations (5.1) of $A_\pm$ and the Cramer formulae to express the canonical variables q, p via $A_\pm$. First calculate

$$\Delta := \begin{vmatrix} A_- & A_+ \\ A_+ & -A_- \end{vmatrix} = -A_-^2 - A_+^2 = -2\sqrt{2H} \neq 0$$

$$\Delta_{\omega q} := \begin{vmatrix} A_+ p_0 & A_+ \\ A_- p_0 & -A_- \end{vmatrix} = -2A_+A_- p_0 = -2\omega q p_0$$

$$\Delta_p := \begin{vmatrix} A_- & A_+ p_0 \\ A_+ & A_- p_0 \end{vmatrix} = A_-^2 p_0 - A_+^2 p_0 = -2pp_0 = -2\omega q p_0$$

Thus we have

$$\omega q = \frac{\Delta_{\omega q}}{\Delta} = -\frac{2\omega q p_0}{-2\sqrt{2H}} \qquad \Longrightarrow \qquad \frac{p_0}{\sqrt{2H}} = 1 \qquad \Longrightarrow \qquad H = p_0^2/2 = E$$

$$p = \frac{\Delta_p}{\Delta} = -\frac{2pp_0}{-2\sqrt{2H}} = \frac{pp_0}{\sqrt{2H}} \qquad \Longrightarrow \qquad \frac{p_0}{\sqrt{2H}} = 1 \qquad \Longrightarrow \qquad H = p_0^2/2 = E$$

Actually, the last implications are possible only at the time moments when $q \neq 0$ and $p \neq 0$, respectively. But q and p can not be simultaneously zero, thus really $H = E$ for all t. □

Thus the evolution of these algebras are generated by the harmonic oscillator, because their multiplications depend on the canonical and quasi-canonical coordinates of the harmonic oscillator.

Remark 6.9. It is interesting to note that if we use in formulae (5.1) the conservation of energy $H = E$, then Theorems 5.11 and 6.7 remain true but we miss Theorem 6.8. Thus, the operadic Lax representations and dynamical deformations of algebras may be useful when searching for the first integrals of the dynamical systems.

6.5 Quantum algebras from Bianchi classification

The material of this section has been published in [31].

By using the structure constants of the 3-dimensional real Lie algebras in the Bianchi classification (Table 6.1), operadic Lax representations (Theorem 5.11) for the harmonic oscillator and relations (6.1) we found evolution (dynamical deformations) of these algebras generated by the harmonic oscillator (see Table 6.2).

In the dynamically deformed Bianchi classification (Table 6.2) the structure functions μ^i_{jk} depend on the canonical and quasi-canonical coordinates of the harmonic oscillator. The quasi-canonical coordinates were defined by constraints (5.1).

Now, by using Table 6.2, we can propose the canonically quantized counterparts of the 3-dimensional real Lie algebras in the Bianchi classification (Table 6.3).

Type	$\hat{\mu}^1_{12}$	$\hat{\mu}^2_{12}$	$\hat{\mu}^3_{12}$	$\hat{\mu}^1_{23}$	$\hat{\mu}^2_{23}$	$\hat{\mu}^3_{23}$	$\hat{\mu}^1_{31}$	$\hat{\mu}^2_{31}$	$\hat{\mu}^3_{31}$
$\mathrm{I}^\hbar$	0	0	0	0	0	0	0	0	0
$\mathrm{II}^\hbar$	0	0	0	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\omega\hat{q}}{2p_0}$	0	$\frac{\omega\hat{q}}{2p_0}$	$\frac{\hat{p}-p_0}{-2p_0}$	0
$\mathrm{VII}^\hbar$	0	0	0	1	0	0	0	1	0
$\mathrm{VI}^\hbar$	0	0	0	$\frac{\hat{p}}{p_0}$	$\frac{\omega\hat{q}}{p_0}$	0	$\frac{\omega\hat{q}}{p_0}$	$-\frac{\hat{p}}{p_0}$	0
$\mathrm{IX}^\hbar$	0	0	1	1	0	0	0	1	0
$\mathrm{VIII}^\hbar$	0	0	-1	1	0	0	0	1	0
$\mathrm{V}^\hbar$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	0	0	0	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	0	0	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{IV}^\hbar$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	1	0	0	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	0	0	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{VII}^\hbar_a$	$\frac{a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-a\hat{A}_+}{\sqrt{2p_0}}$	1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{III}^\hbar_{a=1}$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{VI}^\hbar_{a\neq 1}$	$\frac{a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-a\hat{A}_+}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{A}_+}{\sqrt{2p_0}}$

Table 6.3: Quantum algebras over the harmonic oscillator. Here $p_0 = \sqrt{2E}$

Let us study the Jacobi identities for the quantum algebras from Table 6.3. Denoting $\hat{\mu} := [\cdot,\cdot]_\hbar$, define the quantum analogue of the Jacobiator – the *Jacobi operator* by

$$\hat{J}_\hbar(x;y;z) := [x,[y,z]_\hbar]_\hbar + [y,[z,x]_\hbar]_\hbar + [z,[x,y]_\hbar]_\hbar \tag{6.4}$$

with components $\hat{J}^i_\hbar(x;y;z)$, $i=1,2,3$.

Theorem 6.10 (rigid algebras)**.** *The algebras I, VII, VIII, and IX are rigid over the quantum harmonic oscillator.*

Proof. This is evident from Tables 6.1 and 6.3. □

Theorem 6.11 (quantum Lie algebras)**.** *The algebras $II^\hbar$ and $VI^\hbar$ are Lie algebras.*

Proof. By direct calculations one can show that

$$\hat{J}^1_\hbar(x;y;z) = \hat{J}^2_\hbar(x;y;z) = \hat{J}^3_\hbar(x;y;z) = 0$$

□

Theorem 6.12 (anomalous quantum algebras of the first type)**.** *The algebras $IV^\hbar$ and $V^\hbar$ are non-Lie algebras.*

Proof. By direct calculations (see Appendix A) one can see that

$$\begin{aligned}
\hat{J}^1_\hbar(x;y;z) &= 0 = \hat{J}^2_\hbar(x;y;z)\\
\hat{J}^3_\hbar(x;y;z) &= \frac{(x,y,z)}{p_0}[\hat{A}_+,\hat{A}_-]
\end{aligned}$$

□

Theorem 6.13 (anomalous quantum algebras of the second type)**.** *The algebras $III^\hbar_{a=1}$, $VI^\hbar_{a\neq 1}$, and $VII^\hbar_a$ are non-Lie algebras.*

Proof. Denote

$$\hat{\xi}_\pm := \omega\hat{q}\hat{A}_\mp \pm (\hat{p}\mp p_0)\hat{A}_\pm$$

Then, by direct calculations (see Appendix A) one can check that for the algebras $\mathrm{VI}^\hbar_{a\neq 1}$ and $\mathrm{VII}^\hbar_a$ the Jacobi operator coordinates are

$$\begin{aligned}
\hat{J}^1_\hbar(x;y;z) &= -\frac{a(x,y,z)}{\sqrt{2p_0^3}}\hat{\xi}_+, \quad \hat{J}^2_\hbar(x;y;z) = -\frac{a(x,y,z)}{\sqrt{2p_0^3}}\hat{\xi}_-,\\
\hat{J}^3_\hbar(x;y;z) &= \frac{a^2(x,y,z)}{p_0}[\hat{A}_+,\hat{A}_-]
\end{aligned}$$

and for the algebra $\mathrm{III}^\hbar_{a=1}$ one has the same formulae with $a=1$. □

Thus, the quantum algebras $\mathrm{IV}^\hbar$, $\mathrm{V}^\hbar$, $\mathrm{III}^\hbar_{a=1}$, $\mathrm{VI}^\hbar_{a\neq 1}$, $\mathrm{VII}^\hbar_a$ are *anomalous* in the sense that their Jacobi identities are violated.

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The research of the author has been an essential part of all these **Publications**.

Abstract

In the present thesis, some novel opportunities for introducing dynamics in certain algebraic systems are given.

The main notions and results of the Lie theory are given. The Lie theorems, the Lie and Maurer-Cartan equations are presented.

The canonical formalism for the Lie transformation group $SO(2)$ is developed both in real and complex representation. This group can be seen as a toy model of the Hamilton-Dirac mechanics with constraints. The Lagrangian and Hamiltonian are explicitly constructed and their physical interpretation are given. The crucial observation is that the Euler-Lagrange and Hamilton canonical equations of $SO(2)$ coincide with the Lie equations. It is shown that the constraints satisfy canonical commutation relations. Consistency of the constraints is checked.

A general method for constructing Lagrangians for the Lie transformation groups is explained. As examples the vector Lagrangians for real plane rotations and affine transformations for the real line are constructed.

The second part of the thesis deals with an operadic generalization of the Lax differential equation, modelling the evolution of dynamical systems.

One starts with the notion of an operad and the overview of the Gerstenhaber theory. An operad is an abstract algebraic formulation of composable functions of several variables. Operadic variables satisfy the generalized associativity identities and their time evolution is governed by operadic evolution equation. Based on the Gerstenhaber theory, it is explained how the operadic dynamics may be introduced. The notion of an operadic Lax pair is given. As an example, an operadic (representation for the) harmonic oscillator is proposed.

Operadic Lax representations for the harmonic oscillator are constructed in the following binary real algebras:

- general 2-dimensional algebras,
- 3-dimensional anti-commutative algebras.

Introducing operadic Lax representations can be seen as generating integrable dynamics in algebras. Thus, the results of the thesis give in addition another connection between Hamiltonian and Lax formalisms.

Using the operadic Lax representations for the harmonic oscillator, dynamical deformations are constructed for

- all 3-dimensional real Lie algebras,
- 2-dimensional real associative unital and Lie algebras.

It is shown how the energy conservation of the harmonic oscillator is related to

- the Jacobi identities of the dynamically deformed 3-dimensional real Lie algebras,
- the associativity identities of the dynamically deformed 2-dimensional real associative unital algebras.

Based on this observation, it is proved that the dynamical deformations of 3-dimensional real Lie algebras in the Bianchi classification over the harmonic oscillator are Lie algebras.

Quantum counterparts over the harmonic oscillator are constructed for all 2- and 3-dimensional real Lie algebras. Their Jacobi operators are calculated and studied.

It is discussed how the operadic dynamics in 3-dimensional real Lie algebras over the harmonic oscillator is related to quantization of a 3-dimensional space.

Kokkuvõte

Käesolevas väitekirjas on esitatud mõned uudsed võimalused dünaamika sissetoomiseks teatud tüüpi algebralistes süsteemides, mida realiseeritakse Lie ja operaadide teooria abil.

Töö koosneb kahest osast: peatükid 1-3 ja peatükid 4-6. Esimene peatükk sisaldab Lie teooria põhimõisteid ja -tulemusi. Defineeritakse Lie rühm, Lie algebra ja Lie teisendusrühm. Esitatakse Lie teoreemid, Lie ja Maurer-Cartani võrrandid.

Kanooniline formalism Lie teisendusrühma $SO(2)$ jaoks on arendatud reaalses ja kompleksses esituses. Seda rühma vaadeldakse Hamilton-Diraci seostega mehaanika mudelina. Ilmutatud kujul on konstrueeritud lagranžiaan ja hamiltoniaan ning on antud nende füüsikaline tõlgendus. Osutub, et Euler-Lagrange'i ja Hamiltoni kanoonilised võrrandid langevad Lie võrranditega kokku. Näidatakse, et seosed rahuldavad kanoonilisi kommutatsiooniseoseid. Kontrollitud on seoste kooskõla.

Kolmandas peatükis on selgitatud lagranžiaanide konstrueerimise üldist meetodit Lie teisendusrühmadele. Näidetena on leitud lagranžiaanid tasandi pöörete ning sirge afiinsete teisenduste rühmale.

Väitekirja teises osas tegeldakse dünaamiliste süsteemide evolutsiooni modelleeriva Laxi diferentsiaalvõrrandi operaadüldistusega. Seda ülesannet vaadeldakse esmakordselt. Klassikalises mehaanikas on teatavasti võimalik kirjeldada dünaamilist süsteemi Hamiltoniaaniga $H = H(q,p)$ Hamiltoni võrrandite

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

või nendega ekvivalentse Laxi võrrandiga

$$\frac{dL}{dt} = [M, L]$$

Seega saab mehaanilist süsteemi algebraliselt kirjeldada vektorruumi V lineaarteisenduste, s.t lineaarkujutustega $V \to V$. Väitekirja teise osa põhiideeks on laiendada Laxi meetodit (lineaarsetele) algebralistele operatsioonidele $V^{\otimes n} \to V$.

Kokkuvõte

Alustatakse operaadi mõiste sissetoomise ja ülevaatega Gerstenhaberi teooriast. Operaad on kompositsioonide suhtes kinnine operatsioonide süsteem. Operaadmuutujad rahuldavad üldistatud assotsiatiivsustingimusi ning nende evolutsiooni ajas kirjeldavad Laxi võrrandi operaadüldistused. Defineeritakse Laxi operaadpaari mõiste. Näitena konstrueeritakse harmoonilise ostsillaatori operaadesitused, kus tavalise kommutaatori asemel kasutatakse Gerstenhaberi sulgusid.

Peatükis 5 on leitud harmoonilise ostsillaatori Laxi operaadesitused madalates dimensioonides. Näidetena konstrueeritakse harmooniline operaadostsillaator järgmistes reaalsetes algebrates:

- kolmemõõtmelises antikommutatiivses algebras,
- üldises kahemõõtmelises algebras.

On näidatud, et kahemõõtmelised binaarsed reaalsed assotsiatiivsed ühikuga algebrad säilitavad dünaamilisel deformeerimisel assotsiatiivsuse omaduse.

Kasutades harmoonilise ostsillaatori Laxi operaadesitusi, leitakse peatükis 6 kolmemõõtmeliste reaalsete Lie algebrate dünaamilised deformatsioonid. Näidatakse, kuidas on seotud harmoonilise ostsillaatori energia jäävus dünaamiliselt deformeeritud algebrate Jacobi identsustega. Sellest tähelepanekust lähtudes tõestatakse, et kolmemõõtmeliste reaalsete Lie algebrate dünaamilised deformatsioonid üle harmoonilise ostsillaatori Bianchi klassifikatsioonis on Lie algebrad. Kasutades Bianchi tabeli dünaamilist deformatsiooni, defineeritakse kahe- ja kolmemõõtmeliste reaalsete Lie algebrate kvantanaloogid ning leitakse nende Jacobi operaatorid.

Selgitatakse, kuidas operaadidünaamika kolmemõõtmelistes reaalsetes Lie algebrates üle harmoonilise ostsillaatori on seotud kolmemõõtmelise ruumi kvantimisega.

Laxi operaadesituste sissetoomist saab vaadelda dünaamika genereerimisena algebrates. Käesoleva väitekirja tulemused avavad uue aspekti Laxi formalismi üldistamiseks algebralistele süsteemidele.

Töö tulemuste aprobeerimine

Ilmunud on 9 artiklit (vt. lk. 78). Kõik need on avaldatud rahvusvahelise levikuga eelretsenseeritavates teadusajakirjades (7) ja teaduskogumikes (2), millest enamik (6) on nn. *ISI* väljaanded, ja refereeritakse Ameerika Matemaatikaühingu andmebaasides (*MathScinet*, *Math. Rev*) ning *Zentralblatt Math* poolt. Umbes pooled artiklid on avaldatud füüsikaajakirjades. Lisaks on tulemusi avaldatud ühe preprindina *Preprint ArXiv*'is ning ülevaateartiklina Eesti Matemaatika Seltsi aastaraamatus (vt. lk. 10).

Väitekirja tulemusi on ette kantud mitmetel rahvusvahelistel erialastel konverentsidel ja seminaridel (vt. lk. 10–11).

Appendices

APPENDIX

A

Calculation of the Jacobi operators

In this appendix, we prove Theorems 6.12 and 6.13.

Consider a quantum algebra $\mathcal{A}^\hbar$ with the quantum multiplication $[\cdot,\cdot]_\hbar$ defined by the anti-commutative structure constants (operators)

$$\begin{cases} \hat{\mu}^1_{12} := \frac{a\hat{A}_-}{\sqrt{2p_0}}, & \hat{\mu}^2_{12} := \frac{-a\hat{A}_+}{\sqrt{2p_0}}, & \hat{\mu}^3_{12} := b, \\ \hat{\mu}^1_{23} := -\gamma\frac{\hat{p}-p_0}{2p_0}, & \hat{\mu}^2_{23} := -\beta\frac{\omega\hat{q}}{2p_0}, & \hat{\mu}^3_{23} := \frac{-a\hat{A}_-}{\sqrt{2p_0}} \\ \hat{\mu}^1_{31} := -\beta\frac{\omega\hat{q}}{2p_0}, & \hat{\mu}^2_{31} := \gamma\frac{\hat{p}+p_0}{2p_0}, & \hat{\mu}^3_{31} := \frac{a\hat{A}_+}{\sqrt{2p_0}} \end{cases}$$

with parameters $\beta,\gamma,a,b\in\mathbb{R}$.

Giving specific values to the real–valued parameters β,γ,a,b, one gets some special cases of this algebra. Note that for the particular choice, the result is five quantum algebras (see Table A.1) from Table 6.3.

Quantum algebra	β	γ	a	b
$\mathrm{VII}^\hbar_a$	1	1	a	1
$\mathrm{VI}^\hbar_{a\neq 1}$	1	1	$a\neq 1$	-1
$\mathrm{III}^\hbar_{a=1}$	1	1	1	1
$\mathrm{IV}^\hbar$	0	0	1	1
$\mathrm{V}^\hbar$	0	0	1	0

Table A.1: Special cases of the algebra $\mathcal{A}^\hbar$

These quantum algebras are used in Theorems 6.12-6.13. Let us find the Jacobi operator (6.4) for the algebra $\mathcal{A}^\hbar$.

First, we find the products $[x,y]_\hbar, [y,z]_\hbar, [z,x]_\hbar$ in $\mathcal{A}^\hbar$. Calculate

$$
\begin{aligned}
&[x,y]_\hbar = [x,y]^i_\hbar e_i = \hat{\mu}^i_{jk} x^j y^k e_i \\
&= \left(\hat{\mu}^1_{12}\left(x^1y^2 - x^2y^1\right) + \hat{\mu}^1_{13}\left(x^1y^3 - x^3y^1\right) + \hat{\mu}^1_{23}\left(x^2y^3 - x^3y^2\right)\right) e_1 \\
&+ \left(\hat{\mu}^2_{12}\left(x^1y^2 - x^2y^1\right) + \hat{\mu}^2_{13}\left(x^1y^3 - x^3y^1\right) + \hat{\mu}^2_{23}\left(x^2y^3 - x^3y^2\right)\right) e_2 \\
&+ \left(\hat{\mu}^3_{12}\left(x^1y^2 - x^2y^1\right) + \hat{\mu}^3_{13}\left(x^1y^3 - x^3y^1\right) + \hat{\mu}^3_{23}\left(x^2y^3 - x^3y^2\right)\right) e_3 \\
&= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(x^1y^2 - x^2y^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(x^1y^3 - x^3y^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(x^2y^3 - x^3y^2\right)\right) e_1 \\
&+ \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(x^1y^2 - x^2y^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(x^1y^3 - x^3y^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(x^2y^3 - x^3y^2\right)\right) e_2 \\
&+ \left(\hat{\mu}^3_{12}\left(x^1y^2 - x^2y^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(x^1y^3 - x^3y^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(x^2y^3 - x^3y^2\right)\right) e_3
\end{aligned}
$$

In the same way, one can check that

$$
\begin{aligned}
&[y,z]_\hbar = [y,z]^i_\hbar e_i = \hat{\mu}^i_{jk} y^j z^k e_i \\
&= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(y^1z^3 - y^3z^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(y^2z^3 - y^3z^2\right)\right) e_1 \\
&+ \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(y^1z^3 - y^3z^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(y^2z^3 - y^3z^2\right)\right) e_2 \\
&+ \left(\hat{\mu}^3_{12}\left(y^1z^2 - y^2z^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^3 - y^3z^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^2z^3 - y^3z^2\right)\right) e_3
\end{aligned}
$$

and

$$
\begin{aligned}
&[z,x]_\hbar = [z,x]^i_\hbar e_i = \hat{\mu}^i_{jk} z^j x^j e_i \\
&= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(z^1x^3 - z^3x^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(z^2x^3 - z^3x^2\right)\right) e_1 \\
&+ \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(z^1x^3 - z^3x^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(z^2x^3 - z^3x^2\right)\right) e_2 \\
&+ \left(\hat{\mu}^3_{12}\left(z^1x^2 - z^2x^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^3 - z^3x^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^2x^3 - z^3x^2\right)\right) e_3
\end{aligned}
$$

Now we find the first Jacobi operator coordinate:

$$\hat{J}^1_\hbar(x;y;z) = [x,[y,z]_\hbar]^1_\hbar + [y,[z,x]_\hbar]^1_\hbar + [z,[x,y]_\hbar]^1_\hbar$$

$$
\begin{aligned}
&= \hat{\mu}^1_{jk} x^j [y,z]^k_\hbar + \hat{\mu}^1_{jk} y^j [z,x]^k_\hbar + \hat{\mu}^1_{jk} z^j [x,y]^k_\hbar \\
&= \hat{\mu}^1_{12}\left(x^1[y,z]^2_\hbar - x^2[y,z]^1_\hbar\right) + \hat{\mu}^1_{13}\left(x^1[y,z]^3_\hbar - x^3[y,z]^1_\hbar\right) + \hat{\mu}^1_{23}\left(x^2[y,z]^3_\hbar - x^3[y,z]^2_\hbar\right) \\
&+ \hat{\mu}^1_{12}\left(y^1[z,x]^2_\hbar - y^2[z,x]^1_\hbar\right) + \hat{\mu}^1_{13}\left(y^1[z,x]^3_\hbar - y^3[z,x]^1_\hbar\right) + \hat{\mu}^1_{23}\left(y^2[z,x]^3_\hbar - y^3[z,x]^2_\hbar\right) \\
&+ \hat{\mu}^1_{12}\left(z^1[x,y]^2_\hbar - z^2[x,y]^1_\hbar\right) + \hat{\mu}^1_{13}\left(z^1[x,y]^3_\hbar - z^3[x,y]^1_\hbar\right) + \hat{\mu}^1_{23}\left(z^2[x,y]^3_\hbar - z^3[x,y]^2_\hbar\right) \\
&= \frac{a\hat{A}_-}{\sqrt{2p_0}}\Bigg\{ x^1\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(y^1z^3 - y^3z^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(y^2z^3 - y^3z^2\right)\right) \\
&- x^2\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(y^1z^3 - y^3z^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(y^2z^3 - y^3z^2\right)\right)\Bigg\} \\
&+ \beta\frac{\omega\hat{q}}{2p_0}\Bigg\{ x^1\left(\hat{\mu}^3_{12}\left(y^1z^2 - y^2z^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^3 - y^3z^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^2z^3 - y^3z^2\right)\right) \\
&- x^3\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(y^1z^3 - y^3z^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(y^2z^3 - y^3z^2\right)\right)\Bigg\} \\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\Bigg\{ x^2\left(\hat{\mu}^3_{12}\left(y^1z^2 - y^2z^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^3 - y^3z^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(y^2z^3 - y^3z^2\right)\right) \\
&- x^3\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(y^1z^2 - y^2z^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(y^1z^3 - y^3z^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(y^2z^3 - y^3z^2\right)\right)\Bigg\} \\
&+ \frac{a\hat{A}_-}{\sqrt{2p_0}}\Bigg\{ y^1\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(z^1x^3 - z^3x^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(z^2x^3 - z^3x^2\right)\right) \\
&- y^2\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(z^1x^3 - z^3x^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(z^2x^3 - z^3x^2\right)\right)\Bigg\} \\
&+ \beta\frac{\omega\hat{q}}{2p_0}\Bigg\{ y^1\left(\hat{\mu}^3_{12}\left(z^1x^2 - z^2x^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^3 - z^3x^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^2x^3 - z^3x^2\right)\right) \\
&- y^3\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) + \beta\frac{\omega\hat{q}}{2p_0}\left(z^1x^3 - z^3x^1\right) - \gamma\frac{\hat{p}-p_0}{2p_0}\left(z^2x^3 - z^3x^2\right)\right)\Bigg\} \\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\Bigg\{ y^2\left(\hat{\mu}^3_{12}\left(z^1x^2 - z^2x^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^3 - z^3x^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}}\left(z^2x^3 - z^3x^2\right)\right) \\
&- y^3\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(z^1x^2 - z^2x^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(z^1x^3 - z^3x^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(z^2x^3 - z^3x^2\right)\right)\Bigg\} \\
&+ \frac{a\hat{A}_-}{\sqrt{2p_0}}\Bigg\{ z^1\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\left(x^1y^2 - x^2y^1\right) - \gamma\frac{\hat{p}+p_0}{2p_0}\left(x^1y^3 - x^3y^1\right) - \beta\frac{\omega\hat{q}}{2p_0}\left(x^2y^3 - x^3y^2\right)\right)
\end{aligned}
$$

$$- z^2 \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \left(x^1y^2 - x^2y^1\right) + \beta\frac{\omega\hat{q}}{2p_0} \left(x^1y^3 - x^3y^1\right) - \gamma\frac{\hat{p} - p_0}{2p_0} \left(x^2y^3 - x^3y^2\right) \right) \Bigg\}$$

$$+ \beta\frac{\omega\hat{q}}{2p_0}\Bigg\{ z^1 \left(\hat{\mu}^3_{12} \left(x^1y^2 - x^2y^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}} \left(x^1y^3 - x^3y^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}} \left(x^2y^3 - x^3y^2\right) \right)$$

$$- z^3 \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \left(x^1y^2 - x^2y^1\right) + \beta\frac{\omega\hat{q}}{2p_0} \left(x^1y^3 - x^3y^1\right) - \gamma\frac{\hat{p} - p_0}{2p_0} \left(x^2y^3 - x^3y^2\right) \right) \Bigg\}$$

$$- \gamma\frac{\hat{p} - p_0}{2p_0}\Bigg\{ z^2 \left(\hat{\mu}^3_{12} \left(x^1y^2 - x^2y^1\right) - \frac{a\hat{A}_+}{\sqrt{2p_0}} \left(x^1y^3 - x^3y^1\right) - \frac{a\hat{A}_-}{\sqrt{2p_0}} \left(x^2y^3 - x^3y^2\right) \right)$$

$$- z^3 \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}} \left(x^1y^2 - x^2y^1\right) - \gamma\frac{\hat{p} + p_0}{2p_0} \left(x^1y^3 - x^3y^1\right) - \beta\frac{\omega\hat{q}}{2p_0} \left(x^2y^3 - x^3y^2\right) \right) \Bigg\}$$

By parentheses removal and collecting terms, one gets

$$\begin{aligned} \hat{J}^1_\hbar(x;y;z) = -\frac{a}{2}\frac{\sqrt{2}}{\sqrt{p_0^3}}\Big(& -\beta\omega x^1y^3z^2\hat{q}\hat{A}_- - \gamma x^2y^1z^3\hat{p}\hat{A}_+ + \gamma x^2y^3z^1\hat{p}\hat{A}_+ \\ & - \gamma x^2y^3z^1p_0\hat{A}_+ + \gamma x^2y^1z^3p_0\hat{A}_+ + \beta\omega x^1y^2z^3\hat{q}\hat{A}_- - \beta\omega x^2y^1z^3\hat{q}\hat{A}_- \\ & + \beta\omega x^2y^3z^1\hat{q}\hat{A}_- - \beta\omega x^3y^2z^1\hat{q}\hat{A}_- + \gamma x^3y^1z^3\hat{p}\hat{A}_+ - \gamma x^1y^3z^2\hat{p}\hat{A}_+ \\ & + \gamma x^3y^2z^1p_0\hat{A}_+ - \gamma x^3y^1z^2p_0\hat{A}_+ - \gamma x^3y^2z^1\hat{p}\hat{A}_+ + \beta\omega x^3y^1z^2\hat{q}\hat{A}_- \\ & - \gamma x^1y^2z^3p_0\hat{A}_+ + \gamma x^1y^2z^3\hat{p}\hat{A}_+ + \gamma x^1y^3z^2p_0\hat{A}_+\Big) \end{aligned}$$

Denote the scalar triple product of algebra elements x, y, z by

$$(x,y,z) := \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}$$

Then

$$\hat{J}^1_\hbar(x;y;z) = -\frac{a(x,y,z)}{\sqrt{2p_0^3}}\left(\beta\omega\hat{q}\hat{A}_- + \gamma(\hat{p} - p_0)\hat{A}_+\right)$$

In the same way it is possible to show, that

$$\begin{aligned} \hat{J}^2_\hbar(x;y;z) &= -\frac{a(x,y,z)}{\sqrt{2p_0^3}}\left(\beta\omega\hat{q}\hat{A}_+ - \gamma(\hat{p} + p_0)\hat{A}_-\right) \\ \hat{J}^3_\hbar(x;y;z) &= \quad \frac{a^2(x,y,z)}{p_0}(\hat{A}_+\hat{A}_- - \hat{A}_-\hat{A}_+) = \frac{a^2(x,y,z)}{p_0}[\hat{A}_+, \hat{A}_-] \end{aligned}$$

We have got the Jacobi operator coordinates $\hat{J}^i_\hbar(x;y;z)$, $i = 1,2,3$, for the algebra $\mathcal{A}^\hbar$. Note, that the latter do not depend on the choice of b.

Thus, by definition of the algebra $\mathcal{A}^\hbar$ Theorems 6.12-6.13 are proved.

APPENDIX

$\mathcal{B}$

Discussion: operadic quantization of algebras $\mathbf{VII}_a$, $\mathbf{III}_{a=1}$, $\mathbf{VI}_{a\neq 1}$ over harmonic oscillator

In Section 6.5, the operadic Lax representations for the harmonic oscillator were used to construct the quantum counterparts of 3-dimensional real Lie algebras in the Bianchi classification, and the Jacobi operators of these quantum algebras were calculated.

In this appendix, it is conjectured that the derivative algebras of the quantum algebras $\mathrm{VII}_a^\hbar$, $\mathrm{III}_{a=1}^\hbar$, $\mathrm{VI}_{a\neq 1}^\hbar$ are the Heisenberg algebra. From this it follows that the volume element in $\mathbb{R}^3$ has discrete values: $|(x,y,z)| = 4\sqrt{2}(2n+1)$, $(n=0,1,2,\dots)$.

The material of this appendix is presented in the preprint [29].

B.1 Quasi-canonical quantum conditions

Theorem B.1 (Poisson brackets of quasi-canonical coordinates)**.** *The quasi-canonical coordinates $A_\pm$ satisfy the relations*

$$\{A_+,A_+\} = 0 = \{A_-,A_-\}, \quad \{A_+,A_-\} = \varepsilon := \frac{\omega}{2\sqrt{2H}} \tag{B.1}$$

Proof. While the first two relations in (B.1) are evident, we have only to check the third one. Using several times the Leibniz rule for the Poisson brackets, calculate:

$$\begin{aligned}
2\omega = 2\omega\{p,q\} &= \{A_+^2 - A_-^2, A_+A_-\}\\
&= \{A_+^2, A_+A_-\} - \{A_-^2, A_+A_-\}\\
&= A_+\{A_+^2, A_-\} - \{A_-^2, A_+\}A_-\\
&= A_+\{A_+A_+, A_-\} - \{A_-A_-, A_+\}A_-
\end{aligned}$$

$$= 2(A_+^2 + A_-^2)\{A_+, A_-\}$$
$$= 4\sqrt{2H}\{A_+, A_-\} \qquad \square$$

In what follows, we will use the Schrödinger picture, i.e the operators $\hat{q}, \hat{p}, \hat{H}$ and $\hat{A}_\pm$, acting on a Hilbert space of quantum states, do not depend on time. Denote by $[\cdot,\cdot]$ the ordinary commutator bracketing. Following the canonical quantization prescription, the quantum canonical coordinates satisfy the canonical commutation relations

$$[\hat{q}, \hat{q}] = 0 = [\hat{p}, \hat{p}], \quad [\hat{p}, \hat{q}] = \frac{\hbar}{i}$$

while the quantum quasi-canonical coordinates would satisfy (cf. (5.1)) the constraints

$$\hat{A}_+^2 + \hat{A}_-^2 = 2\sqrt{2\hat{H}}, \quad \hat{A}_+^2 - \hat{A}_-^2 = 2\hat{p}, \quad \hat{A}_+\hat{A}_- + \hat{A}_-\hat{A}_+ = 2\omega\hat{q} \tag{B.2}$$

and the *quasi-canonical commutation relations* (quasi-CCR) as follows:

$$[\hat{A}_+, \hat{A}_+] = 0 = [\hat{A}_-, \hat{A}_-], \quad [\hat{A}_+, \hat{A}_-] = \frac{\hbar}{i}\hat{\varepsilon} := \frac{\hbar}{i}\frac{\omega}{2\sqrt{2\hat{H}}} \tag{B.3}$$

Remark B.2. Recall that in the classical case constraint (5.2) follows from constraints (5.1), thus the system of these constraints is consistent. In the quantum case the consistency of (B.2) is not yet clear. In what follows, we assume all constraints (B.2) hold (see also Final Remark B.10).

B.2 Recapitulation

Theorem B.3. *Let constraints* (B.2) *hold. Then we have:*

$$\hat{J}^1_\hbar(x;y;z) = \frac{a(x,y,z)}{\sqrt{2p_0^3}}\left[\hat{A}_+\left(\sqrt{2E} - \sqrt{2\hat{H}}\right) - \frac{\hbar}{i}\hat{A}_-\frac{\hat{\varepsilon}}{2}\right]$$
$$\hat{J}^2_\hbar(x;y;z) = \frac{a(x,y,z)}{\sqrt{2p_0^3}}\left[\hat{A}_-\left(\sqrt{2E} - \sqrt{2\hat{H}}\right) + \frac{\hbar}{i}\hat{A}_+\frac{\hat{\varepsilon}}{2}\right]$$
$$\hat{J}^3_\hbar(x;y;z) = \frac{\hbar}{i}\frac{a^2(x,y,z)}{p_0}\hat{\varepsilon} \qquad \square$$

Proof. Using relations (B.2) and (B.3) first calculate:

$$\hat{\xi}_+ = \omega\hat{q}\hat{A}_- + (\hat{p} - p_0)\hat{A}_+$$
$$= \frac{1}{2}(\hat{A}_+\hat{A}_- + \hat{A}_-\hat{A}_+)\hat{A}_- + \frac{1}{2}(\hat{A}_+^2 - \hat{A}_-^2)\hat{A}_+ - p_0\hat{A}_+$$

$$
\begin{aligned}
&= \frac{1}{2}(\hat{A}_+\hat{A}_-^2 + \hat{A}_-\hat{A}_+\hat{A}_- + \hat{A}_+^3 - \hat{A}_-^2\hat{A}_+) - p_0\hat{A}_+ \\
&= \frac{1}{2}\left[\hat{A}_-(\hat{A}_+\hat{A}_- - \hat{A}_-\hat{A}_+) + \hat{A}_+(\hat{A}_-^2 + \hat{A}_+^2)\right] - p_0\hat{A}_+ \\
&= \frac{1}{2}\hat{A}_-[\hat{A}_+, \hat{A}_-] + \frac{1}{2}\hat{A}_+(\hat{A}_+^2 + \hat{A}_-^2) - p_0\hat{A}_+ \\
&= \frac{\hbar}{i}\hat{A}_-\frac{\hat{\varepsilon}}{2} + \hat{A}_+\sqrt{2\hat{H}} - \sqrt{2E}\hat{A}_+ \\
&= \frac{\hbar}{i}\hat{A}_-\frac{\hat{\varepsilon}}{2} + \hat{A}_+\left(\sqrt{2\hat{H}} - \sqrt{2E}\right)
\end{aligned}
$$

Next calculate

$$
\begin{aligned}
\hat{\xi}_- &= \omega\hat{q}\hat{A}_+ - (\hat{p} + p_0)\hat{A}_- \\
&= \frac{1}{2}(\hat{A}_+\hat{A}_- + \hat{A}_-\hat{A}_+)\hat{A}_+ - \frac{1}{2}(\hat{A}_+^2 - \hat{A}_-^2)\hat{A}_- - p_0\hat{A}_- \\
&= \frac{1}{2}(\hat{A}_+\hat{A}_-\hat{A}_+ + \hat{A}_-\hat{A}_+^2 - \hat{A}_+^2\hat{A}_- + \hat{A}_-^3) - p_0\hat{A}_- \\
&= \frac{1}{2}\left[\hat{A}_+(\hat{A}_-\hat{A}_+ - \hat{A}_+\hat{A}_-) + \hat{A}_-(\hat{A}_+^2 + \hat{A}_-^2)\right] - p_0\hat{A}_- \\
&= -\frac{1}{2}\hat{A}_+[\hat{A}_+, \hat{A}_-] + \frac{1}{2}\hat{A}_-(\hat{A}_+^2 + \hat{A}_-^2) - p_0\hat{A}_- \\
&= -\frac{\hbar}{i}\hat{A}_+\frac{\hat{\varepsilon}}{2} + \hat{A}_-\sqrt{2\hat{H}} - \sqrt{2E}\hat{A}_- \\
&= -\frac{\hbar}{i}\hat{A}_+\frac{\hat{\varepsilon}}{2} + \hat{A}_-\left(\sqrt{2\hat{H}} - \sqrt{2E}\right)
\end{aligned}
$$

□

Corollary B.4. *Using the energy conservation law $\hat{H} = E$ we obtain*

$$
\begin{aligned}
\hat{J}_\hbar^1(x;y;z) &= -\frac{\hbar}{i}\frac{a(x,y,z)}{\sqrt{(2p_0)^3}}\frac{\omega}{2\sqrt{2E}}\hat{A}_- \\
\hat{J}_\hbar^2(x;y;z) &= +\frac{\hbar}{i}\frac{a(x,y,z)}{\sqrt{(2p_0)^3}}\frac{\omega}{2\sqrt{2E}}\hat{A}_+ \\
\hat{J}_\hbar^3(x;y;z) &= \frac{\hbar}{i}\frac{a^2(x,y,z)}{p_0}\frac{\omega}{2\sqrt{2E}}
\end{aligned}
$$

□

B.3 Operadic quantum conditions

Theorem B.5. *The Jacobi operator coordinates $\hat{J}_\hbar^1, \hat{J}_\hbar^2, \hat{J}_\hbar^3$ of the algebras $VII_a^\hbar$, $III_{a=1}^\hbar$, $VI_{a\neq 1}^\hbar$ satisfy the commutation relations*

$$
[\hat{J}_\hbar^1, \hat{J}_\hbar^3] = 0 = [\hat{J}_\hbar^2, \hat{J}_\hbar^3], \quad [\hat{J}_\hbar^1, \hat{J}_\hbar^2] = C\hat{J}_\hbar^3 \tag{B.4}
$$

where

$$C := -\frac{(x,y,z)}{32}\left(\frac{\hbar\omega}{2E}\right)^2$$

Proof. Use Corollary B.4. □

Definition B.6 (derivative algebra)**.** The anti-commutative algebra given by structure relations (B.4) is called the *derivative algebra* of the algebras $\mathrm{III}^{\hbar}_{a=1}$, $\mathrm{VI}^{\hbar}_{a\neq 1}$, $\mathrm{VII}^{\hbar}_{a}$.

Corollary B.7. *Define the new generators in the derivative algebra:*

$$e_1 = -(x,y,z)\hat{J}^3_{\hbar}, \quad e_2 = -(x,y,z)\hat{J}^1_{\hbar}, \quad e_3 = -(x,y,z)\hat{J}^2_{\hbar}$$

Then

$$[e_1, e_2] = 0 = [e_1, e_3], \quad [e_2, e_3] = \beta^2 e_1$$

where

$$\beta := \frac{\hbar\omega}{2E}\frac{|(x,y,z)|}{4\sqrt{2}}$$

Proof. Calculate:

$$[e_2, e_3] = (x,y,z)(x,y,z)[\hat{J}^1_{\hbar}, \hat{J}^2_{\hbar}] = -C(x,y,z)(x,y,z)\hat{J}^3_{\hbar} = \beta^2 e_1$$ □

Conjecture B.8 (operadic quantum conditions over HO)**.** *The derivative algebra of the algebras $VII^{\hbar}_{a}$, $III^{\hbar}_{a=1}$, $VI^{\hbar}_{a\neq 1}$ is the 3-dimensional real Heisenberg algebra.*

Idea of proof. By elementary calculus one can see that the Jacobi operator of the derivative algebra vanishes. As the only non-vanishing structure constant is $\overset{\circ}{\mu}{}^{1}_{23}$, one can easily see from the Bianchi classification [15] (see Table 6.1) that $\beta = 1$ perfectly suits. □

Corollary B.9. *For the algebras $III^{\hbar}_{a=1}$, $VI^{\hbar}_{a\neq 1}$, $VII^{\hbar}_{a}$ we have*

$$|(x,y,z)| = 4\sqrt{2}(2n+1), \quad n = 0,1,2,\ldots \tag{B.5}$$

Proof. Fix the value of the free parameter E to be the energy eigenvalue of the quantum harmonic oscillator, i.e $E := \hbar\omega(n + 1/2)$ $(n = 0,1,2,\ldots)$. □

Corollary B.9 implies the hypothesis that the harmonic oscillator in the quantum Lie algebras $\mathrm{III}^{\hbar}_{a=1}$, $\mathrm{VI}^{\hbar}_{a\neq 1}$, $\mathrm{VII}^{\hbar}_{a}$ induces discrete spatial coordinates.

Final remark B.10. If system of constraints (B.2) turns out to be inconsistent, it is believed that there may exist some other quantum constraint, which together with two of other constraints (B.2) generates a consistent system. With help of the latter one can try to prove, that instead of (B.5) more sophisticated quantization of spatial coordinates will take place.

APPENDIX

C

On dynamical deformations of some 2d real algebras

In this appendix we briefly study dynamical deformations of 2-dimensional binary real associative unital algebras and Lie algebras. The quantum counterpart of the non-Abelian 2-dimensional real Lie algebra is constructed.

C.1 2-dimensional real associative unital algebras

According to the Malyshev classification [19] of the 2-dimensional algebras, there are only two non-isomorphic 2-dimensional associative unital algebras. Denote them by $F^{\infty}_{1,0,2}$ and ${}'F^{\infty}_{3,-2}$. Define

$$(\overset{\circ}{\mu}{}^1_{jk}) := \begin{pmatrix} \overset{\circ}{\mu}{}^1_{11} & \overset{\circ}{\mu}{}^1_{12} \\ \overset{\circ}{\mu}{}^1_{21} & \overset{\circ}{\mu}{}^1_{22} \end{pmatrix}, \quad (\overset{\circ}{\mu}{}^2_{jk}) := \begin{pmatrix} \overset{\circ}{\mu}{}^2_{11} & \overset{\circ}{\mu}{}^2_{12} \\ \overset{\circ}{\mu}{}^2_{21} & \overset{\circ}{\mu}{}^2_{22} \end{pmatrix}, \quad j,k = 1,2$$

The structure constants of these algebras are introduced in Table C.1.

Algebra	$\overset{\circ}{\mu}{}^1_{jk}$	$\overset{\circ}{\mu}{}^2_{jk}$
$F^{\infty}_{1,0,2}$	$\begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$
${}'F^{\infty}_{3,-2}$	$\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$

Table C.1: Structure constants of $F^{\infty}_{1,0,2}$ and ${}'F^{\infty}_{3,-2}$

Define a parameter

$$\tau := \frac{1}{2\sqrt{2p_0}}$$

Using the Lax representations of the 2-dimensional algebras given in Section 5.2 and following the procedure of dynamical deformation with initial conditions described in Section 6.1, we can find the parameters C_ν ($\nu = 1, 2, \ldots, 8$) and present these in Table C.2.

Algebra	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
$F^{\infty}_{1,0,2}$	0	-5τ	0	3τ	0	-7τ	0	$8\tau^3$
$'F^{\infty}_{3,-2}$	$-9\tau/2$	0	$3\tau/2$	0	$15\tau/2$	0	$12\tau^3$	0

Table C.2: Values of the parameters C_ν for $F^{\infty}_{1,0,2}$ and $'F^{\infty}_{3,-2}$

Now use Theorem 5.7 to get the dynamical deformations of the multiplication μ of these two algebras (see Table C.3).

μ^i_{jk}	$F^{\infty}_{1,0,2,t}$	$'F^{\infty}_{3,-2,t}$
μ^1_{11}	$-7\tau A_+ + 8\tau^3 D_+$	$(15\tau/2)A_- + 12\tau^3 D_-$
μ^1_{12}	$-5\tau A_- + 8\tau^3 D_-$	$(-9\tau/2)A_+ - 12\tau^3 D_+$
μ^1_{21}	$-5\tau A_- + 8\tau^3 D_-$	$(-9\tau/2)A_+ - 12\tau^3 D_+$
μ^1_{22}	$3\tau A_+ - 8\tau^3 D_+$	$(-3\tau/2)A_- - 12\tau^3 D_-$
μ^2_{11}	$3\tau A_- + 8\tau^3 D_-$	$(3\tau/2)A_+ - 12\tau^3 D_+$
μ^2_{12}	$-5\tau A_+ - 8\tau^3 D_+$	$(9\tau/2)A_- - 12\tau^3 D_-$
μ^2_{21}	$-5\tau A_+ - 8\tau^3 D_+$	$(9\tau/2)A_- - 12\tau^3 D_-$
μ^2_{22}	$-7\tau A_- - 8\tau^3 D_-$	$(-15\tau/2)A_+ + 12\tau^3 D_+$

Table C.3: Dynamical deformations of $F^{\infty}_{1,0,2}$ and $'F^{\infty}_{3,-2}$

For every two algebra elements x, y define their product xy by

$$(xy)^i := \mu^i_{jk} x^j y^k$$

Associator is defined by

$$\begin{aligned} A(x;y;z) &:= x \cdot yz - xy \cdot z \\ &= A^1(x;y;z)e_1 + A^2(x;y;z)e_2 \end{aligned} \tag{C.1}$$

Define also five auxiliary functions $\theta_i(p_0, A_\pm)$ ($i = 1, \ldots, 5$):

$$\theta_1 := \quad 16(A_+D_-p_0 - A_-D_+p_0 - 2A_+A_-p_0^2)$$

$$\begin{aligned}
\theta_2 &:= 4(D_+^2 + D_-^2 + 3A_+^2 p_0^2 - 5A_-^2 p_0^2 - 4A_- D_- p_0 - 4A_+ D_+ p_0)\\
\theta_3 &:= -4(D_+^2 + D_-^2 - 5A_+^2 p_0^2 + 3A_-^2 p_0^2 + 4A_- D_- p_0 + 4A_+ D_+ p_0)\\
\theta_4 &:= -4(D_+^2 + D_-^2 - 3A_+^2 p_0^2 + A_-^2 p_0^2 + 2A_- D_- p_0 + 2A_+ D_+ p_0)\\
\theta_5 &:= 4(D_+^2 + D_-^2 + A_+^2 p_0^2 - 3A_-^2 p_0^2 - 2A_- D_- p_0 - 2A_+ D_+ p_0)
\end{aligned}$$

Lemma C.1. *One can express the functions θ_i $(i = 1, \ldots, 5)$ only in terms of quasi-canonical coordinates $A_\pm$ of the harmonic oscillator as follows:*

$$\begin{aligned}
&\theta_1 = 16A_+^3 A_- p_0 + 16A_+ A_-^3 p_0 - 32A_+ A_- p_0^2\\
&\theta_2 = A_-^6 + A_+^6 - 8A_+^4 p_0 + 8A_-^4 p_0 + 3A_-^2 A_+^4 + 3A_-^4 A_+^2 + 12A_+^2 p_0^2 - 20A_-^2 p_0^2\\
&\theta_3 = -A_-^6 - A_+^6 - 8A_+^4 p_0 + 8A_-^4 p_0 - 3A_-^2 A_+^4 - 3A_-^4 A_+^2 + 20A_+^2 p_0^2 - 12A_-^2 p_0^2\\
&\theta_4 = -A_-^6 - A_+^6 - 4A_+^4 p_0 + 4A_-^4 p_0 - 3A_-^2 A_+^4 - 3A_-^4 A_+^2 + 12A_+^2 p_0^2 - 4A_-^2 p_0^2\\
&\theta_5 = A_-^6 + A_+^6 - 4A_+^4 p_0 + 4A_-^4 p_0 + 3A_-^2 A_+^4 + 3A_-^4 A_+^2 + 4A_+^2 p_0^2 - 12A_-^2 p_0^2
\end{aligned}$$

Proof. Use definition of the auxiliary functions $D_\pm$ (see page 59). □

Lemma C.2. *For the algebra $F^{\infty}_{1,0,2,t}$ one has*

$$\begin{aligned}
A^1(x;y;z) &= \frac{1}{16p_0^3}\left(\theta_1(x_2y_1z_1 - x_1y_1z_2) + \theta_2(x_2y_2z_1 - x_1y_2z_2)\right)\\
A^2(x;y;z) &= -\frac{1}{16p_0^3}\left(\theta_1(x_2y_2z_1 - x_1y_2z_2) + \theta_3(x_1y_1z_2 - x_2y_1z_1)\right)
\end{aligned}$$

and for the algebra $'F^{\infty}_{3,-2,t}$ one has

$$\begin{aligned}
A^1(x;y;z) &= -\frac{9}{64p_0^3}\left(\frac{\theta_1}{2}(x_2y_1z_1 - x_1y_1z_2) + \theta_4(x_2y_2z_1 - x_1y_2z_2)\right)\\
A^2(x;y;z) &= \frac{9}{64p_0^3}\left(\frac{\theta_1}{2}(x_2y_2z_1 - x_1y_2z_2) + \theta_5(x_1y_1z_2 - x_2y_1z_1)\right)
\end{aligned}$$

Proof. Direct calculations using defining formula (C.1) in the form

$$A^i(x;y;z) = \mu^i_{jk}\mu^k_{lm}x^j y^l z^m - \mu^i_{jk}\mu^j_{lm}x^l y^m z^k, \quad i = 1, 2$$

and Lemma C.1. □

Lemma C.3. *One has $\theta_i = 0$ $(i = 1, \ldots, 5)$.*

Proof. We use definition of quasi-canonical coordinates (5.1) of the harmonic oscillator and the energy conservation law (6.3). First, calculate

$$p_0 - p - A_-^2 = \frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 - \frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 - A_-^2 = 0$$

and

$$p_0 + p - A_+^2 = \frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 + \frac{1}{2}A_+^2 - \frac{1}{2}A_-^2 - A_+^2 = 0$$

Thus,

$$p_0 \pm p - A_\pm^2 = 0$$

Now, consider the function θ_1:

$$\begin{aligned}\theta_1 &= 16A_+^3A_-p_0 + 16A_+A_-^3p_0 - 32A_+A_-p_0^2\\ &= 16A_+A_-p_0\left(A_+^2 + A_-^2 - 2p_0\right) =\\ &= 16A_+A_-p_0\left(2p_0 - 2p_0\right)\\ &= 0\end{aligned}$$

Next, calculate θ_2:

$$\begin{aligned}\theta_2 &= A_-^6 + A_+^6 - 20A_-^2p_0^2 + 8A_-^4p_0 + 3A_-^2A_+^4 + 3A_-^4A_+^2 + 12A_+^2p_0^2 - 8A_+^4p_0\\ &= \left(A_+^2 + A_-^2\right)^3 + 4p_0^2\left(3A_+^2 - 5A_-^2\right) + 8p_0\left(A_-^4 - A_+^4\right)\\ &= (2p_0)^3 + 4p_0^2\left(3\left(A_+^2 - A_-^2\right) - 2A_-^2\right) + 8p_0\left(A_-^2 - A_+^2\right)\left(A_-^2 + A_+^2\right)\\ &= 8p_0^3 + 4p_0^2\left(6p - 2A_-^2\right) + 8p_0(-2p)(2p_0)\\ &= 8p_0^2\left(p_0 - p - A_-^2\right)\\ &= 8p_0^2 \cdot 0\\ &= 0\end{aligned}$$

Now we find θ_3:

$$\begin{aligned}\theta_3 &= -A_-^6 - A_+^6 + 20A_+^2p_0^2 + 8A_-^4p_0 - 3A_-^2A_+^4 - 3A_-^4A_+^2 - 12A_-^2p_0^2 - 8A_+^4p_0\\ &= -\left(A_+^2 + A_-^2\right)^3 - 4p_0^2\left(3A_-^2 - 5A_+^2\right) + 8p_0\left(A_-^4 - A_+^4\right)\\ &= -(2p_0)^3 - 4p_0^2\left(3\left(A_-^2 - A_+^2\right) - 2A_+^2\right) + 8p_0\left(A_-^2 - A_+^2\right)\left(A_-^2 + A_+^2\right)\\ &= -8p_0^3 - 4p_0^2\left(-6p - 2A_+^2\right) + 8p_0(-2p)(2p_0)\\ &= -8p_0^2\left(p_0 + p - A_+^2\right)\\ &= -8p_0^2 \cdot 0\\ &= 0\end{aligned}$$

Analogously calculate θ_4 and θ_5:

$$\begin{aligned}-\theta_4 &= A_-^6 + A_+^6 + 4A_-^2p_0^2 - 4A_-^4p_0 + 3A_-^2A_+^4 + 3A_-^4A_+^2 - 12A_+^2p_0^2 + 4A_+^4p_0\\ &= \theta_2 + 12p_0\left(A_+^4 - A_-^4 + 2p_0\left(A_-^2 - A_+^2\right)\right)\\ &:= 0 + \theta_4'\\ &= 12p_0\left(\left(A_+^2 + A_-^2\right)\left(A_+^2 - A_-^2\right) - 2p_0\left(A_+^2 - A_-^2\right)\right)\end{aligned}$$

$$= 12p_0 \left((2p_0)(2p) - 2p_0(2p)\right)$$
$$= 12p_0 \cdot 0$$
$$= 0$$

Finally,

$$\begin{aligned}
\theta_5 &= A_-^6 + A_+^6 + 4A_+^2 p_0^2 - 4A_+^4 p_0 + 3A_-^2 A_+^4 + 3A_-^4 A_+^2 - 12A_-^2 p_0^2 + 4A_-^4 p_0 \\
&= -\theta_3 + 12p_0 \left(A_-^4 - A_+^4 + 2p_0 \left(A_+^2 - A_-^2\right)\right) \\
&= 0 - 12p_0 \left(A_+^4 - A_-^4 + 2p_0 \left(A_-^2 - A_+^2\right)\right) \\
&= -\theta_4' \\
&= 0
\end{aligned}$$

□

Theorem C.4. *The dynamical deformations of $F^\infty_{1,0,2}$ and ${}'F^\infty_{3,-2}$ are associative algebras as well.*

Proof. Lemmas C.2 and C.3. □

C.2 2-dimensional real Lie algebras

According to the Malyshev classification [19] of the 2-dimensional algebras, there are only two non-isomorphic 2-dimensional Lie algebras, denoted by F^0 and F. The structure constants of the algebra F are identically zero, and this algebra is evidently dynamically rigid. So we consider only the algebra F^0 (also see Section 5.1 for details on this algebra) with the structure constants

$$(\mathring{\mu}^1_{jk}) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\mathring{\mu}^2_{jk}) := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad j,k = 1,2$$

Using the Lax representations of the 2-dimensional algebras given in Section 5.2 and the procedure of dynamical deformation with initial conditions described in Section 6.1, we can find

$$C_\nu = \begin{cases} \frac{1}{2p_0} & \text{if } \nu = 1, \\ 0 & \text{if } \nu \in \{2,3,\dots,8\}, \end{cases}$$

that implies the dynamical deformation F^0_t of the algebra F^0:

$$(\mu^1_{jk}) = \frac{A_+}{\sqrt{2p_0}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\mu^2_{jk}) = \frac{A_-}{\sqrt{2p_0}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j,k = 1,2$$

Now introduce the quantum counterpart $F^0_\hbar$ of the algebra F^0_t (see also Section 6.5 for details):

$$(\hat{\mu}^1_{jk}) = \frac{\hat{A}_+}{\sqrt{2p_0}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\hat{\mu}^2_{jk}) = \frac{\hat{A}_-}{\sqrt{2p_0}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j,k = 1,2$$

Theorem C.5. *The quantum counterpart $F^0_\hbar$ of the 2-dimensional real Lie algebra F^0 is a Lie algebra.*

Proof. For x, y, z in $F^0_\hbar$, define two functions

$$\zeta_1 := \frac{1}{\sqrt{2p_0}}\left(x^2y^2z^1 - x^2y^1z^2 + x^1y^2z^2 - x^2y^2z^1 + x^2y^1z^2 - x^1y^2z^2\right)$$
$$\zeta_2 := \frac{1}{\sqrt{2p_0}}\left(x^1y^1z^2 - x^1y^2z^1 + x^2y^1z^1 - x^1y^1z^2 + x^1y^2z^1 - x^2y^1z^1\right)$$

that turn out to be identically zero. By definition (6.4) of the Jacobi operator $\hat{J}_\hbar(x; y; z)$ and direct calculations one gets

$$\begin{aligned}\hat{J}^1_\hbar(x; y; z) &= \zeta_1\hat{A}^2_+ + \zeta_2\hat{A}_+\hat{A}_- \\ &= 0 \cdot \hat{A}^2_+ + 0 \cdot \hat{A}_+\hat{A}_- \\ &= 0, \\ \hat{J}^2_\hbar(x; y; z) &= \zeta_2\hat{A}^2_- + \zeta_1\hat{A}_-\hat{A}_+ \\ &= 0 \cdot \hat{A}^2_- + 0 \cdot \hat{A}_-\hat{A}_+ \\ &= 0\end{aligned}$$

□

APPENDIX

D

Curriculum vitae

1. Personal data

   Name: Jüri Virkepu

   Date and place of birth: August 4, 1982; Tallinn (Estonia)

2. Contact information

   Address: Sõpruse pst 242-30, Tallinn 13412, Estonia

   Phone: (+372) 56304794

   E-mail: jvirkepu@staff.ttu.ee

3. Education

Education institution	Graduation year	Education
Tallinn University of Technology	2005	Engineering physics, MSc in natural sciences
Tallinn University of Technology	2003	Engineering physics, BSc in natural sciences
Mustamäe Humanities Gymnasium	2000	Secondary education

4. Language competence

Language	Level
Estonian	Fluent
Russian	Fluent
English	Average
German	Basic

5. Special courses
   -

6. Professional employment

Period	Organisation	Position
2004 – 2008	Tallinn University of Technology	Teaching instructor

7. Scientific work
   A member of the organising committee of the international conference *Baltic-Nordic Workshop on Algebra, Geometry and Mathematical Physics*, Tallinn, October 8, 2005.

8. Defended theses
   Master thesis *Lie theory and its applications.*
   Bachelor thesis *Singular functions of the field theory.*

9. Current research topics
   Lie theory, operads and their applications.

10. Other research projects
    2009 – ... *Modern applied methods of algebra and analysis in the theory of differential and integral equations, mathematical physics and statistics.* Funded by the Ministery of Education and Research.
    2007 – 2010 ETF-6912 *Generalized Lie theory.* Estonian Science Foundation Grant.
    2003 – 2007 *Algebraic structures and applications of mathematical analysis.* Funded by the Ministery of Education and Research.
    2003 – 2006 ETF-5634 *Operads and cohomology.* Estonian Science Foundation Grant.

APPENDIX

$\mathcal{E}$

Elulookirjeldus

1. Isikuandmed

   Ees- ja perekonnanimi: Jüri Virkepu

   Sünniaeg ja -koht: 04.08.1982, Tallinn

   Kodakondsus: Eesti

2. Kontaktandmed

   Aadress: Sõpruse pst 242-30, Tallinn 13412

   Telefon: (+372) 56304794

   E-posti aadress: jvirkepu@staff.ttu.ee

3. Hariduskäik

Õppeasutus	Lõpetamise aeg	Haridus
Tallinna Tehnikaülikool	2005	Tehniline füüsika, loodusteaduste magister
Tallinna Tehnikaülikool	2003	Tehniline füüsika, loodusteaduste bakalaureus
Mustamäe Humanitaargümnaasium	2000	Keskharidus

4. Keelteoskus

Keel	Tase
Eesti	Kõrgtase
Vene	Kõrgtase
Inglise	Kesktase
Saksa	Algtase

5. Täiendusõpe
   -

6. Teenistuskäik

Töötamise aeg	Tööandja nimetus	Ametikoht
2004 – 2008	Tallinna Tehnikaülikool	Õppejõud

7. Teadustegevus
   Rahvusvahelise konverentsi *Baltic-Nordic Workshop on Algebra, Geometry and Mathematical Physics*, 8. oktoober, 2005, Tallinn, korralduskomitee liige.

8. Kaitstud lõputööd
   Magistritöö teemal *Lie teooria ja selle rakendused.*
   Bakalaureusetöö teemal *Väljateooria singulaarsed funktsioonid.*

9. Teadustöö põhisuunad
   Lie teooria, operaadid ja nende rakendused.

10. Teised uurimisprojektid
    2009 – ... SF teema *Algebra ja analüüsi kaasaegsed rakendusmeetodid diferentsiaal- ja integraalvõrrandite teoorias, matemaatilises füüsikas ja statistikas* täitja.
    2007 – 2010 ETF grandi nr. 6912 *Üldistatud Lie teooria* põhitäitja.
    2003 – 2007 SF teema *Algebralised struktuurid ja matemaatilise analüüsi rakendused* täitja.
    2003 – 2006 ETF grandi nr. 5634 *Operaadid ja kohomoloogia* põhitäitja.

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