DOCTORAL THESIS

Peirce's Existential Graphs and the Logic of String Diagrams

Nathan Joseph Haydon

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

Nathan Joseph Haydon ______________________
signature

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Peirce'i eksistentsiaalsed graafid ja nöördiagrammide loogika

NATHAN JOSEPH HAYDON

KIRJASTUS

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List of Publications

The present Ph.D. thesis is based on the following publications that are referred to in the text by Roman numbers.

I N. Haydon and P. Sobociński. Compositional diagrammatic first-order logic. In A.-V. Pietarinen, P. Chapman, L. Bosveld-de Smet, V. Giardino, J. Corter, and S. Linker, editors, *Diagrammatic Representation and Inference*, pages 402–418, Cham, 2020. Springer International Publishing

II N. Haydon and A.-V. Pietarinen. Residuation in existential graphs. In A. Basu, G. Stapleton, S. Linker, C. Legg, E. Manalo, and P. Viana, editors, *Diagrammatic Representation and Inference*, pages 229–237, Cham, 2021. Springer International Publishing

III N. Haydon. C.S. Peirce's early developments in linear logic. 2023

IV F. Bonchi, A. D. Giorgio, N. Haydon, and P. Sobocinski. Diagrammatic algebra of first order logic. To appear at LICS, 2024

V A.-V. Pietarinen, F. Bellucci, A. Bobrova, N. Haydon, and M. Shafiei. The blot. In A.-V. Pietarinen, P. Chapman, L. Bosveld-de Smet, V. Giardino, J. Corter, and S. Linker, editors, *Diagrammatic Representation and Inference*, pages 225–238, Cham, 2020. Springer International Publishing

Author's Contributions to the Publications

I In Article I, the author shared in writing the manuscript with the coauthor, shared in the main contribution of presenting a new set of string diagrammatic inference rules for Peirce's 'cut', and contributed the historical interludes in the text.

II In Article II, the author identified the significance of the topic, developed the main contribution of presenting residuation in Peirce's Beta graphs, and wrote the main sections of the manuscript.

III In Article III, the author was solely responsible for the manuscript, including identifying the proposed perspective in Peirce's work, the significance of the result and historical contributions, and the connection to bilinear and cyclic logic.

IV In Article IV, the theory presented was identified and articulated by the author over the previous three years. Coauthors contributed to the categorical presentation and contributed substantially to the completeness result.

V In Article V, the author contributed a perspective on the interpretation of Peirce's 'blot'. Pietarinen then provided context and summary for each coauthor's contribution.

Abstract

Peirce's Existential Graphs and the Logic of String Diagrams

String diagrams are a viable alternative to more traditional algebraic syntax, often yielding an elegant presentation of the relational features under consideration and one that allows for the treatment of variables and algebraic operations in a compositional manner. Following the pioneering work of Charles S. Peirce, who developed a graphical logic of relations over 100 years ago in his *Existential Graphs*, we treat here and extend the logical aspects of string diagrams. The key developments follow from a renewed emphasis on Peirce's *scroll* — a sign of two nested circles serving at once as an inclusion and an involution — that allows us to capture various logical connectives and other operations. The result is a contemporary graphical relational calculus sufficient to serve as a foundation for large portions of mathematics and for applications to logic and fields like knowledge representation.

Kokkuvõte
Peirce'i eksistentsiaalsed graafid ja nöördiagrammide loogika

Nööridiagrammid on asjalik alternatiiv traditsioonilisemale algebralisele süntaksile. Sageli võimaldavad nad uuritavate relatsiooniliste atribuutide elegantset esitust, kus muutujad ja operatsioonid on käsitletud kompositsiooniliselt. Järgides Charles S. Peirce'i, kes üle saja aasta tagasi oma teedrajavas töös "Eksistentsiaalsed graafid" arendas välja graafilise relatsioonide loogika, käsitleme ja arendame selles töös nööridiagrammide loogikalisi aspekte. Põhilised edasiarendused tulenevad uuendatud rõhuasetusest Peirce'i "rullraamatule", märgile, mis koosneb kahest teineteise sees asetsevast ringist, tähistades ühekorraga sisalduvust ja involutsiooni. "Rullraamat" võimaldab esitada erinevaid loogilisi tehteid ja teisi operatsioone. Tulemuseks on kaasaegne graafiline relatsiooniarvutus, mis on piisav toimimaks matemaatika suurte osade alusena ning rakendusteks loogikas ja valdkondades nagu teadmiste esitamine.

Acknowledgements

This PhD was a rare opportunity. Credit to Pawel, my supervisor, for his initial belief that Peirce's graphs had something to offer contemporary categorical logic. Thanks also to my co-supervisor, Ahti, for his willingness to always chat about Peirce and his philosophical views. It was a rare opportunity indeed to work with two outstanding researchers at the same time. Tallinn is a special place.

To other members of our group — Chad, Diana, Elena, Mario, and Matt — the time spent during retreats, saunas, travel to conferences, and in the coffee room has led to many fond memories. Special thanks to Chad and Matt for hosting dinner parties when I was in town, and Ed who always found time to chat about logic over coffee or dinner. As a researcher, the kindness and patience shown by all of you is not lost on me.

Special thanks also to Nicollò, who spent time reading my work and chatting about related papers and proof theory. Nicollò added general encouragement and motivation that helped the completion of this thesis. Further thanks to Niels for the good company during said dinner parties and outings.

To other members of the department — Amar, Andrea, Cheng-Syuan, Clémence, Ekaterina, Fosco, Michele, Philipp, and Tarmo — though we had fewer conversations in number I remember and appreciate time spent with you all. To everyone, I wish you the best of luck in the future.

And finally, if my family reads this in the future — heya! — you are the best and I love you all.

Introduction

The logic of relations is applicable to a wide range of fields and areas of inquiry. One of the key contributors to the early study of relations, Charles S. Peirce, thought the study of the logic of relations was akin to studying the essence of scientific reasoning. Extensive appeal to relations is also found throughout much of mathematics and the study of natural language. Relational thinking is so pervasive that it is perhaps more accurate to say that very little falls *outside* its scope. This thesis develops and extends a contemporary graphical language for reasoning about relations.

The study of relations goes back to the mid-19th and early 20th centuries with Boole and others in the early algebraic logic tradition such as De Morgan, Peirce, and Schröder. Since that time the explicit study of relations witnessed a resurgence following the work of Tarski and the study of *relation algebras* in the 1940s [70]. Relations have since served as a key setting for algebraic and logical study in the foundation of mathematics [71, 62, 66], in computer science [26, 22, 64, 18], and for advances in concept analysis and cognitive science [42, 73, 69]. Work extending the logic and algebra of relations in this thesis is directly applicable to its use in these areas.

There is furthermore a long history of trying to develop methods of diagrammatic reasoning — imagine Venn diagrams or Feynman diagrams — with the aim of helping us better represent concepts on the page. The broad idea is that better diagrams can help us hone the most important aspects of our conceptions, avoid other extraneous features or calculations, and all the while make the system more intuitive and easier to learn. While the foundational nature of the logic of relations gives rise to an extensive list of applications, this thesis is first and foremost about creating and developing a *diagrammatic syntax*. Along these lines this thesis develops a novel diagrammatic calculus that helps us intuitively and more effectively reason about the logic of relations.

The graphical calculus presented in this thesis is inspired by two directions — one aspect is very recent, employing string diagrams in category theory [57, 8, 9, 7], and another very old, going back to Charles S. Peirce's neglected graphical calculus called the *Existential Graphs*. These two directions combine for a contemporary logic of relations sufficient for the study of algebraic and mathematical theories.

String diagrams — see [67, 1] for background — have increasingly been used to reason about a wide range of theoretical contexts, including electrical circuits [5], control theory [2], concurrency and Petri nets [3], probability theory [37], linear algebra [53], natural language [28], neural networks [32], and concept analysis and cognition [6]. These are instances of *applied category theory (ACT)* and there is good reason to think string diagrams will play important roles in the future of these fields [34].

While the rigorous formalization of string diagrams is a fairly recent development, arising in the late 20th century, the study of relations via string diagrams in a broader logical and algebraic context goes back to the pioneering work of Charles S. Peirce and the development of his *Existential Graphs* (EGs) a century prior.

Peirce's Existential Graphs are noteworthy in several respects:

variables as strings: Peirce understood that variables could be replaced by 'wires' or 'strings' — what he called 'lines of identity' — serving as continuous, bifurcating representations of the identity relation.

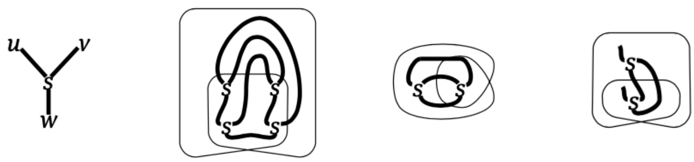
primacy of conjunction: Peirce chose conjunction to be the default connective imbued by the 'sheet of assertion', which is Peirce's term for the page upon which diagrammatic reasoning takes place and graphs are scribed. The 'sheet' possesses the properties of a product with corresponding projections (which Peirce calls 'erasures'), where the unit of conjunction (i.e. ‘True’) is absorbed into the ‘blank’ on the page.

‘cut’: Additional logical connectives are captured by adding one further symbol — a simple circle or ‘cut’, as Peirce calls it — to the string diagrammatic syntax. The ‘cut’ adds a remarkable amount of further logical expressivity. It allows one to capture negation and, when multiple cuts are varyingly nested together, can express relational inclusion and disjunction.

These features give rise to the common presentation of EGs as first-order logic with identity as found in Roberts [61] and Zeman [74] and much of the work on the graphs that followed [68, 29] and [55, see introduction].

Aside from the ‘cut’ symbol, which we return to below, these features are shared by contemporary string diagrams and Peirce should be seen as one of the earliest — if not the first — to present string diagrams as we more or less know them today.

As an example, we show in Figure 1 a series of graphs given by Peirce in 1903. The graphs are reproduced from the recent collection on Peirce’s graphs in [56, p. 156] and are notable for their remarkable similarity to contemporary string diagrams. Peirce uses



(a) *(b)* *(c)* *(d)*

Figure 1 – An early instance of strings diagrams in Peirce’s work that express ‘sum’ operation, associativity and commutativity of ‘sum’, and the existence of negative quantities.

the ‘s’ in the node to represent ‘sum’ or ‘addition’. In Peirce’s own words the meaning of Figure 1(a) expresses: “that w is equal to a result of adding something equal to u to something equal to v”. With the ability to express inequations with nested ‘cuts’, Peirce goes on to express further properties related to addition. In Peirce’s own words Figure 1(c) expresses the “commutativeness of addition,” Figure 1(d) expresses “the existence of negative quantities,” and Figure 1(b) expresses the “associativity of addition”.

One should compare Peirce’s versions of these properties in EGs with the contemporary presentation using string diagrams in Figure 2. We note immediately that the dia-

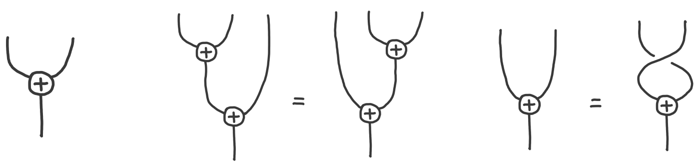


Figure 2 – String diagrams for the ‘addition’ operation, along with the associativity and commutativity of addition.

grams match iconically — i.e. by shape or outline — the contemporary presentation in string diagrams.

We add that the surface of the sheet provides additional **topological freedom** in expressing the same equations. The freedom of the wires allows one to express the 'type'-requirement requisite of reading contemporary equations in string diagrams. The 'minus' operation — often expressed with an additional labeled node, as in Figure 3 — can be expressed in the inclusion with a sort of 'feedback' of the variable wire. Finally, commutativity can be represented — if one wants — with no 'twisting' or 'braiding' of the wires. We find this additional topological freedom and the emphasis on inclusion to be unique to Peirce's graphs.

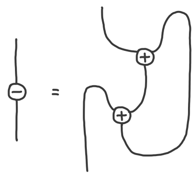


Figure 3 – String diagram for 'minus' operation.

This brings the earliest use of string diagrams to as far back as the 1880s when Peirce began developing the Existential Graphs. Peirce knew the Existential Graphs were of logical and algebraic importance — boldly, he described the graphs as "the logic of the future"— and the main question that occupied the start of this thesis is whether and to what extent contemporary mathematics and the expression of various logics in string diagrams can be aided by Peirce's insights into graphical calculi.

We state upfront our belief — and the thesis goes on to substantially demonstrate the point — that the relatively unknown algebraic studies and graphical syntax found in Peirce's Existential Graphs is a valuable resource for developing and extending a contemporary logic of string diagrams.

Motivation

The connection between Peirce's Existential Graphs and string diagrams has two natural starting points. The first is to connect Peirce's treatment of variables using 'lines of identity' with string diagrams. The second is to add Peirce's 'cut' symbol representing complement or negation — and which, in some sense, is the only additional symbol in the graphs — to contemporary string diagrammatic presentations. The simplified 'cut-as-negation' story has been the predominant story in Peirce's work on EGs and subsequent studies of the graphs. It follows the earliest work of Roberts and Zeman cited above and seemingly Peirce himself, who in his public presentations of the graphs often writes of the importance of the single 'cut'. This is the direction taken in **Article I** ("Compositional Diagrammatic First-Order Logic") and is a key first step in situating Peirce's graphs within contemporary graphical and algebraic terms.

This 'cut-as-negation' story is, however, neither logically (from the perspective of contemporary work) nor historically (from the perspective of the rest of Peirce's philosophical and logical work) the end of the story. Peirce took the 'double cut' — picture two nested 'cuts' as in Figure 4— to be the more primitive logical operation. In fact if one only uses the inference rules Peirce gave for the graphs, then one can seemingly never draw a single cut! On this account the single 'cut-as-negation' story developed in **Article I** is merely an approximation and is a first step towards a more general presentation.

Peirce referred to this 'double cut' as the *scroll* and a main contribution of this thesis is developing the 'scroll' as a key element of the syntax and as a key inferential connective.

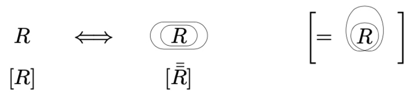


Figure 4 – Peirce's 'double cut' rule (left) and 'scroll' (right)

As an example of the logical connectives in the Beta variant of Peirce's EGs, we show in Figure 5 the Boolean operations and relational inclusion. Of particular interest to us here,

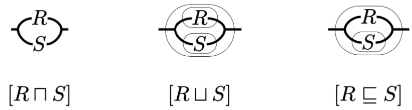


Figure 5 – The Boolean operations of intersection and union (left), along with relational inclusion (right)

is to note how the inclusion shares the same shape as the involution and 'double cut' in Figure 4. Indeed Peirce's 'scroll' serves at one and the same time as an involution and an inclusion.

We can summarize this turn to the 'scroll' as moving in the following directions: (i) as emphasizing the importance of relational inclusion and *inequational* over equational reasoning, (ii) as emphasizing the further function of the 'scroll' as an involution, along with additional connectives that follow from this, and (iii) as stressing additional topological freedoms that follow from the larger surface that is the 'sheet'. In this thesis we connect these features of the 'scroll' to presentations of linear logic and the involution to that of *-autonomous categories. This thesis can be seen as restoring the 'scroll' to its place of primary importance both diagrammatically and as a logical connective.

There is one further motivations that is worth mentioning upfront. Peirce insisted on the importance of triadic relations over mere binary or dyadic relations in syntax and with respect to presenting inference rules. Commenting on the need to go beyond the presentation of binary relations in his 'Note B', Peirce writes:

> The criticism which I make on [my] algebra of dyadic relations, with which I am by no means in love, though I think it is a pretty thing, is that the very triadic relations which it does not recognize, it does itself employ. [CP 8:331]

The emphasis on triadic relations follows from Peirce's insistence on *teridentity* — Peirce's term for forming a branch on the identity relation — and what has been called Peirce's reduction thesis, whereby higher n-ary relations can be reduced to a combination of 1-, 2-, and 3-ary relations [20]. While binary relations have more-or-less persisted as the traditional form of presentation, we take the work here (and contemporary string diagrams more generally) to affirm Peirce's insistence on teridentiy and triadic relations. This is implicitly seen in **Article I** in this thesis, where the syntax and rules employ branching (i.e. triadic) terms, but is also more explicitly employed to yield the key result in **Article IV** ("Diagrammatic Algebra of First Order Logic"), where triadic relations play a key role in generalizing Tarksi's relation algebra to full first-order logic.

What this means for Peirce scholarship is that his assumptions about variables being captured by wires or strings, his emphasis on conjunction as absorbed into the syntax, his turn towards triadic relations and teridentity, his simplified inference rules and emphasis on (relational) composition, and the involution that is the ‘scroll’, are all shared in the motivations found in contemporary categorical logic and the use of string diagrams.

The aim of the thesis is to develop these motivations, to begin to formalize these notions where possible, and — better yet — to extend these features to contemporary applications of string diagrams. Similar to the importance one might place on the (positive) implicational fragment, our take in the end is that Peirce spent a great deal of time in his later studies focusing on and developing the *linear* implicational fragment. In the end, we argue that Peirce understood the relational setting corresponding to the Lambek calculus and the operations from what is called *bilinear logic* [46].

Background and Related Work

We mention two approaches to the logic of relations that followed the century after Peirce’s work. *Relation algebra* has been studied and developed in the context of more classical set-theoretic mathematics. This goes back to Tarski’s seminal [70] — a paper that we like to stress is actually a return to Peirce — and culminates with Tarski and Givant’s [71] in the 1970s (see also [50]). We will not attempt a summary of the last 80+ years of work on relation algebra, and will simply note that relation algebra persists — either explicitly or implicitly — as a predominate field of study in computer science, in the foundations of mathematics, and in logic and cognitive science. A more contemporary direction along these lines is worth particular mention, and that is the work following Schmidt’s ‘Relational Mathematics’ [62]. This includes work in relational mathematics by Winter, Kahl, Berghammer, and others [65, 66]. We come back to this direction again below.

The advent of category theory brought another notable direction in the logic of relations. Two perspectives on the (categorical) study of relations are the relationally-inspired presentation of *allegories* [36] and Carboni and Walters presentation of *cartesian bicategories of relations* (CBRs) [21]. The theory presented here is often directly inspired by the latter. Significant to us here, we note the approach taken in CBRs is (i) closely related to the rules Peirce himself used to describe relations (see **Article I**), (ii) motivates the work on essentially algebraic theories that serve as a further backdrop for the view we employ here [15, 11, 30], and (iii) also serves as the backdrop for further, related approaches in applied category theory [48, 27, 44, 13, 14]. Finally, the CBR axioms are also highly amenable to a string-diagrammatic presentation [12, 35], which is often employed in the applications listed above and will (again) be discussed substantially in the thesis to come.

Freyd and Scedrov’s *allegories* [36] are a similar categorical axiomization of the logic of relations. An aim in developing allegories was to show that a large amount of category theory itself is amenable to a presentation in terms of relations (regular categories, Heyting categories, and toposes each correspond to a development in allegories and so in a categorical presentation of relations). While allegories are often recognized as a significant presentation in the field, it appears to have produced few direct descendants.

Allegories offer an axiomization that can be thought of as halfway between that found in CBR and relation algebras in that the approach emphasizes more traditional relational operations, such as meet, inclusion, and converse. The characteristic feature of allegories is the use of the *modular law*. Given relational composition (;), meet ($\sqcap$), relation inclusion ($\sqsubseteq$), and converse ($\breve{}$), the modular laws states that for three given relations: $Q;R \sqcap S \sqsubseteq Q;(R \sqcap \breve{Q};S)$.

There is substantial overlap in the two categorical approaches above, as cartesian bicategories of relations exactly coincide with the notion of a unitary pretabular allegory. The modular law above can also be derived using the CBR axioms. Both allegories and CBRs should be thought of as characterizing a positive fragment of relations — i.e. relations without a complement operation — which is historically difficult to implement in the categorical setting.

Relation algebras are related to the characterization of allegories and CBRs. The overlap between the (positive) operations emphasized in RAs and the approaches given above means that results in the setting of RAs often have direct correspondence to presentations found in allegories and CBRs. In the context of relation algebra, the modular law is also referred to as the Dedekind equation. As we go on to discuss below, the Dedekind equation and modular law is also closely related to the *Schröder equivalences*. We prefer the presentation in terms of the Schröder equivalences here, as we think residuation — which is characterized by the Schröder equivalences — ought to be stressed as logically essential. As we will see, a key contribution in the thesis is presenting residuation in the graphs.

The inclusion of the complement in relation algebra is a key difference that adds expressive power and often extends to a broader range of (practical) applications. At the same time, and a point we come back to below, more fundamental logics are often associated with omitting negation as a primitive operation.

The connection between relation algebra and categories has been established with the move to heterogenous relation algebras, which allows relational composition (as in categories) to be partially defined (see [63, 64, 43]). In general, the move to heterogenous relation algebras tends to require little update to the syntax and inference rules (needing only to keep track of typing information, which in our graphical syntax, for example, is handled quite naturally). We take this to further solidify the substantial overlap between the relation algebraic and categorical approaches.

We leave it as an open question whether and to what extent the categorical or relation algebraic perspective ought to be taken. Our readers may perhaps side with the categorical, but from another direction the categorical can be seen as a logical culmination of the relation algebraic. We simply refer broadly to a *logic of relations* to refer to the study of relational operations that is shared by both approaches. Regardless, such a core logic of relations offers a rich underlying foundation upon which these fields and their resulting applications rest. We state that the diagrammatic theory here can capture many of the developments from these three approaches.

Each of these three approaches are just starting to develop and leverage the benefits of a graphical syntax. CBRs have gone the farthest in this direction, as the axioms of CBRs have been given a straightforward graphical and string-diagrammatic syntax. Allegories have seen less development since their initial presentation, which despite appealing to traditional relational operations also uses its own, somewhat idiosyncratic, syntax. A notable exception of a graphical treatment of allegories appears in [19], which shows its close connection to the axioms of relation algebra. Recent work on relation algebras emphasize visualizations of relations as matrices, matrix operations, as well as corresponding graphs [64, 62], but the advantages of a further graphical syntax have yet to be realized. With respect to advances in string diagrams, the categorical and relational approaches seem to have functioned largely independently of each other. We note again that the diagrammatic syntax and relational operations given in this thesis, as well as many related fragments, are applicable to each of these areas.

Other graphical treatments found in categorical logic are worth mentioning, such as proof nets and wiring diagrams for linear distributive categories [38, 4, 23]. Alternative graphical approaches that extend string-diagrammatic syntax in other applications include the ZX calculus [72] and dagger linear logic [25]. Examples more explicitly in the logical direction can be found in [33, 35, 45, 54, 1]. These approaches all add various additions (often multiple) to the string diagrammatic syntax. Of interest to us here, is whether and to what extent Peirce's 'scroll' can be employed to better express these directions.

Turning now to Peirce's work, aside from Brady and Trimble's initial work on Peirce's graphs [17, 16] there is little work on the contemporary development of Peirce's graphs in the context of relation algebra and category theory. Again, no approach that we are aware of has focused on the modern advantages accrued by nested 'cuts' using Peirce's 'scroll'. Vaughan Pratt [60, 59] has given a promising initial comparison of Peirce's early algebraic work with linear logic, but this connection has not been situated diagrammatically or in the context of Peirce's later developed EGs. A key direction in the thesis is to make explicit this connection to linear logic in the graphs and in Peirce's work. The work in this thesis solidifies Brady and Trimble's approach, moving as they do to *-autonomy and the multiplicative case. Our approach has the advantage of making explicit the par'd context and linear rules, connecting these directly to Pierce's 'scroll' rather than the single 'cut', and in general connecting these back historically to Peirce's early work on the graphs.

It is also worth mentioning the work of Ahti Pietarinen and collaborators, who have been working for quite some time to extend the philosophical and logical merits of Peirce's graphs [55, 51]. The latter, along with [52], are notable for moving towards intuitionistic variants of the graphs. We address this move but from a very different direction, as we prefer the perspective from linear logic. The intuitionistic case can then in theory be recovered as a fragment, such as by defining intuitionistic implication with residuation and the !-exponential. Work in [51] and more recently in [31] also connects Peirce's rules to deep inference systems. This perspective, too, find support in the direction taken here.

While relation algebra has seen steady advances since the field was resurrected by Tarski, category theory and the turn towards string diagrams offer a powerful, fresh perspective on the subject. Growing interest and accessibility of Peirce's writings — from within philosophy, but also from those interested in formal diagrammatic reasoning and graph rewriting more generally — offers a further timely motivation. This thesis aims at a contemporary logic of relations in accord with the best of these recent developments.

Contribution and Results

Regular logic — employing existential quantification, $\top$ (True), and logical conjunction — is well understood to be represented graphically in string diagrams. When this thesis began, we took regular logic and the corresponding CBR axioms as our starting point. The broad aim was to extend this logic of string diagrams beyond regular logic.

In **Article I** we show that Carboni and Walters' axiomization of CBRs correspond to the positive fragment of Peirce's Beta graphs. We extend the CBR axioms to include a complement or negation operation — by including, in terms of our graphs, Peirce's notion of the 'cut' — and the result extends regular logic to first-order logic. The rules for the behavior of the 'cut' involve Peirce's 'double cut' rule, the principle of contraposition, and a third rule for iteration/deiteration. Two further remaining rules guarantee that 'lines of identity', which serve to keep track of variables in the syntax, do not interact with the 'cut'. The calculus serves as a modern presentation of Peirce's full Beta calculus with the single 'cut' taken as negation.

While adding negation to regular logic is the key step to first-order logic, the presentation in **Article I** has two drawbacks. It defines negation as a unary operation, which limits the potential expressivity towards other logics. It also relies on an 'egg-shell' notation, outside the defined syntax, that serves as a syntactic hack for presenting a generic context inside a 'cut'. While the 'egg-shell' is shown to accomplish its intended purpose later in **Article IV**, its original use and presentation was less than ideal.

Our next step was to move away from the 'cut-as-negation' story and focus specifically on the fragment with the 'scroll' or 'double cut'. A seemingly natural next step — suggested by history and contemporary importance — was to move towards the intuitionistic case and to define negation as $R \to \bot$. Perhaps surprisingly, Peirce already has this connection built directly into the graphs. This is seen clearly with a simple application of

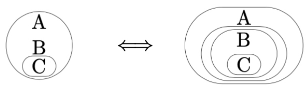
$\neg A$ $\qquad\qquad$ $A \Rightarrow \bot$

Figure 6 – Peirce's 'cut' and intuitionistic negation.

the 'double cut' rule inside the inclusion that is the 'scroll', as in the Figure 6.

Two issues nonetheless arose in the initial pursuit of the intuitionistic case that are worth noting. The first is a broad appreciation of the fact that a standard presentation of the intuitionistic case has more or less been solidified in the literature (such as the requisite "Introduction to Higher Categorical Logic" [47] by Lambek and Scott). We hope — at least as an ideal — that other advantages are accrued by the novel graphical presentation in Peirce's graphs, and so we came to look elsewhere.

The second issue is more significant. Arguably the key feature of the intuitionistic case is that intuitionistic implication is defined (in categorical terms) as right adjoint to conjunction. This is straightforwardly shown in the propositional case seen in Figure 7. Note, also, the similarity to intuitionistic negation, where both consist in adding a 'double



$A \wedge B \to C$ $\qquad\qquad$ $A \to (B \Rightarrow C)$

Figure 7 – Peirce's 'scroll' and intuitionistic implication.

cut' or 'scroll' within the inclusion. The issue is an ambiguity in how this is presented with 'lines of identity', where one can choose between a parallel or sequential presentation. The parallel, or non-relative case (similar to that shown), corresponds to the Booleans, while the sequential case corresponds to relational composition. Much of the work on the intuitionistic case does not distinguish between these, but our graphical syntax in some sense demands it, and we found the direction worth exploring. At the same time we were studying topological equivalences specific to the compositional case and wondering how best to characterize them.

These directions led us to residuation and to the graphical presentation of residuation in **Article II** ("Residuation in Existential Graphs"). In terms of allegories, the addition of residuation yields a corresponding division allegory. In terms of relation algebra, a more concise characterization of relation algebra can also be given by replacing most of the relational axioms with residuation in the form of the *Schröder equivalences*. Graphically, **Article II** shows how residuation and these equivalences are captured by Peirce's 'scroll' and straightforward string deformations. Much of the subsequent work in the thesis can be traced back to this original development. We note, for example, that the key equation in **Article IV** for the linear adjoints and residuation is already given as an example in **Article II**.

Article III ("C.S. Peirce's Early Developments in Linear Logic") and **Article IV** present the full picture of these developments. Both return attention to Peirce's remarkable early presentation of the logic of relations in 'Note B' from 1883. **Article IV** generalizes the presentation in 'Note B' to include triadic relations and to include a contemporary treatment with strings as in Peirce's original Existential Graphs. Following Peirce, the key to the theory is presenting the dual of relational composition, the rules for linear distributivity, and the further equations for linear negation. The paper offers the first instance of linking cartesian with linear structure in this way in string diagrams. In terms of Peirce's early work and relation algebra, these are the links between the Boolean and Peircean, or relative, operations. Significantly, negation then emerges out of the interaction of these other operations. The paper furthermore provides a completeness result worked out in collaboration with coauthors.

Further historical connections to Peirce's early work is left out of **Article IV**, as is the connection to *bilinear logic*, which we find is perhaps the closest modern direction to Peirce's early studies. **Article III** importantly fills in this gap. The article goes on to compare Peirce's graphs to other contemporary notations, such as Cockett et al.'s 'circuit diagrams' in [24], and presents Peirce's innovations towards linear logic.

We end these introductory comments with a brief summary of the directions that Peirce appears to have been right about. For starters, Pierce's adoption of *lines of identity* (LOIs) for capturing variables is confirmed in string-diagrammatic presentations. In particular, 'lines of identity' and variable manipulation obeys the laws of a special Frobenius algebra. Peirce seemed to have understood the basic intuitions behind this structure. The inference rules of the 'sheet' (i.e. evenly enclosed areas or the logical fragment without 'cut') correspond to the rules given by Carboni and Walters for cartesian bicategories of relations. This includes Peirce's rules for 'erasure' and 'iteration'. Within a 'cut' context the directions of these rules are reversed — a key instance of what Peirce refers to as the principle of contraposition, which he stresses as one of the most basic features of the graphs — and this leads to a corresponding presentation of cocartesian structure in these areas. Finally, and returning to Peirce's earlier algebraic work from 'Note B' in 1883, we stress the importance in the graphs of the dual of relational composition and the linear distributive and linear negation laws. The latter leads to stressing additional connectives in the graphs, including the apartness or diversity relation, the combined complement-converse relation that corresponds to linear negation, and the residual that corresponds to linear implication. These correspond in turn to categorical notions and presentations, such as found in linear distributive categories (and linear bicategories), bilinear logic, and other significant fragments, like categorial grammar and the logic of residuation that is the Lambek calculus.

One might ask how such an old theory could be the source of inspiration and even source of progress over a hundred years after its initial development. We will not spend much time on this question, but point out that some of this follows from unavailability of Peirce's original texts and a persistent perception that diagrammatic reasoning or non-traditional forms of graphical reasoning is somehow 'less formal'. Our work here is not historical, but we do cite relevant passages sufficient to defend our point and claims about Peirce. We find Peirce's perspective — in particular his emphasis on compositionality, relational operations, and the importance of good syntax — to be remarkably modern.

Going further still, we believe that Peirce's work will continue to be a worthwhile direction of further study, along with contemporary inspiration and advances for some time to come. While string diagrams are still quite novel, Peirce spent much of his mature intellectual life — some thirty years of study — not only working on the theory but espousing its diagrammatic advantages. Even for a lifetime(s) ago, this rivals the time spent by many of us today on these new issues. Peirce's foresight and conviction in undertaking such a task over a hundred years ago should not go unnoticed.

Outline

The following chapters — more or less in the ordered they have been written — have a natural order. For the reader interested in skipping ahead, we had appropriate short introductions.

Article I is the first approximation of presenting Peirce's Existential Graphs in contemporary string diagrams. Here, we connect the inference rules for Existential Graphs with those of cartesian bicategories. We add to this the notion of negation and in-so-doing extend the existential-conjunctive fragment of regular logic to first-order logic. This requires three rules all motivated by Peirce's presentation in the graphs: a 'double cut' or 'scroll' rule, a principle of contraposition, and an iteration/deiteration rule. To this we add a rule (not stated in Peirce, but understood) that 'lines of identity' pass freely through 'cuts'. The above article takes negation as a primitive operation. We next move to relax this condition.

Article II gives a graphical presentation of residuation. This direction arises from looking at the 'scroll' as an inclusion and other topological features of the 'scroll'. This is substantial for two reasons. The first is historical, showing that Peirce both understood residuation, drew graphs of the residual, and — we add here — appears to have spent much of his later time on the graphs exploring this direction. The second is that the rules for residuation-introduction and -elimination correspond to linear implication, linear negation, and to the adjoint conditions that play a significant role in the last two articles.

Article III serves as a historical interlude and introduction for the fourth. It introduces the key features of Peirce's earlier algebraic work in the context of the graphs and his early presentation of the rules behind *cyclic bilinear logic*. The paper also serves as a summary of the new perspective on Peirce's studies presented here. While the other papers follow a natural historical progression, this paper can alternatively be read as an introduction to the new perspective on Peirce's work that is established in this thesis.

The project outlined in **Article II** and **Article III** is brought to a culmination in **Article IV**, which presents the modern *neo-Peircean calculus of relations*. This includes a completeness result worked out with coauthors. The corresponding string diagrams for the existential-conjunctive fragment is here extended with a dual universal-disjunctive fragment, and these are in turn linked — as in Peirce's original presentation in 'Note B' from 1883 — via linear distributivity and the linear negation laws. Among other advances, we note **Article IV** vindicates Peirce's use of the single 'cut' as negation (negation now being defineable), vindicates his use of the iteration/deiteration rule, and confirms the connection between Peirce's diagrammatic rules and *deep inference* and the calculus of structures (as first suggested in [49]).

Finally, an appendix is included as a **Article V**. This article contains Peirce's work on the diagrammatic presentation of absurdity that Peirce calls the 'blot'. We do not spend as much time on the intuitionistic case — preferring, as we do, the linear one — but this last article fits between the first and second articles as a way of relaxing the 'cut-as-negation' story, where negation is taken as a primitive as in **Article I**.

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1 Article 1 - Compositional Diagrammatic First-Order Logic

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N. Haydon and P. Sobociński. Compositional diagrammatic first-order logic. In A.-V. Pietarinen, P. Chapman, L. Bosveld-de Smet, V. Giardino, J. Corter, and S. Linker, editors, *Diagrammatic Representation and Inference*, pages 402–418, Cham, 2020. Springer International Publishing

Compositional Diagrammatic First-Order Logic

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Abstract. Peirce's β variant of Existential Graphs (EGs) is a diagrammatic formalism, equivalent in expressive power to classical first-order logic. We show that the syntax of EGs can be presented as the arrows of a free symmetric monoidal category. The advantages of this approach are *(i)* that the associated string diagrams share the visual features of EGs while *(ii)* enabling a rigorous distinction between "free" and "bound" variables. Indeed, this diagrammatic language leads to a *compositional* relationship of the syntax with the *semantics* of logic: we obtain models as structure-preserving monoidal functors to the category of relations.

In addition to a diagrammatic syntax for formulas, Peirce developed a sound and complete system of diagrammatic reasoning that arose out of his study of the algebra of relations. Translated to string diagrams we show the implied algebraic structure of EGs sans negation is that of cartesian bicategories of relations: for example, lines of identity obey the laws of special Frobenius algebras. We also show how the algebra of negation can be presented, thus capturing Peirce's full calculus.

1 Introduction

Peirce's Existential Graphs (EGs) arose out of his continued study and development of the algebra of relations. As a diagrammatic calculus, EGs use lines to represent identity, conjunction and existence and nested circles (Peirce's notion of the "cut"[1]) to capture negation. These graphical elements are drawn on the sheet of assertion: the blank page upon which a graph is scribed. Our focus is on the algebra of the β variant of EGs, which we treat as string diagrams. The resulting language, which we call $\mathsf{D}\beta$, shares the same visual features of EGs.

We argue that Peirce's β is closely related to the algebraic structure of cartesian bicategories of relations [7]. Indeed, lines of identity, as string diagrams, obey the laws of special Frobenius algebras, while derivations in the negation-free fragment are the 2-cells of free cartesian bicategories. We identify the additional rules needed to handle negation, which are adapted from Peirce's calculus of

[1] In this paper, we use "cut" in the Peircean sense to mean negation, not the standard notion of cut from proof theory.

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diagrammatic reasoning. Throughout, we argue that Peirce's seminal studies led him to intuitions that suggest that he—at least implicitly—identified the very same algebraic structures.

While $\mathsf{D}\beta$ is visually similar—we joke that a diagram in $\mathsf{D}\beta$ looks like an EG if you squint—it is important to highlight some differences. Making the Frobenius structure explicit in $\mathsf{D}\beta$ imposes more rigour on lines of identity. Relations in $\mathsf{D}\beta$ have left and right wires corresponding to arity/co-arities of the relations. This may actually help the presentation of graphs in EGs as Peirce sometimes imposes an order on relations that is not directly read off the ligatures. An explicit Frobenius structure gives the flexibility of rearranging wires as needed, so expressivity is not lost, but also allows us to have a definite ordering, which is useful in many examples. This amendment, maintaining the visual features while being more definite/exact, may very well be a welcome addition.

Perhaps more significantly, in order to achieve compositionality, the string diagrammatic account forces us to keep track of bound and free variables in a more precise way than in Peirce's original EGs. Indeed the *existential* in the name of EGs means that scribing a graph on the sheet of assertion is to assert the existence (i.e. the quantification) of the respective predicate/variable. EGs have, as Zeman has put it, "implicit quantification" [19]. Treatment of free and bound variables in modified versions of EG (see [4,10]) equip EGs with additional structure. The string diagrammatic language $\mathsf{D}\beta$ makes this treatment quite natural—the result is less cumbersome than the technology of variable management (e.g. α-conversion, capture-avoiding substitution) often waved through at the start of many traditional courses on predicate logic.

Brady and Trimble have previously developed a string diagrammatic account of EGs [2,3], relying as we do on monoidal categories and in particular, the poset-enriched monoidal category of relations as a semantic universe for logic. However, their string diagrams are geometric/topological entities. Instead, we emphasize their *syntactic* nature, which allows, e.g. to define the notion of model as simple inductive procedure, not unlike Tarski's compositional semantics for predicate logic. Moreover, we work in the framework of (poset enriched) props [11], which emphasizes the algebraic structure borne by the underlying monoidal category.

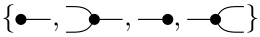
In the discussion below we assume some familiarity with the reading and transformation (i.e. inference) rules of EGs. For a lengthier introduction to Peirce's EGs, and one that includes a description of Peirce's transformation rules, see [17]. Further accounts can be found in [4,9,18], and the introduction in [15]. For an introduction to Peirce's compositional/valental account of relations, see [17, p. 113–118]. A contemporary presentation can be found in [5].

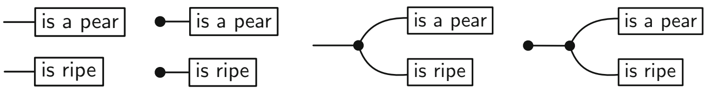
Structure of the Paper. In Sect. 2 we introduce $\mathsf{D}\beta$ and show how to translate it to and from traditional syntax. In Sect. 3 we introduce the structure of cartesian bicategories, which informs the notion of model of the logic, introduced in Sect. 4. We identify iteration laws of this structure with the cut in Sect. 5 and conclude with a worked example of diagrammatic reasoning in Sect. 6.

2 String Diagrams as Syntax

We start with Peirce's *valental* theory of relations, inspired by the theory of valence in chemistry, where elements have open bonds that act as attachment points from which more complex compounds and molecules can be built. Relations are thus seen as having analogous open bonds that can be filled and combined with other relations to form more complex relations.

Consider the 'loves' relation, which in usual FOL syntax is written $\mathsf{loves}(x, y)$. The relation remains indefinite insofar as the objects/subjects of the relation are unspecified, i.e. the variables x and y remain free. Peirce adds "blanks" or "hooks" as graphical placeholders to represent the unspecified objects/subjects, which when filled, "complete the relation". In our example 'loves' is a dyadic relation, and we represent hooks as "dangling" wires, arriving at . Filling in the hooks/connecting the wires in the diagrammatic notation is an analogous operation to passing from *free* to *bound* variables in the usual FOL syntax.

Specific relations are combined by joining free hooks together with what Peirce calls a line of identity. A line of identity asserts the identity of each object/subject at its endpoints. We represent lines of identity with the generators  of a monoid-comonoid pair. Consider the diagrams below.



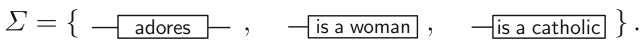
Reading from left to right, the first diagram is the conjunction of the $\mathsf{is\ a\ pear}$ and $\mathsf{is\ ripe}$ relations where the hooks are unfilled/wires are dangling. In usual FOL syntax, $\mathsf{is\ a\ pear}(x) \wedge \mathsf{is\ ripe}(y)$. In the second diagram the hooks are filled/wires are capped off with a unit generator. In usual FOL syntax, $\exists x.\ \mathsf{is\ a\ pear}(x) \wedge \exists y.\ \mathsf{is\ ripe}(y)$. In the third, using the comultiplication generator the two wires have been equated but there is a dangling wire to the left; $\mathsf{is\ a\ pear}(x) \wedge \mathsf{is\ ripe}(x)$. In the final diagram the wire has been capped off: $\exists x.\ \mathsf{is\ a\ pear}(x) \wedge \mathsf{is\ ripe}(x)$.

The syntax of $\mathsf{D}\beta$ below follows Peircean considerations. Let Σ be a monoidal signature: symbols R each with an arity $ar(R) \in \mathbb{N}$ and coarity $coar(R) \in \mathbb{N}$.

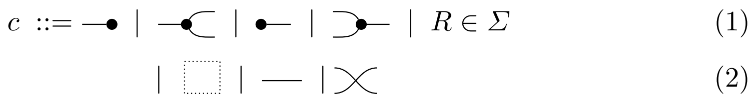
Example 1. The signature for our running example is

$$\Sigma = \{\mathsf{adores},\ \mathsf{is\ a\ woman},\ \mathsf{is\ a\ catholic}\}$$

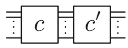
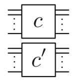
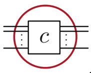
with $ar(\mathsf{adores}) = coar(\mathsf{adores}) = 1$, $ar(\mathsf{is\ a\ woman}) = ar(\mathsf{is\ a\ catholic}) = 1$, $coar(\mathsf{is\ a\ woman}) = coar(\mathsf{is\ a\ catholic}) = 0$. The diagrammatic convention for an element $R \in \Sigma$ is to draw it as a box, with $ar(R)$ wires, ordered from top to bottom, "dangling" on the left and, similarly, $coar(R)$ wires on the right. Thus:



Below we define our recursively defined syntax using BNF notation. These are the basic syntactical elements from which terms in $\mathsf{D}\beta_{\Sigma}$ are constructed.[2]



$$| \; c \oplus c \; | \; c \,;\, c \; | \; c^{-} \tag{3}$$

At this point, the diagrammatic elements of the syntax in (1) and (2) ought to be considered as mere symbols that denote constants. The operations are given in (3): two binary operations ‘;’, ‘$\oplus$’ and one unary operation $\bullet^{-}$. These have their own diagrammatic convention: $c\,;\,c'$ is drawn , $c \oplus c'$ is drawn , and c^{-} is drawn  . Roughly the operations here can again be seen in terms of our relational story from above. ‘$\oplus$’ allows us to scribe relations adjacent to each other (i.e. in parallel) on the sheet, ‘;’ allows us to wire relations together in series (similar to connecting relations via lines of identity), and placing a relation inside a cut expresses its negation/complement.

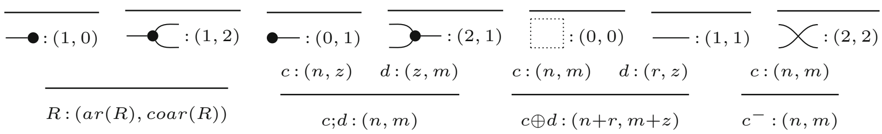
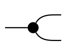


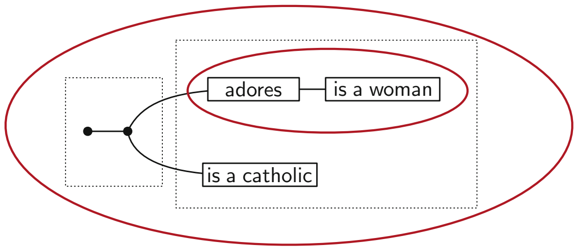
Fig. 1. Sort inference rules.

As opposed to the usual syntax of FOL, ours (1) (2) (3) does not have variables, nor variable binding. The price is an inductive discipline, given in Fig. 1. Intuitively, it keeps track of “dangling” wires—terms are associated with a *sort*, a pair of natural numbers (n, m) that counts the wires on the left and on the right—and ensures that for a term $c\,;\,c'$, c and c' have the right number of wires on their corresponding boundaries so that ‘;’ as “connecting wires” to make sense. It is easy to prove that if a term has a sort, it is unique.

Example 2. The term  ;  has no sort and no diagrammatic depiction. On the other hand  $\oplus$  : $(2, 0)$. Given the signature of Example 1, consider the term (( ; ) ; ((adores ; is a woman)$^{-}$ $\oplus$ is a catholic))$^{-}$ with sort $(0, 0)$.

[2] Henceforward we will not write the subscript Σ, assuming a fixed ambient monoidal signature.

Using the diagrammatic conventions yields the following, where the dotted-line boxes play the role of the parentheses.



It is not difficult to see that sorted terms are in 1-1 correspondence with such diagrams, provided that enough dotted-line boxes are inserted to disambiguate the associativity of ';' and '$\oplus$' and the priority between them.

2.1 Translating to and from Traditional Syntax

The (traditional) syntax below is expressive enough to capture first order logic, containing equality, relation symbols, existential quantification and negation.

$$\Phi \ ::= \top \mid \Phi \wedge \Phi \mid x_i = x_j \mid R(\overrightarrow{x}) \mid \exists x.\Phi \mid \neg\Phi \tag{FOL}$$

To ease the translation between the diagrammatic and the traditional, we introduce a half-way formalism that constraints the syntax FOL with explicit free-variable management. This is a mild extension of a similar calculus in [1, Sec. 2] where an analogous translation is given, albeit without the presence of negation.

$$\frac{}{0 \vdash \top}\,(\top) \qquad \frac{R \in \Sigma \quad ar(R) = n}{n \vdash R(x_0, \ldots, x_{n-1})}\,(\Sigma) \qquad \frac{n \vdash \Phi}{n-1 \vdash \exists x_{n-1}.\Phi}\,(\exists)$$

$$\frac{}{2 \vdash x_0 = x_1}\,(=) \qquad \frac{m \vdash \Phi \quad n \vdash \Psi}{m+n \vdash \Phi \wedge (\Psi[\overrightarrow{x}_{[m,m+n-1]}/\overrightarrow{x}_{[0,x_{n-1}]}])}\,(\wedge) \qquad \frac{n \vdash \Phi}{n \vdash \neg\Phi}\,(\neg)$$

$$\frac{n \vdash \Phi \quad (0 \leq k < n-1)}{n \vdash \Phi[x_{k+1}, x_k/x_k, x_{k+1}]}\,(\mathsf{Sw}_{n,k}) \qquad \frac{n \vdash \Phi}{n-1 \vdash \Phi[x_{n-2}/x_{n-1}]}\,(\mathsf{Id}_n) \qquad \frac{n \vdash \Phi}{n+1 \vdash \Phi}\,(\mathsf{Nu}_n)$$

The idea is that a judgment $n \vdash \Phi$ expresses the fact that Φ is a formula with free variables from the set $\{x_0, x_1, \ldots, x_{n-1}\}$. Indeed, we have the following:

Proposition 1. *A formula Φ with free variables in $\{x_0, x_1, \ldots, x_{n-1}\}$ is derivable from (FOL) if and only if $n \vdash \Phi$.*

Using the above, we can present a translation Θ from (FOL) to $\mathsf{D}\beta$ by induction on the derivation of $n \vdash \Phi$. The rules are given in Fig. 2. A similar translation can be given from $\mathsf{D}\beta$ to (FOL). Another important fact is that the translations respect the underlying semantics of the logics—due to space restrictions we are not able to show this here. We shall introduce the semantics of $\mathsf{D}\beta$ in Sect. 4.

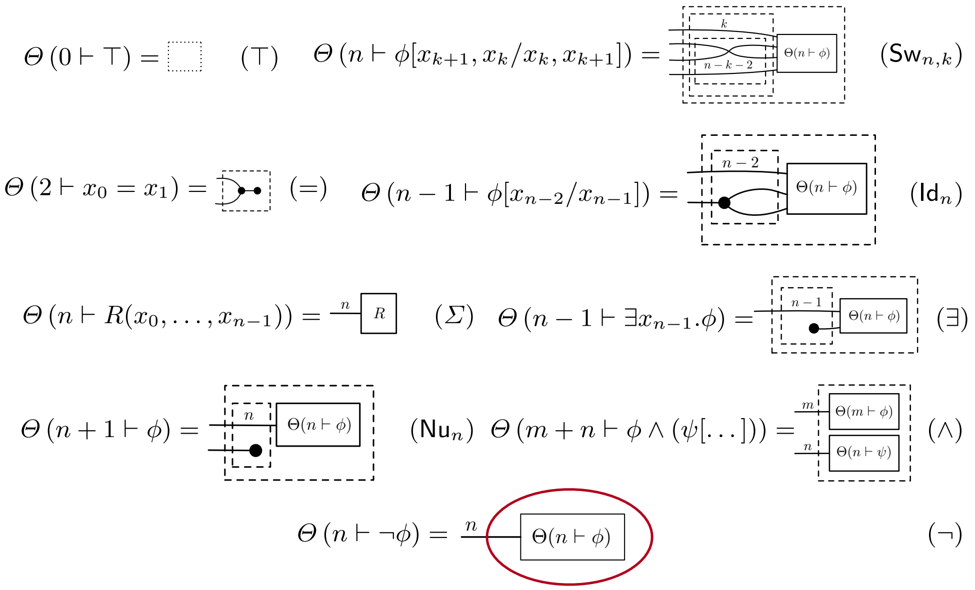


Fig. 2. Translation Θ from FOL to $\mathsf{D}\beta$.

Example 3. Referring to Example 2, the formula expressed by the diagram is

$$\begin{aligned}
&\neg(\exists x.\ \mathsf{is\ a\ catholic}(x) \wedge \neg(\exists y.\ \mathsf{adores}(x,y) \wedge \mathsf{is\ a\ woman}(y)))\\
&\quad \equiv \forall x.\ (\neg\mathsf{is\ a\ catholic}(x) \vee (\exists y.\ \mathsf{adores}(x,y) \wedge \mathsf{is\ a\ woman}(y)))\\
&\qquad \equiv \forall x.\ \mathsf{is\ a\ catholic}(x) \rightarrow \exists y.\ \mathsf{adores}(x,y) \wedge \mathsf{is\ a\ woman}(y).
\end{aligned}$$

2.2 String Diagrams

In order not to clutter diagrams with dotted-line boxes, we will not consider raw terms, but terms quotiented by the laws of symmetric strict monoidal categories [11,12] of a particularly simple nature: the set of objects is the natural numbers and $m \oplus n \stackrel{\text{def}}{=} m + n$. Such categories are called props. Some care has to be taken with the $\bullet^-$ operation, which is not standard: we introduce a simple extension to the usual definition below.

Definition 1. *A prop $\mathbb{X}$ with a unary operation on homsets (*uoh-prop*) is a prop with a family of operations $\overline{}_{m,n} : \mathbb{X}[m,n] \rightarrow \mathbb{X}[m,n]$, where $m, n \in \mathbb{N}$.*

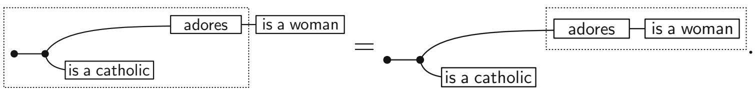
We are ready to define the notion of syntax we will use throughout the paper.

Definition 2. (Syntax). *Let $\mathsf{D}\beta$ be the uoh-prop where arrows $m \rightarrow n$ are (m,n)-sorted terms, modulo the laws of symmetric monoidal categories. The additional unary operation on homsets is given by $\bullet^-$.*

While Definition 2 emphasises the construction of terms from the grammar, $\mathsf{D}\beta$ has an extremely concise mathematical description: it is the *free* uoh-prop on Σ. The characterisation of $\mathsf{D}\beta$ as a free algebraic structure is important: first, it means that our string diagrams are a bona fide notion of syntax, not unlike usual syntax trees. Second, just as syntax admits elegant inductive definitions (not unlike, for instance, Tarski's semantics of first order logic), in order to define a structure preserving translation (homomorphism of uoh-props) from $\mathsf{D}\beta$ to some target semantic universe (some uoh-prop), it suffices to define the target of the constants (1). We shall use this for the concept of *model* in Sect. 4.

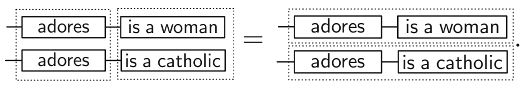
Example 4. For the category-theory uninitiated reader, let us give an intuitive summary of the algebraic structure given by Definition 1, used in Definition 2.

– the two composition operations are strictly associative, e.g.

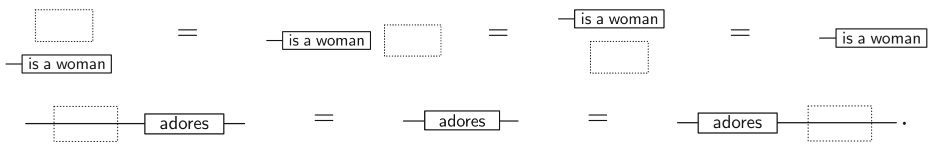


This means the result is the same irrespective of the order we compose, i.e. whether we start with the adored woman or the adoring catholic.

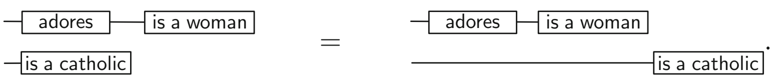
– the two composition operations are compatible, e.g.



– the first two constants of (2) are *identities*; the first the identity on 0, the second the identity on 1. This means, e.g.

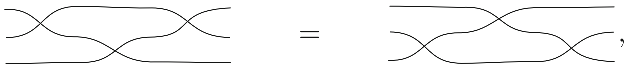


The combination of identity laws and the compatibility of ';' with '$\oplus$' means that unconnected components can be "slid" past each other, e.g.

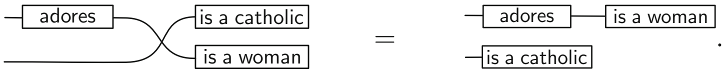


In Peirce's EGs these features are built directly into the conventions of the sheet of assertion. The identities follow from the properties of composition with a blank sheet or with a line of identity. In regards to composition and associativity on the sheet itself, Peirce writes: "If two propositions are written, detached from one another, on the sheet of assertion, both are asserted, regardless of whether one is to the right, to the left, at the top, or at the bottom of the other… If three or more propositions are all written, detached from one another, on the sheet of assertions, the logical relation of any pair of them is the same as that of any other pair" [16, p. 488].

– the last constant of (3) is a *symmetry*. This means that diagrams constructed from it and the identity "behave" as permutations, e.g.



and arbitrary diagrams can "slide" across symmetries[3], e.g.



3 The Algebra of Lines of Identity

In this section we identify some of the algebraic structure of $\mathsf{D}\beta$ that will, in Sect. 5, result in a calculus for diagrammatic reasoning. In addition, the structure introduced here will allow us to specify the correct concept of model in Sect. 4.

Figure 3 depicts the laws of cartesian bicategories (of relations) [7]. Equations (coas), (coco), (counl) say that (,) is a cocommutative comonoid, while (as), (co), (unl) say that (,) is a commutative monoid.

The three equations (fr) are the Frobenius equations. While any two of the three can be used to derive the others, all three are useful in diagrammatic reasoning. The equation (sp) is the so-called "special" law. The equations thus far define what is usually referred to as a (commutative) special Frobenius bimonoid.

It is worth reflecting on how these laws are captured in Peirce's EGs. As mentioned previously, associativity and commutativity are built into the conventions of the sheet of assertion, where the order of composition of relations on the sheet is immaterial. Each of the other rules can be seen as following from the combination of monadic, dyadic, triadic identity elements. (unl) and (counl) are equivalent to being able to add a branch to any line of identity. Peirce called

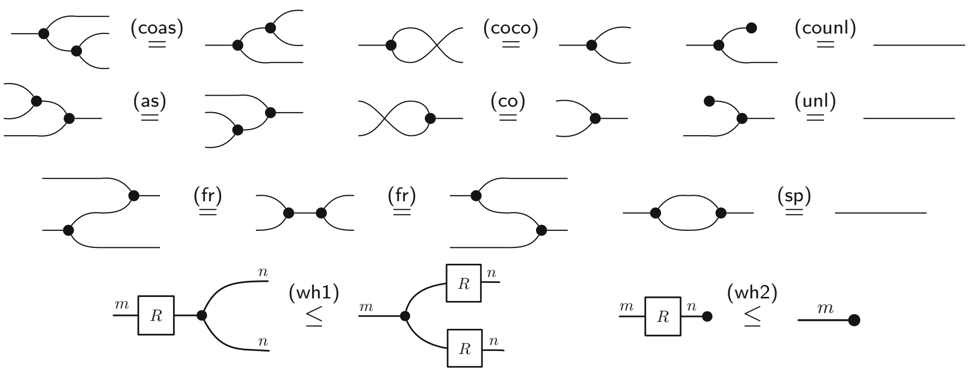
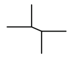
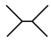


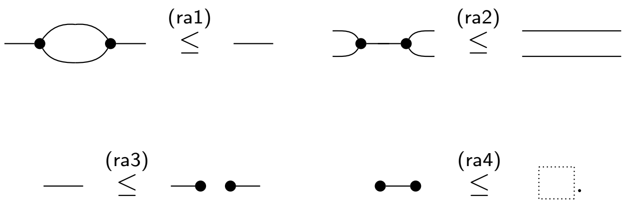
Fig. 3. The laws of cartesian bicategories of relations.

[3] These equations are examples of *naturality* of the symmetry.

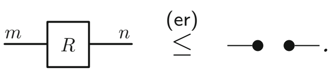
this triadic identity element, where a branch forms a point with three extending wires, the teridentity relation. Peirce's interpretation of this rule in EGs, given in a letter to Lady Welby, is worth quoting: "every line of identity ought to be considered as bristling with microscopic points of teridentity" [14].[4]

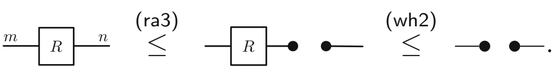
The (fr) and (sp) equations can be seen as observations about the composition of teridentity relations. Two teridentity relations brought together by connecting two of each of the three wires is equivalent to a single (dyadic) line of identity. This yields the (sp) equation. Similarly, the various combinations of two teridentity relations connected through one wire likewise yield the (fr) equalities. Peirce is explicit about the interpretation of this rule in his EGs. He writes: "Quateridentity [Peirce's term for a point with four extending wires] is obviously composed of two teridentities; i.e. This  is  or  or " [14]. Clearly, Peirce had the topological intuitions conveyed by the Frobenius structure.[5]

Notice that (wh1) and (wh2) are not equalities and as such, in subsequent diagrammatic reasoning, derivations can only use them left-to-right. Moreover, they use the diagrammatic convention where a wire with a natural number label m stands for m wires stacked on top of each other. The inequations (wh1) and (wh2) specify that all arrows are weakly homomorphic w.r.t. the comonoid structure. In cartesian bicategories, moreover, the monoid structure is required to be right adjoint to the comonoid structure. This means the following inequalities:

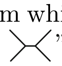


In the context of Frobenius bimonoids that satisfy (wh1) and (wh2), all of (ra1)-(ra4) are redundant. As we will see, (wh1) and (wh2) (along with the redundant (ra1)-(ra4)) give rise to Peirce's transformation rules in EGs. Peirce's assertion, for example, that any graph scribed on the sheet itself (i.e. that is not scribed within a cut) can be erased can be proved as follows.

Lemma 1. 

Proof. 

[4] See, also, [CP 4:583]: "the line of identity... must be understood quite differently. We must hereafter understand it to be *potentially* the graph of teridentity by which means there will virtually be at least one loose end in every graph".

[5] Elsewhere Peirce writes: "There is no need of a point from which four lines of identity proceed; for two triple points answer the same purpose " [16, p. 357].

Remark 1. It is well-known that the Frobenius equations induce a self-dual compact closed structure. Roughly speaking, this allows us to "rewire" diagrams, moving wires between the boundaries. We have used this already in the first diagrams of Example 4, on the $\mathsf{is\ a\ catholic}$ relation.

4 Models

Recal uoh-props, introduced in Definition 1. Below we identify an important class of uoh-props, which together serve as the semantic universe for $\mathsf{D}\beta$.

Definition 3. *Let X be a set. The uoh-prop Rel_X has, as arrows $m \to n$, relations $X^m \to X^n$ (subsets of $X^m \times X^n$), where X^m is the m-fold cartesian product of X. Given a relation $R : X^m \to X^n$, R^- is the (set-theoretical) complement of R as a subset of $X^m \times X^n$.*

Composition in Rel_X is relational composition: given $R : m \to k$ and $S : k \to n$, $R \mathbin{;} S = \{\ (\boldsymbol{x}, \boldsymbol{y}) \mid \exists \boldsymbol{z} \in X^k.\ (\boldsymbol{x}, \boldsymbol{z}) \in R \wedge (\boldsymbol{z}, \boldsymbol{y}) \in S\ \} \subseteq X^m \times X^n$. The monoidal product is cartesian product of relations.

It is well-known that Rel_X is a cartesian bicategory of relations, that is, it satisfies all of the equations of Fig. 3. In the setting of Rel_X, $-\!\bullet\!\!\subset$ is the diagonal relation $\{(x, \binom{x}{x}) \mid x \in X\}$ while $-\!\bullet$ is the relation $\{(x, \star) \mid x \in X\}$, where $\star$ is the unique element of the singleton set X^0. The relations denoted by $\supset\!\!\bullet\!-$ and $\bullet\!-$ are, respectively, the opposite relations. Henceforward we will call these four relations the *canonical Frobenius structure* of Rel_X.

The following is the central definition of this section.

Definition 4. *A* model *for $\mathsf{D}\beta$ consists of a set X and a morphism of uoh-props*

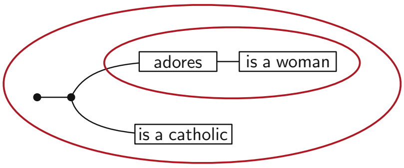
$$[\![-]\!] : \mathsf{D}\beta \to \mathsf{Rel}_X$$

that maps $\{-\!\bullet\!\!\subset, -\!\bullet, \supset\!\!\bullet\!-, \bullet\!-\}$ to the canonical Frobenius structure of Rel_X.

Referring back to the syntax definition (1), to give such a morphism is to give, for each $\sigma : (m,\, n) \in \Sigma$, a relation $[\![\sigma]\!] \subseteq X^m \times X^n$. The rest of the mapping is induced compositionally.

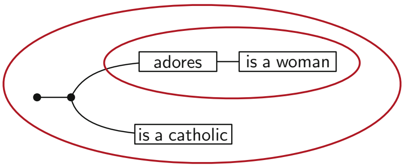
Remark 2. Note that *closed* diagrams, that is those of sort $(0,\, 0)$ map to relations of type $0 \to 0$, that is, subsets of $X^0 \times X^0$. Since X^0 is a singleton, there are precisely two such relations – the empty ($\varnothing$) and the full ($\{(\star, \star)\}$). We identify these with *truth values* – $\varnothing$ with $\perp$ (false) and $\{(\star, \star)\}$ with $\top$ (true).

Example 5. Take the signature of Example 1. Let $X = \{m, w\}$. To define $[\![-]\!] : \mathsf{D}\beta \to \mathsf{Rel}_X$ we need only choose valuations of $-\boxed{\text{adores}}-$, $-\boxed{\text{is a woman}}$, and $-\boxed{\text{is a catholic}}$ as relations. Let $[\![\,-\boxed{\text{is a woman}}\,]\!] \subseteq X^1 \times X^0 = \{(w, \star)\}$. Similarly, let $[\![\,-\boxed{\text{is a catholic}}\,]\!] = \{(m, \star)\}$. If we set $[\![\,-\boxed{\text{adores}}-\,]\!] \subseteq X^1 \times X^1 = \{(m, w)\}$

then $[\![$ 

$]\!] \subseteq X^0 \times X^0 = \{(\star, \star)\} = \top.$

On the other hand, if we assign $[\![-\boxed{\text{adores}}-]\!] = \{(m, m)\}$ then

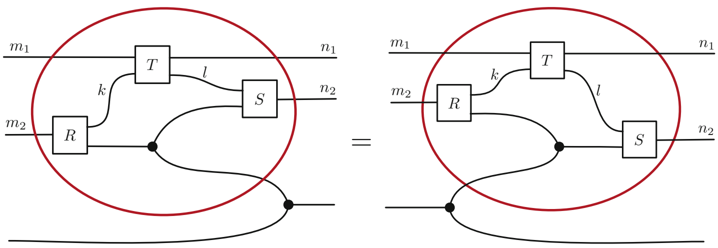
$[\![$ 

$]\!] \subseteq X^0 \times X^0 = \varnothing = \bot.$

Having established the notion of model, we introduce the notions of *soundness*, *completeness* and *logical equivalence*. Two terms t, u of $\mathsf{D}\beta$ are said to be logically equivalent if they have the same semantics in all models, $[\![t]\!] = [\![u]\!]$. An equation is *sound* if it preserves logical equivalence. A calculus is complete if it equates all logically equivalent terms. Note that the fact that Rel_X is a cartesian bicategory of relations means that all of the laws introduced in Sect. 3 are sound.

5 The Algebra of Cut

In Sect. 3 we began the process of axiomatising logical equivalence. Thus far, negation has not played a significant role in our exposition. In Fig. 4 we identify a calculus that is sound, and—taken in conjunction with the laws of Fig. 3—we conjecture to be complete. The equations of Fig. 4 describe the interactions between the algebraic structure of Fig. 3 and Peirce's cut (negation). First, we explain the jagged-line notation, which emphasizes the *local* nature of the interactions. It is shorthand for an arbitrary context inside the cut. For example, (frcut) stands for



for arbitrary R, S and T. Thus with (frcut) we can, roughly speaking, "rewire" a cut to move wires between its left and right boundaries. Indeed (frcut) is a kind of Frobenius law for cuts. In short, the combination of (symcut) and (frcut) means that the cut boundary is permeable to "wiring" and the permutation structure.

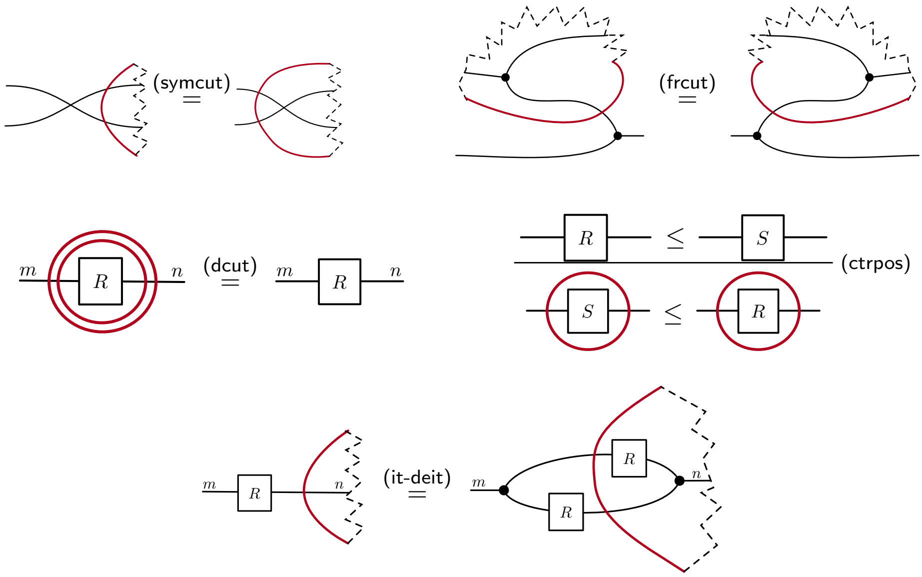


Fig. 4. The algebra of cut.

(dcut) is a diagrammatic representation of Peirce's rule for adding or erasing a double cut around any partial graph. Of course, this is a non-constructive rule; in this paper we only consider classical logic. Some progress has been made recently [13] in the study of how EGs can be used as an intuitionistic logic and we plan to investigate this in our framework in future work.

The (ctrpos) judgement single-handedly captures much of the behavior of the transformation rules within the cut. Peirce explains it as follows: "Of whatever transformation is permissible on the sheet of assertion, the reverse transformation is permissible within a single cut." [16, p. 353]. While our presentation of (ctrpos) represents this point with respect to a single cut, it is worth noting that the reversal continues within subsequent nested cuts. The result is that the same transformation rules that apply on the sheet itself (i.e. to graphs that are not within a cut) also apply to graphs within an even number of cuts.[6] As a rule the

[6] Following the passage quoted above, Peirce writes: "In short, whatever transformation is permissible on the sheet of assertion is permissible on the sheet of assertion within any even number of cuts while the reverse transformation is permissible within any odd number of cuts" [16, p. 353] Or alternatively: "All illative processes are subject to the apagogical principle, or principle of contraposition, which, as applied to graphs, is as follows: If any illative process is valid within an even number of enclosures, its reverse is valid within an odd number, and vice versa" [15, p. 94]. See also [16, p. 257-8, p. 478-9, & p. 539].

principle of contraposition has been markedly absent from other presentations of Peirce's transformation rules in the literature. The latter point is all the more significant in that Peirce often emphasizes the principle at the beginning of his presentations of EGs and often motivates the other transformation rules from it.[7] Our presentation situates the principle in its position of primary importance.

Intuitively, the principle of contraposition captures the symmetry between the valid twin inference rules of modus ponens and modus tollens. If we can infer the transformation from R to S then we can likewise infer from the denial of S the denial of R. In terms of $\mathsf{D}\beta$ and Peirce's EGs, and as stated above, the principle of contraposition allows us to perform the reverse transformations when working within a cut. Our previous proof of the erasure rule, which states that any graph written on the sheet itself (i.e. in an even area) can be erased, can be reversed using (ctrpos) to yield Peirce's insertion rule. Likewise, Peirce's rule that a line of identity can be broken on the sheet itself (ra3) can be reversed using (ctrpos) to yield his rule that a line of identity can be joined in an odd area.[8]

The rule (it-deit) is a statement of Peirce's principle of iteration/deiteration. In Peirce's own words the rule is stated as follows: "…any partial graph, detached or attached, may be iterated within the same or additional cuts provided every line or hook of the iterated graph be attached in the new replica to identically the same ligatures as in the primitive replica; and if a partial graph be already so iterated it can be deiterated by the erasure of one of the replicas which must be within every cut that the replica left standing is within" [16, p. 358]. This rule applies in the same area as the partial graph—i.e. the same rule holds in the case where no cut is present. For us, it is useful to separate the two ideas conceptually, since the latter is implied by the algebraic structure in Sect. 3.

It is worth noting that our (it-deit) rule is similar to Burch's presentation of "Dopplegänger pairs" that form when a line of identity crosses a cut (or two lines of identity abut each other at a cut) [6]. Our rule is more general, as it applies not simply to lines of identity but to relations and partial graphs. Each case is unified under the same rule here.

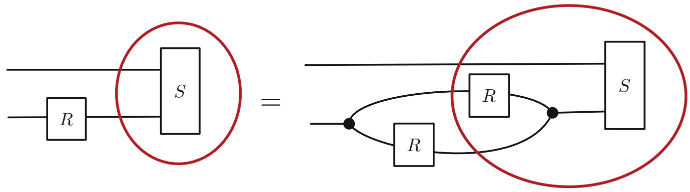
While the soundness of the other rules in Fig. 4 is straightforward, (it-deit) is more involved and less intuitive.

[7] See, for example, the passages in the previous footnote.

[8] In Peirce's words: "…it is to be noted that a line of identity may be broken within an even number of cuts or on the sheet of assertion, while two lines may be joined within an odd number of cuts" [16, p. 358].

Lemma 2. (it-deit) *is sound.*

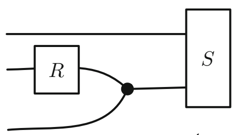
Proof. Since we can "rewire" any cut so that it only has wires on its left boundary, without loss of generality it suffices to show that:



is sound for all possible valuations of R and S. Using traditional syntax, and simplifying somewhat, this is to show the following logical equivalence:

$$\begin{aligned}&\exists \boldsymbol{z}.\ R(\boldsymbol{x_2}, \boldsymbol{z}) \wedge \neg S(\boldsymbol{x_1}, \boldsymbol{z})\\ &\equiv \exists \boldsymbol{z_1}.\ R(\boldsymbol{x_1}, \boldsymbol{z_1}) \wedge \neg(\exists \boldsymbol{z_2}, \boldsymbol{z_3}.\ R(\boldsymbol{x_1}, \boldsymbol{z_2}) \wedge \boldsymbol{z_1} = \boldsymbol{z_3} \wedge \boldsymbol{z_2} = \boldsymbol{z_3} \wedge S(\boldsymbol{x_1}, \boldsymbol{z_3}))\end{aligned}$$

Instead of dealing with the complicated formulas above, we instead directly use the definition of model introduced in Sect. 4. Suppose for some model, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is on the LHS. This happens exactly when there is some y_2 s.t. $x_2 R y_2$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \notin S$.

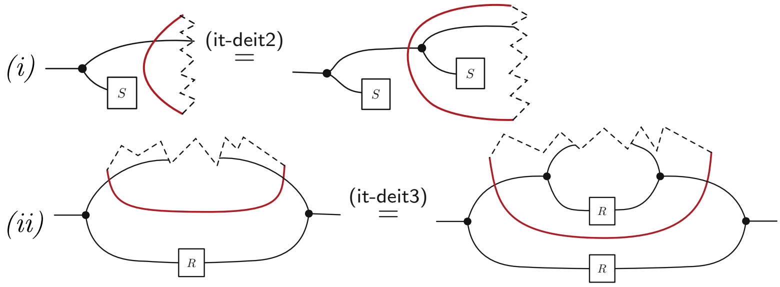
Suppose now that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in$ RHS. This happens exactly when there is some y_2 s.t. $x_2 R y_2$ and $\begin{pmatrix} x_1 \\ x_2 \\ y_2 \end{pmatrix} \notin$ 

. This non-inclusion happens exactly when it is not the case that $x_2 R y_2$ or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \notin S$. Since $x_2 R y_2$ by assumption, it happens precisely when $x_2 R y_2$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \notin S$.

It follows that LHS and RHS denote the same relation in all models. □

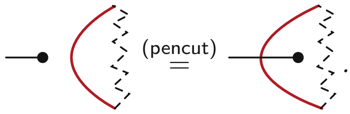
We can use (it-deit) to obtain two similar laws that are useful in diagrammatic proofs. We omit proofs for space reasons but note that Peirce can be seen using an instance of (ii) in his 1903 Lowell Lectures [16, p. 358-9].

Lemma 3.



We can also use (it-deit) to extend a line of identity into a cut. Note that Lemma 4 follows from (it-deit2) when S is the counit.

Lemma 4.



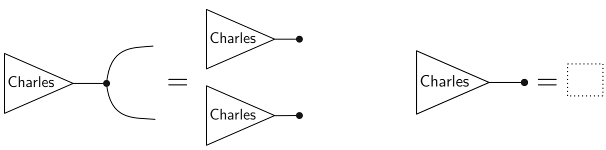
Both Lemma 3 and Lemma 4 show how (it-deit) captures both iteration for a line of identity and for a relation/partial graph.

6 Diagrammatic Reasoning in Action

Example 6. We return to our running example and conclude with a complete diagrammatic derivation of the judgement

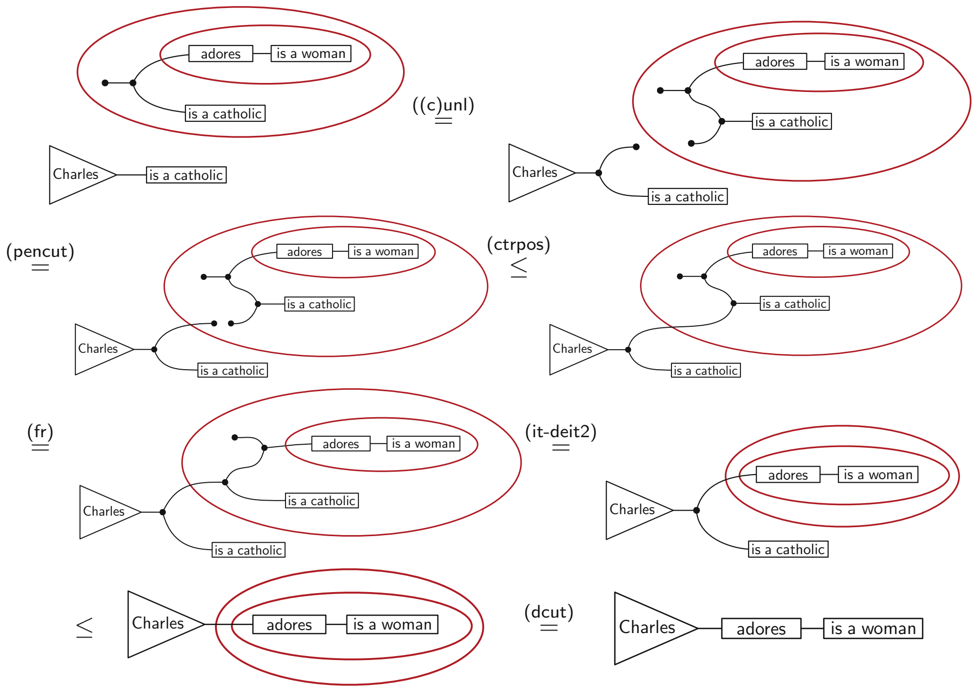
$$\frac{\mathsf{isacatholic}(\mathsf{Charles}) \wedge \forall x.\ \mathsf{isacatholic}(x) \rightarrow \exists y.\ \mathsf{adores}(x, y) \wedge \mathsf{isawoman}(y)}{\exists y.\ \mathsf{adores}(\mathsf{Charles}, y) \wedge \mathsf{isawoman}(y)}.$$

In the derivation we use the triangle notation[9] to denote a *constant* symbol of the logic, that is, a relation that is guaranteed to have singleton models. This (and similarly *function* symbols) are easily encoded in the graphical formalism and do not add expressivity; it suffices to assert that:



9 Borrowed from the notation for states in categorical quantum mechanics [8].

We proceed with the derivation below:



7 Conclusion

Peirce's EGs arose out of his continued study of the algebra of relations and his concern for developing an efficient graphical notation. Seen through contemporary string diagrams, Peirce's lines of identity obey the rules of special Frobenius algebras, while Peirce's inference rules for lines of identity are the axioms of cartesian bicategories of relations. Moreover, diagrammatic reasoning can be extended to cover negation in a straightforward manner.

The category theoretic account of EGs presented here yields a diagrammatic calculus that is as expressive as first-order logic. We summarize the specific benefits of the graphical logical language when we say that it is *compositional*. The syntax is string diagrams, the semantics is Rel_X, and models structure-preserving maps. In particular sub-formulas (sub-diagrams) have their own meaning as relations, with the meaning of the entire formula (diagram) obtained by composing these. In these respects our approach follows Peirce's original intentions.

In regards to Peirce scholarship, our presentation suggests new means of interpreting the transformation rules in EGs. Following Peirce, this presentation showcases contraposition as the governing duality between positive and negative contexts on the sheet. We also clarify the rule of iteration. Robert's presentation [17, pp. 57-8] includes important but fairly ad hoc clauses to the Beta rules of iteration. These clauses, as well as Burch's more recent developments in [6], are unified here with a single principle of iteration. Finally, situating Peirce's EGs in contemporary category theory [2,3] allows for further study and comparisons.

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2 Article 2 - Residuation in Existential Graphs

II

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Residuation in Existential Graphs ⋆

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Abstract. Residuation has become an important concept in the study of algebraic structures and algebraic logic. Relation algebras, for example, are residuated Boolean algebras and residuation is now recognized as a key feature of substructural logics. Early work on residuation can be traced back to studies in the logic of relations by De Morgan, Peirce and Schröder. We know now that Peirce studied residuation enough to have listed equivalent forms that residuals may take and to have given a method for arriving at the different permutations. Here, we present for the first time a graphical treatment of residuation in Peirce's Beta part of Existential Graphs (EGs). Residuation is captured by pairing the ordinary transformations of rules of EGs—in particular those concerning the cuts—with simple topological deformations of lines of identity. We demonstrate the effectiveness and elegance of the graphical presentation with several examples. While there might have been speculation as to whether Peirce recognized the importance of residuation in his later work, or whether residuation in fact appears in his work on EGs, we can now put the matter to rest. We cite passages where Peirce emphasizes the importance of residuation and give examples of graphs Peirce drew of residuals. We conclude that EGs are an effective means of enlightening this concept.

Keywords: Residuation · Existential Graphs · Charles Peirce · Cuts · Lines of Identity.

1 Introduction

As discussed by Pratt [19] and Maddux [13], De Morgan described the residuation laws in the form of Theorem K in 1860 [3, 6]. The resulting equivalences state that given any three relations—a, b, and c—and well known relation operations—relational composition (;), complement ($\bar{\ }$), converse ($\breve{\ }$) and relational containment/inclusion ($\sqsubseteq$)—the following are equivalently defined:

$$a;b \sqsubseteq c \iff \breve{a};\bar{c} \sqsubseteq \bar{b} \iff \bar{c};\breve{b} \sqsubseteq \bar{a} \ . \tag{1}$$

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Residuation can broadly be thought of as an inverse operation, much like how division is inverse to multiplication and subtraction to addition. In the context of relations discussed here, residuation gives a remainder when relational composition is denied or converted (as in the equivalences above).

Another example is found in what is called the residuation property (RES), which shows how residuation also acts like implication:

$$\text{(RES)} \qquad p \wedge q \sqsubseteq r \iff q \sqsubseteq p \to r.$$

This is related to the deduction theorem (and to currying), and the property plays an important role in characterizations of a range of implications from classical to intuitionistic (Heyting algebra) implication. Residuation is now recognized as a key property of substructural logics [14] and, following Lambek, of categorical grammar [7, 8].

Given the significance that has been placed on residuation in the study of relations since, it is perhaps curious that Charles S. Peirce, who studied De Morgan's work closely and who went on to make significant contributions to the algebra of relations (e.g. his 'dual' and 'general' algebras of relatives), seems to have placed little emphasis on it in his later work. Maddux even describes the omission of residuation (De Morgan's Theorem K) in Peirce's later work as "puzzling" [13, p. 435].

We now know that Peirce did emphasize the residuation property above in his characterization of propositional logic as occurs in his 1880 algebra of the copula paper, §4, which presents a calculus of the consequence relation [11]. In it, two meanings of the copula $-\!\!<$ (or $\prec$) are delineated by using two signs: (1) the consequence relation (Peirce's sign of illation) $\Rightarrow$; and (2) the material implication $\to$. An expression of the form $x \Rightarrow y$ is called a *sequent*, according to the proof-theoretic terminology. Then the calculus of the copula (a Boolean algebra) consists of the following axiom and rules:

1. Identity: (Id) $x \Rightarrow x$
2. Peirce's Rule:
$$\frac{x \wedge y \Rightarrow z}{x \Rightarrow y \to z}\,\text{(PR)}$$
3. Rule of Transitivity:
$$\frac{x \Rightarrow y \quad y \Rightarrow z}{x \Rightarrow z}\,\text{(Tr)}$$

The double line in (PR) means that the lower sequent can be derived from the upper sequent and vice versa. The second rule, here renamed as *Peirce's Rule*, is probably the first formulation of the *law of residuation*: that the material implication is a right residual of conjunction.

We also now know that Peirce studied residuation enough to have listed equivalent forms residuals may take and to have given a method for arriving at the different permutations [CP 4.343] [19]. While this helps confirm Peirce's awareness of residuation and to assuage some doubts about the scope of his insights, it does not help explain why Peirce seems to have placed much less emphasis in his later work on a concept whose importance he had—and when looking back on it perhaps should have—so emphasized.

Pursuing Peirce's potential connection to residuation from another direction, it is equally curious that Peirce makes no direct mention of his earlier algebraic studies of residuation in his later presentation of Existential Graphs (EGs). Given that Peirce often cites EGs as the culmination of his earlier work on relations (with his algebraic studies of residuation, no doubt, as one), along with his insistence that EGs should be the "logic of the future" [18], it would be problematic if such a concept was left without representation.

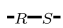
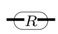
Of course this is not the case. We remedy the seeming omission here by showing how residuation is naturally presented in EGs. Given the relatively sparse syntax of the graphs and that residuation can easily be represented without any changes to the syntax or transformation (i.e., inference) rules, it would seem rather that Peirce's supposed omission might be due to a belief that the other rules of EGs suffice to enlighten the concept. The presentation of residuation in EGs given here is the first of its kind—in particular, the first for its quantificational Beta extension that includes lines of identity.

This paper presents residuation in the context of relations and relational operations and sets aside for the time the functional characterization in terms of Galois connections.[3] The aim is to help situate Peirce's work in the development of residuation (following, in particular, the work of Maddux and Pratt cited above), to present the beginnings of a graphical presentation of residuation, and to address the connection between residuation and Peirce's work on Existential Graphs.

2 Beta Graphs and Relational Operations

We begin with a short introduction to EGs and the diagrammatic presentation of the operations needed to represent residuation in a logic of relations. We assume basic familiarity with the interpretation and transformation rules of EGs. Helpful introductions to Peirce's EGs can be found in [18, 20]. The richer algebraic/categorical framework upon which this work relies can be found in [5]. We save an extended treatment of residuation in the latter context for subsequent work. Here, we stick rather to the perspective from relation algebra, leveraging the more traditional notation for relation algebras found for example in [4, 21, 1]. Though it predates Peirce's EGs, a helpful introduction along these lines is given by Peirce in his "Note B" [15]. What follows can be seen as a graphical treatment *á la* the later EGs of the algebraic work given in this note.

A general binary relation is scribed on the sheet with an ingoing 'wire' serving as a placeholder for the domain of the relation and an outgoing wire signalling the codomain. These wires represent the collection of individuals who might satisfy/stand in the relation presented.

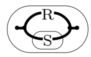
 (a)  (b)  (c) (2)

[3] On the history of residuation (adjunctions) and its relation to Galois connections, see [2].

Operations from relation algebra and their corresponding EGs are given in (2), where (a) is a general relation R, (b) is relational composition $R;S$, and (c) is complement $\bar{R}$. Given the -loves- and -benefits- relations we can for example express 'lovers of benefactors' as -loves—benefits- and 'lovers of non-benefactors' as -loves-(benefits)-.

Relations have a definite order such that reversing the domain and codomain gives a different relation. Changing the domain and codomain of the "x loves y" relation, for example, forms the converse relation "y is loved by x".[4] In [5] it is shown that lines of identity in fact obey the equations of a special Frobenius algebra. Graphically, this involves the addition of 'cups' and 'caps' (,) that serve as markers for keeping explicit track of the bending of wires and the respective domain and codomain for each relation. For example, EGs of (a) a relation R, (b) its converse $\breve{R}$, and (c) relation inclusion/containment $R \sqsubseteq S$ are given below in (3).

(a)


(b)


(c) (3)

The addition of cups and caps are important since the initial presentations of residuation in De Morgan's and Peirce's works depend on tracking the converse (and other) relations. Importantly, whereas the single cut represents complement/negation, a nested cut represents inclusion/containment relation (Peirce's "scroll"). The use of 'cups' and 'caps' as endcaps in this context is to show that the domain of R is preserved in the domain of S and that the codomain of R is preserved in the codomain of S.

More discussion on relation algebras can be found in [4]. For a detailed translation of the EG syntax into first-order logic, relation algebras, and a discussion of the transformation rules in this context, see [5]. With relational composition, complement, converse, and inclusion expressible graphically in the syntax, we have the relational operations needed to present residuation in EGs.

3 Residuation in Existential Graphs

Given relational composition (;), left and right residuals take the form of division.

$$a \sqsubseteq c/b \qquad \iff \qquad a;b \sqsubseteq c \qquad \iff \qquad b \sqsubseteq a \backslash c. \tag{4}$$

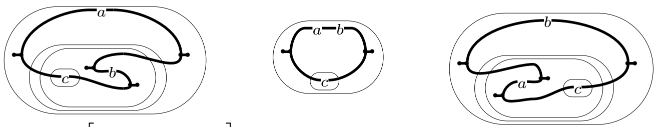
Peirce enumerated several equivalent forms residuals may take (Schröder lists many more in [22]). The list depends on which operations are taken as primitive. We begin by adding to (4) the residuals in terms of complement ($^{-}$) and converse ($\breve{\ }$) relations. This allows us to fairly directly convert the residuation laws into

[4] While Peirce emphasizes this ordering in his algebraic work, he says little about reading such an ordering off the graphs. The few such places on the ordering of the 'hooks' around the relation terms appear in the early drafts of EGs from late 1886, in which Peirce notices how the connections of the lines to the relations should be read "clockwise" or "counterclockwise" (their converses) "beginning at the left/right" of the relation term; see [18, pp.220,263,295,302,303]. A further advantage of the notation in [5] is that the ordering is always explicit.

a form amenable to the syntax of the EGs. We also use (†) to represent what Peirce calls *relative sum*, which is the dual to relative composition.

$$
\begin{array}{lclcll}
a \sqsubseteq c/b & \iff & a;b \sqsubseteq c & \iff & b \sqsubseteq a\backslash c & (4)\\
a \sqsubseteq (\bar{c};\breve{b})^{-} & \iff & a;b \sqsubseteq c & \iff & b \sqsubseteq (\breve{a};\bar{c})^{-} & (5)\\
a \sqsubseteq c \dagger \breve{\bar{b}} & \iff & a;b \sqsubseteq c & \iff & b \sqsubseteq \breve{\bar{a}} \dagger c & (6)\\
\ldots & & \ldots & & \ldots &
\end{array}
$$

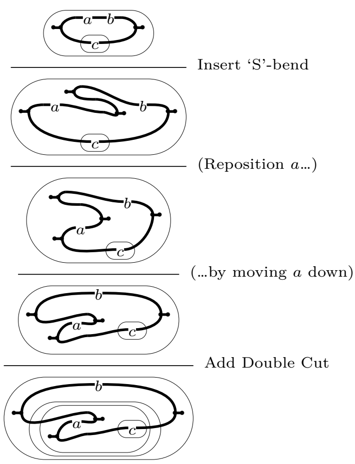
Example 1 (Residuation laws in EGs). Let us begin with EGs of (4) that correspond to the row of equations in (5).



$a \sqsubseteq c/b \;\left[= \; a \sqsubseteq (\bar{c};\breve{b})^{-}\right]$ $\qquad a;b \sqsubseteq c \qquad$ $b \sqsubseteq a\backslash c \;\left[= \; b \sqsubseteq (\breve{a};\bar{c})^{-}\right]$

Relational composition (;) is represented in the graphs by connecting, via a line of identity, the respective outgoing and ingoing wire for the relations a and b. Relational inclusion/containment is captured by nested cuts, i.e. Peirce's scroll, and complement and converse are likewise represented as discussed in Section 2.

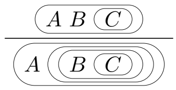
Only two graphical transformations are needed to represent the residuals in the side columns of (5), depicted by the left and right graphs above. One transformation is to add an S- or Z-shaped bend to a line of identity (cf. 'cups' and 'caps' producing converses). The other is to add a double cut around subgraphs. This has the effect of changing the consequent in the newly directed implication (see Remark 2). Both are straightforward transformation rules in EGs.

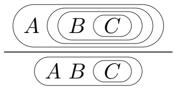


The right residual is derived as follows. The transformation to the final graph begins by inserting an S-shaped bend in the line of identity (employing 'cups' and 'caps') between the composition of a and b. We then grab a and pull it down to the left. Finally, add a double cut around the bottom subgraph to form the consequent of the new implication. Notice that in the process we switch the endcap that preserves the codomain of the relation. This topographical deformation yields the converse on a. Performing similar operations (now with a Z-shaped bend) on b yields the graph on the left in Ex. 1. The simple transformations of bending wires to reposition subgraph relations and adding/removing a double cut are sufficient to capture the long list of equivalences given by Peirce in [CP 3.341].

Remark 1. Peirce's Rule of 1880 (residuation) is of particular importance, as the full distributivity laws can be deduced from it and the standard lattice rules [9]. The rules of Modus Ponens, LEM and *Ex Falso* are deducible similarly.

Remark 2. Peirce's Rule of 1880 has a particularly clear representation in EGs. Graphical representation of logical constants brings out the relation between logical connectives as *adjunctions*. The following rules are provable in EGs (Alpha):

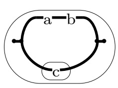

(RG1)


(RG2)

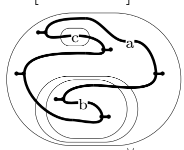
The rule (RG1) is immediate from the observation that it is permissible to add a double cut, namely the scroll with blank areas. The other direction (RG2) follows from the observation that permits removing that scroll. Hence Peirce's Rule is justified by the observational element that entitles the addition/removal of scrolls with blank areas. Indeed according to Peirce, "every copula is so closely connected with a conjunction that the notation should show the connection" [18, p. 428], concluding that "copulas are nothing but conjunctions" [18, p. 426].

Remark 3. On one loose manuscript leaf (RS 104, c.1903) Peirce formulated residuation as the pair of $A\,B \prec C$ and $A \prec C \psi \overline{B}$. Here his algebra of logic notations ψ (aggregate) and $\overline{}$ (vinculum) correspond to logical disjunction and negation, respectively. Above and below these two consequence relations he wrote in the language of EGs: "$AB\ \ C$" and "A C B", respectively. Taking the blank to mean the derivation along the consequence relation, one can move between these two graphs solely by an addition (top-down) or erasure (bottom up) of a blank scroll.

Example 2. We give Peirce's rule for the cyclic permutation of terms [19] in graphs. In Peirce's words the rule is: "the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted" [CP 3.341].



$\Longleftrightarrow$



$a; b \sqsubseteq c$ $\qquad$ $\breve{\bar{c}}; a \sqsubseteq \breve{\bar{b}}$

Again the equivalent expression is captured by the transformation rules in EGs—in this case rotating the subgraphs clockwise by pulling b down and around now also raising ⓒ around and up.

It is worth comparing the topological transformations used in the derivations above with an equivalent derivation using first-order logic.[5] When these moves

[5] One such derivation is [21, p. 42] which we forego here due to space limitations. Schmidt speculates that equational reasoning using predicate logic results in deriva-

are put into graphical notation many steps are found to be either roundabout or to have little content, such as to introduce, label, re-label, and eliminate excess variables.

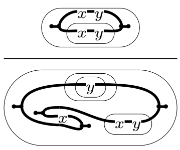
4 Two Further Examples of Residuation in EGs

Two further examples highlight the efficiencies gained by the graphical treatment of residuation for derivations and for further thought.

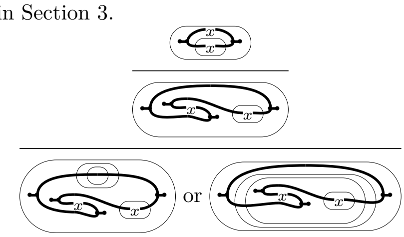
It is known that in axiomizations of relation algebras, such as by [23], Theorem K can replace the axioms governing the rules between involution and distribution over Boolean join [12, p. 25]. One version of the key axiom is $\breve{x};\overline{x;y} \sqsubseteq \bar{y}$.

In “Note B” [15] Peirce also presents the following important equations: $\mathbb{I} \sqsubseteq \overline{\breve{x};\bar{y}}$ and $x;\bar{\breve{x}} \sqsubseteq \bar{\mathbb{I}}$, where $\mathbb{I}$ is the identity relation for relational composition. These equations correspond to the *linear negation operation* in linear logic [19].

Example 3. We show that these equations follow from simple identities on x and the topographical moves described in Section 3.



Graphical transformation of $\breve{x};\overline{x;y} \sqsubseteq \bar{y}$

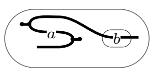


Graphical transformation of $\mathbb{I} \sqsubseteq \overline{\breve{x};\bar{y}}$ or $x;\bar{\breve{x}} \sqsubseteq \bar{\mathbb{I}}$

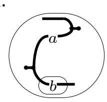
The equations for the right residual, $a\backslash b$, are given by the following [19]:

$$a\backslash b \quad = \quad \bar{\breve{a}} \dagger b \quad = \quad (\breve{a};\bar{b})^{-} \tag{7}$$

Example 4. The rightmost equation is expressed by the EG below left. We clearly see that the residual has the form of an implication.



$(\breve{a};\bar{b})^{-}$



5 Concluding Remarks

This last example brings us back to a common place where Peirce draws specific attention to residuation. He singles out the graph on the right, equivalent to the one on the left and which has a very nice vertical symmetry, to represent the key feature of necessary reasoning. In this sense, residuation is a general logical principle that has a *maximum level of abstractness*. Being maximally abstract means that such principles add nothing to the premises of the inference which they govern (NEM 4:175, 1898).

tions that are six times longer than the corresponding algebraic handling of relations [p. xi].

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3 Article 3 - C.S. Peirce's Early Developments in Linear Logic

III

N. Haydon. C.S. Peirce's early developments in linear logic. 2023

C.S Peirce's Early Developments in Linear Logic

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Abstract. Early developments of linear logic can be traced back to Barr's *-autonomous categories and Lambek's bilinear logic. We show here that C.S. Peirce's early work on the logic of relations should be placed within this tradition. Peirce understood linear distributivity and the linear negation laws, understood linear implication in the form of residuation, and emphasized the dialogical nature of the linear connectives. Much of this can be found in Peirce's early algebraic work on the study of relations going back as early as the 1880s. Peirce eventually went on to develop a diagrammatic calculus — what he called the Existential Graphs — he thought better suited for the purpose. We go on to show graphs corresponding to these notions and confirm that many of Peirce's later studies in the graphs employ these concepts. The result is a dramatic revision of our understanding of the Existential Graphs, as well as Peirce's place in the logic tradition.

Keywords: Linear Logic · Bilinear Logic · Cyclic Linear Logic · Relation Algebra · Existential Graphs · Charles Peirce · History of Logic

Linear logic is one of the most general logics we have. De Morgan duality, often associated with classical logic, is restored in the linear case, while intuitionistic logic, associated in turn with constructive mathematics and the lambda calculus, can be recovered as an important fragment [51]. Further, the ability of linear logic to keep track of resources makes it an attractive logic for explorations in computation and proof theory.

Relations and relation algebra provide a significant setting for linear logic. Relation algebra has a history going back to Tarski's paper on the subject [75] — a paper that, and we come back to this point below, is really a return to the work of C.S. Peirce — and has been developed in many contexts, including the foundation of mathematics [76] and programming theory [68,18,9], and continues in the present day in relational mathematics [69,67], relational algebraic theories [15,29], and in other (relational) categorical settings like allegories [30].

The explicit connection between linear logic and the relational model goes back at least to Pratt [61,62,63], who noticed — and again, it should be noted, after returning to the work of C.S. Peirce — that the combined complement-converse relation is analogous to linear negation and that residuation is analogous to linear implication. The relational model is also given as a key example in Lambek's presentation of bilinear logic [42] and is the motivating example behind Cockett et al.'s introduction of linear bicategories [24]. In such examples one needs to express the dual of relational composition, which importantly serves as the par'd ($\parr$) context in the linear case. The connection between relation algebra (RA) and linear logic (LL) is summarized by Desharnais et al. in Figure 1 [27]. Of note, the additives correspond to the Boolean or non-relative terms, while

LL	RA	LL	RA	LL	RA	LL	RA
1	$\mathbb{I}$	$r^{\perp}$	$\breve{\bar{r}}$	$r \otimes s$	$r ; s$	$r \oplus s$	$r \sqcup s$
$\perp$	$\breve{\bar{\mathbb{I}}}$	$!r$	$!r$	$r \parr s$	$r \dagger s$	$r \& s$	$r \sqcap s$
0	$\perp\!\!\!\perp$	$?r$	$?r$	$r \multimap s$	$r \backslash s$	$r \circ\!\!- s$	r / s
$\top$	$\top\!\!\!\top$						

Fig. 1: Correspondence between linear connectives and relation algebra.

the multiplicatives correspond to the relative terms.[1] The multiplicative units include the identity relation ($\mathbb{I}$) as the unit of relational composition and its linear dual ($\mathbb{I}^{\perp}$, which is the complement-converse of $\mathbb{I}$). The latter corresponds to the *difference* or *diversity* relation, which we will come back to below. The additive units for union ($\sqcup$) and intersection ($\sqcap$) are $\perp\!\!\!\perp$ and $\top\!\!\!\top$, respectively.

A further key feature about the relational model is that it provides a natural setting for non-commutative variants of linear logic [27]. Two points are worth highlighting about this setting. The first is that the sequentiality of relational composition is naturally non-commutative. Swapping the order of composed relations in general leads to a different relation, and the resulting left and right residuals — serving as they do as linear implications — have this directionality and non-commutativity built in. Certain other relational operations, like union and intersection, are commutative. The result is that relation algebra offers a natural setting for combining both commutative and non-commutative operations. The second point is that while sequential composition is generally non-commutative, certain cyclic permutations within inclusions are allowed so long as an overall ordering is preserved. This rule is the significant feature behind what is called *cyclic linear logic* and early examples include the presentation by Yetter [78] and Abrusci [2,3]. The relational model also provides a natural setting for studying this cyclicity.

While precursors of these views can be found earlier, we note that these directions only began to see systematic development in the 1990s. The non-commutative setting, often taken to be more involved (Girard even calling it a 'far-west' [31]) is still arguably finding the adherents and developments it deserves.

[1] Unfortunately, almost all of the sources cited here use different notations and we continue to do so. We tend to follow the categorically-minded presentations of Lambek and Cockett et al., but are still left choosing the relational symbols. Where appropriate we follow Peirce, otherwise we follow Schmidt in [67].

It is remarkable then that C.S. Peirce's studies in the logic of relations as far back as the 1880s led him to understand and present the essential rules for (cyclic) bilinear logic. This includes recognizing the importance of the dual to relational composition, presenting linear distributivity and the linear negation laws, and also recognizing the further cyclicity condition. The subject of this paper is to situate Peirce's early developments and studies in cyclic bilinear logic.

We note upfront that the major developments in this direction are already present in Peirce's 'Note B' [57] from 1883. Proper recognition of this fact requires a dramatic revision of Peirce's subsequent understanding of the logic of relations. Peirce later went on to develop a graphical calculus — what he termed the *Existential Graphs* — that generalized the theory of relations given in 'Note B'. We similarly argue, and this paper goes on to demonstrate the point, for a dramatic revision of Peirce's subsequent graphical calculus.

There are numerous further ramifications for Peirce studies and for his place in the logic tradition. We finish this introduction section by listing three broad themes:

- Historically, this work places the developments of bilinear logic — and so also key aspects of linear logic — significantly earlier than recognized precursors to the view,
- Peirce's early insights into bilinear logic occurred prior to the *Existential Graphs* (EGs) and our readings of the graphs must likewise be revised: we cite, as key examples, an awareness in the graphs of the dual of relational composition, of linear distributivity and linear negation, as well as renewed significance in regarding Peirce's sign of the 'double cut' or 'scroll' as an involution,
- Finally, in terms of contemporary graphical developments, we note that the corresponding EGs still compete favourably with modern graphical presentations of these rules, such as Cockett et al.'s circuit diagrams.

Subsequent sections contain lengthier discussions of these points.

While we give the historical and logical significance of Peirce's developments here, we note that a modern algebraic formulation has been given — with a sound and complete axiomization for full first order-logic — and can be found in [14]. The modern account does not discuss the details of Peirce's work and the historical importance, which is our primary aim here. Neither does that account compare these developments to the cyclic bilinear case, a connection we find important to stress as we find it to be perhaps the closest extant theory to Peirce's own. All of this is to say that Peirce's work continues to reveal itself as remarkably ahead of its time.

As an overview, in Section 1 we give a background of bilinear logic and the connection to Peirce's Existential Graphs. Section 2 introduces the graphical syntax and the corresponding linear connectives in the graphs. Section 3 focuses more specifically on linear implication. Discussion of linear implication shows that Peirce's 'scroll' is an efficient presentation of the equivalences that characterize the bilinear operations. In Section 4 we compare the resulting graphical presentation with Cockett et al.'s circuit diagrams in the literature. The 'scroll' can be seen as providing — as a type of dual presentation — the 'inner workings' of the nodes in a circuit diagram. Some of Peirce's derivations of the linear laws above were motivated by studies of intuitionistic negation, as well as a concern for the refutation clauses of the linear connectives. In Section 5 we give an example of the refutation clauses of the connectives and how some of the linear equivalences are directly captured in the graphs. Finally, in Section 6 we give a few examples of resource sensitivity and the linear modalities.

The work here is just the beginning of this undertaking. Following Tarski and Pratt, we hope this return to Peirce renews interest in this direction of work. The last paragraph of Tarski's seminal paper on relation algebra is still as fitting today as 80 years ago when it was written:

> The aim of this paper has been, not so much to present new results, as to awaken interest in a certain neglected logical theory, and to formulate some new problems concerning this theory. I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it. [75, p. 89]

1 Bilinear Logic: History, Contemporary Context, and Relation to EGs

While the first significant early presentation of linear logic is attributed to Girard's paper from 1987 [32], other forerunners to linear logic can be found in *-autonomous categories [6,7] and the Lambek Calculus [40,41]. *-autonomous categories arose from a categorical study of an involution that behaves like negation does in the classical case. *-autonomy corresponds to the multiplicative fragment of linear logic, which is generally taken to be the most novel and interesting in the linear case. The Lambek calculus is the (non-commutative) logic of linear implication. It is a substructural logic with only residuation.

The connection between *-autonomy and linear logic was not known immediately, as the connection between linear distributivity and linear logic took time to spell out. Cockett and Seely's work on linearly distributive categories (LDCs) in [25] (and again with the addition of Blute and Trimble in [12]) is perhaps the first systematic treatment. The intuitive idea is that linear distributivity mediates between the two multiplicative connectives by giving a rule of inference that preserve linear sequence and resources, i.e. it does not duplicate a premise.

In addition to linear distributivity, the linear negation laws are perhaps just as essential to the characterization of linear logic. The intuitive idea here is that these rules enforce that every introduction of a logical term contains a positive and negative context that keeps track of how the term can be 'used up' as a resource is used up.

While one could take linear distributivity as the more primitive concept, as in Cockett and Seely's early work cited above, linear negation should arguably be taken to be just as logically essential. One reason for this is that while linearly distributive categories can be discussed without linear negation laws, it is known that the latter can be added as a conservative extension [12]. The intuitive (again) idea here is that categorical composition can be broken down into a two step process of linear distributivity followed by a linear negation, so adding linear negation when composition is already presumed adds no further expressivity. A second reason comes from the move to proof theory, where the linear negation laws find renewed significance. This includes the importance such rules have found within deep inference [28] and the calculus of structures [34]. In short, linear distributivity and the linear negations laws are increasingly seen as key ingredients to linear logic.

Following Lambek in [42], we give a summary of the key morphisms characteristic of bilinear logic. We show a first key set of rules in Figure 2. These correspond to the key morphisms of linearly distributive categories with negation and to *-autonomous categories [12]. The rules include (i) associativity and unit

$$
\begin{array}{llcl}
\text{(i)} & (R\otimes S)\otimes T \iff R\otimes(S\otimes T) & & (R\oplus S)\oplus T \iff R\oplus(S\oplus T) \\
 & \mathbb{I}\otimes R \iff R \iff R\otimes\mathbb{I} & & \mathbb{d}\oplus R \iff R \iff R\oplus\mathbb{d} \\
\text{(ii)} & R\otimes(S\oplus T) \implies (R\otimes S)\oplus T & & (R\oplus S)\otimes T \implies R\oplus(S\otimes T) \\
\text{(iii)} & R^{\perp}\otimes R \implies \mathbb{d} & & \mathbb{I} \implies R\oplus R^{\perp} \\
 & R\otimes R^{\perp} \implies \mathbb{d} & & \mathbb{I} \implies R^{\perp}\oplus R
\end{array}
$$

Fig. 2: Key Rules for LDC with Negation

laws for both tensor and co-tensor, (ii) the left and right linear distributivity laws, and (iii) the additional linear negation laws.

We take the additional linear negation laws to be of prime importance here. We add that the aim — or at least our aim in doing so — is not so much to emphasize negation but to emphasize linear implication. The linear negation laws can be thought of as a linear implication introduction rule along with a (linear) contrapositive elimination rule that leads into the dualizing object.

The resulting logical theory, emphasizing as it does both linear implication and its dual, goes by the name *bilinear logic*. The key rules for the two linear implications of bilinear logic are given in Figure 3. Cockett and Seely return to linear negation and these bilinear operations — and now preferably so — in later generalizations of linearly distributive categories into linear bicategories in [24] and for developing the corresponding proof theory in [26]. Lambek similarly comes back to linear negation laws in a later axiomization of the bilinear case in the system BL2 [42].

To these Lambek discusses further rules for cyclic linear logic, which characterizes his system BL3 [42]. The additional rules are given in Figure 4.

$$R \otimes S \implies T \quad \text{iff} \quad R \implies T/S \quad \text{iff} \quad S \implies R\backslash T$$

and

$$T \implies R \oplus S \quad \text{iff} \quad T \oslash S \implies R \quad \text{iff} \quad R \obslash T \implies S$$

Fig. 3: Key Rules for Bilinear Operations

$$A \otimes B \implies \mathbb{d} \quad \text{iff} \quad B \otimes A \implies \mathbb{d}$$

and

$$\mathbb{I} \implies A \oplus B \quad \text{iff} \quad \mathbb{I} \implies B \oplus A$$

Fig. 4: Key Rules for Cyclic Conditions

The rules given in Figures 2-4 will be our main concern here. We refer throughout to this key set of rules as *bilinear logic* and then stress the added *cyclicity* condition when called upon.

We note this system above crops up in close connection to quantales [66], polycategorical composition [74], in characterizing adjunctions [72], in the evolution of quantum systems [13,73], and in linguistics [21]. We believe a strong case can be made that this forms the basis of an extremely important logical system.

The relational setting is often cited as a key example of bilinear logic.[2] In this setting the tensor is relational composition and the dual to relational composition serves as the par or cotensor. Suggestively, we have already written the unit of the tensor above as $\mathbb{I}$ for the identity relation and the unit of the cotensor as $\mathbb{d}$ for the diversity relation. Linear distributivity and the linear negation laws can then be clearly stated via relational inclusion, as can the linear implications in the form of residuation.

Given the emphasis on linear distributivity and linear negation has only seen consistent development since the 1990s, it is (again) remarkable that C.S. Peirce presented all of these rules in his early work on the logic of relations. The key presentation is found in Peirce's 'Note B' from 1883 [57], which also served as the principle inspiration behind Tarski's presentation of relation algebra [75] (for historical context, see [46]). But while Pierce stressed the importance of both linear distributivity and linear negation — going so far to state that they are "highly important" and "so constantly used that hardly anything can be done without them" [57, p. 192 & 190, respectively] — these rules were not emphasized in Tarski's presentation and much of the work that followed. Who knows how the history of logic may have differed without this omission.

Around the same time as the studies around 'Note B,' Peirce expressed linear (and intuitionistic) intuitions about the linear connectives, including game-style semantics of the connectives, and studied linear implication. We show these, almost all for the first time, within the linear context.

[2] See Section 12 in Lambek's [42] and Section 1 in [24].

Peirce continued his work on the logic and algebra of relations for almost thirty years after his original presentation in 'Note B.' Peirce was dissatisfied with his presentation in 'Note B' as it only dealt with binary relations, and went on to develop his *Existential Graphs* (EGs) as an alternative for a calculus of relations. EGs employ a graphical syntax akin to contemporary string diagrams in category theory that emphasize triadic relations over binary relations [36]. EGs employ an additional sign — a circle called the 'cut' — that can surround a graph and that serves as a complement or negation operation. A question is whether Peirce continued to discuss and present these linear rules in his later work on the graphs.

A preliminary response would answer in the negative. Peirce does not mention the linear rules in his most well known descriptions of the inference rules for the Existential Graphs. This is also the predominant story told in the literature. Robert's original presentation on EGs focuses on the classical case of first-order logic with the 'cut' as negation [64], and almost all of the subsequent literature has followed suit (such as in [79,19,70,45] and the overview in [54]). Peirce's presentation of *modus ponens*, which is one of the most common examples Peirce discusses in EGs, also begins by duplicating a premise — and in so doing seemingly goes against the linear rules — and then requires an extra erasure (i.e. projection) to end up with the required result. Finally, many of Peirce's own descriptions take negation or complement as primitive and leave out the more nuanced, linear case.

We show here, however, that this is not the case. The traditional reading of the EGs with 'cut-as-negation' is misguided. Peirce continued to insist on the importance of the linear rules after 'Note B' and continued to develop his understanding of the linear connectives. This becomes clear when we look at Peirce's later EGs, which contain — and even predominately so — linear rules in the form of linear implication and its (linear) dual.

Our judgment of Peirce's later studies in EGs is that, analogous to the importance one might place on the implicational fragment in the classical case, Peirce appears to have made considered effort to emphasize and develop the *linear* implicational fragment. We present for the first time these linear rules in Peirce's later graphs and show corresponding graphs drawn by Peirce. All of this should make us see Peirce's work and his Existential Graphs in a new light.

Several contextual remarks are in order. Much as implication can be seen as the most primitive inferential connective, Peirce insisted on the importance of the 'scroll' — imagine two circles or 'cuts' nested inside the other — as the most primitive connective. We note that this emphasis is often lost in the single 'cut-as-negation' story. In fact, if one only employs the inference rules and graph rewrites as Peirce describes, then one can seemingly never write a single cut! Adding to this perspective here, we return the 'scroll' to its position of prime importance.

Some work has tried to extend Peirce's EGs with 'cut-as-negation' towards the intuitionistic case and towards *-autonomy and LDCs. Extending EGs to the intutionistic case can be found in [48,52]. The work presented here flips this direction around, as the intuitionistic case can be seen as being recovered from the linear one and not as having to extend the classical one. Residuation, as with Peirce's understanding of the sequentiality of negation, already captures the intuitionistic case. We save this development of the intuitionsitic case for future work, but the bilinear case presented here is the groundwork for this approach.

A second noteworthy approach by Brady and Trimble [16,17], and one motivated by categorical considerations, suggests that Peirce's inference rules can be construed by linear strengths (like LDCs) and by an involution along the lines of *-autonomy. The work here affirms this direction and shows that Pierce himself had a much more extensive knowledge of these directions than previously known. We elaborate Peirce's historical developments and connect these to areas in contemporary logic like LDCs and to circuit diagrams.

In terms of other connections to linear logic, we give an example from Peirce of how the 'scroll' mediates between the proof and refutation cases of the connectives. As another example, we show how the linear equivalences are straightforwardly captured — just like the De Morgan duality in propositional EGs — in the graphs.

Finally, we stress that while Peirce appears to have foreseen much of the structure that is cyclic bilinear logic, there are aspects of full linear logic that fall outside this scope. For example, while Peirce seems to have understood some of the key topological intuitions behind the !-exponential (see Section 6), we as of yet know of no place where Peirce draws attention to what would be the significant developments that follow from the linear modalities and to other intuitionistic developments that have been so significant over the last 100+ years. In the end much further study is needed.

Neither do we want to suggest that Peirce understood further aspects of category theory or that motivate linearly distributive categories or the theory of adjunctions. Peirce was concerned with compositionality and took it to be one of the main puzzles EGs helped resolve.[3] Given that category theory is so concerned with compositionality, and that other approaches to category theory, such as allegories, rely so much on relational operations, we find that Peirce's view is surprisingly modern and prescient. A good example is the symmetric quotient and the straightness condition given in Section 3, which do not appear in modern literature until the early 1990s. Our takeaway of Peirce's studies is that it offers a good reminder of where an interest in relational operations, and a concern for both *better syntax* and a better understanding of *topological features*, can lead.

2 Linear Distributivity and Linear Negation

*-autonomous categories provide the semantics for the multiplicative fragment of linear logic. Cockett and Seely showed that *-autonomy can be characterized

[3] See 'Recent Developments of Existential Graphs and their Consequences for Logic' [56, Selection 45].

by linear distributivity and linear negation laws [25,12]. Peirce clearly stated both in his original presentation of a logic of relations in 'Note B' [57] and goes on to give numerous derivations that we now recognize as corresponding to the Lambek Calculus and to the logic of residuation. We discuss this connection to linear implication in the next section and in this section introduce rules for linear distributivity and the linear negation laws. In this section we follow closely the order of Peirce's presentation in 'Note B.' The appropriate conclusion is that Peirce's studies in the logic of relations led him, as far back as 1883, onto the rules that correspond to the multiplicative fragment of linear logic.

We also use this section to introduce the graphical syntax. The syntax is the same as Peirce's Existential Graphs, but the exposition will emphasize relation algebraic operations and the linear laws. This follows the work found in [36] and [35], where the compositional features of the graphs are emphasized. This is a departure from most introductions to Peirce's EGs in terms of the 'cut' as negation and Peirce's descriptions of the graphs and rules in terms of natural language. It has the advantage of being more algebraic and more compositional, as each graph can be decomposed into smaller relational components, e.g. relational composition, intersection, union, etc.

We assume some basic familiarity with EGs as can be found in [36] and [35] (with further background found in [64]). It should (again) be noted that this syntax and corresponding presentation have recently been formalized as a calculus of generalized relations in [14].[4] Though the formal theory mentioned above in [14] extends to n-ary relations, all of our examples here are, for matter of economy, taken from the setting of binary relations. Readers interested in more algebraic treatments are encouraged to look in the citations above.

To begin, a relation is represented as in Figure 5 with a corresponding ingoing and outgoing wire representing the domain and codomain of the relation. The leftmost expression is a relation, between say x and y, such that xRy. We use

$[R]$ $[\bar{R}]$ $[\breve{R}]$

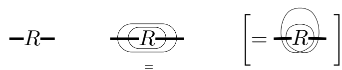
Fig. 5: Primitive (Unary) Relational Operations.

brackets when stating the equivalent relational expression or equivalent graphical expressions. The further relational expressions in Figure 5 show the complement ($^-$) relation, $x\bar{R}y$ that uses the single 'cut' (graphically drawn as ○ around the relation) and the converse ($\breve{}$) relation, $y\breve{R}x$. In regards to the converse, one could wrap the 'wires' around the other way in a 'z'-shape instead of an 's'-shape and

[4] A further difference is worth stating. The general calculus given in [36] uses 'dots' along the wires to allow the treatment of 'free' vs 'bound' variables. The distinction matters little for our purposes here.

achieve the same purpose. We often write a shorthand version of converse that takes the mirror (left-right) image of R as in the rightmost figure. If R is the 'loves' relation then the converse can be read as the 'loved by' relation, reading now the rightmost wire first, which has been wrapped around to the leftmost position, as in 'y is loved by x'. We often omit the typing information (x, y) and just read the relational operations.

There are several graphs we stress that are specific to the linear case. In place of the single 'cut' we emphasize the 'double cut' or 'scroll' in Figure 6. Peirce suggests the 'scroll' as a means of writing continuously and in one motion

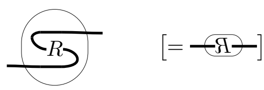


$[R]$ $[\bar{\bar{R}}]$

Fig. 6: Involution in EGs and Peirce's scroll.

two nested 'cuts', as shown in the rightmost graph of Figure 6. The 'scroll' corresponds to an involution — the same motivation and the key feature behind the development of *-autonomous categories — and so it should perhaps come as little surprise that these are related. While many introductions to Peirce's EGs introduce the 'cut' as a key element of the graphical syntax, Peirce himself often states that the 'scroll' or 'double-cut' is of more fundamental importance.[5]

A variant of the single 'cut' remains meaningful in the linear case. The combined complement-converse relation ($\breve{\bar{R}}$) is shown in Figure 7. This corresponds to the linear negation of R, or $R^{\perp}$. The primacy of such an operation over



$[\breve{\bar{R}}]$ $[= R^{\perp}]$

Fig. 7: Linear Negation

compliment or the 'cut' alone may seem surprising. We return to motivate this graphically in Section 3. Related to the comment regarding Figure 5, there is no difference between an 's' or 'z' shaped bending of the wires. We also note that

[5] One such key discussion occurs in 'Selection 8: On Logical Graphs [Euler and EGs] (R 481)' in [54].

whether the converse is outside or inside the cut similarly has not effect, i.e. $(\bar{R})\breve{}$ is the same as $(\breve{R})\bar{}$.[6]

Where a relation is represented with corresponding ingoing and outgoing wires, relational composition (;) is represented in the string diagrammatic syntax by connecting the wire with a shared codomain-domain, such as in Figure 8. As

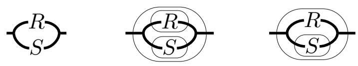
$$-R-S-$$

[R;S]

Fig. 8: Relational Composition

is usual with relational composition, there is an implicit existential quantification over the shared domain wire. We point this out as it will be recalled in the dual case.

We find it convenient to represent graphs of the Booleans vertically, in contrast to the horizontal (sequential) composition of relations above. The way to



$[R \sqcap S]$ $[R \sqcup S]$ $[R \sqsubseteq S]$

Fig. 9: Boolean Operations

read these parallel terms in Figure 9 is that the domain on the left and codomain on the right are shared, i.e. they are the same type. The nexus of the shared branch is Peirce's *tri*-identity or *teridentity* relation (,), which ensures that the shared (co)domain are the same. Sans cut, the way to read the parallel term in $\sqcap$, is that both R and S hold together (i.e. conjunctively), as in $xRy \wedge xSy$. The reading of the dual, given as $\sqcup$, is that both R and S hold disjunctively, as in $xRy \vee xSy$.[7]

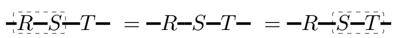
---

6 This is all that is needed for the relational setting described in this paper. We note that these follow, as shown in [36], from the lines of identity obeying the laws of a special Frobenius algebra. In [36] it is also shown that the bending of the wires can freely pass through a 'cut'. Of further note, these laws also allow for commutativity in the parallel connectives but not necessarily the sequential ones, and allow us, as we will observe in Section 5, to derive the linear equivalences.

7 We note that Peirce includes the lattice operations in 'Note B' on p. 109 and that these correspond to the additive or non-relative terms in the linear case. Our discussion will mostly focus on the multiplicatives or relative terms, however, as these tend to be considered most interesting for the linear case.

Much like how implication can be seen as the most primitive inferential connective, the ‘scroll’ serves the same purpose in Peirce’s EGs. The rightmost graph of relational inclusion ($\sqsubseteq$) therefore deserves special mention. Note that the shape of inclusion (omitting the R in Figure 9, for example) is the same shape as the ‘scroll’. This means that the involution R is the same as the blank inclusion R.[8] That the ‘scroll’ manages to be both an involution and an inclusion at one and the same time is perhaps not emphasized enough. We will see that in the linear case this takes on a further meaning, as the ‘scroll’ mediates between the linear contexts and connectives.

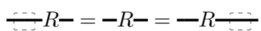
The graphical syntax automatically captures associativity of composition and the unit of composition, which is the identity relation ($\mathbb{I}$) and gives meaning directly to Peirce’s expression ‘the line of identity’. Notice the meaning of the center graph in Figure 10 is the same irrespective of the order of parenthesis (graphically depicted as dashed boxes) used for ordering the composition. The



$[(R;S);T]$ $[R;S;T]$ $[R;(S;T)]$

Fig. 10: Associativity of Relational Composition

identity relation, i.e. the unit of relational composition, is captured directly by the wire as seen in Figure 11. Again, the center graph is the same irrespective of ordering composition with the unit.



$[\mathbb{I};R]$ $[R]$ $[R;\mathbb{I}]$

Fig. 11: Unit of Relational Composition

The two tensors forming the multiplicatives correspond in our relational setting to relational composition and its De Morgan dual, an operation Peirce called *relative sum*. We sometimes, following the linear logic tradition, also refer to this as the par’d (as in ‘$⅋$’) context. Relational composition in EGs is well known, though it is not (perhaps) the way Peirce’s graphs are typically presented. On the other hand, the dual to relational composition is virtually absent in the literature in Peirce’s graphs. This is unfortunate.

[8] One may read the blank in the outer scroll as $\top$, or ‘True’, where a scroll around a relation means primitively that the relation follows from ‘True’ or from whatever follows, as Peirce would say, from the blank that is the sheet of assertion.

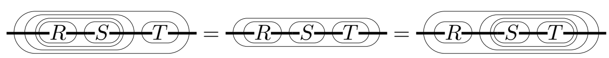
Relative sum, the De Morgan dual of relational composition, is represented in the graphs by adding a 'cut' around each sub-composed term and a 'cut' around the whole expression as in Figure 12. We prefer to see this as a primitive

$[R \dagger S]$

Fig. 12: Relative Sum

n-ary scroll (as in [48]), with multiple inner cuts.[9] While relative composition implicitly existentially quantifies over the shared domain-codomain, relative sum implicitly *universally* quantifies over the shared domain-codomain. One familiar with Peirce's EGs will recognize this reading and will also recognize that relative sum shares the shape as disjunction.[10] Rather than disjunction in parallel, relative sum is disjunction in *sequential* or *compositional* order. The meaning of the expression in Figure 12 is thus: a relation from x to y, such that $\forall z(xRz \vee zSy)$. Just as one can choose to read off an implication from the form of a disjunction, if one has negation then one can also read off a (sequential) implication from the relative sum. An alternative reading of relative sum is then the following: a relation from x to y, such that $\forall z(x\bar{R}z \Rightarrow zSy)$ or a relation from x to y, such that $\forall z(xRz \Leftarrow z\bar{S}y)$.

Like relational composition, the dual is both associative and has a unit. Here we see another hint of the good behavior of the 'double cut' or 'scroll', which captures associativity and a nested hierarchy of the par'd context. As the 'double cut' is an involution it (as usual) is similarly extraneous, and we can capture the meaning by the one graph in the center of Figure 13. As in the case of relational composition, associativity of relative sum is similarly absorbed by the graphical syntax.



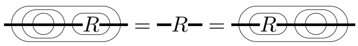
$[(R \dagger S) \dagger T]$ $[R \dagger S \dagger T]$ $[R \dagger (S \dagger T)]$

Fig. 13: Associativity of Relative Sum

[9] The main difference is that we are in the Beta variant of the graph with lines of identity, and we connect n-ary scrolls to the par'd context from multiplicative linear logic. We go further in the next section to draw *generalized entailment relations* consisting in a list of composed terms in the antecedent of the inclusion and a list of dually composed, i.e. par'd, terms in the consequent of the inclusion.

[10] Consider the intersection and join operations, where the latter is De Morgan dual to the former and is read disjunctively as in Figure 9.

Less obvious, but fitting, is that the ‘double cut’ also makes explicit the unit of relative sum. While the unit of relational composition is simply the wire itself,



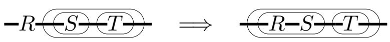
$[\mathbb{d} \dagger R]$ $[R]$ $[R \dagger \mathbb{d}]$

Fig. 14: Unit of Relative Sum

the unit of relative sum is what Peirce calls the *diversity* or *difference* relation ( ) as seen in Figure 14. Following Peirce’s naming we use $\mathbb{d}$ for this relation, as it asserts that two things are not the same. We conclude that the ‘scroll’ as an involution mediates between the twin tensors: it makes explicit a nesting for the associativity of par and makes explicit the par unit. The ‘scroll’ also re-associates in the implicational context, a point we return to in the next section.

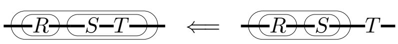
We have now introduced relational composition and its dual. While Peirce stresses the operations in ‘Note B’ and states their importance, we know of no emphasis given to the par’d context in the literature on EGs. As far as we know, the observation that the ‘scroll’ serves as a par-unit introduction and to reassociate within the par’d context has also not been made before.

Once we have relational composition and its dual we can express linear distributivity and the linear negation laws. Peirce does so on p. 190 and p. 192 in ‘Note B’. We express these rules here for the first time in the syntax of EGs. Given relational composition and its dual, left and right linear distributivity are expressed in Figure 15:



(left linear distributivity)

$[R; (S \dagger T) \sqsubseteq (R; S) \dagger T]$



(right linear distributivity)

$[R \dagger (S; T) \sqsupseteq (R \dagger S); T]$

Fig. 15: Equations for linear distributivity.

Peirce sometimes shaded positive and negative regions of the graphs. We show linear distributivity in this way in Figure 16. It perhaps helps to show that linear distributivity takes the outermost composed relation and pushes it inwards to ‘tunnel’ into an inner par’d context.

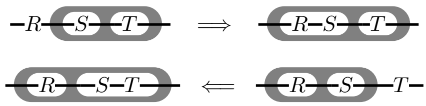
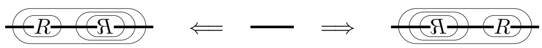


Fig. 16: Equations for linear distributivity with shaded regions.

In addition to the linear distributive laws, Peirce states what we now recognize as the linear negation laws. We first give the introduction rule for left and right linear implication in Figure 17. We call these *introduction* rules because

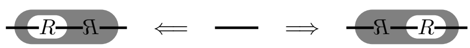


$[R \dagger R^{\perp}]$ $[\,\mathbb{I}\,]$ $[R^{\perp} \dagger R]$

(/-introduction) (\-introduction)

Fig. 17: Equations for linear implication introduction.

taken out of the par'd context they express the residual, which corresponds to linear implication. We show the equivalent residuals in the coloured form in Figure 18. We return to Peirce's study of residuation in the next section. Lastly,



$[R/R]$ $[\,\mathbb{I}\,]$ $[R\backslash R]$

Fig. 18: Equations for linear implication introduction with shaded regions.

we give the corresponding elimination rules that follow directly as the (linear) contrapositive in Figure 19.

Peirce goes on to give a number of examples of the rules (pp. 195-198), all of which we now recognize as examples from categorial grammar and the Lambek calculus. Peirce also demonstrates some of the requisite morphisms in the commuting diagrams for LDCs. Before moving to discuss residuation and linear implication, we give in Figure 20 one paradigmatic example highlighting how linear negation can interact with linear distributivity. This is known as one of the 'zig-zag' equations in categorical logic. Peirce was both aware of this connection, as he employs the steps often in derivations, but also draws explicit mention to it in 'Note B.' Following linear distributivity, he notes that when the

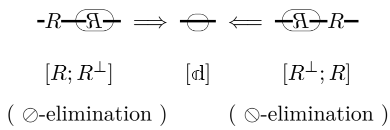


Fig. 19: Additional equations for linear negation.

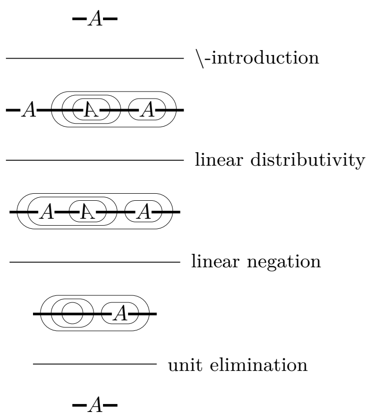


Fig. 20: The ‘zig-zag’ equation in EGs.

relative to be eliminated has been replaced by a unit then it "can often be got rid of" (pp. 193-194).

Finally, we add that Peirce was also aware of admissible cyclic permutations. He observes on p. 194 that the following permutations of the sequent in Figure 21 are equivalent. This is the key feature of the further rule emphasized in cyclic

$$\mathbb{I} \sqsubseteq R \dagger S \quad \Longleftrightarrow \quad \mathbb{I} \sqsubseteq S \dagger R$$

Fig. 21: Allowed (Cyclic) Permutations in the †-context

linear logic in Figure 4. Immediately proceeding this passage, he points out that such cyclicity is not possible when relational composition is in the conclusion of the sequent, for which case only the regular converse operation applies. We will return to these cyclic permutations in the context of the graphs in the next section.

The rules discussed so far — in particular relational composition and its dual, associativity and corresponding units, and left and right linear distributivity — form the foundation of Cockett and Seely's linear distributive categories (LDCs). Adding the linear negation laws, in turn, form the foundation of *-autonomous categories. As mentioned in Section 1, Peirce's characterization of the logic of relations via these rules in 'Note B' places historically the awareness of these rules much earlier.

We also reiterate that Peirce's previous algebraic work on the logic of relations in 'Note B', and in particular his awareness of relative sum and of linear distributivity and linear negation above, has not be been made or addressed in subsequent studies of the graphs. Peirce himself seems to have made little mention of these rules in his later algebraic work (though exceptions can be found, such as in [54, p. 240 & p. 281], where he refers, as linear distributivity is sometimes referred to today, as reassociation.)[11] In subsequent sections we show, however, that many of the graphs Peirce went on to draw employ these concepts.

3 Residuation and Linear Implication

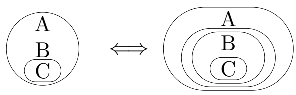
After Peirce introduces linear distributivity and the linear negation laws, he then proceeds in 'Note B' to give a number of examples. Many of these examples show the interplay of linear implication. As has been pointed out in [35], Peirce systematically studied linear implication in the form of residuation and drew corresponding graphs of residuals in EGs. The standard account of the development of residuation traces it back to Dilworth [77] and Birkhoff [10], then

[11] In the latter passage Peirce also presents the rule in EGs and also adds a further operation, graphically motivated, called *fusion*, which adds the 'double-cut' to the outer term first and then 'fuses' the two 'scrolls' together.

on to Lambek [40] and Grishin [33]. The concept of residuation sees substantial development through Galois connections and adjunctions in category theory.

We begin this section with a brief discussion of residuation in the context of conjunction, then move to instances with composition that correspond to the (multiplicative) linear case. The conclusion to draw is that Peirce's developments of residuation and his emphasis on linear implication (along with its dual, both forming the bilinear operations) were far more extensive than previously thought.

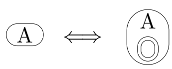
As has been pointed out before in [45], Peirce may have been the first to write about residuation in connection to conjunction. In the propositional case, residuation comes from adding a 'double cut' or 'scroll' inside the inclusion as shown in Figure 22. This is perhaps the simplest expression of 'currying' and 'uncurrying'.[12] For future discussion, we also note in Figure 23 the similar



$A \wedge B \to C$ $\qquad$ $A \to (B \Rightarrow C)$

Fig. 22: Intuitionistic Entailment in Alpha

connection between the operation and intuitionistic negation. The move ensures



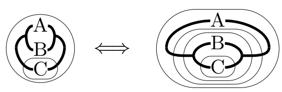
$\neg A$ $\qquad$ $A \Rightarrow \bot$

Fig. 23: Intuitionistic Negation in Alpha

that the intuitionistic implication is analogous to the standard 'cut' as negation.

As we will see, Peirce was explicitly aware of this notion of intuitionistic negation. We add to this story here by noting that the intuitionistic case employing the 'double cut' was actually Peirce's motivation for the single 'cut' as negation and not the other way around. In one of several passages drafted for the purpose, and a passage worth quoting in full, Peirce makes this clear.

[12] A Beta variant is easily given. The difference in the lines of identity is a simple application of associativity.



> [The failing... lies in its] encouraging the idea that negation, or denial, is a relatively simple concept, and that the concept of Consequence is a special composite of two negations. In opposition to that, all my writings upon formal logic have been based on the belief that the idea of sequence in reasoning and in judgment, whether conditional or categorical, could in no wise be replaced by any composition of ideas. Now this view inevitably leads to a negative predication, say "is not P" being regarded as the assertion that upon the supposition of the affirmation "is P" there would be sequent the essence of falsity; and I regard this essence of falsity to consist in permitting the interpreter to opine whatever he may choose. I thus analyze the negation of P into a positing of P as a mere idea together *with the assertion that falsity is sequent upon it.* As a matter of fact, this idea was the starting-point in my mind of the notion of logical graphs;—not merely those of the existential kind, but also of the earlier entitative kind. *The simple Cut is a scroll...* [56, p. 353n10, emphasis added]

The "essence of falsity" Peirce refers to above is the notion of absurdity familiar to us from the intuitionistic case (where anything follows, or, as Peirce says, where the asserter is allowed to "opine whatever").[13] Peirce goes on to suggest two ways this can be interpreted (given the position with respect to the scroll) and to express doubts about how best to iconically capture the process diagrammatical in EGs.

We take the above passage as substantial evidence that the intuitionistic negation is not a further development needing to be added to the theory of EGs, but is rather at the heart of Peirce's understanding of the graphs and has been since the beginning.[14] Further, in the passage Peirce makes clear his belief that negation (at least when taken as a primitive) is not compositional. Regardless, Peirce is emphatic that the single 'cut' really is a 'scroll' and that $\neg P$ is really $P \Rightarrow \bot$.

A natural question is whether and to what extent Peirce understood what we know about intuitionistic negation and its role in constructive mathematics. Further work is needed to answer this question but given that the linear case is more general, and Peirce appears to have had a surprising understanding of the linear case, we suspect the answer may also surprise us.

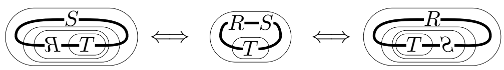
Peirce also understood residuation with respect to relational composition. A preliminary discussion of residuation in this context, i.e. in the Beta variant of EGs, and an overview of Peirce's historical developments along these lines can be found in [35]. We extend this development in three ways here. First, the motivation for adding converse inside the scroll in [35] was given by topological considerations governing the line of identity: a line of identity can freely

13 See also Peirce's discussion of the 'the blot' in [59].

14 See similar passages in [54, p. 169, p. 288, & p. 582] and [56, p.356]. Another noteworthy passage occurs in [CP 4.564] where Peirce describes the single 'cut' as a degenerate implication that lacks a specified consequent. Further connections between Peirce's views and constructivism more generally can be found in [60].

be deformed (into an ‘s’ or ‘z’) and doing so makes the graphs topologically equivalent. Here we give a derivation that shows how these equivalences can also be motivated directly by linear distributivity and the linear negation laws given above. Second, we show that Peirce was aware of coresiduation and so to the operations that give rise to bilinear logic and the multiplicative fragment of non-commutative linear logic. To this we highlight the ability of the ‘scroll’ to represent, in an effective way, these bilinear operations. Third, we add further instances of graphs Peirce drew of residuation and identify them with contemporary notions in the literature.

The important change in the move to the Beta graphs is keeping track of lines of identity for relational composition. One can summarize the equivalences for residuation in EGs with the Schröder equivalences shown in Figure 24.



$[S \sqsubseteq R\backslash T]$ $[R; S \sqsubseteq T]$ $[R \sqsubseteq T/S]$

Fig. 24: Schröder Equivalences in EGs

Note that the left and right residuals can be expressed as in Figure 25. These are the expression in the consequent location — i.e. read from the inner most, concluding ‘cut’ — within the ‘scroll’ from Figure 24. $R\backslash T$ is a relation from

$[R\backslash T]$ $[T/S]$

Fig. 25: Left and Right Residuals

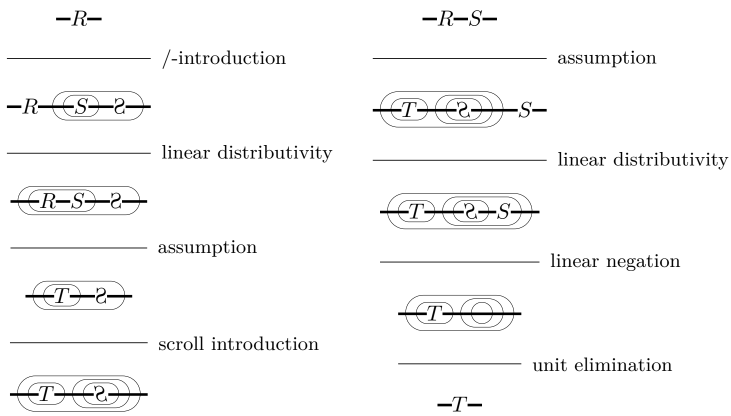
x to y, such that $\forall z(zRx \Rightarrow zTy)$. Peirce sometimes refers to residuation as “the relation of inclusion of correlates”. In his own words (and adapted to our example) “it implies that everything that R stands in any fixed relation to is included among the things to which T stands in that same relation” [55, p. 286].

Peirce was not only aware of the above equivalences but also stated a method for arriving at different permutations in 1882:

> Hence the rule is that having a formula of the form $[R; S \sqsubseteq T]$, the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted. [53, p. 341]

The citation is from around the same time as ‘Note B’ and is before Peirce substantially develops and presents the graphs. Given that linear negation is the combined complement-converse relation (see Figure 7), Peirce is stating exactly the well known rule from linear logic that one can move a term from one side of the entailment to the other by adding a linear negation.

In Figure 26 we show that the equivalences can be derived from linear distributivity and linear negation. We derive both sides of the right-most equivalence in Figure 24. The other side follows from symmetry.



(a) If $R; S \sqsubseteq T$ then $R \sqsubseteq T \dagger S^{\perp}$

(b) If $R \sqsubseteq T \dagger S^{\perp}$ then $R; S \sqsubseteq T$

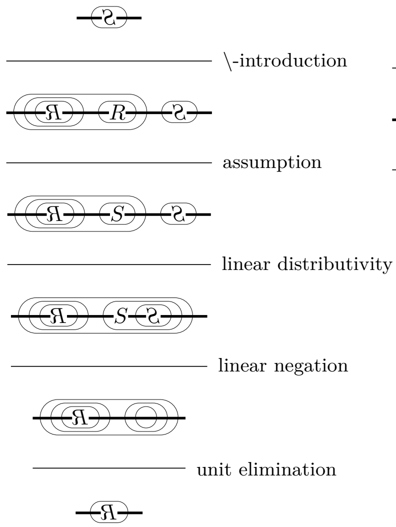
Fig. 26: Derivations of one side of the Schröder Equivalences

We add two additional derivations in Figure 27 to show how these rules work in Peirce's EGs. The first is the linear contrapositive, i.e. If $R \sqsubseteq S$ then $S^{\perp} \sqsubseteq R^{\perp}$. We also derive, as an example of a key inference rule, a modus ponens-like inference rule for the residual, i.e. $R; R\backslash S \sqsubseteq S$.

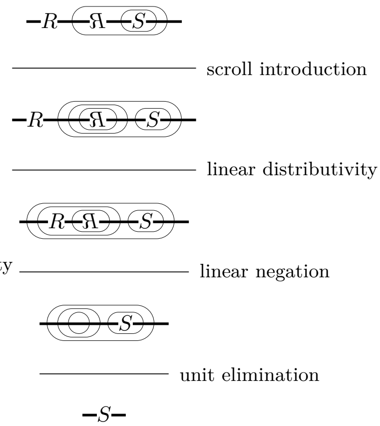
Peirce himself tended to write residuation vertically, as in the right side of Figure 28. Turning back to show the full converse relation, it is clear that these are topologically the same.

Once aware of residuation in the graphs, we find examples often in his later work. In a draft of Lowell Lecture V from 1903, Peirce derives the transitivity of residuation, i.e. $A\backslash C$ from $A\backslash B$ and $B\backslash C$.[15] At one point, Peirce refers to the residual as the "graph of inclusion" and says it has "the shape of necessary reasoning," for necessary reasoning "is that whose conclusion is true of whatever

[15] Reprinted in [55, p.305-307]. We note that Peirce previously derives the transitivity of linear implication in his earlier algebraic studies in [CP 4:94].

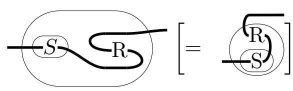


(a) If $R \sqsubseteq S$ then $S^{\perp} \sqsubseteq R^{\perp}$



(b) $R; R\backslash S \sqsubseteq S$

Fig. 27: Further Derivations in EGs



$[S/R]$

Fig. 28: Horizontal and Vertical Presentations of the Residual

state of things there may be in which the premise is true" [55, p. 287].[16] The

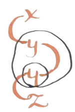


Fig. 29: The Graph of Inclusion in Peirce's hand.

graph, reprinted from a copy in Peirce's hand, is given in Figure 29. In fact, many of the more complicated graphs given by Peirce in Lowell Lecture III and V consist in and can be expressed in terms of residuals.

As a final example, we draw attention to the first three graphs Pierce describes in Lowell Lecture III, here reprinted again in Peirce's hand in Figure 30.[17] The first graph states the existence of a residual, the second is the dual of what

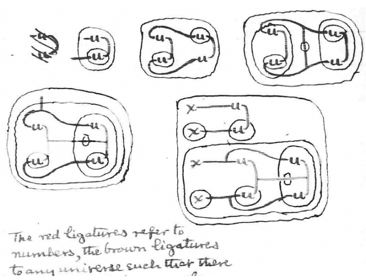


Fig. 30: Graphs Containing Residuals.

is called the symmetric quotient [8] and the third is the straightness condition found in Freyd and Scedrov's "Categories and Allegories" [30]. The symmetric quotient is $(R\backslash S)\sqcap(R\backslash S)^{\smile}$ and the straightness condition is $(R/R)\sqcap(R/R)^{\smile} \sqsubseteq \mathbb{I}$.[18]

[16] Rendering taken from Jukka Nikulainen and can be found in the same section Lowell Lectures V cited above.

[17] Image taken from MS 464-645 as part of the Jeffrey Downard's 'Scalable Peirce Interpretation Network' (SPIN) project at: `http://fromthepage.com/collection/show?collection_id=16`.

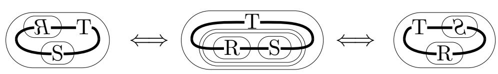
[18] When analyzing the conception of quantity in [CP 4:96], Pierce states an analogous straightness condition and offers the following (contrapositive) reading: "if A and B are not identical, either A can do something B cannot or B can do something A cannot."

Both play roles in domain constructions and, in the latter case, moving from a division allegory towards a power allegory.

As far as we know, Peirce presentation of these relations precedes any other presentation of them in the literature — work on the symmetric quotient in the above works not appearing until the late 1980s. While Peirce describes the meaning of the graphs in the Lowell Lectures, he unfortunately does not describe his means of arriving at them. We are left wondering how he came to place so much importance on the five graphs in Figure 30. More work needs to be done to characterize these and the remaining two graphs.

The logic of residuation, which forms the core of the Lambek calculus, plays a key role in categorial grammar and other non-commutative logics. The non-commutativity comes from having distinct left and right residuals. The intuitive idea behind categorial (or type) grammar is that the words that go on to form a complete sentence do so by either becoming 'more complete' when combined with further words on the left or becoming 'more complete' when combined with further words on the right. Like a chemical molecule, nouns and other phrases are 'built up' into submolecules, and a sentence is complete if the submolecules combine to form a larger compound in the right way. Lambek did not seem to be aware of Peirce's developments early in his studies but eventually cites in [39] features of Peirce's *valental* account of relations, which Peirce based on the same chemical analogy [54, pp. 212-7], as anticipating aspects of these type grammars. This section shows the connection to be much stronger as Peirce understood residuation — a fact almost certainly not known by Lambek at the time — and already drew many graphs and derivations employing residuation.

Finally we show the dual to residuation, what is sometimes called coresiduation, in Figure 31.[19] Whereas the starting point for residuation is when composed relations fall under (i.e. are $\sqsubseteq$) a 'shortcut' relation, the starting point for coresiduation is when a relation falls under dually composed relations. As



$[R \obslash T \sqsubseteq S]$ $\quad$ $[T \sqsubseteq R \dagger S]$ $\quad$ $[T \oslash S \sqsubseteq R]$

Fig. 31: Coresiduation Equivalences

residuation is the right adjoint to relational composition, coresiduation is left adjoint to relative sum. Looking back at Figure 30, one sees that these graphs really contain instances of residuals and coresiduals.

The rule Peirce gives for permuting the terms in the inclusion captures both the case of residuation and of coresiduation. After Peirce states the cyclic rule

[19] As of yet there seems to be no standard name or symbol given to the dual of residuation. Moortgat calls them *right* and *left difference* in [49], which is where we adopt the symbols $\oslash$ and $\obslash$.

given above, he goes on to list over a dozen further examples of which some include the coresiduation case. We also find examples where Peirce specifically studied the par'd context that is the basis for coresiduation. We reproduce two sample graphs from R488 in Figure 32.[20] Given the 'lover of' $(-l-)$, 'servant

Fig. 56 Fig. 57

Fig. 32: Sample graphs from R488

of' $(-s-)$, and 'benefactor of' $(-b-)$ relations, in Peirce's words, sub-Figure 56 reads: "Whoever loves any benefactor of any man serves that man" or "Every lover of any man serves everybody benefitted by that man", or "Any benefactor of any man is loved only by servants of that man". The sub-Figure 57 is the center graph that is the start of coresiduation. In Peirce's words the meaning of the graph reads: "Whoever serves any man loves everybody but the benefactors of that man" or "Any man benefits every man served only by lovers of him". The multiple readings of these graphs shows an awareness that one can freely interpret the orientation within the 'scroll', and shows clearly that Pierce studied the par'd context inside the 'scroll'. These graphs are also related to Cockett et al.'s circuit diagrams that we discuss in the next section.

Coresiduation plays an important role in Moortgat's categorial grammar [49] and in the bilinear case emphasized by Cockett and Seely [26]. It is indeed this second linear implication that is the characteristic feature of bilinear logic. Lambek originally notes coresiduation and the par'd context in the presentation of bilinear logic, but goes on to downplay its linguistic importance. He writes:

> the distinction between the tensor product and its de Morgan dual, called par by Girard, seemed to be irrelevant for linguistic purposes. We therefore decided to drop it (see [22]) and turn to what is now known as compact bilinear logic and its algebraic presentation in the form of pregroups. [43, p. 672]

While it does not add expressivity over the linear implication with linear negation — it is in fact the linear dual of linear implication, i.e. $R \oslash T = (T \backslash R)^{\perp}$ — it does aide as an important connective for the proof theory (e.g. proof nets) and for characterizing inference rules. It appears that Peirce, contra Lambek, would side decisively with Moortgat here.

[20] Reprinted in [54, p. 266]. See also R480 (and again in [54, p. 270].

A question was posed in the last Section 2 about why the combined complement-converse relation (the $\breve{\bar{R}}$ or $R^{\perp}$ shown in Figure 7) could have more fundamental importance than complement itself. These bilinear operations give the answer: taking the inclusion as prime importance, the bilinear operations produce equivalences within the inclusion that induce the combined complement-converse relation.

We add two further points about Pierce's understanding of residuation. First, we repeat a significant passage where Peirce stresses the importance of residuation.

> Yet really, the form $l \dagger \bar{\breve{l}}$ is all-important, inasmuch as it is the basis of all quantitative thought. For the relation expressed by it is a transitive relation…[It] is not only a transitive relation, but it is one which includes identity under it. That is, $\mathbb{I} \prec l \dagger \bar{\breve{l}}$. But it is further demonstrable that every transitive relation which includes identity under it is of the form $l \dagger \bar{\breve{l}}$. [CP 4:94]

Peirce goes on to demonstrate the last claim. The passage shows that Peirce understood that any derivation from the identity can be put in this form.[21] Peirce makes a similar claim on p. 194 in 'Note B':

> When we have only to deal with universal propositions, it will be found convenient so to transpose everything from subject to predicate as to make the subject $\mathbb{I}$. Thus, if we have been given $l \prec b$ we may relatively add $\bar{\breve{l}}$ to both sides; whereupon we have $\mathbb{I} \prec l \dagger \bar{\breve{l}} \prec b \dagger \bar{\breve{l}}$. Every proposition will then be in one of the forms: $\mathbb{I} \prec b \dagger l$ or $\mathbb{I} \prec b; l$.

Note, first, that Peirce is (again) making a claim analogous to moving to the one-sided sequent calculus. Further, Peirce is aware that in this calculus both relational composition and the dual is sufficient to express all the 'universal' propositions. We believe these passages demonstrate a turn towards proof theory, as they suggest i) placing importance on those constructions only from the unit/identity, ii) placing importance, not just on the linear negation rules given above that are so crucial to proof theory, but suggest an awareness of the sufficiency of these rules for such an undertaking.

We summarize to this point. Peirce understood linear distributivity and linear negation laws and states them clearly as far back as 1883 in 'Note B'. At around the same time, Peirce carried out a systematic study of residuation that includes a rule for permuting terms to arrive at various equivalencies. This rule works

[21] The omitted line in the '…' above is also of interest: "By a transitive relation, we mean a relation like that of the copula. If A be so related to B, and B be so related to C, then A is so related to C. The 'copula' is Peirce's sign for illation or inference — it is formally like that of implication — but the passage is significant in that Peirce sides decisively with the linear case and that the sign of inference ought to be based on sequence and not just inclusion. We come back to the importance Peirce places on sequence in Section 6.

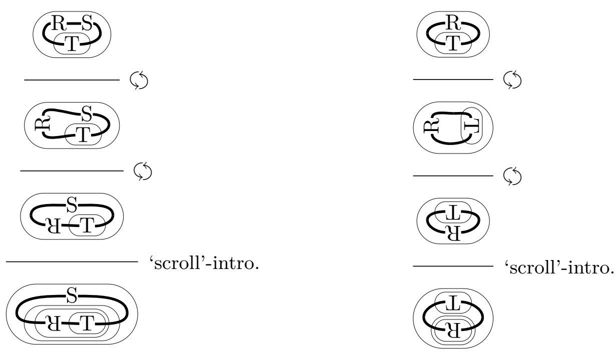
for both residuation and coresiduation and, in an example we return to later in Figure 35, captures the cyclicity condition behind cyclic linear logic. Peirce also states the cyclicity condition in 'Note B' and goes on to give in 'Note B' and elsewhere many examples employing these rules. We also have given numerous examples demonstrating that Peirce continued to develop and draw graphs of residuation. We conclude that Peirce presented and understood the same relation algebraic structure that models cyclic bilinear logic.

We now turn to his Existential Graphs and situate the resulting diagrammatic syntax in more contemporary terms.

4 Existential Graphs and Circuit Diagrams

We showed in the last section how Pierce had an understanding of the bilinear operations corresponding to linear implication and its linear dual. Because the 'scroll' also serves as an inclusion, we suggestively showed — primarily in Figure 24 and Figure 31 — how these equivalences are captured graphically inside the 'scroll'. The residuation equivalences given in [35] were motivated by topological considerations governing Peirce's 'lines of identity'. We go further in this section and make explicit how these equivalences can arise directly from topological features of the 'scroll' itself. As a key source of examples, we go on to compare Peirce's notation to Cockett et al.'s circuit diagrams in the literature.

To begin, we note that the 'scroll' allows the bilinear operations to be efficiently presented as topological moves. Two examples in Figure 33 show how relations can be freely rotated inside the inclusion. These are the same deriva-



(a) \-introduction (b) the linear contrapositive

Fig. 33: Topological derivations inside the 'scroll'

tions as those given in Figure 24(a) and Figure 27(a). The added 'scroll' in each of the last steps makes explicit the change in direction of the inclusion.

Rather than present converse as mirror image (-Я-), in these examples the the relation is 'wrapped around' so that it becomes inverted as in -ꓤ-. One can similarly imagine rotating the page itself and looking at the graphs upside down. In the case of binary relations, this inverted relation is topologically equivalent to the converse in Figure 5 (it is as if one grabs both outermost wires and pulls, leaving the left and rightmost wires the same).[22] Given the derivations from linear distributivity and linear negation above in Figure 26, we can freely associate these re-orientations with derivations. Regardless, we note that Peirce's 'scroll' presents various ways of capturing these logical operations via topological moves.

With these topological considerations, it is worth comparing the ease of presenting the Schröder equivalences with similar diagrammatic approaches in the literature found in [67, p. 159] and [30, p. 259] (and the FOL derivation found, for example, on [67, p. 42]). We find the cyclic presentation within the 'scroll' to be a simple and elegant means of capturing the underlying logical relationship: the 'scroll' allowing clear presentations of the tensor'd and par'd context, along with the variety of linear implications that mediate them in the bilinear context. Given it is well understood that linear sequents can be so permuted, it is perhaps a wonder why a graph with these cyclic connections has not been so used for the purpose.

We summarize all of the relevant operations to this point. Figure 34 offers a shorthand notation for representing residuation and coresiduation via rotations inside the 'scroll'. These moves make clear the various introductions (or

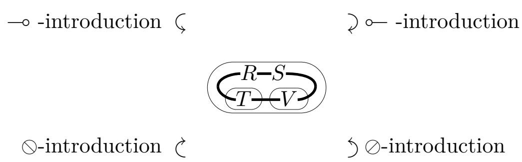


Fig. 34: Summary of Bilinear Operations

eliminations) of bilinear terms while preserving the overall cyclic order.

As a final example, we note that one can use the residuation and coresiduation rules to perform a 'full cycle' and move a par'd context from one side of the sequent to the other side as in the cyclic rules in Figure 4 from Section 1. We show the rule in EGs in Figure 35. At first glance this may look like

[22] One must be more careful in the n-ary case, where the overall top-to-bottom ordering matters. In this more general case it is perhaps better to see the 'scroll' as a sphere, like an 8-ball, where the antecedent is brought around from the back of the 8-ball to the front.

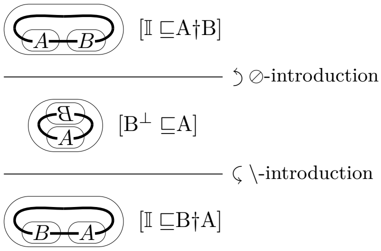


Fig. 35: Cyclic Law via Bilinear Operations

commutativity, as it appears analogous to a (binary) swapping of the order of the consequent. The empty antecedent, however, ensures that the total ordering is (cyclically) preserved. Graphically, this rule corresponds to taking a par'd expression from the bottom right of a 'scroll' and moving it to the upper part (using coresiduation), and then moving it down again to the lower left (now using residuation).[23] This was, as stated in Section 2 and Figure 21, known by Peirce and given algebraically in 'Note B.'

We claim that Peirce's Existential Graphs can represent the 'geometric' graphs of the residuation laws and their respective equivalences in [4] and [5]. It would also be worth comparing the presentation in EGs to the version with quantification in [50]. We will save these developments for another time.

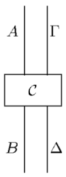
With these topological features of the 'scroll' now emphasized, we return to the contemporary significance of these graphs and moves. Presently we compare the resulting graphs to Cockett et al.'s circuit diagrams for the bilinear case as found in [26].[24]

The key to reading a Cockett et al.'s circuit diagram is that the topmost, incoming wires are implicitly tensored and the output, bottom-most wires are implicitly par'd. See the comparison in Figure 36. In a 'scroll', the inputs are captured by composed terms in the antecedent position, and the outputs are captured by terms in the dual composition, which are captured by separate 'cut' contexts inside the inclusion.

A further, telling comparison is given in Figure 37. Here a standard deduction takes the generic morphism from Figure 36 and introduces a linear implication (think, again, of the example of currying) to the output port. Notice the addition

[23] An involution is introduced during the last step with the residuation introduction, but in our setting the involution is again superfluous. One can also express this rule with left and right (linear) negations. We come back to this point at the end of this section.

[24] Further summaries can be found in [23] and [24].



$A \otimes \Gamma \vdash B \oplus \Delta$

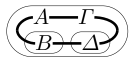
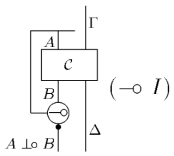


Fig. 36: A comparison of a generic morphism in Cockett et al.'s circuit diagrams (left) and Peirce's EGs (above)



$\Gamma \vdash (A \multimap B) \oplus \Delta$

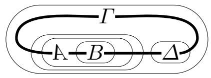


Fig. 37: Another comparison of a morphism in Cockett et al.'s circuit diagrams (left) and Peirce's EGs (above)

of a 'scope box' on the left side in the circuit diagram that keeps track of the original position and type of the wire.

The logical conception that motivated Cockett et al.'s circuit diagrams — almost certainly without any awareness of Peirce — and Peirce's analogous presentation in EGs is remarkable. Even more striking, perhaps, is the description Cockett et al. give of the scope box. They write that "the 'opaque' side of the box ought to be regarded as the wire A bent to join the ⊸ node at the bottom of the box" [24, p. 18]. This is given explicit meaning in the topological features of Peirce's 'scroll'. We give an example of the linear negations below, but we note that each of the component circuits given by Cockett et al. in [26] can be given correspondingly simple expressions in terms of Peirce's 'scroll'.

We draw a further connection to circuit diagrams. Peirce was aware that inclusion graphs like the the 'scroll' can be composed using inference rules for EGs. We give an example in Figure 38 that follows Pierce's graphical depiction of the process in Lowell Lecture II from 1903. [55, p. 201-2]. The steps are as follows. A 'scroll' is first 'iterated' or nested inside the consequent of another. Notice that the iteration that yields the nested 'scrolls' has the effect of lining up the antecedent of the inner scroll with the consequent of the outer scroll. 'Lines of identity' are then similarly 'iterated' and extended inwards to join the shared term. Composition or *cut* (now emphasized and with the usual meaning from the sequent calculus) is then performed to eliminate the middle term. In Pierce's words this involves the shared middle term (i.e. 'A') being 'deiterated' and then 'erased'. The result is a new 'scroll' with the requisite antecedents and consequents.

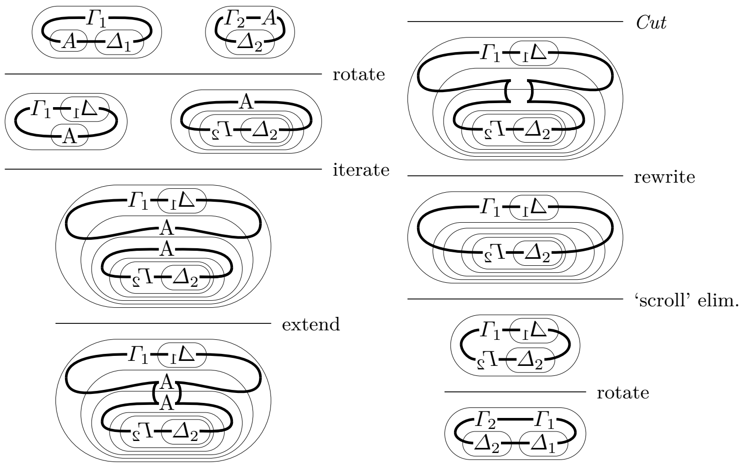


Fig. 38: Derivation of comp_4.

We could also perform the *cut* or composition step by employing linear distributivity followed by a elimination using the linear negation law. Similarly, while Peirce often nests the graphs vertically, as we have shown, we could use a 'one-sided' presentation where we reorient the 'scrolls' as following from the unit (i.e $\mathbb{I}$) and again perform linear distributivity and linear negation.

Four significant variations of *cut* or composition of 'scrolls' are given in Figure 39. These are the rules for planar polycategorical composition (see [23, p.14 & 16]). In terms of the general proof theory, these operations ensure that the respective (planar) ordering of the composed terms is preserved. All the conditions, however, are captured by using the bilinear operations and *cut*. We add that the four variants of planar composition in Figure 39 fall under Pierce's rule that any graph, a 'scroll' included, can be iterated within any 'positive' area. This includes in general the multiple locations allowed by consequents in the par'd context. This is a feature Peirce's Existential Graphs share with *deep inference*.[25]

Cockett et al. refer broadly to this relational setting as the *logic of generalized relations*. The Γ is a placeholder for a list of tensored (or, in our case, relationally composed terms), while the Δ is a placeholder for a list of par'd terms (in the dual presentation). Both are straightforwardly captured by lists in EGs, such as in Figure 40. Given the preceding discussion in this and the previous section, we would argue the Peirce understood this setting.

[25] See discussions of the comparison in [48,44] and in [14]. A further worthwhile direction between deep inference and cyclic linear logic is found in [28].

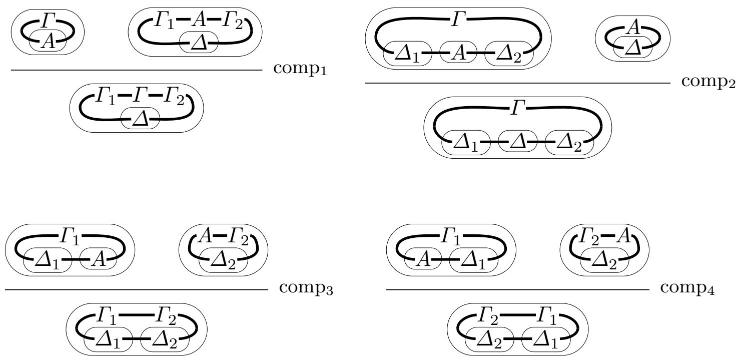


Fig. 39: 'Scroll' (i.e. planar) composition rules.

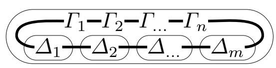
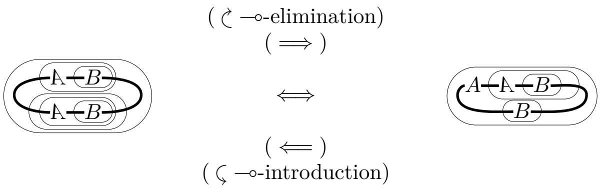


Fig. 40: A Generalized Entailment Relation

An advantage of circuit diagrams, like any wiring diagram, is the ease of expressing composition. A further question is whether nesting 'scrolls' can serve the same function. We believe it can, and that doing so has several advantages, but we are not yet convinced of its efficiency in this particular purpose.

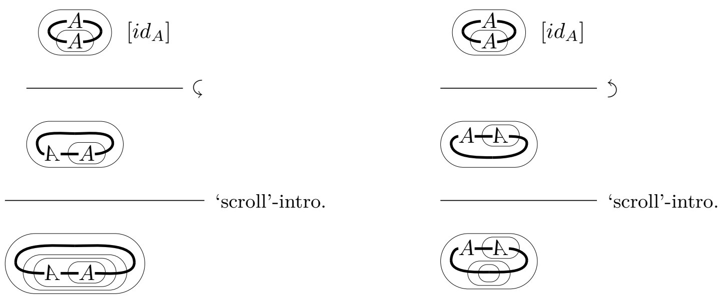
The circuit diagrams from Cockett et al. were created following two major developments: (i) the circuits allowed for treating the units in terms of what are called 'thinning links' that include a variety of coherence conditions and rewrites, and (ii) the scope boxes, again with corresponding coherences and rewrites, were needed to keep track of the bounds of the currying operations in the context of derivations. As hinted at above, Peirce's 'scroll' serves the same function as the scope box. Further, in regards to the use of thinning links to keep track of the units, we note that Peirce's 'lines of identity' not only keep track of the position of the (possible) units but the cyclic presentation meets the exact coherence conditions given for treating the thinning links.

We point out a further advantage the 'scroll' has for linking connectives to their corresponding inference rules. When expressing tautologies for the logical connectives, the resulting introduction and elimination rules are captured (topologically) by these bilinear moves. As an example, we show how the tautology from the residual/linear implication (i.e. $A\backslash B \sqsubseteq A\backslash B$, which we write suggestively as $id_{A\backslash B}$) yields a corresponding modus ponens rule for its elimination in Figure 41. A further example comes from a basic tautology $A \sqsubseteq A$, which leads to the linear negation laws in Figure 42. The requisite inference rules for



$[id_{A\backslash B}]$ $[A; A\backslash B \sqsubseteq B]$

Fig. 41: Corresponding elimination rule from tautology.



$[\mathbb{I} \sqsubseteq A\backslash A]$ $[A; A^{\perp} \sqsubseteq \mathbb{d}]$

Fig. 42: (Topological) Derivations of Linear Negation Laws from $A \sqsubseteq A$

the bilinear case are all fashioned in this way: arising, as they do, out of the residuation operations from tautologies on the connectives.

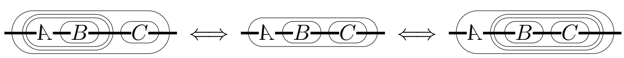
We note these (again) correspond to the required rewrites in Cockett et al.'s circuit diagrams. Figure 43 shows the circuits corresponding to the bottom

$[\mathbb{I} \implies A^\perp \oplus A]$ $[A \otimes A^\perp \implies \bot]$

Fig. 43: A comparison of Cocket et al's circuit diagrams for linear negation circuits.

graphs in 42. Peirce would have been aware of the manipulations that went into these circuit diagrams. Like a dial, the 'scroll' allows us to see the 'inner workings' of the operations that turn wires around in the more general setting.

As mentioned in Section 2, 'scrolls' can be added varyingly around the par'd contexts to reassociate (i.e. nest) par'd terms, add a par'd unit, or (as seen above) reorder an inclusion to yield the linear negation laws. 'Scrolls' can also be added to reassociate implicational terms. We give the following example in Figure 44, which expresses the key morphism for what Cockett et al. call a Bilinear Category [26, p. 103]. The addition of the 'scroll' in each respective location is (again) a

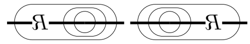


$[(A \multimap B) \dagger C]$ $[A \multimap B \dagger C]$ $[A \multimap (B \dagger C)]$

Fig. 44: Key morphisms for a Bilinear Category

simple way of expressing re-associativity. Cockett et. al recognize the morphisms as such, but we add that this is built into the rules of Peirce's original EGs.

We add a final example for posterity. In the bilinear case the left and right residuals and corresponding duals give rise to four different linear negations [42]. These are expressed, like a sequential variation of the intuitionistic case in Figure 23, by adding a 'scroll' on the requisite side to introduce a par unit. One can imagine, for example, adding these units to the derivations in Figure 42. While the connectives as presented in Figure 45 have the same meaning with the involution, they are distinguishable in the larger cyclic context and proof

$[R\backslash\mathbb{d}]$  $[\mathbb{d}/R]$

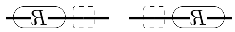
$[R \oslash \mathbb{I}]$  $[\mathbb{I} \oslash R]$

Fig. 45: Bilinear negations.

theory by how they behave, i.e. by keeping track of locations for corresponding expansion and elimination rules.[26] Again, and similar to the question above about whether nesting 'scrolls' can serve the same purpose as proof nets, this is a further direction worth making precise.

Peirce's 'scroll' simultaneously serves as a graph of inclusion and an involution. In this section we have demonstrated a number of further effects from adding a 'scroll' at various locations inside the inclusion. Adding 'scrolls' out of the unit wire or around a unit wire yields the variety of linear negation rules. Adding a 'scroll' can also reassociate in the implicational context, as in Figure 44, yielding the distinguishing morphism of Cockett et al.'s Bilinear Categories. In this section we have shown further that the bilinear operations can take the form of topological moves inside the 'scroll'. We continue demonstrating the graphical efficiencies of Peirce's 'scroll' in the next section, and show how the 'scroll' also keeps track of *refutations*.

5 Proofs and Refutations

While the multiplicative fragment of linear logic is characterized by *-autonomy, and in particular by linear distributivity and linear negation, another common characterization of the 'linear-ity' of linear logic is in terms of separate proof and refutation clauses for the linear connectives. While intuitionistic logic is said to be about proofs, linear logic can be thought of as a dialogical back and forth between a 'prover' and 'refuter'. This can be found, for example, as far back as [11], and also more recently in the anti-thesis interpretation by Shulman [71]. One of the first completeness proofs for linear logic is based on such a prover vs refuter approach in [1].

The intuitive idea behind this approach is that in a derivation the linear negations laws require every atomic expressions to have a linear dual. An appropriate proof then, such as those given in a valid a proof net, has the right number of 'pairs' of atomic expressions and in the right locations to ensure that the linear rules were followed.

Peirce often emphasized the importance of dialogical reasoning — both in thought and in signs [37,58,60].[27] Consider the following passage as one example:

[26] Further discussion on negation in the bilinear context and some further directions can be found in [47].

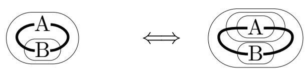
[27] Burch discusses one of the few instances of game semantics in EGs in [20]. See also [65].

> The answer I am reporting now goes on to show, what will hardly be disputed, that all deliberative meditation, or thinking proper, takes the form of a dialogue. The person divides himself into two parties which endeavour to persuade each other. [56, p. 180]

We note in particular that Hintikka's developments in game-theoretic semantics are based in part on the early ideas found in Peirce [37,38,58]. While the emphasis on proofs and refutations was not around at Peirce's time, he was aware of the refutation-clauses of the connectives, including the linear connectives. In fact Peirce's early derivations of linear distributivity were motivated from considerations of the refutation clauses of the connectives [53, see 'On the Logic of Relatives']. We also show, for the first time, how linear contraposition captures the well known linear equivalences in the graphs.

The refutation clauses of the Boolean connectives are well known. What we take to be interesting about the examples in Peirce's graphs is (i) that they show how the 'scroll' is an effective means of moving to the refutation context, and (ii) that the same principle works for the sequential/compositional terms, i.e. for the multiplicatives. We mention again that Peirce clearly employs intuitionistic reasoning about negation in these examples.

As a simple preliminary example, we show a more subtle presentation of the law of contraposition in Figure 46, stating "$A \sqsubseteq B \Leftrightarrow \bar{B} \sqsubseteq \bar{A}$". This is a



$A \sqsubseteq B$ $\qquad$ $\bar{A} \sqsupseteq \bar{B}$

Fig. 46: The Law of Contraposition

straightforward transformation in the graphs that follows from adding a 'double cut' around A. We reverse the direction of implication, $\sqsupseteq$, to signal reading the inclusion from the reverse direction and from bottom to top.

In the examples discussed below, Peirce adds more meaning to contraposition by recognizing that the implication $A \sqsubseteq B$ means that if B were to 'vanish', as he says, then A must also vanish. The vanish here is to go to False or absurdity, as in $B \Rightarrow \bot$ (and graphically: ). Such a meaning can be captured in the graphs, too, and involves making explicit an extra 'double cut' as seen below. We liken this, as in Section 2, to using the 'scroll' to add a par'd (or disjunctive) unit in the context of the graphs — the 'scroll' simply making explicit what is, or *could* be there — but now as a refutation. As in Figure 46, Figure 47 makes this explicit.

A further generalization is needed for the Beta case with sequential composition. We again, though, simply reason in the same manner about what would happen if a consequent were false. We now move to the main example from Peirce

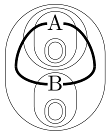


Fig. 47: $(A \Rightarrow \bot) \sqsupseteq (B \Rightarrow \bot)$

[53, p. 338-9] and start with the graph of "every lover of a servant is a benefactor of". Using $-l-$ for the 'loves' relation, $-s-$ for 'serves', and $-b-$ for 'benefactor', this graph is given in the top of Figure 48. Following the discussion above,

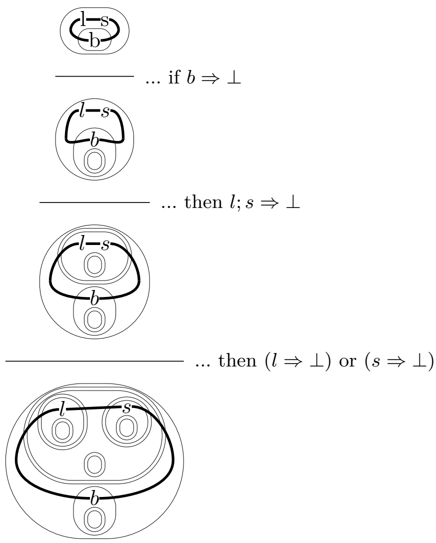


Fig. 48: Establishing the Refutation Clauses

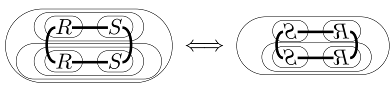
Peirce notes that the meaning of this graph includes that if the consequent b were to vanish, then the antecedent must also vanish. Graphically, this sequence begins in the second derived line in Figure 48, where $b \Rightarrow \bot$ is represented by introducing within the consequent. We next capture the vanishing antecedent by adding further 'scrolls'. The larger 'double cut' around $l; s$ is needed because the orientation of the implication is reversed. The outermost 'cut' signals that we 'peer inside' the first 'cut' to see what the consequent would be, and the consequent now reads (like the b before it) that $l; s \Rightarrow \perp$. Finally, notice that all we have done (and even all we will continue to do) is to add double cuts in or around various subgraphs. This is always appropriate, though the meaning of the graph only changes subtly.

The first important step is done. The next involves thinking about what it means for relational composition to vanish. The composition fails when either of the composed relations vanishes. Again, all we need to express the condition 'Either $l \Rightarrow \perp$ or $s \Rightarrow \perp$' is to add 'double cuts' in or around subgraphs, as in the third line in Figure 48.

At this point we have successfully traced the consequences of the refutation, i.e. what the result would be if the consequent were to vanish and the antecedent were likewise forced to vanish as well. '$l; s \sqsubseteq b$' implies that 'if $b \Rightarrow \perp$ then either ($l \Rightarrow \perp$) or ($s \Rightarrow \perp$)'. And all this is shown in the graphs by making the series of additional 'double cuts' explicit.

The example shows that the 'scroll', standing as it does for an involution, is an effective means of tracking the move to the refutation context. The same principle also extends to relational composition, and its dual, and to the other linear connectives. Peirce goes on to use the same style of reasoning to derive the associativity of the dual of composition, to derive linear distributivity, and to derive the residuation equivalences. For example, when listing the Schröder equivalences Peirce points out that their refutation clauses are all the same and so concludes that these are equivalent expressions (or in our case, equivalent graphs).[28]

As a further example, we show how linear equivalences are captured in the graphs as instances of linear contraposition (as given in the derivation in Figure 42). In the linear case, the contrapositive also induces a converse as in the



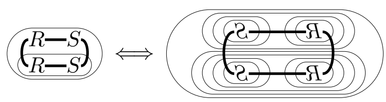
$[id_{R \dagger S}]$

$\left[id_{(R \dagger S)^\perp}\right]$
$\left[= id_{(S^\perp; R^\perp)}\right]$

Fig. 49: $R \dagger S \cong S^\perp; R^\perp$

[28] We mention again that this is for the binary/dyadic relations. The generalization to triadic relations requires that teridentity — the identification of three terms being the same — to be given a refutation where at least one is different. Peirce calls this the *triad of diversity*. This triad involves adding 'scrolls' as well. We also mention that falsity can be replaced with $\perp\!\!\!\perp$, which like $\top\!\!\!\top$ can be thought of as keeping track of the type of the wire. These are employed in the recent neo-Peircean calculus of relations given in [14].

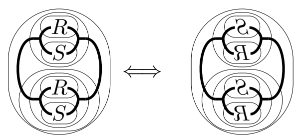
example shown in Figure 27(a). Note that the graph on the left in Figure 49 is the result of transforming the graph on the right by first rotating and inducing the converse, and then by adding a double cut to redirect the order of the inclusion. Following the modified converse in Figure 33, one could similarly turn the page upside down. We give relational composition — the other multiplicative — as another example in Figure 50. Finally, we show one of the additives as well



$[id_{R;S}]$ $\quad$ $\big[id_{(R;S)^{\perp}}\big]$ $\big[= id_{(S^{\perp} \dagger R^{\perp})}\big]$

Fig. 50: $R; S \cong S^{\perp} \dagger R^{\perp}$

in Figure 51. Parallel operations are commutative, and so the final equivalence



$[id_{R \sqcup S}]$ $\quad$ $[id_{(R \sqcup S)^{\perp}}]$ $\big[= id_{(R^{\perp} \sqcap S^{\perp})}\big]$

Fig. 51: $R \sqcup S \cong (R^{\perp} \sqcap S^{\perp})$

follows from an application of commutativity. This is a key difference from the sequential case, where the involution *anti*-distributes rather than distributes. Regardless, the topological features of the scroll captures the difference. The key takeaway is that, given the residuation equations, the 'scroll' displays these topologically as the same graphs.

We list other well-known linear equivalences below in Figure 52. All can be captured in a similar manner topologically in EGs. Note, again, how the cyclic presentation naturally keeps track of the order and any necessary converses. We will come back to the listed exponentials in the next section.

The novelty here is: (i) that these have not been shown in the graphs, (ii) the examples show how the 'scroll' mediates between the proof and refutation contexts, and (iii) that the linear equivalences can be topologically captured by the same graph. The key takeaway is that the De Morgan duality commonly

$$
\begin{array}{ll}
(R \dagger S)^{\perp} \cong S^{\perp} ; R^{\perp} & (R \sqcap S)^{\perp} \cong R^{\perp} \sqcup S^{\perp} \\
(R \,; S)^{\perp} \cong S^{\perp} \dagger R^{\perp} & (R \sqcup S)^{\perp} \cong R^{\perp} \sqcap S^{\perp}
\end{array}
$$

$$
\begin{aligned}
(R^{\perp})^{\perp} &\cong R \\
\mathbb{T}^{\perp} &\cong \perp\!\!\!\perp \\
\mathbb{I}^{\perp} &\cong \mathbb{d}
\end{aligned}
$$

$$
\begin{aligned}
(B \oslash A)^{\perp} &\cong A \multimap B \cong A^{\perp} \multimapinv B^{\perp} \\
(A \oslash B)^{\perp} &\cong B \multimapinv A \cong B^{\perp} \multimap A^{\perp}
\end{aligned}
$$

$$
\begin{aligned}
(!R)^{\perp} &\cong ?(R^{\perp}) \\
(?R)^{\perp} &\cong !(R^{\perp})
\end{aligned}
$$

Fig. 52: Various linear equivalences.

recognized as captured by EGs also extends to the multiplicative (and so linear) case.

We note that these equivalences also follow from the line of identity following the laws of a special Frobenius algebra, and can be derived from those laws. The version of the modern graphical syntax presented in [36,14], based on Peirce's account, develops this view. Of interest here is the addition that the 'scroll' provides; namely, that with residuation the topological features of the scroll *already capture* these equivalences.

We have shown how the 'scroll' reassociates the par'd context and linear implication, the role it plays in capturing the linear negation rules, and we have now shown how the 'scroll' captures the contrapositive and linear contrapositive case. Indeed the only location in the inclusion we have not discussed is the location corresponding to the Boolean contrapositive and the single negation or complement. In [14] it is shown how negation as a unary connective arises out of requisite linear adjunctions as the converse of linear negation.

In this and the previous sections we have greatly generalized the function and purpose of the 'scroll', including its relation to the linear operations and connectives. In the penultimate section we turn towards resource sensitivity.

6 Resource Sensitivity

Linear logic is also described as the logic of resources. The linear negation laws allow for only strict resource production and annihilation, so that specific resources cannot be freely copied or duplicated. As we have made clear, Peirce was well aware of these rules as he arrived at and stated these very laws. We use this last section to mention how Peirce took these notions further and recognized that a notion of *quantity* follows from them. Finally, linear exponentials can be used to add selective copying or duplicating back into the logic. We end with a few remarks on what the linear exponentials look like in the graphs.

Peirce was aware that the linear negation laws, and in particular, the difference or diversity relation, allowed for a notion of quantity. We give an early example from Peirce's algebraic work. Peirce states that from "Some A is B" and "Some A is not-B", we can conclude that "There are at least two A's" [CP 4:88]. This follows from the linear negation laws, where a B composed with a not-B asserts the difference of the remaining terms. The key point Peirce is emphasizing is that some A not being the same as some (other) A means that there are *at least two* A's.

As seen in the quotation about negation in Section 3 and Footnote 21, Peirce often emphasized that logic is a *linear* series of inferences. In the passage with the example given above, Peirce goes on to state that this derivation relies on the principle of contradiction, which he writes as "the non-identity of A and not-A". We note that this appears explicitly to be the multiplicative case, relying as it does on the diversity relation, and Peirce is reading the principle of contradiction specifically in terms of sequential or linear order. Peirce goes on to discuss the transitivity of linear implication, the role it plays in comparative relations, and eventually discusses multitudes, which is Peirce's term for expressing the size of a collection. We remind the reader of the earlier passage where Peirce states that linear implication "form[s] the basis of all quantitative thought". Indeed each of these developments arises from Peirce's study of the linear negation laws (and in particular the diversity relation) and the notions of quantity that he interprets to follow from them.

In the relational model, the linear exponentials are given by the relations in Figure 53.[29] Discussion of the exponentials, some of their properties, and their

$!R$ $[= \mathbb{I} \sqcap R]$ $?R$ $[= \mathbb{d} \sqcup R]$

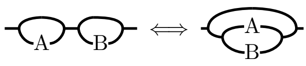
Fig. 53: Exponentials

representation in the relational model can be found in Section 14 in [42]. We know of no place where Peirce calls out these relations and emphasizes their significance as we might, at least with respect to the intuitionistic fragment, motivate them today. One noteworthy passage, however, is worth discussing below.

Perhaps the key feature of the !-exponential is that it converts a multiplicative into an additive. Equationally, this is expressed by the following: $!A \otimes !B \vdash !(A \wedge B)$. We show this graphically in Figure 54. Surprisingly, Peirce makes the same observation in MS 430 and reprinted in part below in Figure 55.[30] Peirce goes on to finish the passage by noting that the last two graphs, equivalent to those in Figure 54, are the same. Peirce is identifying, via topological features, the key

[29] We note again that $\mathbb{d} = \mathbb{I}^{\perp}$ and have left out the converse.

[30] We thank Jukka Nikulainen again for the digitization and presentation in the figure.



$!A; !B$ $\qquad\qquad$ $!(A \sqcap B)$

Fig. 54: Key property of !-exponential.

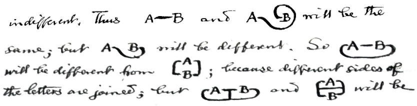

indifferent. Thus A—B and A B will be the same; but A B will be different. So A—B will be different from A B; because different sides of the letters are joined; but A T B and A B will be

Fig. 55: Peirce's identification of a key feature of the !-exponential.

property of the !-exponential. This is a remarkable example of where a concern for graphical syntax and topological features can lead.

A list of well known linear equivalences were given in the previous section in Figure 52. One can show the equivalences for the linear exponentials using the 'scroll' in the same topological manner. We claim that one can express various further properties of exponentials described by Lambek in [42] using the graphs and Pierce's rules.[31] Finally, a key feature of the !-exponential is that it allows one to express intuitionistic implication [42, p. 233]. It is interesting to note how the !-exponential combines with residuation in this case. The additional identity relation has the effect of topologically turning a sequential, linear implication into a graph closer to the vertical (i.e. parallel) inclusion. These directions and their corresponding graphical outcroppings are all worth further study.

7 Conclusion

C.S. Peirce's early work on the logic of the relations has been absorbed into the relation algebraic tradition (following Tarski) and into the development of first-order logic (following his contribution to the discovery of quantifiers, for example). Much of this follows from an awareness of Peirce's earlier algebraic work. At the same time much of the work on Peirce's later Existential Graphs has remained outside the larger logic tradition and has remained, following Robert's initial systematic treatment of the subject, in the confines of first-order logic. This is slowly changing as alternative methods of diagrammatic reasoning — most notably the advent of string diagrams in category theory, but also the focus on graph rewriting more generally — have brought awareness back to Pierce's earlier work in diagrammatic reasoning.

This paper continues in this changing direction and calls attention to Peirce's early developments in linear logic. Peirce's presentation of the calculus of relations from 1883 is noteworthy for emphasizing the dual to relational composition,

[31] Such as, for example, using the rules and syntax found in [14].

linear distributivity, and the linear negation laws. Around the same time Peirce also carried out a systematic study of residuation and stated a rule for how composed terms can be rotated to the dual context on the other side of the inclusion. These are the key rules for bilinear logic and we conclude that Peirce understood these aspects of the relational model of bilinear logic.

This work on the calculus of relations occurred before Peirce's later development of the Existential Graphs. We further showed that these concepts play key roles in his later studies in the graphs. This includes derivations involving linear implications, studies of the par'd context involving the dual of relational composition, and more complicated expressions involving multiple residuals. We have shown some of these graphs, many of which have not been written about in the context of EGs before, and do so for the first time within the context of these bilinear operations.

We reiterate that this is the beginning of such work. Further directions that arose during the exposition include comparisons to the Lambek calculus and categorial grammar, potential generalizations towards proof nets and proof circuits, and new directions for intuitionistic variants of EGs. All of these we believe are worth further study.

We end with a takeaway comment about Peirce scholarship. Peirce emphasized the importance of the 'scroll' both as a key graphical feature of EGs and as a sign of the most primitive logical connective. This work restores the 'scroll' to a place of primary importance in the study of the graphs. The 'scroll' serves at one and the same time *as a sign of involution* and *as a sign of inclusion*. Much of this emphasis on the 'scroll' appears to be vindicated in the *-autonomous and bilinear settings, which emphasize these features.

In the introduction, we wrote of the hope that this work returns interest back to Peirce's early studies and contributions. We add here the hope that this work returns interest to what a concern for diagrammatic reasoning and corresponding topological features is capable of. We suspect Peirce's contribution along these lines to have a bright future.

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4 Article 4 - Diagrammatic Algebra of First Order Logic

IV
F. Bonchi, A. D. Giorgio, N. Haydon, and P. Sobocinski. Diagrammatic algebra of first order logic. To appear at LICS, 2024

Diagrammatic Algebra of First Order Logic

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ABSTRACT

We introduce the calculus of neo-Peircean relations, a string diagrammatic extension of the calculus of binary relations that has the same expressivity as first order logic and comes with a complete axiomatisation. The axioms are obtained by combining two well known categorical structures: cartesian and linear bicategories.

CCS CONCEPTS

• **Theory of computation** → **Logic**; **Categorical semantics**.

KEYWORDS

calculus of relations, string diagrams, deep inference

1 INTRODUCTION

The modern understanding of first order logic (FOL) is the result of an evolution with contributions from many philosophers and mathematicians. Amongst these, particularly relevant for our exposition is the calculus of relations (CR) by Charles S. Peirce [62]. Peirce, inspired by De Morgan [55], proposed a relational analogue of Boole's algebra [12]: a rigorous mathematical language for combining relations with operations governed by algebraic laws.

With the rise of first order logic, Peirce's calculus was forgotten until Tarski, who in [80] recognised its algebraic flavour. In the introduction to [81], written shortly before his death, Tarski put much emphasis on two key features of CR: (a) its lack of quantifiers and (b) its sole deduction rule of substituting equals by equals. The calculus, however, comes with two great shortcomings: (c) it is strictly less expressive than FOL and (d) it is *not* axiomatisable.

Despite these limitations, CR had —and continues to have— a great impact in computer science, e.g., in the theory of databases [20] and in the semantics of programming languages [2, 38, 45, 47, 74]. Indeed, the lack of quantifiers avoids the usual burden of bindings, scopes and capture-avoid substitutions (see [25, 30, 33, 40, 68, 70] for some theories developed to address specifically the issue of bindings). This feature, together with purely equational proofs, makes CR particularly suitable for proof assistants [43, 71, 72].

Less influential in computer science, there are two others quantifiers-free alternatives to FOL that are worth mentioning: first, *predicate functor logic* (PFL) [75] that was thought by Quine as the first order logic analogue of combinatory logic [22] for the λ-calculus; second, Peirce's *existential graphs* (EGs) [77] and, in particular, its fragment named *system β*. In this system FOL formulas are *diagrams* and the deduction system consists of rules for their manipulation. Peirce's work on EGs remained unpublished during his lifetime.

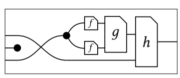
Diagrams have been used as formal entities since the dawn of computer science, e.g. in the Böhm-Jacopini theorem [3]. More recently, the spatial nature of mobile computations led Milner to move from the traditional term-based syntax of process calculi to bigraphs [53]. Similarly, the impossibility of copying quantum information and, more generally, the paradigm-shift of treating data as a physical resource (see e.g. [31, 59]), has led to the use [1, 5, 6, 10, 21, 26, 27, 32, 56, 69] of *string diagrams* [42, 79] as syntax. String diagrams, formally arrows of a freely generated symmetric (strict) monoidal category, combine the rigour of traditional terms with a visual and intuitive graphical representation. Like traditional terms, they can be equipped with a compositional semantics.

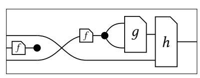
In this paper, we introduce the calculus of *neo-Peircean relations*, a string diagrammatic account of FOL that has several key features:

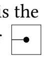
(1) Its diagrammatic syntax is closely related to Peirce's EGs, but it can also be given through a context free grammar equipped with an elementary type system;
(2) It is quantifier-free and, differently than FOL, its compositional semantics can be given by few simple rules: see (8);
(3) Terms and predicates are not treated as separate syntactic and semantic entities;
(4) Its sole deduction rule is substituting equals by equals, like CR, but differently, it features a complete axiomatisation;
(5) The axioms are those of well-known algebraic structures, also occurring in different fields such as linear algebra [11] or quantum foundations [21];
(6) It allows for compositional encodings of FOL, CR and PFL;
(7) String diagrams disambiguate interesting corner cases where traditional FOL encounters difficulties. One perk is that we allow empty models —forbidden in classical treatments— leading to (slightly) more general Gödel completeness;
(8) The corner case of empty models coincides with *propositional* models and in that case our axiomatisation simplifies to the deep inference Calculus of Structures [15, 34].

By returning to the algebraic roots of logic we preserve CR's benefits (a) and (b) while overcoming its limitations (c) and (d).

Cartesian syntax. To ease the reader into this work, we show how traditional terms appear as string diagrams. Consider a signature Σ consisting of a unary symbol f and two binary symbols g and h. The term $h(\, g(f(x_3), f(x_3)), x_1\,)$ corresponds to the string diagram on the left below.





A difference wrt traditional syntax tree is the explicit treatment of copying and discarding. The discharger  informs us that the

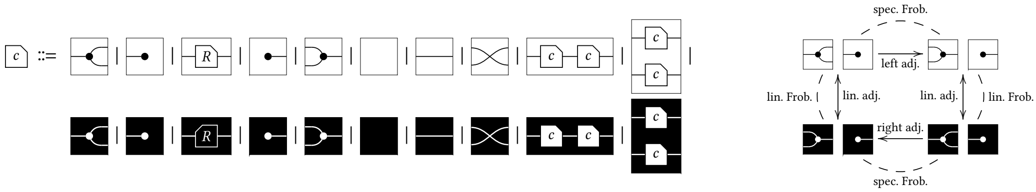
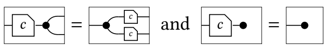


Figure 1: Diagrammatic syntax of NPR_Σ (left) and a summary of its axioms (right)

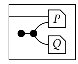
variable x_2 does not appear in the term; the copier makes clear that the variable x_3 is shared by two sub-terms. The string diagram on the represents the same term if one admits the equations


(Nat)

Fox [28] showed that (Nat) together with axioms asserting that copier and discard form a *comonoid* (($\blacktriangleleft^\circ$-as), ($\blacktriangleleft^\circ$-un), ($\blacktriangleleft^\circ$-co) in Fig. 2) force the monoidal category of string diagrams to be *cartesian* ($\otimes$ is the categorical product): actually, it is the *free* cartesian category on Σ.

Functorial semantics. The work of Lawvere [48] illustrates the deep connection of syntax with semantics, explaining why cartesian syntax is so well-suited to functional structures, but also hinting at its limitations when denoting other structures, e.g. relations. Given an algebraic theory $\mathbb{T}$ in the universal algebraic sense, i.e., a signature Σ with a set of equations E, one can freely generate a cartesian category $\mathbf{L}_\mathbb{T}$. *Models* –in the standard algebraic sense– are in one-to-one correspondence with cartesian functors $\mathcal{M}$ from $\mathbf{L}_\mathbb{T}$ to **Set**, the category of sets and functions. More generally, models of the theory in any cartesian category **C** are cartesian functors $\mathcal{M}\colon \mathbf{L}_\mathbb{T} \to \mathbf{C}$. By taking **C** to be $\mathbf{Rel}^\circ$, the category of sets and relations, one could wish to use the same approach for relational theories but any such attempt fails immediately since the cartesian product of sets is not the categorical product in $\mathbf{Rel}^\circ$.

Cartesian bicategories. An evolution of Lawvere's approach for relational structures is developed in [7, 9, 78]. Departing from cartesian syntax, it uses string diagrams generated by the *first* row of the grammar in Fig. 1, where R is taken from a monoidal signature Σ – a set of symbols equipped with both an arity and also a *coarity* – and can be thought of as akin to relation symbols of FOL. The diagrams are subject to the laws of cartesian bicategories [16] in Fig. 2: and form a comonoid, but the category of diagrams is not cartesian since the equations in (Nat) hold laxly (($\blacktriangleleft^\circ$-nat), ($!^\circ$-nat)). The diagrams and form a *monoid* (($\blacktriangleright^\circ$-as), ($\blacktriangleright^\circ$-un), ($\blacktriangleright^\circ$-co)) and are right *adjoint* to copier and discard. Monoids and comonoids together satisfy *special Frobenius* equations ((S°),(F°)). The category of diagrams $\mathbf{CB}_\Sigma$ is the free cartesian bicategory generated by Σ and, like in Lawvere's functorial semantics, models are morphisms of cartesian bicategories $\mathcal{M}\colon \mathbf{CB}_\Sigma \to \mathbf{Rel}^\circ$. Importantly, the laws of cartesian bicategories provide a complete axiomatisation for $\mathbf{Rel}^\circ$, meaning that c, d in $\mathbf{CB}_\Sigma$ are provably equal with the laws of cartesian bicategories iff $\mathcal{M}(c) = \mathcal{M}(d)$ for all models $\mathcal{M}$.



The (co)monoid structures allow one to express existential quantification: for instance, the FOL formula $\exists x_2.P(x_1,x_2) \wedge Q(x_2)$ is depicted as the diagram on the right. The expressive power of $\mathbf{CB}_\Sigma$ is, however, limited to the existential-conjunctive fragment of FOL.

Cocartesian bicategories. To express the universal-disjunctive fragment, we consider the category $\mathbf{CB}_\Sigma$ (white on black) of string diagrams generated by the *second* row of the grammar in Fig. 1 and subject to the laws of cocartesian bicategories in Fig. 3: those of cartesian bicategories but with the reversed order $\geq$. The diagrams of $\mathbf{CB}_\Sigma$ (white on black) are photographic negative of those in $\mathbf{CB}_\Sigma$. To explain this change of colour, note that sets and relations form *another* category: $\mathbf{Rel}^\bullet$. Composition $\,⨟\,$ (black) in $\mathbf{Rel}^\bullet$ is the De Morgan dual of the usual relational composition: $R ⨟ S \stackrel{\text{def}}{=} \{(x,z) \mid \exists y.(x,y) \in R \wedge (y,z) \in S\}$ while $R ⨟ S \stackrel{\text{def}}{=} \{(x,z) \mid \forall y.(x,y) \in R \vee (y,z) \in S\}$. While $\mathbf{Rel}^\circ$ is a cartesian bicategory, $\mathbf{Rel}^\bullet$ is *cocartesian*. Interestingly, the "black" composition ⨟ was used in Peirce's approach [61] to relational algebra.

Just as $\mathbf{CB}_\Sigma$ is complete with respect to $\mathbf{Rel}^\circ$, dually, $\mathbf{CB}_\Sigma$ (white on black) is complete wrt $\mathbf{Rel}^\bullet$. The former accounts for the existential-conjunctive fragment of FOL; the latter for its universal-disjunctive fragment. This raises a natural question:

How do the white and black structures combine to form a complete account of first order logic?

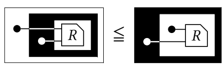
Linear bicategories. Although $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$ have the same objects and arrows, there are two different compositions (⨟ and ⨟). The appropriate categorical structures to deal with these situations are *linear bicategories* introduced in [17] as a horizontal categorification of linearly distributive categories [19, 23]. The laws of linear bicategories are in Fig. 4: the key law is *linearly distributivity* of ⨟ over ⨟ ((δ_l), (δ_r)), that was already known to hold for relations since the work of Peirce [61]. Another crucial property observed by Peirce is that for any relation $R \subseteq X \times Y$, the relation $R^\perp \subseteq Y \times X \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \notin R\}$ is its *linear adjoint*. This operation has an intuitive graphical representation: given c, take its mirror image and then its photographic negative. For instance, the linear adjoint of R is R (white on black).

First order bicategories. The final step is to characterise how cartesian, cocartesian and linear bicategories combine: *(i)* white and black (co)monoids are linear adjoints that *(ii)* satisfy a "linear" version of the Frobenius law. We dub the result *first order bicategories*. We shall see that this is a complete axiomatisation for first order logic, yet all of the algebraic machinery is compactly summarised at the right of Fig. 1.

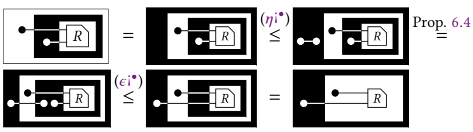
Functorial semantics for first order theories. In the spirit of functorial semantics, we take the free first order bicategory $\mathbf{FOB}_{\mathbb{T}}$ generated by a theory $\mathbb{T}$ and observe that models of $\mathbb{T}$ in a first order bicategory C are morphisms $\mathcal{M}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$. Taking $\mathbf{C} = \mathbf{Rel}$, the first order bicategory of sets and relations, these are models in the sense of FOL with one notable exception: in FOL models with the empty domain are forbidden. As we shall wee, theories with empty models are exactly the propositional theories.

Completeness. We prove that the laws of first order bicategories provide a complete axiomatisation for first order logic. The proof is a diagrammatic adaptation of Henkin's proof [37] of Gödel's completeness theorem. However, in order to properly consider models with an empty domain, we make a slight additional step to go beyond Gödel completeness.

A taste of diagrammatic logic. Before we introduce the calculus of neo-Peircean relations, we start with a short worked example to give the reader a taste of using the calculus to prove a non-trivial result of first order logic. Doing so lets us illustrate the methodology of proof within the calculus, which is sometimes referred to as diagrammatic reasoning or string diagram surgery.



Let R be a symbol with arity 2 and coarity 0. The two diagrams on the right correspond to FOL formulas $\exists x.\, \forall y.\, R(x, y)$ and $\forall y.\, \exists x.\, R(x, y)$: see § 9 for a dictionary of translating between FOL and diagrams. It is well-known that $\exists x.\, \forall y.\, R(x, y) \models \forall y.\, \exists x.\, R(x, y)$, i.e. in any model, if the first formula evaluates to true then so does the second. Within our calculus, this statement is expressed as the above inequality. This can be proved by mean of the axiomatisation we introduce in this work:



(1)

The central step relies on the particularly good behaviour of *maps*, intuitively those relations that are functional. In particular  is an example. The details are not important at this stage.

Synopsis. We begin by recalling CR in § 2. The calculus of neo-Peircean relations is introduced in § 3, together with the statement of our main result (Theorem 3.2). We recall (co)cartesian and linear bicategories in § 4 and § 5. The categorical structures most important for our work are first-order bicategories, introduced in § 6. In § 7 we consider first order theories, the diagrammatic version of the deduction theorem (Theorem 7.7) and some subtle differences with FOL that play an important role on the proof of completeness in § 8. Translations of CR and FOL into the calculus of neo-Peircean relations are given in § 8.1 and § 9. The encoding of PFL and additional material omitted due to space restrictions are in Appendix B. All proofs are in the remaining appendices.

2 THE CALCULUS OF BINARY RELATIONS

The calculus of binary relations, in an original presentation given by Peirce in [61], features two forms of relational compositions $⨾$ and $\mathbin{;\!\!\!\bullet}$, defined for all relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ as

$$R ⨾ S \stackrel{\text{def}}{=} \{(x, z) \mid \exists y \in Y \,.\, (x, y) \in R \wedge (y, z) \in S\} \subseteq X \times Z \text{ and}$$
$$R \mathbin{;\!\!\!\bullet} S \stackrel{\text{def}}{=} \{(x, z) \mid \forall y \in Y \,.\, (x, y) \in R \vee (y, z) \in S\} \subseteq X \times Z \qquad (2)$$

with units the equality and the difference relations respectively, defined for all sets X as

$$id^{\circ}_X \stackrel{\text{def}}{=} \{(x,y) \mid x = y\} \subseteq X \times X \text{ and } id^{\bullet}_X \stackrel{\text{def}}{=} \{(x,y) \mid x \neq y\} \subseteq X \times X. \quad (3)$$

Beyond the usual union $\cup$, intersection $\cap$, and their units $\bot$ and $\top$, the calculus also features two unary operations $(\cdot)^\dagger$ and $\overline{(\cdot)}$ denoting the opposite and the complement: $R^\dagger \stackrel{\text{def}}{=} \{(y, x) \mid (x, y) \in R\}$ and $\overline{R} \stackrel{\text{def}}{=} \{(x, y) \mid (x, y) \notin R\}$. In summary, its syntax is given by the following context free grammar

$$\begin{array}{rclllllllllll} E & ::= & R & \mid & id^\circ & \mid & E ⨾ E & \mid & id^\bullet & \mid & E \mathbin{;\!\!\!\bullet} E & \mid \\ & & E^\dagger & \mid & \top & \mid & E \cap E & \mid & \bot & \mid & E \cup E & \mid & \overline{E} \end{array} \qquad (\mathsf{CR}_\Sigma)$$

where R is taken from a given set Σ of generating symbols. The semantics is defined wrt a *relational interpretation* $\mathcal{I}$, that is, a set X together with a binary relation $\rho(R) \subseteq X \times X$ for each $R \in \Sigma$.

$$\begin{array}{lll} \langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \rho(R) & \langle id^\circ \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\circ_X & \langle E_1 ⨾ E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} ⨾ \langle E_2 \rangle_{\mathcal{I}} \\ \langle E^\dagger \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E \rangle_{\mathcal{I}}^\dagger & \langle id^\bullet \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^\bullet_X & \langle E_1 \mathbin{;\!\!\!\bullet} E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{;\!\!\!\bullet} \langle E_2 \rangle_{\mathcal{I}} \\ \langle \overline{E} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} & \langle \bot \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \varnothing & \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} \\ & \langle \top \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} X \times X & \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}} \end{array} \qquad (4)$$

Two expressions E_1, E_2 are said to be *equivalent*, written $E_1 \equiv_{\mathsf{CR}} E_2$, if and only if $\langle E_1 \rangle_{\mathcal{I}} = \langle E_2 \rangle_{\mathcal{I}}$, for all interpretations $\mathcal{I}$. Inclusion, denoted by $\leq_{\mathsf{CR}}$, is defined analogously by replacing $=$ with $\subseteq$. For instance, the following inclusions hold, witnessing the fact that $⨾$ *linearly distributes* over $\mathbin{;\!\!\!\bullet}$.

$$R ⨾ (S \mathbin{;\!\!\!\bullet} T) \leq_{\mathsf{CR}} (R ⨾ S) \mathbin{;\!\!\!\bullet} T \qquad (R \mathbin{;\!\!\!\bullet} S) ⨾ T \leq_{\mathsf{CR}} R \mathbin{;\!\!\!\bullet} (S ⨾ T) \qquad (5)$$

Along with the boolean laws, in 'Note B' [61] Peirce states (5) and stresses its importance. However, since $R \mathbin{;\!\!\!\bullet} S \equiv_{\mathsf{CR}} \overline{\overline{R} ⨾ \overline{S}}$ and $id^\bullet \equiv_{\mathsf{CR}} \overline{id^\circ}$, both $\mathbin{;\!\!\!\bullet}$ and $id^\bullet$ are often considered redundant, for instance by Tarski [80] and much of the later work.

Tarski asked whether $\equiv_{\mathsf{CR}}$ can be axiomatised, i.e., is there a basic set of laws from which one can prove all the valid equivalences? Unfortunately, there is no finite complete axiomatisations for the whole calculus [54] nor for several fragments, e.g., [4, 29, 39, 73, 76].

Our work returns to the same problem, but from a radically different perspective: we see the calculus of relations as a sub-calculus of a more general system for arbitrary (i.e. not merely binary) relations. The latter is strictly more expressive than CR_Σ – actually it is as expressive as first order logic (FOL)– but allows for an elementary complete axiomatisation based on the interaction of two influential algebraic structures: that of linear bicategories and cartesian bicategories.

3 NEO-PEIRCEAN RELATIONS

Here we introduce the calculus of *neo-Peircean relations* (NPR_Σ).

The first step is to move from binary relations $R \subseteq X \times X$ to relations $R \subseteq X^n \times X^m$ where, for any $n \in \mathbb{N}$, X^n denotes the set of row vectors $(x_1, \ldots, x_n)$ with all $x_i \in X$. In particular, X^0 is the one

Table 1: Typing rules (top); inductive definitions of syntactic sugar (middle); structural congruence (bottom)

$$
\begin{array}{c}
id^{\circ}_0 : 0 \to 0 \quad id^{\circ}_1 : 1 \to 1 \quad \sigma^{\circ}_{1,1} : 2 \to 2 \qquad \dfrac{ar(R) = n \quad coar(R) = n}{R^{\circ} : n \to m} \qquad \dfrac{ar(R) = n \quad coar(R) = m}{R^{\bullet} : m \to n} \qquad \dfrac{c\colon n_1 \to m_1 \quad d\colon n_2 \to m_2}{c \otimes d\colon n_1 + n_2 \to m_1 + m_2} \qquad \dfrac{c\colon n \to m \quad d\colon m \to o}{c \fatsemi d\colon n \to o} \\
\blacktriangleleft^{\circ}_1 : 1 \to 2 \quad !^{\circ}_1 : 1 \to 0 \quad \blacktriangleright^{\circ}_1 : 2 \to 1 \quad \text{¡}^{\circ}_1 : 0 \to 1
\end{array}
$$

$$
\begin{array}{llll}
\blacktriangleleft^{\circ}_0 = id^{\circ}_0 \quad \blacktriangleleft^{\circ}_{n+1} = (\blacktriangleleft^{\circ}_1 \otimes \blacktriangleleft^{\circ}_n) \fatsemi (id^{\circ}_1 \otimes \sigma^{\circ}_{1,n} \otimes id^{\circ}_n) & !^{\circ}_0 = id^{\circ}_0 \quad !^{\circ}_{n+1} = !^{\circ}_1 \otimes !^{\circ}_n & id^{\circ}_0 = id^{\circ}_0 & \sigma^{\circ}_{0,0} = id^{\circ}_0 \qquad \sigma^{\circ}_{1,0} = \sigma^{\circ}_{0,1} = id^{\circ}_1 \\
\blacktriangleright^{\circ}_0 = id^{\circ}_0 \quad \blacktriangleright^{\circ}_{n+1} = (id^{\circ}_1 \otimes \sigma^{\circ}_{1,n} \otimes id^{\circ}_n) \fatsemi (\blacktriangleright^{\circ}_1 \otimes \blacktriangleright^{\circ}_n) & \text{¡}^{\circ}_0 = id^{\circ}_0 \quad \text{¡}^{\circ}_{n+1} = \text{¡}^{\circ}_1 \otimes \text{¡}^{\circ}_n & id^{\circ}_{n+1} = id^{\circ}_1 \otimes id^{\circ}_n & \sigma^{\circ}_{1,n+1} = (\sigma^{\circ}_{1,n} \otimes id^{\circ}_1) \fatsemi (id^{\circ}_n \otimes \sigma^{\circ}_{1,1}) \qquad \sigma^{\circ}_{m+1,n} = (id^{\circ}_1 \otimes \sigma^{\circ}_{m,n}) \fatsemi (\sigma^{\circ}_{1,n} \otimes id^{\circ}_m)
\end{array}
$$

$$
a \fatsemi (b \fatsemi c) = (a \fatsemi b) \fatsemi c \quad id^{\circ}_n \fatsemi c = c = c \fatsemi id^{\circ}_m \quad (a \otimes b) \otimes c = a \otimes (b \otimes c) \quad id^{\circ}_0 \otimes c = c = id^{\circ}_0 \otimes c \quad (a \otimes b) \fatsemi (c \otimes d) = (a \fatsemi c) \otimes (b \fatsemi d) \quad \sigma^{\circ}_{1,1} \fatsemi \sigma^{\circ}_{1,1} = id^{\circ}_2 \quad (c \otimes id^{\circ}_o) \fatsemi \sigma^{\circ}_{m,o} = \sigma^{\circ}_{n,o} \fatsemi (id^{\circ}_o \otimes c)
$$

element set $\mathbb{1} \stackrel{\text{def}}{=} \{\star\}$. Considering this kind of relations allows us to identify two novel fundamental constants: the *copier* $\blacktriangleleft^{\circ}_X \subseteq X \times X^2$ which is the diagonal function $\langle id^{\circ}_X, id^{\circ}_X \rangle \colon X \to X \times X$ (considered as a relation) and the *discharger* $!^{\circ}_X \subseteq X \times \mathbb{1}$ which is, similarly, the unique function from X to $\mathbb{1}$. By combining them with opposite and complement we obtain, in total, 8 basic relations.

$$
\begin{array}{ll}
\blacktriangleleft^{\circ}_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x = y \wedge x = z\} & !^{\circ}_X \stackrel{\text{def}}{=} \{(x,\star) \mid x \in X\} \\
\blacktriangleright^{\circ}_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x = y \wedge x = z\} & \text{¡}^{\circ}_X \stackrel{\text{def}}{=} \{(\star,x) \mid x \in X\} \\
\hline
\blacktriangleleft^{\bullet}_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x \neq y \vee x \neq z\} & !^{\bullet}_X \stackrel{\text{def}}{=} \varnothing \\
\blacktriangleright^{\bullet}_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x \neq y \vee x \neq z\} & \text{¡}^{\bullet}_X \stackrel{\text{def}}{=} \varnothing
\end{array}
\tag{6}
$$

Together with id°_X and $id^{\bullet}_X$ and the compositions $\fatsemi$ and $\mathbin{;^{\bullet}}$ from (3), there are black and white *symmetries*: $\sigma^{\circ}_{X,Y} \stackrel{\text{def}}{=} \{(\ (x,y),(y,x)\ ) \mid x \in X, y \in Y\}$ and $\sigma^{\bullet}_{X,Y} \stackrel{\text{def}}{=} \overline{\sigma^{\circ}_{X,Y}}$. The calculus does *not* feature the boolean operators nor the opposite and the complement: these can be derived using the above structure and two *monoidal products* $\otimes$ and $\circledast$, defined for $R \subseteq X \times Y$ and $S \subseteq V \times W$ as

$$
\begin{array}{rcl}
R \otimes S & \stackrel{\text{def}}{=} & \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \wedge (v,w) \in S\} \\
R \circledast S & \stackrel{\text{def}}{=} & \{(\ (x,v),(y,w)\ ) \mid (x,y) \in R \vee (v,w) \in S\}\ .
\end{array}
\tag{7}
$$

Syntax. Terms are defined by the following context free grammar

$$
\begin{array}{rl}
c ::= & \blacktriangleleft^{\circ}_1 \mid !^{\circ}_1 \mid R^{\circ} \mid \text{¡}^{\circ}_1 \mid \blacktriangleright^{\circ}_1 \mid id^{\circ}_0 \mid id^{\circ}_1 \mid \sigma^{\circ}_{1,1} \mid c \fatsemi c \mid c \otimes c \mid \\
& \blacktriangleleft^{\bullet}_1 \mid !^{\bullet}_1 \mid R^{\bullet} \mid \text{¡}^{\bullet}_1 \mid \blacktriangleright^{\bullet}_1 \mid id^{\bullet}_0 \mid id^{\bullet}_1 \mid \sigma^{\bullet}_{1,1} \mid c \mathbin{;^{\bullet}} c \mid c \circledast c
\end{array}
\tag{NPR_Σ}
$$

where R, like in CR_Σ, belongs to a fixed set Σ of *generators*. Differently than in CR_Σ, each $R \in \Sigma$ comes with two natural numbers: arity $ar(R)$ and coarity $coar(R)$. The tuple $(\Sigma, ar, coar)$, usually simply Σ, is a *monoidal signature*. Intuitively, every $R \in \Sigma$ represents some relation $R \subseteq X^{ar(R)} \times X^{coar(R)}$.

In the first row of (NPR_Σ) there are eight constants and two operations: white composition ($\fatsemi$) and white monoidal product ($\otimes$). These, together with identities (id°_0 and id°_1) and symmetry ($\sigma^{\circ}_{1,1}$) are typical of symmetric monoidal categories. Apart from R°, the constants are the copier ($\blacktriangleleft^{\circ}_1$), discharger ($!^{\circ}_1$) and their opposite cocopier ($\blacktriangleright^{\circ}_1$) and codischarger ($\text{¡}^{\circ}_1$). The second row contains the "black" versions of the same constants and operations. Note that our syntax does not have variables, no quantifiers, nor the usual associated meta-operations like capture-avoiding substitution.

We shall refer to the terms generated by the first row as the *white fragment*, while to those of second row as the *black fragment*. Sometimes, we use the gray colour to be agnostic wrt white or black. The rules in top of Table 1 assigns to each term at most one type $n \to m$. We consider only those terms that can be typed. For all $n, m \in \mathbb{N}$, $id^{\circ}_n \colon n \to n$, $\sigma^{\circ}_{n,m} \colon n + m \to m + n$, $\blacktriangleleft^{\circ}_n \colon n \to n + n$, $\blacktriangleright^{\circ}_n \colon n + n \to n$, $!^{\circ}_n \colon n \to 0$ and $\text{¡}^{\circ}_n \colon 0 \to n$ are as in middle of Table 1.

Semantics. As for CR_Σ, the semantics of NPR_Σ needs an interpretation $\mathcal{I} = (X, \rho)$: a set X, the *semantic domain*, and $\rho(R) \subseteq X^{ar(R)} \times X^{coar(R)}$ for each $R \in \Sigma$. The semantics of terms is:

$$
\begin{array}{llll}
\mathcal{I}^\sharp(\blacktriangleleft^{\circ}_1) \stackrel{\text{def}}{=} \blacktriangleleft^{\circ}_X & \mathcal{I}^\sharp(!^{\circ}_1) \stackrel{\text{def}}{=} !^{\circ}_X & \mathcal{I}^\sharp(\blacktriangleright^{\circ}_1) \stackrel{\text{def}}{=} \blacktriangleright^{\circ}_X & \mathcal{I}^\sharp(\text{¡}^{\circ}_1) \stackrel{\text{def}}{=} \text{¡}^{\circ}_X \\
\mathcal{I}^\sharp(id^{\circ}_0) \stackrel{\text{def}}{=} id^{\circ}_{\mathbb{1}} & \mathcal{I}^\sharp(id^{\circ}_1) \stackrel{\text{def}}{=} id^{\circ}_X & \mathcal{I}^\sharp(\sigma^{\circ}_{1,1}) \stackrel{\text{def}}{=} \sigma^{\circ}_{X,X} & \mathcal{I}^\sharp(R^{\circ}) \stackrel{\text{def}}{=} \rho(R) \\
\mathcal{I}^\sharp(c \fatsemi d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \fatsemi \mathcal{I}^\sharp(d) & \mathcal{I}^\sharp(c \otimes d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) & \mathcal{I}^\sharp(R^{\bullet}) \stackrel{\text{def}}{=} \overline{\rho(R)}^{\dagger} &
\end{array}
\tag{8}
$$

The constants and operations appearing on the right-hand-side of the above equations are amongst those defined in (2), (3), (6), (7). A simple inductive argument confirms that $\mathcal{I}^\sharp$ maps terms c of type $n \to m$ to relations $R \subseteq X^n \times X^m$. In particular, $id^{\circ}_0 \colon 0 \to 0$ is sent to $id^{\circ}_{\mathbb{1}} \subseteq \mathbb{1} \times \mathbb{1}$, since $X^0 = \mathbb{1}$ independently of X. Note that there are only two relations on the singleton set $\mathbb{1} = \{\star\}$: the relation $\{(\star,\star)\} \subseteq \mathbb{1} \times \mathbb{1}$ and the empty relation $\varnothing \subseteq \mathbb{1} \times \mathbb{1}$. These are, by (3), $id^{\circ}_{\mathbb{1}}$ and $id^{\bullet}_{\mathbb{1}}$, embodying *truth* and *falsity*.

Example 3.1. Take Σ with two symbols R and S with arity and coarity 1. From Table 1, the two terms below have type $1 \to 1$.

$$
!^{\circ}_1 \fatsemi \text{¡}^{\circ}_1 \qquad \blacktriangleleft^{\circ}_1 \fatsemi ((R^{\circ} \otimes S^{\circ}) \fatsemi \blacktriangleright^{\circ}_1)
\tag{9}
$$

For any interpretation $\mathcal{I} = (X, \rho)$, $\mathcal{I}^\sharp(!^{\circ}_1 \fatsemi \text{¡}^{\circ}_1)$ is the top $X \times X$:

$$
\begin{aligned}
\mathcal{I}^\sharp(!^{\circ}_1 \fatsemi \text{¡}^{\circ}_1) = !^{\circ}_X \fatsemi \text{¡}^{\circ}_X &= \{(x,\star) \mid x \in X\} \fatsemi \{(\star,x) \mid x \in X\} \\
&= \{(x,y) \mid x,y \in X\} = X \times X = \langle \top \rangle_{\mathcal{I}}.
\end{aligned}
$$

Similarly, $\mathcal{I}^\sharp(\blacktriangleleft^{\circ}_1 \fatsemi ((R^{\circ} \otimes S^{\circ}) \fatsemi \blacktriangleright^{\circ}_1)) = \rho(R) \cap \rho(S) = \langle R \cap S \rangle_{\mathcal{I}}$.

Remark 1. NPR_Σ *is as expressive as* FOL. *We draw the reader's attention to the simplicity of the inductive definition of semantics compared to the traditional* FOL *approach where variables and quantifiers make the definition more involved. Moreover, in traditional accounts, the domain of an interpretation is required to be a non-empty set. In our calculus this is unnecessary and it is* not *a mere technicality: in § 7 we shall see that empty models capture the propositional calculus.*

Two terms $c, d \colon n \to m$ are *semantically equivalent*, written $c \equiv d$, if and only if $\mathcal{I}^\sharp(c) = \mathcal{I}^\sharp(d)$, for all interpretations $\mathcal{I}$. *Semantic inclusion* ($\leqq$) is defined analogously replacing $=$ with $\subseteq$.

By definition $\equiv$ and $\leqq$ only relate terms of the same type. Throughout the paper, we will encounter several relations amongst terms of the same type. To avoid any confusion with the relations denoted by the terms, we call them *well-typed relations* and use symbols $\mathbb{I}$ rather than the usual R, S, T. In the following, we write $c \mathbb{I} d$ for $(c, d) \in \mathbb{I}$ and $\mathsf{pc}(\mathbb{I})$ for the smallest precongruence (w.r.t. $\fatsemi$, $\mathbin{;^{\bullet}}$, $\otimes$ and $\circledast$) generated by $\mathbb{I}$, i.e., the relation inductively generated as

$$
\begin{array}{c}
\dfrac{c \mathbb{I} d}{c \,\mathsf{pc}(\mathbb{I})\, d}\ (id) \qquad \dfrac{-}{c \,\mathsf{pc}(\mathbb{I})\, c}\ (r) \qquad \dfrac{a \,\mathsf{pc}(\mathbb{I})\, b \quad b \,\mathsf{pc}(\mathbb{I})\, c}{a \,\mathsf{pc}(\mathbb{I})\, c}\ (t) \\
\dfrac{c_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathsf{pc}(\mathbb{I})\, d_2}{c_1 \fatsemi d_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \fatsemi d_2}\ (\fatsemi) \qquad \dfrac{c_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathsf{pc}(\mathbb{I})\, d_2}{c_1 \otimes d_1 \,\mathsf{pc}(\mathbb{I})\, c_2 \otimes d_2}\ (\otimes)
\end{array}
\tag{10}
$$

Axioms. Fig. 9 in App. B illustrates a complete system of axioms for $\leqq$. Let $\mathbb{FOB}$ be the well-typed relation obtained by substituting a, b, c, d in Fig. 9 with terms of the appropriate type and and call its precongruence closure *syntactic inclusion*, written $\lesssim$. In symbols $\lesssim = \mathsf{pc}(\mathbb{FOB})$. We will also write $\cong \stackrel{\text{def}}{=} \lesssim \cap \gtrsim$. Our main result is:

THEOREM 3.2. *For all terms $c, d \colon n \to m$, $c \lesssim d$ iff $c \leqq d$.*

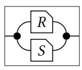
The axiomatisation is far from minimal and is redundant in several respects. We chose the more verbose presentation in order to emphasise both the underlying categorical structures and the various dualities that we will highlight in the next sections.

Diagrams. We confined the complete axiomatisation to the appendix because the axioms in Fig. 9 appear also in Figs. 2, 3, 4, 5 in diagrammatic form. This allows a more principled, staged presentation and place each axiom in its proper context, highlighting their provenance from one of the categorical structures involved.

Diagrams, inspired by string diagrams [42, 79], take centre stage in our presentation. A term $c \colon n \to m$ is drawn as a diagram with n ports on the left and m ports on the right; $⨾$ is depicted as horizontal composition while $\otimes$ by vertically "stacking" diagrams. The two compositions $⨾$ and $\bullet$ and two monoidal products $\otimes$ and $\circledast$ are distinguished with different colours. All constants in the white fragment have white background, mutatis mutandis for the black fragment: for instance id°_1 and $id^\bullet_1$ are drawn  and . Indeed, the diagrammatic version of (NPR$_\Sigma$) is the grammar in Fig.1.

To better grasp the correspondence between terms and diagrams, the reader may compare the diagrammatic version of the axioms (Fig.s 2, 3, 4, 5) with the term-based one (in Figure 9).

Note that one diagram may correspond to more than one term: for instance the diagram on the right does not only represent the rightmost term in (9), namely $◀^\circ_1 ⨾ ((R^\circ \otimes S^\circ) ⨾ ▶^\circ_1)$, but also $(◀^\circ_1 ⨾ (R^\circ \otimes S^\circ)) ⨾ ▶^\circ_1$. Indeed, it is clear that traditional term-based syntax carries more information than the diagrammatic one (e.g. associativity). From the point of view of the semantics, however, this bureaucracy is irrelevant and is conveniently discarded by the diagrammatic notation. To formally show this, we recall that diagrams capture only the axioms of symmetric monoidal categories [42, 79], illustrated in Table 1; we call *structural congruence*, written $\approx$, the well-typed congruence generated by such axioms and we observe that $\approx \subseteq \equiv$.

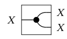
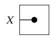
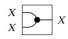
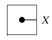


Proofs as diagrams rewrites. Proofs in NPR$_\Sigma$ are rather different from those of traditional proof systems: since the only inference rules are those in (10), any proof of $c \lesssim d$ consists of a sequence of applications of axioms. As an example consider (1) from the Introduction (see App. B.1 for a proof not using Prop. 6.4). Note that, when applying axioms, we are in fact performing diagram rewriting: an instance of the left hand side of an axiom is found within a larger diagram and replaced with the right hand side. Since such rewrites can happen anywhere, there is a close connection between proofs in NPR$_\Sigma$ and work on *deep inference* [15, 34, 41] – see Ex. 7.6.

4 (CO)CARTESIAN BICATEGORIES

Although the term bicategory might seem ominous, the beasts considered in this paper are actually quite simple. We consider *poset enriched symmetric monoidal categories*: every homset carries a partial order $\leq$, and composition $⨾$ and monoidal product $\otimes$ are monotone. That is, if $a \leq b$ and $c \leq d$ then $a ⨾ c \leq b ⨾ d$ and $a \otimes c \leq b \otimes d$. A *poset enriched symmetric monoidal functor* is a (strong, and usually strict) symmetric monoidal functor that preserves the order $\leq$. The notion of *adjoint arrows*, which will play a key role, amounts to the following: for $c \colon X \to Y$ and $d \colon Y \to X$, c is *left adjoint* to d, or d is *right adjoint* to c, written $d \vdash c$, if $id^\circ_X \leq c ⨾ d$ and $d ⨾ c \leq id^\circ_Y$.

For a symmetric monoidal bicategory $(\mathbf{C}, \otimes, I)$, we will write $\mathbf{C}^{\mathsf{op}}$ for the bicategory having the same objects as $\mathbf{C}$ but homsets $\mathbf{C}^{\mathsf{op}}[X, Y] \stackrel{\text{def}}{=} \mathbf{C}[Y, X]$: ordering, identities and monoidal product are defined as in $\mathbf{C}$, while composition $c ⨾ d$ in $\mathbf{C}^{\mathsf{op}}$ is $d ⨾ c$ in $\mathbf{C}$. Similarly, we will write $\mathbf{C}^{\mathsf{co}}$ to denote the bicategory having the same objects and arrows of $\mathbf{C}$ but equipped with the reversed ordering $\geq$. Composition, identities and monoidal product are defined as in $\mathbf{C}$. In this paper, we will often tacitly use the facts that, by definition, both $(\mathbf{C}^{\mathsf{op}})^{\mathsf{op}}$ and $(\mathbf{C}^{\mathsf{co}})^{\mathsf{co}}$ are $\mathbf{C}$ and that $(\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is $(\mathbf{C}^{\mathsf{op}})^{\mathsf{co}}$.

All monoidal categories considered throughout this paper are tacitly assumed to be strict [50], i.e. $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $I \otimes X = X = X \otimes I$ for all objects X, Y, Z. This is harmless: strictification [50] allows to transform any monoidal category into a strict one, enabling the sound use of string diagrams. These will be exploited in this and the next two sections to describe the categorical structures of interest. In particular, in the following definition $◀^\circ_X \colon X \to X \otimes X$, $!^\circ_X \colon X \to I$, $▶^\circ_X \colon X \otimes X \to X$ and $¡^\circ_X \colon I \to X$ are drawn, respectively, as , ,  and .

Definition 4.1. A *cartesian bicategory* $(\mathbf{C}, \otimes, I, ◀^\circ, !^\circ, ▶^\circ, ¡^\circ)$, shorthand $(\mathbf{C}, ◀^\circ, ▶^\circ)$, is a poset enriched symmetric monoidal category $(\mathbf{C}, \otimes, I)$ and, for every object X in $\mathbf{C}$, arrows $◀^\circ_X \colon X \to X \otimes X$, $!^\circ_X \colon X \to I$, $▶^\circ_X \colon X \otimes X \to X$, $¡^\circ_X \colon I \to X$ s.t.

1. $(◀^\circ_X, !^\circ_X)$ is a comonoid and $(▶^\circ_X, ¡^\circ_X)$ a monoid (i.e., ($◀^\circ$-as), ($◀^\circ$-un), ($◀^\circ$-co) and ($▶^\circ$-as), ($▶^\circ$-un), ($▶^\circ$-co) in Fig. 2 hold);
2. arrows $c \colon X \to Y$ are lax comonoid morphisms (($◀^\circ$-nat), ($!^\circ$-nat));
3. $(◀^\circ_X, !^\circ_X)$ are left adjoints to $(▶^\circ_X, ¡^\circ_X)$ (($\eta ◀^\circ$), ($\epsilon ◀^\circ$), ($\eta !^\circ$), ($\epsilon !^\circ$));
4. $(◀^\circ_X, !^\circ_X)$ and $(▶^\circ_X, ¡^\circ_X)$ form special Frobenius algebras ((F°), (S°));
5. $(◀^\circ_X, !^\circ_X)$ and $(▶^\circ_X, ¡^\circ_X)$ satisfy the coherence conditions:[1]

$$
\begin{array}{llll}
◀^\circ_I = id^\circ_I & ◀^\circ_{X\otimes Y} = (◀^\circ_X \otimes ◀^\circ_Y) ⨾ (id^\circ_X \otimes \sigma^\circ_{X,Y} \otimes id^\circ_Y) & & \\
▶^\circ_I = id^\circ_I & ▶^\circ_{X\otimes Y} = (id^\circ_X \otimes \sigma^\circ_{X,Y} \otimes id^\circ_Y) ⨾ (▶^\circ_X \otimes ▶^\circ_Y) & & \\
!^\circ_I = id^\circ_I & !^\circ_{X\otimes Y} = !^\circ_X \otimes !^\circ_Y & ¡^\circ_I = id^\circ_I & ¡^\circ_{X\otimes Y} = ¡^\circ_X \otimes ¡^\circ_Y
\end{array}
$$

$\mathbf{C}$ is a *cocartesian bicategory* if $\mathbf{C}^{\mathsf{co}}$ is a cartesian bicategory. A *morphism of (co)cartesian bicategories* is a poset enriched strong symmetric monoidal functor preserving monoids and comonoids.

The archetypal example of a cartesian bicategory is $(\mathbf{Rel}^\circ, ◀^\circ, ▶^\circ)$. $\mathbf{Rel}^\circ$ the bicategory of sets and relations ordered by inclusion $\subseteq$ with white composition $⨾$ and identities id° defined as in (2) and (3). The monoidal product on objects is the cartesian product of sets with unit I the singleton set $\mathbb{1}$. on arrows, $\otimes$ is defined as in (7). It is immediate to check that, for every set X, the arrows $◀^\circ_X$, $!^\circ_X$ defined in (6) form a comonoid in $\mathbf{Rel}^\circ$, while $▶^\circ_X$, $¡^\circ_X$ a monoid. Simple computations also proves all the (in)equalities in Fig. 2.

[1] Note that the coherence conditions are not in Fig. 2 since they hold in NPR$_\Sigma$, given the inductive definitions of Tab. 1.

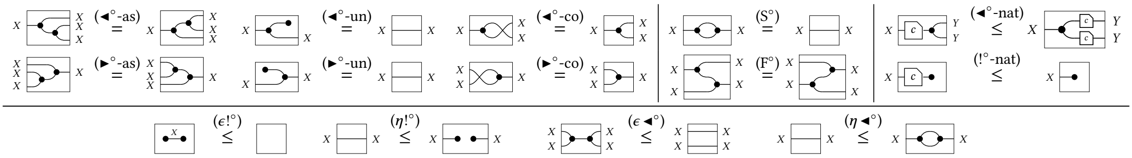


Figure 2: Axioms of cartesian bicategories

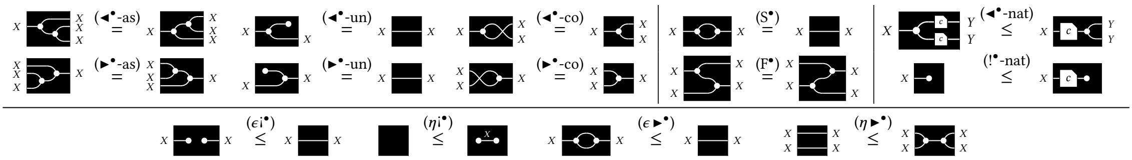
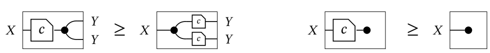


Figure 3: Axioms of cocartesian bicategories

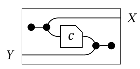
The fact that relations are lax comonoid homomorphisms is the most interesting to show: since $R ⨾ \blacktriangleleft^{\circ}_Y = \{(x,(y,y)) \mid (x,y) \in R\}$ is included in $\{(x,(y,z)) \mid (x,y) \in R \wedge (x,z) \in R\} = \blacktriangleleft^{\circ}_X ⨾ (R \otimes R)$ and $R ⨾ !^{\circ}_Y = \{(x,\star) \mid \exists y \in X \,.\, (x,y) \in R\}$ in $\{(x,\star) \mid x \in X\} = !^{\circ}_X$ for any relation $R \subseteq X \times Y$, ($\blacktriangleleft^{\circ}$-nat) and ($!^{\circ}$-nat) hold. The reversed inclusions are interesting to consider: $R ⨾ \blacktriangleleft^{\circ}_Y \supseteq \blacktriangleleft^{\circ}_X ⨾ (R \otimes R)$ holds iff the relation R is single valued, while $R ⨾ !^{\circ}_Y \supseteq !^{\circ}_X$ iff R a total. That is, the two inequalities in Definition 4.1.(2) are equalities iff the relation R is a function. This justifies the following:

Definition 4.2. An arrow $c\colon X \to Y$ is a *map* if



It is easy to see that maps form a monoidal subcategory of $\mathbf{C}$ [16], hereafter denoted by $\mathsf{Map}(\mathbf{C})$. In fact, it is cartesian.

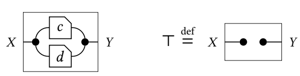
Given a cartesian bicategory $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$, one can take $\mathbf{C}^{\mathsf{op}}$, swap monoids and comonoids and thus, obtain a cartesian bicategory $(\mathbf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ})$. Most importantly, there is an identity on objects isomorphism $(\cdot)^{\dagger}\colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ defined for all arrows $c\colon X \to Y$ as

$c^{\dagger} \stackrel{\text{def}}{=}$  (11)

Proposition 4.3. *$(\cdot)^{\dagger}\colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ is an isomorphism of cartesian bicategories, namely the laws in the first three rows of Table 2.(a) hold.*

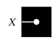
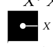
Hereafter, we write c for $c^{\dagger}$ and we call it the *mirror image* of c. Note that in § 2, we used the same symbol $(\cdot)^{\dagger}$ to denote the converse relation. This is no accident: in the cartesian bicategory $(\mathbf{Rel}^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$, $R^{\dagger}$ as in (11) is exactly $\{(y,x) \mid (x,y) \in R\}$.

In a cartesian bicategory, one can also define, for all arrows $c,d\colon X \to Y$, $c \sqcap d$ and $\top$ as follows.

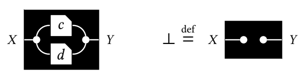
$c \sqcap d \stackrel{\text{def}}{=}$ $\qquad \top \stackrel{\text{def}}{=}$  (12)

We have already seen in Example 3.1 that these terms, when interpreted in $\mathbf{Rel}^{\circ}$, denote respectively intersection and top. It is easy to show that in any cartesian bicategory $\mathbf{C}$, $\sqcap$ is associative, commutative, idempotent and has $\top$ as unit. Namely, $\mathbf{C}[X,Y]$ is a meet-semilattice with top. However, $\mathbf{C}$ is usually *not* enriched over meet-semilattices since $⨾$ distributes only laxly over $\sqcap$. Indeed, in $\mathbf{Rel}^{\circ}$, $R ⨾ (S \cap T) \subseteq (R ⨾ S) \cap (R ⨾ T)$ holds but the reverse does not.

Let us now turn to *co*cartesian bicategories. Our main example is $(\mathbf{Rel}^{\bullet}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$. $\mathbf{Rel}^{\bullet}$ is the bicategory of sets and relations ordered by $\subseteq$ with composition $\bullet$, identities $id^{\bullet}$ and $\otimes$ defined as in (2), (3) and (7). Comonoids $(\blacktriangleleft^{\bullet}_X, !^{\bullet}_X)$ and monoids $(\blacktriangleright^{\bullet}_X, ¡^{\bullet}_X)$ are those of (6). To see that $\mathbf{Rel}^{\bullet}$ is a cocartesian bicategory, observe that the complement $\overline{(\cdot)}$ is a poset-enriched symmetric monoidal isomorphism $\overline{(\cdot)}\colon (\mathbf{Rel}^{\circ})^{\mathsf{co}} \to \mathbf{Rel}^{\bullet}$ preserving (co)monoids.

We draw arrows of cocartesian bicategories in black: $\blacktriangleleft^{\bullet}_X, !^{\bullet}_X$, $\blacktriangleright^{\bullet}_X$ and $¡^{\bullet}_X$ are drawn , ,  and . Following this convention, the axioms of cocartesian bicategories are in Fig. 3; they can also be obtained from Fig. 2 by inverting both the colours and the order.

It is not surprising that in a cocartesian bicategory $\mathbf{C}$, every homset $\mathbf{C}[X,Y]$ carries a join semi-lattice with bottom, where $c \sqcup d$ and $\bot$ are defined for all arrows $c,d\colon X \to Y$ as follows.

$c \sqcup d \stackrel{\text{def}}{=}$ $\qquad \bot \stackrel{\text{def}}{=}$  (13)

5 LINEAR BICATEGORIES

We have seen that $\mathbf{Rel}^{\circ}$ forms a cartesian, and $\mathbf{Rel}^{\bullet}$ a cocartesian bicategory. Categorically, they are remarkably similar — as evidenced by the isomorphism $\overline{(\cdot)}$ — but from a logical viewpoint they represent two complimentary parts of FOL: $\mathbf{Rel}^{\circ}$ the existential conjunctive fragment, and $\mathbf{Rel}^{\bullet}$ the universal disjunctive fragment. To discover the full story, we must merge them into one entity and study the algebraic interactions between them. However, the coexistence of two different compositions $⨾$ and $\bullet$ brings us out of the realm of ordinary categories. The solution is linear bicategories [17]. Here $⨾$ linearly distributes over $\bullet$, as in Pierce's calculus. To keep

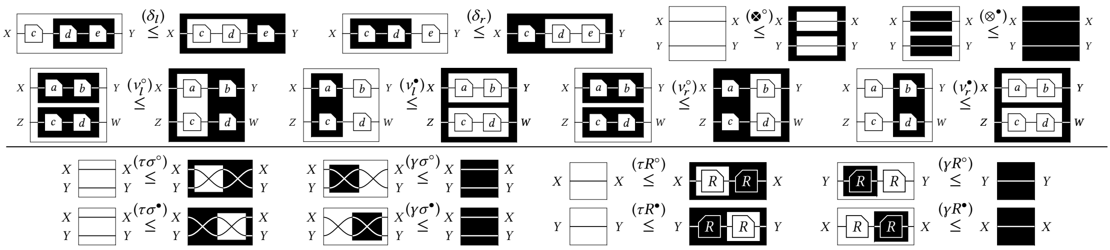


Figure 4: Axioms of closed symmetric monoidal linear bicategories

our development easier, we stick to the poset enriched case and rely on diagrams, using white and black to distinguish $\mathbin{\mathring{;}}$ and $\mathbin{\dot{;}}$.

Definition 5.1. A *linear bicategory* $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ}, \mathbin{\dot{;}}, id^{\bullet})$ consists of two poset enriched categories $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ})$ and $(\mathbf{C}, \mathbin{\dot{;}}, id^{\bullet})$ with the same objects, arrows and orderings but possibly different identities and compositions such that $\mathbin{\mathring{;}}$ linearly distributes over $\mathbin{\dot{;}}$ (i.e., (δ_l) and (δ_r) in Fig. 4 hold). A *symmetric monoidal linear bicategory* $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ}, \mathbin{\dot{;}}, id^{\bullet}, \otimes, \sigma^{\circ}, \circledast, \sigma^{\bullet}, I)$, shortly $(\mathbf{C}, \otimes, \circledast, I)$, consists of a linear bicategory $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ}, \mathbin{\dot{;}}, id^{\bullet})$ and two poset enriched symmetric monoidal categories $(\mathbf{C}, \otimes, I)$ and $(\mathbf{C}, \circledast, I)$ such that $\otimes$ and $\circledast$ agree on objects, i.e., $X \otimes Y = X \circledast Y$, share the same unit I and
1. there are linear strengths for $(\otimes, \circledast)$, (i.e., (ν_l°), (ν_r°), $(\nu_l^{\bullet})$, $(\nu_r^{\bullet})$);
2. $\circledast$ preserves id° colaxly and $\otimes$ preserves $id^{\bullet}$ laxly ($(\otimes^{\bullet})$, $(\circledast^{\circ})$).

A *morphism of symmetric monoidal linear bicategories* $\mathcal{F} : (\mathbf{C}_1, \otimes, \circledast, I) \to (\mathbf{C}_2, \otimes, \circledast, I)$ consists of two poset enriched symmetric monoidal functors $\mathcal{F}^{\circ} : (\mathbf{C}_1, \otimes, I) \to (\mathbf{C}_2, \otimes, I)$ and $\mathcal{F}^{\bullet} : (\mathbf{C}_1, \circledast, I) \to (\mathbf{C}_2, \circledast, I)$ that agree on objects and arrows: $\mathcal{F}^{\circ}(X) = \mathcal{F}^{\bullet}(X)$ and $\mathcal{F}^{\circ}(c) = \mathcal{F}^{\bullet}(c)$.

Remark 2. *In the literature $\mathbin{\mathring{;}}$, id°, $\mathbin{\dot{;}}$ and $id^{\bullet}$ are written with the linear logic notation $\otimes, \top, \oplus$ and $\perp$. Modulo this, the traditional notion of linear bicategory (Definition 2.1 in [17]) coincides with the one in Definition 5.1 whenever the 2-structure is collapsed to a poset.*

Monoidal products on linear bicategories are not much studied although the axioms in Definition 5.1.1 already appeared in [57]. They are the linear strengths of the pair $(\otimes, \circledast)$ seen as a linear functor (Definition 2.4 in [17]), a notion of morphism that crucially differs from ours on the fact that the $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$ may not coincide on arrows. Instead the inequalities $(\otimes^{\bullet})$ and $(\circledast^{\circ})$ are, to the best of our knowledge, novel. Beyond being natural, they are crucial for Lemma 5.2 below.

All linear bicategories in this paper are symmetric monoidal. We therefore omit the adjective *symmetric monoidal* and refer to them simply as linear bicategories. For a linear bicategory $(\mathbf{C}, \otimes, \circledast, I)$, we will often refer to $(\mathbf{C}, \otimes, I)$ as the *white structure*, shorthand $\mathbf{C}^{\circ}$, and to $(\mathbf{C}, \circledast, I)$ as the *black structure*, $\mathbf{C}^{\bullet}$. Note that a morphism $\mathcal{F}$ is a mapping of objects and arrows that preserves the ordering, the white and black structures; thus we write $\mathcal{F}$ for both $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$.

If $(\mathbf{C}, \otimes, \circledast, I)$ is linear bicategory then $(\mathbf{C}^{\mathrm{op}}, \otimes, \circledast, I)$ is a linear bicategory. Similarly $(\mathbf{C}^{\mathrm{co}}, \circledast, \otimes, I)$, the bicategory obtained from $\mathbf{C}$ by reversing the ordering and swapping the white and the black structure, is a linear bicategory.

Our main example is the linear bicategory **Rel** of sets and relations ordered by $\subseteq$. The white structure is the symmetric monoidal category $(\mathbf{Rel}^{\circ}, \otimes, \mathbb{1})$, introduced in the previous section and the black structure is $(\mathbf{Rel}^{\bullet}, \circledast, \mathbb{1})$. Observe that the two have the same objects, arrows and ordering. The white and black monoidal products $\otimes$ and $\circledast$ agree on objects and are the cartesian product of sets. As common unit object, they have the singleton set $\mathbb{1}$. We already observed in (5) that the white composition $\mathbin{\mathring{;}}$ distributes over $\mathbin{\dot{;}}$ and thus (δ_l) and (δ_r) hold. By using the definitions in (2), (3) and (7), the reader can easily check also the inequalities in Definition 5.1.1,2.

Lemma 5.2. *Let $(\mathbf{C}, \otimes, \circledast, I)$ be a linear bicategory. For all arrows a, b, c the following hold:*

(1) $id_I^{\bullet} \leq id_I^{\circ}$ (2) $a \otimes b \leq a \circledast b$ (3) $(a \circledast b) \otimes c \leq a \circledast (b \otimes c)$

Remark 3. *As $\otimes$ linearly distributes over $\circledast$, it may seem that symmetric monoidal linear bicategories of Definition 5.1 are linearly distributive [19, 23]. Moreover (1), (2) of Lemma 5.2 may suggest that they are mix categories [18]. This is not the case: functoriality of $\otimes$ over $\mathbin{\dot{;}}$ and of $\circledast$ over $\mathbin{\mathring{;}}$ fails in general.*

Closed linear bicategories. In § 4, we recalled adjoints of arrows in bicategories; in linear bicategories one can define *linear* adjoints. For $a : X \to Y$ and $b : Y \to X$, a is *left linear adjoint* to b, or b is *right linear adjoint* to a, written $b \Vdash a$, if $id_X^{\circ} \leq a \mathbin{\dot{;}} b$ and $b \mathbin{\mathring{;}} a \leq id_Y^{\bullet}$.

Next we discuss some properties of right linear adjoints. Those of left adjoints are analogous but they do not feature in our exposition since in the categories of interest — in next section — left and right linear adjoint coincide. As expected, linear adjoints are unique.

Lemma 5.3. *If $b \Vdash a$ and $c \Vdash a$, then $b = c$.*

$$\begin{tikzcd} X \arrow[r, "a"] & Y \\ Z \arrow[u, bend left, "c"] \arrow[u, "b \mathbin{\dot{;}} a^{\perp}"'] \arrow[ur, "b"'] & \end{tikzcd}$$

By virtue of the above result we can write $a^{\perp} : Y \to X$ for *the* right linear adjoint of $a : X \to Y$. With this notation one can write the *left residual* of $b : Z \to Y$ by $a : X \to Y$ as $b \mathbin{\dot{;}} a^{\perp} : Z \to X$. The left residual is the greatest arrow $Z \to X$ making the diagram on the right commute laxly in $\mathbf{C}^{\circ}$, namely if $c \mathbin{\mathring{;}} a \leq b$ then $c \leq b \mathbin{\dot{;}} a^{\perp}$. This can be equivalently expressed as:

Lemma 5.4 (Residuation). $a \leq b$ *iff* $id_X^{\circ} \leq b \mathbin{\dot{;}} a^{\perp}$.

Definition 5.5. A linear bicategory $(\mathbf{C}, \otimes, \circledast, I)$ is said to be *closed* if every $a : X \to Y$ has both a left and a right linear adjoint and the white symmetry is both left and right linear adjoint to the black symmetry, i.e. $(\tau\sigma^{\circ})$, $(\gamma\sigma^{\circ})$, $(\tau\sigma^{\bullet})$ and $(\gamma\sigma^{\bullet})$ in Fig. 4 hold.

Rel is a a closed linear bicategory: both left and right linear adjoints of a relation $R \subseteq X \times Y$ are given by $\overline{R}^{\dagger} = \{(y, x) \mid (x, y) \notin R\} \subseteq Y \times X$. With this, it is easy to see that $\sigma^{\bullet} \Vdash \sigma^{\circ} \Vdash \sigma^{\bullet}$ in **Rel**.

Observe that if a linear bicategory $(\mathbf{C}, \otimes, \otimes\!\!\!\!\bullet, I)$ is closed, then also $(\mathbf{C}^{\mathsf{op}}, \otimes, \otimes\!\!\!\!\bullet, I)$ and $(\mathbf{C}^{\mathsf{co}}, \otimes\!\!\!\!\bullet, \otimes, I)$ are closed. The assignment $a \mapsto a^{\perp}$ gives rise to an identity on objects functor $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$.

Proposition 5.6. *$(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is a morphism of linear bicategories, i.e., the laws in the first two columns of Table 2.(b) hold.*

Hereafter, the diagram obtained from $\boxed{c}$, by taking its mirror image $\boxed{c}$ and then its photographic negative $\blacksquare$ will denote $\boxed{c}^{\perp}$.

6 FIRST ORDER BICATEGORIES

Here we focus on the most important and novel part of the axiomatisation. Indeed, having introduced the two main ingredients, cartesian and linear bicategories, it is time to fire up the Bunsen burner. The remit of this section is to understand how the cartesian and the linear bicategory structures interact in the context of relations. We introduce *first order bicategories* that make these interactions precise. The resulting axioms echo those of cartesian bicategories but in the linear bicategory setting: recall that in a cartesian bicategory the monoid and comonoids are adjoint and satisfy the Frobenius law. Here, the white and black (co)monoids are again related, but by *linear* adjunctions; moreover, they also satisfy appropriate "linear" counterparts of the Frobenius equations.

Definition 6.1. A *first order bicategory* $(\mathbf{C}, \otimes, \otimes\!\!\!\!\bullet, I, \blacktriangleleft^{\circ}, !^{\circ}, \blacktriangleright^{\circ}, ¡^{\circ}, \blacktriangleleft^{\bullet}, !^{\bullet}, \blacktriangleright^{\bullet}, ¡^{\bullet})$, shorthand *fo-bicategory* $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$, consists of

1. a closed linear bicategory $(\mathbf{C}, \otimes, \otimes\!\!\!\!\bullet, I)$,
2. a cartesian bicategory $(\mathbf{C}, \otimes, I, \blacktriangleleft^{\circ}, !^{\circ}, \blacktriangleright^{\circ}, ¡^{\circ})$ and
3. a cocartesian bicategory $(\mathbf{C}, \otimes\!\!\!\!\bullet, I, \blacktriangleleft^{\bullet}, !^{\bullet}, \blacktriangleright^{\bullet}, ¡^{\bullet})$, such that
4. the white comonoid $(\blacktriangleleft^{\circ}, !^{\circ})$ is left and right linear adjoint to black monoid $(\blacktriangleright^{\bullet}, ¡^{\bullet})$ and $(\blacktriangleright^{\circ}, ¡^{\circ})$ is left and right linear adjoint to $(\blacktriangleleft^{\bullet}, !^{\bullet})$, i.e. the inequalities on the left of Figure 5 hold;
5. white and black (co)monoids satisfy the linear Frobenius laws, i.e. the equalities on the right of Fig. 5 hold.

A *morphism of fo-bicategories* is a morphism of linear bicategories *and* of (co)cartesian bicategories.

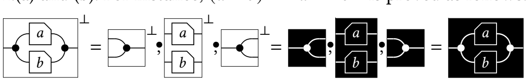
We have seen that **Rel** is a closed linear bicategory, $\mathbf{Rel}^{\circ}$ a cartesian bicategory and $\mathbf{Rel}^{\bullet}$ a cocartesian bicategory. Given (6), it is easy to confirm linear adjointness and linear Frobenius.

Now if $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ is a fo-bicategory then $(\mathbf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$ and $(\mathbf{C}^{\mathsf{co}}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ are fo-bicategories: the laws of Fig. 5 are closed under mirror-reflection and photographic negative. The fourth condition in Definition 6.1 entails that the linear bicategory morphism $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ (see Prop. 5.6) is a morphism of fo-bicategories and, similarly, the fifth condition that also $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ (Prop. 4.3) is a morphism of fo-bicategories.

Proposition 6.2. *Let $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ be a fo-bicategory. Then $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ and $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ are isomorphisms of fo-bicategories, namely the laws in Table 2.(a) and (b) hold.*

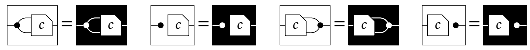
Corollary 6.3. *The laws in Table 2.(c) hold.*

The corollary follows from (12) and (13) and the laws in Tables 2.(a) and (b). For instance, $(a \sqcap b)^{\perp} = a^{\perp} \sqcup b^{\perp}$ is proved as follows.



The next result about maps (Definition 4.2) plays a crucial role.

Proposition 6.4. *For all maps $f \colon X \to Y$ and arrows $c \colon Y \to Z$, $f ; c = (f^{\dagger})^{\perp} \,\bullet\, c$ and thus*



For fo-bicategory $\mathbf{C}$, we have the four isomorphisms in the diagram on the right, which commutes by Corollary 6.3. We can thus define the complement as the diagonal of the square, namely $\overline{(\cdot)} \stackrel{\text{def}}{=} ((\cdot)^{\perp})^{\dagger}$.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "(\cdot)^{\dagger}"] \arrow[d, "(\cdot)^{\perp}"'] & \mathbf{C}^{\mathsf{op}} \arrow[d, "(\cdot)^{\perp}"] \\ (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}} \arrow[r, "(\cdot)^{\dagger}"'] & \mathbf{C}^{\mathsf{co}} \end{tikzcd}$$

In diagrams, given $\boxed{c}$, its negation is $(\boxed{c}^{\perp})^{\dagger} = \blacksquare^{\dagger} = \blacksquare$.

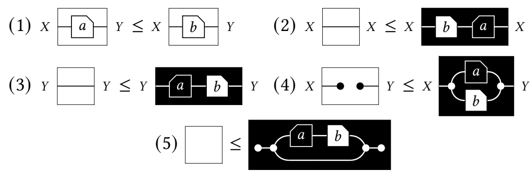
Clearly $\overline{(\cdot)} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{co}}$ is an isomorphism of fo-bicategories. Moreover, it induces a Boolean algebra on each homset of $\mathbf{C}$.

Proposition 6.5. *Let $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ be a fo-bicategory. Then every homset of $\mathbf{C}$ is a Boolean algebra: the laws in Tab. 2.(d) hold. Further, $(\mathbf{C}, \otimes, I)$ is monoidally enriched over $\sqcup$-semilattices with $\perp$, while $(\mathbf{C}, \otimes\!\!\!\!\bullet, I)$ over $\sqcap$-semilattices with $\top$: the laws in Tab. 2.(e) hold.*

The monoidal enrichment is interesting: as we mentioned in § 4, the white structure is not enriched over $\sqcap$, but it *is* enriched over $\sqcup$. In **Rel**, this is the fact that $R ; (S \cup T) = (R ; S) \cup (R ; T)$.

We conclude with a result that extends Lemma 5.4 with five different possibilities to express the concept of logical entailment.

Lemma 6.6. *In a fo-bicategory, the following are equivalent:*



6.1 The calculus of neo-Peircean relations as a freely generated first order bicategory

We now return to NPR_{Σ}. Recall that $\lesssim$ is the precongruence obtained from the axioms in Fig.s 2, 3, 4 and 5. Its soundness (half of Theorem 3.2) is immediate since **Rel** is a fo-bicategory.

Proposition 6.7. *For all terms $c, d \colon n \to m$, if $c \lesssim d$ then $c \leqq d$.*

Next, we show how NPR_{Σ} gives rise to a fo-bicategory $\mathbf{FOB}_{\Sigma}$. Objects are natural numbers and monoidal products $\otimes$ are defined as addition with unit object 0. Arrows from n to m are terms $c \colon n \to m$ modulo syntactic equivalence $\cong$, namely $\mathbf{FOB}_{\Sigma}[n, m] \stackrel{\text{def}}{=} \{[c]_{\cong} \mid c \colon n \to m\}$. Observe that this is well defined since $\cong$ is well-typed. Since $\cong$ is a congruence, the operations $;$ and $\otimes$ on terms are well defined on equivalence classes: $[t_1]_{\cong} ; [t_2]_{\cong} \stackrel{\text{def}}{=} [t_1 ; t_2]_{\cong}$ and $[t_1]_{\cong} \otimes [t_2]_{\cong} \stackrel{\text{def}}{=} [t_1 \otimes t_2]_{\cong}$. By fixing as partial order the syntactic inclusion $\lesssim$, one can easily prove the following.

Proposition 6.8. *$\mathbf{FOB}_{\Sigma}$ is a first order bicategory.*

A useful consequence is that, for any interpretation $\mathcal{I} = (X, \rho)$, the semantics $\mathcal{I}^{\sharp}$ gives rise to a morphism $\mathcal{I}^{\sharp} \colon \mathbf{FOB}_{\Sigma} \to \mathbf{Rel}$ of fo-bicategories: it is defined on objects as $n \mapsto X^n$ and on arrows by the inductive definition in (8). To see that it is a morphism, note that, by (8), all the structure of (co)cartesian bicategories and of

Table 2: Properties of first order bicategories.

(a) Properties of $(\cdot)^{\dagger} : (\mathsf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to (\mathsf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$			
if $c \leq d$ then $c^{\dagger} \leq d^{\dagger}$		$(c^{\dagger})^{\dagger} = c$	
$(c ;^{\circ} d)^{\dagger} = d^{\dagger} ;^{\circ} c^{\dagger}$	$(id^{\circ}_X)^{\dagger} = id^{\circ}_X$	$(\blacktriangleright^{\circ}_X)^{\dagger} = \blacktriangleleft^{\circ}_X$	$(¡^{\circ}_X)^{\dagger} = !^{\circ}_X$
$(c \otimes d)^{\dagger} = c^{\dagger} \otimes d^{\dagger}$	$(\sigma^{\circ}_{X,Y})^{\dagger} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\dagger} = \blacktriangleright^{\circ}_X$	$(!^{\circ}_X)^{\dagger} = ¡^{\circ}_X$
$(c ;^{\bullet} d)^{\dagger} = d^{\dagger} ;^{\bullet} c^{\dagger}$	$(id^{\bullet}_X)^{\dagger} = id^{\bullet}_X$	$(\blacktriangleright^{\bullet}_X)^{\dagger} = \blacktriangleleft^{\bullet}_X$	$(¡^{\bullet}_X)^{\dagger} = !^{\bullet}_X$
$(c \otimes^{\bullet} d)^{\dagger} = c^{\dagger} \otimes^{\bullet} d^{\dagger}$	$(\sigma^{\bullet}_{X,Y})^{\dagger} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\dagger} = \blacktriangleright^{\bullet}_X$	$(!^{\bullet}_X)^{\dagger} = ¡^{\bullet}_X$

(b) Properties of $(\cdot)^{\perp} : (\mathsf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to ((\mathsf{C}^{\mathsf{co}})^{\mathsf{op}}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ})$			
if $c \leq d$ then $c^{\perp} \geq d^{\perp}$		$(c^{\perp})^{\perp} = c$	
$(c ;^{\circ} d)^{\perp} = d^{\perp} ;^{\bullet} c^{\perp}$	$(id^{\circ}_X)^{\perp} = id^{\bullet}_X$	$(\blacktriangleright^{\circ}_X)^{\perp} = \blacktriangleleft^{\bullet}_X$	$(¡^{\circ}_X)^{\perp} = !^{\bullet}_X$
$(c \otimes d)^{\perp} = c^{\perp} \otimes^{\bullet} d^{\perp}$	$(\sigma^{\circ}_{X,Y})^{\perp} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\perp} = \blacktriangleright^{\bullet}_X$	$(!^{\circ}_X)^{\perp} = ¡^{\bullet}_X$
$(c ;^{\bullet} d)^{\perp} = d^{\perp} ;^{\circ} c^{\perp}$	$(id^{\bullet}_X)^{\perp} = id^{\circ}_X$	$(\blacktriangleright^{\bullet}_X)^{\perp} = \blacktriangleleft^{\circ}_X$	$(¡^{\bullet}_X)^{\perp} = !^{\circ}_X$
$(c \otimes^{\bullet} d)^{\perp} = c^{\perp} \otimes d^{\perp}$	$(\sigma^{\bullet}_{X,Y})^{\perp} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\perp} = \blacktriangleright^{\circ}_X$	$(!^{\bullet}_X)^{\perp} = ¡^{\circ}_X$

(c) Interaction of $\cdot^{\dagger}$ and $\cdot^{\perp}$ with $\sqcap$ and $\sqcup$	
$(c \sqcap d)^{\dagger} = c^{\dagger} \sqcap d^{\dagger}$	$\top^{\dagger} = \top$
$(c \sqcup d)^{\dagger} = c^{\dagger} \sqcup d^{\dagger}$	$\bot^{\dagger} = \bot$
$(c \sqcap d)^{\perp} = c^{\perp} \sqcup d^{\perp}$	$(\top)^{\perp} = \bot$
$(c \sqcup d)^{\perp} = c^{\perp} \sqcap d^{\perp}$	$(\bot)^{\perp} = \top$
$(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$	

(d) Laws of Boolean algebras	
$c \sqcap (d \sqcup e) = (c \sqcap d) \sqcup (c \sqcap e)$	
$c \sqcup (d \sqcap e) = (c \sqcup d) \sqcap (c \sqcup e)$	
$\overline{(c \sqcap d)} = \overline{c} \sqcup \overline{d}$	$\overline{\top} = \bot$
$\overline{(c \sqcup d)} = \overline{c} \sqcap \overline{d}$	$\overline{\bot} = \top$
$c \sqcap \overline{c} = \bot$	$c \sqcup \overline{c} = \top$

(e) Enrichment over join-meet semilattices						
	$c ;^{\circ} (d \sqcup e) = (c ;^{\circ} d) \sqcup (c ;^{\circ} e)$	$(d \sqcup e) ;^{\circ} c = (d ;^{\circ} c) \sqcup (e ;^{\circ} c)$	$c ;^{\circ} \bot = \bot = \bot ;^{\circ} c$	$c \otimes (d \sqcup e) = (c \otimes d) \sqcup (c \otimes e)$	$(d \sqcup e) \otimes c = (d \otimes c) \sqcup (e \otimes c)$	$c \otimes \bot = \bot = \bot \otimes c$
	$c ;^{\bullet} (d \sqcap e) = (c ;^{\bullet} d) \sqcap (c ;^{\bullet} e)$	$(d \sqcap e) ;^{\bullet} c = (d ;^{\bullet} c) \sqcap (e ;^{\bullet} c)$	$c ;^{\bullet} \top = \top = \top ;^{\bullet} c$	$c \otimes^{\bullet} (d \sqcap e) = (c \otimes^{\bullet} d) \sqcap (c \otimes^{\bullet} e)$	$(d \sqcap e) \otimes^{\bullet} c = (d \otimes^{\bullet} c) \sqcap (e \otimes^{\bullet} c)$	$c \otimes^{\bullet} \top = \top = \top \otimes^{\bullet} c$

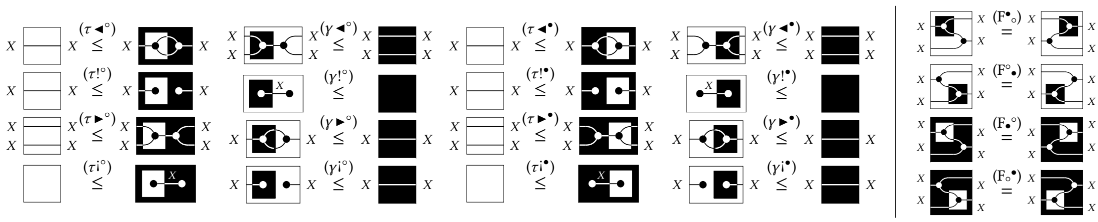


Figure 5: Additional axioms for fo-bicategories

linear bicategories is preserved (e.g. $\mathcal{I}^{\sharp}(\blacktriangleleft^{\circ}_1) = \blacktriangleleft^{\circ}_X$). Moreover, the ordering is preserved by Prop. 6.7. Note that, by construction,

$$\mathcal{I}^{\sharp}(1) = X \text{ and } \mathcal{I}^{\sharp}(R^{\circ}) = \rho(R) \text{ for all } R \in \Sigma. \tag{14}$$

Actually, $\mathcal{I}^{\sharp}$ is the unique such morphism of fo-bicategories. This is a consequence of a more general universal property: **Rel** can be replaced with an arbitrary fo-bicategory **C**. To see this, we first need to generalise the notion of interpretation.

Definition 6.9. Let Σ be a monoidal signature and **C** a first order bicategory. An *interpretation* $\mathcal{I} = (X, \rho)$ of Σ in **C** consists of an object X of **C** and an arrow $\rho(R) : X^n \to X^m$ for each $R \in \Sigma[n, m]$.

With this definition, we can state that $\mathbf{FOB}_{\Sigma}$ is the fo-bicategory freely generated by Σ.

Proposition 6.10. *Let Σ be a monoidal signature,* **C** *a first order bicategory and $\mathcal{I} = (X, \rho)$ an interpretation of Σ in* **C**. *There exists a unique morphism of fo-bicategories $\mathcal{I}^{\sharp} : \mathbf{FOB}_{\Sigma} \to \mathbf{C}$ such that $\mathcal{I}^{\sharp}(1) = X$ and $\mathcal{I}^{\sharp}(R^{\circ}) = \rho(R)$ for all $R \in \Sigma$.*

7 DIAGRAMMATIC FIRST ORDER THEORIES

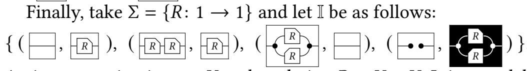
Here we take the first steps towards completeness and show that for first order theories, fo-bicategories play an analogous role to cartesian categories in Lawvere's functorial semantics [48].

A *first order theory* $\mathbb{T}$ is a pair $(\Sigma, \mathbb{I})$ where Σ is a signature and $\mathbb{I}$ is a set of *axioms*: pairs (c, d) for $c, d : n \to m$ in $\mathbf{FOB}_{\Sigma}$. A *model* of $\mathbb{T}$ is an interpretation $\mathcal{I}$ of Σ where if $(c, d) \in \mathbb{I}$, then $\mathcal{I}^{\sharp}(c) \subseteq \mathcal{I}^{\sharp}(d)$.

Example 7.1. The simplest case is $\Sigma = \mathbb{I} = \varnothing$. An interpretation is a set: all sets, including the empty set $\varnothing$, are models.

Next take $\Sigma = \varnothing$ and $\mathbb{I} = \{($, $)\}$. An interpretation $\mathcal{I}$ is a set X. By (8), $\mathcal{I}^{\sharp}($ $) = \{(\star, x) \mid x \in X\} ; \{(x, \star) \mid x \in X\}$, so $\mathcal{I}^{\sharp}($ $) = \{(\star, \star)\}$ if $X \neq \varnothing$, but $\varnothing$ if $X = \varnothing$. Instead, $\mathcal{I}^{\sharp}($ $) = \{(\star, \star)\}$ always, since X^0 is always $\mathbb{1}$. Succinctly, $\mathcal{I}^{\sharp}($ $) \subseteq \mathcal{I}^{\sharp}($ $)$ iff $X \neq \varnothing$: models are *non-empty sets.*

Finally, take $\Sigma = \{R : 1 \to 1\}$ and let $\mathbb{I}$ be as follows:



An interpretation is a set X and a relation $R \subseteq X \times X$. It is a model iff R is an order, i.e., reflexive, transitive, antisymmetric and total.

Monoidal signatures Σ, differently from usual FOL alphabets, do not have function symbols. The reason is that, by adding the axioms below to $\mathbb{I}$, one forces a symbol $f : n \to 1 \in \Sigma$ to be a function.

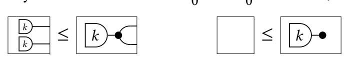
$$(\mathbb{M}_f)$$

Indeed, as we remarked in § 4, $f \subseteq X^n \times X$ satisfies $\mathbb{M}_f$ if and only if it is single valued and total, i.e. a function. We depict functions as and constants, being $0 \to 1$ functions, as .

The axioms of a theory together with $\lesssim$ form a deduction system. Formally, the *deduction relation* induced by $\mathbb{T} = (\Sigma, \mathbb{I})$ is the closure (see (10)) of $\lesssim \cup\, \mathbb{I}$, i.e. $\lesssim_{\mathbb{T}} \stackrel{\text{def}}{=} \mathsf{pc}(\lesssim \cup\, \mathbb{I})$. We write $\cong_{\mathbb{T}}$ for $\lesssim_{\mathbb{T}} \cap \gtrsim_{\mathbb{T}}$.

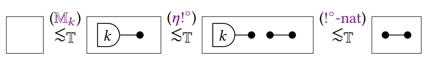
Proposition 7.2. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. If $c \lesssim_{\mathbb{T}} d$, then $\mathcal{I}^{\sharp}(c) \subseteq \mathcal{I}^{\sharp}(d)$ for all models $\mathcal{I}$.*

Example 7.3. Consider the theory $\mathbb{T}$ with $\Sigma = \{k : 0 \to 1\}$ and axioms $\mathbb{M}_k$. By the definitions of $\blacktriangleleft^{\circ}_0$ and $!^{\circ}_0$ in Tab. 1, these are:



$$(\mathbb{M}_k)$$

An interpretation $\mathcal{I}$ of Σ consists of a set X and a relation $k \subseteq \mathbb{1} \times X$. An interpretation is a model iff k is a function of type $\mathbb{1} \to X$. One can easily prove that in all models the domain is non-empty:



$$\tag{15}$$

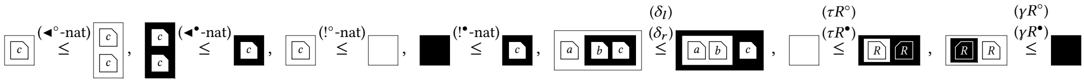
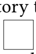


Figure 6: The axioms in Figures 2, 3 and 4 reduce to those above for diagrams of type $I \to I$

Contradictory vs trivial theories. The distinction between contradictory and trivial theories is so subtle that, as shown in Remark 5, it is invisible in FOL. Let us start with the definition.

Definition 7.4. A theory $\mathbb{T}$ is *contradictory* if  $\lesssim_{\mathbb{T}}$ . It is *trivial* if  $\lesssim_{\mathbb{T}}$ .

Triviality implies all models have domain $\varnothing$: $\mathcal{I}^\sharp($$) = \{(\star, x) \mid x \in X\}$ is included in $\varnothing = \mathcal{I}^\sharp($$)$ iff $X = \varnothing$. On the other hand, contradictory theories cannot have a model, not even when $X = \varnothing$: since $\mathcal{I}^\sharp($$) = \{(\star,\star)\}$ and $\mathcal{I}^\sharp($$) = \varnothing$ independently of X. Every contradictory theory is trivial (see Prop. F.1 in App. F).

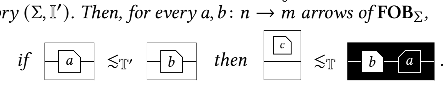
In trivial theories diagrams of type $0 \to 0$ can be quite interesting (see Example 7.6), while those with a different type collapse:

Lemma 7.5. *Let $\mathbb{T}$ be a trivial theory and $c\colon n \to m{+}1$, $d\colon m{+}1 \to n$ be arrows in $\mathbf{FOB}_\Sigma$. Then $\top \lesssim_{\mathbb{T}} c \lesssim_{\mathbb{T}} \bot$ and $\top \lesssim_{\mathbb{T}} d \lesssim_{\mathbb{T}} \bot$.*

Example 7.6 (The trivial theory of propositional calculus). Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be the theory where Σ contains only symbols $P, Q, R \dots$ of type $0 \to 0$ and $\mathbb{I} = \{($, $)\}$. In any model of $\mathbb{T}$, the domain X must be $\varnothing$, because of the only axiom in $\mathbb{I}$. A model is a mapping of each of the symbols in Σ to either $\{(\star,\star)\}$ or $\varnothing$. In other words, $P, Q, R, \dots$ act as propositional variables and any model is just an assignment of boolean values. By Lemma 7.5 all arrows collapse, with the exception of those of type $0 \to 0$, that are exactly propositional formulas (see Prop. B.1 in App. B.2). Our axiomatisation reduces to the one in Fig. 6. The reader can check App. B.2 to see that this is the deep inference system SKSg in [15].

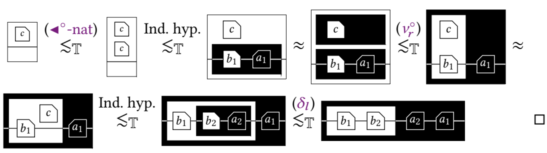
Diagrams $c\colon 0 \to 0$, which can be thought of as closed formulas of FOL, also play an important role in the following result: a diagrammatic analogue of the deduction theorem (the reader may check App. F.1 for a detailed comparison with theories in FOL).

Theorem 7.7 (Deduction theorem). *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c\colon 0 \to 0$ in $\mathbf{FOB}_\Sigma$. Let $\mathbb{I}' = \mathbb{I} \cup \{(id_0^\circ, c)\}$ and let $\mathbb{T}'$ denote the theory $(\Sigma, \mathbb{I}')$. Then, for every $a, b\colon n \to m$ arrows of $\mathbf{FOB}_\Sigma$,*



Proof. By induction on the rules of (10). We show only the case for (⨾). The remaining ones are in App. F.

Assume $a = a_1 ⨾ a_2$ and $b = b_1 ⨾ b_2$ for some $a_1, b_1\colon n \to l$, $a_2, b_2\colon l \to m$ such that $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$. By induction hypothesis $c \otimes id_n^\circ \lesssim_{\mathbb{T}} b_1 ⨟ a_1^\perp$ and $c \otimes id_n^\circ \lesssim_{\mathbb{T}} b_2 ⨟ a_2^\perp$. Thus:



□

Corollary 7.8. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory, $c\colon 0 \to 0$ in $\mathbf{FOB}_\Sigma$ and $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id_0^\circ, \bar{c})\})$. Then $id_0^\circ \lesssim_{\mathbb{T}} c$ iff $\mathbb{T}'$ is contradictory.*

7.1 Functorial semantics for first order theories

Recall that the notion of interpretation of a signature Σ in **Rel** has been generalised in Definition 6.9 to an arbitrary fo-bicategory. As expected, the same is possible also with the notion of model.

Definition 7.9. Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $\mathbf{C}$ a first order bicategory. An interpretation $\mathcal{I}$ of Σ in $\mathbf{C}$ is a model iff, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d)$.

For any theory $\mathbb{T} = (\Sigma, \mathbb{I})$, one can build a fo-bicategory $\mathbf{FOB}_\mathbb{T}$: this is like $\mathbf{FOB}_\Sigma$, but homsets are now $\mathbf{FOB}_\mathbb{T}[n, m] = \{[d]_{\cong_\mathbb{T}} \mid d \in \mathbf{FOB}_\Sigma[n, m]\}$ ordered by $\lesssim_\mathbb{T}$. Since, by definition, $\lesssim \subseteq \lesssim_\mathbb{T}$, $\mathbf{FOB}_\mathbb{T}$ is a fo-bicategory. Thus, one can take an interpretation $\mathcal{Q}_\mathbb{T}$ of Σ in $\mathbf{FOB}_\mathbb{T}$: the domain X is 1 and $\rho(R) = [R^\circ]_{\cong_\mathbb{T}}$ for all $R \in \Sigma$. By Prop. 6.10, $\mathcal{Q}_\mathbb{T}$ induces a fo-bicategory morphism $\mathcal{Q}_\mathbb{T}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_\mathbb{T}$.

Proposition 7.10. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory, $\mathbf{C}$ a fo-bicategory and $\mathcal{I}$ an interpretation of Σ in $\mathbf{C}$. $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$ iff $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ factors uniquely through $\mathcal{Q}_\mathbb{T}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_\mathbb{T}$.*

In other words, there is a unique fo-bicategory morphism $\mathcal{I}_\mathbb{T}^\sharp\colon \mathbf{FOB}_\mathbb{T} \to \mathbf{C}$ s.t. the diagram on the right commutes. The assignment $\mathcal{I} \mapsto \mathcal{I}_\mathbb{T}^\sharp$ yields a 1-to-1 correspondence between models and morphisms.

$$\begin{tikzcd} \mathbf{FOB}_\Sigma \arrow[r, "\mathcal{Q}_\mathbb{T}^\sharp"] \arrow[dr, "\mathcal{I}^\sharp"'] & \mathbf{FOB}_\mathbb{T} \arrow[d, dotted, "\mathcal{I}_\mathbb{T}^\sharp"] \\ & \mathbf{C} \end{tikzcd}$$

Corollary 7.11. *To give a model of $\mathbb{T}$ in $\mathbf{C}$ is to give a fo-bicategory morphism $\mathbf{FOB}_\mathbb{T} \to \mathbf{C}$.*

By virtue of the above, we can tacitly identify models and morphisms. Proposition 7.10 can also be used to obtain the following result, useful for showing completeness in the next section.

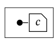
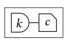
Lemma 7.12. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ and $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be theories s.t. $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$. Then there exists an identity on objects fo-bicategory morphism $\mathcal{F}\colon \mathbf{FOB}_\mathbb{T} \to \mathbf{FOB}_{\mathbb{T}'}$ mapping each d of $\mathbf{FOB}_\mathbb{T}$ to $[d]_{\cong_{\mathbb{T}'}}$.*

8 BEYOND GÖDEL'S COMPLETENESS

Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. First, we prove Gödel completeness

$$\text{if } \mathbb{T} \text{ is non-trivial, then } \mathbb{T} \text{ has a model} \qquad \text{(Gödel)}$$

by adapting Henkin's [37] proof to NPR_Σ. We begin with two additional definitions. Note that when referring to arrows in the context of $\mathbb{T}$, we mean arrows of $\mathbf{FOB}_\mathbb{T}$ (or of $\mathbf{FOB}_\Sigma$, it is immaterial).

Definition 8.1. $\mathbb{T}$ is *syntactically complete* if for all $c\colon 0 \to 0$ either $id_0^\circ \lesssim_\mathbb{T} c$ or $id_0^\circ \lesssim_\mathbb{T} \bar{c}$. $\mathbb{T}$ has *Henkin witnesses* if for all $c\colon 1 \to 0$ there is a map $k\colon 0 \to 1$ s.t.  $\lesssim_\mathbb{T}$ .

These properties do not hold for the theories we have considered so far. In terms of FOL, syntactic completeness means that closed formulas either hold in all models of the theory or in none. A Henkin witness is a term k such that $c(k)$ holds: a theory has Henkin witnesses if for every true formula $\exists x.c(x)$, there exists such a k. We shall see in Theorem 8.3 that non-trivial theories can be expanded to have Henkin witnesses, be non-contradictory and syntactically complete. The key idea of Henkin's proof, Theorem 8.6, is that these three properties yield a model.

To add a witness for $c\colon 1 \to 0$, one adds a constant $k\colon 0 \to 1$ and the axiom $\mathbb{W}^c_k$, asserting that k is a witness. This preserves non-triviality.

$$\mathbb{W}^c_k \stackrel{\text{def}}{=} \{(\square, \blacksquare)\}$$

Lemma 8.2 (Witness Addition). *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and consider an arbitrary $c\colon 1 \to 0$. Let $\mathbb{T}' = (\Sigma \cup \{k\colon 0 \to 1\}, \mathbb{I} \cup \mathbb{M}_k \cup \mathbb{W}^c_k)$. If $\mathbb{T}$ is non-trivial then $\mathbb{T}'$ is non-trivial.*

Remark 4. *Observe that the distinction between trivial and contradictory theories is essential for the above development. Indeed, under the conditions of Lemma 8.2, it does* not *hold that*

if $\mathbb{T}$ is non-contradictory, then $\mathbb{T}'$ is non-contradictory.

As counter-example, take as $\mathbb{T}$ the theory consisting only of the trivialising axiom $(tr) \stackrel{\text{def}}{=} (\square, \blacksquare)$. By definition $\mathbb{T}$ is trivial but non-contradictory. Instead $\mathbb{T}'$ is contradictory:

$$\square \stackrel{(15)}{\lesssim_{\mathbb{T}}} \square \stackrel{(tr)}{\lesssim_{\mathbb{T}}} \blacksquare \stackrel{(\gamma!^\circ)}{\lesssim_{\mathbb{T}}} \blacksquare \tag{16}$$

This shows that adding Henkin witnesses to a non-contradictory theory may end up in a contradictory theory. Therefore, the usual Henkin proof for FOL *works just for our* non-trivial *theories.*

By iteratively using Lemma 8.2, one can transform a non-trivial theory into a non-trivial theory with Henkin witnesses. To obtain a syntactically complete theory, we use the standard argument featuring Zorn's Lemma (see Prop. G.4 in App. G). In summary:

Theorem 8.3. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-trivial theory. There exists a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ such that $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$; $\mathbb{T}'$ has Henkin witnesses; $\mathbb{T}'$ is syntactically complete; $\mathbb{T}'$ is non-contradictory.*

Before introducing Henkin's interpretation, observe that any map $c\colon 0 \to n$ can be decomposed as $k_1 \otimes \ldots \otimes k_n$ where each $k_i\colon 0 \to 1$ is a map (see Prop. G.1 in App. G). We thus write such c as $\vec{k}$, depicted as $\boxed{\vec{k}}\!-\,n$, to make explicit its status as a vector.

Definition 8.4. Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. The *Henkin interpretation $\mathcal{H}$* of Σ, consists of a set $X \stackrel{\text{def}}{=} \mathtt{Map}(\mathbf{FOB}_{\mathbb{T}})[0, 1]$ and a function ρ, defined for all $R\colon n \to m \in \Sigma$ as:

$$\rho(R) \stackrel{\text{def}}{=} \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{k}-R-\vec{l}}\}$$

The domain is the set of constants of the theory. Then $R\colon n \to m$ is mapped to all pairs $(\vec{k}, \vec{l})$ of vectors that make R true in $\mathbb{T}$. The following characterisation of $\mathcal{H}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ is crucial.

Proposition 8.5. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory, syntactically complete theory with Henkin witnesses. Then, for any $c\colon n \to m$, $\mathcal{H}^\sharp(c) = \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{k}-c-\vec{l}}\}$.*

Theorem 8.6. *If $\mathbb{T}$ is non-contradictory, syntactically complete with Henkin witnesses, then $\mathcal{H}$ is a model.*

Proof. We show that $c \lesssim_{\mathbb{T}} d$ gives $\mathcal{H}^\sharp(c) \subseteq \mathcal{H}^\sharp(d)$. If $(\vec{k}, \vec{l}) \in \mathcal{H}^\sharp(c)$ then $\square \lesssim_{\mathbb{T}} \boxed{\vec{k}-c-\vec{l}}$ by Prop. 8.5. Since $c \lesssim_{\mathbb{T}} d$, $\square \lesssim_{\mathbb{T}} \boxed{\vec{k}-c-\vec{l}} \lesssim_{\mathbb{T}} \boxed{\vec{k}-d-\vec{l}}$ and by Prop. 8.5, $(\vec{k}, \vec{l}) \in \mathcal{H}^\sharp(d)$. □

Theorems 8.3 and 8.6 give us a proof for (Gödel).

Proof of (Gödel). Let $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be obtained via Theorem 8.3. Since $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$, by Lemma 7.12, we have $\mathcal{F}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{FOB}_{\mathbb{T}'}$. Since $\mathbb{T}'$ has Henkin witnesses, is syntactically complete and non-contradictory, Theorem 8.6 gives $\mathcal{H}^\sharp_{\mathbb{T}'}\colon \mathbf{FOB}_{\mathbb{T}'} \to \mathbf{Rel}$. We thus have a morphism $\mathbf{FOB}_{\mathbb{T}} \to \mathbf{Rel}$. □

Now, we would like to conclude Theorem 3.2 by means of (Gödel), but this is not possible since, for the former one needs a model for all non-contradictory theories, while (Gödel) provides it only for non-trivial ones. Thankfully, the Henkin interpretation $\mathcal{H}$ gives us, once more, a model (see Prop. in App. G) that allows us to prove

$$\text{if } \mathbb{T} \text{ is trivial and non-contradictory, then } \mathbb{T} \text{ has a model.} \tag{Prop}$$

From (Prop) and (Gödel) we can prove general completeness

$$\text{if } \mathbb{T} \text{ is non-contradictory, then } \mathbb{T} \text{ has a model} \tag{General}$$

and thus deduce our main result.

Proof of (General) and Theorem 3.2. To prove (General) take $\mathbb{T}$ to be a non-contradictory theory. If $\mathbb{T}$ is trivial, then it has a model by (Prop). Otherwise, it has a model by (Gödel). Now, by means of traditional FOL arguments exploiting Corollary 7.8, one can show that (General) entails Theorem 3.2 (see Prop.G.14 in App. G). □

8.1 The Calculus of Binary Relations (revisited)

The map $\mathcal{E}(\cdot)$ defined in Table 3 is an econding of the calculus of relations into NPR_Σ. Since $\mathcal{E}(\cdot)$ preserves the semantics (see Prop. G.15 in App.G.4), from Theorem 3.2 follows that one can prove inclusions of expressions of CR_Σ by translating them into NPR_Σ via $\mathcal{E}(\cdot)$ and then using the axioms in Fig.s 2, 3, 4 and 5.

Corollary 8.7. *For all E_1, E_2, $E_1 \leq_{\mathsf{CR}} E_2$ iff $\mathcal{E}(E_1) \lesssim \mathcal{E}(E_2)$.*

9 FIRST ORDER LOGIC WITH EQUALITY

As we already mentioned in the introduction the white fragment of NPR_Σ is as expressive as the existential-conjunctive fragment of first order logic with equality (FOL). The semantic preserving encodings between the two fragments are illustrated in [9]. From the fact that the full NPR_Σ can express negation, we get immediately semantic preserving encodings between NPR_Σ and the full FOL. In this section we illustrate anyway a translation $\mathcal{E}(\cdot)\colon \mathsf{FOL} \to \mathsf{NPR}_\Sigma$ to emphasise the subtle differences between the two. To go in the other way, the reader is referred to App. B.4.

To ease the presentation, we consider FOL formulas φ to be typed in the context of a list of variables that are allowed (but not required) to appear in φ. Fixing $\mathbf{x}_n \stackrel{\text{def}}{=} \{x_1, \ldots, x_n\}$ we write $n\colon\varphi$ if all free variables of φ are contained in $\mathbf{x}_n$. It is standard to present FOL in two steps: first terms and then formulas. For every function symbol f of arity m in FOL, we have a symbol $f\colon m \to 1$ in the signature Σ together with the equations $\mathbb{M}_f$ forcing f to be interpreted as

Table 3: The encoding $\mathcal{E}(\cdot)\colon \mathsf{CR}_\Sigma \to \mathsf{NPR}_\Sigma$

$\mathcal{E}(R) \stackrel{\text{def}}{=} R^\circ$	$\mathcal{E}(id^\circ) \stackrel{\text{def}}{=} id_1^\circ$	$\mathcal{E}(E_1 ⨾ E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) ⨾ \mathcal{E}(E_2)$	$\mathcal{E}(\top) \stackrel{\text{def}}{=} !_1^\circ ⨾ ¡_1^\circ$	$\mathcal{E}(E_1 \cap E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\circ ⨾ (\mathcal{E}(E_1) \otimes \mathcal{E}(E_2)) ⨾ \blacktriangleright_1^\circ$	$\mathcal{E}(\overline{E}) \stackrel{\text{def}}{=} \overline{\mathcal{E}(E)}$
$\mathcal{E}(E^\dagger) \stackrel{\text{def}}{=} \mathcal{E}(E)^\dagger$	$\mathcal{E}(id^\bullet) \stackrel{\text{def}}{=} id_1^\bullet$	$\mathcal{E}(E_1 \mathbin{;} E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) \mathbin{;} \mathcal{E}(E_2)$	$\mathcal{E}(\bot) \stackrel{\text{def}}{=} !_1^\bullet \mathbin{;} ¡_1^\bullet$	$\mathcal{E}(E_1 \cup E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\bullet \mathbin{;} (\mathcal{E}(E_1) \mathbin{\otimes^\bullet} \mathcal{E}(E_2)) \mathbin{;} \blacktriangleright_1^\bullet$	

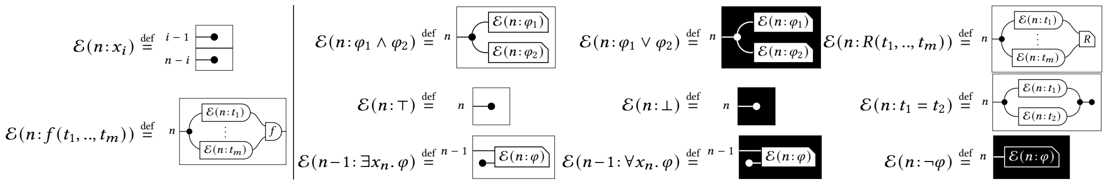


Figure 7: FOL **encoding in** NPR_Σ**.**

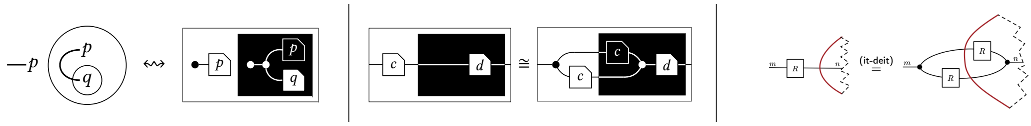


Figure 8: An EG **and its encoding in** NPR_Σ **(left); Peirce's (de)iteration rule in** NPR_Σ **(middle) and in [36] (right).**

a function. The translation of $n\colon t$ to an NPR_Σ diagram $n \to 1$ is given inductively in the left part of Fig. 7.

Formulas $n\colon \varphi$ translate to NPR_Σ diagrams $n \to 0$. For every n-ary predicate symbol R in FOL there is a symbol $R\colon n \to 0 \in \Sigma$. In order not to over-complicate the presentation with bureaucratic details, we assume that it is always the last variable that is quantified over. Additional variable manipulation can be introduced: see App. B.3 for an encoding of Quine's predicate functor logic.

The full encoding in Fig. 7 should give the reader the spirit of the correspondence between NPR_Σ and traditional syntax. There is one aspect of the above translation that merits additional attention.

Remark 5. *By the definition of $!_n^\circ$ in Table 1, we have that:*

$\mathcal{E}(0\colon \top) \stackrel{def}{=}$  $\qquad \mathcal{E}(0\colon \bot) \stackrel{def}{=}$ 

Thus $\top$ and $\bot$ translate to, respectively id_0°, $id_0^\bullet$ in the absence of free variables or to $!_n^\circ$, $!_n^\bullet$, respectively, when $n > 0$. This can be seen as an ambiguity in the traditional FOL syntax, which obscures the distinction between inconsistent and trivial theories in traditional accounts, and as a side effect requires the assumption on non-empty models in formal statements of Gödel completeness. Instead, the syntax of NPR_Σ ensures that this pitfall is side-stepped.

10 CONCLUDING REMARKS

The diagrammatic notation of NPR_Σ is closely related to system β of Peirce's EGs [64–66, 77]. Consider the two diagrams on the left of Fig. 8 corresponding to the closed FOL formula $\exists x.\, p(x) \land \forall y.\, p(y) \to q(y)$. In existential graph notation the circle enclosure (dubbed 'cut' by Peirce) signifies negation. To move from EGs to diagrams of NPR_Σ it suffices to treat lines and predicate symbols in the obvious way and each cut as a color switch.

A string diagrammatic approach to existential graphs appeared in [36]. This exploits the white fragment of NPR_Σ with a primitive negation operator rendered as Peirce's cut, namely a circle around diagrams. However, this inhibits a fully compositional treatment since, for instance, negation is not functorial. As an example consider Peirce's (de)iteration rule in Fig. 8: in NPR_Σ on the center, and in [36] on the right. Note that the diagrams on the right require open cuts, a notational trick, allowing to express the rule for arbitrary contexts, i.e. any diagram eventually appearing inside the cut. In NPR_Σ this ad-hoc treatment of contexts is not needed as negation is not a primitive operation, but a derived one. A proof of the law in the middle of Fig. 8 can be found in App. B.1.

Other diagrammatic calculi of Peirce's EGs appear in [52] and [14]. The categorical treatment goes, respectively, through the notions of chiralities and doctrines. The formers consider a pair of categories $(\mathbf{Rel}_\bullet, \mathbf{Rel}_\circ)$ that are significantly different from our $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$: to establish a formal correspondence, it might be convenient to first focus on doctrines. To this aim, we plan to exploit the equivalence in [8] between cartesian bicategories and certain doctrines (elementary existential with comprehensive diagonals and unique choice [51]). Preliminary attempts suggests the same equivalence restrict to fo-bicategories and boolean hyperdoctrines but many details have to be carefully checked. The connection with allegories [29] is also worth to be explored: since cartesian bicategories are equivalent to unitary pretabular allegories, Prop. 6.5 suggests that fo-bicategories are closely related to Peirce allegories [58].

Through the Introduction, we have already emphasized the key features of the calculus of neo-Peircean relations. We hope that the reader has also appreciated its beauty. Quoting Dijkstra [24]:

> "When we recognize the battle against chaos, mess and unmastered complexity as one of computing science's major challenges, we must admit that Beauty is our Business."

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A A TRIBUTE TO CHARLES S. PEIRCE

We have chosen the name "Neo-Peircean Relations" to emphasize several connections with the work of Charles S. Peirce. First of all, NPR_Σ and the calculus of relations in 'Note B' [61] share the same underlying philosophy: they both propose a relational analogue to Boole's algebra of classes.

Second, Peirce's presentation in 'Note B' contains already several key ingredients of NPR_Σ. As we have stressed, it singles out the two forms of composition ($\fatsemi$ and $\bullet$), presents linear distributivity ((δ_l) and (δ_r)) and linear adjunctions (($\tau\sigma^\circ$), ($\tau\sigma^\bullet$), ($\gamma\sigma^\circ$), and ($\gamma\sigma^\bullet$)), and even the cyclic conditions of Lemma 6.6.(2)-(3). With respect to the rules for linear distributivity and linear adjunction, Peirce states that the latter are "highly important" and that the former are "so constantly used that hardly anything can be done without them" (p. 192 & 190).

At around the same time as 'Note B' Pierce gave a systematic study of residuation [60, see "On the Logic of Relatives"] and listed a set of equivalent expressions that includes the discussion given after Lemma 5.3, where $c \fatsemi a \leq b$ iff $c \leq b \bullet a^\perp$. In Peirce's words:

> Hence the rule is that having a formula of the form $[c \fatsemi a \leq b]$, the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted. [60, p. 341]

Peirce took the principal defect of the presentation in 'Note B' to be its focus on binary relations [63, 8:831]. He went on to emphasize the *teri-* or *tri*-identity relation, arising from adding a 'branch' to the identity relation, as the key to moving from binary to arbitrary relations. Having the advantage now of "treating triadic and higher relations as easily as dyadic relations... it's superiority to the human mind as an instrument of logic", he writes, "is overwhelming" [67, p. 173].

By moving from binary to arbitrary relations, Peirce felt the importance of a graphical syntax and developed the existential graphs.

> "One of my earliest works was an enlargement of Boole's idea so as to take into account ideas of relation, — or at least of all ideas of existential relation… *I was finally led to prefer what I call a diagrammatic syntax.* It is a way of setting down on paper any assertion, however intricate… " [MS 515, emphasis in original, 1911]

We refer the reader to [36] for a detailed explanation of Peirce's topological intuitions behind the Frobenius equations and the correspondence of some inference rules for EGs with those of (co)cartesian bicategories. Moreover, we now know that Peirce continued to study and draw graphs of residuation [35] and — as affirmed in Fig. 6 — we know the rules for propositional EGs comprise a deep inference system [49].

In short, Peirce's development of EGs shares many of the features that NPR_Σ has over other approaches, such as Tarski's presentation of relation algebra. We like to think that if Peirce had known category theory then he would have presented NPR_Σ.

$$
\begin{array}{rcl}
\blacktriangleleft^{\circ}_{1} \fatsemi (id^{\circ}_{1} \otimes \blacktriangleleft^{\circ}_{1}) & \overset{(\blacktriangleleft^{\circ}\text{-as})}{=} & \blacktriangleleft^{\circ}_{1} \fatsemi (\blacktriangleleft^{\circ}_{1} \otimes id^{\circ}_{1}) \\
\blacktriangleleft^{\circ}_{1} \fatsemi (id^{\circ}_{1} \otimes !^{\circ}_{1}) & \overset{(\blacktriangleleft^{\circ}\text{-un})}{=} & id^{\circ}_{1} \\
\blacktriangleleft^{\circ}_{1} \fatsemi \sigma^{\circ}_{1,1} & \overset{(\blacktriangleleft^{\circ}\text{-co})}{=} & \blacktriangleleft^{\circ}_{1}
\end{array}
\qquad
\begin{array}{rcl}
(id^{\circ}_{1} \otimes \blacktriangleright^{\circ}_{1}) \fatsemi \blacktriangleright^{\circ}_{1} & \overset{(\blacktriangleright^{\circ}\text{-as})}{=} & (\blacktriangleright^{\circ}_{1} \otimes id^{\circ}_{1}) \fatsemi \blacktriangleright^{\circ}_{1} \\
(id^{\circ}_{1} \otimes ¡^{\circ}_{1}) \fatsemi \blacktriangleright^{\circ}_{1} & \overset{(\blacktriangleright^{\circ}\text{-un})}{=} & id^{\circ}_{1} \\
\sigma^{\circ}_{1,1} \fatsemi \blacktriangleright^{\circ}_{1} & \overset{(\blacktriangleright^{\circ}\text{-co})}{=} & \blacktriangleright^{\circ}_{1}
\end{array}
$$

$$
(\blacktriangleleft^{\circ}_{1} \otimes id^{\circ}_{1}) \fatsemi (id^{\circ}_{1} \otimes \blacktriangleright^{\circ}_{1}) \overset{(\mathrm{F}^{\circ})}{=} (id^{\circ}_{1} \otimes \blacktriangleleft^{\circ}_{1}) \fatsemi (\blacktriangleright^{\circ}_{1} \otimes id^{\circ}_{1})
\qquad
\blacktriangleleft^{\circ}_{1} \fatsemi \blacktriangleright^{\circ}_{1} \overset{(\mathrm{S}^{\circ})}{=} id^{\circ}_{1}
$$

$$
\begin{array}{rcl}
¡^{\circ}_{1} \fatsemi !^{\circ}_{1} & \overset{(\epsilon !^{\circ})}{\leq} & id^{\circ}_{0} \\
id^{\circ}_{1} & \overset{(\eta !^{\circ})}{\leq} & !^{\circ}_{1} \fatsemi ¡^{\circ}_{1}
\end{array}
\qquad
\begin{array}{rcl}
\blacktriangleright^{\circ}_{1} \fatsemi \blacktriangleleft^{\circ}_{1} & \overset{(\epsilon \blacktriangleleft^{\circ})}{\leq} & (id^{\circ}_{1} \otimes id^{\circ}_{1}) \\
id^{\circ}_{1} & \overset{(\eta \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_{1} \fatsemi \blacktriangleright^{\circ}_{1}
\end{array}
\qquad
\begin{array}{rcl}
c \fatsemi \blacktriangleleft^{\circ}_{m} & \overset{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} & \blacktriangleleft^{\circ}_{n} \fatsemi (c \otimes c) \\
c \fatsemi !^{\circ}_{m} & \overset{(!^{\circ}\text{-nat})}{\leq} & !^{\circ}_{n}
\end{array}
$$

$$
\begin{array}{rcl}
\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} (id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} \blacktriangleleft^{\bullet}_{1}) & \overset{(\blacktriangleleft^{\bullet}\text{-as})}{=} & \blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} (\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{1}) \\
\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} (id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} !^{\bullet}_{1}) & \overset{(\blacktriangleleft^{\bullet}\text{-un})}{=} & id^{\bullet}_{1} \\
\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{1,1} & \overset{(\blacktriangleleft^{\bullet}\text{-co})}{=} & \blacktriangleleft^{\bullet}_{1}
\end{array}
\qquad
\begin{array}{rcl}
(id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} \blacktriangleright^{\bullet}_{1}) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} & \overset{(\blacktriangleright^{\bullet}\text{-as})}{=} & (\blacktriangleright^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{1}) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} \\
(id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} ¡^{\bullet}_{1}) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} & \overset{(\blacktriangleright^{\bullet}\text{-un})}{=} & id^{\bullet}_{1} \\
\sigma^{\bullet}_{1,1} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} & \overset{(\blacktriangleright^{\bullet}\text{-co})}{=} & \blacktriangleright^{\bullet}_{1}
\end{array}
$$

$$
(\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{1}) \mathbin{\boldsymbol{;}} (id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} \blacktriangleright^{\bullet}_{1}) \overset{(\mathrm{F}^{\bullet})}{=} (id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} \blacktriangleleft^{\bullet}_{1}) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{1})
\qquad
\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} \overset{(\mathrm{S}^{\bullet})}{=} id^{\bullet}_{1}
$$

$$
\begin{array}{rcl}
!^{\bullet}_{1} \mathbin{\boldsymbol{;}} ¡^{\bullet}_{1} & \overset{(\epsilon ¡^{\bullet})}{\leq} & id^{\bullet}_{1} \\
id^{\bullet}_{0} & \overset{(\eta ¡^{\bullet})}{\leq} & ¡^{\bullet}_{1} \mathbin{\boldsymbol{;}} !^{\bullet}_{1}
\end{array}
\qquad
\begin{array}{rcl}
\blacktriangleleft^{\bullet}_{1} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{1} & \overset{(\epsilon \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_{1} \\
\blacktriangleright^{\bullet}_{1} \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_{1} & \overset{(\eta \blacktriangleright^{\bullet})}{\leq} & (id^{\bullet}_{1} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{1})
\end{array}
\qquad
\begin{array}{rcl}
\blacktriangleleft^{\bullet}_{n} \mathbin{\boldsymbol{;}} (c \mathbin{\boldsymbol{\otimes}} c) & \overset{(\blacktriangleleft^{\bullet}\text{-nat})}{\leq} & c \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_{m} \\
!^{\bullet}_{n} & \overset{(!^{\bullet}\text{-nat})}{\leq} & c \mathbin{\boldsymbol{;}} !^{\bullet}_{m}
\end{array}
$$

$$
a \fatsemi (b \mathbin{\boldsymbol{;}} c) \overset{(\delta_{l})}{\leq} (a \fatsemi b) \mathbin{\boldsymbol{;}} c
\qquad
(a \mathbin{\boldsymbol{;}} b) \fatsemi c \overset{(\delta_{r})}{\leq} a \mathbin{\boldsymbol{;}} (b \fatsemi c)
$$

$$
\begin{array}{rcl}
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\circ})}{\leq} & \sigma^{\circ}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{m,n} \\
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\bullet})}{\leq} & \sigma^{\bullet}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\circ}_{m,n}
\end{array}
\quad
\begin{array}{rcl}
\sigma^{\bullet}_{n,m} \fatsemi \sigma^{\circ}_{m,n} & \overset{(\gamma\sigma^{\circ})}{\leq} & id^{\bullet}_{n+m} \\
\sigma^{\circ}_{n,m} \fatsemi \sigma^{\bullet}_{m,n} & \overset{(\gamma\sigma^{\bullet})}{\leq} & id^{\bullet}_{n+m}
\end{array}
\quad
\begin{array}{rcl}
id^{\circ}_{n} & \overset{(\tau R^{\circ})}{\leq} & R^{\circ} \mathbin{\boldsymbol{;}} R^{\bullet} \\
id^{\circ}_{m} & \overset{(\tau R^{\bullet})}{\leq} & R^{\bullet} \mathbin{\boldsymbol{;}} R^{\circ}
\end{array}
\quad
\begin{array}{rcl}
R^{\bullet} \fatsemi R^{\circ} & \overset{(\gamma R^{\circ})}{\leq} & id^{\bullet}_{m} \\
R^{\circ} \fatsemi R^{\bullet} & \overset{(\gamma R^{\bullet})}{\leq} & id^{\bullet}_{n}
\end{array}
$$

$$
\begin{array}{rcl}
id^{\circ}_{n+m} & \overset{(\boldsymbol{\otimes}^{\circ})}{\leq} & id^{\circ}_{n} \mathbin{\boldsymbol{\otimes}} id^{\circ}_{m} \\
(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) & \overset{(\nu^{\circ}_{l})}{\leq} & (a \otimes c) \mathbin{\boldsymbol{;}} (b \mathbin{\boldsymbol{\otimes}} d) \\
(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) & \overset{(\nu^{\circ}_{r})}{\leq} & (a \mathbin{\boldsymbol{\otimes}} c) \mathbin{\boldsymbol{;}} (b \otimes d)
\end{array}
\qquad
\begin{array}{rcl}
id^{\bullet}_{n} \otimes id^{\bullet}_{m} & \overset{(\otimes^{\bullet})}{\leq} & id^{\bullet}_{n+m} \\
(a \mathbin{\boldsymbol{\otimes}} c) \fatsemi (b \otimes d) & \overset{(\nu^{\bullet}_{l})}{\leq} & (a \fatsemi b) \mathbin{\boldsymbol{\otimes}} (c \fatsemi d) \\
(a \otimes c) \fatsemi (b \mathbin{\boldsymbol{\otimes}} d) & \overset{(\nu^{\bullet}_{r})}{\leq} & (a \fatsemi b) \mathbin{\boldsymbol{\otimes}} (c \fatsemi d)
\end{array}
$$

$$
\begin{array}{rcl}
id^{\circ}_{n} & \overset{(\tau \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_{n} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_{n} \\
id^{\circ}_{n} & \overset{(\tau !^{\circ})}{\leq} & !^{\circ}_{n} \mathbin{\boldsymbol{;}} ¡^{\bullet}_{n} \\
id^{\circ}_{n} & \overset{(\tau \blacktriangleleft^{\bullet})}{\leq} & \blacktriangleleft^{\bullet}_{n} \mathbin{\boldsymbol{;}} \blacktriangleright^{\circ}_{n} \\
id^{\circ}_{n} & \overset{(\tau !^{\bullet})}{\leq} & !^{\bullet}_{n} \mathbin{\boldsymbol{;}} ¡^{\circ}_{n}
\end{array}
\quad
\begin{array}{rcl}
\blacktriangleright^{\bullet}_{n} \fatsemi \blacktriangleleft^{\circ}_{n} & \overset{(\gamma \blacktriangleleft^{\circ})}{\leq} & id^{\bullet}_{n+n} \\
¡^{\bullet}_{n} \fatsemi !^{\circ}_{n} & \overset{(\gamma !^{\circ})}{\leq} & id^{\bullet}_{0} \\
\blacktriangleright^{\circ}_{n} \fatsemi \blacktriangleleft^{\bullet}_{n} & \overset{(\gamma \blacktriangleleft^{\bullet})}{\leq} & id^{\bullet}_{n+n} \\
¡^{\circ}_{n} \fatsemi !^{\bullet}_{n} & \overset{(\gamma !^{\bullet})}{\leq} & id^{\bullet}_{0}
\end{array}
\quad
\begin{array}{rcl}
id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\circ})}{\leq} & \blacktriangleright^{\circ}_{n} \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_{n} \\
id^{\circ}_{0} & \overset{(\tau ¡^{\circ})}{\leq} & ¡^{\circ}_{n} \mathbin{\boldsymbol{;}} !^{\bullet}_{n} \\
id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\bullet})}{\leq} & \blacktriangleright^{\bullet}_{n} \mathbin{\boldsymbol{;}} \blacktriangleleft^{\circ}_{n} \\
id^{\circ}_{0} & \overset{(\tau ¡^{\bullet})}{\leq} & ¡^{\bullet}_{n} \mathbin{\boldsymbol{;}} !^{\circ}_{n}
\end{array}
\quad
\begin{array}{rcl}
\blacktriangleleft^{\bullet}_{n} \fatsemi \blacktriangleright^{\circ}_{n} & \overset{(\gamma \blacktriangleright^{\circ})}{\leq} & id^{\bullet}_{n} \\
!^{\bullet}_{n} \fatsemi ¡^{\circ}_{n} & \overset{(\gamma ¡^{\circ})}{\leq} & id^{\bullet}_{n} \\
\blacktriangleleft^{\circ}_{n} \fatsemi \blacktriangleright^{\bullet}_{n} & \overset{(\gamma \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_{n} \\
!^{\circ}_{n} \fatsemi ¡^{\bullet}_{n} & \overset{(\gamma ¡^{\bullet})}{\leq} & id^{\bullet}_{0}
\end{array}
$$

$$
\begin{array}{rcl}
(\blacktriangleleft^{\bullet}_{n} \otimes id^{\circ}_{n}) \fatsemi (id^{\circ}_{n} \otimes \blacktriangleright^{\circ}_{n}) & \overset{(\mathrm{F}^{\bullet}{}_{\circ})}{=} & (id^{\circ}_{n} \otimes \blacktriangleleft^{\circ}_{n}) \fatsemi (\blacktriangleright^{\bullet}_{n} \otimes id^{\circ}_{n}) \\
(\blacktriangleleft^{\circ}_{n} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{n}) \mathbin{\boldsymbol{;}} (id^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} \blacktriangleright^{\bullet}_{n}) & \overset{(\mathrm{F}_{\bullet}{}^{\circ})}{=} & (id^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} \blacktriangleleft^{\bullet}_{n}) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\circ}_{n} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{n})
\end{array}
\quad
\begin{array}{rcl}
(\blacktriangleleft^{\circ}_{n} \otimes id^{\circ}_{n}) \fatsemi (id^{\circ}_{n} \otimes \blacktriangleright^{\bullet}_{n}) & \overset{(\mathrm{F}^{\circ}{}_{\bullet})}{=} & (id^{\circ}_{n} \otimes \blacktriangleleft^{\bullet}_{n}) \fatsemi (\blacktriangleright^{\circ}_{n} \otimes id^{\circ}_{n}) \\
(\blacktriangleleft^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{n}) \mathbin{\boldsymbol{;}} (id^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} \blacktriangleright^{\circ}_{n}) & \overset{(\mathrm{F}_{\circ}{}^{\bullet})}{=} & (id^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} \blacktriangleleft^{\circ}_{n}) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_{n} \mathbin{\boldsymbol{\otimes}} id^{\bullet}_{n})
\end{array}
$$

Figure 9: Axioms for NPR_{Σ}. Here a, b, c, d are diagrams of the appropriate type.

B ADDITIONAL MATERIAL

In Figure 9 we give a term-based version of the axioms of NPR_Σ. In the rest of this appendix we give some additional diagrammatic proofs; some more details on the trivial theory of Propositional Calculus (Example 7.6); an encoding of Quine's PFL_Σ in NPR_Σ; and a translation of NPR_Σ diagrams into (typed) FOL formulas.

B.1 Additional proofs

In Figure 10 we give a completely axiomatic proof of the inclusion in (1). In Figure 11 we prove Peirce's (de)iteration rule (Figure 8), showing the two inclusions separately.

B.2 The trivial theory of Propositional Calculus (Example 7.6)

In this appendix we revisit the propositional case shortly illustrated in Example 7.6.

First, we better details why the axioms of fo-bicategories (in Fig.s 2, 3, 4, 5) collapse to those in Fig. 6 for arrows of type $0 \to 0$. Consider for instance ($\blacktriangleleft^\circ$-nat): by definition of $\blacktriangleleft_0^\circ$ in Tab. 1, the two diagrams of ($\blacktriangleleft^\circ$-nat) in Fig. 2 reduce to those in Fig. 6. The rules (ν_l°), (ν_r°), $(\nu_l^\bullet)$ and $(\nu_r^\bullet)$ become redundant since, by the axioms of symmetric monoidal categories, $\,;\,$ and $\otimes$ coincide on diagrams $0 \to 0$ and are associative, commutative and with unit id_0°.

Then, we draw reader attention toward the correspondence with [15]: this is illustrated in Figure 12. We expect that there exists also a strong connection with Peirce's system α and its categorical treatment given in [13] by means of *-autonomy.

We conclude with the following proposition ensuring that diagrams $0 \to 0$ are exactly propositional formulas.

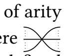
Proposition B.1. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be the theory of Example 7.6. For every diagram $a \colon 0 \to 0$ in $\mathbf{FOB}_\Sigma$ there exists a $\cong_\mathbb{T}$-equivalent diagram generated by the following grammar where $R \in \Sigma$.*

Proof. By induction on $a \colon 0 \to 0$. Observe that there are only four base cases: id_0°, $id_0^\bullet$, R° and $R^\bullet$. These already appear in the grammar above. We have the usual four inductive cases:

(1) $a = c \,;\, d$. There are two sub-cases: either $c, d \colon 0 \to 0$ or $c \colon 0 \to n+1$ and $d \colon n+1 \to 0$. In the former we can use the inductive hypothesis to get c' and d' generated by the above grammar such that $c' \cong_\mathbb{T} c$ and $d' \cong_\mathbb{T} d$. Thus a is $\cong_\mathbb{T}$-equivalent to $c' \,;\, d'$ that is generated by the above grammar.
Consider now the case where $c \colon 0 \to n+1$ and $d \colon n+1 \to 0$. By Lemma 7.5, $c \cong_\mathbb{T} ¡_{n+1}^\circ$ and $d \cong_\mathbb{T} !_{n+1}^\bullet$. By axiom $(\gamma !^\bullet)$, $¡_{n+1}^\circ \,;\, !_{n+1}^\bullet \cong id_0^\bullet$. Thus $a \cong id_0^\bullet$.
(2) $a = c \otimes d$. Note that, in this case both c and d must have type $0 \to 0$. Thus we can use the inductive hypothesis to get c' and d' generated by the above grammar such that $c' \cong_\mathbb{T} c$ and $d' \cong_\mathbb{T} d$. Thus $a \cong_\mathbb{T} c' \otimes d' \approx c' \,;\, d'$. Note that $c' \,;\, d'$ is generated by the above grammar.
(3) $a = c \,⨾^\bullet\, d$. The proof follows symmetrical arguments to the case $c \,;\, d$.
(4) $a = c \,\otimes^\bullet\, d$. The proof follows symmetrical arguments to the case $c \otimes d$. □

B.3 Quine's predicate functor logic

Inspired by combinatory logic, Quine [75] introduced *predicate functor logic*, PFL_Σ for short, as a quantifier-free treatment of first order logic with equality. Several flavours of the logic have been proposed by Quine and others, here we focus on the treatment by Kuhn [44]. Using the terminology of that thread of research, for each $n \geq 0$ there is a collection of atomic n-ary predicates, corresponding to traditional FOL predicate symbols together with an additional binary predicate I (identity). The term (predicate) constructors are called *functors* – here the terminology is unrelated to the notion of functor in category theory. These are divided into unary operations $\mathbf{p}, \mathbf{P}, [,]$ called *combinatory functors* that, in the absence of explicit variables, capture the combinatorial aspects of handling variable lists as well as (existential) quantification. To get full expressivity of FOL, there are two additional *alethic functors*: a binary conjunction and unary negation.

The syntax is reported on the top of Table 4 where R belong to Σ, a set of symbols with an associated arity. Similarly to NPR_Σ, only the predicates that can be typed according to the rules in Table 4 are considered. The semantics, on the bottom, is defined w.r.t. an interpretation $\mathcal{I}$ consisting of a *non-empty* set X and a set $\rho(R) \subseteq X^n$ for all $R \in \Sigma$ of arity n. For all predicates P, $\langle P \rangle_\mathcal{I}$ is a subset of $X^\omega \stackrel{\text{def}}{=} \{\tau_1 \cdot \tau_2 \cdots \mid \tau_i \in X \text{ for all } i \in \mathbb{N}^+\}$. From $\mathcal{I} = (X, \rho)$, one can define an interpretation of NPR_Σ $\mathcal{I}_p \stackrel{\text{def}}{=} (X, \rho_p)$ where $\rho_p(R) \stackrel{\text{def}}{=} \{(x, \star) \mid x \in \rho(R)\} \subseteq X^n \times \mathbb{1}$ for all $R \in \Sigma$ of arity n. The encoding of PFL_Σ into NPR_Σ is given in Table 5 where  is a suggestive representation for the permutation formally defined as $\sigma_{1,n-1}^\circ \,;\, (\sigma_{n-2,1}^\circ \otimes id_1^\circ)$ for $n \geq 2$, id_n° for $n < 2$.

Proposition B.2. *Let $P \colon n$ be a predicate of PFL_Σ. Then $\langle P \rangle_\mathcal{I} = \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}$.*

Proof. The proof goes by induction on the typing rules:
Base cases:

- $I \colon 2$. By definition $\langle I \rangle_\mathcal{I} = \{\tau \mid \tau_1 = \tau_2\}$ and $\mathcal{I}_p^\sharp(\mathcal{E}(I)) = \{((x_1, x_2), \star) \mid x_1 = x_2\}$. Thus $\langle I \rangle_\mathcal{I} = \{\tau \mid ((\tau_1, \tau_2), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(I))\}$.
- $R \colon n$. Assume $ar(R) = n$. By definition $\langle R \rangle_\mathcal{I} = \{\tau \mid (\tau_1, \dots, \tau_n) \in \rho(R)\}$ and $\mathcal{I}_p^\sharp(\mathcal{E}(R)) = \{((x_1, \dots, x_n), \star) \mid (x_1, \dots x_n) \in \rho(R)\}$. Thus $\langle R \rangle_\mathcal{I} = \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(R))\}$.

The inductive cases follow always the same argument. We report below only the most interesting ones.

- $P_1 \cap P_2$. Assume $P_1 \colon n$, $P_2 \colon m$ and $n \geq m$.

$$
\begin{aligned}
& \langle P_1 \cap P_2 \rangle_\mathcal{I} \\
= \;& \langle P_1 \rangle_\mathcal{I} \cap \langle P_2 \rangle_\mathcal{I} && (\text{def. } \langle \cdot \rangle_\mathcal{I}) \\
= \;& \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1))\} \\
& \quad \cap \{\tau \mid ((\tau_1, \dots, \tau_m), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_2))\} && (\text{ind. hyp.}) \\
= \;& \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1)) \\
& \quad \wedge ((\tau_1, \dots, \tau_m), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_2))\} \\
= \;& \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1 \cap P_2))\} && (\text{def. } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot))
\end{aligned}
$$

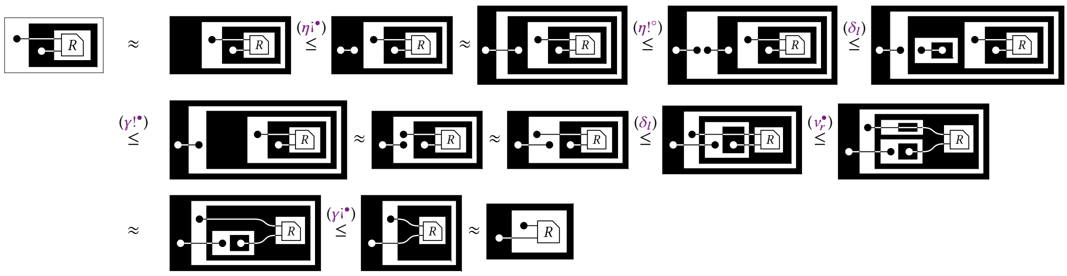


Figure 10: Completely axiomatic proof of (1).

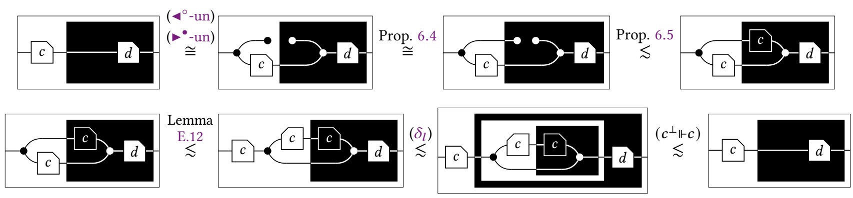


Figure 11: Proof of Peirce's (de)iteration rule in Figure 8.

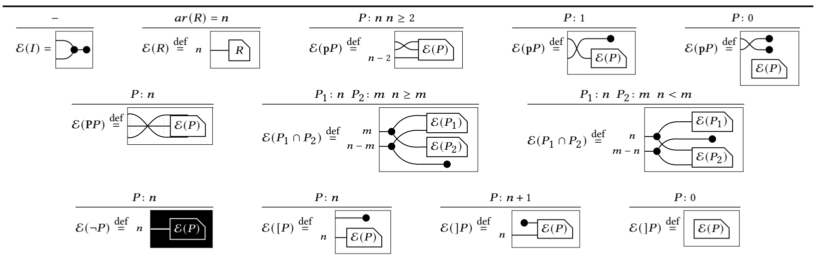
Table 4: PFL_Σ: (top) syntax; (mid) typing rules; (bottom) semantics w.r.t. an interpretation $\mathcal{I} = (X, \rho)$.

$$P \quad ::= \quad R \mid I \mid \mathbf{p}P \mid \mathbf{P}P \mid [P \mid]P \mid P \cap P \mid \neg P, \quad \text{where } R \in \Sigma$$

$$\frac{-}{I : 2} \quad \frac{ar(R) = n}{R : n} \quad \frac{P : n \;\; n \geq 2}{\mathbf{p}P : n} \quad \frac{P : 1}{\mathbf{p}P : 2} \quad \frac{P : 0}{\mathbf{p}P : 2} \quad \frac{P : n}{\mathbf{P}P : n} \quad \frac{P_1 : n \;\; P_2 : m \;\; n \geq m}{P_1 \cap P_2 : n} \quad \frac{P_1 : n \;\; P_2 : m \;\; n < m}{P_1 \cap P_2 : m} \quad \frac{P : n}{\neg P : n} \quad \frac{P : n}{[P : n+1} \quad \frac{P : n+1}{]P : n} \quad \frac{P : 0}{]P : 0}$$

$$\begin{array}{lll}
\langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid (\tau_1, \dots, \tau_n) \in \rho(R)\} & \langle I \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_1 = \tau_2\} & \langle]P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_2 \cdot \tau_3 \cdots \in \langle P \rangle_{\mathcal{I}})\} \\
\langle P_1 \cap P_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle P_1 \rangle_{\mathcal{I}} \cap \langle P_2 \rangle_{\mathcal{I}} & \langle \neg P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau \notin \langle P \rangle_{\mathcal{I}}\} & \langle [P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{x_0 \cdot \tau_1 \cdot \tau_2 \cdots \mid x_0 \in X, \tau_1 \cdot \tau_2 \cdots \in \langle P \rangle_{\mathcal{I}}\} \\
\langle \mathbf{P}P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_n \cdot \tau_2 \cdots \tau_{n-1} \cdot \tau_1 \cdot \tau_{n+1} \cdots \in \langle P \rangle_{\mathcal{I}}\} & & \langle \mathbf{p}P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_2 \cdot \tau_1 \cdots \in \langle P \rangle_{\mathcal{I}}\}
\end{array}$$

Table 5: The encoding $\mathcal{E}(\cdot) : \mathsf{PFL}_\Sigma \to \mathsf{NPR}_\Sigma$



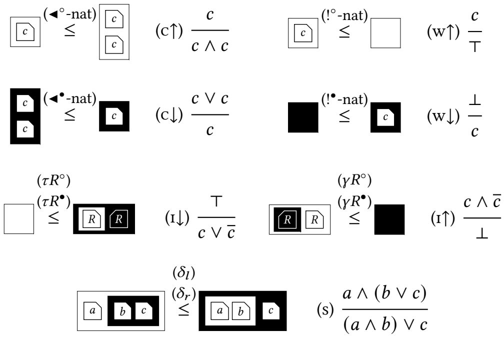


Figure 12: Correspondence between axioms in Figure 6 and rules of SKSg in [15]. By the laws of symmetric monoidal categories $\fatsemi$ and $\otimes$ coincide: they both correspond to $\wedge$. Moreover they are associative, commutative and with unit id_I°, corresponding to $\top$. Symmetrically $\bullet$ and $\otimes$ coincide and correspond to $\vee$.

- $\mathbf{p}P$: 2. Assume P: 1.

$$
\begin{aligned}
\langle \mathbf{p}P\rangle_{\mathcal{I}} &= \{\tau \mid \tau_2, \tau_1, \tau_3, \tau_4 \cdots \in \langle P\rangle_{\mathcal{I}}\} && (\text{def. } \langle\cdot\rangle_{\mathcal{I}})\\
&= \{\tau \mid \tau_2, \tau_1, \cdots \in \{\tau \mid (\tau_1, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\ \} && (\text{ind. hyp.})\\
&= \{\tau \mid (\tau_2, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\\
&= \{\tau \mid ((\tau_1, \tau_2), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(\mathbf{p}P))\} && (\text{def. } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot))
\end{aligned}
$$

- $]P$: 0. Assume P: 0.

$$
\begin{aligned}
\langle]P\rangle_{\mathcal{I}} &= \{\tau \mid \tau_2, \tau_3, \cdots \in \langle P\rangle_{\mathcal{I}}\} && (\text{def. } \langle\cdot\rangle_{\mathcal{I}})\\
&= \{\tau \mid \tau_2, \tau_3, \cdots \in \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\ \} && (\text{ind. hyp.})\\
&= \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\\
&= \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(]P))\} && (\text{def. } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot))
\end{aligned}
$$

□

B.4 Translation from NPR_Σ to FOL

In § 9 we show how to translate typed formulas of FOL into diagrams of NPR_Σ. Here we show the translation in the other direction.

Note that in general terms of NPR_Σ feature "dangling" wires both on the left and on the right of a term. While this is inconsequential from the point of view of expressivity, since terms can always be "rewired" using the compact closed structure of cartesian bicategories, this separation is convenient for composing terms in a flexible manner. Therefore, in the translation in Figure 13, we keep two separate lists of free variables in the context, denoted as $n; m$, where n and m are the lenghts of the two lists.

C PROOFS OF SECTION 4

Lemma C.1. *Let $(\mathsf{C}, \blacktriangleleft^\circ, \blacktriangleright^\circ)$ be a cartesian bicategory. The following holds*

1. *For all objects X, $id_X^\circ \colon X \to X$, $\blacktriangleleft_X^\circ \colon X \to X \otimes X$ and $!_X^\circ \colon X \to I$ are maps;*
2. *For maps a and b properly typed, $a \fatsemi b$ and $a \otimes b$ are maps;*
3. *If $a \colon I \to I$ is a map, then $a = id_I^\circ$;*
4. *If $a \colon I \to X \otimes Y$ is a map, then there exist maps $c \colon I \to X$ and $d \colon I \to Y$ such that $a = c \otimes d$.*

Proof. See Theorem 1.6 in [16]. □

Proof of Proposition 4.3. See Theorem 2.4 in [16]. □

Lemma C.2. *Let $\mathcal{F} \colon \mathbf{C_1} \to \mathbf{C_2}$ be a morphism of cartesian bicategories. Then, for all $c \colon X \to Y$, $\mathcal{F}(c)^\dagger = \mathcal{F}(c^\dagger)$.*

Proof. See Remark 2.9 in [16]. □

Proof of Proposition C.3. See Lemma 2.5 in [16]. □

The following generalises the well-known fact that R is a function iff it is left adjoint to $R^\dagger$.

Proposition C.3. *In a cartesian bicategory, an arrow $c \colon X \to Y$ is a map iff $c^\dagger \vdash c$.*

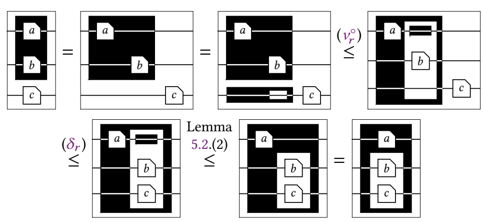
D PROOFS OF SECTION 5

Several results stated in §5 (e.g., Lemmas 5.3, 5.4 and D.1) are well-known from [17]. However, for convenience of the reader, we group in this appendix the proofs of all the results stated in §5.

Proof of Lemma 5.2. The proof of (1) is on the left and (2) on the right:

$$
\begin{aligned}
id_I^\bullet &= id_I^\bullet \fatsemi id_I^\circ\\
&= id_I^\bullet \fatsemi (id_I^\bullet \bullet id_I^\circ)\\
&\leq (id_I^\bullet \fatsemi id_I^\bullet) \bullet id_I^\circ && (\delta_l)\\
&= (id_I^\bullet \otimes id_I^\bullet) \bullet id_I^\circ && (\text{SMC})\\
&\leq (id_I^\bullet \circledast id_I^\bullet) \bullet id_I^\circ && (\circledast^\bullet)\\
&= id_I^\circ
\end{aligned}
\qquad
\begin{aligned}
&a \otimes b\\
&= (a \bullet id^\bullet) \otimes (b \bullet id^\bullet)\\
&\leq (a \circledast b) \bullet (id^\bullet \otimes id^\bullet) && (\nu_r^\circ)\\
&\leq (a \circledast b) \bullet (id^\bullet \circledast id^\bullet) && (\otimes^\bullet)\\
&= a \circledast b
\end{aligned}
$$

The proof of (3) is given diagrammatically as follows:



□

Proof of Lemma 5.3. By the following two derivations.

$$
\begin{aligned}
b &= b \fatsemi id_X^\circ\\
&\leq b \fatsemi (a \bullet c) && (c \Vdash a)\\
&\leq (b \fatsemi a) \bullet c && (\delta_l)\\
&\leq id_Y^\bullet \bullet c && (b \Vdash a)\\
&= c
\end{aligned}
\qquad
\begin{aligned}
c &= c \fatsemi id_X^\circ\\
&\leq c \fatsemi (a \bullet b) && (b \Vdash a)\\
&\leq (c \fatsemi a) \bullet b && (\delta_l)\\
&\leq id_Y^\bullet \bullet b && (c \Vdash a)\\
&= b
\end{aligned}
$$

□

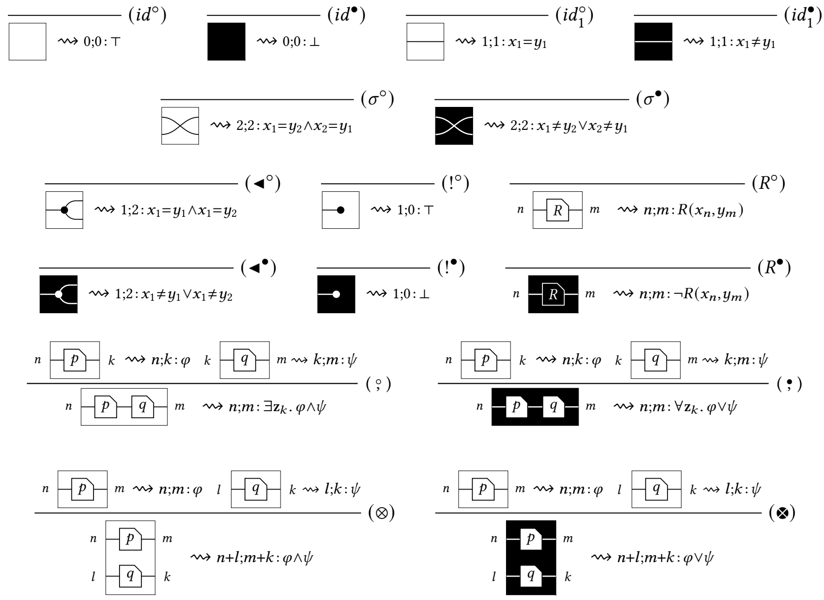


Figure 13: Encoding of NPR_Σ diagrams as FOL formulas.

Proof of Lemma 5.4. In the leftmost derivation we prove $a \leq b \Rightarrow id_X^\circ \leq b \,\bullet\, a^\perp$ and in the rightmost $a \leq b \Leftarrow id_X^\circ \leq b \,\bullet\, a^\perp$

$$\begin{aligned} id_X^\circ &\leq a \,\bullet\, a^\perp && (a^\perp \Vdash a)\\ &\leq b \,\bullet\, a^\perp && (a \leq b)\end{aligned}$$

$$\begin{aligned} a &= id_X^\circ \fatsemi a \\ &\leq (b \,\bullet\, a^\perp) \fatsemi a && (id_X^\circ \leq b \,\bullet\, a^\perp)\\ &\leq b \,\bullet\, (a^\perp \fatsemi a) && (\delta_r)\\ &\leq b \,\bullet\, id_Y^\bullet && (a^\perp \Vdash a)\\ &= b \end{aligned}$$

□

Lemma D.1. *Let $\mathcal{F}\colon \mathbf{C_1} \to \mathbf{C_2}$ be a morphism of closed linear bicategories. Then, for all $a\colon X \to Y$, $\mathcal{F}(a)^\perp = \mathcal{F}(a^\perp)$.*

Proof. Consider the following two derivations witnessing that $F(a^\perp)$ is right linear adjoint to $F(a)$.

$$\begin{aligned} id_X^\circ &= F(id_X^\circ)\\ &\leq F(a \,\bullet\, a^\perp) && (a^\perp \Vdash a)\\ &= F(a) \,\bullet\, F(a^\perp)\end{aligned}$$

$$\begin{aligned} &F(a^\perp) \fatsemi F(a)\\ =\;& F(a^\perp \fatsemi a)\\ \leq\;& F(id_Y^\bullet) && (a^\perp \Vdash a)\\ =\;& id_Y^\bullet\end{aligned}$$

Thus, by Lemma 5.3, $(F(a))^\perp = F(a^\perp)$. □

Proof of Proposition 5.6. First, we prove that for all $a, b\colon X \to Y$ it holds

(0) if $a \leq b$ then $a^\perp \geq b^\perp$

The proof is illustrated below.

(0)

$$\begin{aligned} b^\perp &= b^\perp \fatsemi id_Y^\circ\\ &\leq b^\perp \fatsemi (a \,\bullet\, a^\perp) && (a^\perp \Vdash a)\\ &\leq (b^\perp \fatsemi a) \,\bullet\, a^\perp && ((\delta_l))\\ &\leq (b^\perp \fatsemi b) \,\bullet\, a^\perp && (a \leq b)\\ &\leq id_Y^\bullet \,\bullet\, a^\perp && (b^\perp \Vdash b)\\ &= a^\perp\end{aligned}$$

We next illustrate that for all $a\colon X \to Y$ and $b\colon Y \to Z$

(1) $(id_X^\circ)^\perp = id_X^\bullet$
(2) $(id_X^\bullet)^\perp = id_X^\circ$
(3) $(a \fatsemi b)^\perp = b^\perp \,\bullet\, a^\perp$
(4) $(a \,\bullet\, b)^\perp = b^\perp \fatsemi a^\perp$

The proofs are dispayed below.

(1) Observe that $id_X^\circ = id_X^\circ \,\bullet\, id_X^\bullet$ and $id_X^\bullet \fatsemi id_X^\circ = id_X^\bullet$. Thus, by Lemma 5.3, $(id_X^\circ)^\perp = id_X^\bullet$.
(2) Similarly, $id_X^\circ = id_X^\bullet \,\bullet\, id_X^\circ$ and $id_X^\circ \fatsemi id_X^\bullet = id_X^\bullet$. Again, by Lemma 5.3, $(id_X^\bullet)^\perp = id_X^\circ$.
(3) The following two derivations

id°_X	$(b^{\perp} \mathbin{;^{\bullet}} a^{\perp}) \mathbin{;^{\circ}} (a \mathbin{;^{\circ}} b)$
$\leq a \mathbin{;^{\bullet}} a^{\perp} \quad (a^{\perp} \Vdash a)$	$= ((b^{\perp} \mathbin{;^{\bullet}} a^{\perp}) \mathbin{;^{\circ}} a) \mathbin{;^{\circ}} b$
$= (a \mathbin{;^{\circ}} id^{\circ}_Y) \mathbin{;^{\bullet}} a^{\perp}$	$\leq (b^{\perp} \mathbin{;^{\bullet}} (a^{\perp} \mathbin{;^{\circ}} a)) \mathbin{;^{\circ}} b \quad (\delta_r)$
$\leq (a \mathbin{;^{\circ}} (b \mathbin{;^{\bullet}} b^{\perp})) \mathbin{;^{\bullet}} a^{\perp} \quad (b^{\perp} \Vdash b)$	$\leq (b^{\perp} \mathbin{;^{\bullet}} id^{\bullet}_Y) \mathbin{;^{\circ}} b \quad (a^{\perp} \Vdash a)$
$\leq ((a \mathbin{;^{\circ}} b) \mathbin{;^{\bullet}} b^{\perp}) \mathbin{;^{\bullet}} a^{\perp} \quad (\delta_l)$	$= b^{\perp} \mathbin{;^{\circ}} b$
$= (a \mathbin{;^{\circ}} b) \mathbin{;^{\bullet}} (b^{\perp} \mathbin{;^{\bullet}} a^{\perp})$	$\leq id^{\bullet}_Z \quad (b^{\perp} \Vdash b)$

show that $(b^{\perp} \mathbin{;^{\bullet}} a^{\perp}) \Vdash (a \mathbin{;^{\circ}} b)$. Thus, by Lemma 5.3, $(a \mathbin{;^{\circ}} b)^{\perp} = b^{\perp} \mathbin{;^{\bullet}} a^{\perp}$.

(4) The following two derivations

id°_X	$(b^{\perp} \mathbin{;^{\circ}} a^{\perp}) \mathbin{;^{\circ}} (a \mathbin{;^{\bullet}} b)$
$\leq a \mathbin{;^{\bullet}} a^{\perp} \quad (a^{\perp} \Vdash a)$	$= b^{\perp} \mathbin{;^{\circ}} (a^{\perp} \mathbin{;^{\circ}} (a \mathbin{;^{\bullet}} b))$
$= a \mathbin{;^{\bullet}} (id^{\circ}_Y \mathbin{;^{\circ}} a^{\perp})$	$= b^{\perp} \mathbin{;^{\circ}} ((a^{\perp} \mathbin{;^{\circ}} a) \mathbin{;^{\bullet}} b) \quad (\delta_l)$
$\leq a \mathbin{;^{\bullet}} ((b \mathbin{;^{\bullet}} b^{\perp}) \mathbin{;^{\circ}} a^{\perp}) \quad (b^{\perp} \Vdash b)$	$\leq b^{\perp} \mathbin{;^{\circ}} (id^{\bullet}_Y \mathbin{;^{\bullet}} b) \quad (a^{\perp} \Vdash a)$
$\leq a \mathbin{;^{\bullet}} (b \mathbin{;^{\bullet}} (b^{\perp} \mathbin{;^{\circ}} a^{\perp})) \quad (\delta_r)$	$= b^{\perp} \mathbin{;^{\circ}} b$
$= (a \mathbin{;^{\bullet}} b) \mathbin{;^{\bullet}} (b^{\perp} \mathbin{;^{\circ}} a^{\perp})$	$\leq id^{\bullet}_Z \quad (b^{\perp} \Vdash b)$

show that $(b^{\perp} \mathbin{;^{\circ}} a^{\perp}) \Vdash (a \mathbin{;^{\bullet}} b)$. Thus, by Lemma 5.3, $(a \mathbin{;^{\bullet}} b)^{\perp} = b^{\perp} \mathbin{;^{\circ}} a^{\perp}$.

Next, we illustrate that for all $a\colon X_1 \to Y_1$ and $b\colon X_2 \to Y_2$

(5) $(a \otimes b)^{\perp} = a^{\perp} \mathbin{\otimes^{\bullet}} b^{\perp}$
(6) $(a \mathbin{\otimes^{\bullet}} b)^{\perp} = a^{\perp} \otimes b^{\perp}$
(7) $(\sigma^{\circ})^{\perp} = \sigma^{\bullet}$
(8) $(\sigma^{\bullet})^{\perp} = \sigma^{\circ}$

The proofs are shown below.

(5) The following two derivations

$id^{\circ}_{X_1 \otimes X_2}$	$(a^{\perp} \mathbin{\otimes^{\bullet}} b^{\perp}) \mathbin{;^{\circ}} (a \otimes b)$
$= id^{\circ}_{X_1} \otimes id^{\circ}_{X_2}$	$\leq (a^{\perp} \mathbin{;^{\circ}} a) \mathbin{\otimes^{\bullet}} (b^{\perp} \mathbin{;^{\circ}} b) \quad (\nu^{\bullet}_l)$
$\leq (a \mathbin{;^{\bullet}} a^{\perp}) \otimes (b \mathbin{;^{\bullet}} b^{\perp}) \quad (a^{\perp} \Vdash a\,,\ b^{\perp} \Vdash b)$	$\leq id^{\bullet}_{Y_1} \mathbin{\otimes^{\bullet}} id^{\bullet}_{Y_2} \quad (a^{\perp} \Vdash a\,,\ b^{\perp} \Vdash b)$
$\leq (a \otimes b) \mathbin{;^{\bullet}} (a^{\perp} \mathbin{\otimes^{\bullet}} b^{\perp}) \quad (\nu^{\circ}_r)$	$= id^{\bullet}_{Y_1 \otimes^{\bullet} Y_2}$

show that $(a^{\perp} \mathbin{\otimes^{\bullet}} b^{\perp}) \Vdash (a \otimes b)$. Thus, by Lemma 5.3, $(a \otimes b)^{\perp} = b^{\perp} \mathbin{\otimes^{\bullet}} a^{\perp}$.

(6) The following two derivations

$id^{\circ}_{X_1 \otimes X_2}$	$(a^{\perp} \otimes b^{\perp}) \mathbin{;^{\circ}} (a \mathbin{\otimes^{\bullet}} b)$
$= id^{\circ}_{X_1} \otimes id^{\circ}_{X_2}$	$\leq (a^{\perp} \mathbin{;^{\circ}} a) \mathbin{\otimes^{\bullet}} (b^{\perp} \mathbin{;^{\circ}} b) \quad (\nu^{\bullet}_l)$
$\leq (a \mathbin{;^{\bullet}} a^{\perp}) \otimes (b \mathbin{;^{\bullet}} b^{\perp}) \quad (a^{\perp} \Vdash a\,,\ b^{\perp} \Vdash b)$	$\leq id^{\bullet}_{Y_1} \mathbin{\otimes^{\bullet}} id^{\bullet}_{Y_2} \quad (a^{\perp} \Vdash a\,,\ b^{\perp} \Vdash b)$
$\leq (a \mathbin{\otimes^{\bullet}} b) \mathbin{;^{\bullet}} (a^{\perp} \otimes b^{\perp}) \quad (\nu^{\circ}_r)$	$= id^{\bullet}_{Y_1 \otimes^{\bullet} Y_2}$

show that $(a^{\perp} \otimes b^{\perp}) \Vdash (a \mathbin{\otimes^{\bullet}} b)$. Thus, by Lemma 5.3, $(a \mathbin{\otimes^{\bullet}} b)^{\perp} = b^{\perp} \otimes a^{\perp}$.

(7) By axioms $(\tau\sigma^{\circ})$ and $(\gamma\sigma^{\circ})$.
(8) By axioms $(\tau\sigma^{\bullet})$ and $(\gamma\sigma^{\bullet})$.

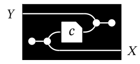
□

E PROOF OF SECTION 6

E.1 Proofs of Proposition 6.2

In this appendix we illustrate several results to prove Proposition 6.2. We first focus on $(\cdot)^{\dagger}\colon \mathsf{C} \to \mathsf{C}^{\mathsf{op}}$ (Lemma E.6) and then $(\cdot)^{\perp}\colon \mathsf{C} \to (\mathsf{C}^{\mathsf{co}})^{\mathsf{op}}$ (Lemma E.8).

In order to prove that $(\cdot)^{\dagger}\colon \mathsf{C} \to \mathsf{C}^{\mathsf{op}}$ is a morphism of fo-bicategories, it is convenient to define, for all arrows $c\colon X \to Y$, $c^{\ddagger}\colon Y \to X$ as

$c^{\ddagger} \stackrel{\text{def}}{=}$ 

The assignment $c \mapsto c^{\ddagger}$ gives rise to an identity on object functor $(\cdot)^{\ddagger}\colon \mathsf{C} \to \mathsf{C}^{\mathsf{op}}$ which preserves the stucture of *co*cartesian bicategories.

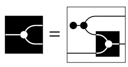
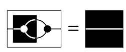
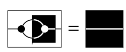
Proposition E.1. *$(\cdot)^{\ddagger}\colon \mathsf{C} \to \mathsf{C}^{\mathsf{op}}$ is an isomorphism of cocartesian bicategories, that is the rules in the first three rows of Table 6 hold.*

Proof. See Theorem 2.4 in [16]. □

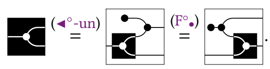
Table 6: Properties of $\cdot^{\ddagger}\colon \mathsf{C} \to \mathsf{C}^{\mathsf{op}}$

if $c \leq d$ then $c^{\ddagger} \leq d^{\ddagger}$		$(c^{\ddagger})^{\ddagger} = c$	
$(c \mathbin{;^{\circ}} d)^{\ddagger} = d^{\ddagger} \mathbin{;^{\circ}} c^{\ddagger}$	$(id^{\circ}_X)^{\ddagger} = id^{\circ}_X$	$(\blacktriangleright^{\circ}_X)^{\ddagger} = \blacktriangleleft^{\circ}_X$	$(¡^{\circ}_X)^{\ddagger} = !^{\circ}_X$
$(c \otimes d)^{\ddagger} = c^{\ddagger} \otimes d^{\ddagger}$	$(\sigma^{\circ}_{X,Y})^{\ddagger} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\ddagger} = \blacktriangleright^{\circ}_X$	$(!^{\circ}_X)^{\ddagger} = ¡^{\circ}_X$
$(c \mathbin{;^{\bullet}} d)^{\ddagger} = d^{\ddagger} \mathbin{;^{\bullet}} c^{\ddagger}$	$(id^{\bullet}_X)^{\ddagger} = id^{\bullet}_X$	$(\blacktriangleright^{\bullet}_X)^{\ddagger} = \blacktriangleleft^{\bullet}_X$	$(¡^{\bullet}_X)^{\ddagger} = !^{\bullet}_X$
$(c \mathbin{\otimes^{\bullet}} d)^{\ddagger} = c^{\ddagger} \mathbin{\otimes^{\bullet}} d^{\ddagger}$	$(\sigma^{\bullet}_{X,Y})^{\ddagger} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\ddagger} = \blacktriangleright^{\bullet}_X$	$(!^{\bullet}_X)^{\ddagger} = ¡^{\bullet}_X$

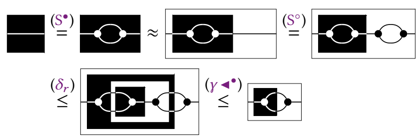
Lemma E.2. *The following hold:*

(1)  (2)  (3) 

Proof. Point (1) is proved by the following derivation:



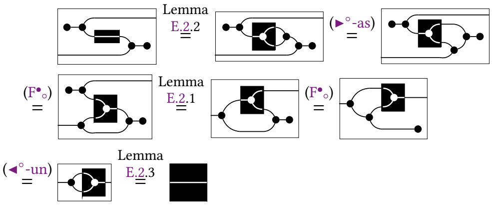
For point (2) observe that the left to right inclusion is $(\gamma\blacktriangleright^{\circ})$ and the other inclusion is proved as follows:



Proof of point (3) is analogous to the one above, except that one exploits (δ_r) and $(\gamma\blacktriangleright^{\bullet})$. □

Lemma E.3. *The following hold: $id^{\bullet}_X = (id^{\bullet}_X)^{\dagger}$ and $id^{\circ}_X = (id^{\circ}_X)^{\ddagger}$*

Proof. Here we show only $id^{\bullet}_X = (id^{\bullet}_X)^{\dagger}$, the other follows a similar reasoning.



□

Lemma E.4. *For all $a\colon X \to Y$ it holds $(a^{\dagger})^{\perp} = (a^{\perp})^{\ddagger}$*

Proof. The proof follows from the fact that $(\blacktriangleleft^{\circ}, !^{\circ})$ is right linear adjoint to $(\blacktriangleright^{\bullet}, ¡^{\bullet})$, Proposition 5.6 and the definition of $(\cdot)^{\dagger}$ and $(\cdot)^{\ddagger}$. □

Lemma E.5. *For all $a\colon X \to Y$ it holds $a^{\dagger} = a^{\ddagger}$*

Proof. We prove the inclusion $a^{\dagger} \leq a^{\ddagger}$ (left) by means of Lemma 5.4 and the other inclusion (right) directly:

$$\begin{array}{lr} (a^{\ddagger} \mathbin{\boldsymbol{;}} (a^{\dagger})^{\perp}) & \\ = a^{\ddagger} \mathbin{\boldsymbol{;}} (a^{\perp})^{\ddagger} & \text{(Lemma E.4)} \\ = (a^{\perp} \mathbin{\boldsymbol{;}} a)^{\ddagger} & \text{(Table 6)} \\ \geq (id^{\circ}_Y)^{\ddagger} & (a^{\perp} \Vdash a) \\ = id^{\circ}_Y & \text{(Lemma E.3)} \end{array}$$

$$\begin{array}{lr} a^{\ddagger} & \\ = ((a^{\dagger})^{\dagger})^{\ddagger} & ((\cdot)^{\dagger} \text{ is an iso}) \\ \leq ((a^{\ddagger})^{\dagger})^{\ddagger} & (a^{\dagger} \leq a^{\ddagger}) \\ = a^{\dagger} & ((\cdot)^{\ddagger} \text{ is an iso}) \end{array}$$

□

Lemma E.6. *$(\cdot)^{\dagger}\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ is an isomorphisms of fo-bicategories, namely all the laws in Table 2.(a) hold.*

Proof. Follows from Lemma E.5 and the fact that $(\cdot)^{\dagger}$ preserves the positive structure (Proposition 4.3) and $(\cdot)^{\ddagger}$ preserve the negative structure (Proposition E.1). For instance, to prove that $(a \mathbin{\boldsymbol{;}} b)^{\dagger} = b^{\dagger} \mathbin{\boldsymbol{;}} a^{\dagger}$, it is enough to observe that $(a \mathbin{\boldsymbol{;}} b)^{\dagger} = (a \mathbin{\boldsymbol{;}} b)^{\ddagger}$ and that $(a \mathbin{\boldsymbol{;}} b)^{\ddagger} = b^{\ddagger} \mathbin{\boldsymbol{;}} a^{\ddagger}$. □

Lemma E.7. *For all $a\colon X \to Y$*

(1) $(a^{\perp})^{\perp} = a$

Proof. The following two derivations

$$\begin{array}{lr} id^{\circ}_Y & \\ =(id^{\circ}_Y)^{\dagger} & \text{(Proposition 4.3)} \\ \leq(a^{\dagger} \mathbin{\boldsymbol{;}} (a^{\dagger})^{\perp})^{\dagger} & ((a^{\dagger})^{\perp} \Vdash a^{\dagger}) \\ =(a^{\dagger} \mathbin{\boldsymbol{;}} (a^{\perp})^{\dagger})^{\dagger} & \text{(Corollary 6.3)} \\ =((a^{\perp} \mathbin{\boldsymbol{;}} a)^{\dagger})^{\dagger} & \text{(Lemma E.6)} \\ =a^{\perp} \mathbin{\boldsymbol{;}} a & \text{(Proposition 4.3)} \end{array}$$

$$\begin{array}{lr} id^{\bullet}_X & \\ =(id^{\bullet}_X)^{\dagger} & \text{(Lemma E.6)} \\ \geq((a^{\dagger})^{\perp} \fatsemi a^{\dagger})^{\dagger} & ((a^{\dagger})^{\perp} \Vdash a^{\dagger}) \\ =((a^{\perp})^{\dagger} \fatsemi a^{\dagger})^{\dagger} & \text{(Corollary 6.3)} \\ =((a \fatsemi a^{\perp})^{\dagger})^{\dagger} & \text{(Proposition 4.3)} \\ =a \fatsemi a^{\perp} & \text{(Proposition 4.3)} \end{array}$$

prove that the right linear adjoint of $a^{\perp}$ is a. Thus, by Lemma 5.3, $(a^{\perp})^{\perp} = a$.

□

Lemma E.8. *$(\cdot)^{\perp}\colon \mathbf{C} \to (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$ is an isomorphisms of fo-bicategories, namely all the laws in Table 2.(b) hold.*

Proof. By Proposition 5.6, $(\cdot)^{\perp}\colon \mathbf{C} \to (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$ is a morphism of linear bicategories. Observe that $(\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$ carries the structure of a cartesian bicategory where the positive comonoid is $(\blacktriangleright^{\bullet}, ¡^{\bullet})$ and the positive monoid is $(\blacktriangleleft^{\bullet}, !^{\bullet})$. By Definition 5.1.4, one has that $(\blacktriangleleft^{\circ})^{\perp} = \blacktriangleright^{\bullet}$, $(!^{\circ})^{\perp} = ¡^{\bullet}$ and $(\blacktriangleright^{\circ})^{\perp} = \blacktriangleleft^{\bullet}$, $(¡^{\circ})^{\perp} = !^{\bullet}$. Thus $(\cdot)^{\perp}\colon \mathbf{C} \to (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$ is a morphism of cartesian bicategories.

By Lemma E.7, we also immediately know that $(\blacktriangleleft^{\bullet})^{\perp} = \blacktriangleright^{\circ}$, $(!^{\bullet})^{\perp} = ¡^{\circ}$ and $(\blacktriangleright^{\bullet})^{\perp} = \blacktriangleleft^{\circ}$, $(¡^{\bullet})^{\perp} = !^{\circ}$. Thus, $(\cdot)^{\perp}\colon \mathbf{C} \to (\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}$ is a morphism of cocartesian bicategories. Thus, it is a morphism of fo-bicategories.

The fact that it is an isomorphism is immediate by Lemma E.7.

□

Proof of Proposition 6.2. By Lemmas E.6 and E.8. □

E.2 Proofs of the other results

Proof of Corollary 6.3. $(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$ is immediate from Proposition 6.2 and Lemma D.1. The other rules follow from the definitions of $\sqcap, \top, \sqcup, \bot$ in (12) and (12), and the laws in Tables 2.(a) and 2.(b). For instance $(\bot)^{\perp} = (!^{\bullet} \mathbin{\boldsymbol{;}} ¡^{\bullet})^{\perp} = (¡^{\bullet})^{\perp} \fatsemi (!^{\bullet})^{\perp} = !^{\circ} \fatsemi ¡^{\circ} = \top$. □

Proof of Proposition 6.4. Recall that, by Proposition C.3 an arrow $f\colon X \to Y$ is a map iff it is a left adjoint, namely

$$id^{\circ}_X \leq f \fatsemi f^{\dagger} \qquad f^{\dagger} \fatsemi f \leq id^{\circ}_Y \tag{17}$$

The following two derivations prove the two inclusion.

$$\begin{array}{lr} f \fatsemi c & \\ = id^{\circ}_X \fatsemi f \fatsemi c & \\ \leq ((f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} f^{\dagger}) \fatsemi f \fatsemi c & \\ & (f^{\dagger} \Vdash (f^{\dagger})^{\perp}) \\ \leq (f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} (f^{\dagger} \fatsemi f \fatsemi c) & (\delta_r) \\ \leq (f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} (id^{\circ}_Y \fatsemi c) & (17) \\ = (f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} c & \end{array}$$

$$\begin{array}{lr} f \fatsemi c & \\ = f \fatsemi (id^{\bullet}_X \mathbin{\boldsymbol{;}} c) & \\ \geq f \fatsemi ((f^{\dagger} \fatsemi (f^{\dagger})^{\perp}) \mathbin{\boldsymbol{;}} c) & \\ & ((f^{\dagger}) \Vdash (f^{\dagger})^{\perp}) \\ \geq f \fatsemi f^{\dagger} \fatsemi ((f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} c) & (\delta_l) \\ \geq id^{\circ}_X \fatsemi ((f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} c) & (17) \\ = (f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} c & \end{array}$$

Note that $f^{\dagger} \Vdash (f^{\dagger})^{\perp}$ holds since, by Proposition 6.2, in any fo-bicategory left and right linear adjoint coincide (namely $(a^{\perp})^{\perp} = a$).

To check the four equivalences, first observe that

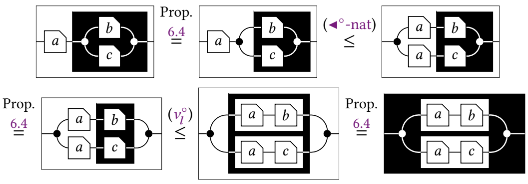
$$c \fatsemi f^{\dagger} = (f \fatsemi c)^{\dagger} = ((f^{\dagger})^{\perp} \mathbin{\boldsymbol{;}} c)^{\dagger} = c \mathbin{\boldsymbol{;}} f^{\perp}$$

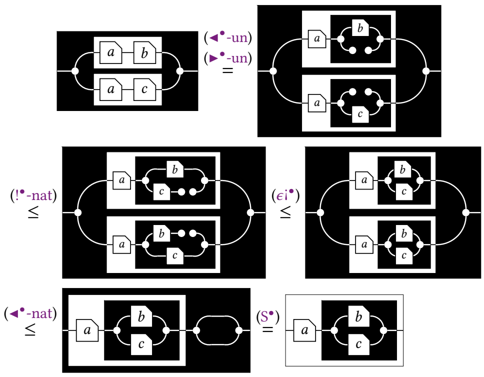
and conclude by taking as map f either $\blacktriangleleft^{\circ}$ or $!^{\circ}$. □

Lemma E.9. *Let $\mathbf{C}$ be a fo-bicategory. Then, $(\mathbf{C}, \fatsemi, \otimes)$ and $(\mathbf{C}, \mathbin{\boldsymbol{;}}, \boldsymbol{\otimes})$ are monoidally enriched over $\sqcup$-semilattices with $\bot$ and $\sqcap$-semilattices with $\top$, respectively. Namely, the following hold:*

(1) $a \fatsemi (b \sqcup c) = (a \fatsemi b) \sqcup (a \fatsemi c)$ *and* $(b \sqcup c) \fatsemi a = (b \fatsemi a) \sqcup (c \fatsemi a)$
(2) $a \mathbin{\boldsymbol{;}} (b \sqcap c) = (a \mathbin{\boldsymbol{;}} b) \sqcap (a \mathbin{\boldsymbol{;}} c)$ *and* $(b \sqcap c) \mathbin{\boldsymbol{;}} a = (b \mathbin{\boldsymbol{;}} a) \sqcap (c \mathbin{\boldsymbol{;}} a)$
(3) $a \fatsemi \bot = \bot = \bot \fatsemi a$
(4) $a \mathbin{\boldsymbol{;}} \top = \top = \top \mathbin{\boldsymbol{;}} a$
(5) $a \otimes (b \sqcup c) = (a \otimes b) \sqcup (a \otimes c)$ *and* $(b \sqcup c) \otimes a = (b \otimes a) \sqcup (c \otimes a)$
(6) $a \boldsymbol{\otimes} (b \sqcap c) = (a \boldsymbol{\otimes} b) \sqcap (a \boldsymbol{\otimes} c)$ *and* $(b \sqcap c) \boldsymbol{\otimes} a = (b \boldsymbol{\otimes} a) \sqcap (c \boldsymbol{\otimes} a)$
(7) $a \otimes \bot = \bot = \bot \otimes a$
(8) $a \boldsymbol{\otimes} \top = \top = \top \boldsymbol{\otimes} a$

Proof. We prove the two inclusions of the first equation in (1) separately.



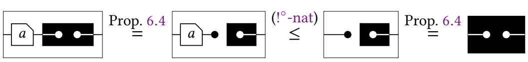


For the second equation, namely the one with the composition on the right, it suffices to apply the properites of $(\cdot)^\dagger$ in Tables 2.(a) and 2.(c) and the drivation above to get that:

$$\begin{aligned}(b \sqcup c) ; a &= (((b \sqcup c) ; a)^\dagger)^\dagger = (a^\dagger ; (b^\dagger \sqcup c^\dagger))^\dagger \\ &= ((a^\dagger ; b^\dagger) \sqcup (a^\dagger ; c^\dagger))^\dagger = (((b ; a) \sqcup (a ; c))^\dagger)^\dagger \\ &= (b ; a) \sqcup (a ; c)\end{aligned}$$

The proofs for (2) are analogous to those of (1).

We prove the left to right inclusion of the first equation in (3). The other inclusion holds since $\bot$ is the bottom element.



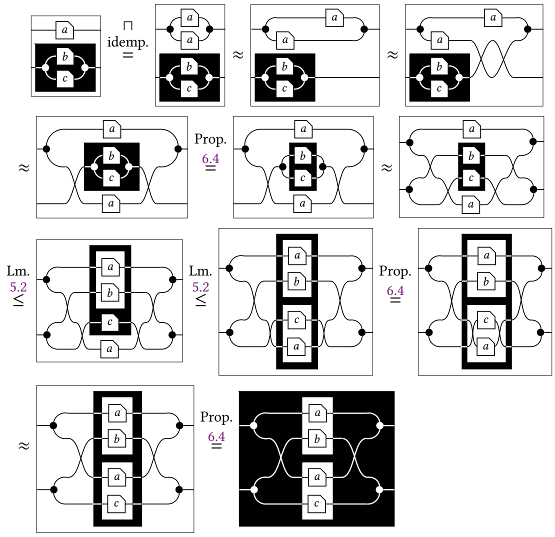
For the second equation, namely the one with the composition on the right, it suffices to apply the properites of $(\cdot)^\dagger$ in Tables 2.(a) and 2.(c) and the drivation above to get that:

$$\bot ; a = ((\bot ; a)^\dagger)^\dagger = (a^\dagger ; \bot^\dagger)^\dagger = (a^\dagger ; \bot)^\dagger = \bot^\dagger = \bot$$

The proofs for (4) are analogous to those of (3).

The right to left inclusion of the first equation in (5) is proved by the universal property of $\sqcup$, namely: if $a \otimes b = a \otimes (b \sqcup \bot) \leq a \otimes (b \sqcup c)$ and $a \otimes c = a \otimes (\bot \sqcup c) \leq a \otimes (b \sqcup c)$, then $(a \otimes b) \sqcup (a \otimes c) \leq a \otimes (b \sqcup c)$.

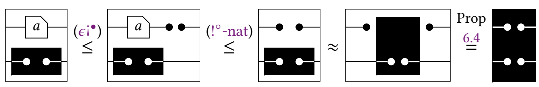
For the other inclusion, the following holds:



For the second equation, namely $(b \sqcup c) \otimes a = (b \otimes a) \sqcup (c \otimes a)$, the proof follows the exact same reasoning.

The proofs for (6) are analogous to those of (5).

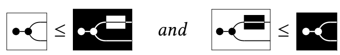
We prove the left to right inclusion of the first equation in (7). The other inclusion holds since $\bot$ is the bottom element.



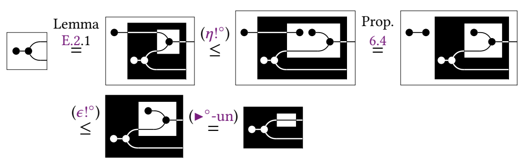
For the second equation, namely $\bot = \bot \otimes a$, the proof follows the exact same reasoning.

The proofs for (8) are analogous to those of (7). □

Lemma E.10. *The following hold:*

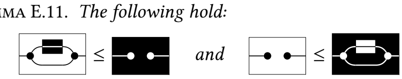


Proof.

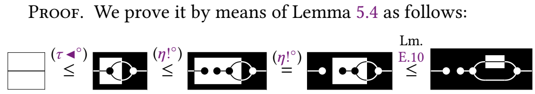


The proof of the other inequality is analogous. □

Lemma E.11. *The following hold:*



Proof. We prove it by means of Lemma 5.4 as follows:



The proof of the other inequality is analogous. □

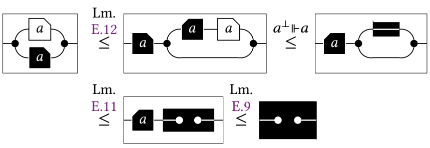
Lemma E.12. *The following hold:*

Proof. The inclusion on the left is usually known as "wrong way" and it holds in any cartesian bicategory. See for example [7] for a detailed proof. The inclusion on the right is the "negated" version holding in any cocartesian bicategory. □

Lemma E.13. *The following hold:*

(1) $a \sqcap \overline{a} \leq \bot$
(2) $\top \leq a \sqcup \overline{a}$

Proof. We prove (1). The proof for (2) is analogous.



□

Proof of Proposition 6.5. The enrichments have been proved in Lemma E.9.

The first six laws of Boolean algebras in Table 2.(d) are proved below:

$$\overline{c \sqcap d} \overset{\text{Def. }\overline{(\cdot)}}{=} ((c \sqcap d)^{\perp})^{\dagger} \overset{\text{Cor. 6.3}}{=} (c^{\perp})^{\dagger} \sqcup (d^{\perp})^{\dagger} \overset{\text{Def. }\overline{(\cdot)}}{=} \overline{c} \sqcup \overline{d},$$

$$\overline{\top} \overset{\text{Def. }\overline{(\cdot)}}{=} (\top^{\perp})^{\dagger} \overset{\text{Cor. 6.3}}{=} \bot,$$

$$\overline{c \sqcup d} \overset{\text{Def. }\overline{(\cdot)}}{=} ((c \sqcup d)^{\perp})^{\dagger} \overset{\text{Cor. 6.3}}{=} (c^{\perp})^{\dagger} \sqcap (d^{\perp})^{\dagger} \overset{\text{Def. }\overline{(\cdot)}}{=} \overline{c} \sqcap \overline{d},$$

$$\overline{\bot} \overset{\text{Def. }\overline{(\cdot)}}{=} (\bot^{\perp})^{\dagger} \overset{\text{Cor. 6.3}}{=} \top,$$

$$\begin{aligned} a \sqcup (b \sqcap c) &\overset{(12)}{=} \blacktriangleleft^{\bullet} \mathbin{;} (a \otimes (b \sqcap c)) \mathbin{;} \blacktriangleright^{\bullet} \\ &\overset{\text{Table 2.}(e)}{=} \blacktriangleleft^{\bullet} \mathbin{;} ((a \otimes b) \sqcap (a \otimes c)) \mathbin{;} \blacktriangleright^{\bullet} \\ &\overset{\text{Table 2.}(e)}{=} (\blacktriangleleft^{\bullet} \mathbin{;} (a \otimes b) \mathbin{;} \blacktriangleright^{\bullet}) \sqcap (\blacktriangleleft^{\bullet} \mathbin{;} (a \otimes c) \mathbin{;} \blacktriangleright^{\bullet}) \\ &\overset{(12)}{=} (a \sqcup b) \sqcap (a \sqcup c), \end{aligned}$$

$$\begin{aligned} a \sqcap (b \sqcup c) &\overset{(12)}{=} \blacktriangleleft^{\circ} \mathbin{;} (a \otimes (b \sqcup c)) \mathbin{;} \blacktriangleright^{\circ} \\ &\overset{\text{Table 2.}(e)}{=} \blacktriangleleft^{\circ} \mathbin{;} ((a \otimes b) \sqcup (a \otimes c)) \mathbin{;} \blacktriangleright^{\circ} \\ &\overset{\text{Table 2.}(e)}{=} (\blacktriangleleft^{\circ} \mathbin{;} (a \otimes b) \mathbin{;} \blacktriangleright^{\circ}) \sqcup (\blacktriangleleft^{\circ} \mathbin{;} (a \otimes c) \mathbin{;} \blacktriangleright^{\circ}) \\ &\overset{(12)}{=} (a \sqcap b) \sqcup (a \sqcap c) \end{aligned}$$

The remaining two laws are proved in Lemma E.13. □

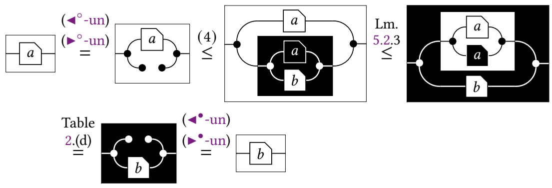
It is worth emphasising that the following result stands at the core of our proofs. Once again, the diagrammatic approach proves to be an enhancement over the classical syntax. In this specific case we are looking at five (of many) different possibilities to express the ubiquitous concept of logical entailment. (1) expresses a implies b as a direct rewriting of the former into the latter. We have already seen that (2) corresponds to residuation. (3) corresponds to right residuation. (4) asserts the validity of the formula $\neg a \vee b$, thus it corresponds to the classical implication. Finally, (5) may look eccentric but it is actually a closed version of (3) that comes in handy if one has to consider closed diagrams.

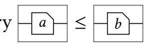
Proof of Lemma 6.6. (1) iff (2) is Lemma 5.4.

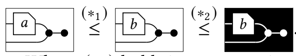
(1) iff (3) is proved as follows: $a \leq b$ iff $b^{\perp} \leq a^{\perp}$ by the property of $(\cdot)^{\perp}$ in Table 2.(b). By Lemma 5.4, $b^{\perp} \leq a^{\perp}$ iff $id^{\circ}_{Y} \leq a^{\perp} \mathbin{;} (b^{\perp})^{\perp}$ where $(b^{\perp})^{\perp} = b$ by the property of $(\cdot)^{\perp}$ in Table 2.(b).

(1) implies (4) follows from the fact that every homset carries a Boolean algebra: $\overline{a} \sqcup b \overset{(1)}{\geq} \overline{a} \sqcup a \overset{\text{Table 2.}(d)}{=} \top$.

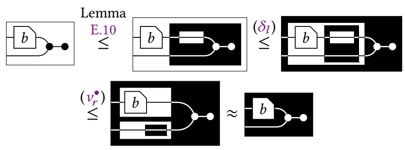
(4) implies (1) is proved by the following derivation:

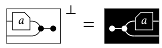


(1) iff (5): observe that in any fo-bicategory  iff

.

Where $(*_1)$ holds in any cartesian bicategory and $(*_2)$ is proved below:



Thus, we conclude from (1) iff (3) and . □

E.3 Proofs of Section 6.1

Proof of Proposition 6.7. Let $\mathcal{I} = (X, \rho)$ be an interpretation of Σ. Recall that $\lesssim$ is defined as $\mathsf{pc}(\mathbb{FOB})$. We prove by induction on the rules in (10), that

$$\text{if } c \lesssim d \text{ then } \mathcal{I}^{\sharp}(c) \subseteq \mathcal{I}^{\sharp}(d).$$

By definition of $\lesssim$, the above statement is equivalent to the proposition.

The proof for the rules (r) and (t) is trivial. For the rule $(\mathbin{;})$, suppose that $c = c_1 \mathbin{;} c_2$ and $d = d_1 \mathbin{;} d_2$ with $c_1 \lesssim d_1$ and $c_2 \lesssim d_2$. Then

$$\begin{aligned} \mathcal{I}^{\sharp}(c) &= \mathcal{I}^{\sharp}(c_1 \mathbin{;} c_2) \\ &= \mathcal{I}^{\sharp}(c_1) \mathbin{;} \mathcal{I}^{\sharp}(c_2) && (8) \\ &\subseteq \mathcal{I}^{\sharp}(d_1) \mathbin{;} \mathcal{I}^{\sharp}(d_2) && \text{(ind. hyp.)} \\ &= \mathcal{I}^{\sharp}(d_1 \mathbin{;} d_2) && (8) \\ &= \mathcal{I}^{\sharp}(d) \end{aligned}$$

The proof for ($\otimes$) is analogous to the one above. The only interesting case is the rule (id): we should prove that if $(c,d) \in \mathbb{FOB}$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$. However, we have already done most of the work: since all the axioms in $\mathbb{FOB}$ – with the only exception of the four stating $R^\bullet \Vdash R^\circ \Vdash R^\bullet$ (axioms (τR°), (γR°), ($\tau R^\bullet$) and ($\gamma R^\bullet$) in Figure 4) – are those of fo-bicategories and since **Rel** is a fo-bicategory, it only remains to show the soundness of those stating $R^\bullet \Vdash R^\circ \Vdash R^\bullet$. Note however that this is trivial by definition of $\mathcal{I}^\sharp(R^\bullet)$ as $\rho(R)^\perp = (\mathcal{I}^\sharp(R^\circ))^\perp$. □

In order to prove Proposition 6.8 is convenient to use the following function on diagrams and then prove that it maps every diagram in its right (Lemma E.15) and left (Lemma E.18) linear adjoint.

Definition E.14. The function $\alpha\colon \mathsf{NPR}_\Sigma \to \mathsf{NPR}_\Sigma$ is inductively defined as follows.

$$\alpha(id_0^\circ) \stackrel{\text{def}}{=} id_0^\bullet \quad \alpha(id_1^\circ) \stackrel{\text{def}}{=} id_1^\bullet \quad \alpha(R^\circ) \stackrel{\text{def}}{=} R^\bullet \quad \alpha(\sigma_{1,1}^\circ) \stackrel{\text{def}}{=} \sigma_{1,1}^\bullet$$
$$\alpha(\blacktriangleleft_1^\circ) \stackrel{\text{def}}{=} \blacktriangleright_1^\bullet \quad \alpha(!_1^\circ) \stackrel{\text{def}}{=} ¡_1^\bullet \quad \alpha(\blacktriangleright_1^\circ) \stackrel{\text{def}}{=} \blacktriangleleft_1^\bullet \quad \alpha(¡_1^\circ) \stackrel{\text{def}}{=} !_1^\bullet$$
$$\alpha(c \mathbin{;} d) \stackrel{\text{def}}{=} \alpha(d) \mathbin{;}^\bullet \alpha(c) \qquad \alpha(c \otimes d) \stackrel{\text{def}}{=} \alpha(c) \mathbin{\otimes^\bullet} \alpha(d)$$

$$\alpha(id_0^\bullet) \stackrel{\text{def}}{=} id_0^\circ \quad \alpha(id_1^\bullet) \stackrel{\text{def}}{=} id_1^\circ \quad \alpha(R^\bullet) \stackrel{\text{def}}{=} R^\circ \quad \alpha(\sigma_{1,1}^\bullet) \stackrel{\text{def}}{=} \sigma_{1,1}^\circ$$
$$\alpha(\blacktriangleleft_1^\bullet) \stackrel{\text{def}}{=} \blacktriangleright_1^\circ \quad \alpha(!_1^\bullet) \stackrel{\text{def}}{=} ¡_1^\circ \quad \alpha(\blacktriangleright_1^\bullet) \stackrel{\text{def}}{=} \blacktriangleleft_1^\circ \quad \alpha(¡_1^\bullet) \stackrel{\text{def}}{=} !_1^\circ$$
$$\alpha(c \mathbin{;}^\bullet d) \stackrel{\text{def}}{=} \alpha(d) \mathbin{;} \alpha(c) \qquad \alpha(c \mathbin{\otimes^\bullet} d) \stackrel{\text{def}}{=} \alpha(c) \otimes \alpha(d)$$

Lemma E.15. *For all terms $c\colon n \to m$ in NPR_Σ, $id_n^\circ \lesssim c \mathbin{;}^\bullet \alpha(c)$ and $\alpha(c) \mathbin{;} c \lesssim id_m^\bullet$.*

Proof. The proof goes by induction on c. For the base cases of black and white (co)monoid, it is immediate by the axioms in the first block of Figure 5.For R°, $R^\bullet$, σ° and $\sigma^\bullet$, it is immediate by the axioms in the bottom Figure 4. For id° and $id^\bullet$ is trivial. For the inductive cases of $;$, $;^\bullet$, $\otimes$ and $\otimes^\bullet$ one can reuse exactly the proof of Proposition 5.6. □

Lemma E.16. *For all term $c\colon n \to m$ in NPR_Σ, $\alpha(\alpha(c)) = c$.*

Proof. The proof goes by induction on c. For the base cases, it is immediate by Definition E.14. For the inductive cases, one have just to use the definition and the inductive hypothesis. For instance $\alpha(\alpha(a \mathbin{;} b))$ is, by Definition E.14, $\alpha(\alpha(a) \mathbin{;}^\bullet \alpha(b))$ which, by Definition E.14, is $\alpha(\alpha(a)) \mathbin{;} \alpha(\alpha(b))$ that, by induction hypothesis, is $a \mathbin{;} b$. □

Lemma E.17. *For all terms $c, d\colon n \to m$ in NPR_Σ, if $c \lesssim d$, then $\alpha(d) \lesssim \alpha(c)$.*

Proof. Observe that the axioms in Figures 2, 3, 4 and 5 are closed under α, namely if $c \leq d$ is an axiom also $\alpha(d) \leq \alpha(c)$ is an axiom. □

Lemma E.18. *For all terms $c\colon n \to m$ in NPR_Σ, $id_m^\circ \lesssim \alpha(c) \mathbin{;}^\bullet c$ and $c \mathbin{;} \alpha(c) \lesssim id_n^\bullet$.*

Proof. By Lemma E.15, it holds that

$$id_n^\circ \lesssim c \mathbin{;}^\bullet \alpha(c) \text{ and } \alpha(c) \mathbin{;} c \lesssim id_m^\bullet.$$

By Lemma E.17, one can apply α to all the sides of the two inequalities to get

$$\alpha(c \mathbin{;}^\bullet \alpha(c)) \lesssim \alpha(id_n^\circ) \text{ and } \alpha(id_m^\bullet) \lesssim \alpha(\alpha(c) \mathbin{;} c).$$

That, by Definition E.14 gives exactly

$$\alpha(\alpha(c)) \mathbin{;} \alpha(c) \lesssim id_n^\bullet \text{ and } id_m^\circ \lesssim \alpha(c) \mathbin{;}^\bullet \alpha(\alpha(c)).$$

By Lemma E.16, one can conclude that

$$c \mathbin{;} \alpha(c) \lesssim id_n^\bullet \text{ and } id_m^\circ \lesssim \alpha(c) \mathbin{;}^\bullet c.$$

□

Proof of Proposition 6.8. By Lemmas E.15 and E.18, the diagram $\alpha(c)$ is both the right and the left linear adjoint of any diagram c. Thus $\mathbf{FOB}_\Sigma$ is a closed linear bicategory.

Next, we show that $(\mathbf{FOB}_\Sigma^\circ, \blacktriangleleft^\circ, \blacktriangleright^\circ)$ is a cartesian bicategory: for all objects $n \in \mathbb{N}$, $\blacktriangleleft_n^\circ$, $!_n^\circ$, $\blacktriangleright_n^\circ$ and $¡_n^\circ$ are inductively defined as in Table 1. Observe that such definitions guarantees that the coherence conditions in Definition 4.1.(5) are satisfied. The conditions in Definition 4.1.(1).(2).(3).(4) are the axioms in Figure 2 (and appear in the term version in Figure 9) that we have used to generate $\lesssim$.

Similarly, $(\mathbf{FOB}_\Sigma^\bullet, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$ is a cocartesian bicategory: for all objects $n \in \mathbb{N}$, $\blacktriangleleft_n^\bullet$, $!_n^\bullet$, $\blacktriangleright_n^\bullet$ and $¡_n^\bullet$ are inductively defined as in Table 1. Again, the coherence conditions are satisfied by construction. The other conditions are the axioms in Figure 3 (and appear in the term version in Figure 9) that, by construction, are in $\lesssim$. To conclude that $\mathbf{FOB}_\Sigma$ is a first order bicategory we have to check that the conditions in Definition 6.1.(4),(5). But these are exactly the axioms in Figure 5 (and appear in the term version in Figure 9). □

Proof of Proposition 6.10. Observe that the rules in (8) defining $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ also defines $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ for an interpretation $\mathcal{I}$ of Σ in $\mathbf{C}$ by fixing $\mathcal{I}^\sharp(R^\bullet) = (\mathcal{I}^\sharp(R^\circ))^\perp$. To prove that $\mathcal{I}^\sharp$ preserve the ordering, one can use exactly the same proof of Proposition 6.7. All the structure of (co)cartesian bicateries and linear bicategories is preserved by definition of $\mathcal{I}^\sharp$. Thus, $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ is a morphism of fo-bicategories. By definition, it also holds that $\mathcal{I}^\sharp(1) = X$ and $\mathcal{I}^\sharp(R^\circ) = \rho(R)$.

To see that it is unique, observe that a morphism $\mathcal{F}\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ should map the object 0 into I (the unit object of $\otimes$) and any other natural number n into $\mathcal{F}(1)^n$. Thus the only degree of freedom for the objects is the choice of where to map the natural number 1. Similarly, for arrows, the only degree of freedom is where to map R° and $R^\bullet$. However, the axioms in $\mathbb{FOB}$ obliges $R^\bullet$ to be mapped into the right linear adjoint of R°. Thus, by fixing $\mathcal{F}(1) = X$ and $\mathcal{F}(R^\circ) = \rho(R)$, $\mathcal{F}$ is forced to be $\mathcal{I}^\sharp$. □

F PROOFS OF SECTION 7

Proof of Proposition 7.2. By induction on (10). For the rule (id), we have two cases: either $(c,d) \in \lesssim$ or $(c,d) \in \mathbb{I}$. For $\lesssim$, we conclude immediately by Proposition 6.7. For $(c,d) \in \mathbb{I}$, the inclusion $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$ holds by definition of model. The proofs for the other rules are trivial. □

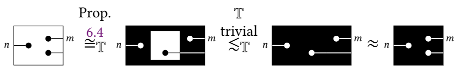
Lemma F.1. *Let $\mathbb{T}$ be a theory. If $\mathbb{T}$ is contradictory then it is trivial.*

Proof. Assume $\mathbb{T}$ to be contradictory and consider the following derivation.

$$
\begin{aligned}
\mathsf{i}_1^{\circ} &= id_0^{\circ} \,;\, \mathsf{i}_1^{\circ} \\
&\leq id_0^{\bullet} \,;\, \mathsf{i}_1^{\circ} && (\mathbb{T}\text{ contradictory}) \\
&= id_0^{\bullet} \,;\, \mathsf{i}_1^{\bullet} && (\text{Proposition 6.4}) \\
&= \mathsf{i}_1^{\bullet}
\end{aligned}
$$

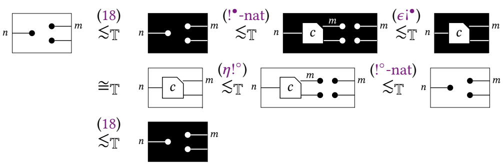
□

Proof of Lemma 7.5. First observe that the following holds:



(18)

Then, a simple derivation proves the statement:



The proof for $!^{\circ}_{n+1} \,;\, \mathsf{i}^{\circ}_{m} \lesssim_{\mathbb{T}} d \lesssim_{\mathbb{T}} !^{\bullet}_{n+1} \,;\, \mathsf{i}^{\bullet}_{m}$ follows a similar reasoning. □

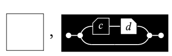
F.1 Theories in FOL and NPR$_\Sigma$

Once a first order alphabet is fixed, a theory in FOL is usually defined as a set $\mathcal{T}$ of *closed* formulas that must be considered true. Intuitively, closed formulas corresponds in our language to diagrams d of type $0 \to 0$. Indeed the semantics $\mathcal{I}^{\sharp}$ assigns to such diagrams a relation $R \subseteq \mathbb{1} \times \mathbb{1}$: either $\{(\star,\star)\}$ (i.e., $id_{\mathbb{1}}^{\circ}$) representing true or $\varnothing$ (i.e., $id_{\mathbb{1}}^{\bullet}$) representing false. The fact that d must hold in any model is forced by requiring $(id_0^{\circ}, d) \in \mathbb{I}$. This motivates the following definition.

Definition F.2. A theory $\mathbb{T} = (\Sigma, \mathbb{I})$ is said to be *closed* if all the pairs $(c, d) \in \mathbb{I}$ are of the form (id_0°, d).

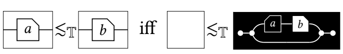
For instance, the theory of sets and the theory of non-empty sets in Example 7.1 are closed, while the third theory – the one of order – is not closed. By means of Lemma 6.6, one can always translate an arbitrary theory $\mathbb{T} = (\Sigma, \mathbb{I})$ into a closed theory $\mathbb{T}^c = (\Sigma, \mathbb{I}^c)$ where

$$\mathbb{I}^c \stackrel{\text{def}}{=} \Big\{ \Big(\quad , \quad \Big) \mid (c,d) \in \mathbb{I} \Big\}.$$



Proposition F.3. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $a, b \colon n \to m$ be diagrams in* **FOB**$_\Sigma$*. Then $a \lesssim_{\mathbb{T}} b$ iff $a \lesssim_{\mathbb{T}^c} b$.*

Proof. By induction on the rules in (10). The base case (id) is given by means of Lemma 6.6 and in particular from the fact that:



for any $(a, b) \in \mathbb{I}$.

The base case (r) and the inductive cases are trivial. □

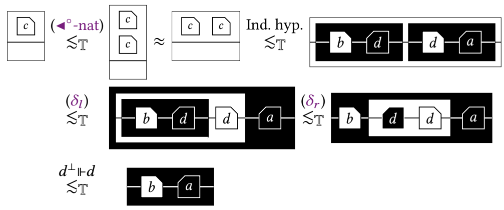
This result allows us to safely restrict our attention to closed theories, but this fact is not used in our proof of completeness. More interestingly, it tells us that while theories as introduced in §7 appear to be rather different from the usual FOL theories, they can always be translated into closed theories which are essentially the same as the FOL ones. Indeed from a closed theory $\mathbb{I}$, one can obtain the of set of closed formulas $\{d \mid (id_0^{\circ}, d) \in \mathbb{I}\}$ and, from a set of closed formulas $\mathcal{T}$ one can take $\mathbb{I}$ as $\{(id_0^{\circ}, d) \mid d \in \mathcal{T}\}$.

The fact that a closed formula d is derivable in $\mathcal{T}$, usually written as $\mathcal{T} \vdash d$, translates in NPR$_\Sigma$ to $id_0^{\circ} \lesssim_{\mathbb{T}} d$. In particular, when d is an implication $c \Rightarrow b$, we have $id_0^{\circ} \lesssim_{\mathbb{T}} b \,;\, c^{\perp}$ that, by Lemma 5.4, is equivalent to $c \lesssim_{\mathbb{T}} b$. In FOL it is trivial – by modus ponens – that if $\mathcal{T} \vdash c \Rightarrow b$ then $\mathcal{T} \cup \{c\} \vdash b$. In NPR$_\Sigma$, this fact follows by transitivity of $\lesssim_{\mathbb{T}}$: fix $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id^{\circ}, c)\})$ and observe that $id_0^{\circ} \lesssim_{\mathbb{T}'} c \lesssim_{\mathbb{T}'} b$. The converse implication, namely if $\mathcal{T} \cup \{c\} \vdash b$ then $\mathcal{T} \vdash c \Rightarrow b$, is known in FOL as deduction theorem. It can be generalised in NPR$_\Sigma$ as in Theorem 7.7.

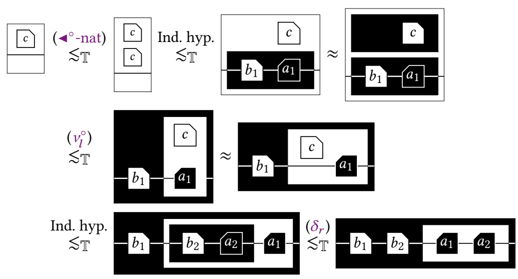
F.2 Deduction Theorem

Proof of Theorem 7.7. The base cases are trivial. We show the case for $(;)$ in the main text. We show here the remaining inductive cases:

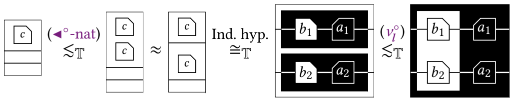
(t) Assume $a \lesssim_{\mathbb{T}'} d$ and $d \lesssim_{\mathbb{T}'} b$ for some $d \colon n \to m$. Observe that $a \lesssim_{\mathbb{T}'} b$ by (t) and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} d \,;\, a^{\perp}$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b \,;\, d^{\perp}$ by inductive hypothesis. To conclude we need to show:



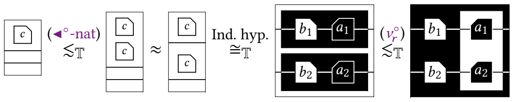
$(;)$ Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 \,;\, a_2$ and $b = b_1 \,;\, b_2$ for some $a_1, b_1 \colon n \to l, a_2, b_2 \colon l \to m$. Observe that $a_1 \,;\, a_2 \lesssim_{\mathbb{T}'} b_1 \,;\, b_2$ by $(;)$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_1 \,;\, a_1^{\perp}$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_2 \,;\, a_2^{\perp}$ by inductive hypothesis. To conclude we need to show:



$(\otimes)$ Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$ for some $a_1, b_1 \colon n' \to m', a_2, b_2 \colon n'' \to m''$. Observe that $a_1 \otimes a_2 \lesssim_{\mathbb{T}'} b_1 \otimes b_2$ by $(\otimes)$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_1 \,;\, a_1^{\perp}$ and $c \otimes id_n^{\circ} \lesssim_{\mathbb{T}} b_2 \,;\, a_2^{\perp}$ by inductive hypothesis. To conclude we need to show:



(⊗) Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 ⊗ a_2$ and $b = b_1 ⊗ b_2$ for some $a_1, b_1 \colon n' \to m', a_2, b_2 \colon n'' \to m''$. Observe that $a_1 ⊗ a_2 \lesssim_{\mathbb{T}'} b_1 ⊗ b_2$ by (⊗) and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_1 ; a_1^\perp$ and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_2 ; a_2^\perp$ by inductive hypothesis. To conclude we need to show:



□

Proof of Corollary 7.8. Suppose that $\mathbb{T}'$ is contradictory, namely $id^\circ_0 \lesssim_{\mathbb{T}'} id^\bullet_0$. By the deduction theorem (Theorem 7.7), $\overline{c} \lesssim_{\mathbb{T}} id^\bullet_0$ and thus $\overline{id^\bullet_0} \lesssim_{\mathbb{T}} \overline{\overline{c}}$, that is $id^\circ_0 \lesssim_{\mathbb{T}} c$. The the other direction is trivial: since $id^\circ_0 \lesssim_{\mathbb{T}'} c$ and $id^\circ_0 \lesssim_{\mathbb{T}'} \overline{c}$, then $id^\circ_0 \lesssim_{\mathbb{T}'} c \sqcap \overline{c} \lesssim_{\mathbb{T}'} \perp = id^\bullet_0$. □

F.3 Proofs of Section 7.1

Proof of Proposition 7.10. First, observe that a simple inductive argument allows to prove that, for all diagrams c in $\mathbf{FOB}_\Sigma$,

$$Q^\sharp_{\mathbb{T}}(c) = [c]_{\cong_{\mathbb{T}}}. \tag{19}$$

Now, suppose that there exists $\mathcal{I}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ making commutes the following diagram

$$\begin{tikzcd}
\mathbf{FOB}_\Sigma \arrow[r, "Q^\sharp_{\mathbb{T}}"] \arrow[dr, "\mathcal{I}^\sharp"'] & \mathbf{FOB}_{\mathbb{T}} \arrow[d, dotted, "\mathcal{I}^\sharp_{\mathbb{T}}"] \\
& \mathbf{C}
\end{tikzcd}$$

and consider $(c, d) \in \mathbb{I}$. By definition, $c \lesssim_{\mathbb{T}} d$ and, by (19),

$$Q^\sharp_{\mathbb{T}}(c) \lesssim_{\mathbb{T}} Q^\sharp_{\mathbb{T}}(d). \tag{20}$$

Then, the following derivation confirms that $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$.

$$\begin{aligned}
\mathcal{I}^\sharp(c) &= \mathcal{I}^\sharp_{\mathbb{T}}(Q^\sharp_{\mathbb{T}}(c)) && (\mathcal{I}^\sharp = Q^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}}) \\
&\leq \mathcal{I}^\sharp_{\mathbb{T}}(Q^\sharp_{\mathbb{T}}(d)) && ((20) \text{ and } \mathcal{I}^\sharp_{\mathbb{T}} \text{ is a morphism}) \\
&= \mathcal{I}^\sharp(d) && (\mathcal{I}^\sharp = Q^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}})
\end{aligned}$$

Viceversa, suppose that $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$. Then by definition of model, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d)$. A simple inductive argument on the rules in (10) confirms that, for all diagrams c, d in $\mathbf{FOB}_\Sigma$,

$$\text{if } c \lesssim_{\mathbb{T}} d \text{ then } \mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d).$$

In particular, if $c \cong_{\mathbb{T}} d$ then $\mathcal{I}^\sharp(c) = \mathcal{I}^\sharp(d)$. Therefore, we are allowed to define $\mathcal{I}^\sharp_{\mathbb{T}}([c]_{\cong_{\mathbb{T}}}) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c)$ for all arrows $[c]_{\cong_{\mathbb{T}}}$ of $\mathbf{FOB}_{\mathbb{T}}$ and $\mathcal{I}^\sharp_{\mathbb{T}}(n) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(n)$ for all objects n of $\mathbf{FOB}_{\mathbb{T}}$. The fact that $\mathcal{I}^\sharp_{\mathbb{T}}$ preserves the ordering follows immediately from the above implication. The fact that $\mathcal{I}^\sharp_{\mathbb{T}}$ preserves the structure of fo-bicategories follows easily from the fact that $\mathcal{I}^\sharp$ is a morphism. Therefore $\mathcal{I}^\sharp_{\mathbb{T}}$ is a morphism of fo-bicategories. The fact that the above diagram commutes is obvious by definition of $\mathcal{I}^\sharp_{\mathbb{T}}$ and (19).

Uniqueness follows immediately from the fact that $Q^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_{\mathbb{T}}$ is an epi, namely all objects and arrows of $\mathbf{FOB}_{\mathbb{T}}$ are in the image of $Q^\sharp_{\mathbb{T}}$. □

Proof of Corollary 7.11. To go from models to morphisms we use the assignment $\mathcal{I} \mapsto \mathcal{I}^\sharp_{\mathbb{T}}$ provided by Proposition 7.10.

To transform morphisms into models, we need a slightly less straightforward assignment. Take a morphism of fo-bicategories $\mathcal{F} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ and consider $Q^\sharp_{\mathbb{T}}; \mathcal{F} \colon \mathbf{FOB}_\Sigma \to \mathbf{C}$. This gives rise to the interpretation $\mathcal{I}_{\mathcal{F}}$ defined as

$$\text{the domain } X \text{ is } Q^\sharp_{\mathbb{T}}; \mathcal{F}(1) \text{ and } \rho(R) \text{ is } Q^\sharp_{\mathbb{T}}; \mathcal{F}(R^\circ) \text{ for all } R \in \Sigma.$$

By Proposition 6.10, $\mathcal{I}^\sharp_{\mathcal{F}} = Q^\sharp_{\mathbb{T}}; \mathcal{F}$ and thus, by Proposition 7.10, $\mathcal{I}_{\mathcal{F}}$ is a model.

Since $\mathcal{I}^\sharp_{\mathcal{F}} = Q^\sharp_{\mathbb{T}}; \mathcal{F}$, by the uniqueness provided by Proposition 7.10, $(\mathcal{I}_{\mathcal{F}})^\sharp_{\mathbb{T}} = \mathcal{F}$.

To conclude, we only need to prove that $\mathcal{I}_{(\mathcal{I}^\sharp_{\mathbb{T}})} = \mathcal{I}$. Since $Q^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}} = \mathcal{I}^\sharp$, then $\mathcal{I}_{(\mathcal{I}^\sharp_{\mathbb{T}})}(R^\circ) = Q^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}}(R^\circ) = \mathcal{I}^\sharp(R^\circ) = \rho(R)$ for all $R \in \Sigma$. Similarly for the domain X.

□

Proof of Lemma 7.12. By Proposition 7.10, it is enough to give a model of $\mathbb{T}$ in $\mathbf{FOB}_{\mathbb{T}'}$. Define the interpretation $\mathcal{I}$ having as domain X the object 1 of $\mathbf{FOB}_{\mathbb{T}'}$ and $\rho(R) \stackrel{\text{def}}{=} [R^\circ]_{\cong_{\mathbb{T}'}}$ for each $R \in \Sigma$. A simple inductive arguments confirms that $\mathcal{I}^\sharp(c) = [c]_{\cong_{\mathbb{T}'}}$ for all diagrams c in $\mathbf{FOB}_\Sigma$. Since $\mathbb{I} \subseteq \mathbb{I}'$ is obvious that, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \lesssim_{\mathbb{T}'} \mathcal{I}^\sharp(d)$. Thus $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{FOB}_{\mathbb{T}'}$. □

G PROOFS OF SECTION 8

Proposition G.1. *In any cartesian bicategory an n-ary map $\vec{k} \colon 0 \to n$ can always be decomposed as:*

$$\vec{k} = k_1 \otimes k_2 \otimes \ldots \otimes k_n \quad \text{where each } k_i \colon 0 \to 1 \text{ is a map.}$$

Proof. Follows from Lemma C.1.(4). □

Lemma G.2. *For any $c \colon n \to m$ in $\mathbf{FOB}_\Sigma$ the following hold*

$$\mathcal{H}^\sharp(c^\dagger) = (\mathcal{H}^\sharp(c))^\dagger, \quad \mathcal{H}^\sharp(c^\perp) = (\mathcal{H}^\sharp(c))^\perp, \quad \mathcal{H}^\sharp(\overline{c}) = \overline{(\mathcal{H}^\sharp(c))}$$

Proof. Since $\mathcal{H}^\sharp$ is a morphism of fo-bicategory the proof for $(\cdot)^\dagger$ and $(\cdot)^\perp$ follows from Lemma D.1 and Lemma C.2.

Negation is preserved as well, since $\overline{(\cdot)} = (\cdot^\dagger)^\perp$.

□

Proposition G.3. *Let I be a linearly ordered set and for all $i \in I$ let $\mathbb{T}_i = (\Sigma_i, \mathbb{I}_i)$ be first order theories such that if $i \leq j$, then $\Sigma_i \subseteq \Sigma_j$ and $\mathbb{I}_i \subseteq \mathbb{I}_j$. Let $\mathbb{T}$ be the theory $(\bigcup_{i \in I} \Sigma_i, \bigcup_{i \in I} \mathbb{I}_i)$.*

(1) *If all $\mathbb{T}_i$ are non-contradictory, then $\mathbb{T}$ is non-contradictory.*
(2) *If all $\mathbb{T}_i$ are non-trivial, then $\mathbb{T}$ is non-trivial.*

Proof. By using the well-known fact that $\mathrm{pc}(\cdot)$ preserves chains, one can easily see that

$$\lesssim_{\mathbb{T}} = \bigcup_{i \in I} \lesssim_{\mathbb{T}_i} \tag{21}$$

The interested reader can find all the details in Appendix H.1, Lemma H.12.

(1) Suppose that $\mathbb{T}$ is contradictory. By definition $id_0^\circ \lesssim_{\mathbb{T}} id_0^\bullet$ and then, by (21), $(id_0^\circ, id_0^\bullet) \in \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$. Thus there exists an $i \in I$ such that $id_0^\circ \lesssim_{\mathbb{T}_i} id_0^\bullet$. Against the hypothesis.
(2) Suppose that $\mathbb{T}$ is trivial. By definition $\mathsf{i}_1^\circ \lesssim_{\mathbb{T}} \mathsf{i}_1^\bullet$ and then, by (21), $(\mathsf{i}_1^\circ, \mathsf{i}_1^\bullet) \in \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$. Thus there exists an $i \in I$ such that $\mathsf{i}_1^\circ \lesssim_{\mathbb{T}_i} \mathsf{i}_1^\bullet$. Against the hypothesis.

□

Proposition G.4. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory theory. There exists a theory $\mathbb{T}' = (\Sigma, \mathbb{I}')$ that is syntactically complete, non-contradictory and $\mathbb{I} \subseteq \mathbb{I}'$.*

Proof of Proposition G.4. The proof of this proposition relies on Zorn Lemma [82] which states that if, in a non empty poset poset L every chain has a least upper bound, then L has at least one maximal element.

We consider the set Γ of all non-contradictory theories on Σ that include $\mathbb{I}$, namely

$$\Gamma \stackrel{\text{def}}{=} \{\mathbb{T} = (\Sigma, \mathbb{J}) \mid \mathbb{T} \text{ is non-contradictory and } \mathbb{I} \subseteq \mathbb{J}\}.$$

Observe that the set Γ is non empty since there is at least $\mathbb{T}$ which belongs to Γ.

Let $\Lambda \subseteq \Gamma$ be a chain, namely $\Lambda = \{\mathbb{T}_i = (\Sigma, \mathbb{J}_i) \in \Gamma \mid i \in I\}$ for some linearly ordered set I and if $i \leq j$, then $\mathbb{J}_i \subseteq \mathbb{J}_j$. By Proposition G.3, the theory $(\Sigma, \bigcup_{i\in I} \mathbb{J}_i)$ is non-contradictory and thus it belongs to Γ.

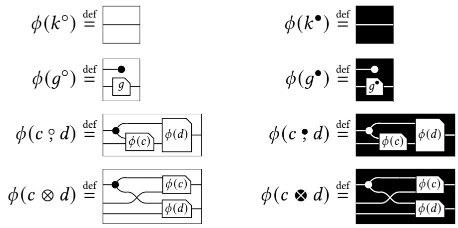
We can thus use Zorn Lemma: the set Γ has a maximal element $\mathbb{T}' = (\Sigma, \mathbb{I}')$. By definition of Γ, $\mathbb{I} \subseteq \mathbb{I}'$ and, moreover, $\mathbb{T}'$ is non-contradictory.

We only need to prove that $\mathbb{T}'$ is syntactically complete, i.e., for all $c \colon 0 \to 0$, either $id_0^\circ \lesssim_{\mathbb{T}'} c$ or $id_0^\circ \lesssim_{\mathbb{T}'} \overline{c}$. Assume that $id_0^\circ \not\lesssim_{\mathbb{T}} c$. Thus $\mathbb{I}'$ is *strictly* included into $\mathbb{I}' \cup \{(id_0^\circ, c)\}$. By maximality of $\mathbb{T}'$ in Γ, we have that the theory $\mathbb{T}'' = (\Sigma, \mathbb{I}' \cup \{(id_0^\circ, c)\})$ is contradictory, i.e., $id_0^\circ \lesssim_{\mathbb{T}''} id_0^\bullet$. By the deduction theorem (Theorem 7.7), $c \lesssim_{\mathbb{T}'} id_0^\bullet$. Therefore $id_0^\circ \lesssim_{\mathbb{T}'} \overline{c}$. □

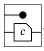
G.1 Proofs for Lemma 8.2 and Theorem 8.3

In order to prove Lemma 8.2 and then Theorem 8.3, we need to showing that *adding* constants to a non-trivial theory results in a non-trivial theory. To do this, it is useful to have a procedure for *erasing* constants. This is defined as follows.

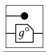
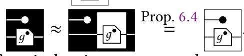
Definition G.5. Let Σ be a signature and $\Sigma' = \Sigma \cup \{k \colon 0 \to 1\}$. The function $\phi \colon \mathbf{FOB}_{\Sigma'}[n, m] \to \mathbf{FOB}_{\Sigma}[1 + n, m]$ is inductively defined as follows:



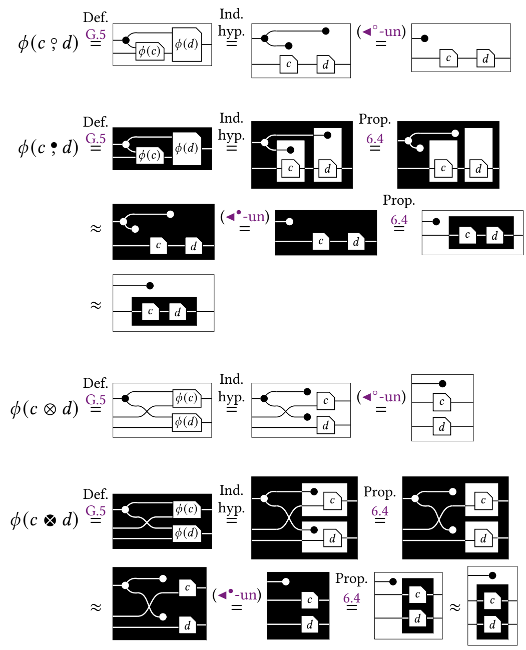
where $g^\circ \in \{\blacktriangleleft_1^\circ, !_1^\circ, R^\circ, \mathsf{i}_1^\circ, \blacktriangleright_1^\circ, id_0^\circ, id_1^\circ, \sigma_{1,1}^\circ\}$ and $g^\bullet \in \{\blacktriangleleft_1^\bullet, !_1^\bullet, R^\bullet, \mathsf{i}_1^\bullet, \blacktriangleright_1^\bullet, id_0^\bullet, id_1^\bullet, \sigma_{1,1}^\bullet\}$.

Lemma G.6. *Let $c \colon n \to m$ be a diagram of $\mathbf{FOB}_\Sigma$, then $\phi(c) =$* .

Proof. The proof goes by induction on the syntax.

The base cases are split in two groups. For all generators g° in $\mathrm{NPR}_\Sigma^\circ$, $\phi(g^\circ) =$  by definition, while for those $g^\bullet$ in $\mathrm{NPR}_\Sigma^\bullet$, $\phi(g^\bullet) =$ .

The four inductive cases are shown below:

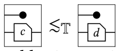


□

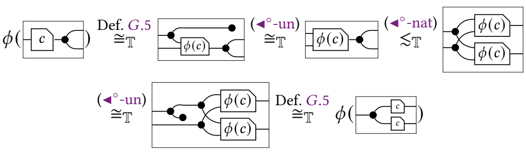
Lemma G.7 (Constant Erasion). *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be the theory where $\Sigma' = \Sigma \cup \{k \colon 0 \to 1\}$ and $\mathbb{I}' = \mathbb{I} \cup \mathbb{M}_k$. Then, for any $c, d \colon n \to m$ in $\mathbf{FOB}_{\Sigma'}$ if $c \lesssim_{\mathbb{T}'} d$ then $\phi(c) \lesssim_{\mathbb{T}} \phi(d)$.*

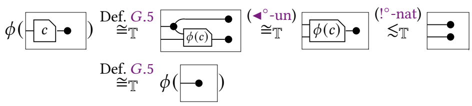
Proof. The proof goes by induction on the rules in (10).

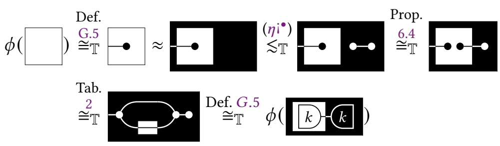
For the rule (id) we have three cases: either $(c, d) \in \mathbb{I}$ or $(c, d) \in \lesssim_{\Sigma'}$ or $(c, d) \in \mathbb{M}_k$.

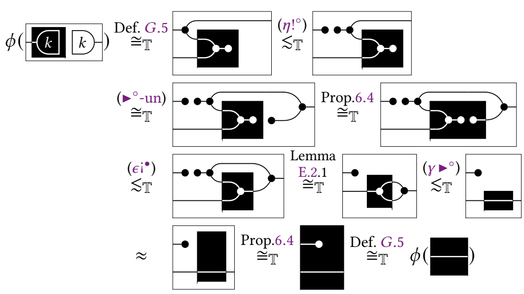
If $(c, d) \in \mathbb{I}$ then, by Lemma G.6, $\phi(c) =$  $= \phi(d)$.

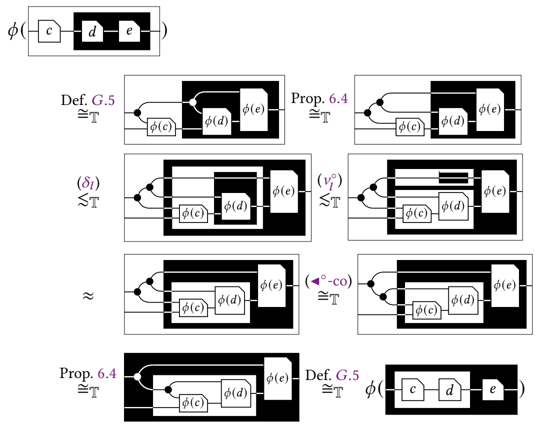
If $(c, d) \in \lesssim_{\Sigma'}$ then (c, d) has been obtained by instantiating the axioms in Figures 2,3 and 4 with diagrams containing k. Therefore, we need to show that ϕ preserves these axioms. In the following we show only ($\blacktriangleleft^{\circ}$-nat), ($!^{\circ}$-nat), ($\tau R^{\circ}$), ($\gamma R^{\circ}$), ($\delta_l$) and ($\nu_r^{\circ}$). The remaining ones follow similar reasonings.

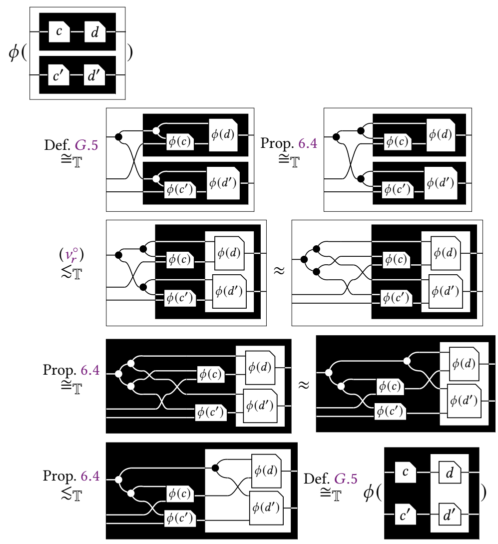




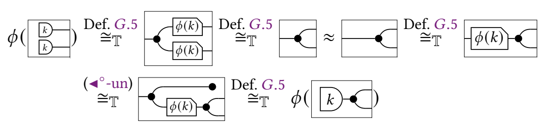


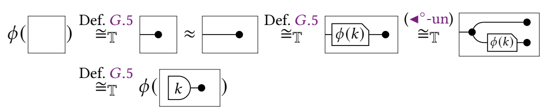






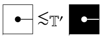
Similar to the previous argument, if $(c, d) \in \mathbb{M}_k$ then it is enough to show that ϕ preserves the axioms in $\mathbb{M}_k$.



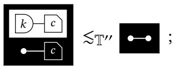


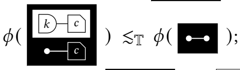
The base case (r) is trivial, while the proof for the remaining rules follows a straightforward inductive argument. □

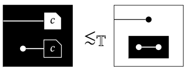
Proof of Lemma 8.2. We prove that if $\mathbb{T}'$ is trivial, then also $\mathbb{T}$ is trivial. Let $\mathbb{T}'' = \{\Sigma \cup k, \mathbb{I} \cup \mathbb{M}_k\}$ and assume $\mathbb{T}'$ to be trivial, i.e.



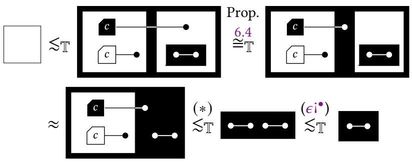
, then:

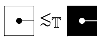
(1) by the Deduction Theorem (7.7) we have  ;

(2) thus, by Lemma G.7,  ;

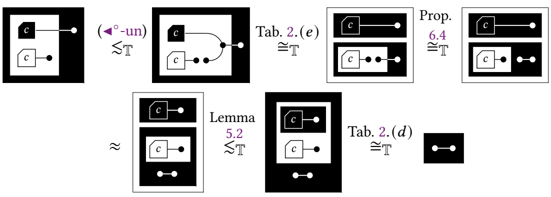
(3) and, by Def. G.5 and Lemma G.6,  .

To conclude, apply Lemma 6.6 and observe:



which, by Lemma 6.6 again, is exactly that . Namely $\mathbb{T}$ is trivial.

Note that in the step $(*)$ above we used the following derivation which holds for any $c\colon 0 \to 1$:



□

Proof of Theorem 8.3. This proof reuses the well-known arguments reported e.g. in [46].

We first illustrate a procedure to add Henkin witnesses without losing the property of being non-trivial.

Take an enumeration of diagrams in $\mathbf{FOB}_\Sigma[1,0]$ and write c_i for the i-th diagram.

For all natural numbers $n \in \mathbb{N}$, we define

$$\Sigma^n \stackrel{\text{def}}{=} \Sigma \cup \{k_i\colon 0 \to 1 \mid i \leq n\} \qquad \mathbb{I}^n \stackrel{\text{def}}{=} \mathbb{I} \cup \mathbb{M}_{k_i} \cup \textstyle\bigcup_{i\leq n} \mathbb{W}^{c_i}_{k_i}$$
$$\mathbb{T}^n \stackrel{\text{def}}{=} (\Sigma^n, \mathbb{I}^n)$$

By applying Lemma 8.2 n-times, one has that $\mathbb{T}^n$ is non-trivial. Define now

$$\Sigma_0 \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \Sigma^i \qquad \mathbb{I}_0 \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \mathbb{I}^j \qquad \mathbb{T}_0 \stackrel{\text{def}}{=} (\Sigma_0, \mathbb{I}_0)$$

Since $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \cdots \subseteq \mathbb{T}^n \subseteq \ldots$ are all non-trivial, then by Proposition G.3.2, we have that $\mathbb{T}_0$ is non-trivial. One must not jump to the conclusion that $\mathbb{T}_0$ has Henkin witnesses: all the diagrams in $\mathbf{FOB}_\Sigma[1,0]$ have Henkin witnesses, but in $\mathbb{T}_0$ we have more diagrams, since we have added the constants k_i to Σ_0.

We thus repeat the above construction, but now for diagrams in $\mathbf{FOB}_{\Sigma_0}[1,0]$. We define

$$\Sigma_1 \stackrel{\text{def}}{=} \Sigma_0 \cup \{k_c \mid c \in \mathbf{FOB}_{\Sigma_0}[1,0]\} \qquad \mathbb{I}_1 \stackrel{\text{def}}{=} \mathbb{I}_0 \cup \mathbb{M}_{k_c} \cup \mathbb{W}^c_{k_c}$$
$$\mathbb{T}_1 \stackrel{\text{def}}{=} (\Sigma_1, \mathbb{I}_1)$$

The theory $\mathbb{T}_1$ is non-trivial but has Henkin witnesses only for the diagrams in $\mathbf{FOB}_{\Sigma_0}$.

Thus, for all natural numbers $n \in \mathbb{N}$, we define

$$\Sigma_{n+1} \stackrel{\text{def}}{=} \Sigma_n \cup \{k_c \mid c \in \mathbf{FOB}_{\Sigma_n}[1,0]\} \qquad \mathbb{I}_{n+1} \stackrel{\text{def}}{=} \mathbb{I}_n \cup \mathbb{M}_{k_c} \cup \mathbb{W}^c_{k_c}$$
$$\mathbb{T}_{n+1} \stackrel{\text{def}}{=} (\Sigma_{n+1}, \mathbb{I}_{n+1})$$

and

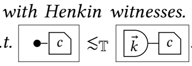
$$\Sigma' \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \Sigma_i \qquad \mathbb{I}' \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \mathbb{I}_i \qquad \mathbb{T}' \stackrel{\text{def}}{=} (\Sigma', \mathbb{I}')$$

Since $\mathbb{T}_0 \subseteq \mathbb{T}_1 \subseteq \cdots \subseteq \mathbb{T}_n \subseteq \ldots$ are all non-trivial, then by Proposition G.3.2, we have that $\mathbb{T}'$ is also non-trivial. Now $\mathbb{T}'$ has Henkin witnesses: if $c \in \mathbf{FOB}_{\Sigma'}[0,1]$, then there exists $n \in \mathbb{N}$ such that $c \in \mathbf{FOB}_{\Sigma_n}[0,1]$. By definition of $\mathbb{I}_n$, it holds that $\mathbb{W}^c_{k_c} \subseteq \mathbb{I}_{n+1}$ and thus $\mathbb{W}^c_{k_c} \subseteq \mathbb{I}'$.

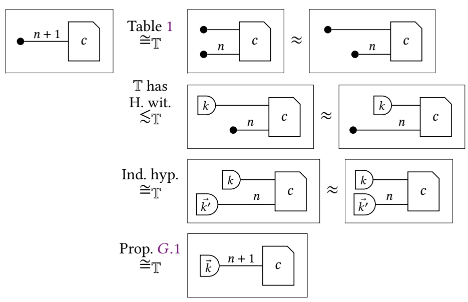
Summarising, we manage to build a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ that has Henkin witnesses and it is non-trivial. By Lemma F.1, $\mathbb{T}'$ is non-contradictory. We can thus use Proposition G.4, to obtain a theory $\mathbb{T}'' = (\Sigma', \mathbb{I}'')$ that is syntactically complete and non-contradictory. Observe that $\mathbb{T}''$ has Henkin witnesses, since the signature Σ' is the same as in $\mathbb{T}'$ and $\mathbb{I}' \subseteq \mathbb{I}''$. □

G.2 Proofs for Proposition 8.5

Proposition 8.5 is the second key to prove Gödel completeness. Before illustrating its proof, we need an additional lemma.

Lemma G.8. *Let $\mathbb{T}$ be a theory with Henkin witnesses. For all $c\colon n \to 0$ there is a map $\vec{k}\colon n \to 1$ s.t.* .

Proof. The proof goes by induction on n. For $n = 0$, take id°_0 as $\vec{k}$. For $n+1$, we have the following:



□

Proof of Proposition 8.5. The proof goes by induction on c. The white base cases are easy, we show three representative cases below.

$$\begin{aligned}
\mathcal{H}^\sharp(\boxed{\ }) &\overset{\text{Def. }\mathcal{H}^\sharp}{=} id_{\mathbb{1}}^\circ = \{(\star,\star)\in\mathbb{1}\times\mathbb{1}\}\\
&= \{(\star,\star)\in\mathbb{1}\times\mathbb{1} \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\ }\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^\sharp(\boxed{-}) &\overset{\text{Def. }\mathcal{H}^\sharp}{=} id_X^\circ = \{(k,k)\in X\times X\}\\
&\overset{\text{Prop. C.3}}{=} \{(k,k)\in X\times X \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{k - k}\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}^\sharp(\boxed{-\!\!\bullet\!\!<}) &\overset{\text{Def. }\mathcal{H}^\sharp}{=} \blacktriangleleft_X^\circ = \{(k,\begin{pmatrix}k\\k\end{pmatrix})\in X\times X^2\}\\
&\overset{\text{Prop. C.3}}{=} \{(k,\begin{pmatrix}k\\k\end{pmatrix})\in X\times X^2 \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\begin{matrix}k - k\\ k - k\end{matrix}}\}\\
&\overset{(\mathbb{M}_k)}{=} \{(k,\begin{pmatrix}k\\k\end{pmatrix})\in X\times X^2 \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{k \bullet \begin{matrix}k\\k\end{matrix}}\}
\end{aligned}$$

For the base case $\blacksquare$ suppose that there are maps $k,l\colon 0\to 0$ such that $\boxed{\ } \lesssim_{\mathbb{T}} \boxed{k\ \blacksquare\ l}$. However the only map of type $0\to 0$ is $\boxed{\ }$ and thus we have that $\boxed{\ } \lesssim_{\mathbb{T}} \boxed{\blacksquare} \approx \blacksquare$ which contradicts the hypothesis that $\mathbb{T}$ is non-contradictory. Therefore, $\{(k,l)\in\mathbb{1}\times\mathbb{1} \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{k\ \blacksquare\ l}\} = \varnothing \overset{\text{Def. }\mathcal{H}^\sharp}{=} \mathcal{H}^\sharp(\blacksquare)$.

The proof of the remaining base cases follows a recurring pattern. For this reason we show only the case of ${}_m\blacksquare\!\!\boxed{R}_n$.

$$\begin{aligned}
\mathcal{H}^\sharp(\blacksquare\!\boxed{R}) &\overset{\text{Def. }\mathcal{H}^\sharp}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid (\vec{k},\vec{l})\notin\mathcal{H}^\sharp(\boxed{R})\}\\
&\overset{\mathcal{H}^\sharp(R^\circ)}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid \boxed{\ } \not\lesssim_{\mathbb{T}} \boxed{\vec{k}\,R\,\vec{l}}\}\\
&\overset{\mathbb{T}\text{ is s.c.}}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid \boxed{\ } \lesssim_{\mathbb{T}} \blacksquare\!\boxed{\vec{k}\,R\,\vec{l}}\}\\
&\overset{\text{Table 2.(a)}}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid (\boxed{\ })^\dagger \lesssim_{\mathbb{T}} (\blacksquare\!\boxed{\vec{k}\,R\,\vec{l}})^\dagger\}\\
&\overset{\text{Table 2.(a)}}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid \boxed{\ } \lesssim_{\mathbb{T}} \blacksquare\!\boxed{\vec{l}\,R\,\vec{k}}\}\\
&\overset{\text{Prop. 6.4}}{=} \{(\vec{l},\vec{k})\in X^m\times X^n \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{l}\ \blacksquare\!\boxed{R}\ \vec{k}}\}
\end{aligned}$$

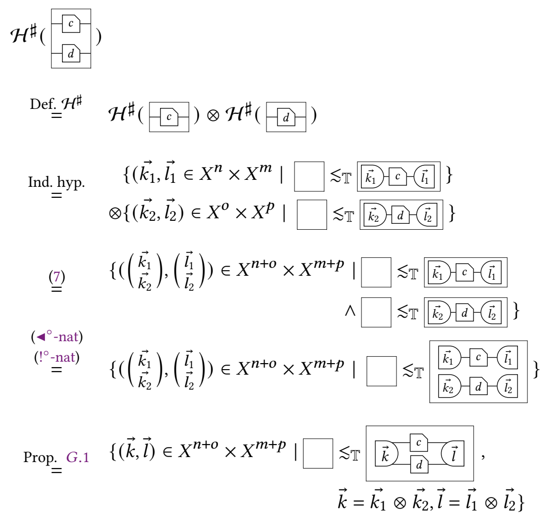
For the inductive case $c\,;d$ we prove the two inclusions separately. Suppose $c\colon n\to o$ and $d\colon o\to m$, then

$$\begin{aligned}
&\mathcal{H}^\sharp(\boxed{c\,d})\\
&\overset{\text{Def. }\mathcal{H}^\sharp}{=} \mathcal{H}^\sharp(\boxed{c})\,;\mathcal{H}^\sharp(\boxed{d})\\
&\overset{\text{Ind. hyp.}}{=} \{(\vec{k},\vec{t})\in X^n\times X^o \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,\vec{t}}\}\\
&\qquad ;\{(\vec{t},\vec{l})\in X^o\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,d\,\vec{l}}\}\\
&\overset{(2)}{=} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,\vec{t}}\\
&\qquad \wedge \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,d\,\vec{l}}\}\\
&\overset{\substack{(\blacktriangleleft^\circ\text{-nat})\\(!^\circ\text{-nat})}}{=} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\begin{matrix}\vec{k}\,c\,\vec{t}\\ \vec{t}\,d\,\vec{l}\end{matrix}}\}\\
&\approx \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,\vec{t}\quad \vec{t}\,d\,\vec{l}}\}\\
&\overset{\text{Prop. C.3}}{\subseteq} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,d\,\vec{l}}\}
\end{aligned}$$

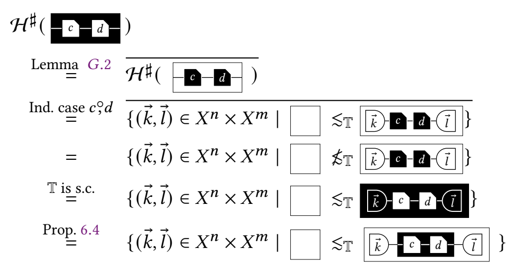
For the other inclusion the following holds:

$$\begin{aligned}
&\{(\vec{k},\vec{l})\in X^n\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,d\,\vec{l}}\}\\
&\overset{(11)}{=} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\bullet\!-\!\bullet \begin{matrix}c\,\vec{k}\\ d\,\vec{l}\end{matrix}}\}\\
&\overset{\text{Lemma G.8}}{\subseteq} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\bullet \begin{matrix}c\,\vec{k}\\ d\,\vec{l}\end{matrix}}\}\\
&\overset{(\mathbb{M}_k)}{=} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\begin{matrix}\vec{t}\,c\,\vec{k}\\ \vec{t}\,d\,\vec{l}\end{matrix}}\}\\
&\overset{\substack{(\blacktriangleleft^\circ\text{-nat})\\(!^\circ\text{-nat})}}{=} \{(\vec{k},\vec{l})\in X^n\times X^m \mid \exists\vec{t}\ \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,c\,\vec{k}}\\
&\qquad \wedge \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,d\,\vec{l}}\}\\
&\overset{(2)}{=} \{(\vec{k},\vec{t})\in X^n\times X^o \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,c\,\vec{k}}\}\\
&\qquad ;\{(\vec{t},\vec{l})\in X^o\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,d\,\vec{l}}\}\\
&\overset{\text{Table 2.(a)}}{=} \{(\vec{k},\vec{t})\in X^n\times X^o \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{k}\,c\,\vec{t}}\}\\
&\qquad ;\{(\vec{t},\vec{l})\in X^o\times X^m \mid \boxed{\ } \lesssim_{\mathbb{T}} \boxed{\vec{t}\,d\,\vec{l}}\}\\
&\overset{\text{Ind. hyp.}}{=} \mathcal{H}^\sharp(\boxed{c})\,;\mathcal{H}^\sharp(\boxed{d})\\
&\overset{\text{Def. }\mathcal{H}^\sharp}{=} \mathcal{H}^\sharp(\boxed{c\,d})
\end{aligned}$$

The inductive case $c\otimes d$ is proved as follows: Suppose $c\colon n\to m$ and $d\colon o\to p$, then



The inductive case $c \,;\, d$ is proved as follows:
Suppose $c\colon n \to o$ and $d\colon o \to m$, then



The proof above relies on Lemma G.2 and the previous inductive case of $c \,⨾\, d$. The case of $c \,\bullet\, d$ follows the exact same reasoning but, as expected, this time one has to exploit the proof of $c \otimes d$. $\square$

G.3 Proofs from Gödel completeness to Theorem 3.2

After having proved (Gödel), we show how to obtain a proof for Theorem 3.2. The main step is to prove (Prop).

Lemma G.9. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory that is trivial and non-contradictory and let $\mathcal{H}$ be the Henkin interpretation of Σ. Then, the domain X of $\mathcal{H}$ is $\varnothing$ and $\rho(R) = \{(\star,\star)\}$ if $id_0^\circ \lesssim_{\mathbb{T}} R^\circ$ and $\varnothing$ otherwise.*

Proof. Recall by Definition 8.4, that the domain X of $\mathcal{H}$ is defined as the set $\mathsf{Map}(\mathbf{FOB}_{\mathbb{T}})[0,1]$. This set should be necessarily empty since, if there exists some map $k\colon 0 \to 1$, then by (16), $\mathbb{T}$ would be contradictory, against the hypothesis. Thus $\mathsf{Map}(\mathbf{FOB}_{\mathbb{T}})[0,1] = \varnothing$. By Proposition G.1, one has also that $\mathsf{Map}(\mathbf{FOB}_{\mathbb{T}})[0,n+1] = \varnothing$. We thus have only one map in $\mathbf{FOB}_{\mathbb{T}}$, that is $id_0^\circ\colon 0 \to 0$ (depicted as $\boxed{}$).

Recall that by Definition 8.4, $\rho(R) = \{(\vec{k},\vec{l}) \in X^n{\times}X^m \mid \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\text{-}R\text{-}\vec{l}}\}$ for all $R \in \Sigma$. Since our only map is $id_0^\circ\colon 0 \to 0$, we have that $\rho(R) = \{(\star,\star) \in \mathbb{1}\times\mathbb{1} \mid id_0^\circ \lesssim_{\mathbb{T}} R^\circ\}$. $\square$

Lemma G.10. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and let $c\colon n \to m+1$ and $d\colon n+1 \to m$ be arrows of $\mathbf{FOB}_{\mathbb{T}}$. Thus $\mathcal{H}^\sharp(c) = \varnothing$ and $\mathcal{H}^\sharp(d) = \varnothing$.*

Proof. Recall that for any interpretation $\mathcal{I}$, $\mathcal{I}^\sharp(c) \subseteq X^n \times X^{m+1} = X^n \times X^m \times X$. For $\mathcal{H}$, $X = \varnothing$ by Lemma G.9 and thus $\mathcal{H}^\sharp(c) \subseteq \varnothing \times \varnothing^n \times \varnothing^m$, i.e., $\mathcal{H}^\sharp(c) = \varnothing$. The proof for $\mathcal{H}^\sharp(d)$ is identical. $\square$

Lemma G.11. *Let $\mathbb{T}$ be a trivial theory that is syntactically complete. Let $c\colon 0 \to 0$ be an arrow of $\mathbf{FOB}_{\mathbb{T}}$. If $\mathcal{H}^\sharp(c) = \{(\star,\star)\}$ then $c =_{\mathbb{T}} id_0^\circ$.*

Proof. We proceed by induction on c.
For the base cases, there are only four constants $c\colon 0 \to 0$.

- $c = id_0^\circ$. Then, it is trivial.
- $c = id_0^\bullet$. Then $\mathcal{H}^\sharp(c) = \varnothing$ against the hypothesis.
- $c = R^\circ$. If $\mathcal{H}^\sharp(R^\circ) = \{(\star,\star)\}$, then by definition of $\mathcal{H}$, $id_0^\circ =_{\mathbb{T}} R^\circ$.
- $c = R^\bullet$. If $\mathcal{H}^\sharp(R^\circ) = \{(\star,\star)\}$, then by definition of $\mathcal{H}^\sharp$, $\{(\star,\star)\} \notin \rho(R)$. Thus, by definition of $\mathcal{H}$, $id_0^\circ \not\lesssim_{\mathbb{T}} R^\circ$. Since $\mathbb{T}$ is syntactically complete $id_0^\circ \lesssim_{\mathbb{T}} R^\bullet$.

We now consider the usual four inductive cases.

- $c = c_1 \otimes c_2$. Since $c\colon 0 \to 0$, then also c_1 and c_2 have type $0 \to 0$. By definition, $\mathcal{H}^\sharp(c) = \mathcal{H}^\sharp(c_1) \otimes \mathcal{H}^\sharp(c_2)$. By definition of $\otimes$ in **Rel** both $\mathcal{H}^\sharp(c_1)$ and $\mathcal{H}^\sharp(c_2)$ must be $\{(\star,\star)\}$. We can thus apply the inductive hypothesis to deduce that $c_1 =_{\mathbb{T}} id_0^\circ$ and $c_2 =_{\mathbb{T}} id_0^\circ$. Therefore $c = c_1 \otimes c_2 =_{\mathbb{T}} id_0^\circ \otimes id_0^\circ =_{\mathbb{T}} id_0^\circ$.
- $c = c_1 \,⨾\, c_2$. There are two possible cases: either $c_1\colon 0 \to n+1$ and $c_2\colon n+1 \to 0$, or $c_1\colon 0 \to 0$ and $c_2\colon 0 \to 0$. In the former case, we have by Lemma G.10, that $\mathcal{H}^\sharp(c) = \mathcal{H}^\sharp(c_1) ⨾ \mathcal{H}^\sharp(c_2) = \varnothing ⨾ \varnothing = \varnothing$. Against the hypothesis. Thus the second case should hold: $c_1\colon 0 \to 0$ and $c_2\colon 0 \to 0$. In this case we just observe that $c_1 \,⨾\, c_2$ is, by the laws of symmetric monoidal categories, equal to $c_1 \otimes c_2$. We can thus reuse the proof of the point above.
- $c = c_1 \,\bullet\, c_2$. Since $c\colon 0 \to 0$, then also c_1 and c_2 have type $0 \to 0$. Consider the case where $\mathcal{H}^\sharp(c_1) = \varnothing = \mathcal{H}^\sharp(c_2)$. Thus $\mathcal{H}^\sharp(c) = \varnothing$, against the hypothesis. Therefore either $\mathcal{H}^\sharp(c_1) = \{(\star,\star)\}$ or $\mathcal{H}^\sharp(c_2) = \{(\star,\star)\}$. If $\mathcal{H}^\sharp(c_1) = \{(\star,\star)\}$, then by induction hypothesis $c_1 =_{\mathbb{T}} id_0^\circ$. Therefore $c = c_1 \bullet c_2 = c_1 \sqcup c_2 =_{\mathbb{T}} id_0^\circ \sqcup c_2 =_{\mathbb{T}} \top \sqcup c_2 =_{\mathbb{T}} \top =_{\mathbb{T}} id_0^\circ$. The case for $\mathcal{H}^\sharp(c_2) = \{(\star,\star)\}$ is symmetric.
- $c = c_1 \,;\, c_2$. There are two possible cases: either $c_1\colon 0 \to n+1$ and $c_2\colon n+1 \to 0$, or $c_1\colon 0 \to 0$ and $c_2\colon 0 \to 0$. In the former case, we have by Lemma 7.5 that $c_1 =_{\mathbb{T}} ¡_{n+1}^\bullet$ and $c_2 =_{\mathbb{T}} !_{n+1}^\circ$. Thus $c =_{\mathbb{T}} ¡_{n+1}^\bullet \,;\, !_{n+1}^\circ =_{\mathbb{T}} id_0^\circ$. For the last equivalence observe that $id_0^\circ \lesssim_{\mathbb{T}} ¡_{n+1}^\bullet \,;\, !_{n+1}^\circ$ since $(¡_{n+1}^\bullet)^\perp = !_{n+1}^\circ$. The

other inclusion is $¡^{\bullet}_{n+1} \mathbin{;} !^{\circ}_{n+1} \cong_{\mathbb{T}} (¡^{\bullet}_{n+1} \mathbin{;} !^{\circ}_{n+1}) \fatsemi id^{\circ}_0 \overset{\text{Def. }!^{\circ}}{\cong_{\mathbb{T}}} (¡^{\bullet}_{n+1} \mathbin{;} !^{\circ}_{n+1}) \fatsemi !^{\circ}_0 \overset{(!^{\circ}\text{-nat})}{\cong_{\mathbb{T}}} !^{\circ}_0 \overset{\text{Def. }!^{\circ}}{\cong_{\mathbb{T}}} id^{\circ}_0$. Consider now the case where $c_1 \colon 0 \to 0$ and $c_2 \colon 0 \to 0$. In this case $c_1 \mathbin{;} c_2$ is, by the laws of symmetric monoidal categories, equal to $c_1 \circledast c_2$. We can thus reuse the proof of the point above.

□

Lemma G.12. *Let $\mathbb{T}$ be a trivial theory that is syntactically complete. Let $c \colon 0 \to 0$ be an arrow of* $\mathbf{FOB}_{\mathbb{T}}$. *If $\mathcal{H}^{\sharp}(c) = \varnothing$ then $c =_{\mathbb{T}} id^{\bullet}_0$.*

Proof. If $\mathcal{H}^{\sharp}(c) = \varnothing$, then by Lemma G.2, $\mathcal{H}^{\sharp}(\overline{c}) = \overline{\varnothing} = \{(\star,\star)\}$. Thus by Lemma G.11, $\overline{c} =_{\mathbb{T}} id^{\circ}_0$ and thus $c =_{\mathbb{T}} id^{\bullet}_0$. □

Proposition G.13. *if $\mathbb{T}$ is trivial, syntactically complete and non-contradictory, then $\mathcal{H}$ is a model. Namely, for all $c, d \colon n \to m$ in* $\mathbf{FOB}_{\Sigma}$, *if $c \lesssim_{\mathbb{T}} d$, then $\mathcal{H}^{\sharp}(c) \subseteq \mathcal{H}^{\sharp}(d)$.*

Proof. Recall that by definition $\mathcal{H}$ is a model iff for all $c, d \colon n \to m$ in $\mathbf{FOB}_{\Sigma}$, if $c \lesssim_{\mathbb{T}} d$, then $\mathcal{H}^{\sharp}(c) \subseteq \mathcal{H}^{\sharp}(d)$. We prove that if $\mathcal{H}^{\sharp}(c) \not\subseteq \mathcal{H}^{\sharp}(d)$, then $c \not\lesssim_{\mathbb{T}} d$.

If $c \colon n \to m{+}1$ or $c \colon n{+}1 \to m$, then by Lemma G.10, $\mathcal{H}^{\sharp}(c) = \varnothing$ and thus it is not the case that $\mathcal{H}^{\sharp}(c) \not\subseteq \mathcal{H}^{\sharp}(d)$. Thus we need to consider only the case where $c, d \colon 0 \to 0$.

For $c, d \colon 0 \to 0$ if $\mathcal{H}^{\sharp}(c) \not\subseteq \mathcal{H}^{\sharp}(d)$, then $\mathcal{H}^{\sharp}(c) = \{(\star,\star)\}$ and $\mathcal{H}^{\sharp}(d) = \varnothing$. By Lemmas G.11 and G.12, we thus have that $c =_{\mathbb{T}} id^{\circ}_0$ and $d =_{\mathbb{T}} id^{\bullet}_0$. Since $\mathbb{T}$ is non-contradictory, then $c \not\lesssim_{\mathbb{T}} d$. □

Proof of (Prop). Since $\mathbb{T} = (\Sigma, \mathbb{I})$ is non-contradictory, by Proposition G.4 there exists a syntactically complete non-contradictory theory $\mathbb{T}' = (\Sigma, \mathbb{I}')$ such that $\mathbb{I} \subseteq \mathbb{I}'$. Since $¡^{\circ}_1 \lesssim_{\mathbb{T}} ¡^{\bullet}_1$, then $¡^{\circ}_1 \lesssim_{\mathbb{T}'} ¡^{\bullet}_1$, $\mathbb{T}'$ is also trivial. We can thus use Proposition G.13, to deduce that $\mathbb{T}'$ has a model. Since $\mathbb{I} \subseteq \mathbb{I}'$, then by Lemma 7.12, also $\mathbb{T}$ has a model. □

Proposition G.14. (General) *entails Theorem 3.2.*

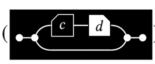
Proof. Assuming that (General) holds, one can prove that, for all theories $\mathbb{T} = (\Sigma, \mathbb{I})$ and diagrams $c \colon 0 \to 0$ in $\mathbf{FOB}_{\Sigma}$,

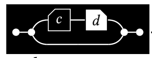
$$\text{if, for all models } \mathcal{I} \text{ of } \mathbb{T}, \{(\star,\star)\} \subseteq \mathcal{I}^{\sharp}(c) \text{ then } id^{\circ}_0 \lesssim_{\mathbb{T}} c. \tag{22}$$

Suppose indeed that $id^{\circ}_0 \not\lesssim_{\mathbb{T}} c$. Then, by Corollary 7.8, $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id^{\circ}_0, \overline{c})\})$ is non-contradictory. Thus, by (General), $\mathbb{T}'$ has a model, namely, there exists a morphism of fo-bicategories $\mathcal{G} \colon \mathbf{FOB}_{\mathbb{T}'} \to \mathbf{Rel}$. By Lemma 7.12, we have a morphism $\mathcal{F} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{FOB}_{\mathbb{T}'}$ and thus we have a model $\mathcal{F};\mathcal{G} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{Rel}$. Observe that since $\mathcal{G}$ is a model of $\mathbb{T}'$, then $\mathcal{G}(\overline{[c]_{\cong_{\mathbb{T}'}}}) = \{(\star,\star)\}$ and, by construction of $\mathcal{F}$, $\mathcal{F};\mathcal{G}(\overline{[c]_{\cong_{\mathbb{T}}}}) = \{(\star,\star)\}$. By Lemmas C.2 and D.1, $\mathcal{F};\mathcal{G}([c]_{\cong_{\mathbb{T}}}) = \varnothing$. Thus, there is a model assigning $\varnothing$ to c, against the hypothesis of (22).

By (22) and Lemma 6.6 one can easily conclude Theorem 3.2.

Consider a theory $\mathbb{T} = (\Sigma, \emptyset)$ for some monoidal signature Σ. Let $c, d \colon n \to m$ be diagrams in $\mathbf{FOB}_{\Sigma}$. For any interpretation $\mathcal{I}$, if $\mathcal{I}^{\sharp}(c) \subseteq \mathcal{I}^{\sharp}(d)$ then, $\mathbf{Rel}$ is a fo-bicategory and Lemma 6.6, it holds that

$$\{(\star,\star)\} \subseteq \mathcal{I}^{\sharp}(\boxed{}) \subseteq \mathcal{I}^{\sharp}($$  $$).$$

If, for all $\mathcal{I}$, $\mathcal{I}^{\sharp}(c) \subseteq \mathcal{I}^{\sharp}(d)$ then, by (22), $\boxed{} \lesssim_{\mathbb{T}}$ . Again, by Lemma 6.6, $c \lesssim_{\mathbb{T}} d$. Since $\mathbb{T} = (\Sigma, \emptyset)$, $c \lesssim d$. □

G.4 Proofs for Corollary 8.7

Proposition G.15. *For all expressions E of* CR_{Σ} *and interpretations $\mathcal{I}$, $\langle E \rangle_{\mathcal{I}} = \mathcal{I}^{\sharp}(\mathcal{E}(E))$.*

Proof. The proof is by induction on E. The base cases are trivial. The inductive cases are shown below.

$$\begin{aligned}
\mathcal{I}^{\sharp}(\mathcal{E}(E_1 \fatsemi E_2)) &\overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}(\mathcal{E}(E_1) \fatsemi \mathcal{E}(E_2)) \\
&\overset{(8)}{=} \mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \fatsemi \mathcal{I}^{\sharp}(\mathcal{E}(E_2)) \\
&\overset{\text{Ind. hyp.}}{=} \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}} \\
&\overset{(4)}{=} \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{\sharp}(\mathcal{E}(E_1 \mathbin{;} E_2)) &\overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}(\mathcal{E}(E_1) \mathbin{;} \mathcal{E}(E_2)) \\
&\overset{(8)}{=} \mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \mathbin{;} \mathcal{I}^{\sharp}(\mathcal{E}(E_2)) \\
&\overset{\text{Ind. hyp.}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{;} \langle E_2 \rangle_{\mathcal{I}} \\
&\overset{(4)}{=} \langle E_1 \mathbin{;} E_2 \rangle_{\mathcal{I}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{\sharp}(\mathcal{E}(E_1 \cap E_2)) &\overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}(\blacktriangleleft^{\circ}_1 \fatsemi (\mathcal{E}(E_1) \otimes \mathcal{E}(E_2)) \fatsemi \blacktriangleright^{\circ}_1) \\
&\overset{(8)}{=} \mathcal{I}^{\sharp}(\blacktriangleleft^{\circ}_1) \fatsemi (\mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \otimes \mathcal{I}^{\sharp}(\mathcal{E}(E_2))) \fatsemi \mathcal{I}^{\sharp}(\blacktriangleright^{\circ}_1) \\
&\overset{(8)}{=} \blacktriangleleft^{\circ}_X \fatsemi (\mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \otimes \mathcal{I}^{\sharp}(\mathcal{E}(E_2))) \fatsemi \blacktriangleright^{\circ}_X \\
&\overset{\text{Ind. hyp.}}{=} \blacktriangleleft^{\circ}_X \fatsemi (\langle E_1 \rangle_{\mathcal{I}} \otimes \langle E_2 \rangle_{\mathcal{I}}) \fatsemi \blacktriangleright^{\circ}_X \\
&\overset{(12)}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}} \\
&\overset{(4)}{=} \langle E_1 \cap E_2 \rangle_{\mathcal{I}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{\sharp}(\mathcal{E}(E_1 \cup E_2)) &\overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}(\blacktriangleleft^{\bullet}_1 \mathbin{;} (\mathcal{E}(E_1) \circledast \mathcal{E}(E_2)) \mathbin{;} \blacktriangleright^{\bullet}_1) \\
&\overset{(8)}{=} \mathcal{I}^{\sharp}(\blacktriangleleft^{\bullet}_1) \mathbin{;} (\mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \circledast \mathcal{I}^{\sharp}(\mathcal{E}(E_2))) \mathbin{;} \mathcal{I}^{\sharp}(\blacktriangleright^{\bullet}_1) \\
&\overset{(8)}{=} \blacktriangleleft^{\bullet}_X \mathbin{;} (\mathcal{I}^{\sharp}(\mathcal{E}(E_1)) \circledast \mathcal{I}^{\sharp}(\mathcal{E}(E_2))) \mathbin{;} \blacktriangleright^{\bullet}_X \\
&\overset{\text{Ind. hyp.}}{=} \blacktriangleleft^{\bullet}_X \mathbin{;} (\langle E_1 \rangle_{\mathcal{I}} \circledast \langle E_2 \rangle_{\mathcal{I}}) \mathbin{;} \blacktriangleright^{\bullet}_X \\
&\overset{(13)}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} \\
&\overset{(4)}{=} \langle E_1 \cup E_2 \rangle_{\mathcal{I}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{\sharp}(\mathcal{E}(E^{\dagger})) &\overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}((\mathcal{E}(E))^{\dagger}) \\
&\overset{\text{Lemma C.2}}{=} (\mathcal{I}^{\sharp}(\mathcal{E}(E)))^{\dagger} \\
&\overset{\text{Ind. hyp.}}{=} \langle E \rangle^{\dagger}_{\mathcal{I}} \\
&\overset{(4)}{=} \langle E^{\dagger} \rangle_{\mathcal{I}}
\end{aligned}$$

$$\mathcal{I}^{\sharp}(\mathcal{E}(\overline{E})) \overset{\text{Table 3}}{=} \mathcal{I}^{\sharp}(\overline{(\mathcal{E}(E))})$$

$$\begin{aligned}
&\overset{\text{Def. }\overline{(\cdot)}}{=} \mathcal{I}^\sharp(((\mathcal{E}(E))^\perp)^\dagger)\\
&\overset{\text{Lemmas C.2,D.1}}{=} (\mathcal{I}^\sharp(\mathcal{E}(E))^\perp)^\dagger\\
&\overset{\text{Ind. hyp.}}{=} ((\langle E\rangle_{\mathcal{I}})^\perp)^\dagger\\
&\overset{\text{Def. }\overline{(\cdot)}}{=} \overline{\langle E\rangle_{\mathcal{I}}}\\
&\overset{(4)}{=} \langle \overline{E}\rangle_{\mathcal{I}}
\end{aligned}$$

□

Proof of Corollary 8.7.

$$\begin{aligned}
E_1 \leq_{\mathrm{CR}} E_2 &\iff \forall \mathcal{I}.\ \langle E_1\rangle_{\mathcal{I}} \subseteq \langle E_2\rangle_{\mathcal{I}} && \text{(Def. of } \leq_{\mathrm{CR}})\\
&\iff \forall \mathcal{I}.\ \mathcal{I}^\sharp(\mathcal{E}(E_1)) \subseteq \mathcal{I}^\sharp(\mathcal{E}(E_2)) && \text{(Prop. G.15)}\\
&\iff \mathcal{E}(E_1) \leqq \mathcal{E}(E_2) && \text{(Def. of } \leqq)\\
&\iff \mathcal{E}(E_1) \lesssim \mathcal{E}(E_2) && \text{(Theorem 3.2)}
\end{aligned}$$

□

H SOME WELL KNOWN FACTS ABOUT CHAINS IN A LATTICE

A *chain* on a complete lattice $(L, \sqsubseteq)$ is a family $\{x_i\}_{i\in I}$ of elements of L indexed by a linearly oredered set I such that $x_i \sqsubseteq x_j$ whenever $i \leq j$. A monotone map $f : L \to L$ is said to *preserve chains* if

$$f(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f(x_i)$$

We write $id : L \to L$ for the identity function and $f \sqcup g : L \to L$ for the pointwise join of $f : L \to L$ and $g : L \to L$, namely $f \sqcup g(x) \overset{\text{def}}{=} f(x) \sqcup g(x)$ for all $x \in L$. For all natural numbers $n \in \mathbb{N}$, we define $f^n : L \to L$ inductively as $f^0 = id$ and $f^{n+1} = f^n ; f$. We fix $f^\omega \overset{\text{def}}{=} \bigsqcup_{n\in\mathbb{N}} f^n$.

Lemma H.1. *Let $f, g : L \to L$ be monotone maps preserving chains. Then*

1. *$id : L \to L$ preserves chains;*
2. *$f \sqcup g : L \to L$ preserves chains;*
3. *$f^\omega : L \to L$ preserves chains.*

Proof. (1) Trivial.

(2) By hypothesis we have that $f(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f(x_i)$ and $g(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} g(x_i)$. Thus

$$\begin{aligned}
f \sqcup g(\bigsqcup_{i\in I} x_i) &= f(\bigsqcup_{i\in I} x_i) \sqcup g(\bigsqcup_{i\in I} x_i)\\
&= \bigsqcup_{i\in I} f(x_i) \sqcup \bigsqcup_{i\in I} g(x_i)\\
&= \bigsqcup_{i\in I} (f(x_i) \sqcup g(x_i))\\
&= \bigsqcup_{i\in I} (f \sqcup g)(x_i)
\end{aligned}$$

(3) We prove $f^n(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f^n(x_i)$ for all $n \in \mathbb{N}$. We proceed by induction on n.
For $n = 0$, $f^0(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} x_i = \bigsqcup_{i\in I} f^0(x_i)$.

For $n + 1$, we use the hypothesis that f preserves chain and thus

$$\begin{aligned}
f^{n+1}((\bigsqcup_{i\in I} x_i) &= f(\, f^{n+1}((\bigsqcup_{i\in I} x_i)\,)\\
&= f(\bigsqcup_{i\in I} f^n(x_i)\,) && \text{(induction hypothesis)}\\
&= \bigsqcup_{i\in I} f(\, f^n(x_i)\,)\\
&= \bigsqcup_{i\in I} f^{n+1}(x_i)
\end{aligned}$$

□

Lemma H.2. *Let $f, g : L \to L$ be monotone maps preserving chains such that $g \sqsubseteq f$. Then $f^\omega ; g \sqsubseteq f^\omega$*

Proof. For all $x \in L$, $f^\omega ; g(x) = g(\bigsqcup_{n\in\mathbb{N}} f^n(x)) = \bigsqcup_{n\in\mathbb{N}} g(f^n(x)) \sqsubseteq \bigsqcup_{n\in\mathbb{N}} f^{n+1}(x) \sqsubseteq \bigsqcup_{n\in\mathbb{N}} f^n(x) = f^\omega(x)$. □

Lemma H.3. *Let $f : L \to L$ be a monotone map preserving chains. Thus $f^\omega = f^\omega ; f^\omega$*

Proof. $f^\omega = f^\omega ; id \sqsubseteq f^\omega ; f^\omega$. For the other direction we prove that $f^\omega ; f^n \sqsubseteq f^\omega$ for all $n \in \mathbb{N}$. We proceed by induction on n. For $n = 0$ is trivial. For $n + 1$, $f^\omega ; f^{n+1} = f^\omega ; f^n ; f \sqsubseteq f^\omega ; f \sqsubseteq f^\omega$. For the last inequality we use Lemma H.2. □

Lemma H.4. *Let $f, g : L \to L$ be monotone maps preserving chains. Then $(f \sqcup g)^\omega = (f^\omega \sqcup g)^\omega$*

Proof. Since $f = f^1 \sqsubseteq f^\omega$ and since $(\cdot)^\omega$ is monotone, it holds that $(f \sqcup g)^\omega \sqsubseteq (f^\omega \sqcup g)^\omega$.

For the other inclusion, we prove that $(f^\omega \sqcup g)^n \sqsubseteq (f \sqcup g)^\omega$ for all $n \in \mathbb{N}$. We proceed by induction on $n \in \mathbb{N}$. For $n = 0$, $(f^\omega \sqcup g)^0 = id \sqsubseteq (f \sqcup g)^\omega$.

For $n + 1$, observe that $f^\omega \sqsubseteq (f \sqcup g)^\omega$ and than $g \sqsubseteq (f \sqcup g)^\omega$. Thus

$$(f^\omega \sqcup g) \sqsubseteq (f \sqcup g)^\omega \tag{23}$$

We conclude with the following derivation.

$$\begin{aligned}
(f^\omega \sqcup g)^{n+1} &= (f^\omega \sqcup g)^n ; (f^\omega \sqcup g)\\
&\sqsubseteq (f \sqcup g)^\omega ; (f^\omega \sqcup g) && \text{(Induction Hypothesis)}\\
&\sqsubseteq (f \sqcup g)^\omega ; (f \sqcup g)^\omega && \text{((23))}\\
&= (f \sqcup g)^\omega && \text{(Lemma H.3)}
\end{aligned}$$

□

H.1 Some well known facts about precongruence closure

Let $X = \{X[n, m]\}_{n,m\in\mathbb{N}}$ be a family of sets indexes by pairs of natural numbers $(n, m) \in \mathbb{N}\times\mathbb{N}$. A well-typed relation $\mathbb{R}$ is a family of relation $\{R_{n,m}\}_{n,m\in\mathbb{N}}$ such that each $R_{n,m} \subseteq X[n, m] \times X[n, m]$. We consider the set WTRel_X of well typed relations over X. It is easy to see that WTRel_X forms a complete lattice with join given by union $\cup$. Hereafter we fix an arbitrary well-typed relation $\mathbb{I}$ and the well-typed identity relation Δ.

We define the following monotone maps for all $\mathbb{R} \in \mathrm{WTRel}_X$:

- $(id) : \mathrm{WTRel}_X \to \mathrm{WTRel}_X$ defined as the identity function;

- $(\mathbb{I})\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as the constant function $\mathbb{R} \mapsto \mathbb{I}$;
- $(r)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as the constant function $\mathbb{R} \mapsto \Delta$;
- $(t)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x,z) \mid \exists y.(x,y) \in \mathbb{R} \wedge (y,z) \in \mathbb{R}\}$;
- $(s)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x,y) \mid (y,x) \in \mathbb{R}\}$;
- $(\fatsemi)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x_1 \fatsemi y_1, x_2 \fatsemi y_2) \mid (x_1,x_2) \in \mathbb{R} \wedge (y_1,y_2) \in \mathbb{R}\}$;
- $(\otimes)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x_1 \otimes y_1, x_2 \otimes y_2) \mid (x_1,x_2) \in \mathbb{R} \wedge (y_1,y_2) \in \mathbb{R}\}$;

Observe that the function (id), (r), (t), $(\fatsemi)$ and $(\otimes)$ are exactly the inference rules used in the definition of $\mathsf{pc}(\cdot)$ given in (10). Indeed the function $\mathsf{pc}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ can be decomposed as

$$\mathsf{pc}(\cdot) = (\, (id) \cup (r) \cup (t) \cup (\fatsemi) \cup (\otimes) \,)^{\omega}$$

where f^{ω} stands the ω-iteration of a map f defined in the standard way (see Appendix H for a definition).

Similarly the congruence closure $\mathsf{c}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ can be decomposed as

$$\mathsf{c}(\cdot) = (\, (id) \cup (r) \cup (t) \cup (s) \cup (\fatsemi) \cup (\otimes) \,)^{\omega}$$

These decompositions allow us to prove several facts in a modular way. For instance, to prove that $\mathsf{pc}(\cdot)$ preserves chains is enough to prove the following.

Lemma H.5. *The monotone maps (id), $(\mathbb{I})$, (r), (s), (t), $(\fatsemi)$ and $(\otimes)$ defined above preserve chains.*

Proof. All the proofs are straightforward, we illustrate as an example the one for $(\otimes)$.

Let I be a linearly ordered set and $\{\mathbb{R}_i\}_{i\in I}$ be a family of well-typed relations such that if $i \le j$, then $R_i \subseteq R_j$. We need to prove that $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) = \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$.

The inclusion $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) \supseteq \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$ trivially follows from monotonicity of $(\otimes)$ and the universal property of union. For the inclusion $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) \subseteq \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$, we take an arbitrary $(a,b) \in (\otimes)(\bigcup_{i\in I}\mathbb{R}_i)$. By definition of $(\otimes)$, there exist x_1, x_2, y_1, y_2 such that

$$a = x_1 \otimes y_1 \quad b = x_2 \otimes y_2 \quad (x_1,x_2) \in \bigcup_{i\in I}\mathbb{R}_i \quad (y_1,y_2) \in \bigcup_{i\in I}\mathbb{R}_i$$

By definition of union, there exist $i,j \in I$ such that $(x_1,y_1) \in R_i$ and $(x_2,y_2) \in R_j$. Since I is linearly ordered, there are two cases: either $i \le j$ or $i \ge j$.

If $i \le j$, then $R_i \subseteq R_j$ and thus $(x_1,y_1) \in R_j$. By definition of $(\otimes)$, we have $(x_1 \otimes x_2, y_1 \otimes y_2) \in R_j$ and thus $(a,b) \in R_j$. Since $R_j \subseteq \bigcup_{i\in I}\mathbb{R}_i$, then $(a,b) \in \bigcup_{i\in I}\mathbb{R}_i$. The case for $j \le i$ is symmetric. ◻

Proposition H.6. *The monotone maps $\mathsf{pc}(\cdot), \mathsf{c}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ preserve chains.*

Proof. Follows immediately from Lemma H.5 and Lemma H.1 in Appendix H. ◻

Lemma H.7. *For all well-typed relations $\mathbb{J}$, the map $\mathsf{pc}(\mathbb{J} \cup \cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ preserves chains.*

Proof. Follows immediately from Lemma H.5 and Lemma H.1 in Appendix H. ◻

Lemma H.8. *For all well-typed relations $\mathbb{I}$ and $\mathbb{J}$, $\mathsf{pc}(\mathbb{I} \cup \mathbb{J}) = \mathsf{pc}(\mathsf{pc}(\mathbb{I}) \cup \mathbb{J})$*

Proof. Let $(\mathbb{J})\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ be the constant function to $\mathbb{J}$ and define $f,g\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ as

$$f \stackrel{\text{def}}{=} (id) \cup (r) \cup (t) \cup (\fatsemi) \cup (\otimes) \qquad g \stackrel{\text{def}}{=} (\mathbb{J})$$

From Lemma H.5 and Lemma H.1, both f and g preserve chains. Observe that $f^{\omega}(\mathbb{I}) = \mathsf{pc}(\mathbb{I})$, that $(f \cup g)^{\omega} = \mathsf{pc}(\mathbb{I} \cup \mathbb{J})$ and that $(f^{\omega} \cup g)^{\omega}(\mathbb{I}) = \mathsf{pc}(\mathsf{pc}(\mathbb{I}) \cup \mathbb{J})$. Conclude with Lemma H.4 in Appendix H. ◻

Lemma H.9. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a first order theory. Then $\lesssim_{\mathbb{T}} = \mathsf{pc}(\mathbb{FOB} \cup \mathbb{I})$*

Proof. By definition $\lesssim_{\mathbb{T}} = \mathsf{pc}(\lesssim \cup \mathbb{I})$. Recall that $\lesssim = \mathsf{pc}(\mathbb{FOB})$. Thus $\lesssim_{\mathbb{T}} = \mathsf{pc}(\mathsf{pc}(\mathbb{FOB}) \cup \mathbb{I})$. By Lemma H.8, $\lesssim_{\mathbb{T}} = \mathsf{pc}(\mathbb{FOB} \cup \mathbb{I})$. ◻

Lemma H.10. *Let I be a linearly ordered set and, for all $i \in I$, let $\mathbb{T}_i = (\Sigma, \mathbb{I}_i)$ be first order theories such that if $i \le j$, then $\mathbb{I}_i \subseteq \mathbb{I}_j$. Let $\mathbb{T}$ be the theory $(\Sigma, \bigcup_{i\in I}\mathbb{I}_i)$. Then $\lesssim_{\mathbb{T}} = \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$.*

Proof. By definition $\lesssim_{\mathbb{T}} = \mathsf{pc}(\lesssim \cup \bigcup_{i\in I}\mathbb{I}_i)$. Since $\mathbb{I}_i$ form a chain, by Lemma H.7, $\mathsf{pc}(\lesssim \cup \bigcup_{i\in I}\mathbb{I}_i) = \bigcup_{i\in I}\mathsf{pc}(\lesssim \cup \mathbb{I}_i)$. The latter is, by definition, $\bigcup_{i\in I} \lesssim_{\mathbb{I}_i}$. ◻

Lemma H.11. *Let I be a linearly ordered set and, for all $i \in I$, let $\mathbb{T}_i = (\Sigma_i, \mathbb{I})$ be first order theories such that if $i \le j$, then $\Sigma_i \subseteq \Sigma_j$. Let $\mathbb{T}$ be the theory $(\bigcup_{i\in I}\Sigma_i, \mathbb{I})$. Then $\lesssim_{\mathbb{T}} = \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$.*

Proof. By Lemma H.5, the monotone map $\mathsf{pcr}(\cdot) \stackrel{\text{def}}{=} (\, (id) \cup (\mathbb{I}) \cup (t) \cup (\fatsemi) \cup (\otimes) \,)^{\omega}$ preserves chains. Let Δ_i be the well-typed identity relation on $\mathbf{FOB}_{\Sigma_i}$. Observe that $\lesssim_{\mathbb{T}_i} = \mathsf{pcr}(\Delta_i)$ and that $\lesssim_{\mathbb{T}} = \mathsf{pcr}(\bigcup_{i\in I}\Delta_i)$. To summarise:

$$\begin{aligned}
\lesssim_{\mathbb{T}} &= \mathsf{pcr}(\bigcup_{i\in I}\Delta_i) \\
&= \bigcup_{i\in I}\mathsf{pcr}(\Delta_i) && \text{(preserve chains)} \\
&= \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}
\end{aligned}$$

◻

Lemma H.12. *Let I be a linearly ordered set and, for all $i \in I$, let $\mathbb{T}_i = (\Sigma_i, \mathbb{I}_i)$ be first order theories such that if $i \le j$, then $\Sigma_i \subseteq \Sigma_j$ and $\mathbb{I}_i \subseteq \mathbb{I}_j$. Let $\mathbb{T}$ be the theory $(\bigcup_{i\in I}\Sigma_i, \bigcup_{i\in I}\mathbb{I}_i)$. Then $\lesssim_{\mathbb{T}} = \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$.*

Proof. Immediate by Lemma H.11 and Lemma H.10. ◻

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Appendix (Article 5)

V

A.-V. Pietarinen, F. Bellucci, A. Bobrova, N. Haydon, and M. Shafiei. The blot. In A.-V. Pietarinen, P. Chapman, L. Bosveld-de Smet, V. Giardino, J. Corter, and S. Linker, editors, *Diagrammatic Representation and Inference*, pages 225–238, Cham, 2020. Springer International Publishing

The Blot

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Abstract. The blot is a sign in Peirce's diagrammatic syntax of existential graphs that has hitherto been neglected in the literature on logical graphs. It is needed in order to trigger the cut-as-negation to come out from the scroll, namely from the implicational sign of a positive implicational (paradisiacal) logic. Since the cut-as-negation presupposes the blot and the scroll, what does the blot represent? On the one hand, it stands for constant absurdity, but on the other hand, Peirce takes it to be an affirmative sign. This paper explores the blot and its logical and conceptual properties from the multiple perspectives of notation, rules of transformation, icons, and scriptibility of graphs. It explains the apparent conflict in the blot's meaning in its capacity of giving rise to the pseudo-graph that exploits positive character of absurdity. In effect, the blot is the mirror image of the sheet of assertion, not its complementation. On the sheet, it acts as a non-juxtaposable singularity.

Keywords: Blot · Pseudo-graph · Scroll · Existential graphs · Absurdity · Scriptibility

1 Introduction

The blot is a constant logical sign (the pseudograph) in Peirce's diagrammatic syntax of existential graphs. Studies of its nature and even the very existence have hitherto been neglected in the literature on logical graphs (with the sole exception of Roberts 1973, p. 36). The blot is needed in order to trigger the cut-as-negation to come out from the

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scroll, namely from the implicational sign of a positive implicational (what Peirce calls *paradisiacal*) logic. Since the cut-as-negation presupposes the blot and the scroll, what does the blot represent? On the one hand, it stands for constant absurdity, but on the other hand, Peirce takes it to be an affirmative sign. Either way, it is a pseudo-graph because it ought to be an "expression to which the interpreter shall be free to give any propositional meaning he pleases" (R 492, 1903). A pseudo-graph represents no possible or conceivable state of the universe.

This paper explores the blot and its logical and conceptual properties from the multiple perspectives of diagrammatic notation, rules of transformation, icons, and scriptibility of graphs. It explains the apparent conflict in the blot's meaning by its capacity of giving rise to the pseudo-graph that exploits the positive character of absurdity. In effect, the blot is the mirror image of the sheet of assertion, not its complementation. On the sheet, it acts as a non-juxtaposable singularity.

2 Peirce on the Blot

The blot was a new addition to Peirce's theory of existential graphs introduced during his preparation of the 1903 Lowell Lectures. In the unpublished "Logical Tracts. No. 1" (R 491) he described it (without yet naming it as the blot) as a pseudograph which "is a construction out of elements like those of graphs, but which, owing to the way in which these are put together, has no meaning as a diagram of the system to which it belongs". The need for it arises from the need of depicting absurdity in graphs in some suitably quasi-diagrammatic fashion. Substitute a pseudo-graph "What is false is true" in place of *c* in Fig. 4, and it may be read, "If *b* is true the false is true". This, Peirce states, "reduces *b* to absurdity, and is equivalent to a denial of *b*". He proposes to simplify the scribing of these graphs by making the inner enclosure "indefinitely small, or be suppressed; so that Fig. 2 denies *b*; and generally, a single enclosure has the effect of denying the whole graph which it contains". Hence, Peirce tells, "Fig. 3 asserts that *b* is true and *c* false; while Fig. 4 denies this, that is, asserts that either *b* is false or *c* is true, or, in other words, that if *b* is true, so is *c*" (*ibid*).[1]

Fig. 2

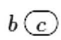


Fig. 3

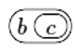


Fig. 4

In a long follow-up treatise also produced during 1903, entitled "Logical Tracts. No. 2" (R 492), Peirce explains the procedure by introducing "alogoid" conditional propositions, namely those that express "If anything, then everything":

> Whichever method of expressing conditionals be used, it will sometimes be desirable to place in one of the compartments a proposition either absurd or well-understood between the graphist and his interpreter to be false, which may be called an *alogoid* proposition (I prefer this form, because *alogous* might be wanted to mean logically absurd). If we say that two propositions which will always be true or false together are *equivalent*, then any alogoid proposition is equivalent to "If anything, then everything". For logic has no purpose

[1] The caption numberings in quotations preserve those in Peirce's original writings.

unless some consequence is false; and therefore this must be well-understood between the graphist and his interpreter.

Alogoid propositions are expressed by blackening the respective compartment within which the alogoid proposition is located:

In order to express an alogoid proposition, therefore, we need only an expression to which the interpreter shall be free to give any propositional meaning he pleases. Such an expression, introduced into our system of graphs, will not be a graph because it does not represent any possible state of the universe. I shall call it *the pseudograph*; for, however it be written, it remains the same in its equivalence. Since it is the assertion of all propositions, nothing can be added to it; and therefore it may be represented by blackening the whole compartment within which it is placed. Let this convention be adopted. The compartment so blackened may then be made very small or thin. Thus … Fig. 8 and Fig. 9 will express "If *a* is true, everything is true"; that is, "*a* is not true".

Fig. 8

Fig. 9

Fig. 10


Fig. 11

In practice, Fig. 10 would naturally be drawn in place of either Fig. 8 or Fig. 9. Following this practice, Fig. 11 will in either system be another way of writing the pseudograph. (*ibid.*)

Peirce soon formulates this idea as a specific convention of existential graphs:

Convention No. 10. *The* pseudograph*, or expression in this system of a proposition implying that every proposition is true, may be drawn as a black spot entirely filling the close in which it is.* Since the size of signs has no significance, the blackened close may be drawn invisibly small. Thus Fig. 33 as in Fig. 34, or even as in Fig. 35, Fig. 36, or lastly as in Fig. 37.

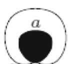


Fig. 33

Fig. 34

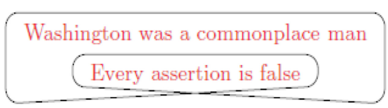
Fig. 35

Fig. 36

Fig. 37

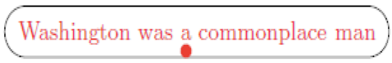
Interpretational Corollary 1. *A scroll with its contents having the pseudograph in the inner close is equivalent to the precise denial of the contents of the outer close.* (*ibid.*)

In the lecture notes related to this convention, Peirce had characterised the writing of the pseudograph on the sheet of assertion as "equivalent to burning up the sheet, since the sheet only exists, as such, in the minds of the graphist and the interpreter, and *that* by virtue of the agreement which the writing of the pseudograph destroys". He notes that it is nevertheless "useful to write the pseudograph in the inner close of a graph" (R 450). For example, the graph



says "If Washington was a commonplace man, then every assertion is false",[2] which is the same as to say that "Washington was not a commonplace man". Convention 10 tells that filling up a close leaves no room in it, which means that the pseudograph is inserted in the close. To deny that Washington was a commonplace man, Peirce scribes the corresponding graph as follows:

Since the size of a sep (the inner loop) is not a significant feature, Peirce scribes this equivalently as



"Making the loop infinitesimal", Peirce continues, "we shall understand a sep as denying what is written in its close" (R 450). In the related 1903 text "Syllabus of Logic" (R 478) Peirce described the "filling up of any entire area with whatever writing material (ink, chalk, etc.)" to amount to "*obliterating* that area". Notice that it is the area that is obliterated, not the loop itself. It follows from the obliteration as a corollary that, "[s]ince an obliterated area may be made indefinitely small, a single cut will have the effect of denying the entire graph in its area. For to say that if a given proposition is true, everything is true, is equivalent to denying that proposition" (R 478).

The pseudograph is the sign of nothing. Yet is asserts that "everything is true" (R 455). Peirce explains: "Were every graph asserted to be true, there would be nothing that could be added to that assertion", and that accordingly, "our expression for it may very appropriately consist in completely filling up the area on which it is asserted" (*ibid.*). Here (and this happens during his second Lowell Lecture), Peirce introduces the term "blot" for the first time: "Such filling up of an area may be termed a *blot*". We can learn from his notes that there are thus "two peculiar graphs": the *blank place* "which asserts only what is already well understood between us to be true, and the *blot* which asserts something well understood to be false" (*ibid.*). In addition, in the Alpha graphs one then only needs "two signs which are not graphs." First, "the putting of two graph-replicas upon the same area," where (recall that a blank is a graph), includes "the scribing of a single graph as a special case". Peirce rightly takes the idea that "scribing a graph is a transformation of a graph already accepted" to be a "very useful one" (*ibid.*). Second, the other sign is the *scroll*.

There are only a few further occasions in which Peirce revisits the blot and provides some further analyses and explanations of it. The idea of the blot surfaces in R 693 (1904) and in the related glossary of graphs in terms of *oppleted* graphs: "An area is said to be *oppleted*, or *opplete*, when it is virtually quite filled up, all graphs having replicas upon it. This is represented by completely blackening it. An enclosure whose area is

[2] The consequent should be "…then every assertion is true". The meaning of the "red blot" as "…then every assertion is false" comes from an earlier lecture draft (R 450), which Peirce soon in his next draft (R 455) corrects to the original meaning of the blot as in Convention 10.

opplete is equivalent to a blank". The last couple of definitions (33–40) in the glossary of 40 technical terms relate to this filling up of areas:

> An area so affected is said to be *opplete* (33) or to be *oppleted* (34) (from *opplëere*, to stuff up). Or we may prefer to say that it is the *annulus* (35), or annular space, comprising all that area except that occupied by the replica that effects the *oppletion* (36) that is *oppleted* (37). Or again, we may say that the enclosure in the area of which the *opplent* (38) replica occurs is *opplete* (39). Connected with this conception is that of a vacant *enclosure* (40), which is an enclosure whose area is entirely blank. (R S-26)

Another occasion is found among the many copy-texts and segments prepared for his 1906 "Prolegomena" paper but not included in the published version (R S-30, "Copy T"):

> [T]he Scroll affords me no other means of denying any Graph, say A, than by scribing that if A be true, everything is true. Now since it is impossible by any addition to increase Everything, this I can suitably express by completely filling with a blot the Inner Close of a Scroll that carries only A (and the Blank) in its Outer Close, so that there shall be no more room in that Inner Close for anything else.
>
> I can then make this blackened Inner Close as small as I please, at least, so long as I can still see it there, whether with my outer eye or in my mind's eye (Horatio). Can I not make it quite invisibly small, even to my mind's eye? "No", you will say, "for then it would not be scribed at all". You are right. Yet since confession will be good for my soul, and since it will be well for you to learn how like walking on smooth ice this business of reasoning about logic is,—so much so that I have often remarked that nobody commits what is called a "logical fallacy", or hardly ever does so, except logicians; and they are slumping into such stuff continually,—it is my duty to say that this error of assuming that, because the blackened Inner Close can be made indefinitely small, therefore it can be struck out entirely, like an infinitesimal. That led me to say that a Cut around a graph-instance has the effect of denying it. I retract: it only does so if the Cut encloses also a blot, however small, to represent iconically the blackened Inner Close. I was partly misled by the fact that in the Conditional *de inesse* the Cut may be considered as denying the contents of its Area. That is true, so long as the entire Scroll is on the Place. But that does not prove that a single Cut, without an Inner Close, has this effect. On the contrary, a single Cut, enclosing only A and a blank, merely says: "If A", or "If A, then" and there stops. If what? You ask. It does not say. "Then something follows", perhaps; but there is no assertion at all. This can be proved, too. For if we scribe on the Phemic Sheet the Graph expressing "If A is true, Something is true", we shall have a Scroll with A alone in the Outer Close, and with nothing but a Blank in the Inner Close. Now this Blank is an Iterate of the Blank-instance that is always present on the Phemic Sheet; and this may, according to the rule, be deiterated by removing the Blank in the inner close. This will do, what the blot would not; namely, it will cause the collapse of the Inner Close, and thus leaves A in a single cut. We thus see that a Graph, A, enclosed in a single Cut that contains nothing else but a Blank has no signification that is not implied in the proposition, "If A is true, Something is true".

This long passage from "Copy T" deserves a comment, in part because its reading of the single cut differs from the standard presentation (i.e. as simple negation) in existential graphs. The conditional is used in denying a graph, A, by scribing that "If A be true, everything is true". The consequent cannot be represented in Alpha graphs, because in

Alpha graphs there is no way to assert “everything”. In Beta (first-order) graphs, on the other hand, there is no way of quantifying over assertions. Peirce’s solution is to have the whole area of the inner close of the scroll saturated by the blot, which conveys the idea that nothing else can be added to that area (Fig. 1). The “blotted area” signifies that what is placed in it “is true”, but since nothing else can be added to the blotted area, nothing else in it is true, namely everything in it is true. Now “everything in it” amounts to “everything”, since no further specification needs to be given. The filling of the area of the inner close of the scroll is therefore an *icon* of the assertion “everything is true”.

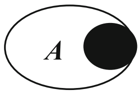


Fig. 1.

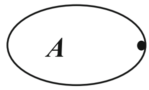


Fig. 2.

Peirce then explains that the “blotted inner close” of a scroll can be made infinitesimally small (Fig. 2) though it never completely disappears (it leaves two opposite turning points on the boundary, and so is not the “unknotted knot” in the sense of knot theory). The reason, he explains, is that a single cut (here taken in the sense of a simple closed boundary curve with no intersection points, that is, as the “trivial knot”) does not signify negation; negation can only be signified by a “blotted cut” (a scroll with a blotted inner close, however small). To show the difference between the single cut and the blotted cut he imagines a scroll like in Fig. 3:

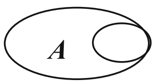


Fig. 3.

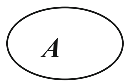


Fig. 4.

The inner close of the scroll in Fig. 3 contains a blank, which may be considered as the result of an iteration of the blank that lies outside of the scroll, which here as always may be the blank of the sheet. This is the new thought that Peirce develops in “Copy T”: The sheet means “Something is true,” and so does any portion of it that is the result of an application of the rule of iteration. The graph in Fig. 3 therefore means “If A, something is true.” But since it is iterated, the blank in the inner close of that scroll can also be de-iterated. What would be the result of such de-iteration? Peirce says that this will do something that the blot does not do: “it will cause the collapse of the Inner Close, and thus leaves A in a single cut” (R S-30). The de-iteration of the blank from the inner close of the scroll does not turn the graph in Fig. 3 into that of Fig. 1. It causes the *collapse* of the inner close, turning Fig. 3 into Fig. 4, that is, into the single cut. But then the single cut does not signify negation. It only signifies what the graph in Fig. 3 signifies before the de-iteration, namely “If A, then something is true.” This, Peirce suggests, amounts to the truncated statements “If A…” or “If A, then…” These are not complete assertions but non-well-formed, deformed parts that violate the grammar of the diagrammatic syntax. They do not mean the same as the negation of A, which is properly represented by the graphs as depicted in Figs. 1 and 2.

We may then restate the argument above in "Copy T" as follows. The primary notational function of the oval is to group propositions together. That is, it is a *collectional sign* like parentheses are in a non-diagrammatic syntax (R 430, 1902; R 670, 1911; Bellucci and Pietarinen 2016a, 2016b). In a system whose primitive operations are those of *conjunction* and *conditional*, collectional signs are only needed to distinguish the antecedents of the conditionals from their consequents (for conjunction is associative). The collectional oval is only needed in Alpha graphs in this role. In a scroll, the outer loop marks the area of the antecedent and the inner cut marks the area of the consequent. Thus the meaning of the graph in Fig. 4 is simply "If A, then…", because since there is no inner cut there is no consequent.

The meaning of the single cut is *purely collectional*. In a complete scroll, with the blot (the pseudograph or absurdum) appearing in the inner close, the meaning of negation is *added to* the collectional meaning of the cut, and this results in a sign of negation (the blotted cut). In other words, Peirce realised around 1906 that negation is represented in existential graphs by the blotted cut, and that the single cut simply functions as a collectional sign devoid of truth-functional meaning.

3 Positive and Negative Absurdity

We are not done yet. If there is a difference between a single cut and a blotted cut, however small or invisible this blot may be, what justifies using them interchangeably in the diagrammatic system, one of whose aims is to make the differences observable? Further analysis is needed in order to clarify the meaning of absurdity and accordingly the iconic generation of the cut.

First of all, the notion of absurdity is supposed to be a basis for that of negation; thus it itself has to be formed in a positive manner. When the system has only positive forms, that is, contains no notion of falsity, and no sign for negation, either, how can one express that a proposition, A, is false? Peirce's answer was to go on to assert, "If A is true, then everything is true". Such conditionals have no negation as a constituent. But notice that to say that "A is true only if everything is true" is also a rather strong *refutation* of the possibility of A being the case. Thus, to form negative propositions is to assign a sign for the proposition "Everything is true". This is what the blot does. Does it need to fill up the entire area then? As any instance of a graph in an area means its presence everywhere on that area (graphs can be scribed at any position in an area, i.e. all those positions are isotopy-equivalent), both the graphs in Figs. 5 and 6 equally express that absurdity implies P.

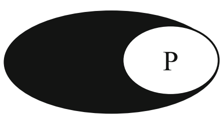


Fig. 5.

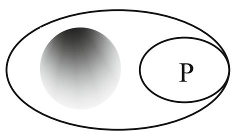


Fig. 6.

The blot as shown in Fig. 6 is the preferred notation, however, for one should distinguish between the blot and the scroll with both areas filled with black. Also, the blot should not be confused with a cut filled with black stuff. The blot with radiating, blurred boundaries is, we propose, to be preferred as the notation for it. This is consistent with Peirce's Convention 10, since we fill the inner close *with* a blot. A blot is *in* the inner close, it is not the filled inner close itself. The loop around the blot is not part of the blot. Roberts (1973, p. 36) describes the pseudo-graph to be "a cut entirely filled in, or blackened", but this is not the best possible choice of words.

The sign for negation, namely the scroll with a blot in its inner close, may be considered as a simple oval, the cut, since "the blackened close may be drawn invisibly small". The justification of this is not to be derived from the behavior of the permissive, deductive rules of transformation, since there is no cut in such language as yet. The blackened inner close remains on the boundary. However, if the aim of this language is to sustain diagrammatic syntax and the virtues of the iconization of reasoning (Bellucci and Pietarinen 2017), we should be wary of *apropos* conventions and remain mindful of the genealogy of the cut. What is it that justifies the equation between the cut and the scroll with a blot in its inner close? A further look at the notion of absurdity may be helpful here.

The definition of "negation of P" as "P implies absurdity" was known to Peirce since his 1885 "Algebra of Logic" paper. In the context of the further development of algebra into graphs, we find reasons for defining negation as a shorthand for P $\rightarrow$ $\perp$ becoming increasingly clear precisely when Peirce is moving on to an interpretation of absurdity as "Everything is true." This "positive" characterization of absurdity is one of Peirce's profound insights into negation. If the negation of P is to be understood as P implying any absurdity, this "negative" sense of absurdity, namely absurdity understood as any false (or necessarily false) proposition, introduces no real insight into the embryonic development of the idea of negation. In some sense, it *presupposes* negation, while at the same time being that from which negation is developed.

However, in order to explore all possible iconic possibilities we have to consider other conceptions of absurdity. An alternative to absurdity (taken as a proposition) is the statement "There is no truth" or "Nothing is true". How can we state "Nothing is true" in existential graphs? We have the sheet of assertion, which represents the truth. When nothing is scribed on any position on the sheet, the blank asserts "Something is true". As the sheet is the place for truths, perhaps we can show that there is no place for truth by "closing off" the sheet of assertion or some parts of it. There are two problems: how can one denote the collapse of the sheet of assertion? As soon as that is somehow done, one would be *asserting* that "Nothing is true". But we also need that as a proposition to be used in other graphs. Therefore, we need to separate the scopes and then collapse one of them. According to the rule of the scroll, the scroll with blank outer and inner closes can be scribed and erased around any graph. Thus any part of the sheet of assertion enclosed within such a scroll would mean the same as it did before. The graph  says "If something is true then something is true".

Now since the inner close is the place of truth, if we were to completely obliterate it, it would diagrammatise the state in which there is no place for truth. This would result in the proposition, "If something is true then nothing is true". The following sequence is intended to show how a proposition "If P then something is true" morphs into the proposition "If P then nothing is true" by obliterating the inner close (Fig. 7):

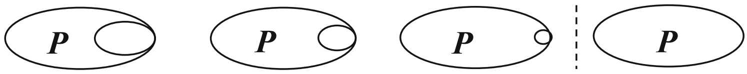


Fig. 7.

As far as the inner close exists and thus possesses a blank area, however small, the graph still means “If P then something is true”. But when the inner close is completely dissolved, the meaning changes to “If P then nothing is true”. Hence there is an equivocation in the above sequence; it does not display a meaning-preserving process of transformations.

Now we have a graph for “If P then nothing is true” but not a graph for “Nothing is true”. But “Nothing is true” is equivalent to “If something is true, then nothing is true”. Therefore the cut with a blank area is read “Nothing is true”. Provided that “If P, then nothing is true” is synonymous with the negation of P, then a cut with P in its scope means not-P, as does the scroll with P in the outer close and the blot in the inner close, which states “If P, then everything is true”.

To show something of the nature of absurdity by closing off some scopes is a realization of what Peirce had termed “unscriptibility” of some graphs in another slightly earlier and unpublished work of his (R 501, 1901; Ma and Pietarinen 2019). We propose to endow absurdity with this meaning. Both the blot and the collapsed inner close partake of the character of unscriptibility. Nothing is scriptible in a collapsed close: no space exists in a collapsed close at all. Nothing is scriptible in a close with the blot, either, since everything is already scribed in that blackened area. Miniaturising the inner close would not affect the character of unscriptiblity, since even if it were to dissolve into the boundary, the character of unscriptibility will be preserved. Therefore, although the sequence of graphs in Fig. 7 is not a meaning-preserving transformation, the one in Fig. 8 is:[3]

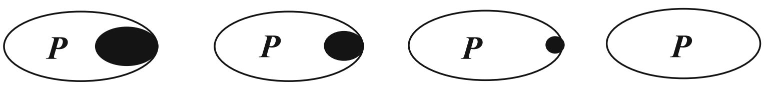


Fig. 8.

There are now two ways to introduce the cut. In the first, absurdity is “Everything is true”, in the other it is “Nothing is true”. Peirce’s preference lies with the former,

[3] In Peirce’s hand, a similar sequence looked like this (R 455(s)):

Peirce intended this to show that “the impossibility [that exists within the inloop] destroys the cut and all it contains” (*ibid.*). By this, Peirce is preparing ground for his decidability operations for the Alpha system (Roberts 1997).

since it *analyses* falsity and negation without assuming it. The proposition “Everything is true” will do that work well. Absurdity should be of the nature of affirmation, not denial. “Everything is true” is absurdity as an affirmative, “There is no truth” as a denial. Indeed “affirmation is psychically the simpler”, confirms Peirce, and “I therefore make the blot an affirmation”. That is, he makes the absurdity an affirmation and then equates it with the blot, namely “Everything is true”.

Taking absurdity as “Everything is true” has some other conceptual and formal advantages that we briefly list. (1) It explains *ex falso*: If everything is true then P also is true. There is no need for an axiom or a rule and no need to appeal to proofs by disjunctive syllogisms, which are known to be circular. (2) The Law of Excluded Middle (LEM) and the elimination of double cut are laws not inherent in the nature of negation, which is a desirable feature intuitionistically (Peirce came close to intuitionistic logic in many related senses).[4] (3) The double cut rule is to be derived, if justified, from more primitive, observational considerations. If the cut were defined as reversing its area, then the double cut rule would be immediate by symmetry. But symmetry, though advantageous in calculus, is an unfavorable guideline when the purpose is logical analysis (CP 4.375).

On the other hand, although the absurdities “Everything is true” and “There is no truth” are semantically equivalent, the latter is gotten from the former: rules like the elimination of double cut are not eligible at this level of analysis. From “Everything is true” it follows that “It is true that there is no truth”. But from “There is no truth” it follows, for example, that “It is wrong that something is wrong”, which means that everything is true. However, we need an extra move here. Thus from negative absurdity we cannot directly derive the positive absurdity (its justification would need another rule or an axiom, such as LEM). But from positive absurdity other facets of absurdity follow.

Another candidate for the meaning of absurdity is *unassertibility*: It is irrational to assert absurdity. Or, one may say that absurdity is whatever is rationally unassertible. How can we scribe such absurdity in graphs? How can we assert the unassertible? A meaningless sign or nothing would not do because they express nothing; we want to express absurdity when its assertion is rationally forbidden. It is not meaningless activity: it just has a meaning that is to be avoided at all costs. One has a right to be irrational, but penalties will be visited upon one who chooses to be so. Asserting the unassertible is possible but risky. In existential graphs, three candidates could be thought of: (1) to close off a loop by collapsing an area so that no space remains for any assertion, (2) to fill the area so that no assertion can fit there, (3) to police an area by flagging it, such as a cross mark $\times$, that forbids any assertion in that area. The last option is not that promising as one has to use an *ad hoc* mark for unassertibility, yielding little iconic harvest. The first two resort to diagrammatic unscriptibility to effect unassertibility. Peirce’s option was the second. Maybe something can be reaped from (1), too, as it preserves diagrammatic results and features no further conventions.

4 See Oostra (2010) on Alpha System with the scroll that agrees with propositional intuitionistic logic. In this case, new graph for disjunction needs to be introduced as in intuitionistic logic, logical connectives are not interdefinable. How such modifications demonstrate the potential insights of Peirce’s EGs has been discussed in Shafiei (2019). Moreover, Ma and Pietarinen (2018) have offered an EGs version for intuitionistic logic analyzing the nature of deep inference.

Considering the pragmatistic office of existential graphs, yet another conception for the absurdity may be proposed. In comparison to the pragmatistic motto "do not block the way of enquiry", we might say in existential graphs, and in logic as such, we have the principle: "do not block the way of inference". The way of inference would be blocked when we have a graph from which no consequences follow or a graph that cannot be antecedently motivated, i.e. that cannot be taken to be the consequence of an inference. This situation is exactly that of absurdity. Therefore, in order to show absurdity we show a case when inference is blocked. The smallest part of inference, an illation, is diagrammatized by the scroll. To obliterate the inner close would prevent the consequence being asserted. Therefore, if we put P in the outer close and close off the inner close it means "From P nothing follows" (not even itself), and this amounts to taking P as an expression of absurdity. The case has been discussed above. On the other hand, placing P in the inner close and closing off the outer oval, which results in a cut enclosing P, means that P follows from nothing, or that P is true under no assumptions. This equally amounts to taking P as absurd. Notice that here we are not saying "P is true under *any* assumption" which would be to consider P as a logical truth; rather we say "P is true under *no* assumption". In order to iconize the former one should leave the outer close, i.e. the place of assumption, blank and receptive for any graph; for the latter case one should totally close down the place of assumption, which is to dissolve the outer close in the boundaries of the inner close. Such new analysis now leads to a different type of a generation of the cut, but results in the same meaning as the previous ones did. In this new respect, the cut is the inner close of the scroll in whose boundaries the outer oval has been dissolved. (Peirce had a similar argument in another "Copy Text" of R S-30 not quoted in the previous section.) Notice that this case states that "Under no assumption P is true", which is different from "Under the assumption of nothing, P is true", for it is the place of assumptions that is obliterated instead of being filled with nothingness or absurdity.

4 The Blot and the Sheet of Assertion

This returns us back to some basic questions about notation. Why can a blank scroll be made to appear and disappear on the sheet? The sheet is a tautology and the blank scroll does not make any transformations of it. The sheet embraces all tautologies and true propositions that may ever be scribed on it.

There is one more element in the genealogy of negation to be pointed out. As briefly mentioned, Peirce proposes the original element of reasoning to be *paradisiacal* (R 493, c.1899; R 669, 1911): only the scroll is presented on the sheet. This positive, proto-reasoning operates without the presence of falsity or negation. Anything implies anything. Take one-valued logic (Hamblin 1967) or positive implicational logic as similarly paradisiacal proposals. There is not even any juxtaposition and hence no conjunction in positive implicational logic. In existential graphs, proto-reasoning has a paradisiacal scroll, in which the inner scroll may be blackened to contain all possible assertions, so that nothing could be added to it. Paradisiacal reasoning is in a highly unstable state, however, since "it will soon be recognized that not every assertion is true; and that once recognized, as soon as one notices that if a certain thing were true, every assertion would be true, one at once rejects the antecedent that lead to that absurd consequence" (R 669; Pietarinen 2015, p. 920). Any small perturbation and the blackened area atrophies to the first, primordial cut; a serpent appears in the paradise of pure reason.

Since the blot has the power of tipping the sheet off the equilibrium, the result is a scroll that promulgates cuts endowed with the meaning of negation. The blot is strictly speaking then not part of the logical vocabulary of the theory at all. It operates prior to the formation of logical systems (such as classical or some non-standard Alpha, Beta, etc.). The blot generates falsity and loses its signification and power (which are now hidden) in the process. Contrast this with the sheet of assertion. The blot has an opposite behavior to that of the sheet: white–black, blank–filled, scriptible–non-scriptible (see Ma and Pietarinen 2019). This area signifies the space of all possible consequences, which means that "non-scriptibility" is not identical to falsity or negation, and scriptibility is not identical to that of truth. These are Peirce's proposed generalisations of values (R 501). Likewise, the blot is not a logical complementation of the sheet. Both areas are positive. One more confirmation of this affirmative nature of the blot is found in a fragment of Peirce's late letter to J. Kehler 1911:

The simplest part of speech which this syntax contemplates, which, as scribed, I shall term a blot is itself an assertion. Ought it to be an affirmation or a denial? A denial is logically the simpler, because it implies merely that the utterer recognizes, however vaguely, some discrepancy between the fact and the speech, while an affirmation implies that he has examined all the implications of the latter and finds no discrepancy with the fact. This is a circumstance to be borne in mind; but since the denial implies recognition of the affirmation, while the affirmation is so far from implying recognition of the denial, that one might imagine a paradisaic state of innocence in which men never had the idea of falsity, and yet might reason, we must admit that affirmation is psychically the simpler. Now I think that upon this point we must prefer psychical to logical simplicity. I therefore make the blot an affirmation. (RL 376, 1911)

Strictly the blot is not placed on the sheet at all and thus is not to be asserted. Rather it is a mirror image of the sheet of assertion: assertible/non-assertible. The blot may appear to the field of vision from within the sheet, but it does so only when confined to the areas of the scroll. The sheet alone has no blots in it. What is more, any juxtaposition of a graph with the blot would result in an annihilation of that graph, including any blank graph such as the sheet. On the dark side of the sheet, there are no juxtapositions.[5]

Zaitsev and Grigoriev (2011) have proposed a generalisation of logical values beyond a Cartesian divide of them as either epistemological or ontological. Something of this sort is happening at Peirce's paradisiacal level of existential graphs. Zaitsev and Shramko (2013) call the truth-values from the ontological perspective "referential", and the truth-values treated as characteristic of statements involved in reasoning "inferential", which "means that a sentence is *taken as* (i.e., considered) true (and thus accepted) or false (and thus rejected)" (Zaitsev and Shramko 2013, p. 1302). A combination of two sets of truth-values has one of them interpreted referentially by $2^{\mathrm{T}} = \{\mathrm{T}, \mathrm{F}\}$ and the other inferentially by $2^{1} = \{1, 0\}$. In Peirce's case, this project should be read as a way (may

5 When things are unscriptible, it is even not clear whether deduction works as the right mode of reasoning in that dark realm (Peirce once talked about the mode of reasoning of "correction", which is not "deduction" when all propositions are unscriptible (Ma and Pietarinen 2019).

not be the only one) to clarify how the initial paradisiacal logic, which is implicational without juxtaposition, operates. These values are initially limited to two singletons {T} and {1}, since the paradisiacal state of mind is not acquainted with falsity. Now the value {1} may be assigned to the sheet while the value {T} may be assigned to the blot. These values are not juxtaposable, although the areas to which they are assigned are not unrelated: Something that is considered as true {1} is objectively true {T}. The consequence reminds one of stereotypical thinking in which agents' objective truth is aligned with anything they are about to observe. Paradisiacal scroll relates these two values, stating that anything that is scribed and *considered* true implies anything that has to be *objectively* true. The scroll is the connection between the white and the black sheet. The border of the scroll is the place where two types of truth meet, and can be treated as the limit case of {T, 1}.

5 Conclusion

This paper surveyed Peirce's notion of "blot" and explained some of its main characteristics: logical notation, two kinds of absurdities, paradisiacal logic of the scroll, obliterating loops, and the relation of the blot to the rules of transformations. A few ways forward along the last point may be added. As it stands, negation in existential graphs is often treated as an ordinary graph-instance where the other inference/transformation rules still apply (such as iteration and de-iteration, or modus ponens and modus tollens). This is fine if negation is meant as a type of complement or inversion of truth-values, but in the case of absurdity a pragmatic elucidation would have to go further, because absurdity suggests that the rules themselves, including both the permissive transformations and the conventions, begin to break down. This is where the notion of absurdity begins to have a deeper meaning, and it is here that the blot gives us a valuable way forward. One interesting feature of the blot is that because of its nature of being non-vacant, completely occupied area, adding a scroll (or scribing a scroll on top of the blot as it were) does nothing to the graph. What this means is that the ordinary dualities and symmetries start to degrade. Maybe the idea of deduction has to go over the board, too. What would be interesting is to trace this effect to the origins of the inference rules to see at which stage, when retrogressing towards our proto-logical paradise, the rules themselves start to degrade.

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2020–*The Blot* with Ahti-Veikko Pietarinen, Francesco Bellucci, Angeline Bobrova, and Mohammad Shafiei. Proceedings of Diagrammatic Representation and Inference: 11th International Conference Diagrams 2020. Springer. p. 225-238.

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