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**REDUCING PAIRWISE COMPARISONS IN ANALYTIC
HIERARCHY PROCESS USING TRANSITIVE PROPERTY**

Master's Thesis

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PhD

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**ANALÜÜTILISTE HIERARHIATE PROTSESSI
PAARISVÕRDLUSTE VÄHENDAMINE TRANSITIIVSE
OMADUSE ABIL**

Magistritöö

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PhD

Tallinn 2023

Author's Declaration of Originality

I hereby certify that I am the sole author of this thesis. All the used materials, references to the literature and the work of others have been referred to. This thesis has not been presented for examination anywhere else.

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10.05.2023

Abstract

The aim of this master's thesis is to determine the number of comparisons that could be calculated with the transitive property in analytical hierarchy process (AHP). Additionally, the thesis aims to evaluate the impact of using the transitive property on the quality metrics of AHP results, including consistency ratio and weights of alternatives.

Analytic hierarchy process, developed by Thomas L. Saaty, is one of the most widely used multi-criteria decision making methods. The major drawback with the AHP is the amount of work which is required in making all of the necessary pairwise comparisons, especially if there are more than recommended nine alternatives. For example, if 17 Sustainable Development Goals are compared, a total of 136 pairwise comparisons must be made. Depending on the complexity of the comparisons each pairwise comparison can take from seconds to minutes to evaluate. A business need arises to reduce the number of comparisons for various reasons. Firstly, it becomes more difficult and less likely for decision-maker to stay consistent with comparisons. Secondly, long surveys and repetitive questions in the AHP survey process can cause survey fatigue, which can lead to a decline in response quality towards the end of the process. Thirdly, there may be instances where decision-maker lacks the necessary knowledge or is unwilling to make a decision.

Based on the analysis of synthetic data sets using descriptive data analysis, it was concluded that depending on the number of alternatives, around 25 to 40% of comparisons could be missing in most situations, while still being able to calculate all the missing elements with the transitive property. Secondly, as the number of elements calculated using the transitive property increased, there was a corresponding increase of changes in the prioritization of alternatives and a decrease in the consistency ratio.

Researching these areas supports anyone who wants to use AHP for decision-making, especially in cases where it's necessary to compare more than nine alternatives.

Keywords: Analytic hierarchy process, pairwise comparisons, transitive property, simulations, synthetic data.

The thesis is written in English and is 59 pages long, including 5 chapters, 22 figures and 5 tables.

Annotatsioon

Analüütiliste hierarhiate protsessi paarisvõrdluste vähendamine transitiivse omaduse abil

Magistritöö eesmärk on selgitada välja, kui suur hulk analüütiliste hierarhiate protsessi (AHP) paarisvõrdlusi on võimalik arvutada transitiivse omadusega. Lisaks, on töö eesmärk hinnata transitiivse omaduse rakendamise mõju AHP tulemuste kvaliteedimõõdikutele, milleks on kooskõlaindeks ja alternatiivide kaalud.

Analüütiliste hierarhiate protsess, loodud Thomas L. Saaty poolt, on üks enimkasutatud hulgikriteeriumite analüüsi meetodeid. AHP rakendamise suureks takistuseks on töö maht, mis kulub kõikide paarisvõrdluste tegemiseks, eriti olukordades, kus soovitakse võrrelda rohkem alternatiive, kui meetodi soovituslik maksimaalne üheksa. Näiteks, 17 säästva arengu eesmärgi paarisvõrdlemiseks, tuleb kokku teha 136 paarisvõrdlust. Sõltuvalt paarisvõrdluste keerukusest, võib iga paarisvõrdlus aega võtta sekunditest kuni minutiteni. Sellest tulenevalt tekib mitmel põhjusel äriiline vajadus paarisvõrdluste vähendamiseks. Esiteks, mida rohkem on paarisvõrdlusi, seda keerulisem on otsustajal olla järjepidev oma vastustes. Teiseks, mahukas paarisvõrdluste arv ja sarnase ülesehitusega paarisvõrdlused võivad põhjustada küsimustiku täitmise väsimust, mistõttu vastuste kvaliteet võib ajas vähenema hakata. Kolmandaks, võib tekkida olukordi, kus otsustajal ei ole piisavalt informatsiooni paarisvõrdluse tegemiseks või ei ole nõus paarisvõrdlust tegema. Sünteetilistest andmekogudest ja kirjeldava statistika tulemustest järeldati, et sõltuvalt alternatiivide arvust, on transitiivse omadusega enamikel juhudel võimalik arvutada ligikaudu 25 kuni 40% puuduolevaid väärtuseid. Teiseks, mida rohkem elemente arvutatakse transitiivse omadusega, seda enam hakkavad muutuma alternatiivide kaalud ning kooskõlaindeks. Käesoleva magistritöö võib aidata kõiki neid, kes soovivad kasutada AHP otsuste langetamiseks, eelkõige olukordades, kus on vaja võrrelda üheksat või enam alternatiivi.

Märksõnad: Analüütiliste hierarhiate protsess, paarisvõrdlused, transitiivne omadus, simulatsioonid, sünteetilised andmed

Lõputöö on kirjutatud inglise keeles ning sisaldab teksti 59 leheküljel, 5 peatükki, 22 joonist, 5 tabelit.

List of Abbreviations and Terms

AHP	Analytic Hierarchy Process
ANP	Analytic Network Process
BPMSG	Business Performance Management Singapore
CI	Consistency Index
CR	Consistency Ratio
DSS	Decision Support Systems
FLTR	First-Level Transitivity Rule
GMIBM	Geometric Mean Induced Bias Matrix
MCDM	Multi-Criteria Decision Making
MLP	Multi-Layer Perceptron
NGT	Nominal Group Technique
RCI	Random Consistency Index
Alternative	One of AHP matrix elements that is being compared
Saaty Scale	Scale that ranges from 1/9 to 9 and is used in AHP
Survey fatigue	A lack or loss of interest in completing surveys
Synthetic database	Information that is artificially manufactured rather than generated by real-world events

Table of Contents

1	Introduction	9
1.1	Research problem and purpose	9
1.2	Experiment questions	10
1.3	Outline of the thesis	11
2	Theoretical framework	12
2.1	Analytic hierarchy process overview	12
2.1.1	Pairwise comparison process overview	13
2.1.2	Consistency ratio	13
2.2	Reasons to reduce alternatives and pairwise comparisons	14
2.2.1	Consistency	15
2.2.2	Time	16
2.2.3	Human capacity to process information	17
2.2.4	Survey fatigue	18
2.2.5	Decision-maker knowledge and behavior	18
2.3	Literature review	19
2.3.1	First-level transitivity rule method	19
2.3.2	Geometric mean induced bias matrix	19
2.3.3	Optimization methods	20
2.3.4	Assigning the alternatives into subsets	20
2.3.5	Decision to stop making pairwise comparisons	21
2.3.6	Balanced incomplete block design technique	21
2.3.7	Analytic network process	22
2.3.8	Integrating AHP with NGT	22
2.3.9	Neural network-based method	23
2.3.10	Revised geometric mean method	24
3	Methodology	25
3.1	Data collection method	25
3.1.1	Overview of research object	26
3.1.2	Creating synthetic data sets	26
3.1.3	Finding an acceptable number of elements to remove	28
3.1.4	Defining rounding ranges for Saaty scale's reciprocal values	30
3.1.5	Removing the elements and applying transitive property	31
3.1.6	Calculating the alternatives weights	33

3.1.7	Preparing the data for descriptive analysis	33
3.2	Data analysis method	34
4	Experiments results and findings	35
4.1	Number of comparisons that could be missing	35
4.2	Situations when to use transitive property	36
4.3	Consistency ratio change in case of 3 to 10 alternatives	37
4.4	Alternatives weights change in case of 3 to 10 alternatives	39
4.5	Consistency ratio change in case of 11 to 20 alternatives	40
4.6	Alternatives weights change in case of 11 to 20 alternatives	42
4.7	Data validation	44
5	Summary	48
5.1	Further research recommendations	49
	References	50
	Appendix 1 – Non-Exclusive License for Reproduction and Publication of a Graduation Thesis	53
	Appendix 2 – R Code for database creation	54
	Appendix 3 – R Code for removing and recalculating AHP matrix elements	55
	Appendix 4 – Various R codes to calculate attributes	57
	Appendix 5 – Various R codes for data visualization	58

List of Figures

1	<i>Graph of first differences in random inconsistency.</i>	15
2	<i>Graph of number of pairwise comparisons based on AHP alternatives. . .</i>	17
3	<i>Formula to compute a column vector L and a row vector R.</i>	20
4	<i>Figure of tibble that stored necessary information about initial matrices. .</i>	27
5	<i>Figure of second tibble that stored necessary information about initial matrices and matrices where random number of elements had been removed and transitive property had been applied.</i>	32
6	<i>Figure of data frame that stored necessary information for data analysis. .</i>	34
7	<i>Initial, theoretical maximum and optimal number of pairwise comparisons</i>	36
8	<i>Overview of 3 to 10 alternatives data set.</i>	37
9	<i>Overview of 3 to 10 alternatives CR categories distribution.</i>	37
10	<i>Overview of 3 to 10 alternatives CR change in relation to number of removed values.</i>	38
11	<i>Overview of 3 to 10 alternatives weights change in relation to number of removed values.</i>	39
12	<i>Overview of 11 to 20 alternatives data set.</i>	40
13	<i>Overview of 11 to 20 alternatives data set in CR 0.5-1.0 category.</i>	40
14	<i>Overview of 11 to 20 alternatives data set in CR ≥ 1.0 category.</i>	41
15	<i>Overview of CR change in relation to group of removed values of 11 to 20 alternatives in CR 0.5-1.0 category.</i>	41
16	<i>Overview of CR change in relation to group of removed values of 11 to 20 alternatives in CR ≥ 1.0 category.</i>	42
17	<i>Overview of weights change in relation to group of removed values of 11 to 20 alternatives in CR 0.5-1.0 category.</i>	43
18	<i>Overview of weights change in relation to group of removed values of 11 to 20 alternatives in CR ≥ 1.0 category.</i>	44
19	<i>Distribution overview of 3x3 to 10x10 AHP matrices and removed values.</i>	45
20	<i>Distribution overview of 11x11 to 20x20 AHP matrices and removed values.</i>	45
21	<i>CR categories distribution overview of 3x3 to 10x10 AHP matrices.</i>	46
22	<i>CR categories distribution overview of 11x11 to 20x20 AHP matrices. . .</i>	46

List of Tables

1	<i>Saaty's scale of relative importance.</i>	12
2	<i>T. L. Saaty random-like matrix value for matrices of different sizes.</i>	14
3	<i>Theoretical maximum and optimal number of missing comparisons.</i>	29
4	<i>Total combinations of theoretical maximum plus one from initial pairwise comparisons.</i>	30
5	<i>The fundamental Saaty's scale reciprocal values start and end ranges for transitive property</i>	31

1. Introduction

Analytic hierarchy process (AHP), developed by Thomas L. Saaty, is one of the most widely used multi-criteria decision making (MCDA) methods [1]. The AHP is a MCDA method aimed at supporting decision-making processes in individual and group contexts. The core of AHP lies in presenting the problem as a hierarchy and comparing the hierarchical elements in a pairwise manner using Saaty's 9-point scale to express the importance of one element over another in regards to the element in the higher level [2].

Even though the method was developed in 1970s, Fountzoula and Aravossis [3] conducted a literature review on the use of MCDM methods in 2022 and they concluded that, during the period of 2010 to 2020, MCDM methods were commonly used in public sector decision-making and AHP was the most frequent one when counting both simple and integrated methods [3]. AHP usage is highly extended in several areas, such as: logistic [1], strategic planning [1], technological investment evaluation [4], supplier selection [5], analysis of financial parameters [6], inventory classification [7], Internet access technology [8], IT project selection [9] and even for re-engineering of the health-care system [10]. According to Google Scholar statistics AHP has been cited 16 200 times, since 2020 until May of 2023.

In 2023, Innar et al. [11] mention that it has become a recurring necessity and exercise for corporations to assess the alignment of their corporate strategy and goals with United Nations Sustainable Development Goals (SDGs). Such an assessment is a highly complex task, full of inconsistencies and subjective opinions of internal and external stakeholders, which eventually influences the formal processes of strategy making and strategic choices. A customized version of AHP, which was custom tailored for SDG assessment to structure and organize the decision process and find and eliminate inconsistencies of group decision-making was presented by Innar et al. [11].

1.1 Research problem and purpose

Effective decision-making at strategic levels requires correct identification of factors that can influence business in future time periods. The ideal situation would be to include all variables in the decision-making process, but such an inflow of information would obfuscate our perception and finally make the distinction between important and unimportant variables impossible. According to Thomas L. Saaty the number of criteria and

alternatives analyzed in the decision-making process should not exceed 7 (+/- 2), which gives 9 alternatives maximum [12].

The two advantages which the AHP has over the other multi-criteria methods are the ease of use and the ability to handle inconsistencies in judgements. People, acting unilaterally, are seldom consistent in their judgements; groups are even less likely to be consistent. The AHP does not force an individual or a group to be consistent when making pairwise comparisons, but incorporates the inconsistencies into the process [13].

The major drawback with the AHP is the amount of work which is required in making all of the necessary pairwise comparisons [13]. For example, if 17 SDGs are compared, a total of 136 pairwise comparisons must be made. Depending on the complexity of the comparisons each pairwise comparison can take from seconds to minutes to evaluate. The more alternatives the more comparisons have to be done. Generally all the $\frac{n*(n-1)}{2}$ judgements must be made, where n is the number of alternatives to be compared. The total time spent making pairwise comparisons increases with the number of decision-makers.

The number of pairwise comparisons starts to increase quickly, especially in cases when there are more alternatives than recommended maximum of nine. A business need arises to reduce the number of comparisons for various reasons. Firstly, it becomes more difficult and less likely for decision-maker to stay consistent with comparisons. In 1956, G. A. Miller [14] conjectured that there is an upper limit on our capacity to process information on simultaneously interacting elements with reliable accuracy and with validity. This limit is seven plus or minus two elements. Secondly, long surveys and repetitive questions in the AHP survey process can cause survey fatigue, which can lead to a decline in response quality towards the end of the process. Thirdly, there may be instances where decision-maker lacks the necessary knowledge or is unwilling to make a decision.

The purpose of the thesis is to determine the number of comparisons that can be calculated using the transitive property and to determine the amount of excess information that is generated if all the comparisons are done. Additionally, the thesis aims to evaluate the impact of using the transitive property on the quality metrics of AHP results, including consistency ratio and weights of alternatives.

1.2 Experiment questions

The thesis has two focuses: determining the number of comparisons that can be (in most cases) be calculated with transitive property and if transitive property is used to calculate missing information, then what's the effect on AHP results from consistency ratio and

alternatives prioritization perspective. Researching these areas supports anyone who wants to use AHP for decision-making, especially in cases where it's necessary to compare more than nine alternatives.

To achieve this objective, the author proposes the following experiment questions and sub-questions:

1. What's the number of comparisons that can be calculated with the transitive property?
 - (a) In what situations it makes sense to use transitive property?
 - (b) What should be considered when using transitive property?
2. What's the effect on AHP results, in case of 3 to 20 alternatives, from consistency ratio and alternatives prioritization perspective if transitive property is used to calculate missing information?

1.3 Outline of the thesis

The thesis is divided into five chapters. The first chapter gives an overview of the research purpose, problem and provides the experiment questions. The second chapter introduces the theoretical framework of the AHP, provides an overview of pairwise comparison process and presents five reasons to justify the importance of reducing the number of pairwise comparisons. Under literature review ten approaches are presented that previous researchers have used to reduce the number of pairwise comparisons or how they have handled situations when missing pairwise comparisons. The following third chapter describes the methodology used for data collection and analysis, which involves the experimental quantitative data analysis method. The experiment design is elaborated, including how the data was prepared for descriptive analysis. The fourth chapter presents the experiment's results and findings, including how the results were validated. Finally, the last chapter concludes the thesis by summarizing the main findings and recommendations for further research are provided.

2. Theoretical framework

This theoretical framework aims to provide a comprehensive understanding of analytic hierarchy process, how it's conducted and what are the main reasons to reduce pairwise comparisons.

2.1 Analytic hierarchy process overview

The analytic hierarchy process is a multicriteria method aimed at supporting decision-making processes in individual and group contexts. The core of AHP lies in presenting the problem as a hierarchy and comparing the hierarchical elements in a pairwise manner using Saaty's 9-point scale to express the importance of one element over another in regards to the element in the higher level [2]. Scale is presented in Table 1 [15].

Table 1. *Saaty's scale of relative importance.*

Intensity of importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Weak importance of one over another	Experience and judgment slightly favor one activity over another
5	Essential of strong importance	Experience and judgment strongly favor one activity over another
7	Demonstrated importance	An activity is strongly favored and its dominance demonstrated in practice
9	Absolute importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgment	When compromise is needed

The two advantages which the AHP has over the other multi-criteria methods are the ease of use and the ability to handle inconsistencies in judgements. People, acting unilaterally, are seldom consistent in their judgements; groups are even less likely to be consistent. The AHP does not force an individual or a group to be consistent when making pairwise comparisons, but incorporates the inconsistencies into the process [13]. Even physical measurements are never exact in a mathematical sense; and hence, allowance must be made for deviations. Secondly, in human judgments, deviations are considerably larger [16].

2.1.1 Pairwise comparison process overview

Without losing generality, the problem can be formalized by considering the prioritization of n elements E_1, E_2, \dots, E_n at a given level of hierarchy. The decision-maker semantically compares any two elements E_i and E_j indirectly (verbally) or directly (numerically), using the scale in Table 1, assigns the value a_{ij} that represents a judgment of the relative importance of decision element E_i over E_j . If element E_i is of the same importance for the decision-maker, then $a_{ij} = 1$, and if E_i is preferred to E_j , then $a_{ij} > 1$. The reciprocal property $a_{ij} = 1 / a_{ji}$ by assumption always holds, and $a_{ii} = 1$ for all $i = 1, 2, \dots, n$. If n elements of one level of the hierarchy are compared regarding the element in the upper level, a comparison matrix A has the following quadratic form:

$$A = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a_{n1}} & \frac{1}{a_{n2}} & \dots & 1 \end{pmatrix} \quad (2.1)$$

Each matrix element a_{ij} provided by the decision-maker is a subjective judgment of the mutual importance of the two elements, i and j . If the decision-maker is fully consistent, then the transition rule $a_{ij}a_{jk} = a_{ik}$ should apply for all i, j , and k in the range 1 to n [2].

2.1.2 Consistency ratio

The consistency ratio (CR) is calculated as a part of the standard AHP procedure. First, the consistency index (CI) is calculated using the following equation:

$$CI = \frac{\lambda_{max} - n}{n - 1} \quad (2.2)$$

where λ_{max} is the principal eigenvalue of matrix (2.1). Knowing the consistency index and random consistency index (RCI) defined also by Saaty and shown in Table 2, the consistency ratio is obtained:

$$CR = \frac{CI}{RCI} \quad (2.3)$$

Where RCI value depends on the size of the matrix.

Table 2. *T. L. Saaty random-like matrix value for matrices of different sizes.*

Order	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RCI	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59
First-Order Differences	0	0.52	0.37	0.22	0.14	0.10	0.05	0.05	0.04	0.03	0.02	0.02	0.02	0.01

Saaty suggested considering the maximum level of the decision-maker's inconsistency to be 0.10; that is, CR should be less or equal to 0.10 [2]. Allowable consistency ratio should be not more than about 0.10. The requirement of 10% cannot be made smaller such as 1% or 0.1% without trivializing the impact of inconsistency. But inconsistency itself is important because without it, new knowledge that changes preference cannot be admitted. Assuming that all knowledge should be consistent contradicts experience that requires continued revision of understanding [16].

The quality of response to stimuli is determined by three factors. Accuracy or validity, consistency, and efficiency or amount of information generated. Our judgment is much more sensitive and responsive to large perturbations. With perturbations is meant a numerical change from consistent ratios obtained from priorities. The larger the inconsistency, and hence, also the larger the perturbations in priorities, the greater is our sensitivity to make changes in the numerical values assigned. Conversely, the smaller the inconsistency, the more difficult it is for us to know where the best changes should be made to produce not only better consistency but also better validity of the outcome. Once near consistency is attained, it becomes uncertain which coefficients should be perturbed by small amounts to transform a near consistent matrix to a consistent one. If such perturbations were forced, they could be arbitrary and thus, distort the validity of the derived priority vector in representing the underlying decision [16].

2.2 Reasons to reduce alternatives and pairwise comparisons

Effective decision-making at strategic levels requires correct identification of factors that can influence business in future time periods. The ideal situation would be to include all variables in the decision-making process, but such an inflow of information would obfuscate our perception and finally make the distinction between important and unimportant variables impossible. According to T. L. Saaty the number of criteria and alternatives analyzed in the decision-making process should not exceed 7 (+/- 2), which gives 9 alternatives maximum [12].

There are various reasons why making all the pairwise comparisons or adding all the alternatives into the analysis isn't always the best approach, for example:

- Consistency
- Time
- Human capacity to process information
- Survey fatigue
- Decision-maker knowledge and behavior

2.2.1 Consistency

Consistency is necessary to measure in order to preserve integrity that the outcomes are trustful. Standard AHP uses eigenvector, the prioritization method, and the consistency coefficient consistency ratio to indicate the inconsistency of the decision-maker [2]. The consistency ratio of a pairwise comparison matrix is the ratio of its consistency index: to the corresponding random index value in Table 2. Figure 1 is a plot of "Order" and "First-Order Differences" of Table 2. It shows the asymptotic nature of random inconsistency [16].

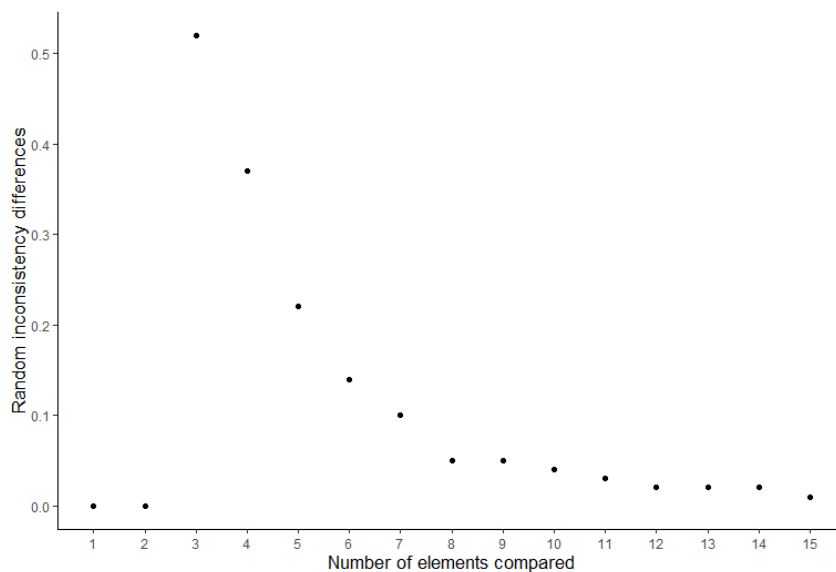


Figure 1. *Graph of first differences in random inconsistency.*

The measure of random inconsistency reveals that as the number of elements being compared is increased the measure of inconsistency decreases so slowly that there is insufficient room for improving the judgments and, therefore, also consistency. From Figure 2, can be concluded that to serve both consistency and redundancy, it is best to keep the number of elements seven or less. It appears that Miller's seven plus or minus two is indeed a limit, a channel capacity on our ability to process information [16].

If the CR is larger than desired, Saaty and Ozdemir [16] recommend doing following three things:

1. Find the most inconsistent judgment in the matrix.
2. Determine the range of values to which that judgment can be changed corresponding to which the inconsistency would be improved.
3. Ask the decision-maker to consider, whether the decision-maker can change his or her judgment to a plausible value in that range. If the decision-maker is unwilling, then try with the second most inconsistent judgment and so on. If no judgment is changed the decision is postponed until better understanding of the stimuli is obtained.

Decision-makers who understand the theory are always willing to revise their judgments often not the full value but partially and then examine the second most inconsistent judgment and so on. It can happen that a decision-maker's knowledge does not permit one to improve his or her consistency and more information is required to improve the consistency of judgments [16]. Nowadays, decision support systems (DSS) are used to assist the decision-maker and, accordingly, to provide more self-confidence for decision tasks. There are many types of DSS and they are used with personal through managerial and enterprise purposes, assisting either individual or group decisions and implementing diverse techniques on stand-alone and web-based architectures [1].

2.2.2 Time

Depending on the complexity of the comparisons each pairwise comparison can take from seconds to minutes to evaluate. As shown in Figure 2, the more alternatives the more comparisons have to be done. The total time spent making pairwise comparisons increases with the number of decision-makers.

The number of comparisons can be calculated with the following formula:

$$\frac{n * (n - 1)}{2} \quad (2.4)$$

where n is a number of alternatives to compare.

When the judgements are perfectly consistent, only the first $(n - 1)$ comparisons need to be made to calculate the weights. However, when inconsistencies are present, the entire top

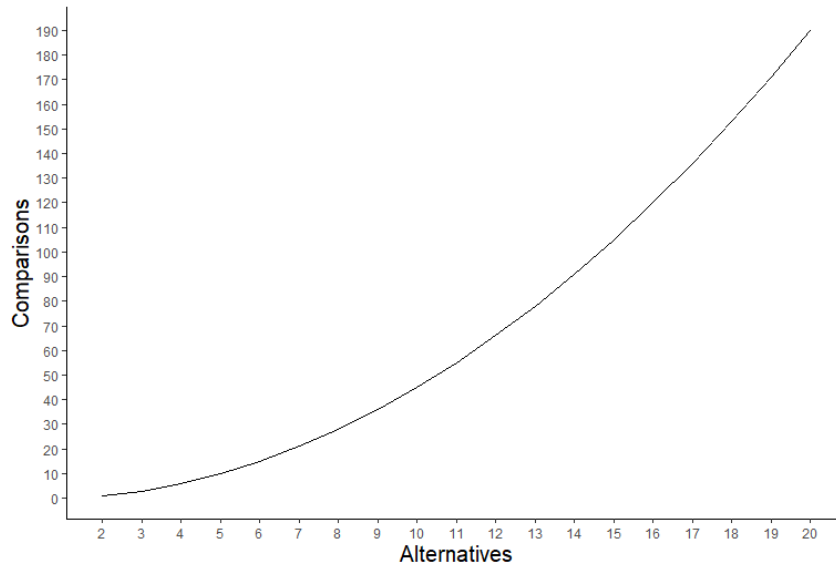


Figure 2. Graph of number of pairwise comparisons based on AHP alternatives.

triangular portion of the matrix, $\frac{n*(n-1)}{2}$ judgements, must be completed since the various paths between two vertices will have different intensities. On the other hand, one would not want to make only $(n - 1)$ comparisons since a certain amount of redundancy is necessary to "correct" any errors in the judgements [13].

The major drawback with the AHP is the amount of work which is required in making all of the necessary pairwise comparisons. For example, if nine objects according to five criteria is compared, a total of 190 pairwise comparisons must be made [13]. Even if there is an unlimited amount of time, then there are other reasons why having too many alternatives, and due to that having more pairwise comparisons, isn't a good approach.

2.2.3 Human capacity to process information

In 1956, G. A. Miller [14] conjectured that there is an upper limit on our capacity to process information on simultaneously interacting elements with reliable accuracy and with validity. This limit is seven plus or minus two elements. The reason is founded in the consistency of information derived from relations among the elements. When the number of elements increases past seven, the resulting increase in inconsistency is too small for the mind to single out the element that causes the greatest inconsistency to scrutinize and correct its relation to the other elements, and the result is confusion to the mind from the existing information [16].

The AHP as a theory of measurement has a basic way to obtain a measure of inconsistency for any such set of pairwise judgments. When the number of elements is seven or less

the inconsistency measurement is relatively large with respect to the number of elements involved; when the number is more it is relatively small. The most inconsistent judgment is easily determined in the first case and the individual providing the judgments can change it in an effort to improve the overall inconsistency. In the second case, as the inconsistency measurement is relatively small, improving inconsistency requires only small perturbations and the judge would be hard put to determine what that change should be, and how such a small change could be justified for improving the validity of the outcome. The mind is sufficiently sensitive to improve large inconsistencies but not small ones. And the implication of this is that the number of elements in a set should be limited to seven plus or minus two [16].

2.2.4 Survey fatigue

For any researcher who has observed a survey that takes for several hours, it is clear that some respondents disengage as the survey drags on, because they are exhausted, bored, or because their attention wanders. As a result, response quality during the later part of a long survey may suffer, a phenomenon known as survey fatigue. An additional hour of survey time increases the probability that a respondent skips a question by 10% to 64%. Similar effect sizes were found within phone surveys in which respondents were already familiar with questions, suggesting that cognitive burden may be a key driver of survey fatigue. Past research suggests that survey fatigue may be driven by people deliberately choosing to not answer questions in order to expedite the end of the survey, or if people become more likely to inadvertently make mistakes as they become tired [17]. Researchers have also conjectured that, over time, respondents learn that answering "no" to a question often invokes a "skip code" that will allow them to skip a number of follow-up questions. This behavior is known as "satisficing" [18].

2.2.5 Decision-maker knowledge and behavior

There could be a situation where a decision-maker is unwilling to make a direct comparison between two particular alternatives or a decision-maker is unsure about some comparisons. Transitive property could be used to calculate the value of a missing pairwise comparison, though in these cases Saaty and Ozdemir [16] recommend to postpone the decision until better understanding is obtained.

2.3 Literature review

Under literature review a brief overview of ten approaches is given, that previous researchers have used to reduce the number of pairwise comparisons or how they have handled situations when missing pairwise comparisons.

2.3.1 First-level transitivity rule method

Srdjevic et al. [2] propose first-level transitivity rule (FLTR) based on screening matrix entries in the neighborhood of a missing one. Scaling (where necessary) and geometric averaging of screened entries allows filling of the gap in the matrix and later prioritization of involved decision elements by the eigenvector, or any other known method. The FLTR assures coherency of the generating process in a sense that all numeric values in a matrix (original entries, plus one generated) come from the same ratio scale and have correct element-wise semantic equivalents. FLTR involves geometric averaging and scaling of inner products that realize the transition rule, and the final matching of computed numerical to the scale used for all other pairwise comparisons of decision elements in a hierarchy. The proposed method can generate only one missing judgment in any local matrix and in turn enables prioritization for a given matrix, as well as the final AHP synthesis.

2.3.2 Geometric mean induced bias matrix

The geometric mean induced bias matrix (GMIBM) only requires the original information of the incomplete comparison matrix and is independent of the way of deriving priority weights from a pairwise matrix. Specifically, the missing judgments are first filled in by unknown variables, and then the adapted GMIBM is applied to obtain a revised "complete" pairwise matrix. To keep the global consistency and estimate the missing judgments, the least absolute error method and the least square method are used to optimize the objective function and find the optimal solution of missing comparisons [19].

The specific steps [19] of the missing comparisons estimation include:

- Step 1: Fill in the missing comparisons with unknown variables x_1 and $\frac{1}{x_1}$; x_2 and $\frac{1}{x_2}$; etc.
- Step 2: Construct the GMIBM ϵ by the following three sub-steps.
 - Step 2.1: Compute a column vector L and a row vector R (see Figure 3).
 - Step 2.2: Compute the geometric mean matrix by formula $\bar{A} = L * R$.
 - Step 2.3: Compute GMIBM ϵ .

- Step 3: Establish an overdetermined system of equations by minimizing all entries in the error matrix ϵ , i.e. let $\epsilon_{ij}(x_1, x_2, \dots, x_p, a_{ij}) = 0, i, j = 1, 2, \dots, n$ hold.
- Step 4: Solve the overdetermined system of equations.
- Step 5: Test the revised comparison matrix A by replacing the missing comparisons with the estimated values.

The results show that the proposed models are not only capable of completing missing values, but also can efficiently improve the matrix consistency at the same time [19].

$$C = LR \circ A^T = (c_{ij}) = \left(\sqrt[n]{\prod_{k=1}^n a_{ik}} \cdot \sqrt[n]{\prod_{k=1}^n a_{kj}} \cdot a_{ji} \right) = U \quad \text{if } a_{ik}a_{kj} = a_{ij},$$

where $L = \left(\sqrt[n]{\prod_{k=1}^n a_{ik}} \right)_{n \times 1}$ ($i = 1, \dots, n$), represents an n -by-one column matrix composed of geometric mean of rows in matrix A .
 $R = \left(\sqrt[n]{\prod_{k=1}^n a_{kj}} \right)_{1 \times n}$ denotes an n -by-one row matrix composed of geometric mean of columns in matrix A .

Figure 3. Formula to compute a column vector L and a row vector R .

2.3.3 Optimization methods

Optimization models start with a handful of pairwise comparisons only. The remaining pairwise comparisons are estimated using optimization algorithms by taking advantage of the matrix properties of A (2.1). Starting with a of minimum $n - 1$ comparisons, a gradient descent method is proposed to select the next pairwise comparison that would have the biggest information gain [20]. The methodology by Bozoki, Fulop, and Ronyai [21] uses nonlinear optimization with exponential scaling to estimate the missing pairwise comparisons from available ones. However, all possible combinations of connecting paths must be considered. The number of combinations exponentially grows as the number of missing comparisons increases and thus would be inefficient to solve.

2.3.4 Assigning the alternatives into subsets

Assign the alternatives into subsets based on a subjective absolute scale in which alternatives that have close "magnitudes" are grouped together. By using several pivot alternatives that is common to at most two subsets, the global priorities of the alternatives are then obtained. Note, the definition of close "magnitudes" is not well defined and is highly subjective. Furthermore, no guidelines are provided to determine which alternatives are assigned to which subset [22].

2.3.5 Decision to stop making pairwise comparisons

Harker [13] suggests three approaches when to stop making pairwise comparisons:

1. The first is to let the decision maker decide whether or not to continue with the questioning. In fact, this option is always available under the other two stopping rules.
2. The second rule states that if the maximum absolute difference in the attribute weights from one question to the next is less than or equal to α , where α is a given constant, then one should stop the questioning since the new information did not have a major influence on the weights. This rule is "liberal" in the sense that further questioning may drastically alter the weights even if there is no major change at the present time. However, the decision-maker always has veto power over the rule's choice to stop and hence, this rule may work well in practice.
3. The third stopping rule is very conservative in the sense that comparisons will continue to be made until one is sure that the ordinal rank will not be reversed. The weights w are cardinal rankings of the alternatives which, of course, create an ordinal ranking. By answering more questions, the cardinal ranking in w may be slightly altered but the ordinal ranking implied by w could remain the same. The third stopping criterion simply states that the next question will be asked if it appears that the ordinal ranking could be reversed. This rule is very conservative in the sense that two alternatives may have low but almost equal weights and this rule would not terminate the comparisons. Alternatives with low weights are not important and thus one would like to ignore a possible rank reversal in this situation.

2.3.6 Balanced incomplete block design technique

Weiss and Rao [23] presented a balanced incomplete block design technique to reduce the number of pairwise comparisons. Their work was different from Harker's (1987a, 1987b) in the sense that they developed a factorial design of the comparisons for large scale problems. The proposal was to allocate the appropriate portions of the hierarchy to different respondents rather than having each respondent working on the entire hierarchy. The proposal was also based on the assumption that people who make large numbers of comparisons become less consistent. They also suggested deleting attributes that were identical to one another. However, their guidelines for deletion of attributes seemed arbitrary [23] [24].

2.3.7 Analytic network process

Analytic Network Process (ANP) is a more general form of AHP, structuring a decision problem as a network, as opposed to a hierarchy. As a multicriteria theory of measurement, its main use is deriving relative priority scales of absolute numbers from individual judgements. Alternatively, it can also use actual measurements normalized to a relative form. These judgements also belong to a fundamental scale of absolute numbers and represent the relative influence between elements in a pairwise comparison. Based on an underlying control criterion, it examines whether one of two elements in the pairwise comparison influences a third element in the system [25]. Adding the interdependencies to AHP makes consistent pairwise comparison even more confusing and difficult. Even if the existence of such interdependencies can be justified, it is hard for the decision maker to consider these relations while doing pairwise comparisons [26]. For the ANP, based on Formula 2.4, n^2 comparisons are needed to estimate the inner dependencies of the criteria [27].

2.3.8 Integrating AHP with NGT

Islam and Abdullah [24] proposed to exclude less important criteria at the beginning of the exercise. The question was how it's possible to know in the beginning which criteria are important and which are not? A simple solution to this problem is to apply Nominal Group Technique (NGT) at the beginning of the exercise. NGT is a highly useful, structured brainstorming technique that is used to produce a large number of ideas pertaining to an issue/problem while ensuring balanced participation among the group members [28]. NGT not only generates a large number of ideas, but also it prioritises those ideas using certain voting techniques. After the voting session, the ideas that receive a higher number of votes are generally considered superior or important ideas [24].

Nominal group technique requires a group of about eight people. This group is facilitated by someone who is expected to have prior experience in conducting some nominal group sessions. The group members meet in a meeting room equipped with a marker board and marker pen and each of them has a few sheets of paper. The steps [24] to be followed are as follows:

1. Step 1: silent generation of criteria in writing. Participants are given about ten minutes to write down as many criteria as possible pertaining to the decision making problem.
2. Step 2: round-robin recording of criteria. The facilitator asks every participant (start-

ing from one end of the room) to provide the most important criterion (participants have to judge which one is the most important criterion) from his or her list. If there are eight participants then there should be eight important criteria recorded in the first round. After this, the facilitator goes for the second round, collecting the second most important criterion from the lists of all the participants. Subsequent rounds are carried out till all the criteria are exhausted from the participants lists. By this time, all the relevant criteria are written down on the marker board. The list generated is usually called the master list.

3. Step 3: voting to select the most important criteria. Each participant is asked to identify the most important five criteria from the master list on the board and rank them using a 1 to 5 scale according to their importance. The most important criterion is assigned a rating of 5 and the least among these five criteria receives the rating of 1. When all the participants finish the ranking task, cards are collected from them and votes are written against the criteria on the board. When the votes are aggregated, it is easy to single out the important criteria. If there are, say, 25 criteria in the master list and we want to consider the most important ten, then the ten criteria that receive higher votes are to be chosen.

When Islam and Abdullah [24] used only the dominant criteria, then they managed to save 1223 pairwise comparisons from total of 2081, which was 58.77% saving.

2.3.9 Neural network-based method

Neural networks are widely used in function approximation problems, pattern recognition and clustering. Such networks can be used to represent hierarchical preferences [29] and (broadly speaking) they can be divided into two main groups: those that use supervised learning and those that use unsupervised learning. The Multi-Layer Perceptron (MLP) belongs to the first group and its training algorithm is called the backpropagation algorithm [30]. The MLP architecture contains neurons in layers: one input layer, one or more hidden layers and one output layer. Learning patterns are pairs (input, desired output) used to train the network. In the learning process, the neural network adapts the connection weights, attempting to minimize the difference between the network output and the desired output. When an acceptable average training error is reached, the network is able to recognize incomplete patterns. This architecture is easy to implement, the training process is fast and perhaps most significantly, after the training process, the network gives the most similar learned pattern associated to the distorted input [31].

2.3.10 Revised geometric mean method

The revised geometric mean method [32] is used to fill in the missing elements in a reciprocal multiplicative matrix A . The process consists of adding one to the main diagonal for each missing element in the row. Then, missing elements are replaced by zeros, that is, the elements r_{ij} of the transformed matrix, R , are calculated as follows:

$$r_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \text{ is not a missing value, with } i \neq j \\ 0 & \text{if } a_{ij} \text{ is a missing value, with } i \neq j \\ 1 + c_i & \text{with } i = j \end{cases} \quad (2.5)$$

where c_i is the number of missing values in row i .

After that, the eigenvector associated to the largest eigenvalue of matrix R is calculated and every missing element of A is replaced by the correspond ratio. That is to say, if $w = [w_1, w_2, \dots, w_n]$ is the eigenvector associated to the largest eigenvalue of the matrix R , then each missing element in matrix A , a_{ij} , is replaced by:

$$a_{ij} = \frac{W_i}{W_j} \quad (2.6)$$

3. Methodology

To understand how applying transitive property to AHP matrix randomly removed elements affects the matrix consistency ratio and alternatives weights, the experimental quantitative data analysis method was adopted as the methodological approach. It involves data collection, in current thesis context doing experiments with AHP matrices in R Studio and gathering data into synthetic data sets, and then applying mathematical and statistical techniques to identify patterns, trends and relationships among variables. Descriptive data analysis was used to analyze and interpret numerical data. The goal is to draw objective conclusions to experiment questions. Considering the aim and experiment questions of the thesis, the experimental quantitative data analysis method was considered as the best approach.

Experiments process consisted of four phases:

1. Based on academic literature, understanding of AHP matrix properties and quality metrics were acquired.
2. After that synthetic data sets were created and generated matrices attributes, consistency ratio and weights, were calculated and validated.
3. Descriptive analysis was done to understand what happened with consistency ratio and alternatives weights once transitive property was applied to recalculate the AHP matrix randomly removed elements.
4. Conclusions were drawn from descriptive analysis results.

3.1 Data collection method

Synthetic data sets were created by the author. Synthetic data sets were necessary to be able to conduct the experiments and afterwards data analysis. Author created functions in R Studio to generate the data sets (Appendices 2, 3 and 4). If randomness was necessary then "*set.seed(123)*" was used to be able to have reproducible results. Business Performance Management Singapore (BPMSG) AHP calculator [33], developed by Klaus D. Goepel, was used to validate created R functions outputs.

Since AHP matrices need to meet certain criteria, listed under Section 3.1.1, it was possible to create the data sets synthetically. A data set of 100 000 random AHP matrices was created to analyze 11x11 to 20x20 matrices and a 10 000 matrices data set was created to

analyze 3x3 to 10x10 matrices. Once the data sets were created, it was validated if there is a uniform distribution of different matrix sizes and removed elements. That was the case. An overview of matrices sizes distribution and an overview of removed elements distribution is presented in Subsection 4.7.

3.1.1 Overview of research object

In order to generate synthetic data sets, it was necessary to understand the properties of AHP matrices. Following AHP process and matrix properties were concluded from academic literature:

- AHP pairwise comparisons results can be presented in a quadratic matrix form, where main diagonal elements are 1 and the reciprocal property $a_{ij} = \frac{1}{a_{ji}}$ by assumption always holds [2], as shown on Formula 2.1.
- The transition rule $a_{ij}a_{jk} = a_{ik}$ should apply for all i, j , and k in the range 1 to n [2].
- The matrix elements represents decision-maker estimate on the importance of one alternative over another and the importance is selected from one of the 17 pre-defined Saaty scale values: 1, 2, 3, 4, 5, 6, 7, 8, 9, plus reciprocal values: 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9. The explanations of Saaty scale values are presented in Table 1.
- The consistency ratio is used to measure how consistent decision maker judgements were. If the matrix is fully transitive, then consistency ratio is 0. The consistency ratio is obtained with Formula 2.3.
- The number of comparisons can be calculated with Formula 2.4. If all pairwise comparisons were perfectly consistent, then only the first $(n - 1)$ comparisons need to be made to calculate the weights [13].
- The weights depend on the number of criteria, the maximum weight or maximum priority $w_{max} = \frac{M}{n+M-1}$ [34]. Generally approximate eigenvector method is used to calculate alternatives weights. For example largest eigenvector, geometric mean and fuzzy geometric mean could be also used.

3.1.2 Creating synthetic data sets

Author couldn't find any existing data sets that would meet the set requirements, having big enough sample of 3x3 to 20x20 matrices with a wide range of consistency ratios and alternatives weights. It could be argued that smaller data sets would have given similar results, but considering the law of big numbers [35], author thought the bigger data sets would be better. Especially considering the fact, that data sets creation functions had been created and it was just a matter of time to generate bigger data sets. Since AHP matrices

need to meet certain criteria, listed under Section 3.1.1, it was possible to create the data sets synthetically.

A function (Appendix 2) was created to generate a random AHP matrix. Function, that takes a value of what size of a AHP matrix needs to be generated, works in following steps:

1. First a matrix is generated where all the values are 0.
2. After that main diagonal elements are set to 1.
3. Then a random sample of values are drawn from the predefined list, where all the allowed AHP matrix values are listed. The number of values drawn is based on the number of elements in the matrix upper triangular.
4. Then these values are added to the matrix upper triangular.
5. Lastly, upper triangular reciprocal values are calculated to the matrix lower triangular.
6. Additional step, having a limitation of maximum consistency ratio, was added to generate 3x3 to 10x10 matrices. After a matrix has been generated, then it's checked with "*calculate_CR*" function (Appendix 4) that calculates a matrix consistency ratio. If generated matrix consistency ratio is within acceptable limitations, in this case less than or equal to 0.5, then a matrix is returned and added into a list, if not, then process is repeated until a matrix that meets the consistency ratio criteria is met.

R Studio "*for loop*" was used to run the "*generate_ahp_matrix*" function (Appendix 2) as many times as necessary to have required number of AHP matrices.

Generated AHP matrix was initially stored into a list and then stored into a tibble before starting generating next matrix. Due to the reason that generated matrices as "*Matrix*" and weights as "*Weights*" were stored in a list, tibble was necessary to be used, instead of R built in data frame, since a tibble can have columns that are lists. In addition, initial matrix consistency ratio as "*CR*" and size of the matrix as "*MatrixSize*" were also stored into the tibble. Example of the first tibble structure and values are shown on Figure 4.

```
> head(matrices_tibble)
# A tibble: 6 x 4
  Matrix          CR weights MatrixSize
  <list>         <dbl> <list>    <list>
1 <dbl [9 x 9]>  0.462 <dbl [9]> <int [1]>
2 <dbl [3 x 3]>  0.0301 <dbl [3]> <int [1]>
3 <dbl [9 x 9]>  0.449 <dbl [9]> <int [1]>
4 <dbl [10 x 10]> 0.476 <dbl [10]> <int [1]>
5 <dbl [6 x 6]>  0.465 <dbl [6]> <int [1]>
6 <dbl [3 x 3]>  0.103 <dbl [3]> <int [1]>
```

Figure 4. Figure of tibble that stored necessary information about initial matrices.

3.1.3 Finding an acceptable number of elements to remove

Before starting to create second data set, that would store information about matrices which elements have been randomly removed and recalculated with transitive property, it was necessary to understand how many elements could be potentially removed from different size of matrices. To proceed further, the author gives definition to following four terms:

1. "*Theoretical maximum*" as maximum number of missing elements in an AHP matrix in case it's always possible to recalculate all the missing elements in the AHP matrix with transitive property. Removing theoretical maximum plus one element could cause a situation where all the AHP matrix elements couldn't be recalculated with transitive property.
2. "*Theoretical minimum comparisons limitation*" as difference between the number of initial pairwise comparisons and theoretical maximum. It's always possible to recalculate all the AHP matrix missing elements with transitive property if at least this number of pairwise comparisons has been done.
3. "*Critical elements*" as minimum number of AHP matrix elements, based on either 10 000 or 100 000 simulations, that are necessary to calculate all the missing elements with transitive property. This number of elements allows to recalculate all the missing AHP matrix elements in most cases, but not always.
4. "*Optimal number of missing comparisons*" as number of comparisons information that, based on either 10 000 or 100 000 simulations, in most cases are fine to be missing from a given AHP matrix. It must be kept in mind that if theoretical maximum plus one judgement information contains within this number, then recalculating all the the AHP matrix elements isn't possible.

To validate the acceptable number of removed elements 10 000 simulations were run for 4x4 to 6x6 matrices and 100 000 simulations were run for 7x7 to 20x20 matrices. Simulations were repeated until there wasn't any failure, after either 10 000 or 100 000 simulations, to recalculate all the AHP matrix missing elements while always missing information about the optimal number of missing comparisons. 10 000 simulations for 4x4 to 6x6 was enough due to smaller number of combinations how theoretical maximum plus one elements could be removed. Maximum number of removed elements for matrices 2x2 and 3x3 came from a calculation, where initial pairwise comparisons was subtracted with theoretical minimum comparisons limitation. Overview of theoretical maximum and optimal number of missing comparisons are presented in Table 3 and overview of total combinations are presented in Table 4.

Table 3. *Theoretical maximum and optimal number of missing comparisons.*

Number of alternatives	Initial pairwise comparisons	Theoretical minimum comparisons limitation	Difference aka theoretical maximum	Optimal number of missing comparisons	Percentage of missing pairwise comparisons
1	0	0	0	0	0%
2	1	1	0	0	0%
3	3	2	1	1	33%
4	6	4	2	2	33%
5	10	7	3	3	30%
6	15	11	4	4	27%
7	21	16	5	5	24%
8	28	22	6	7	25%
9	36	29	7	10	28%
10	45	37	8	13	29%
11	55	46	9	16	29%
12	66	56	10	22	33%
13	78	67	11	28	36%
14	91	79	12	34	37%
15	105	92	13	38	36%
16	120	106	14	42	35%
17	136	121	15	50	37%
18	153	137	16	60	39%
19	171	154	17	75	44%
20	190	172	18	95	50%

In reality more pairwise comparisons, from matrices 8x8 and bigger, could be removed than theoretical maximum since it's rather unlikely that exactly all the elements containing in theoretical maximum plus one elements gets removed. Simulations proved that statement. For matrices 3x3 to 18x18 around 30 to 40% of comparisons can be missing to be able to recalculate an AHP matrix missing elements with transitive property. Even more comparisons information could be missing with bigger number of alternatives, in case of 19 and 20 alternatives, up to 44 to 50% of random comparisons could be missing, while still managing to run successfully 100 000 simulations.

Most cases it would be probably fine if additional 1 to 5 or even more comparisons information is missing, depending on the number of alternatives, the more alternatives there are, the more additional elements could be potentially missing. This on the other hand, increases the likelihood of not being able to calculate all the AHP matrix missing elements with transitive property. The least number of comparisons that are necessary is $n - 1$, where n is number of alternatives, but requires having information about a set of very specific comparisons. Out of all possible combinations, there exists only one such combination. On the other hand, then there wouldn't be any additional information to

"correct" any errors in the judgements of other comparisons, as explained in Subsection 2.2.2.

Table 4. *Total combinations of theoretical maximum plus one from initial pairwise comparisons.*

Number of alternatives	Initial pairwise comparisons	Theoretical maximum plus one	Combin(Initial pairwise comparisons, Theoretical maximum plus one)
1	0	1	-
2	1	1	1
3	3	2	3
4	6	3	20
5	10	4	210
6	15	5	3 003
7	21	6	54 264
8	28	7	1 184 040
9	36	8	30 260 340
10	45	9	886 163 135
11	55	10	2.9249E+10
12	66	11	1.0741E+12
13	78	12	4.3431E+13
14	91	13	1.9173E+15
15	105	14	9.1749E+16
16	120	15	4.7305E+18
17	136	16	2.6143E+20
18	153	17	1.5416E+22
19	171	18	9.6612E+23
20	190	19	6.4123E+25

Information gathered from simulations enabled to create a *removed elements index*, which consisted of values: 1, 2, 3, 4, 5, 7, 10, 13, 16, 22, 28, 34, 38, 42, 50, 60, 75, 95, representing the maximum number of elements that can be removed from 3x3 to 20x20 matrices. *Removed elements index* was used in a function to remove up to allowed number of elements from a matrix before applying transitive property to recalculate missing elements, that's explained in detail in the Subsection 3.1.5.

3.1.4 Defining rounding ranges for Saaty scale's reciprocal values

Before starting to remove and recalculating the elements, the transition rule states that $a_{ij}a_{jk} = a_{ik}$, meaning two AHP matrix elements have to be multiplied to find a missing element. This can lead to a situation where a multiplication gives a value that is outside of Saaty scale, if that was the case, then the Saaty scale maximum value 9 was assigned, for

example if both elements to calculate a missing element were 9. The calculated missing elements had to be converted into allowed Saaty scale values, unless scale normalization is being done, which wasn't the case in current research. Author defined following ranges for Saaty scale values between 0 and 1.

Table 5. *The fundamental Saaty's scale reciprocal values start and end ranges for transitive property*

Saaty scale value	Reciprocal value	Start range	End range
1	1/1	0.75000	1.00000
2	1/2	0.41667	0.75000
3	1/3	0.29167	0.41667
4	1/4	0.22500	0.29167
5	1/5	0.18333	0.22500
6	1/6	0.15476	0.18333
7	1/7	0.13393	0.15476
8	1/8	0.11806	0.13393
9	1/9	0.00000	0.11806

If the multiplication result was higher than 1, then standard rounding rules to nearest integer were applied.

3.1.5 Removing the elements and applying transitive property

A function (Appendix 3) was created to remove randomly random number of elements from generated matrices. Removed elements were recalculated with transitive property. The maximum number of elements that could be removed from certain size of a matrix was defined in *removed elements index*. Code, that removed elements and recalculated the missing elements and used first data set matrices as input, works in following steps:

1. Duplicate of the initial matrix is made and stored as *recalc_ahp_matrix*. Following steps are applied to the duplicated AHP matrix.
2. The number of elements that are removed is selected randomly. The maximum number of elements that can be removed from given matrix is defined in *removed elements index*.
3. Elements, and their reciprocal elements, are being removed until the necessary number from previous point is reached. There is a condition that checks if the chosen element hasn't been already removed or it isn't a diagonal element, if one of the conditions is met, then a new element is chosen, until suitable element is found. Once enough elements has been removed, then the number of elements removed is stored in a list.

4. Then a *for loop* is used to scan missing elements from the matrix, if a missing element is found, then transitive property is used to calculate the missing element and missing element reciprocal value. First suitable available elements are used. In case the multiplication is less than 1, then the rounding ranges defined in Section 3.1.4 were used.
5. Once the matrix has been restored, CR and weights are calculated and stored into a list.

R Studio "*for loop*" was used to run the code (Appendix 4) as many times as necessary to have from all matrices at least one element removed and recalculated with transitive property. In the process information about the same attributes as for the initial matrices were gathered and stored initially in a list, but then moved to the second tibble.

Second tibble was created to contain the same information as the first one, but in addition had following attributes:

- "*CR_recalc*" as consistency ratio after a random number of elements had been removed and recalculated with transitive property. Created R function is added under Appendix 4.
- "*CR_dif*" as difference between recalculated and initial matrix consistency ratio.
- "*Weights_recalc*" as weights of alternatives in recalculated matrix. Created R function is added under Appendix 4.
- "*Values_rm*" as how many elements were removed randomly and recalculated from initial matrix.

Example of the second tibble structure and values are shown on Figure 5.

```
> head(matrices_tibble2)
# A tibble: 6 × 8
  Matrix              CR CR_recalc  CR_dif weights  weights_recalc values_rm MatrixSize
  <list>             <dbl>   <dbl>   <dbl> <list>   <list>         <int> <list>
1 <dbl [9 × 9]>    0.462   0.470  0.00795 <dbl [9]> <dbl [9]>         8 <int [1]>
2 <dbl [3 × 3]>    0.0301  0      -0.0301 <dbl [3]> <dbl [3]>         1 <int [1]>
3 <dbl [9 × 9]>    0.449   0.435  -0.0139 <dbl [9]> <dbl [9]>        10 <int [1]>
4 <dbl [10 × 10]> 0.476   0.403  -0.0727 <dbl [10]> <dbl [10]>        11 <int [1]>
5 <dbl [6 × 6]>    0.465   0.407  -0.0575 <dbl [6]> <dbl [6]>         2 <int [1]>
6 <dbl [3 × 3]>    0.103   0.0187 -0.0838 <dbl [3]> <dbl [3]>         1 <int [1]>
```

Figure 5. *Figure of second tibble that stored necessary information about initial matrices and matrices where random number of elements had been removed and transitive property had been applied.*

The reason why second tibble contained first tibble data was to have a back-up if something should happen with the second tibble. It took about 30 minutes to generate 10 000 matrices with size between 3x3 and 10x10 and with consistency ratio equal to or less than 0.5. And less than minute to remove randomly elements and recalculate missing elements with

transitive property. Creating 100 000 11x11 and 20x20 matrices without consistency ratio cap took two and half minutes. Having a consistency ratio cap had a significant impact. And around 30 minutes to remove randomly elements and recalculate missing elements with transitive property.

3.1.6 Calculating the alternatives weights

R Studio "Ahp" package was used to calculate alternatives weights. Kendall rank correlation coefficient was calculated to understand how close alternatives weights of initial generated AHP matrix and weights of recalculated AHP matrix were.

A function (Appendix 4) was created that took generated AHP matrix as an input and as an output gave the list of alternatives weights. This data was later used in descriptive data analysis.

3.1.7 Preparing the data for descriptive analysis

Then from the second tibble a data frame was created for data analysis. This data frame contained following information:

- Attributes *CR*, *CR_recalc*, *CR_dif*, *MatrixSize*, *Values_rm* were taken from the second tibble.
- *CR_cat* was calculated based on initial CR. Three consistency ratio categories were created to be able to analyse different categories separately:
 1. $CR < 0.5$ - where initial consistency ratio was less than 0.5.
 2. $CR 0.5-1.0$ - where initial consistency ratio was equal to or greater than 0.5 and less than 1.0.
 3. $CR \geq 1.0$ - where initial consistency ratio was equal to or greater than 1.0.
- *Kendall* as Kendall rank correlation coefficient which was calculated with "*cor.test*" R function, where *method* = "*kendall*" and based on weights of initially generated AHP matrix and weights of recalculated AHP matrix. Values of Kendall's tau range from -1 (100% negative association, or perfect inversion) to +1 (100% positive association, or perfect agreement). A value of zero indicates the absence of association. The higher the Kendall's tau, the better is the similarity between the two compared rankings [36].

CR categories were created based on CR median values and distributions. Distributions are presented under Section 4.7. The category "CR < 0.5" was used to analyse only 3x3 to

10x10 matrices data and categories "CR 0.5-1.0" and "CR >= 1.0" were used to analyse 11x11 to 20x20 matrices data.

Overview of the data frame that stored initial information for data analysis is presented on Figure 6.

```
> head(matrices_data)
  CR CR_cat CR_recalc CR_dif Kendall Matrix_size values_rm
1 0.46194294 CR < 0.5 0.46988834 0.007945399 0.7777778      9      8
2 0.03008419 CR < 0.5 0.00000000 -0.030084194 1.0000000      3      1
3 0.44908604 CR < 0.5 0.43517860 -0.013907437 1.0000000      9     10
4 0.47620278 CR < 0.5 0.40349473 -0.072708045 0.8666667     10     11
5 0.46462571 CR < 0.5 0.40710061 -0.057525093 0.7333333      6      2
6 0.10254436 CR < 0.5 0.01873269 -0.083811664 1.0000000      3      1
```

Figure 6. Figure of data frame that stored necessary information for data analysis.

Line graphs, boxplots and histograms were used to present overview of the data and the results of the data analysis. Result from descriptive analysis, for which ggplot2 was used, are presented in the Section 4.

3.2 Data analysis method

Descriptive data analysis provides valuable insights into the patterns and trends in the data. It was used to summarize and describe the information gathered from created synthetic data sets and relevant calculated attributes: *CR_dif* to show change in initial CR and CR after random number of matrix elements had been removed and recalculated. *Kendall* to show correlation in alternatives weights after random number of matrix elements had been removed and recalculated. Kendall rank correlation coefficient tau was chosen to measure the correlation. R Studio package "ggplot2" was used for descriptive analysis data visualization. Data visualization codes are given under Appendix 5 and results are presented and explained in Section 4.

4. Experiments results and findings

The main objective of this chapter is to analyze and interpret the data that has been collected from the synthetic data sets, with the aim of drawing conclusions and making recommendations based on the research problem and experiment questions.

4.1 Number of comparisons that could be missing

The least number of comparisons necessary for the AHP is the number of alternatives minus one. Though in that case, there exists only one such combination of pairwise comparisons out of all possible combinations. Unless this combination is purposefully selected and used, it's recommended to have as many comparisons made as possible, since depending on the number of alternatives, it's very unlikely to get this combination by chance. On the other hand, there wouldn't be any additional information to "correct" any errors in the judgements of other comparisons [13].

For this thesis context two terms: *theoretical maximum* and *optimal number of missing comparisons*, were introduced in Subsection 3.1.3 to present certain boundaries of the number of comparisons that information could be calculated with transitive property. The Figure 7 gives an overview of these boundaries. From three alternatives and onward theoretical maximum boundary could be calculated as $n - (n - 2)$ where n is the number of alternatives. There needs to exist at least one combination of two elements which would allow to calculate the missing element. Since the likelihood of having exactly theoretical maximum plus one elements removed by chance is very unlikely, as presented in Table 4, based on 10 000 and 100 000 simulations around 30 to 40% of comparisons could be missing instead, as presented in Table 3, while in most situations still being able to calculate all the missing elements with transitive property. Even more comparisons information could be missing with higher number of alternatives, in case of 19 and 20 alternatives, up to 44 to 50% of random comparisons information could be missing, while still managing to run successfully 100 000 simulations.

In case of 10 and more alternatives 1 to 5 extra comparisons information could be missing, on top of optimal number of missing comparisons, and still managing to calculate all the missing elements. Though the simulations started failing to calculate all the missing elements when the optimal number of missing comparisons was exceeded. As a potential solution, that could be considered in case an error occurs to calculate some alternative

elements, is to exclude this alternative from the analysis, to be able to produce at least some results. It's recommended to get back to decision maker to provide necessary information, so all the alternatives could be considered in the final results.

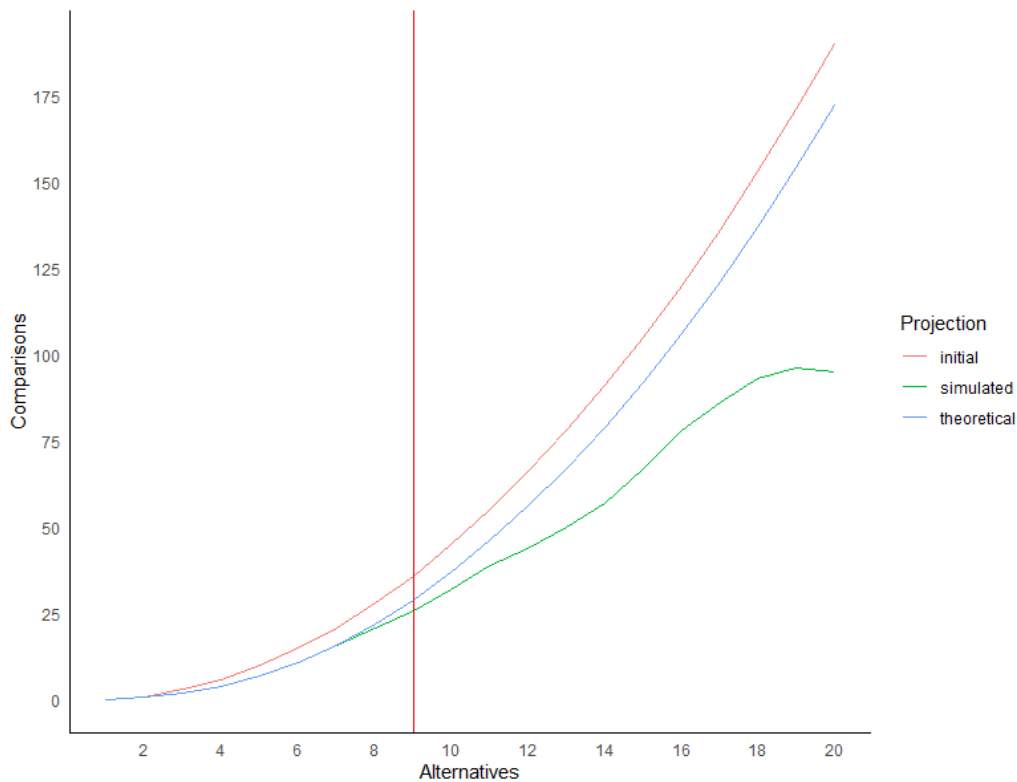


Figure 7. Initial, theoretical maximum and optimal number of pairwise comparisons

4.2 Situations when to use transitive property

Since the amount of comparisons is relatively low in cases of less than 9 alternatives (up to 28 comparisons), then for the best results the author recommends making all the pairwise comparisons with decision-maker and if necessary adjusting inconsistencies afterwards. Transitive property could be used to calculate a missing comparison information that was skipped by accident or calculating the comparison information that the decision-maker is unwilling to make.

Across the number of alternatives around 25 to 40% of elements could be calculated with transitive property, though based on Figure 7 noticeable effect comes from 9 alternatives and onward, especially if there are 13 or more alternatives. In case of 3 to 8 alternatives transitive property can be used to calculate around 25-30% of missing elements as well, but the actual number of elements is rather low, ranging between 1 to 7, it's recommended just to take the extra time and make all the comparisons, because using transitive property can have negative side effects to the results as presented in Section 4.3 and 4.4. The actual number of elements that could be calculated with transitive property in case of 9 to 20

alternatives ranges from 10 to 95, this is the situation when it might be worth considering using transitive property to reduce the number of pairwise comparisons, because it can reduce time to conduct the AHP and the final results could be better than in normal process due to survey fatigue.

Now, the theoretical maximum and optimal number of elements boundaries are known, in the next sections it's presented how consistency ratio and alternatives priorities weights are impacted if certain number of pairwise comparisons have been randomly removed and the missing elements are recalculated with transitive property.

4.3 Consistency ratio change in case of 3 to 10 alternatives

Overview of the data set that consisted of 10 000 matrices and was used to analyze matrices of 3 to 10 alternatives is presented on Figure 8. Median change for CR as CR_{dif} was -0.02814 or -2.8% meaning that the consistency ratio remained on average almost the same. Negative change in this context means that the CR improved and the results became more consistent, which was expected because fully transitive matrix CR is 0. It can be seen that CR ranged from 0 to 0.5, because the maximum allowed CR for 3 to 10 alternatives was set to 0.5. The median CR value was 0.4147, which proves the statement mentioned in Section 4.7, that it's difficult to generate low CR AHP matrices randomly.

```
> summary(matrices_data)
  CR          CR_cat          CR_recalc          CR_dif          Kendall          Matrix_size          values_rm
Min. :0.0000 Length:10000 Min. :0.0000 Min. :-0.49844 Min. :-0.6000 Min. : 3.000 Min. : 1.000
1st Qu.:0.3023 Class :character 1st Qu.:0.1882 1st Qu.: -0.10337 1st Qu.: 0.7333 1st Qu.: 5.000 1st Qu.: 1.000
Median :0.4147 Mode :character Median :0.3526 Median :-0.02814 Median : 0.8667 Median : 6.000 Median : 2.000
Mean :0.3674 Mean :0.3157 Mean :0.05168 Mean : 0.8379 Mean : 6.486 Mean : 3.384
3rd Qu.:0.4681 3rd Qu.:0.4497 3rd Qu.: 0.00946 3rd Qu.: 1.0000 3rd Qu.: 8.000 3rd Qu.: 4.000
Max. :0.5000 Max. :1.1243 Max. : 0.68412 Max. : 1.0000 Max. :10.000 Max. :13.000
```

Figure 8. Overview of 3 to 10 alternatives data set.

The actual distribution of AHP matrices and generated matrices CR's is presented in Section 4.7, but for analysis purposes all the CR categories were combined under one " $CR \leq 0.5$ " category. Only 13.3% of matrices had CR less than 0.2, due to low sample size of lower CR matrices, as presented on Figure 9 and Figure 21, lower categories weren't analyzed separately, because there wasn't enough or any data for analysis. For further research ideas how to create lower CR matrices are presented under Section 5.1.

```
> table(CR_cat2, Matrix_size)
      Matrix_size
CR_cat2  3  4  5  6  7  8  9 10
CR < 0.05 352 30  2  0  0  0  0  0
CR 0.05-0.1 137 77 18  3  0  0  0  0
CR 0.1-0.2 236 282 137 43 15  1  1  0
CR 0.2-0.3 185 279 323 203 105 30  8  2
CR 0.3-0.4 166 290 369 400 369 247 154 74
CR 0.4-0.5 175 285 423 589 798 975 1079 1138
```

Figure 9. Overview of 3 to 10 alternatives CR categories distribution.

On Figure 10 the graphs are split by the number of alternatives and graphs show CR change (y-axis) depending on the number of elements being recalculated (x-axis) with transitive property. General trend shows that the more number of elements were recalculated, the more CR value reduced, which is expected due to the nature of transitive property. Mean change was -0.05 or -5%. It must be also noticed that outliers existed and in some cases significant CR change should be expected, which can distort the AHP results due to the amplitude of CR difference.

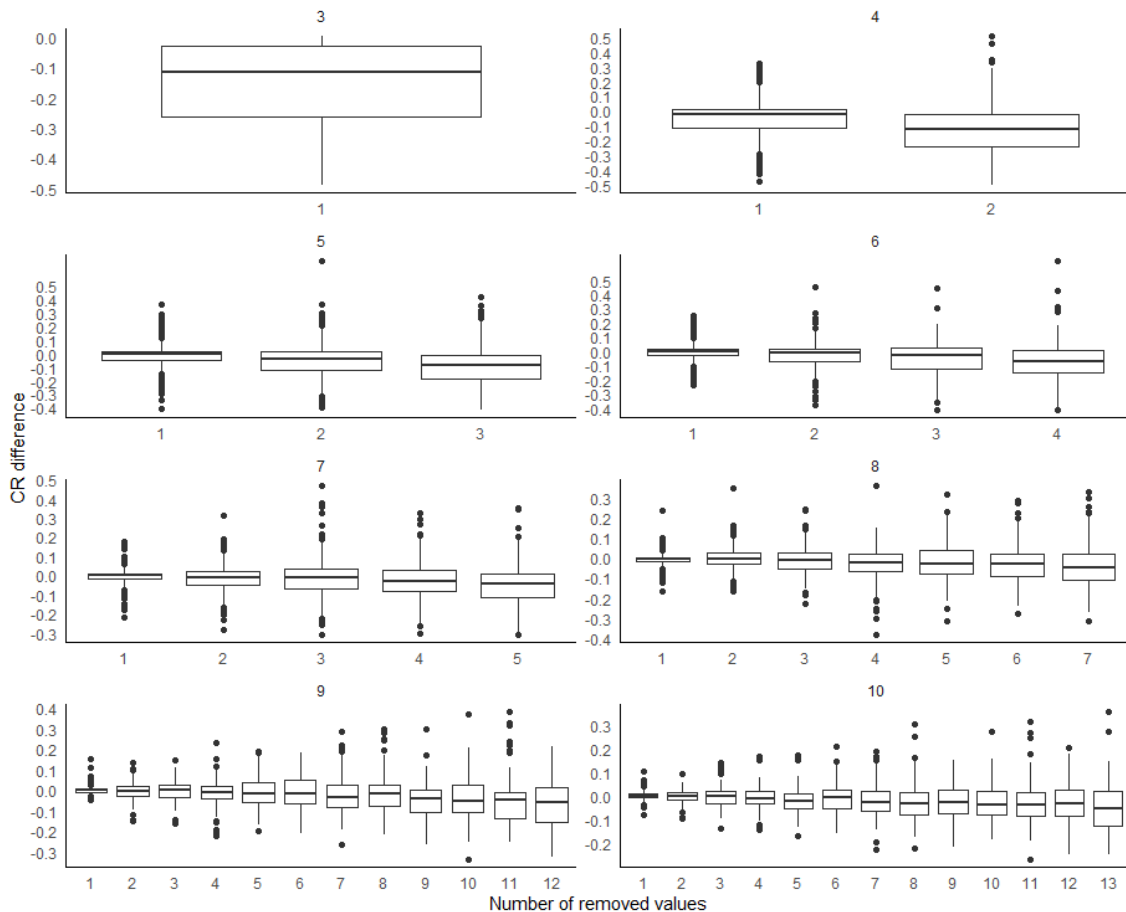


Figure 10. Overview of 3 to 10 alternatives CR change in relation to number of removed values.

In case of 5 to 10 alternative, on average there isn't almost any impact to CR, if 1 to 3 elements are calculated with the transitive property, of course the outliers can't be ignored and it must be kept in mind that depending on initial CR and missing elements, the impact can distort the AHP results. Though, in case of a missing element or decision-maker unwillingness to make some pairwise comparisons, the transitive property could be used to calculate the missing elements without having much or any impact on the final results, in most cases. Next section validates the approach from alternatives weights perspective that the weights are likely to stay the same in case of calculating 1 to 3 elements with the transitive property.

4.4 Alternatives weights change in case of 3 to 10 alternatives

Kendall rank correlation coefficient as *Kendall* was used to analyze change in alternatives weights. Coefficient ranges are explained in Subsection 3.1.7. Figure 8 shows that the median value of *Kendall* was 0.8667 and mean value was 0.8379, the weights remained on average almost the same, as it was also the case for CR. It must be noticed, that there exists values where coefficient was around zero or below zero, meaning either having negative association or absence of association. The p-value of *CR_dif* and *Kendall* was $2.2e-16$ meaning the results are statistically highly significant.

On Figure 11 the graphs are split by the number of alternatives and graphs show *Kendall* change (y-axis) depending on the number of elements being recalculated (x-axis) with transitive property. General trend shows that the more number of elements were recalculated, the more *Kendall* value decreased. Mean change was -0.1621. It must be also noticed that outliers existed and in some cases significant changes in weights should be expected, which can distort the AHP results due to the amplitude of *Kendall* difference.

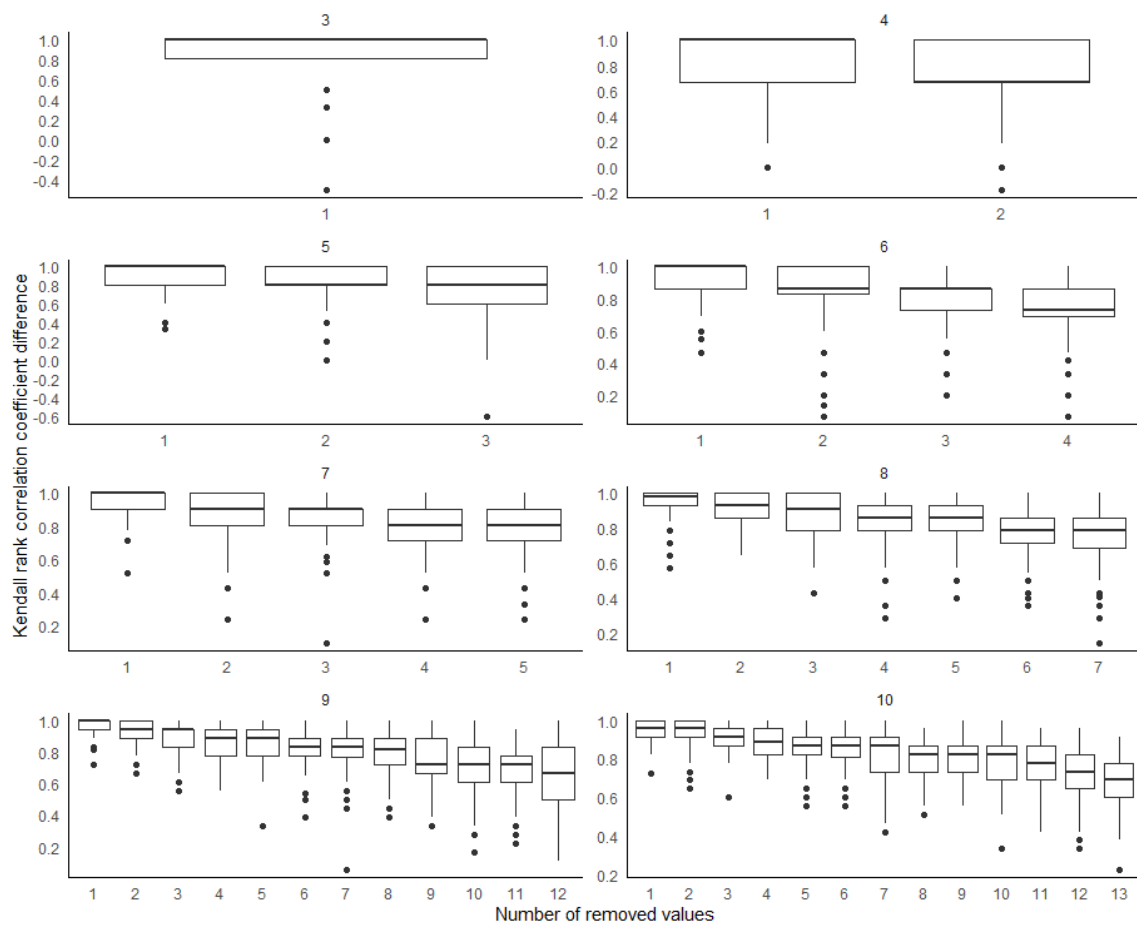


Figure 11. Overview of 3 to 10 alternatives weights change in relation to number of removed values.

4.5 Consistency ratio change in case of 11 to 20 alternatives

Overview of the data set that consisted of 100 000 matrices and was used to analyze matrices of 11 to 20 alternatives is presented on Figure 12. The CR didn't change significantly, considering the median CR value of 1.0027 or 100.27%. Median change for CR as *CR_dif* was -0.047420 or -4.7%. Negative change in this context means that the CR improved and the results became more consistent, which was expected because fully transitive matrix CR is 0. It can be seen that CR ranged from 0.5269 to 1.4487, there wasn't any restriction set for maximum or minimum CR when generating AHP matrices with 11 to 20 alternatives. From the median value it can be concluded, that if generating 11 to 20 alternatives AHP matrices randomly, the median CR is around 1.0. It's also worth noticing that the lowest AHP matrix from 100 000 simulations was with 0.5269 CR, validating the statement mentioned in Section 4.7, about difficulty of generating low CR AHP matrices randomly. Distribution of AHP matrices and generated matrices CR's is presented in Section 4.7.

```
> summary(matrices_data11x20)
  CR          CR_cat      CR_recalc      CR_dif      Kendall      MatrixSize      Values_rm      Values_rm_cat
Min. :0.5269 Length:100000 Min. :0.3776 Min. : -0.579811 Min. : -0.4545 Min. :11.00 Min. : 1.00 Length:100000
1st Qu.:0.9498 Class :character 1st Qu.:0.8695 1st Qu.: -0.110635 1st Qu.: 0.4421 1st Qu.:13.00 1st Qu.: 9.00 Class :character
Median :1.0027 Mode :character   Median :0.9438 Median : -0.047420 Median : 0.6000 Median :15.00 Median :19.00 Mode :character
Mean :1.0030 Mean :0.9388 Mean : -0.064128 Mean : 0.5903 Mean :15.48 Mean :23.51
3rd Qu.:1.0559 3rd Qu.:1.0120 3rd Qu.: -0.005323 3rd Qu.: 0.7516 3rd Qu.: 0.7516 3rd Qu.:18.00 3rd Qu.:33.00
Max. :1.4487 Max. :1.4168 Max. : 0.341713 Max. : 1.0000 Max. :20.00 Max. :95.00
```

Figure 12. Overview of 11 to 20 alternatives data set.

The number of elements that could be removed from 11 to 20 alternatives AHP matrices and recalculated started to increase quickly, as presented on Figure 2. In order to present the results, the removed elements were categorized into groups of five. Secondly, for the analysis of 11 to 20 alternatives two groups of CR: "*CR 0.5-1.0*" and "*CR >=1.0*", was created and are compared against each other.

As presented on Figure 13, in case of *CR 0.5-1.0* category median change of CR was -0.039704 or -3.97% while in case of *CR >= 1.0* category the median change was -0.055270 or -5.52%, as presented on Figure 14. It can be concluded, that transitive property impact on CR is greater if the initial CR is higher.

```
> summary(matrices_data11x20CRhalf)
  CR          CR_cat      CR_recalc      CR_dif      Kendall      MatrixSize      Values_rm      Values_rm_cat
Min. :0.5269 Length:48617 Min. :0.3776 Min. : -0.579811 Min. : -0.3590 Min. :11.00 Min. : 1.00 Length:48617
1st Qu.:0.9082 Class :character 1st Qu.:0.8264 1st Qu.: -0.098661 1st Qu.: 0.4667 1st Qu.:13.00 1st Qu.:10.00 Class :character
Median :0.9480 Mode :character   Median :0.8925 Median : -0.039704 Median : 0.6190 Median :16.00 Median :19.00 Mode :character
Mean :0.9349 Mean :0.8800 Mean : -0.054862 Mean : 0.6110 Mean :15.58 Mean :23.74
3rd Qu.:0.9756 3rd Qu.:0.9455 3rd Qu.: -0.001001 3rd Qu.: 0.7692 3rd Qu.: 0.7692 3rd Qu.:18.00 3rd Qu.:33.00
Max. :1.0000 Max. :1.2464 Max. : 0.341713 Max. : 1.0000 Max. :20.00 Max. :95.00
```

Figure 13. Overview of 11 to 20 alternatives data set in *CR 0.5-1.0* category.

Must be also noticed, that the difference of median CR for *CR 0.5-1.0* and *CR >= 1.0* category is -0.106. Considering that perfectly transitive matrix CR is 0, it seems logical, that transitive property impact is greater in case the initial CR is higher.

CR	CR_cat	CR_recalc	CR_dif	Kendall	MatrixSize	Values_rm	Values_rm_cat
Min. :1.000	Length:51383	Min. :0.5121	Min. :-0.546565	Min. :-0.4545	Min. :11.00	Min. :1.00	Length:51383
1st Qu.:1.025	Class :character	1st Qu.:0.9360	1st Qu.: -0.122310	1st Qu.: 0.4182	1st Qu.:13.00	1st Qu.: 9.00	Class :character
Median :1.054	Mode :character	Median :1.0022	Median : -0.055270	Median : 0.5758	Median :15.00	Median :18.00	Mode :character
Mean :1.067		Mean :0.9945	Mean : -0.072895	Mean : 0.5707	Mean :15.39	Mean :23.28	
3rd Qu.:1.094		3rd Qu.:1.0562	3rd Qu.: -0.009973	3rd Qu.: 0.7353	3rd Qu.:18.00	3rd Qu.:32.00	
Max. :1.449		Max. :1.4168	Max. : 0.283058	Max. : 1.0000	Max. :20.00	Max. :95.00	

Figure 14. Overview of 11 to 20 alternatives data set in $CR \geq 1.0$ category.

On Figure 15 and 16 the graphs are split by the number of alternatives and graphs show CR change (y-axis) depending on the number of elements being recalculated (x-axis) with transitive property. General trend shows that the more number of elements were recalculated, the more CR value reduced, which is expected due to the nature of transitive property. It must be also noticed that outliers existed and in some cases significant CR change should be expected, which can distort the AHP results due to the amplitude of CR difference.

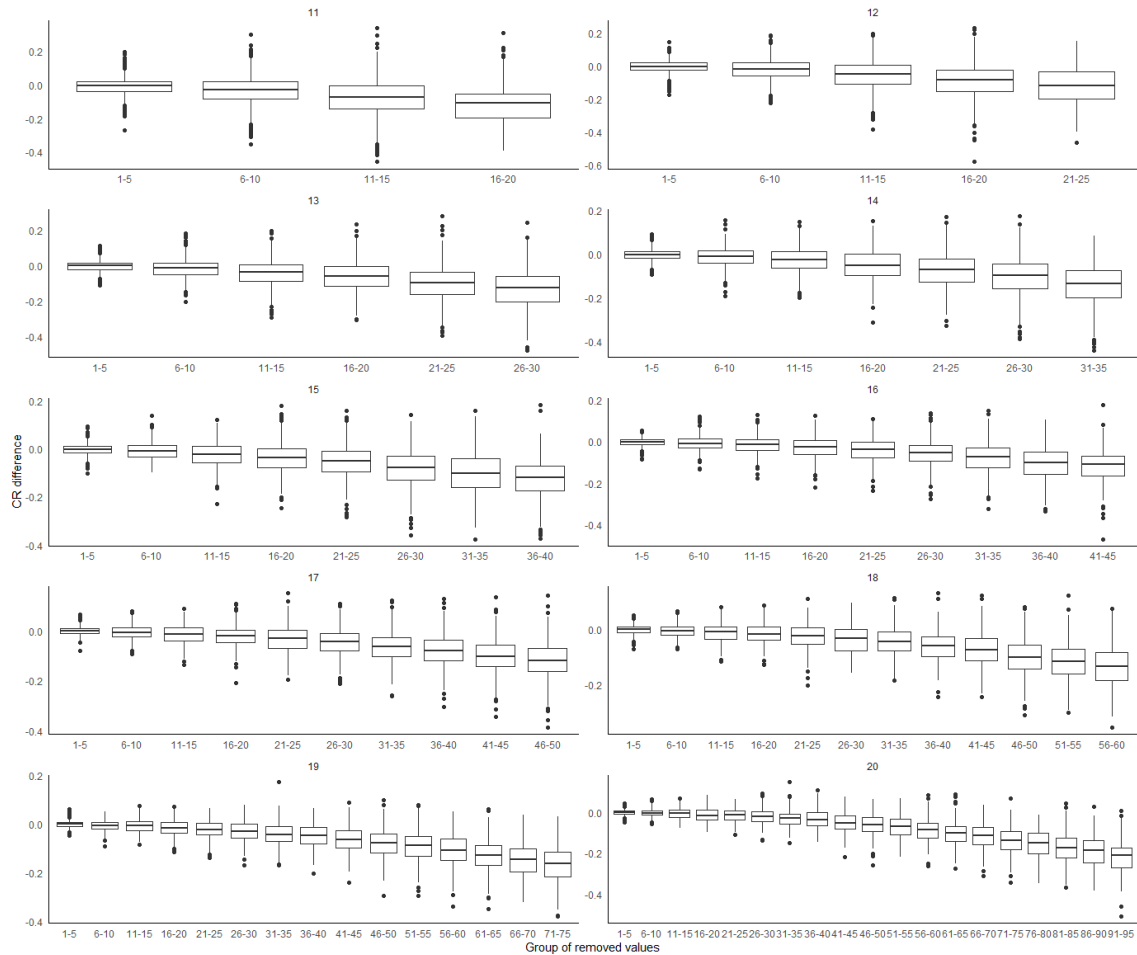


Figure 15. Overview of CR change in relation to group of removed values of 11 to 20 alternatives in $CR 0.5-1.0$ category.

In case of $CR \geq 1.0$ category the CR decreases slightly quicker, but as seen from median values the difference is rather insignificant, the percentage difference is -1.55%. In case of both categories, there isn't almost any change in CR, if number of theoretical maximum elements, presented in Table 3, are calculated with transitive property, which account for 9

to 16% of total elements, depending on the number of alternatives, the higher the number of alternatives, the lower the percentage. Figures 15 and 16 indicate that approximately half of the optimal number of missing comparisons can be computed using the transitive property, provided that the maximum permitted effect on the initial CR is roughly 0.1.

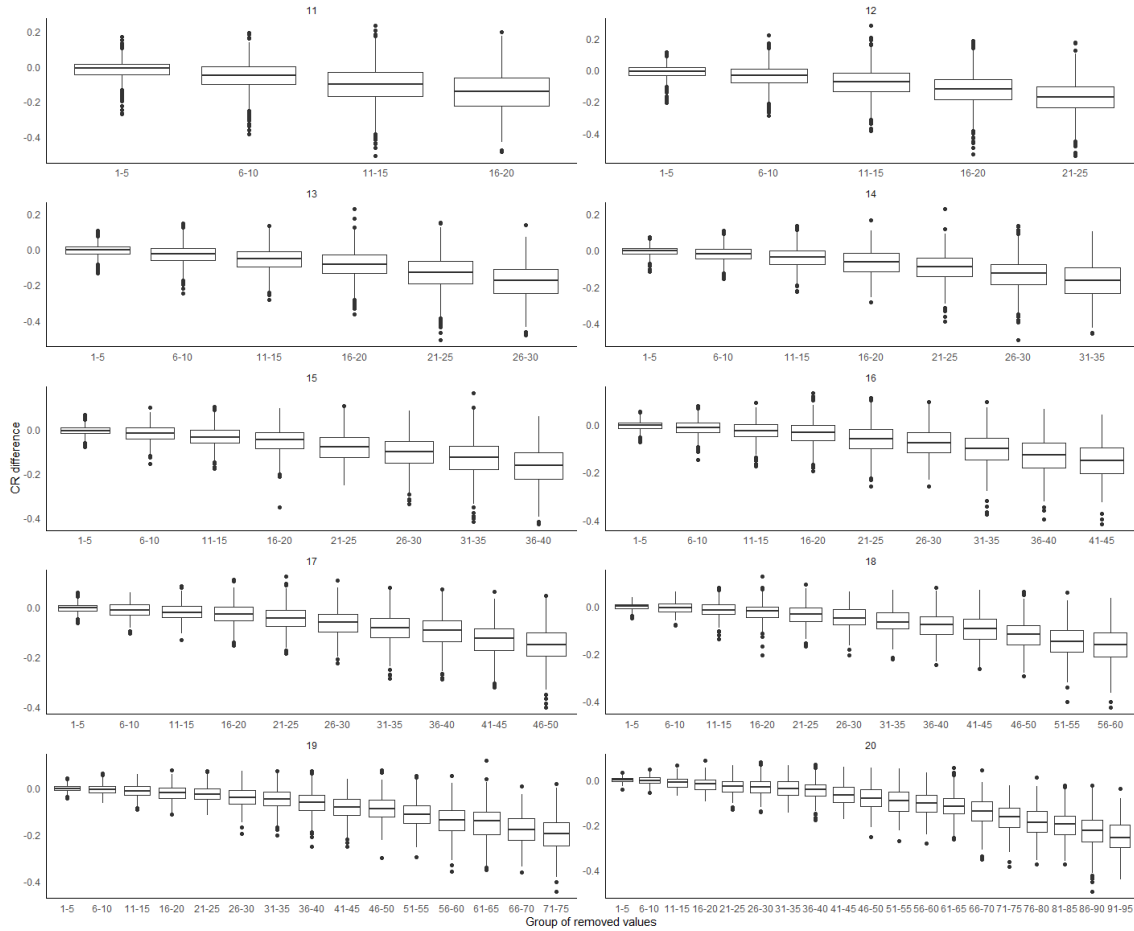


Figure 16. Overview of CR change in relation to group of removed values of 11 to 20 alternatives in $CR \geq 1.0$ category.

In the next section the alternatives weights change is being analyzed and it will be concluded what is the optimal amount of reduced elements based on *Kendall*.

4.6 Alternatives weights change in case of 11 to 20 alternatives

Figure 12 shows that the median value of *Kendall* was 0.6000 and mean value was 0.5903, it can be concluded that if the transitive property is applied, then it's highly probable that there is changes in the prioritization of alternatives. It must be noticed, that there exists values were coefficient was around zero or below zero, meaning either having negative association or absence of association.

As presented on Figure 13 in case of *CR 0.5-1.0* category median value of *Kendall* was 0.6190 while in case of $CR \geq 1.0$ category the median value was 0.5758, as presented on Figure 14. It can be concluded that transitive property impact on *Kendall* is higher if the initial *CR* is higher, meaning the prioritization of alternatives is more likely to change and at higher scale after transitive property has been used to calculate missing elements.

On Figure 17 the graphs are split by the number of alternatives and graphs show *Kendall* change (y-axis) depending on the number of elements being recalculated (x-axis) with transitive property. General trend shows that the more number of elements were recalculated, the more *Kendall* value decreased. Mean change was -0.389. It must be also noticed that outliers existed and in some cases significant changes in weights should be expected, which can distort the AHP results due to the amplitude of *Kendall* difference.

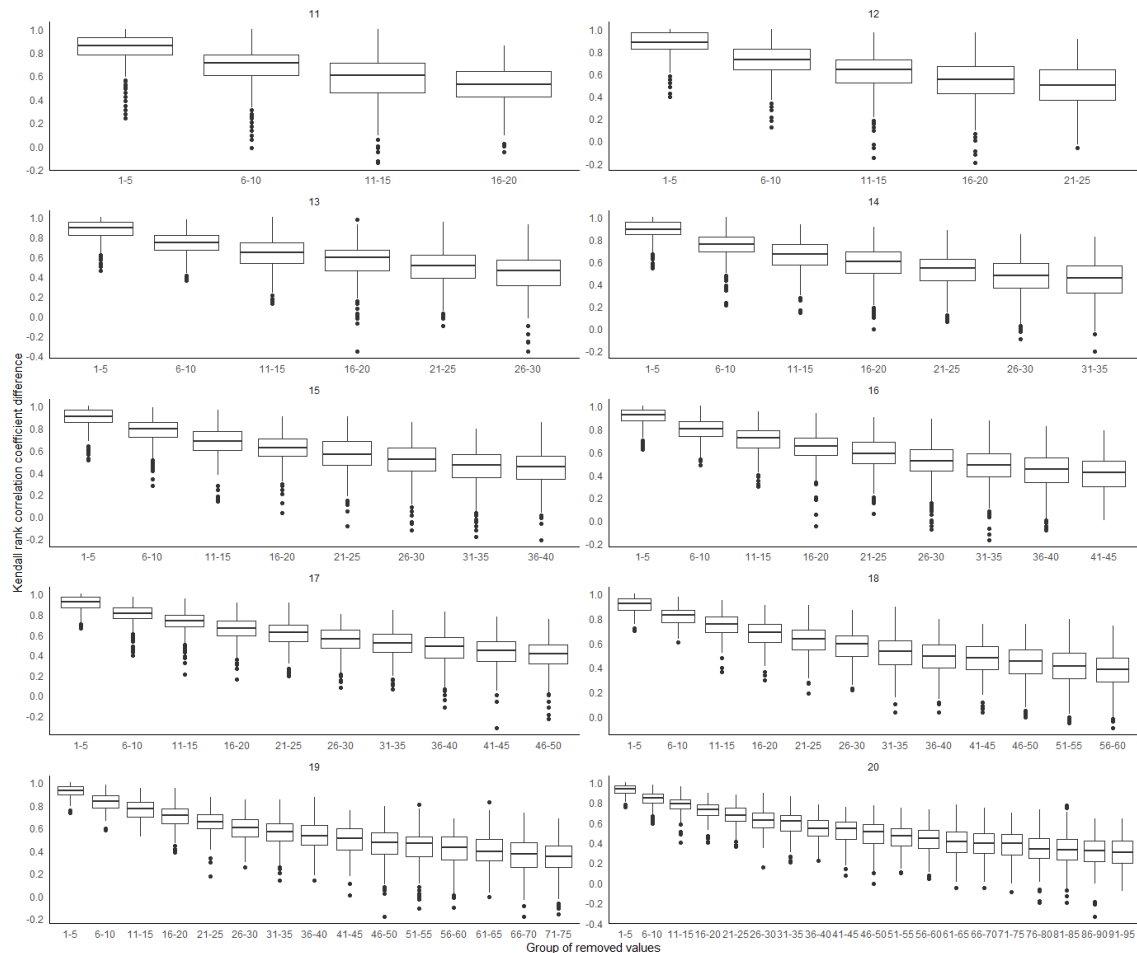


Figure 17. Overview of weights change in relation to group of removed values of 11 to 20 alternatives in *CR 0.5-1.0* category.

From Figure 18 same trend can be noticed, that the more number of elements were recalculated, the more *Kendall* value decreased. Mean change was -0.4293. Important to notice, that in case of higher initial *CR*, the amplitude of *Kendall* difference is higher.

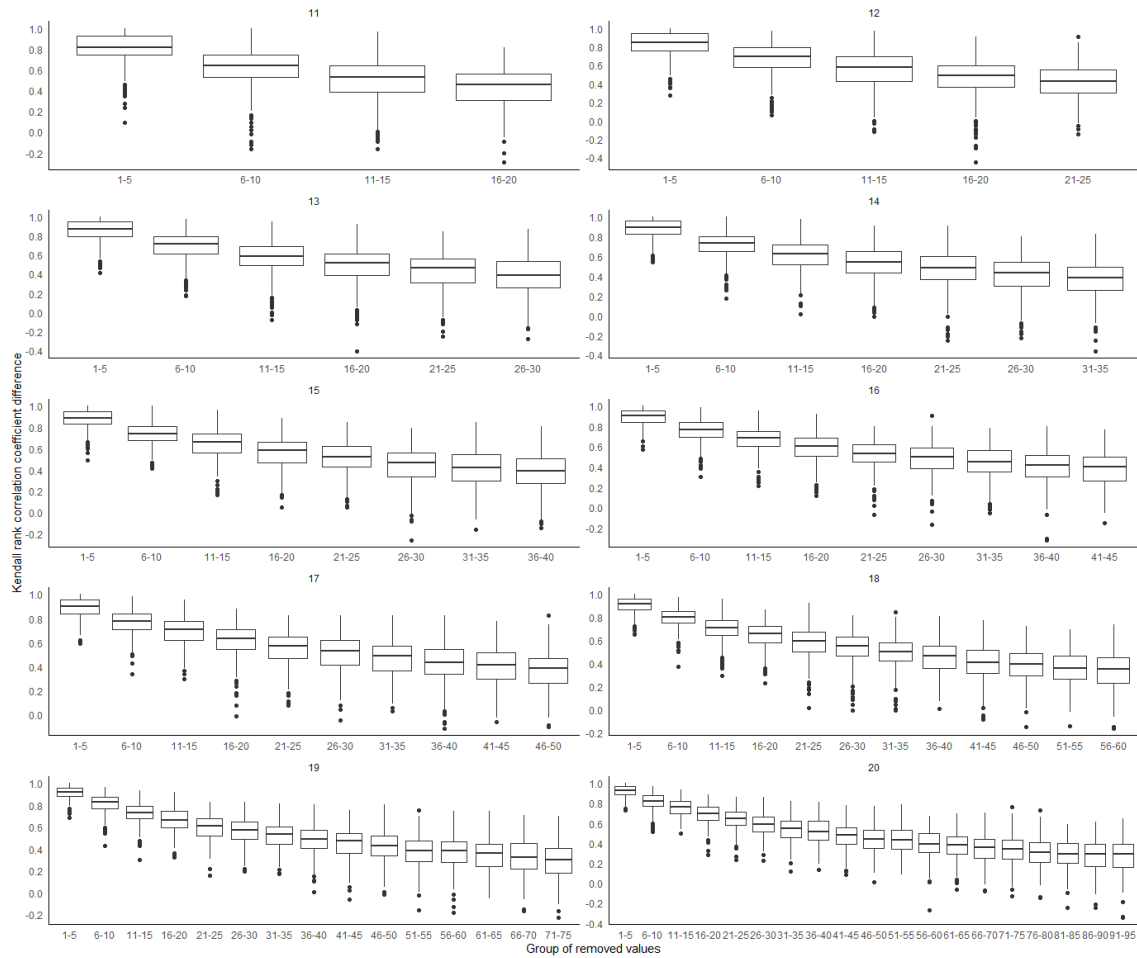


Figure 18. Overview of weights change in relation to group of removed values of 11 to 20 alternatives in $CR \geq 1.0$ category.

This concludes the descriptive data analysis part and in the next chapter data validation approaches are described and overview of different distributions are presented.

4.7 Data validation

Different approaches were used to validate the data. Firstly, it was validated if the randomness had worked properly, when synthetic data sets had been created. For that the distribution of matrices across the alternatives was checked, data should be distributed uniformly. That was the case and is presented on the Figures 19 and 20. "*set.seed(123)*" was used to be able to have reproducible results. Secondly, consistency ratio distribution was checked in the same way and results are presented on Figures 21 and 22.

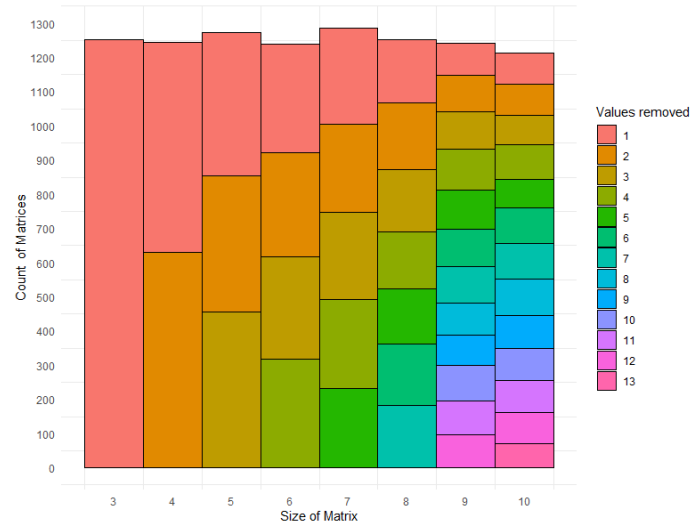


Figure 19. *Distribution overview of 3x3 to 10x10 AHP matrices and removed values.*

For 3 to 10 alternatives total 10 000 AHP matrices were created. Distribution across the number of alternatives was uniformal, around 1 250 matrices for each alternative. The number of values removed was distributed uniformly as well.

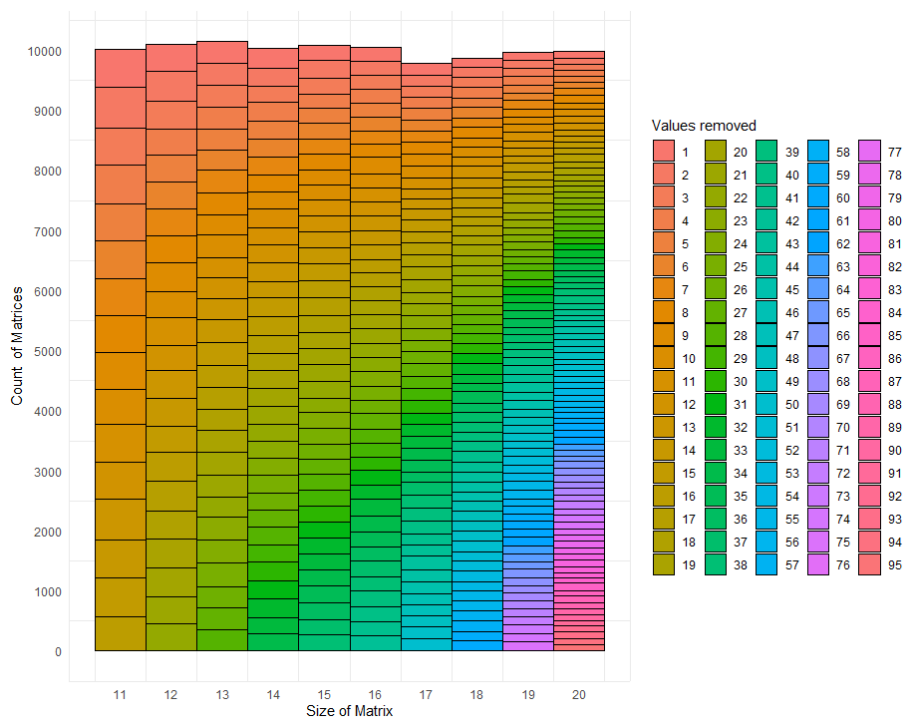


Figure 20. *Distribution overview of 11x11 to 20x20 AHP matrices and removed values.*

For 11 to 20 alternatives total 100 000 AHP matrices were created. Distribution across the number of alternatives was uniformal, around 10 000 matrices for each alternative. The number of values removed was distributed uniformly as well.

From Figure 21 shows that the likelihood of creating randomly an AHP matrix with less

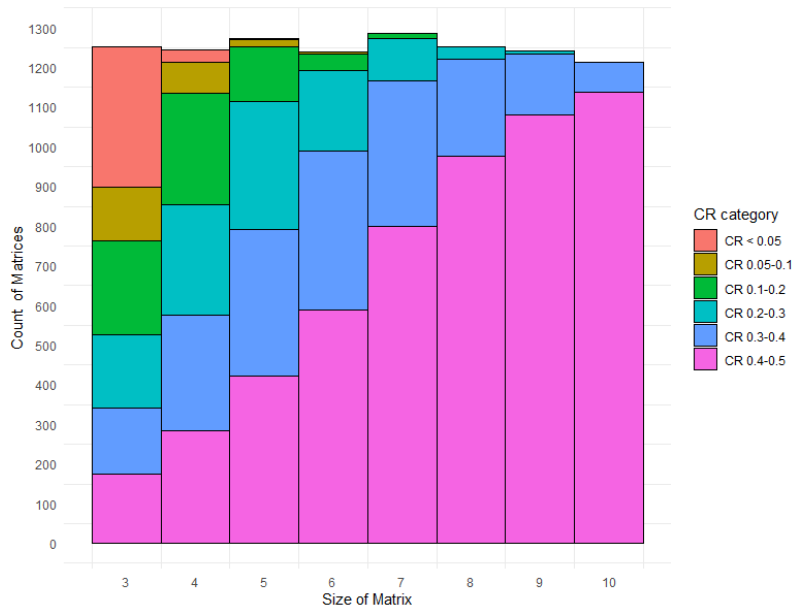


Figure 21. CR categories distribution overview of 3x3 to 10x10 AHP matrices.

than 0.4 CR started to decrease very quickly. Due to that reason one general CR group "CR ≤ 0.5" was used in the analysis.

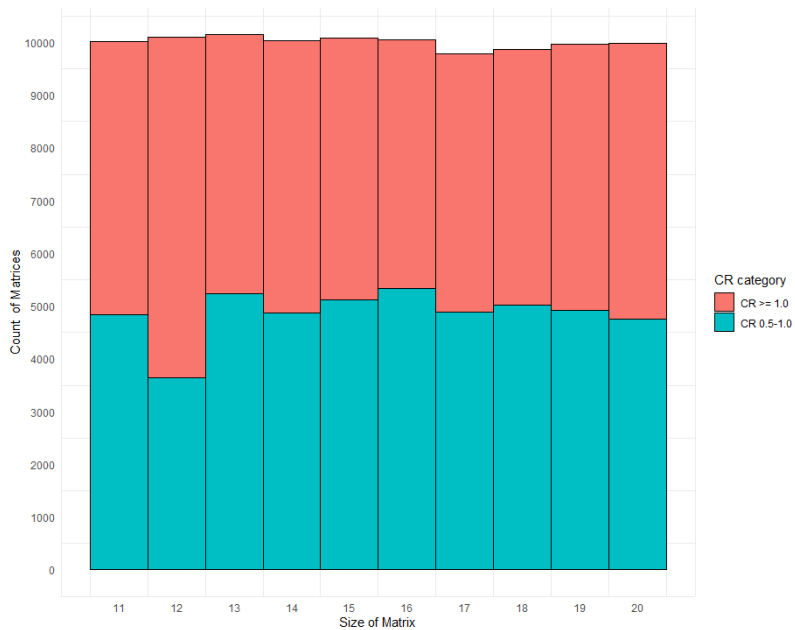


Figure 22. CR categories distribution overview of 11x11 to 20x20 AHP matrices.

It was similar case for 11 to 20 alternatives, though the median CR was even higher if there were more alternatives. CR categories distributed quite uniformly, in case of 12 alternatives it can be noticed that there was slightly more matrices in "CR ≥ 1.0" category, but that's most likely due to randomness.

Lastly, to validate the proposed optimal amount of missing comparisons presented in Table

3, depending on the number of alternatives either 10 000 or 100 000 simulations were run to validate the number. Kendall rank correlation coefficient was used to check if correlation between *CR_dif* and *Kendell* was statistically highly significant, which was the case and presented in Section 4.4. As mentioned under Section 3.1, BPMSG AHP calculator [33], developed by Klaus D. Goepel, was used to validate created R functions outputs.

5. Summary

Analytic hierarchy process, developed by Thomas L. Saaty, is one of the most widely used multi-criteria decision making methods. The major drawback with the AHP is the amount of work which is required in making all of the necessary pairwise comparisons, especially if there are more than recommended nine alternatives.

The purpose of the thesis was to determine the number of comparisons that could be calculated with the transitive property. Additionally, the thesis aimed to evaluate the impact of using the transitive property on the quality metrics of AHP results, including consistency ratio and weights of alternatives. The experimental quantitative data analysis method was adopted as the methodological approach, which involved creation of synthetic data sets as data collection method. Numerical data collected was analyzed and interpreted using descriptive data analysis.

Main findings of the thesis:

- Depending on the number of alternatives, around 25 to 40% of comparisons could be missing in most situations and still being able to calculate all the missing elements with the transitive property.
- As the number of elements calculated using the transitive property increased, there was a corresponding increase of changes in the prioritization of alternatives and a decrease in the consistency ratio.
- If no criteria were defined for the generated matrices consistency ratio, the average consistency ratio for 11 to 20 alternatives was around 1.0.

The actual number of elements that could be calculated with the transitive property in case of 9 to 20 alternatives ranges from 10 to 95, then it might be worth considering using the transitive property to reduce the number of pairwise comparisons, because it can reduce time to conduct the AHP and the final results could be better than in normal process due to survey fatigue. Otherwise, author recommends using the transitive property if missing a comparison information that was skipped by accident or calculating the comparison information that the decision-maker is unwilling to make.

All the experiment questions got answers, though further research could be done on the topic. List of further ideas are presented under further research recommendations.

5.1 Further research recommendations

In this chapter list of further ideas are presented.

- Instead of using simulations to find out the optimal number of missing comparisons, perhaps it could be solved mathematically. Might be worth looking into hypergeometric distribution and Monte Carlo simulations.
- Trying to generate a synthetic data sets by limiting maximum CR and then still calculating matrices randomly, worked somewhat in case of 3 to 10 alternatives and with limitation of maximum CR of 0.5. For example if 0.2 CR was set as maximum value, then in cases of 10 alternatives, it often took hours to find one such AHP matrix by chance. Author couldn't create any matrices that had less than 0.1 CR with 20 alternatives, even if the function tried to calculate one AHP matrix over 8 hours.
- An idea how to create better CR matrices: having a sample of fully transitive AHP matrices (CR is 0) for different number of alternatives, then starting to replace random elements with random values and checking CR after each change. The elements could be removed until defined maximum CR is crossed, then one step is taken back and the matrix is stored.
- Other idea to have better CR matrices is to still generate matrices randomly, but creating another function that could improve the random matrix until suitable CR is achieved.
- Improving the given code's, so the data sets creating would be quicker and has more customization possibilities in regards of CR. There are also some workarounds in the code, that could be improved.
- Using a data set where median CR would be lower (for example 0.1) to investigate what would be the impact on CR change and alternatives prioritization.
- Investigating deeper what changes in the in priorities are happening, for example in case of 11 to 20 alternatives, when calculating close to optimal amount of missing pairwise comparisons, did top 3 alternatives change or stayed the same? What's the number of elements that could be removed if the goal is not to change prioritization of top 3 alternatives?

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Appendix 2 - R code for synthetic database creation

```
# Function to create a random AHP matrix with predefined maximum CR value.

generate_ahp_matrix <- function(size_of_matrix) {
  # Create a size_of_matrix times size_of_matrix matrix with all zeros.
  ahp_matrix <- matrix(0, size_of_matrix, size_of_matrix)

  # Set matrix diagonal elements to 1.
  diag(ahp_matrix) <- 1

  # Pre-defined list of values that can be assigned into the matrix.
  ahp_matrix_values <- c(1, .2, .3, .4, .5, .6, .7, .8, 9, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9)

  # Set matrix upper triangular entries to random values from ahp_matrix_values.
  ahp_matrix[upper.tri(ahp_matrix)] <- sample(ahp_matrix_values, size = sum(upper.tri(ahp_matrix)), replace = TRUE)

  # Calculate lower triangular values as reciprocal values of the upper triangular.
  ahp_matrix[lower.tri(ahp_matrix)] <- 1/t(ahp_matrix)[lower.tri(ahp_matrix)]

  # Set matrix initial consistency ratio (CR) to 0 and then calculate generated matrix CR with calculate_CR function.
  matrix_CR <- 0
  matrix_CR <- calculate_CR(ahp_matrix)

  # Re-generate a matrix, if generated matrix CR value didn't meet the defined expectation, until condition is met.
  while (matrix_CR > 0.5) {
    ahp_matrix[upper.tri(ahp_matrix)] <- sample(ahp_matrix_values, size = sum(upper.tri(ahp_matrix)), replace = TRUE)
    ahp_matrix[lower.tri(ahp_matrix)] <- 1/t(ahp_matrix)[lower.tri(ahp_matrix)]
    matrix_CR <- calculate_CR(ahp_matrix)
  }
  return(ahp_matrix)
}
```


Appendix 3 – R code for removing and recalculating AHP matrix elements

```

set.seed(123)

removed_elements_index <- c(0, 0, 1, 2, 3, 4, 5, 7, 10, 13, 16, 22, 28, 34, 38, 42, 50, 60, 75, 95)
values_removed <- list()
recalc_matrices <- list()
recalc_ahp_matrices <- list()
recalc_consistency_ratios <- list()
recalc_ahp_priorities <- list()

for (matrix in 1:length(matrices_tibble$Matrix)) {

  # Matrix is restructured from matrices_tibble and assigned to recalc_ahp_matrix for further calculations
  recalc_ahp_matrix <- t(matrix(c(unlist(matrices_tibble$Matrix[matrix])), nrow =
    as.integer(matrices_tibble$MatrixSize[matrix]), byrow = TRUE))

  # Depending on the size of a matrix random amount of values within given range will be removed.
  removed_value <- 0
  values_to_remove <- sample(1:removed_elements_index[nrow(recalc_ahp_matrix)],1)

  # While loop will remove a random amount of values from a given matrix.
  while (removed_value < values_to_remove) {
    row <- sample(1:nrow(recalc_ahp_matrix),1)
    column <- sample(1:nrow(recalc_ahp_matrix),1)

    # Condition to check if element is not a diagonal element and element is not already NA.
    # If these conditions are met, then matrix element and it's reciprocal element is replaced with NA.
    if (row != column & is.na(recalc_ahp_matrix[row, column])==FALSE) {
      recalc_ahp_matrix[row, column] <- NA
      recalc_ahp_matrix[column, row] <- NA
    }

    # Counter to calculate how many values were removed from a matrix.
    removed_value = removed_value + 1
  }

  # A list to keep track of removed values from different matrices.
  values_removed <- append(values_removed, removed_value)

  # For loop is re-calculating removed elements based on transitivity rule:  $a_{ij} = a_{ik} * a_{kj}$ .
  for (i in 1:nrow(recalc_ahp_matrix)) {
    for (j in 1:ncol(recalc_ahp_matrix)) {
      if (is.na(recalc_ahp_matrix[i, j])) {
        for (k in 1:nrow(recalc_ahp_matrix)) {
          # Checking if selected i and j are not NA's. Both values are necessary to re-calculate a missing value.
          if (is.na(recalc_ahp_matrix[i, k])==FALSE & is.na(recalc_ahp_matrix[k, j])==FALSE) {

            # If calculated element is higher than maximum value in Saaty's scale (9), then it's replaced with
            # maximum value (9).
            if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] > 9) {
              recalc_ahp_matrix[i, j] = 9
              recalc_ahp_matrix[j, i] = 1/9
            }

            # If calculated element is lower than 1, then Saaty scale's reciprocal values ranges are used (Section 5.4)
            else if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] < 1) {

              if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] < 0.11806) {
                recalc_ahp_matrix[i, j] = 1/9
                recalc_ahp_matrix[j, i] = 9
              }
              else if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] < 0.13393) {
                recalc_ahp_matrix[i, j] = 1/8
                recalc_ahp_matrix[j, i] = 8
              }
              else if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] < 0.15476) {
                recalc_ahp_matrix[i, j] = 1/7
                recalc_ahp_matrix[j, i] = 7
              }
              else if (recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j] < 0.18333) {
                recalc_ahp_matrix[i, j] = 1/6
                recalc_ahp_matrix[j, i] = 6
              }
            }
          }
        }
      }
    }
  }
}

```

```

else if(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k,j] < 0.22500) {
  recalc_ahp_matrix[i, j] = 1/5
  recalc_ahp_matrix[j, i] = 5
}
else if(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k,j] < 0.29167) {
  recalc_ahp_matrix[i, j] = 1/4
  recalc_ahp_matrix[j, i] = 4
}
else if(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k,j] < 0.41667) {
  recalc_ahp_matrix[i, j] = 1/3
  recalc_ahp_matrix[j, i] = 3
}
else if(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k,j] < 0.75000) {
  recalc_ahp_matrix[i, j] = 1/2
  recalc_ahp_matrix[j, i] = 2
}
else if(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k,j] < 1.00000) {
  recalc_ahp_matrix[i, j] = 1
  recalc_ahp_matrix[j, i] = 1
}
}
}
# Otherwise the value is calculated based on transitivity rule and rounded to a nearest integer.
else {
  recalc_ahp_matrix[i, j] = round(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j], 0)
  recalc_ahp_matrix[j, i] = 1/round(recalc_ahp_matrix[i, k] * recalc_ahp_matrix[k, j], 0)
}
break
}
}
}
}
}
# Once the matrix has been restored, then it is added into a list and CR and weights are calculated and added into
# corresponding lists.
recalc_ahp_matrices [[matrix]] <- recalc_ahp_matrix
recalc_consistency_ratios [[matrix]] <- calculate_CR(recalc_ahp_matrix)
recalc_ahp_priorities [[matrix]] <- calculate_prios(recalc_ahp_matrix)
}

```

Appendix 4 – Various R codes to calculate attributes

```
# Function to calculate ahp_matrix consistency ratio

calculate_CR <- function(ahp_matrix) {

  # Calculate the row sums and divide each element by the corresponding row sum
  R <- rowSums(ahp_matrix)
  ahp_matrix_norm <- ahp_matrix / R

  # Calculate the maximum eigenvector of M matrix
  suppressWarnings(maxeigenvalue <- max(as.numeric(eigen(ahp_matrix)$values)))

  # Calculate the consistency index
  consistency_index <- (maxeigenvalue - nrow(ahp_matrix)) / (nrow(ahp_matrix) - 1)

  # Select a random index for the ahp_matrix
  random_index <- c(0, 0, 0.58, 0.90, 1.12, 1.24, 1.32, 1.41, 1.45, 1.49, 1.51, 1.48, 1.56, 1.57, 1.59, 1.605, 1.61,
    1.615, 1.62, 1.625)
  random_consistency_index <- random_index[nrow(ahp_matrix)]

  # Calculate the consistency ratio
  consistency_ratio <- consistency_index / random_consistency_index

  return(consistency_ratio)
}
```

```
# Calculating ahp_matrix priorities weights with ahp package

calculate_prios <- function(ahp_matrix) {

  weights <- ahp::PrioritiesFromPairwiseMatrixGeometricMean(ahp_matrix)$priority
}
```

```
# Calculate ahp_matrix matrix size

calculate_matrix_size <- function(ahp_matrix) {

  return(nrow(ahp_matrix))
}
```

Appendix 5 – Various R codes for data visualization

```
# How many matrices there are with different sizes

ggplot(matrices_data , aes(x = Matrix_size , fill=factor(Values_rm))) +
  geom_histogram(color="black", bins=8) + #color = "black", fill = "white",
  theme_minimal() +
  scale_x_continuous(breaks = seq(1,10, by = 1)) +
  scale_y_continuous(breaks = seq(0, 10000, by = 100)) +
  theme(panel.grid.major.x = element_blank(), panel.grid.major.y = element_blank()) +
  xlab("Size of Matrix") +
  ylab("Count of Matrices") +
  scale_fill_discrete("Values removed")
```

```
# CR_cat distribution

ggplot(matrices_data , aes(x = Matrix_size , fill=factor(CR_cat2))) +
  geom_histogram(color="black", bins=8) +
  theme_minimal() +
  scale_x_continuous(breaks = seq(1,10, by = 1)) +
  scale_y_continuous(breaks = seq(0, 10000, by = 100)) +
  theme(panel.grid.major.x = element_blank(), panel.grid.major.y = element_blank()) +
  xlab("Size of Matrix") +
  ylab("Count of Matrices") +
  scale_fill_discrete("CR category")
```

```
# Line graph to show how number of pair-wise comparisons is in relation to alternatives

xValue <- 2:20
yValue <- (xValue * (xValue - 1)) / 2
line1_data <- data.frame(xValue , yValue)

ggplot(line1_data , aes(x=xValue , y=yValue)) +
  geom_line() +
  theme_minimal() +
  theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
        panel.background = element_blank(), axis.line = element_line(colour = "black")) +
  scale_x_continuous(breaks = seq(0,20, by = 2)) +
  scale_y_continuous(breaks = seq(0,200, by = 20)) +
  xlab("Alternatives") +
  ylab("Comparisons")
```

```
# Line graph to show difference between initial comparisons and maximum optimal amount of removed values

line2_data <- data.frame(alternatives = c(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20),
                        initial = c(0,1,3,6,10,15,21,28,36,45,55,66,78,91,105,120,136,153,171,190),
                        simulated = c(0,1,2,4,7,11,16,21,26,32,39,44,50,57,67,78,86,93,96,95),
                        theoretical = c(0,1,2,4,7,11,16,22,29,37,46,56,67,79,92,106,121,137,154,172))

line2_data <- line2_data %>% pivot_longer(cols=c('initial', 'simulated', 'theoretical'),
                                         names_to='Projection',
                                         values_to='value')

ggplot(line2_data , aes(x=alternatives , y=value)) +
  theme_minimal() +
  geom_line(aes(color=Projection)) +
  theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
        panel.background = element_blank(), axis.line = element_line(colour = "black")) +
  scale_x_continuous(breaks = seq(0,20, by = 2)) +
  scale_y_continuous(breaks = seq(0,200, by = 25)) +
  xlab("Alternatives") +
  ylab("Comparisons") +
  geom_vline(xintercept = 9, col="red", lwd=0.5)
```

```
# Saaty plot of random inconsistency

xRI <- 1:15 #seq(0, 1.8, by=0.2)
yRI <- c(0,0,0.52,0.89,1.11,1.25,1.35,1.40,1.45,1.49,1.52,1.54,1.56,1.58,1.59)
lineRI_data <- data.frame(xValue2 , yValue2)

ggplot(lineRI_data , aes(x=xRI , y=yRI)) +
```

```

geom_point() +
theme_minimal() +
theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
      panel.background = element_blank(), axis.line = element_line(colour = "black")) +
scale_x_continuous(breaks = seq(0,15, by = 1)) +
scale_y_continuous(breaks = seq(0,1.8, by = 0.2)) +
xlab("Number of elements compared") +
ylab("Random inconsistency")

```

```

# Boxplot that show's how CR changed in relation to removed values. Split by matrix sizes.

```

```

matrices_data11x20 %>%
  filter(MatrixSize < 17) %>%
  filter(CR_cat == "CR 0.5-1.0") %>%
  ggplot(aes(x=factor(Values_rm_cat, level = rm_cat_order), y=CR_dif))+
  geom_boxplot() +
  theme_minimal() +
  theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
        panel.background = element_blank(), axis.line = element_line(colour = "black")) +
  facet_wrap(~MatrixSize, scale="free", nrow = 5)+
  scale_y_continuous(breaks=seq(-1, 1, by = 0.2)) +
  xlab("Group of removed values") +
  ylab("CR difference")

```

```

# Boxplot that show's how weights changed in relation to removed values. Split by matrix sizes.

```

```

matrices_data11x20 %>%
  filter(MatrixSize < 17) %>%
  filter(CR_cat == "CR >= 1.0") %>%
  ggplot(aes(x=factor(Values_rm_cat, level = rm_cat_order), y=Kendall))+
  geom_boxplot() +
  theme_minimal() +
  theme(panel.grid.major = element_blank(), panel.grid.minor = element_blank(),
        panel.background = element_blank(), axis.line = element_line(colour = "black")) +
  facet_wrap(~MatrixSize, scale="free", nrow = 5)+
  scale_y_continuous(breaks=seq(-1, 1, by = 0.2)) +
  xlab("Group of removed values") +
  ylab("Kendall rank correlation coefficient difference")

```