DOCTORAL THESIS

Proof Theory of Semi-Substructural Logics

Cheng-Syuan Wan

TALLINNA TEHNIKAÜLIKOOL
TALLINN UNIVERSITY OF TECHNOLOGY
TALLINN 2025

TALLINN UNIVERSITY OF TECHNOLOGY
DOCTORAL THESIS
43/2025

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TALTECH PRESS

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The dissertation was accepted for the defence of the degree of Doctor of Philosophy (Information Technology) on 2 June 2025

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Defence of the thesis: 25 June 2025, Tallinn

Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

Cheng-Syuan Wan ______________________
signature

ISSN 2585-6898 (publication)
ISBN 978-9916-80-324-0 (publication)
ISSN 2585-6901 (PDF)
ISBN 978-9916-80-325-7 (PDF)
DOI https://doi.org/10.23658/taltech.43/2025
Printed by EVG Print

Wan, C.-S. (2025). *Proof Theory of Semi-Substructural Logics* [TalTech Press]. https://doi.org/10.23658/taltech.43/2025

TALLINNA TEHNIKAÜLIKOOL
DOKTORITÖÖ
43/2025

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TALTECH KIRJASTUS

Contents

List of Publications

The present Ph.D. thesis is based on the following publications.

I T. Uustalu, N. Veltri, and C.-S. Wan. Proof theory of skew non-commutative MILL. In A. Indrzejczak and M. Zawidzki, editors, *Proceedings of 10th International Conference on Non-classical Logics: Theory and Applications, NCL 2022*, volume 358 of *Electronic Proceedings in Theoretical Computer Science*, pages 118–135. Open Publishing Association, 2022 (Chapter 2)

II N. Veltri and C.-S. Wan. Semi-substructural logics with additives. In T. Kutsia, D. Ventura, D. Monniaux, and J. F. Morales, editors, *Proceedings of 18th International Workshop on Logical and Semantic Frameworks, with Applications and 10th Workshop on Horn Clauses for Verification and Synthesis, LSFA/HCVS 2023*, volume 402 of *Electronic Proceedings in Theoretical Computer Science*, pages 63–80. Open Publishing Association, 2024 (Chapter 5)

III C.-S. Wan. Semi-substructural logics à la Lambek. In A. Indrzejczak and M. Zawidzki, editors, *Proceedings of 11th International Conference on Non-classical Logics: Theory and Applications, NCL 2024*, volume 415 of *Electronic Proceedings in Theoretical Computer Science*, pages 195–213. Open Publishing Association, 2024 (Chapter 6)

IV N. Veltri and C.-S. Wan. Craig interpolation for a semi-substructural logic. *Studia Logica*, to appear (Chapters 2 and 3)

Chapter 4 and the second half of Chapter 6 (starting from Section 6.4) are not covered by the publications above.

The work on this thesis also involved significant proof assistant formalization effort. The Agda code associated to the thesis is available in my GitHub repository `https://github.com/cswphilo/code-PhD-thesis`.

Author's Contributions to the Publications

I My contribution to this work was to identify the research problem, to formulate the main definitions, to prove the theorems, write the manuscript, and present the paper at the corresponding conference.

II My contribution to this work was to identify the research problem, to formulate the main definitions, to prove the theorems, implement proofs in Agda, write the manuscript, and present the paper at the corresponding conference.

III I was the sole author. The results of the paper are my own, and I presented the paper at the corresponding conference.

IV My contribution to this work was to identify the research problem, to formulate the main definitions, to prove the theorems, and write the manuscript.

1 My contribution to the underlying material of Chapter 4, a collaborative work with Niccolò Veltri, was to identify the research problem, to formulate the main definitions, to prove the theorems, write the manuscript.

2 I was the sole author of the underlying material of the second half of Chapter 6.

Chapter 1

Introduction and Background

Substructural logics can be understood through the lens of sequent calculus as logical systems that omit or modify one or more structural rules. In the sequent calculi for classical and intuitionistic logic with explicit structural rules, three key structural rules govern how assumptions are handled: weakening, contraction, and exchange. These rules permit assumptions to be discarded, duplicated, or reordered freely, as shown below:

$$\frac{\Gamma \vdash C}{A, \Gamma \vdash C}\ \mathsf{wk} \qquad \frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C}\ \mathsf{ctr} \qquad \frac{\Gamma_0, A, B, \Gamma_1 \vdash C}{\Gamma_0, B, A, \Gamma_1 \vdash C}\ \mathsf{ex}$$

These structural rules effectively transform the list-like structure of antecedents into sets, allowing arbitrary duplication and reordering of elements. Substructural logics emerge when we selectively remove these rules, resulting in systems where the antecedents maintain more rigid algebraic structures than sets. This restriction creates logics that can more accurately model resource-sensitive or order-dependent reasoning.

A canonical example is Lambek's syntactic calculus [43], which eliminates weakening, contraction, and exchange entirely. This system treats antecedents as ordered lists of formulae, making it particularly suited for analyzing linguistic structures. Different variations and extensions have emerged, including the non-associative Lambek calculus [52], which relaxes the structure of antecedents. In this system, antecedents are structured as binary trees rather than flat lists, meaning $(A, B), C$ and $A, (B, C)$ are distinguished (see Section 2.2.2 for a formal definition of trees). The structural rules governing associativity are formalized as follows:

$$\frac{T[U_0, (U_1, U_2)] \vdash C}{T[(U_0, U_1), U_2] \vdash C}\ \mathsf{assoc}_1 \qquad \frac{T[(U_0, U_1), U_2] \vdash C}{T[U_0, (U_1, U_2)] \vdash C}\ \mathsf{assoc}_2$$

Another significant development in substructural logic is Girard's linear logic [28], which offers a more nuanced approach to resource management. While it generally prohibits weakening and contraction, it introduces modalities (also called exponentials) that can selectively restore these capabilities for specific formulae. This flexibility has proven valuable in both theoretical studies and practical applications.

The deep connections between these systems become apparent when we examine their logical connectives. The fragment of intuitionistic linear logic without exponentials shares its connectives with Lambek calculus (augmented with unit and exchange), revealing fundamental structural similarities. This correspondence extends to various non-commutative and non-associative variants, though these variations are typically studied more in the context of Lambek calculus than in linear logic. When non-commutative or non-associative linear logic is discussed, it often includes (sub)exponentials that allow a modal formula to swap its position with adjacent formula or allow a grouping of modal formulae to change the way it is bracketed, see [10] for an example.

1.1 Substructural Logics and Categorical Semantics

In this section and the rest of the thesis, we always use the `typewriter` font to refer to a logic, understood as the set of judgements (sequents) provable in its canonical proof system (ignoring that different proof systems for the same logic may use slightly different judgement forms). For example, the non-commutative multiplicative intuitionistic linear logic is denoted as `NMILL`.

We then use subscripts to denote specific formulations of proof systems for these logics where `S` indicates sequent calculus (with and without stoup), `T` indicates tree sequent calculus, and `A` indicates axiomatic calculus. For example, $\texttt{NMILL}_{\texttt{S}}$ refers to the sequent calculus of non-commutative multiplicative intuitionistic linear logic.

1.1.1 Substructural Logics

As a foundational substructural logic, let us consider the sequent calculus for the tensor-unit fragment of non-commutative multiplicative intuitionistic linear logic ($\texttt{NMILL}^{\mathsf{I},\otimes}$):

$$\frac{}{A \vdash A}\ \mathsf{ax} \qquad \frac{}{\vdash \mathsf{I}}\ \mathsf{IR} \qquad \frac{\Gamma_0, \Gamma_1 \vdash C}{\Gamma_0, \mathsf{I}, \Gamma_1 \vdash C}\ \mathsf{IL}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \qquad \frac{\Gamma_0, A, B, \Gamma_1 \vdash C}{\Gamma_0, A \otimes B, \Gamma_1 \vdash C}\ \otimes\mathsf{L}$$

A sequent $\Gamma \vdash C$ consists of antecedents forming the context (a list of formulae) Γ and a single formula C as the succedent. Reading $\Gamma \vdash C$ tells us we can derive C from the resources in Γ.

Stronger substructural logics are built incrementally from $\texttt{NMILL}^{\mathsf{I},\otimes}$ by adding rules corresponding to new connectives, see Figure 1.1.

The first extension of $\texttt{NMILL}^{\mathsf{I},\otimes}$ comes from adding linear implications ($\multimap$ and $\circ\!-$), also known as right and left residuations in Lambek calculus (written as $/$ and $\backslash$ respectively). When we interpret antecedents as concatenation of resources, the right rule for $\multimap$ can be understood in this way: if we can deduce B from Γ, A, then using only Γ (meaning Γ has been divided from the right by A), we can deduce a resource $A \multimap B$. This can be read as "B that is divided by A from the right". The interpretation for $\circ\!-$ follows the same pattern in the opposite direction. Notice that two right rules of implication are invertible in any extension constructed from the rules in Figure 1.1. In a proof tree, we sometimes use f to

(NMILL$^{\mathsf{I},\otimes}$)

$$\frac{}{A \vdash A}\ \mathsf{ax} \qquad \frac{}{\vdash \mathsf{I}}\ \mathsf{IR} \qquad \frac{\Gamma_0, \Gamma_1 \vdash C}{\Gamma_0, \mathsf{I}, \Gamma_1 \vdash C}\ \mathsf{IL}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \qquad \frac{\Gamma_0, A, B, \Gamma_1 \vdash C}{\Gamma_0, A \otimes B, \Gamma_1 \vdash C}\ \otimes\mathsf{L}$$

(linear implications)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \multimap\mathsf{R} \qquad \frac{\Gamma \vdash A \quad \Delta_0, B, \Delta_1 \vdash C}{\Delta_0, A \multimap B, \Gamma, \Delta_1 \vdash C}\ \multimap\mathsf{L}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash B \circ\!- A}\ \circ\!-\mathsf{R} \qquad \frac{\Gamma \vdash A \quad \Delta_0, B, \Delta_1 \vdash C}{\Delta_0, \Gamma, B \circ\!- A, \Delta_1 \vdash C}\ \circ\!-\mathsf{L}$$

(additives)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge\mathsf{R} \qquad \frac{\Gamma_0, A_i, \Gamma_1 \vdash C}{\Gamma_0, A_1 \wedge A_2, \Gamma_1 \vdash C}\ \wedge\mathsf{L}_i$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ \vee\mathsf{R}_i \qquad \frac{\Gamma_0, A, \Gamma_1 \vdash C \quad \Gamma_0, B, \Gamma_1 \vdash C}{\Gamma_0, A \vee B, \Gamma_1 \vdash C}\ \vee\mathsf{L}$$

(exchange)

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathsf{ex}_{A,B}$$

Figure 1.1: Rules for building sequent calculi of multiplicative and additive intuitionistic linear logics through NMILL$^{\mathsf{I},\otimes}$

denote the derivation ends with a certain sequent. Therefore the invertibility of two right rules of implication are expressed as two admissible rules below, which are both proved by structural induction on the derivation f.

$$\frac{\overset{f}{\Gamma \vdash A \multimap B}}{\Gamma, A \vdash B}\ \multimap\mathsf{R}^{-1} \qquad \frac{\overset{f}{\Gamma \vdash B \circ\!- A}}{A, \Gamma \vdash B}\ \circ\!-\mathsf{R}^{-1}$$

We can develop the additive extension of NMILL$^{\mathsf{I},\otimes}$ independently by incorporating rules for additive connectives $\wedge$ and $\vee$ into the calculus. These connectives $\wedge$ and $\vee$ carry their standard meaning from classical logic. An important distinction lies in how the rules handle contexts: two-premises rules like $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$ split the context (called context-splitting rules), dividing antecedents between the two premises. In contrast, $\wedge\mathsf{R}$ and $\vee\mathsf{L}$ are context-sharing rules where all antecedent resources remain identical in both premises. This sharing behavior is precisely why we call them "additive" rules.

The final extension leads to a commutative system by adding the rule ex (we sometimes omit the subscript when there is no ambiguity). For any extension constructed from the commutative NMILL$^{\mathsf{I},\otimes}$ with the rules in Figure 1.1, the admissibility of the following general permutation rules always holds:

$$\frac{\Gamma, A, \Lambda, \Delta \vdash C}{\Gamma, \Lambda, A, \Delta \vdash C}\ \mathsf{exs}_{A,\Lambda} \qquad \frac{\Gamma, \Lambda, A, \Delta \vdash C}{\Gamma, A, \Lambda, \Delta \vdash C}\ \mathsf{exs}_{\Lambda,A}$$

They are both proved by induction on the complexity of Λ.

The commutative extensions of NMILL$^{\mathsf{I},\otimes}$ have two noteworthy features:

1. When the commutative extension includes linear implications, the two implications collapse into one. Specifically, formulae $A \multimap B$ and $B \circ\!- A$ become

logically equivalent.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \multimap B \vdash A \multimap B}\ \mathsf{ax}}{A \multimap B, A \vdash B}\ {\multimap}\mathsf{R}^{-1}}{A, A \multimap B \vdash B}\ \mathsf{ex}_{A \multimap B, A}}{A \multimap B \vdash B \circ\!\!- A}\ {\circ\!\!-}\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B \circ\!\!- A \vdash B \circ\!\!- A}\ \mathsf{ax}}{A, B \circ\!\!- A \vdash B}\ {\circ\!\!-}\mathsf{R}^{-1}}{B \circ\!\!- A, A \vdash B}\ \mathsf{ex}_{A, B \circ\!\!- A}}{B \circ\!\!- A \vdash A \multimap B}\ {\multimap}\mathsf{R}$$

2. In non-commutative logics, when we look at left rules, principal formulae always appear sandwiched between two contexts in the antecedents, since the order of formulae matters and cannot be freely rearranged like in classical logic. With the general exchange rule in place, we can place principal formulae of left rules at either the leftmost or the rightmost position of antecedents. For example, consider the (left) fixed version of $\otimes\mathsf{L}$:

$$\dfrac{A, B, \Gamma \vdash C}{A \otimes B, \Gamma \vdash C}\ \otimes\mathsf{L}'$$

The rule $\otimes\mathsf{L}'$ together with the general permutation rules could recover the original $\otimes\mathsf{L}$, i.e.

$$\dfrac{\dfrac{f}{\Gamma, A, B, \Delta \vdash C}}{\Gamma, A \otimes B, \Delta \vdash C}\ \otimes\mathsf{L} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f}{\Gamma, A, B, \Delta \vdash C}}{A, \Gamma, B, \Delta \vdash C}\ \mathsf{exs}_{\Gamma, A}}{A, B, \Gamma, \Delta \vdash C}\ \mathsf{exs}_{\Gamma, B}}{A \otimes B, \Gamma, \Delta \vdash C}\ \otimes\mathsf{L}'}{\Gamma, A \otimes B, \Delta \vdash C}\ \mathsf{exs}_{A \otimes B, \Gamma}$$

Other left rules follow a similar pattern. As a consequence, the commutative extensions of $\mathtt{NMILL}^{!,\otimes}$ admit equivalent variations where the ex rule becomes implicit, meaning there is no formal ex rule presented and the antecedents become multisets rather than lists.

As mentioned above, linear logic allows exponentials that selectively restore weakening and contraction for specific formulae. The exponential operator ! (often read as "of course" or "bang") marks formulae that can be treated more like reusable/disposable resources, allowing them to be duplicated or discarded. While a detailed discussion of exponentials lies beyond the scope of this thesis, readers seeking comprehensive coverage can find it in Girard's seminal work on linear logic [28], while Abrusci's work [1] specifically addresses non-commutative linear logic. For reference, we present the rules here (note that the calculus is implicitly commutative):

$$\dfrac{A, \Gamma \vdash C}{!A, \Gamma \vdash C}\ !\mathsf{L} \qquad \dfrac{!\Gamma \vdash C}{!\Gamma \vdash !C}\ !\mathsf{R} \qquad \dfrac{\Gamma \vdash C}{!A, \Gamma \vdash C}\ \mathsf{wk} \qquad \dfrac{!A, !A, \Gamma \vdash C}{!A, \Gamma \vdash C}\ \mathsf{ctr}$$

These rules capture how exponential formulae behave from a bottom-up perspective: they can be derelicted (!L, removing ! from a banged formula in the antecedent), promoted (!R, removing ! from the succedent formula when the formulae in the antecedent are all banged, in particular also when the antecedent is empty), weakened (wk, discarding a resource), or contracted (ctr, duplicating a resource). Exponentials provide the flexibility when we need some formulae to behave like formulae in classical/intuitionistic logic within our linear system.

1.1.2 Categorical Semantics

The sound and complete categorical models of $\mathtt{NMILL}^{\mathsf{I},\otimes}$ are monoidal categories, which provide a natural semantic framework for understanding these logical systems. Monoidal categories are defined as follows:

Definition 1.1.1. A *monoidal category* $\mathbb{C}$ is a category with a unit object I and a functor $\otimes : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ with three natural isomorphisms λ, ρ, α typed $\lambda_A : \mathsf{I}\otimes A \cong A$, $\rho_A : A \cong A \otimes \mathsf{I}$ and $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, satisfying Mac Lane's coherence conditions [46]:

$$
\text{(m1)}\quad
\begin{array}{ccc}
 & \mathsf{I}\otimes\mathsf{I} & \\
{\scriptstyle\rho_{\mathsf{I}}}\nearrow & & \searrow{\scriptstyle\lambda_{\mathsf{I}}} \\
\mathsf{I} & = = = & \mathsf{I}
\end{array}
\qquad
\text{(m2)}\quad
\begin{array}{ccc}
(A\otimes\mathsf{I})\otimes B & \xrightarrow{\alpha_{A,\mathsf{I},B}} & A\otimes(\mathsf{I}\otimes B) \\
{\scriptstyle\rho_A\otimes B}\uparrow & & \downarrow{\scriptstyle A\otimes\lambda_B} \\
A\otimes B & = = = & A\otimes B
\end{array}
$$

$$
\text{(m3)}\quad
\begin{array}{ccc}
(\mathsf{I}\otimes A)\otimes B & \xrightarrow{\alpha_{\mathsf{I},A,B}} & \mathsf{I}\otimes(A\otimes B) \\
{\scriptstyle\lambda_A\otimes B}\searrow & & \swarrow{\scriptstyle\lambda_{A\otimes B}} \\
 & A\otimes B &
\end{array}
\qquad
\text{(m4)}\quad
\begin{array}{ccc}
(A\otimes B)\otimes\mathsf{I} & \xrightarrow{\alpha_{A,B,\mathsf{I}}} & A\otimes(B\otimes\mathsf{I}) \\
{\scriptstyle\rho_{A\otimes B}}\nwarrow & & \nearrow{\scriptstyle A\otimes\rho_B} \\
 & A\otimes B &
\end{array}
$$

$$
\text{(m5)}\quad
\begin{array}{ccccc}
(A\otimes(B\otimes C))\otimes D & & \xrightarrow{\alpha_{A,B\otimes C,D}} & & A\otimes((B\otimes C)\otimes D) \\
{\scriptstyle\alpha_{A,B,C}\otimes D}\uparrow & & & & \downarrow{\scriptstyle A\otimes\alpha_{B,C,D}} \\
((A\otimes(B\otimes C))\otimes D & \xrightarrow[\alpha_{A\otimes B,C,D}]{} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow[\alpha_{A,B,C\otimes D}]{} & A\otimes(B\otimes(C\otimes D))
\end{array}
$$

Kelly [39] observed that equations (m1), (m3), and (m4) can be derived from (m2) and (m5).

An important property of monoidal categories is Mac Lane's coherence theorem [47] stating that, in any monoidal category, given any two parallel maps, if they are constructed only via the identity, composition, the tensor, the two unitors and the associator, then they are equal. In other words, in the free monoidal category over a given set of objects, any two maps with the same domain and codomain are equal, which also means that there is at most one element in the set of morphisms between any two objects, i.e. the free monoidal category is *thin*.

Similar to the construction in the last section, categorical models of extensions of $\mathtt{NMILL}^{\mathsf{I},\otimes}$ are also constructed modularly from monoidal categories.

The categorical models of $\mathtt{NMILL}^{\mathsf{I},\otimes}$ with rules for linear implications are monoidal bi-closed categories:

Definition 1.1.2. A *monoidal bi-closed category* $\mathbb{C}$ is a monoidal category in addition with two functors $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ and $\multimapinv : \mathbb{C} \times \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ forming two adjunctions $- \otimes B \dashv B \multimap -$ and $B \otimes - \dashv - \multimapinv B$ natural in B, respectively [44].

Remark 1.1.3. Lambek calculus with unit is also modelled by monoidal bi-closed categories, while the original Lambek calculus (without unit) is modelled by monoidal bi-closed categories without the unit I.

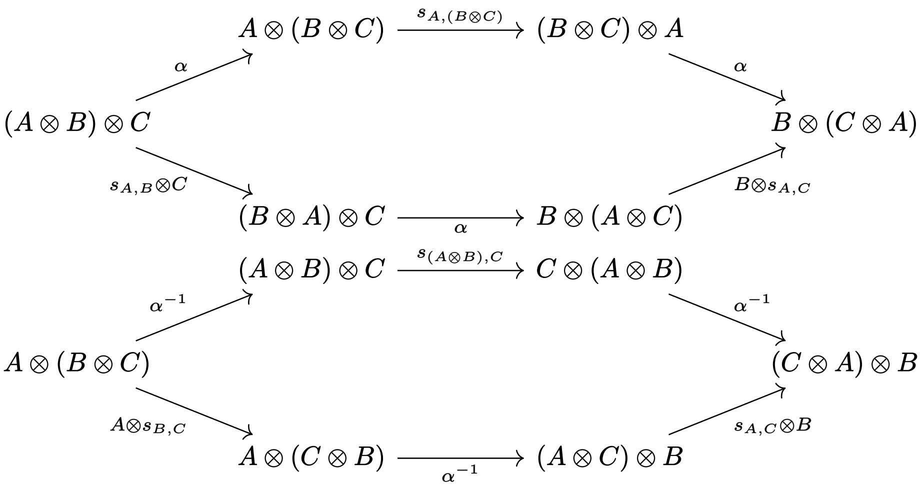
The categorical models of $\mathtt{NMILL}^{\mathsf{I},\otimes}$ with rules of $\wedge$ and $\vee$ are distributive monoidal categories with binary products.

Definition 1.1.4. A *distributive monoidal category with binary products* $\mathbb{C}$ is a monoidal category in addition with binary products $(\times)$ and coproducts $(+)$ that are distributive, i.e. the canonical morphisms, $(A \otimes C) + (B \otimes C) \to (A + B) \otimes C$ and $(C \otimes A) + (C \otimes B) \to C \otimes (A + B)$, have inverses $l : (A + B) \otimes C \to (A \otimes C) + (B \otimes C)$ and $r : C \otimes (A + B) \to (C \otimes A) + (C \otimes B)$, respectively [6] (notice that the usual definition of distributive monoidal categories does not include binary products).

The distributivity is required to correctly capture the provable sequents $(A \vee B) \otimes C \vdash (A \otimes C) \vee (B \otimes C)$ and $C \otimes (A \vee B) \vdash (C \otimes A) \vee (C \otimes B)$ in the extensions of $\mathtt{NMILL}^{!,\otimes}$ with $\wedge$ and $\vee$.

The categorical models of $\mathtt{NMILL}^{!,\otimes}$ with ex are symmetric monoidal categories:

Definition 1.1.5. A *braided monoidal category* $\mathbb{C}$ is a monoidal category with an additional natural isomorphism $s_{A,B} : A \otimes B \to B \otimes A$, called the *braiding*, satisfying Joyal and Street's hexagon identities [37]:



For any braiding s on a monoidal category, s^{-1} is also a braiding [37].
A braided monoidal category is *symmetric* if s satisfies $s_{B,A} \circ s_{A,B} = \mathsf{id}_{A \otimes B}$.

Despite our approach of adding one structure at a time, we can also construct arbitrary combinations of the aforementioned categories, which serve as categorical semantics for their corresponding extensions of $\mathtt{NMILL}^{!,\otimes}$. For instance, symmetric monoidal closed categories provide categorical models of $\mathtt{NMILL}^{!,\otimes}$ extended with both linear implications and exchange.

Definition 1.1.6. A *symmetric monoidal closed category* $\mathbb{C}$ is a symmetric monoidal category (Definition 1.1.5) equipped with a functor $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ natural in B [25].

As noted earlier, the two implications collapse into one in the commutative case, hence we only need one additional functor and adjunction to model this extension.

The exponentials in linear logic are modelled through the addition of a linear exponential comonad (!) on symmetric monoidal closed categories [8]. This comonad serves as a bridge between linear and cartesian worlds, transforming linear (resource-sensitive) formulae into cartesian (resource-insensitive) formulae. Furthermore, intuitionistic linear logic with ! is equivalent to the linear-non-linear logic of Benton (LNL [7]). LNL can be modelled via a symmetric monoidal adjunction between a symmetric monoidal closed category and a cartesian closed category. This adjunction provides a controlled mechanism for movement between these categories. Since a detailed discussion of exponentials lies beyond the scope of this thesis, we direct interested readers to Benton et al.'s papers [8, 7] for an in-depth treatment of the linear exponential comonad and adjunction. For broader perspectives and comprehensive surveys of categorical semantics, readers may consult [22, 49].

1.2 Semi-Substructural Logics

1.2.1 Motivation

Left skew monoidal categories [61] are a weaker variant of Mac Lane's monoidal categories where the structural morphisms of associativity and unitality are not required to be bidirectional, they are natural transformations with a particular orientation. Therefore, they can be seen as *semi-associative* and *semi-unital* variants of monoidal categories.

Definition 1.2.1. A *left skew monoidal category* $\mathbb{C}$ is a category with a unit object I and a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with three *natural transformations* λ, ρ, α typed $\lambda_A : \mathsf{I} \otimes A \to A$, $\rho_A : A \to A \otimes \mathsf{I}$ and $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, satisfying the Mac Lane axioms in Definition 1.1.1.

In contrast with Definition 1.1.1, equations (m1), (m3), and (m4) cannot be derived from (m2) and (m5), making all five equations essential in the definition of left skew monoidal categories.

Left skew monoidal categories arise naturally in the semantics of programming languages [4], while the concept of semi-associativity is connected with combinatorial structures like the Tamari lattice and Stasheff associahedra [74, 51]. Left skew monoidal categories quickly attracted significant attention from category theorists, leading to numerous developments [41, 60, 42, 16, 11, 13, 14].

In the remainder of the thesis, we sometimes omit the word "left" when discussing left skew monoidal categories and related structures (such as left skew monoidal closed categories and symmetric left skew monoidal categories) when no right skew variants are under discussion.

The coherence problem for left skew monoidal categories differs significantly from that of traditional monoidal categories. Unlike monoidal categories, where the coherence theorem guarantees uniqueness of parallel morphisms in the free monoidal category, the free skew monoidal category can have multiple distinct morphisms between the same pair of objects.

Example 1.2.2. These diagrams illustrate two distinct morphisms between the same objects in the free skew monoidal category. In each diagram, one path is the identity morphism while the other is the combination of λ, ρ, and α. These diagrams show the key feature of the free skew monoidal category that all of its characterizations should respect

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\rho_A \otimes B} & (A \otimes \mathsf{I}) \otimes B \\ {\scriptstyle A\otimes\lambda_B}\uparrow & \neq & \downarrow{\scriptstyle \alpha_{A,\mathsf{I},B}} \\ A \otimes (\mathsf{I} \otimes B) & = & A \otimes (\mathsf{I} \otimes B) \end{array}$$

$$\begin{array}{ccc} A \otimes (\mathsf{I} \otimes B) & \xrightarrow{A \otimes \lambda_B} & A \otimes B \\ {\scriptstyle \alpha_{A,\mathsf{I},B}}\uparrow & \neq & \downarrow{\scriptstyle \rho_A\otimes B} \\ (A \otimes \mathsf{I}) \otimes B & = & (A \otimes \mathsf{I}) \otimes B \end{array}$$

The non-uniqueness of morphisms distinguishes the free skew monoidal category from the free monoidal category. Specifically, some pairs of objects in the free skew monoidal category have multiple morphisms between them. To address the coherence problem, researchers have developed several approaches:

- Through rewriting techniques, Uustalu [62] showed that there is at most one map between an object and its normal form.
- Through categorical methods, Lack and Street [42] established a faithful functor to a category of ordinals, later elaborated by Bourke and Lack [12] with explicit morphism descriptions.
- Through the focused proof system by Uustalu, Veltri, and Zeilberger [67], which we will introduce in Section 1.2.3.

Skew monoidal categories and their variants lead us to consider coherence problems in a broader context. We call the logical correspondent to these variants of skew monoidal categories *semi-substructural* logics. We use this term because these are intermediate logics between non-associative and associative, as well as between non-unital and unital variants of intuitionistic linear logic (or Lambek calculus), offering finer control over structural properties like associativity and unitality.

1.2.2 Sequent Calculus for Skew Monoidal Categories

To formalize the logic of skew monoidal categories, Uustalu et al. [67] develop a sequent calculus framework that captures the semi-associativity and semi-unitality of skew monoidal categories. In this section, we review the sequent calculus in their work [67].

Formulae are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes B$, where X comes from a fixed set At of atoms, I is a multiplicative verum, $\otimes$ is a multiplicative conjunction.

The key feature is the modified sequent structure of the form $S \mid \Gamma \vdash A$, where:

- S is an optional formula, called the stoup (following Girard's terminology [29]). It will be written as $-$ when S is empty.
- Γ is an ordered list of formulae, called the context.
- A is a single formula, call the succedent.

The derivations in the sequent calculus are generated inductively by the following rules:

$$\frac{}{A \mid \;\vdash A}\;\mathsf{ax} \qquad \frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\;\mathsf{pass}$$

$$\frac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\;\mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\;\otimes\mathsf{L}$$

$$\frac{}{- \mid \;\vdash \mathsf{I}}\;\mathsf{IR} \qquad \frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\;\otimes\mathsf{R}$$

The stoup plays a crucial role in representing the semi-associativity and semi-unitality. By carefully restricting how introduction rules can interact with the stoup and the context Γ, we can encode semi-associativity and semi-unitality in a natural way. This framework allows only the specific direction of associativity and unitality transformations presented in Definition 1.2.1, reflecting the "skew" nature of the corresponding categorical structures.

The inference rules in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ are reminiscent of the ones in $\mathtt{NMILL}^{\mathsf{I},\otimes}$, but there are some crucial differences.

1. The left logical rules IL and $\otimes\mathsf{L}$, read bottom-up, are only allowed to be applied on the formula in the stoup position. In particular, there is no general way to remove a unit I nor decompose a tensor $A \otimes B$ if these formulae are located in the context and not in the stoup.

2. The right tensor rule $\otimes\mathsf{R}$, read bottom-up, splits the antecedent of the conclusion between the two premises whereby the formula in the stoup, in case such a formula is present, has to be moved to the stoup of the first premise. In particular, the stoup formula of the conclusion cannot be moved to the antecedent of the second premise even if Γ is chosen to be empty.

3. The presence of the stoup implies a distinction between antecedents of forms $A \mid \Gamma$ and $- \mid A, \Gamma$. The structural rule pass (for 'passivation'), read bottom-up, allows the moving of the leftmost formula in the context to the stoup position whenever the stoup is initially empty.

The restrictions in 1–3 are essential for precisely capturing all the features of skew monoidal closed categories and nothing more. Notice also that, similarly to the case of $\mathtt{NMILL}^{\mathsf{I},\otimes}$, all structural rules of exchange, contraction, and weakening are absent. We give names to derivations and we write $f : S \mid \Gamma \vdash A$ when f is a particular derivation of the sequent $S \mid \Gamma \vdash A$. We refer readers to Section 2.1 to see how the design of the sequent calculus captures the natural transformation λ, ρ, and α in Definition 1.2.1 while forbidding their inverses.

To address the coherence problem of skew monoidal categories, Uustalu et al. define a congruence relation $\overset{\circ}{=}$ on the sets of derivations in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$, generated by the pairs of derivations in Figure 1.2.

The sequent calculus $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ is cut-free:

$$\frac{}{\mathsf{I} \mid \ \vdash \mathsf{I}}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{}{- \mid \ \vdash \mathsf{I}}\ \mathsf{IR}}{\mathsf{I} \mid \ \vdash \mathsf{I}}\ \mathsf{IL}$$

$$\frac{}{A \otimes B \mid \ \vdash A \otimes B}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax} \quad \dfrac{\dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{- \mid B \vdash B}\ \mathsf{pass}}{A \mid B \vdash A \otimes B}\ \otimes\mathsf{R}}{A \otimes B \mid \ \vdash A \otimes B}\ \otimes\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A}}{- \mid A', \Gamma \vdash A}\ \mathsf{pass} \quad \overset{g}{- \mid \Delta \vdash B}}{- \mid A', \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{- \mid A', \Gamma, \Delta \vdash A \otimes B}\ \mathsf{pass}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A}}{\mathsf{I} \mid \Gamma \vdash A}\ \mathsf{IL} \quad \overset{g}{- \mid \Delta \vdash B}}{\mathsf{I} \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{- \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{\mathsf{I} \mid \Gamma, \Delta \vdash A \otimes B}\ \mathsf{IL}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A}}{A' \otimes B' \mid \Gamma \vdash A}\ \otimes\mathsf{L} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \otimes B' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \doteq \quad \dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \mid B', \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{A' \otimes B' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{L}$$

Figure 1.2: Equivalence of derivations in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$

Proposition 1.2.3 ([67, Lemma 5])**.** *The sequent calculus enjoys cut admissibility: the following two cut rules are admissible.*

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}$$

Proposition 1.2.4 ([67, Lemma 10])**.** $\otimes\mathsf{L}$ *and* IL *are invertible, i.e. the following rules are admissible:*

$$\dfrac{\overset{f}{A \otimes B \mid \Gamma \vdash C}}{A \mid B, \Gamma \vdash C}\ \otimes\mathsf{L}^{-1} \qquad \dfrac{\overset{f}{\mathsf{I} \mid \Gamma \vdash C}}{- \mid \Gamma \vdash C}\ \mathsf{IL}^{-1}$$

Moreover, the invertible rules $\otimes\mathsf{L}^{-1}$ and IL^{-1} are compatible with the equivalence relation in Figure 1.2, i.e.

$$\begin{array}{rclcrcl} \otimes\mathsf{L}^{-1}(\otimes\mathsf{L}\ f) & = & f & \quad & \otimes\mathsf{L}(\otimes\mathsf{L}^{-1}\ f) & \doteq & f \\ \mathsf{IL}^{-1}(\mathsf{IL}\ f) & = & f & & \mathsf{IL}(\mathsf{IL}^{-1}\ f) & \doteq & f \end{array}$$

Uustalu et al. proved that ($\mathtt{SkNMILL}^{\mathsf{I},\otimes}$) is sound and complete with respect to skew monoidal categories. They demonstrated this by establishing its equivalence to the following axiomatic (or Hilbert-style) calculus [67, Theorems 1 and 2]:

$$\frac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id} \quad \frac{A \vdash_{\mathsf{A}} B \quad B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} C}\ \mathsf{comp} \quad \frac{A \vdash_{\mathsf{A}} C \quad B \vdash_{\mathsf{A}} D}{A \otimes B \vdash_{\mathsf{A}} C \otimes D}\ \otimes$$
$$\frac{}{\mathsf{I} \otimes A \vdash_{\mathsf{A}} A}\ \lambda \quad \frac{}{A \vdash_{\mathsf{A}} A \otimes \mathsf{I}}\ \rho \quad \frac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} A \otimes (B \otimes C)}\ \alpha \tag{1.1}$$

The axiomatic calculus captures the free skew monoidal category generated by a set At. The morphisms between formulae A and B are derivations of the sequent $A \vdash_{\mathsf{A}} B$, quotiented by the congruence relation $\doteq$ given in Figure 2 of [67]. Importantly, this equivalence applies not only to the provability of sequents but also to their proofs: for any two derivations f and g in the axiomatic calculus, if $f \doteq g$, then their corresponding derivations f' and g' are also equivalent in the sequent calculus $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$, meaning $f' \overset{\circ}{=} g'$, and vice versa.

The proof of equivalence is covered by the one of Theorem 2.2.5, so we do not reproduce it here.

1.2.3 The Coherence Problem of Skew Monoidal Categories

When we consider the congruence relation $\overset{\circ}{=}$ from Figure 1.2 as a term rewrite system (interpreting equations from left to right), we observe two key properties: local confluence and strong normalization. These properties together ensure that the system is confluent and has unique normal forms. These normal forms can be described by a focused subsystem of the sequent calculus, which differs from but shares key similarities with the system introduced by Andreoli [5] for classical linear logic.

The derivations of the focused subsystem are generated inductively by the rules:

(left phase)
$$\frac{- \mid \Gamma \vdash_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{L}} C}\ \mathsf{IL} \quad \frac{A \mid B, \Gamma \vdash_{\mathsf{L}} C}{A \otimes B \mid \Gamma \vdash_{\mathsf{L}} C}\ \otimes\mathsf{L} \quad \frac{A \mid \Gamma \vdash_{\mathsf{L}} C}{- \mid A, \Gamma \vdash_{\mathsf{L}} C}\ \mathsf{pass}$$

(phase switch)
$$\frac{T \mid \Gamma \vdash_{\mathsf{R}} C}{T \mid \Gamma \vdash_{\mathsf{L}} C}\ \mathsf{R2L} \tag{1.2}$$

(right phase)
$$\frac{}{X \mid \quad \vdash_{\mathsf{R}} X}\ \mathsf{ax} \quad \frac{}{- \mid \quad \vdash_{\mathsf{R}} \mathsf{I}}\ \mathsf{IR}$$
$$\frac{T \mid \Gamma \vdash_{\mathsf{R}} A \quad - \mid \Delta \vdash_{\mathsf{L}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{R}} A \otimes B}\ \otimes\mathsf{R}$$

The metavariable T indicates an *irreducible* stoup, i.e. $T \neq \mathsf{I}$ and $T \neq A \otimes B$ (T can also be empty).

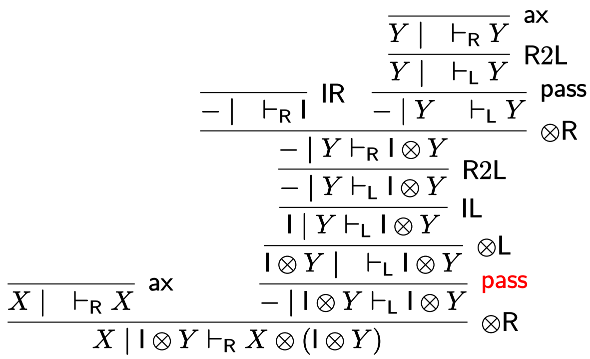
The focused subcalculus (1.2) has similar rules to $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ but with additional *mode* annotations. The bottom-up proof search strategy in (1.2) proceeds as following:

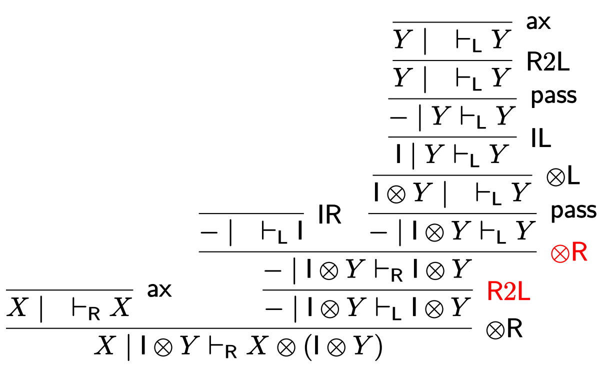
($\vdash_{\mathsf{L}}$) Proof search begins in the left phase where we eagerly apply left invertible rules ($\otimes$L and IL) to the stoup formula. When the stoup becomes an atomic formula, we switch to the right phase by applying R2L. When the stoup becomes empty, we encounter a choice point: we can either apply pass to move the head formula (if it exists) from the context into the stoup, or apply R2L to switch the sequent's phase. This choice point plays a crucial role in capturing the essential properties of the free skew monoidal category, which we will elaborate upon later.

($\vdash_{\mathsf{R}}$) Within the right phase, we proceed by applying one of three rules as appropriate: ax, IR, or ⊗R. It is important to note that when applying ⊗R, its right premise transitions back to the left phase, enabling continued proof search over the formulae in context Δ.

Although the focused calculus aims to provide a deterministic proof search strategy by prioritizing left rules over right rules (with ax classified as a right rule), some *essential* non-determinism remains within the calculus. This non-determinism is necessary to capture the fundamental nature of the free skew monoidal category, as demonstrated in Example 1.2.2. In particular, there are two types of essential non-determinism:

1. In the left phase, when the stoup is empty, there is a choice of whether to apply pass or R2L and sometimes both options lead to a derivation. For example, the sequent $X \mid \mathsf{I} \otimes Y \vdash_{\mathsf{R}} X \otimes (\mathsf{I} \otimes Y)$ has two distinct derivations in the focused system and the corresponding derivations in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ are not $\doteq$-equivalent.





   Notice the colored rules in the derivations above. These two derivations demonstrate different choices when the stoup is empty: in the first derivation, we apply the pass rule, which leads to a proof. In contrast, the second derivation immediately switches phase and then applies the ⊗R rule.

2. In phase R, the ⊗R rule splits the context in two parts and sends them to different premises. Sometimes several splits can lead to a valid derivation.

For example, the sequent $X \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (X \otimes \mathsf{I}) \otimes Y$ has two distinct derivations.

$$
\dfrac{
  \dfrac{}{X \mid\ \vdash_{\mathsf{R}} X}\,\mathsf{ax}
  \quad
  \dfrac{\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{R}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{R2L}}{\mathsf{I} \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{IL}}{- \mid \mathsf{I} \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{pass}
}{X \mid {\color{red}\mathsf{I}} \vdash_{\mathsf{R}} X \otimes \mathsf{I}}\,\otimes\mathsf{R}
\quad
\dfrac{\dfrac{\dfrac{}{Y \mid\ \vdash_{\mathsf{R}} Y}\,\mathsf{ax}}{Y \mid\ \vdash_{\mathsf{L}} Y}\,\mathsf{R2L}}{- \mid Y \vdash_{\mathsf{L}} Y}\,\mathsf{pass}
\Bigg/ \; X \mid {\color{red}\mathsf{I}}, Y \vdash_{\mathsf{R}} (X \otimes \mathsf{I}) \otimes Y \;\; \otimes\mathsf{R}
$$

$$
\dfrac{
  \dfrac{\dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{R}} X}\,\mathsf{ax} \quad \dfrac{}{- \mid\ \vdash_{\mathsf{R}} \mathsf{I}}\,\mathsf{IR}}{X \mid\ \vdash_{\mathsf{R}} X \otimes \mathsf{I}}\,\otimes\mathsf{R}}{X \mid\ \vdash_{\mathsf{R}} X \otimes \mathsf{I}}\,\otimes\mathsf{R}
  \quad
  \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid\ \vdash_{\mathsf{R}} Y}\,\mathsf{ax}}{Y \mid\ \vdash_{\mathsf{L}} Y}\,\mathsf{R2L}}{- \mid Y \vdash_{\mathsf{L}} Y}\,\mathsf{pass}}{\mathsf{I} \mid Y \vdash_{\mathsf{L}} Y}\,\mathsf{IL}}{- \mid {\color{red}\mathsf{I}}, Y \vdash_{\mathsf{L}} Y}\,\mathsf{pass}
}{X \mid {\color{red}\mathsf{I}}, Y \vdash_{\mathsf{R}} (X \otimes \mathsf{I}) \otimes Y}\,\otimes\mathsf{R}
$$

Notice that in the derivations above, the red I in the endsequent are sent to different premises. Both derivations are valid in the focused calculus.

Theorem 1.2.5. *The focused calculus (1.2) is equivalent to the unfocused calculus* $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$*, meaning that the two statements below are true:*

- *For any derivation* $f : S \mid \Gamma \vdash C$*, there exists a derivation* $\mathsf{focus}\ f : S \mid \Gamma \vdash_{\mathsf{L}} C$.
- *For any derivation in* $f : S \mid \Gamma \vdash_{ph} C$*, there exists a derivation* $\mathsf{emb}_{ph} : S \mid \Gamma \vdash C$*, for all* $ph \in \{\mathsf{L}, \mathsf{R}\}$.

The second statement is immediate: the emb_{ph} functions can be defined by mutual recursion where all functions erase the all phase annotations.

Lemma 1.2.6. *All rules in phase* R *are admissible in phase* L*:*

$$
\dfrac{}{A \mid\ \vdash_{\mathsf{L}} A}\,\mathsf{ax}^{\mathsf{L}}
\qquad
\dfrac{}{- \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{IR}^{\mathsf{L}}
\qquad
\dfrac{\overset{f}{S \mid \Gamma \vdash_{\mathsf{L}} A} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{L}} B}}{S \mid \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{L}}
$$

Proof.
Case IR^{L}.

$$
\dfrac{}{- \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{IR}^{\mathsf{L}}
\quad = \quad
\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{R}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\,\mathsf{R2L}
$$

Case $\otimes\mathsf{R}^{\mathsf{L}}(f, g)$. The proof proceeds by induction on f.
If $f = \otimes\mathsf{L}\ f'$, then

$$
\dfrac{\dfrac{\overset{f'}{A' \mid B', \Gamma \vdash_{\mathsf{L}} A}}{A' \otimes B' \mid \Gamma \vdash_{\mathsf{L}} A}\,\otimes\mathsf{L} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{L}} B}}{A' \otimes B' \mid \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{R}}
$$

$$
= \dfrac{\dfrac{\overset{f'}{A' \mid B', \Gamma \vdash_{\mathsf{L}} A} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{L}} B}}{A' \mid B', \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{R}}}{A' \otimes B' \mid \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\,\otimes\mathsf{L}
$$

The cases of IL and pass are similar.
If $f = \mathsf{R2L}\ f'$, then

$$\dfrac{\dfrac{\overset{f'}{T \mid \Gamma \vdash_{\mathsf{R}} A}}{T \mid \Gamma \vdash_{\mathsf{L}} A}\ \mathsf{R2L} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{L}} B}}{T \mid \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\ \otimes\mathsf{R}^{\mathsf{R}} \quad = \quad \dfrac{\dfrac{\overset{f'}{T \mid \Gamma \vdash_{\mathsf{R}} A} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{L}} B}}{T \mid \Gamma, \Delta \vdash_{\mathsf{R}} A \otimes B}\ \otimes\mathsf{R}}{T \mid \Gamma, \Delta \vdash_{\mathsf{L}} A \otimes B}\ \mathsf{R2L}$$

Notice that this is the base case of $\otimes\mathsf{R}^{\mathsf{L}}$.
Case ax^{L}. The proof proceeds by induction on formula A.
If $A = X$, for some $X \in \mathsf{At}$, then

$$\dfrac{}{X \mid\ \vdash_{\mathsf{L}} X}\ \mathsf{ax}^{\mathsf{L}} \quad = \quad \dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{R}} X}\ \mathsf{ax}}{X \mid\ \vdash_{\mathsf{L}} X}\ \mathsf{R2L}$$

If $A = \mathsf{I}$, then

$$\dfrac{}{\mathsf{I} \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\ \mathsf{ax}^{\mathsf{L}} \quad = \quad \dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{R}} \mathsf{I}}\ \mathsf{IR}}{- \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\ \mathsf{R2L}}{\mathsf{I} \mid\ \vdash_{\mathsf{L}} \mathsf{I}}\ \mathsf{IL}$$

If $A = A' \otimes B'$, then

$$\dfrac{}{A' \otimes B' \mid\ \vdash_{\mathsf{L}} A' \otimes B'}\ \mathsf{ax}^{\mathsf{L}} \quad = \quad \dfrac{\dfrac{\dfrac{}{A' \mid\ \vdash_{\mathsf{L}} A'}\ \mathsf{ax}^{\mathsf{L}} \quad \dfrac{\dfrac{}{B' \mid\ \vdash_{\mathsf{R}} B'}\ \mathsf{ax}^{\mathsf{L}}}{- \mid B' \vdash_{\mathsf{L}} B'}\ \mathsf{pass}}{A' \mid B' \vdash_{\mathsf{L}} A' \otimes B'}\ \otimes\mathsf{R}^{\mathsf{L}}}{A' \otimes B' \mid\ \vdash_{\mathsf{L}} A' \otimes B'}\ \otimes\mathsf{L}$$

□

Lemma 1.2.6 allows the construction of the function $\mathsf{focus} : S \mid \Gamma \vdash A \to S \mid \Gamma \vdash_{\mathsf{L}} A$, replacing applications of each rule in the sequent calculus $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ with inferences by the corresponding (admissible) focused rule in phase L.

Theorem 1.2.7. *The functions* $\mathsf{emb}_{\mathsf{L}}$ *and* focus *define a bijective correspondence between the set of derivations of* $S \mid \Gamma \vdash A$ *quotiented by the equivalence relation* $\doteq$ *and the set of derivations of* $S \mid \Gamma \vdash_{\mathsf{L}} A$*:*

- *For all* $f, g : S \mid \Gamma \vdash A$*, if* $f \doteq g$ *then* $\mathsf{focus}\ f = \mathsf{focus}\ g$.
- *For all* $f : S \mid \Gamma \vdash A$*,* $\mathsf{emb}_{\mathsf{L}}\ (\mathsf{focus}\ f) \doteq f$.
- *For all* $f : S \mid \Gamma \vdash_{\mathsf{L}} A$*,* $\mathsf{focus}\ (\mathsf{emb}_{\mathsf{L}}\ f) = f$.

The focused sequent calculus solves the *coherence problem* for skew monoidal categories by ensuring that all $\doteq$-equivalent derivations in the unfocused calculus become syntactically identical. For any two morphisms in the free skew monoidal category, we can determine their equality as follows: first, we interpret them into the unfocused sequent calculus and then apply the focus function. If the two resulting derivations are identical, then by Theorem 1.2.7, the unfocused derivations are $\doteq$-equivalent. Since $\doteq$ is chosen to capture morphism equality in the free skew monoidal category, this implies that the original two morphisms are equal.

As discussed above, there have been different strategies to characterize morphisms in the free skew monoidal category. Lack and Street [41] established normal forms where morphisms decompose into three sequential phases: (1) shrink morphisms, where units are removed from the domain object by applications of λ and some "necessary" applications of α ("necessary" in the sense that the reassociations make a new I appear on the left of a tensor, which can then be removed by an application of λ, for example, reassociating $(A \otimes \mathsf{I}) \otimes B$ to $A \otimes (\mathsf{I} \otimes B)$ via α so that λ becomes applicable); (2) Tamari morphisms (consisting solely of α morphisms) that move brackets from left to right; and (3) swell morphisms (consisting of ρ morphisms and "necessary" α morphisms) that add units. The shape of morphisms is well-behaved under composition: when a sequence of shrink, Tamari, and swell morphisms is composed, it can be rewritten and rearranged to maintain this three-phase shape using naturality and the Mac Lane axioms.

Lack and Street's normal forms are distinct from derivations in the focused sequent calculus. Consider, for example, morphisms from $X \otimes Y \to X \otimes Y$ (where $X, Y \in \mathsf{At}$) in the free skew monoidal category. Only one morphism exists. In Lack and Street's presentation, the morphism is composed of two parts: the identity shrink morphism (where no unit is removed) followed by the identity swell morphism (where no unit is added). In contrast, the sequent $X \otimes Y \vdash_{\mathsf{L}} X \otimes Y$ has a unique focused derivation:

$$
\dfrac{\dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{R}} X}\ \mathsf{ax} \qquad \dfrac{\dfrac{\dfrac{}{Y \mid\ \vdash_{\mathsf{R}} Y}\ \mathsf{ax}}{Y \mid\ \vdash_{\mathsf{L}} Y}\ \mathsf{R2L}}{- \mid Y \vdash_{\mathsf{L}} Y}\ \mathsf{pass}}{\dfrac{X \mid Y \vdash_{\mathsf{R}} X \otimes Y}{X \mid Y \vdash_{\mathsf{L}} X \otimes Y}\ \mathsf{R2L}}\ \otimes\mathsf{R}}{X \otimes Y \mid\ \vdash_{\mathsf{L}} X \otimes Y}\ \otimes\mathsf{L}
$$

When translated back to the axiomatic calculus (1.1), this derivation becomes:

$$
\dfrac{\dfrac{\dfrac{}{X \vdash_{\mathsf{A}} X \otimes \mathsf{I}}\ \rho \quad \dfrac{}{Y \vdash_{\mathsf{A}} Y}\ \mathsf{id}}{X \otimes Y \vdash_{\mathsf{A}} (X \otimes \mathsf{I}) \otimes Y}\ \otimes \qquad \dfrac{\dfrac{}{(X \otimes \mathsf{I}) \otimes Y \vdash_{\mathsf{A}} X \otimes (\mathsf{I} \otimes Y)}\ \alpha \quad \dfrac{\dfrac{}{X \vdash_{\mathsf{A}} X}\ \mathsf{id} \quad \dfrac{}{\mathsf{I} \otimes Y \vdash_{\mathsf{A}} Y}\ \lambda}{X \otimes (\mathsf{I} \otimes Y) \vdash_{\mathsf{A}} X \otimes Y}\ \otimes}{(X \otimes \mathsf{I}) \otimes Y \vdash_{\mathsf{A}} X \otimes Y}\ \mathsf{comp}}{X \otimes Y \vdash_{\mathsf{A}} X \otimes Y}\ \mathsf{comp}
$$

This derivation corresponds to the longer route $(X \otimes \lambda_Y \circ \alpha_{X,\mathsf{I},Y} \circ \rho_X \otimes Y)$ in the second diagram of Mac Lane's axiom in Definition 1.2.1, which is $\doteq$-equivalent to the identity morphism in the axiomatic calculus.

1.3 Focusing

The goal of focusing is to eliminate redundant proofs that are fundamentally equivalent. For example, the $\stackrel{\circ}{=}$ relation introduced in the previous section is a way to classify fundamentally equivalent proofs in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$. Andreoli [5] approached this problem for classical linear logic by first dividing the inference rules into two groups: asynchronous and synchronous. Asynchronous rules are those where the inverse of the rule is admissible. In $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$, examples of asynchronous rules include $\otimes\mathsf{L}$ and IL. Conversely, synchronous rules are those that are non-invertible, such as $\otimes\mathsf{R}$ and IR in $\texttt{SkNMILL}^{\mathsf{I},\otimes}$. For consistency, we call asynchronous rules invertible and synchronous rules non-invertible in the remainder of the section.

Let us now examine an example of a focused calculus for $\texttt{MILL}^{\mathsf{I},\otimes}$ with linear implication (notice the dropped $\texttt{N}$ for non-commutative), which we have adapted from Howe's PhD thesis [35].

(right invertible)
$$\frac{\Gamma \Uparrow \Lambda, A \Rightarrow B \Uparrow}{\Gamma \Uparrow \Lambda \Rightarrow A \multimap B \Uparrow}\multimap\mathsf{R} \qquad \frac{\Gamma \Uparrow \Lambda \Rightarrow P \Downarrow}{\Gamma \Uparrow \Lambda \Rightarrow P \Uparrow}\Uparrow\mathsf{R}$$

(left invertible)
$$\frac{\Gamma \Uparrow \Lambda, A, B \Rightarrow P \Downarrow}{\Gamma \Uparrow \Lambda, A \otimes B \Rightarrow P \Downarrow}\otimes\mathsf{L} \qquad \frac{\Gamma \Uparrow \Lambda \Rightarrow P \Downarrow}{\Gamma \Uparrow \Lambda, \mathsf{I} \Rightarrow P \Downarrow}\mathsf{IL}$$
$$\frac{\Gamma, N \Uparrow \Lambda \Rightarrow P \Downarrow}{\Gamma \Uparrow \Lambda, N \Rightarrow P \Downarrow}\mathsf{Pop}$$

(choice)
$$\frac{\Gamma \Downarrow \quad \Rightarrow P \Uparrow}{\Gamma \Uparrow \quad \Rightarrow P \Downarrow}\Downarrow\mathsf{R} \qquad \frac{\Gamma \Downarrow N \Rightarrow P \Downarrow}{\Gamma, N \Uparrow \quad \Rightarrow P \Downarrow}\mathsf{Push}$$

(right focusing)
$$\frac{\Gamma \Downarrow \quad \Rightarrow A \Uparrow \qquad \Delta \Downarrow \quad \Rightarrow B \Uparrow}{\Gamma, \Delta \Downarrow \quad \Rightarrow A \otimes B \Uparrow}\otimes\mathsf{R} \qquad \frac{}{\Downarrow \quad \Rightarrow \mathsf{I} \Uparrow}\mathsf{IR}$$
$$\frac{\Gamma \Uparrow \quad \Rightarrow N \Uparrow}{\Gamma \Downarrow \quad \Rightarrow N \Uparrow}\Downarrow\mathsf{L}_1$$

(left focusing)
$$\frac{}{\Downarrow A \Rightarrow A \Downarrow}\mathsf{ax} \qquad \frac{\Gamma \Downarrow \quad \Rightarrow A \Uparrow \qquad \Delta \Downarrow B \Rightarrow P \Downarrow}{\Gamma, \Delta \Downarrow A \multimap B \Rightarrow P \Downarrow}\multimap\mathsf{L}$$
$$\frac{\Gamma \Uparrow P \Rightarrow P' \Downarrow}{\Gamma \Downarrow P \Rightarrow P' \Downarrow}\Downarrow\mathsf{L}_2$$

In the rules above, the metavariable P denotes a *non-negative* formula, i.e. $P \neq A \multimap B$, while metavariable N indicates a *non-positive* formula, i.e. $N \neq \mathsf{I}$ and $N \neq A \otimes B$. Notice that atomic formulae are both non-negative and non-positive.

There are four types of sequents, which have different roles in the bottom-up proof search strategy:

($\Uparrow\Rightarrow\Uparrow$) Proof search begins with sequents of the form $\Gamma \Uparrow \Lambda \Rightarrow C \Uparrow$ where Γ is a multiset and Λ is a list. The right invertible rule $\multimap\mathsf{R}$ is applied eagerly to decompose the succedent formulae until the succedent becomes non-negative. At this point, we transition to the left invertible phase using the rule $\Uparrow\mathsf{R}$, which changes the up arrow to a down arrow in the succedent.

($\Uparrow\Rightarrow\Downarrow$) In this phase, left invertible rules $\otimes\mathsf{L}$ and IL are applied eagerly to the list of formulae Λ starting from the rightmost position. When the rightmost formula is non-positive, it moves to the left side of the downarrow in the antecedent. Once Λ becomes empty, we can either focus on the succedent using $\Downarrow\mathsf{R}$ or select a specific formula from Γ using Push to focus on the antecedent. This choice determines whether the sequent enters the right or left focusing phase.

($\Downarrow\Rightarrow\Uparrow$) In the right focusing phase, the right side of the downarrow in the antecedent remains empty. We apply $\otimes\mathsf{R}$ or IR based on the structure of the succedent formula. In the case of $\otimes\mathsf{R}$, focusing continues on the subformulae of $A \otimes B$. When we encounter a non-positive succedent, we return to the right invertible phase using the rule ${\Downarrow}\mathsf{L}_1$.

(${\Downarrow}{\Rightarrow}{\Downarrow}$) In the left focusing phase, we can close the derivation using ax when the sequent is in the correct shape. Otherwise, we can apply ${\multimap}\mathsf{L}$. The right premise of ${\multimap}\mathsf{L}$ maintains the left focusing phase, while the left premise transitions to the right focusing phase. This exemplifies the core principle of focusing: once a formula is selected for focus, we continue decomposing it and its subformulae until either the derivation closes or we must necessarily switch to decomposing other formulae.

Example 1.3.1. Consider the sequent $\Uparrow X \otimes (X \multimap Y) \Rightarrow Z \multimap (Y \otimes Z) \Uparrow$, the focused proof is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow X \Rightarrow X \Downarrow}\,\mathsf{ax}}{X \Uparrow \;\Rightarrow X \Downarrow}\,\mathsf{Push}}{X \Uparrow \;\Rightarrow X \Uparrow}\,{\Uparrow}\mathsf{R}}{X \Downarrow \;\Rightarrow X \Uparrow}\,{\Downarrow}\mathsf{L}_1 \quad \dfrac{}{\Downarrow Y \Rightarrow Y \Downarrow}\,\mathsf{ax}}{X \Downarrow X \multimap Y \Rightarrow Y \Downarrow}\,{\multimap}\mathsf{L}}{X, X \multimap Y \Uparrow \;\Rightarrow Y \Downarrow}\,\mathsf{Push}}{X, X \multimap Y \Uparrow \;\Rightarrow Y \Uparrow}\,{\Uparrow}\mathsf{R}}{X, X \multimap Y \Downarrow \;\Rightarrow Y \Uparrow}\,{\Downarrow}\mathsf{L}_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow Z \Rightarrow Z \Downarrow}\,\mathsf{ax}}{Z \Uparrow \;\Rightarrow Z \Downarrow}\,\mathsf{Push}}{Z \Uparrow \;\Rightarrow Z \Uparrow}\,{\Uparrow}\mathsf{R}}{Z \Downarrow \;\Rightarrow Z \Uparrow}\,{\Downarrow}\mathsf{L}_1}{Z, X \multimap Y, X \Downarrow \;\Rightarrow Y \otimes Z \Uparrow}\,\otimes\mathsf{R}}{Z, X \multimap Y, X \Uparrow \;\Rightarrow Y \otimes Z \Downarrow}}\,\color{red}{{\Downarrow}\mathsf{R}}}{Z, X \multimap Y \Uparrow X \Rightarrow Y \otimes Z \Downarrow}\,\mathsf{Pop}}{Z \Uparrow X, X \multimap Y \Rightarrow Y \otimes Z \Downarrow}\,\mathsf{Pop}}{Z \Uparrow X \otimes (X \multimap Y) \Rightarrow Y \otimes Z \Downarrow}\,\otimes\mathsf{L}}{\Uparrow X \otimes (X \multimap Y), Z \Rightarrow Y \otimes Z \Downarrow}\,\mathsf{Pop}}{\Uparrow X \otimes (X \multimap Y), Z \Rightarrow Y \otimes Z \Uparrow}\,{\Uparrow}\mathsf{R}}{\Uparrow X \otimes (X \multimap Y) \Rightarrow Z \multimap (Y \otimes Z) \Uparrow}\,{\multimap}\mathsf{R}
$$

Notice the red sequent $Z, X \multimap Y, X \Uparrow \;\Rightarrow Y \otimes Z \Downarrow$ is where we can choose either focus on the succedent or focus on one of the formulae in context. In the derivation above, we focus on succedent via the rule ${\Downarrow}\mathsf{R}$. Alternatively, we can focus on $X \multimap Y$ in the antecedent via the rule Pop, which would lead to a different proof:

$$
\color{red}{\dfrac{\color{black}{\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow X \Rightarrow X \Downarrow}\,\mathsf{ax}}{X \Uparrow \;\Rightarrow X \Downarrow}\,\mathsf{Push}}{X \Uparrow \;\Rightarrow X \Uparrow}\,{\Uparrow}\mathsf{R}}{X \Downarrow \;\Rightarrow X \Uparrow}\,{\Downarrow}\mathsf{L}_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow Y \Rightarrow Y \Downarrow}\,\mathsf{ax}}{Y \Uparrow \;\Rightarrow Y \Downarrow}\,\mathsf{Push}}{Y \Uparrow \;\Rightarrow Y \Uparrow}\,{\Uparrow}\mathsf{R}}{Y \Downarrow \;\Rightarrow Y \Uparrow}\,{\Downarrow}\mathsf{L}_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow Z \Rightarrow Z \Downarrow}\,\mathsf{ax}}{Z \Uparrow \;\Rightarrow Z \Downarrow}\,\mathsf{Push}}{Z \Uparrow \;\Rightarrow Z \Uparrow}\,{\Uparrow}\mathsf{R}}{Z \Downarrow \;\Rightarrow Z \Uparrow}\,{\Downarrow}\mathsf{L}_1}{Z, Y \Downarrow \;\Rightarrow Y \otimes Z \Uparrow}\,\otimes\mathsf{R}}{Z, Y \Uparrow \;\Rightarrow Y \otimes Z \Downarrow}\,{\Downarrow}\mathsf{R}}{Z \Uparrow Y \Rightarrow Y \otimes Z \Downarrow}\,\mathsf{Pop}}{Z \Downarrow Y \Rightarrow Y \otimes Z \Downarrow}\,{\Downarrow}\mathsf{L}_2}{Z, X \Uparrow X \multimap Y \Rightarrow Y \otimes Z \Downarrow}\,{\multimap}\mathsf{L}}}{Z, X \multimap Y, X \Uparrow \;\Rightarrow Y \otimes Z \Downarrow}}\,\mathsf{Pop}
$$

Non-determinism appears in the above focused calculus in two ways: (i) the choice between left and right focusing, and (ii) the selection of which formula to focus on from the multiset Γ during left focusing. This means proof normal forms are unique only up to permutation of non-invertible rules, a property known as weak focusing. Chaudhuri et al. [19] strengthened this approach by introducing multi-focusing, where multiple formulae can be focused and decomposed simultaneously. Their system produces maximally multi-focused proofs that select the largest possible set of formulae to focus at each step, achieving maximal parallelism. For unit-free multiplicative classical linear logic, these proofs are equivalent to proof nets. This approach has since been extended to other calculi [18, 15, 59, 55].

The focusing strategy introduced by Uustalu et al. [67] and employed throughout this thesis proves to be a strong focusing strategy (see the first bullet of Theorem 1.2.7). The stoup sequents in the focused calculus (1.2) eliminate choices in antecedents. However, as we add more connectives in later chapters, designing a strong focusing strategy becomes challenging. In our approach, we prioritize left non-invertible rules over right ones. As discussed in Section 2.4.1, this strict ordering would result in an incomplete focused calculus. We address this issue by introducing additional annotations on sequents and formulae. These annotations ensure that left non-invertible rules are applied before right ones when reading proofs bottom-up, except in cases where the proof requires a different order.

Strictly speaking, our strategy deviates from traditional focusing since we release the focus on the focused formula immediately after applying any non-invertible rule, returning to the invertible phase. Nevertheless, we retain the term “focused calculus” as our approach draws significant inspiration from the focusing principles.

1.4 Interpolation Properties for Substructural Logics

Craig interpolation is a fundamental result in first-order logic, named after the logician William Craig [20]. A logic $\mathcal{L}$ has the *Craig interpolation property* if, for any formula $A \to C$ provable in $\mathcal{L}$ (where $\to$ is the implication connective in $\mathcal{L}$), there exists a formula D such that $A \to D$ and $D \to C$ are provable in $\mathcal{L}$, satisfying the variable condition: $\mathsf{var}(D) \subseteq \mathsf{var}(A) \cap \mathsf{var}(C)$, where $\mathsf{var}(A)$ is the set of atomic formulae appearing in A. Craig interpolation has been mostly employed to prove model-theoretical results, including Beth's definability theorem [9], but more recently it has found applications in other areas, e.g. in model checking [34].

Craig interpolation for substructural logics has been extensively studied, using either algebraic or proof-theoretic techniques.

For substructural logics that lack a cut-free sequent calculus, such as arbitrary extensions of the full Lambek calculus with exchange ($\mathtt{FL_e}$), Craig interpolation is established using algebraic methods such as amalgamation. For further details on this approach, see [26] and for the relationship between amalgamation and interpolation properties in substructural logics, see [40].

For substructural logics that admit a cut-free sequent calculus, Craig interpolation is typically proven by adapting Maehara's method [48], which originally aimed to prove interpolation for $\mathtt{LK}$, a sequent calculus for classical logic. This includes the full Lambek calculus ($\mathtt{FL}$) and its extensions that incorporate various combinations of weakening, exchange, and contraction. In the case of $\mathtt{FL}$, for instance, the proof starts by establishing a stronger form of interpolation which we call *Maehara interpolation property* (MIP) [53]. The latter property states:

(MIP for FL) Given $f : \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist a formula D and two derivations $g : \Gamma_1 \vdash D$ and $h : \Gamma_0, D, \Gamma_2 \vdash C$, and $\mathsf{var}(D) \subseteq \mathsf{var}(\Gamma_1) \cap \mathsf{var}(\Gamma_0, \Gamma_2, C)$

Being a partition simply means that the ordered list of formulae Γ is equal to the concatenation of Γ_0, Γ_1 and Γ_2, i.e. $\Gamma = \Gamma_0, \Gamma_1, \Gamma_2$. Maehara interpolation also holds for the commutative variant of FL, called $\mathtt{FL_e}$. In the commutative case, Γ is an unordered list of formulae, i.e. a finite multiset, and it is partitioned as a pair of multisets instead of a triple of lists. This simplification is allowed by the fact that the order of formulae in the antecedent is irrelevant, and therefore Γ_0 and Γ_2 can be combined together into a single multiset.

FL without additive connectives enjoys a stronger variant of Maehara interpolation where the variable condition is replaced by a variable *multiplicity* condition [52]. Let $\sigma_X(A)$ be the number of occurrences of the atomic formula X in the formula A, and $\sigma_X(\Gamma)$ be the number of occurrences of X in the list of formulae Γ. The stronger variant of Maehara interpolation states:

(MIP for FL with variable multiplicity condition) Given $f : \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist a formula D and two derivations $g : \Gamma_1 \vdash D$ and $h : \Gamma_0, D, \Gamma_2 \vdash C$ such that $\sigma_X(D) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D) \leq \sigma_X(\Gamma_0, \Gamma_2, C)$ for all atomic X.

Notice that Craig interpolation is a property of a logic (with a notion of implication), while Maehara interpolation is a property of a deductive system in which it is possible to appropriately partition antecedents.

Maehara interpolation is a stronger form of the so-called *deductive interpolation property*. A logic $\mathcal{L}$ has the deductive interpolation property if, for any formulae A and C, whenever $A \vdash C$ (where $\vdash$ is the consequence relation of $\mathcal{L}$), then there exists a formula B such that $A \vdash B$ and $B \vdash C$ while also satisfying the usual variable condition. Furthermore, if the sequent calculus of $\mathcal{L}$ admits the invertibility of implication-right rules (as is the case in FL for both left and right implication), Craig interpolation follows immediately as a consequence of deductive interpolation.

While Maehara's method is often applicable to extensions of FL, it does not work for some of its fragment, which therefore do not enjoy Maehara interpolation. This is the case for fragments lacking multiplicative and/or additive conjunction, such as the product-free (multiplicative-conjunction-free) Lambek calculus [54] (with only left and right implications as connectives) and the implicational fragment of intuitionistic logic [38]. The variant of Maehara interpolation satisfied by the product-free Lambek calculus, which we call *Maehara multi-interpolation property* (MMIP), is particularly relevant for our work. Here is its statement, which we have slightly modified to better align with our forthcoming discussion:

(MMIP for product-free Lambek calculus) Given $f : \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist

- a partition $\langle \Delta_1, \dots, \Delta_n \rangle$ of Γ_1,
- a list of interpolant formulae $D_1, \dots, D_n$,
- derivations $g_i : \Delta_i \vdash D_i$ for all $i \in [1, \dots, n]$,

  - a derivation $h : \Gamma_0, D_1, \ldots, D_n, \Gamma_2 \vdash C$, such that
  - $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_0, \Gamma_2, C)$ for all atomic formulae X.

Differently from Maehara interpolation, in the above property we look for a list of interpolants instead of a single formula. This adjustment allows us to overcome the difficulty caused by the absence of conjunction.

In Section 3, we will prove a Craig interpolation theorem for the sequent calculus of skew monoidal closed categories via a skew variant of MMIP. Regarding interpolation for the sequent calculus $\mathtt{SkNMILL}^{!,\otimes}$, please refer to Remark 3.2.4.

1.5 Ternary Relational Semantics for Lambek Calculus

In this thesis, beyond categorical models, we study ternary relational models for semi-substructural logics. Ternary relational semantics is a classical approach for studying non-associative Lambek calculus and its extensions [24]. The correspondence theorem between frame conditions and logical properties is of interest in this semantics. Because of its clarity and modularity, we can build semantics for extensions of non-associative Lambek calculus by requiring additional frame conditions. As we will show in Section 6.3, this modularity allows us to provide a proof for the poset version of the main theorems concerning the interdefinability of a series of skew categories discussed in [64].

In this section, we recapitulate the basics of ternary relational semantics.

Consider the (associative) Lambek calculus, where formulae (Fma) are inductively generated by the grammar $A, B ::= X \mid A \otimes B \mid A \diagdown B \mid B \diagup A$, where X comes from a fixed set At of atoms, $\otimes$ is a multiplicative conjunction, and $\diagdown$ ($\diagup$) is left (right) residuation. Derivations are generated inductively by the following rules:

$$
\begin{gathered}
\frac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{A}} B \quad B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} C}\ \mathsf{comp} \qquad \frac{A \vdash_{\mathsf{A}} C \quad B \vdash_{\mathsf{A}} D}{A \otimes B \vdash_{\mathsf{A}} C \otimes D}\ \otimes \\
\frac{C \vdash_{\mathsf{A}} A \quad B \vdash_{\mathsf{A}} D}{A \diagdown B \vdash_{\mathsf{A}} C \diagdown D}\ \diagdown \qquad \frac{C \vdash_{\mathsf{A}} A \quad B \vdash_{\mathsf{A}} D}{B \diagup A \vdash_{\mathsf{A}} D \diagup C}\ \diagup \\
\frac{A \otimes B \vdash_{\mathsf{A}} C}{B \vdash_{\mathsf{A}} A \diagdown C}\ \diagdown_{\mathsf{RES}} \qquad \frac{B \vdash_{\mathsf{A}} A \diagdown C}{A \otimes B \vdash_{\mathsf{A}} C}\ \diagdown_{\mathsf{RES}}^{-1} \\
\frac{A \otimes B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} C \diagup B}\ \diagup_{\mathsf{RES}} \qquad \frac{A \vdash_{\mathsf{A}} C \diagup B}{A \otimes B \vdash_{\mathsf{A}} C}\ \diagup_{\mathsf{RES}}^{-1} \\
\frac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} A \otimes (B \otimes C)}\ \alpha \qquad \frac{}{A \otimes (B \otimes C) \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\ \alpha^{-1}
\end{gathered}
\tag{1.3}
$$

The rules shown in black constitute the non-associative Lambek calculus. Adding the rules shown in red yields the associative Lambek calculus.

A ternary frame is a pair $\langle W, \mathbb{R} \rangle$, where W is a nonempty set and $\mathbb{R}$ is a ternary relation on W. For $\mathbb{R}abc$ where $a, b, c \in W$, we interpret c as the "implicit" multiplication of a and b. A function $v : \mathsf{Fma} \to \mathcal{P}(W)$ on a ternary frame is a valuation if it satisfies:

$$\begin{aligned} v(A \otimes B) &= \{c : \exists a \in v(A), b \in v(B),\ \mathbb{R}abc\} \\ v(A \setminus B) &= \{c : \forall a \in v(A), b \in W,\ \mathbb{R}acb \Rightarrow b \in v(B)\} \\ v(B / A) &= \{c : \forall a \in v(A), b \in W,\ \mathbb{R}cab \Rightarrow b \in v(B)\} \end{aligned}$$

A ternary relational model of non-associative Lambek calculus is a frame with a valuation $\langle W, \mathbb{R}, v\rangle$ and the following theorem holds:

Theorem 1.5.1. *The non-associative Lambek calculus is sound and complete with respect to ternary relational models.*

Regarding the associative Lambek calculus, one should consider adding frame conditions on ternary frames, analogous to adding frame conditions to Kripke frames for semantics of modal logics:

Left Skew Associativity (LSA) $\quad \forall a, b, c, d, x \in W, \mathbb{R}abx \ \&\ \mathbb{R}xcd$
$\longrightarrow \exists y \in W$ such that $\mathbb{R}bcy \ \&\ \mathbb{R}ayd.$

Right Skew Associativity (RSA) $\quad \forall a, b, c, d, x \in W, \mathbb{R}bcx \ \&\ \mathbb{R}axd$
$\longrightarrow \exists y \in W$ such that $\mathbb{R}aby \ \&\ \mathbb{R}ycd.$

Therefore, the ternary relation models $\langle W, \mathbb{R}, v\rangle$ for the associative Lambek calculus are the models of the non-associative Lambek calculus where $\mathbb{R}$ further satisfies both LSA and RSA.

Theorem 1.5.2. *The associative Lambek calculus is sound and complete with respect to ternary relational models satisfying both LSA and RSA.*

Notice that the frame conditions LSA and RSA are strongly connected to the associativity axioms α and α^{-1}. In Section 6, we will explore this connection in depth through correspondence theorems (Theorems 6.3.7 and 6.4.11) that relate various frame conditions to different presentations of structural axioms.

Ternary relational semantics were introduced by Meyer and Routley [57] within the context of relevant logics. The Lambek calculus and its extensions have several other semantics. For example, residuated lattices [27] provide a general and modular semantic framework, establishing algebraic semantics for systems ranging from the associative Lambek calculus to the full Lambek calculus. Phase semantics [2, 30], although originating from linear logic, also applies to the Lambek calculus because of the equivalence between the Lambek calculus and specific fragments of linear logic. For a brief survey of semantics of the Lambek calculus, we refer readers to [24]. For a detailed and comprehensive introduction to the Lambek calculus, we refer readers to [52].

1.6 Contributions

This thesis makes several contributions to the proof-theoretical study of semi-substructural logics and their categorical models.

First, we develop sequent calculi that correspond to several variants of skew monoidal categories:

(i) skew monoidal closed categories (Definition 2.3.1),

(ii) symmetric skew monoidal closed categories (Definition 4.2.1),

(iii) distributive skew monoidal categories with binary products (Definition 5.2.1),

(iv) distributive symmetric skew monoidal categories with binary products (Section 5.4), and

(v) distributive skew monoidal closed categories with binary products (Section 5.5).

For each logical system, we construct a corresponding focused calculus with additional annotations on sequents and formulae that guarantee strong focusing, which not only provides a decision procedure for derivability but also solves the coherence problem for the corresponding categorical models. The construction of this hierarchy of focused calculi requires careful consideration since strongly focused calculi are sensitive to the addition of new connectives and new sequent inference rules. This proof-theoretical investigation extends the work of [67] to a broader family of logics that can be viewed as skew variants of intuitionistic linear logic.

Through our investigations, we discovered that these logics possess interesting properties that set them apart from traditional substructural logics. In particular, the proof techniques typically used for establishing Craig interpolation in linear logic do not extend directly to semi-substructural logics, requiring us to develop new approaches. We overcome this challenge, using ideas similar to those of Kanazawa [38] and Pentus [54], to prove Craig interpolation for the logic of skew monoidal closed categories (Theorem 3.2.3). We also demonstrate that the sequent calculus for the logic enjoys proof-relevant interpolation (Theorem 3.4.1), a concept originally discussed by Čubrić [68] for intuitionistic logics and recently by Saurin [58] for classical and intuitionistic linear logic.

While the sequent calculus with stoup (using sequents of the form $S \mid \Gamma \vdash C$) successfully characterizes many semi-substructural logics, we found it unsuitable for certain logics arising from specific variants of skew monoidal categories, e.g. skew monoidal bi-closed categories (Section 6.1) and symmetric skew monoidal bi-closed categories (Section 6.4). For these systems, we employ axiomatic and tree sequent calculi, an approach inspired by studies on non-associative Lambek calculus. We also study the sound and complete ternary relational semantics, for which we prove correspondence theorems linking frame conditions to structural axioms (Theorems 6.3.7 and 6.4.11). Studying such systems addresses a gap in the literature: to our knowledge, extensions of non-associative Lambek calculus that feature either a single-sided associativity axiom (α) or skew unitality (λ and ρ) as structural rules have not been examined. Yet, it is precisely such rules that characterize the semi-substructural logics explored in this thesis. This gap likely exists because traditional treatments have naturally concentrated on well-established algebraic structures, such as semigroups for associative Lambek calculus and monoids for its unital variant. Consequently, semi-associative and semi-unital cases, which represent less conventional algebraic alternatives, have received less attention.

1.7 Thesis Structure

The remainder of this thesis is organized as follows:

- Chapter 2, based on [63] and part of [72], lays the groundwork by introducing skew non-commutative multiplicative intuitionistic linear logic (SkNMILL). We present its sequent calculus formulation $\mathtt{SkNMILL}_{\mathtt{S}}$, establish relationships between various calculi (2.2), and explore both categorical semantics through skew monoidal closed categories (Section 2.3) and proof-theoretic semantics via focusing (Section 2.4).

- Chapter 3, based on [72], investigates Craig interpolation for SkNMILL, revealing why traditional Maehara interpolation fails (Section 3.1) and using a variation of Maehara interpolation for establishing (proof-relevant) interpolation properties.

- Chapter 4[1] extends our proof-theoretical analysis to SkMILL, a skew commutative extension of SkNMILL. In this sequent calculus, formulae can be exchanged within the context, but the formula in the stoup cannot be exchanged with any formula in the context.

- Chapter 5, based on [71], broadens the scope by incorporating additive connectives to the sequent calculus $\mathtt{SkNMILL}^{!,\otimes}$ and investigating two further extensions including skew exchange (Section 5.4) and linear implication (Section 5.5).

- Chapter 6, based on [73] and an unpublished manuscript, moves beyond the stoup-based approach to investigate semi-substructural logics requiring different frameworks. It first explores the logic of skew monoidal bi-closed categories (Section 6.2) and the corresponding relational semantics (Section 6.3), and then extends the analysis to symmetric skew monoidal bi-closed categories (Section 6.4).

- Chapter 7 concludes by summarizing our contributions and discussing future research directions.

1.8 Formalization

Given the complexity of proofs in this domain and their reliance on structural induction, we have formalized key results using the proof assistant Agda (version 2.6.4.1). Our formalization includes:

- The sequent calculi $\mathtt{SkMILL}_{\mathtt{S}}$, $\mathtt{SkNMILL}_{\mathtt{A}}$, and their focused versions, with equivalence relations on derivations.

- The focusing procedures for both $\mathtt{SkMILL}_{\mathtt{S}}$ and $\mathtt{SkNMILL}_{\mathtt{A}}$ (Sections 4.3 and 5.3).

Formalizations for other additive extensions and the tree sequent calculus were deemed too complex to complete at this stage and are left for future work. The

[1] This chapter is based on a collaborative unpublished work with Niccolò Veltri.

Agda code is available in my GitHub repository `https://github.com/cswphilo/code-PhD-thesis`.

Additionally, Veltri has formalized related results for $\texttt{SkNMILL}_{\texttt{S}}$, including the Craig, Maehara, and proof-relevant interpolation properties. The associated code is available at `https://github.com/niccoloveltri/code-skewmonclosed/tree/interpolation`.

Chapter 2

Skew Non-Commutative Multiplicative Intuitionistic Linear Logic

We begin from a skew variant of non-commutative intuitionistic linear logic (SkMILL), where the associativity and both unit laws hold only in one direction. We present this logic through a sequent calculus $\mathtt{SkNMILL_S}$, which not only provides a proof system but also precisely captures the categorical notion of skew monoidal closed categories. We first develop the basic proof theory, then establish a focused system that will be essential for solving the coherence problem for these categories.

2.1 Sequent calculus

We begin by introducing a sequent calculus $\mathtt{SkNMILL_S}$ that formalizes a skew variant of non-commutative multiplicative intuitionistic linear logic (NMILL), which we call SkNMILL. While elements of $\mathtt{SkNMILL_S}$ were briefly discussed in Section 1.2.2 since it is extended from the sequent calculus of $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ by adding the linear implication $\multimap$, we now present it in full detail to establish a foundation for the extensions that follow.

Formulae are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes B \mid A \multimap B$, where X comes from a fixed set At of atoms, I is a multiplicative verum, $\otimes$ is a multiplicative conjunction and $\multimap$ is a linear implication.

A sequent is a triple of the form $S \mid \Gamma \vdash A$, where the succedent A is a single formula (as in the sequent calculus for NMILL) and the antecedent is divided in two parts: an optional formula S, called *stoup* [29], and an ordered list of formulae Γ, called *context*. The peculiar design of sequents, involving the presence of the stoup in the antecedent, comes from previous work on deductive systems with skew structure by Uustalu, Veltri and Zeilberger [67, 66, 65, 69]. The metavariable S always denotes a stoup, i.e., S can be a single formula or empty, in which case we write $S = -$, and X, Y, Z are always names of atomic formulae.

Derivations of a sequent $S \mid \Gamma \vdash A$ are inductively generated by the following rules:

$$\frac{}{A \mid \; \vdash A}\ \mathsf{ax} \qquad \frac{A \mid \Gamma \vdash C}{- \mid A, \Gamma \vdash C}\ \mathsf{pass}$$

$$\frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C}\ {\multimap}\mathsf{L} \qquad \frac{- \mid \Gamma \vdash C}{\mathsf{I} \mid \Gamma \vdash C}\ \mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash C}{A \otimes B \mid \Gamma \vdash C}\ {\otimes}\mathsf{L}$$

$$\frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} \qquad \frac{}{- \mid \; \vdash \mathsf{I}}\ \mathsf{IR} \qquad \frac{S \mid \Gamma \vdash A \quad - \mid \Delta \vdash B}{S \mid \Gamma, \Delta \vdash A \otimes B}\ {\otimes}\mathsf{R}$$

These inference rules in $\mathtt{SkNMILL_S}$ are reminiscent of the ones in the sequent calculus for `NMILL` [1], but there are some crucial differences.

1. The left logical rules IL, $\otimes\mathsf{L}$ and $\multimap\mathsf{L}$, read bottom-up, are only allowed to be applied on the formula in the stoup position. In particular, there is no general way to remove a unit I nor decompose a tensor $A \otimes B$ if these formulae are located in the context and not in the stoup (we will see in (2.3) that something can actually be done to deal with implications $A \multimap B$ in the context).

2. The right tensor rule $\otimes\mathsf{R}$, read bottom-up, splits the antecedent of the conclusion between the two premises whereby the formula in the stoup, in case such a formula is present, has to be moved to the stoup of the first premise. In particular, the stoup formula of the conclusion cannot be moved to the antecedent of the second premise even if Γ is chosen to be empty.

3. The presence of the stoup implies a distinction between antecedents of forms $A \mid \Gamma$ and $- \mid A, \Gamma$. The structural rule pass (for 'passivation'), read bottom-up, allows the moving of the leftmost formula in the context to the stoup position whenever the stoup is initially empty.

4. The logical connectives of `NMILL` typically include two ordered implications $\multimap$ and $\multimapinv$, which are two variants of linear implication arising from the removal of the exchange rule from intuitionistic linear logic. In `SkNMILL` only one of the ordered implications (the right implication $\multimap$) is present.

The restrictions in 1–4 are essential for precisely capturing all the features of skew monoidal closed categories and nothing more, as we discuss in Section 2.3. Notice also that, similarly to the case of `NMILL`, all structural rules of exchange, contraction, and weakening are absent. We give names to derivations and we write $f : S \mid \Gamma \vdash A$ when f is a particular derivation of the sequent $S \mid \Gamma \vdash A$.

Examples of valid derivations in the sequent calculus, corresponding to the structural laws λ, ρ and α of skew monoidal closed categories (see Definition 2.3.1) are given below.

$$
(\lambda)\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \mid\ \vdash A}\,\mathsf{ax}}{- \mid A \vdash A}\,\mathsf{pass}}{\mathsf{I} \mid A \vdash A}\,\mathsf{IL}}{\mathsf{I}\otimes A \mid\ \vdash A}\,\otimes\mathsf{L}
\qquad\qquad
(\rho)\qquad \dfrac{\dfrac{}{A \mid\ \vdash A}\,\mathsf{ax} \quad \dfrac{}{- \mid\ \vdash \mathsf{I}}\,\mathsf{IR}}{A \mid\ \vdash A \otimes \mathsf{I}}\,\otimes\mathsf{R}
$$

$$
(\alpha)\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \mid\ \vdash A}\,\mathsf{ax} \quad \dfrac{\dfrac{\dfrac{}{B \mid\ \vdash B}\,\mathsf{ax} \quad \dfrac{\dfrac{}{C \mid\ \vdash C}\,\mathsf{ax}}{- \mid C \vdash C}\,\mathsf{pass}}{B \mid C \vdash B \otimes C}\,\otimes\mathsf{R}}{- \mid B, C \vdash B \otimes C}\,\mathsf{pass}}{A \mid B, C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{R}}{A \otimes B \mid C \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L}}{(A \otimes B) \otimes C \mid\ \vdash A \otimes (B \otimes C)}\,\otimes\mathsf{L} \tag{2.1}
$$

Examples of non-derivable sequents include the "inverses" of the conclusions in (2.1), obtained by swapping the stoup formula with the succedent formula. More precisely, given any three atomic formulae X, Y, and Z, the following three sequents $X \mid\ \vdash \mathsf{I} \otimes X$, $X \otimes \mathsf{I} \mid\ \vdash X$ and $X \otimes (Y \otimes Z) \mid\ \vdash (X \otimes Y) \otimes Z$ do not have a derivation. All possible attempts of constructing a valid derivation for each of them end in failure (a double question mark ?? means that no rule can be applied to close the derivation).

$$
(\lambda^{-1})\qquad \dfrac{\dfrac{??}{X \mid\ \vdash \mathsf{I}} \quad \dfrac{??}{- \mid\ \vdash X}}{X \mid\ \vdash \mathsf{I} \otimes X}\,\otimes\mathsf{R}
\qquad\qquad
(\rho^{-1})\qquad \dfrac{\dfrac{??}{X \mid \mathsf{I} \vdash X}}{X \otimes \mathsf{I} \mid\ \vdash X}\,\otimes\mathsf{L}
$$

($\otimes$R sends X to 1st premise) (IL does not act on I in context)

$$
(\alpha^{-1})\qquad \dfrac{\dfrac{\dfrac{??}{X \mid Y \otimes Z \vdash X \otimes Y} \quad \dfrac{??}{- \mid\ \vdash Z}}{X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z}\,\otimes\mathsf{R}}{X \otimes (Y \otimes Z) \mid\ \vdash (X \otimes Y) \otimes Z}\,\otimes\mathsf{L}
\qquad
\dfrac{\dfrac{\dfrac{??}{X \mid\ \vdash X \otimes Y} \quad \dfrac{??}{- \mid Y \otimes Z \vdash Z}}{X \mid Y \otimes Z \vdash (X \otimes Y) \otimes Z}\,\otimes\mathsf{R}}{X \otimes (Y \otimes Z) \mid\ \vdash (X \otimes Y) \otimes Z}\,\otimes\mathsf{L}
$$

($\otimes$L does not act on $\otimes$ in context, so only $\otimes$R is applicable)

Analogously, the sequents $\mathsf{I} \multimap A \mid\ \vdash A$ and $(A \otimes B) \multimap C \mid\ \vdash A \multimap (B \multimap C)$ are derivable, while generally their "inverses" are not. Also, a derivation of $A \mid\ \vdash B$ always yields a derivation of $\mathsf{I} \mid\ \vdash A \multimap B$, but there are A, B such that $\mathsf{I} \mid\ \vdash A \multimap B$ is derivable while $A \mid\ \vdash B$ is not (take, e.g., $A = X$, $B = \mathsf{I} \otimes X$).

Sets of derivations are quotiented by a congruence relation $\stackrel{\circ}{=}$, generated by the pairs of derivations in Figures 2.1 and 2.2.

The three equations in Figure 2.1 are η-conversions, completely characterizing the ax rule on non-atomic formulae. The remaining equations in Figure 2.2 are permutative conversions. The congruence $\stackrel{\circ}{=}$ has been carefully chosen to serve as the proof-theoretic counterpart of the equational theory of skew monoidal closed

$$\frac{}{\mathsf{I} \mid \ \vdash \mathsf{I}}\ \mathsf{ax} \quad \doteq \quad \frac{\dfrac{}{- \mid \ \vdash \mathsf{I}}\ \mathsf{IR}}{\mathsf{I} \mid \ \vdash \mathsf{I}}\ \mathsf{IL}$$

$$\frac{}{A \otimes B \mid \ \vdash A \otimes B}\ \mathsf{ax} \quad \doteq \quad \frac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax} \quad \dfrac{\dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{- \mid B \vdash B}\ \mathsf{pass}}{A \mid B \vdash A \otimes B}\ \otimes\mathsf{R}}{A \otimes B \mid \ \vdash A \otimes B}\ \otimes\mathsf{L}$$

$$\frac{}{A \multimap B \mid \ \vdash A \multimap B}\ \mathsf{ax} \quad \doteq \quad \frac{\dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax}}{- \mid A \vdash A}\ \mathsf{pass} \quad \dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{A \multimap B \mid A \vdash B}\ \multimap\mathsf{L}}{A \multimap B \mid \ \vdash A \multimap B}\ \multimap\mathsf{R}$$

Figure 2.1: Equivalence of derivations in $\mathtt{SkNMILL_S}$*:* η*-conversions*

categories, introduced in Definition 2.3.1. The subsystem of equations involving only $(\mathsf{I}, \otimes)$ originated in [67] while the subsystem involving only $\multimap$ is from [65].

Theorem 2.1.1. *The sequent calculus enjoys cut admissibility: the following two cut rules are admissible.*

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}$$

Proof. The proof proceeds by induction on the height of derivations and the complexity of cut formulae. Specifically, for scut, we first perform induction on the first premise f, and if necessary, we perform subinduction on g or the complexity of the cut formula A. For ccut, we start by performing induction on the right premise g instead. The cases other than $\multimap\mathsf{L}$ and $\multimap\mathsf{R}$ have been discussed in [67, Lemma 5], so we will only elaborate on the cases of $\multimap$.

We first deal with scut. If $f = \multimap\mathsf{L}(f', f'')$, then we permute scut up, i.e.

$$\frac{\dfrac{\overset{f'}{- \mid \Gamma \vdash A'} \quad \overset{f''}{B' \mid \Delta \vdash A}}{A' \multimap B' \mid \Gamma, \Delta \vdash A}\ \multimap\mathsf{L} \quad \overset{g}{A \mid \Lambda \vdash C}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash C}\ \mathsf{scut}$$

$$= \quad \frac{\overset{f'}{- \mid \Gamma \vdash A'} \quad \dfrac{\overset{f''}{B' \mid \Delta \vdash A} \quad \overset{g}{A \mid \Lambda \vdash C}}{B' \mid \Delta, \Lambda \vdash C}\ \mathsf{scut}}{A' \multimap B' \mid \Gamma, \Delta, \Lambda \vdash C}\ \multimap\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A}}{- \mid A', \Gamma \vdash A}\ \mathsf{pass} \quad \overset{g}{- \mid \Delta \vdash B}}{- \mid A', \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{- \mid A', \Gamma, \Delta \vdash A \otimes B}\ \mathsf{pass}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A}}{\mathsf{I} \mid \Gamma \vdash A}\ \mathsf{IL} \quad \overset{g}{- \mid \Delta \vdash B}}{\mathsf{I} \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{- \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{\mathsf{I} \mid \Gamma, \Delta \vdash A \otimes B}\ \mathsf{IL}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A}}{A' \otimes B' \mid \Gamma \vdash A}\ \otimes\mathsf{L} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \otimes B' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \mid B', \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{R}}{A' \otimes B' \mid \Gamma, \Delta \vdash A \otimes B}\ \otimes\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash C} \quad \overset{g}{D \mid \Delta \vdash A}}{C \multimap D \mid \Gamma, \Delta \vdash A}\ \multimap\mathsf{L} \quad \overset{h}{- \mid \Lambda \vdash B}}{C \multimap D \mid \Gamma, \Delta, \Lambda \vdash A \otimes B}\ \otimes\mathsf{R} \quad \circeq \quad \dfrac{\overset{f}{- \mid \Gamma \vdash C} \quad \dfrac{\overset{g}{D \mid \Delta \vdash A} \quad \overset{h}{- \mid \Lambda \vdash B}}{D \mid \Delta, \Lambda \vdash A \otimes B}\ \otimes\mathsf{R}}{C \multimap D \mid \Gamma, \Delta, \Lambda \vdash A \otimes B}\ \multimap\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A \vdash B}}{A' \mid \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}}{- \mid A', \Gamma \vdash A \multimap B}\ \mathsf{pass} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A \vdash B}}{- \mid A', \Gamma, A \vdash B}\ \mathsf{pass}}{- \mid A', \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma, A \vdash B}}{- \mid \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}}{\mathsf{I} \mid \Gamma \vdash A \multimap B}\ \mathsf{IL} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{- \mid \Gamma, A \vdash B}}{\mathsf{I} \mid \Gamma, A \vdash B}\ \mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma, A \vdash B}}{A' \mid B', \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}}{A' \otimes B' \mid \Gamma \vdash A \multimap B}\ \otimes\mathsf{L} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma, A \vdash B}}{A' \otimes B' \mid \Gamma, A \vdash B}\ \otimes\mathsf{L}}{A' \otimes B' \mid \Gamma \vdash A \multimap B}\ \multimap\mathsf{R}$$

$$\dfrac{\overset{f}{- \mid \Gamma \vdash A'} \quad \dfrac{\overset{g}{B' \mid \Delta, A \vdash B}}{B' \mid \Delta \vdash A \multimap B}\ \multimap\mathsf{R}}{A' \multimap B' \mid \Gamma, \Delta \vdash A \multimap B}\ \multimap\mathsf{L} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A'} \quad \overset{g}{B' \mid \Delta, A \vdash B}}{A' \multimap B' \mid \Gamma, A \vdash B}\ \multimap\mathsf{L}}{A' \multimap B' \mid \Gamma, \Delta \vdash A \multimap B}\ \multimap\mathsf{R}$$

Figure 2.2: Equivalence of derivations in $\mathtt{SkNMILL_S}$*: permutative conversions*

If $f = {\multimap}\mathsf{R}\ f'$, then we perform a subinduction on g. If $g = {\multimap}\mathsf{L}(g', g'')$, then

$$\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} \qquad \dfrac{\overset{g'}{- \mid \Delta \vdash A} \quad \overset{g''}{B \mid \Lambda \vdash C}}{A \multimap B \mid \Delta, \Lambda \vdash C}\ {\multimap}\mathsf{L}}{S \mid \Gamma, \Delta, \Lambda \vdash C}\ \mathsf{scut}$$

$$= \dfrac{\overset{g'}{- \mid \Delta \vdash A} \qquad \dfrac{\overset{f'}{S \mid \Gamma, A \vdash B} \quad \overset{g''}{B \mid \Lambda \vdash C}}{S \mid \Gamma, A, \Lambda \vdash C}\ \mathsf{scut}}{S \mid \Gamma, \Delta, \Lambda \vdash C}\ \mathsf{ccut}$$

where the complexity of the cut formulae is reduced. For the rules other than ax, we permute scut up. For example, if $g = {\multimap}\mathsf{R}\ g'$, then

$$\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} \qquad \dfrac{\overset{g'}{A \multimap B \mid \Delta, A' \vdash B'}}{A \multimap B \mid \Delta \vdash A' \multimap B'}\ {\multimap}\mathsf{R}}{S \mid \Gamma, \Delta \vdash A' \multimap B'}\ \mathsf{scut}$$

$$= \dfrac{\dfrac{\dfrac{\overset{f'}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} \qquad \overset{g'}{A \multimap B \mid \Delta, A' \vdash B'}}{S \mid \Gamma, \Delta, A' \vdash B'}\ \mathsf{scut}}{S \mid \Gamma, \Delta \vdash A' \multimap B'}\ {\multimap}\mathsf{R}$$

For ccut, if $g = {\multimap}\mathsf{R}\ g'$, then we permute ccut up. If $g = {\multimap}\mathsf{L}(g', g'')$, we permute ccut up as well, but depending on where the cut formula is placed, we either apply ccut on f and g' or f and g''. □

Moreover, more equations of derivations hold in $\mathtt{SkNMILL}_{\mathtt{S}}$ due to the cut-elimination procedures defined in [67] and the proof of Theorem 2.1.1. This set of equations fully describe the possible interactions between cut rules. The first set of equations in Figure 2.3 shows that parallel composition of cut rules is commutative. The second set of equations in Figure 2.4 shows that sequential composition of cut rules is associative. Analogous equations have been proved in [67] for the fragment of $\mathtt{SkNMILL}_{\mathtt{S}}$ without linear implication. Notice that the each pair of derivations in these equations are *strictly* equal, not merely $\circeq$-related.

Proposition 2.1.2. *The commutativity equations in Figure 2.3 and the associativity equations in Figure 2.4 are admissible.*

Proof. The proof proceeds by mutual induction on the structure of derivations. There are many cases to consider. We do not include the long proof here and refer the interested reader to consult Veltri's Agda formalization, `https://github.com/niccoloveltri/code-skewmonclosed/blob/interpolation/Equations.agda`. Heavy proofs by pattern matching like this one is where the employment of a proof assistant becomes very helpful, in our experience. □

Here are some other admissible rules relevant for the metatheory of this calculus.

- The left rules for I and $\otimes$ are invertible up to $\circeq$, and similarly the right rule for $\multimap$. No other rule is invertible; in particular, the passivation rule pass is

$$\dfrac{\overset{f}{S \mid \Gamma_0 \vdash A} \quad \dfrac{\overset{g}{- \mid \Gamma_2 \vdash B} \quad \overset{h}{A \mid \Gamma_1, B, \Gamma_3 \vdash C}}{A \mid \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{ccut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{scut} = \dfrac{\overset{g}{- \mid \Gamma_2 \vdash B} \quad \dfrac{\overset{f}{S \mid \Gamma_0 \vdash A} \quad \overset{h}{A \mid \Gamma_1, B, \Gamma_3 \vdash C}}{S \mid \Gamma_0, \Gamma_1, B, \Gamma_3 \vdash C}\,\mathsf{scut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{ccut}$$

$$\dfrac{\overset{f}{- \mid \Gamma_1 \vdash A} \quad \dfrac{\overset{g}{- \mid \Gamma_3 \vdash B} \quad \overset{h}{S \mid \Gamma_0, A, \Gamma_2, B, \Gamma_4 \vdash C}}{S \mid \Gamma_0, A, \Gamma_2, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut} = \dfrac{\overset{g}{- \mid \Gamma_3 \vdash B} \quad \dfrac{\overset{f}{- \mid \Gamma_1 \vdash A} \quad \overset{h}{S \mid \Gamma_0, A, \Gamma_2, B, \Gamma_4 \vdash C}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, B, \Gamma_4 \vdash C}\,\mathsf{ccut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut}$$

Figure 2.3: Commutativity of cut

not.

$$\dfrac{\overset{f}{A \otimes B \mid \Gamma \vdash C}}{A \mid B, \Gamma \vdash C}\,\otimes\mathsf{L}^{-1} \qquad \dfrac{\overset{f}{\mathsf{I} \mid \Gamma \vdash C}}{- \mid \Gamma \vdash C}\,\mathsf{IL}^{-1} \qquad \dfrac{\overset{f}{S \mid \Gamma \vdash A \multimap B}}{S \mid \Gamma, A \vdash B}\,\multimap\mathsf{R}^{-1}$$

$$\begin{array}{rclrcl} \otimes\mathsf{L}^{-1}(\otimes\mathsf{L}\ f) & = & f & \otimes\mathsf{L}(\otimes\mathsf{L}^{-1}\ f) & \circeq & f \\ \mathsf{IL}^{-1}(\mathsf{IL}\ f) & = & f & \mathsf{IL}(\mathsf{IL}^{-1}\ f) & \circeq & f \\ \multimap\mathsf{R}^{-1}(\multimap\mathsf{R}\ f) & = & f & \multimap\mathsf{R}(\multimap\mathsf{R}^{-1}\ f) & \circeq & f \end{array}$$

- Applications of the invertible left logical rules can be iterated, and similarly for the invertible right $\multimap\mathsf{R}$ rule, resulting in the two admissible rules

$$\dfrac{S \mid \Gamma, \Delta \vdash C}{[\![S \mid \Gamma]\!]_{\otimes} \mid \Delta \vdash C}\,\mathsf{L}^{\star} \qquad \dfrac{S \mid \Gamma, \Delta \vdash C}{S \mid \Gamma \vdash [\![\Delta \mid C]\!]_{\multimap}}\,\multimap\mathsf{R}^{\star} \tag{2.2}$$

  The interpretation of antecedents $[\![S \mid \Gamma]\!]_{\otimes}$ in (2.2) is the formula obtained by substituting the separator | and the commas with tensors, $[\![S \mid A_1, \ldots, A_n]\!]_{\otimes} = (\ldots(([\![S\langle\!\langle \otimes A_1) \otimes A_2) \ldots) \otimes A_n$, where the interpretation of stoups is defined by $[\![-\langle\!\langle = \mathsf{I}$ and $[\![A\langle\!\langle = A$. Dually, the formula $[\![\Delta \mid C]\!]_{\multimap}$ in (2.2) is obtained by substituting | and commas with implications: $[\![A_1, \ldots, A_n \mid C]\!]_{\multimap} = A_1 \multimap (A_2 \multimap (\ldots \multimap (A_n \multimap C)))$.

- Another left implication rule, acting on a formula $A \multimap B$ in the context, is

$$\dfrac{\overset{f}{S \mid \Gamma_0 \vdash A} \qquad \dfrac{\overset{g}{A \mid \Gamma_1 \vdash B} \quad \overset{h}{B \mid \Gamma_2 \vdash C}}{A \mid \Gamma_1, \Gamma_2 \vdash C}\,\mathsf{scut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2 \vdash C}\,\mathsf{scut}
= \dfrac{\dfrac{\overset{f}{S \mid \Gamma_0 \vdash A} \quad \overset{g}{A \mid \Gamma_1 \vdash B}}{S \mid \Gamma_0, \Gamma_1 \vdash B}\,\mathsf{scut} \qquad \overset{h}{B \mid \Gamma_2 \vdash C}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2 \vdash C}\,\mathsf{scut}$$

$$\dfrac{\overset{f}{- \mid \Gamma_1 \vdash A} \qquad \dfrac{\overset{g}{S \mid \Gamma_0, A, \Gamma_2 \vdash B} \quad \overset{h}{B \mid \Gamma_3 \vdash C}}{S \mid \Gamma_0, A, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{scut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{ccut}
= \dfrac{\dfrac{\overset{f}{- \mid \Gamma_1 \vdash A} \quad \overset{g}{S \mid \Gamma_0, A, \Gamma_2 \vdash B}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2 \vdash B}\,\mathsf{ccut} \qquad \overset{h}{B \mid \Gamma_3 \vdash C}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \vdash C}\,\mathsf{scut}$$

$$\dfrac{\overset{f}{- \mid \Gamma_2 \vdash A} \qquad \dfrac{\overset{g}{- \mid \Gamma_1, A, \Gamma_3 \vdash B} \quad \overset{h}{S \mid \Gamma_0, B, \Gamma_4 \vdash C}}{S \mid \Gamma_0, \Gamma_1, A, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut}
= \dfrac{\dfrac{\overset{f}{- \mid \Gamma_2 \vdash A} \quad \overset{g}{- \mid \Gamma_1, A, \Gamma_3 \vdash B}}{S \mid \Gamma_1, \Gamma_2, \Gamma_3 \vdash B}\,\mathsf{ccut} \qquad \overset{h}{S \mid \Gamma_0, B, \Gamma_4 \vdash C}}{S \mid \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \vdash C}\,\mathsf{ccut}$$

Figure 2.4: Associativity of cut rules

derivable using cut:

$$\frac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{g}{S \mid \Delta_0, B, \Delta_1 \vdash C}}{S \mid \Delta_0, A \multimap B, \Gamma, \Delta_1 \vdash C} \multimap \mathsf{L_C}$$

$$= \frac{\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \dfrac{}{B \mid \ \vdash B}\mathsf{ax}}{A \multimap B \mid \Gamma \vdash B} \multimap \mathsf{L}}{- \mid A \multimap B, \Gamma \vdash B} \mathsf{pass} \quad \overset{g}{S \mid \Delta_0, B, \Delta_1 \vdash C}}{S \mid \Delta_0, A \multimap B, \Gamma, \Delta_1 \vdash C} \mathsf{ccut} \tag{2.3}$$

SkNMILL as a Logic of Resources Like MILL and NMILL, SkNMILL can be interpreted through the lens of resource management. In this interpretation, formulae in the sequent calculus $\mathtt{SkNMILL_S}$ represent different types of resources. The primitive resources are represented by atomic formulae. The compound formula $A \otimes B$ denotes having resource A followed by resource B, while I is "nothing". A formula of the form $A \multimap B$ describes a transformation method that converts resource A into resource B when $A \multimap B$ is available just before A whereby the resource $A \multimap B$ itself is also consumed. As with other substructural logics that lack weakening and contraction rules, each resource can be used exactly once.

In a sequent, the antecedent lists the available resources, while the succedent specifies the desired output resource. A derivation represents a specific method for transforming the input resources into the target resource. This resource-oriented perspective naturally leads to reading and constructing derivations from the conclusion upward to the premises. The antecedent maintains an ordered sequence of resources, indicating their required consumption order—if resource A appears before resource B in the antecedent, then A must be used before B. When the stoup position contains a resource, it indicates the resource that is immediately available for use. Resources in the context must wait until after the resource in the stoup has been consumed. In proof trees, time flows from bottom to top (conclusion to premises), with the left premise always being processed temporally before the right premise.

The context's behavior resembles that of a stack (or queue) data structure. When no resource is immediately available for consumption (i.e. the stoup is empty), the next resource can be promoted to immediate availability through the pass rule, which pops the leftmost element from the context and moves it to the stoup. The $\multimap$R rule enables pushing new resources to the rightmost position of the context. Through the $\otimes$L rule, an immediately available resource $A \otimes B$ can be split into its components: A remains immediately available in the stoup, while B is pushed to the top of the context for subsequent use. An immediately available I resource can be discarded via the IL rule, allowing the next context resource to be accessed. The IR rule permits generation of I without cost. The $\multimap$L rule handles situations where an immediately available resource $A \multimap B$ must be used to produce C. This makes B accessible, but only after generating A using some portion of the available resources. This process splits the context into Γ and Δ, where Γ is used to produce A. Once accomplished, the process continues with B immediately available and Δ reserved for later use. In the $\otimes$R rule, a succedent of form $A \otimes B$ requires first producing A and then B. This necessitates another context split: the earlier-usable resources in Γ produce A in the left premise, while the remaining resources in Δ subsequently produce B in the right premise. A key "skew" aspect of SkNMILL manifests here: an immediately available stoup resource S must contribute to producing A, even if A could be produced without resources.

The second fundamental "skew" characteristic of SkNMILL restricts left rules to operating solely on stoup formulae. In resource terms, it means that decomposing resources is not permitted while they remain in the context.

2.2 Equivalent calculi of SkNMILL

We now present two equivalent calculi of SkNMILL. Both presentations illustrate different aspects of the logic. We first present an axiomatic (Hilbert-style) calculus that corresponds to the categorical semantics in a more direct manner. Then we introduce a tree sequent calculus that treats structural rules explicitly and shows that the peculiar antecedent of SkNMILL should be thought of as a tree associating to the left rather than as a list with a distinguished first element. Each of these formulations has its own emphasis, and their equivalence ensures we can freely move between them while preserving the same logic.

2.2.1 Axiomatic calculus

The sequent calculus $\mathtt{SkNMILL_S}$ has an equivalent axiomatic calculus $\mathtt{SkNMILL_A}$, also called the Hilbert-style deductive system (similar correspondence has also been shown in [67, 66, 65, 69]). Both calculi share the same formulae, but $\mathtt{SkNMILL_A}$ uses simpler sequents of the form $A \vdash_{\mathsf{A}} B$ where both A and B are single formulae. The derivation rules of $\mathtt{SkNMILL_A}$, shown below, directly mirror the structure of skew monoidal closed categories from Definition 2.3.1, with π and π^{-1} corresponding to the residuation operation of $\otimes$ and $\multimap$.

$$\frac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{A}} B \quad B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

$$\frac{A \vdash_{\mathsf{A}} C \quad B \vdash_{\mathsf{A}} D}{A \otimes B \vdash_{\mathsf{A}} C \otimes D}\ \otimes \qquad \frac{C \vdash_{\mathsf{A}} A \quad B \vdash_{\mathsf{A}} D}{A \multimap B \vdash_{\mathsf{A}} C \multimap D}\ \multimap$$

$$\frac{}{\mathsf{I} \otimes A \vdash_{\mathsf{A}} A}\ \lambda \qquad \frac{}{A \vdash_{\mathsf{A}} A \otimes \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} A \otimes (B \otimes C)}\ \alpha$$

$$\frac{A \otimes B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} B \multimap C}\ \pi \qquad \frac{A \vdash_{\mathsf{A}} B \multimap C}{A \otimes B \vdash_{\mathsf{A}} C}\ \pi^{-1}$$

The derivations in $\mathtt{SkNMILL_A}$ are quotiented by a congruence relation $\doteq$. This relation is generated by the pairs of derivations shown in Figure 2.5, which state that $\multimap$ is a functor and π is an isomorphism and natural on three arguments together with the equations for skew monoidal categories from [67, Figure 2].

We show the equivalence between $\mathtt{SkNMILL_S}$ and $\mathtt{SkNMILL_A}$ with the following lemmata.

Lemma 2.2.1. *Given a formula A and contexts Γ and Δ, $[\![[\![A \mid \Gamma]\!]_\otimes \mid \Delta]\!]_\otimes = [\![A \mid \Gamma, \Delta]\!]_\otimes$.*

Proof. The proof proceeds by induction on Γ. If $\Gamma = [\,]$, then $[\![[\![A \mid [\,]]\!]_\otimes \Delta]\!]_\otimes = [\![A \mid \Gamma]\!]_\otimes = [\![A \mid [\,], \Delta]\!]_\otimes$.
If $\Gamma = (B, \Gamma')$, then

$$\dfrac{\dfrac{}{A \vdash_{\mathsf{A}} A}\,\mathsf{id} \quad \dfrac{}{B \vdash_{\mathsf{A}} B}\,\mathsf{id}}{A \multimap B \vdash_{\mathsf{A}} A \multimap B}\,{\multimap} \quad \doteq \quad \dfrac{}{A \multimap B \vdash_{\mathsf{A}} A \multimap B}\,\mathsf{id}$$

$$\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{A}} B} \quad \overset{g}{B \vdash_{\mathsf{A}} C}}{A \vdash_{\mathsf{A}} C}\,\mathsf{comp} \quad \dfrac{\overset{h}{D \vdash_{\mathsf{A}} E} \quad \overset{k}{E \vdash_{\mathsf{A}} F}}{D \vdash_{\mathsf{A}} F}\,\mathsf{comp}}{C \multimap D \vdash_{\mathsf{A}} A \multimap F}\,{\multimap}$$

$$\doteq \dfrac{\dfrac{\overset{g}{B \vdash_{\mathsf{A}} C} \quad \overset{h}{D \vdash_{\mathsf{A}} E}}{C \multimap D \vdash_{\mathsf{A}} B \multimap E}\,{\multimap} \quad \dfrac{\overset{f}{A \vdash_{\mathsf{A}} B} \quad \overset{k}{E \vdash_{\mathsf{A}} F}}{B \multimap E \vdash_{\mathsf{A}} A \multimap F}\,{\multimap}}{C \multimap D \vdash_{\mathsf{A}} A \multimap F}\,\mathsf{comp}$$

$$\dfrac{\overset{f}{A \vdash_{\mathsf{A}} A'} \quad \dfrac{\overset{g}{A' \otimes B \multimap C}}{A' \vdash_{\mathsf{A}} B \multimap C}\,\pi}{A \vdash_{\mathsf{A}} B \multimap C}\,\mathsf{comp}$$

$$\doteq \dfrac{\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{A}} A'} \quad \dfrac{}{B \vdash_{\mathsf{A}} B}\,\mathsf{id}}{A \otimes B \vdash_{\mathsf{A}} A' \otimes B}\,\otimes \quad \overset{g}{A' \otimes B \vdash_{\mathsf{A}} C}}{A \otimes B \vdash_{\mathsf{A}} \multimap C}\,\mathsf{comp}}{A \vdash_{\mathsf{A}} B \multimap C}\,\pi$$

$$\dfrac{\dfrac{\overset{f}{A \otimes B \vdash_{\mathsf{A}} C} \quad \overset{g}{C \vdash_{\mathsf{A}} C'}}{A \otimes B \vdash_{\mathsf{A}} C'}\,\mathsf{comp}}{A \vdash_{\mathsf{A}} B \multimap C'}\,\pi$$

$$\doteq \dfrac{\dfrac{\overset{f}{A \otimes B \vdash_{\mathsf{A}} C}}{A \vdash_{\mathsf{A}} B \multimap C}\,\pi \quad \dfrac{\dfrac{}{B \vdash_{\mathsf{A}} B}\,\mathsf{id} \quad \overset{g}{C \vdash_{\mathsf{A}} C'}}{B \multimap C \vdash_{\mathsf{A}} B \multimap C'}\,{\multimap}}{A \vdash_{\mathsf{A}} B \multimap C'}\,\mathsf{comp}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{A}} A}\,\mathsf{id} \quad \overset{f}{B \vdash_{\mathsf{A}} B'}}{A \otimes B \vdash_{\mathsf{A}} A \otimes B'}\,\otimes}{A \vdash_{\mathsf{A}} B \multimap (A \otimes B')}\,\pi$$

$$\doteq \dfrac{\dfrac{\dfrac{}{A \otimes B' \vdash_{\mathsf{A}} A \otimes B'}\,\mathsf{id}}{A \vdash_{\mathsf{A}} B' \multimap (A \otimes B')}\,\pi \quad \dfrac{\overset{f}{B \vdash_{\mathsf{A}} B'} \quad \dfrac{}{A \otimes B' \vdash_{\mathsf{A}} A \otimes B'}\,\mathsf{id}}{B' \multimap (A \otimes B') \vdash_{\mathsf{A}} B \multimap (A \otimes B')}\,{\multimap}}{A \vdash_{\mathsf{A}} B \multimap (A \otimes B')}\,\mathsf{comp}$$

$$\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{A}} B \multimap C}}{A \otimes B \vdash_{\mathsf{A}} C}\,\pi^{-1}}{A \vdash_{\mathsf{A}} B \multimap C}\,\pi \quad \doteq \quad \overset{f}{A \vdash_{\mathsf{A}} B \multimap C}$$

$$\dfrac{\dfrac{\overset{f}{A \otimes B \vdash_{\mathsf{A}} C}}{A \vdash_{\mathsf{A}} B \multimap C}\,\pi}{A \otimes B \vdash_{\mathsf{A}} C}\,\pi^{-1} \quad \doteq \quad \overset{f}{A \otimes B \vdash_{\mathsf{A}} C}$$

Figure 2.5: Additional equations on derivations of $\mathtt{SkNMILL}_{\mathsf{A}}$

$[\![[\![A \mid B,\Gamma']\!]_\otimes \mid \Delta]\!]_\otimes = [\![[\![A \otimes B \mid \Gamma']\!]_\otimes \mid \Delta]\!]_\otimes \overset{\text{I.H.}}{=} [\![A \otimes B \mid \Gamma', \Delta]\!]_\otimes = [\![A \mid B, \Gamma', \Delta]\!]_\otimes$ □

Lemma 2.2.2. *Given a context* Γ *and a derivation* $f : A \vdash_{\mathsf{A}} B$*, the following rule is admissible:*

$$\frac{\overset{f}{A \vdash_{\mathsf{A}} B}}{[\![A \mid \Gamma]\!]_\otimes \vdash_{\mathsf{A}} [\![B \mid \Gamma]\!]_\otimes}\;[\![f \mid \Gamma]\!]_\otimes$$

Proof. The proof proceeds by induction on Γ. If $\Gamma = [\,]$, then f is the desired derivation.
If $\Gamma = (C, \Gamma')$, then we construct the desired derivations as follows:

$$\frac{\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{A}} B} \quad \dfrac{}{C \vdash_{\mathsf{A}} C}\,\mathsf{id}}{A \otimes C \vdash_{\mathsf{A}} B \otimes C}\,\otimes}{[\![A \otimes C \mid \Gamma']\!]_\otimes \vdash_{\mathsf{A}} [\![B \otimes C \mid \Gamma']\!]_\otimes}\,[\![f \otimes \mathsf{id} \mid \Gamma']\!]_\otimes}{[\![A \mid C, \Gamma']\!]_\otimes \vdash_{\mathsf{A}} [\![B \mid C, \Gamma']\!]_\otimes}\,=$$

The double-line inference rule denotes an equality of sequents. □

Lemma 2.2.3. *Given two formulae* A *and* B *and a context* Γ*, there exists a derivation of the sequent* $[\![A \mid B, \Gamma]\!]_\otimes \vdash_{\mathsf{A}} A \otimes [\![B \mid \Gamma]\!]_\otimes$*.*

Proof. The proof proceeds by induction on Γ. If $\Gamma = [\,]$, then the desired derivation is $\mathsf{id} : A \otimes B \vdash_{\mathsf{A}} A \otimes B$.
If $\Gamma = (C, \Gamma')$, then we construct the desired derivations as follows:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} A \otimes (B \otimes C)}\,\alpha}{[\![(A \otimes B) \otimes C \mid \Gamma']\!]_\otimes \vdash_{\mathsf{A}} [\![A \otimes (B \otimes C) \mid \Gamma']\!]_\otimes}\,[\![\alpha \mid \Gamma']\!]_\otimes}{[\![A \mid B, C, \Gamma']\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid B \otimes C, \Gamma']\!]_\otimes}\,= \quad \overset{\text{I.H.}}{[\![A \mid B \otimes C, \Gamma']\!]_\otimes \vdash_{\mathsf{A}} A \otimes [\![B \otimes C \mid \Gamma']\!]_\otimes}}{[\![A \mid B, C, \Gamma']\!]_\otimes \vdash_{\mathsf{A}} A \otimes [\![B \otimes C \mid \Gamma']\!]_\otimes}\,\mathsf{comp}}{[\![A \mid B, C, \Gamma']\!]_\otimes \vdash_{\mathsf{A}} A \otimes [\![B \mid C, \Gamma']\!]_\otimes}\,=$$

□

Lemma 2.2.4. *Given a formula* A *and two contexts* Γ *and* Δ*, there exists a derivation of the sequent* $[\![A \mid \Gamma, \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes [\![- \mid \Delta]\!]_\otimes$*.*

Proof. The desired derivation is constructed as follows:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\![A \mid \Gamma]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes \mathsf{I}}\,\rho}{[\![[\![A \mid \Gamma]\!]_\otimes \mid \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![[\![A \mid \Gamma]\!]_\otimes \otimes \mathsf{I} \mid \Delta]\!]_\otimes}\,[\![\rho \mid \Delta]\!]_\otimes}{[\![[\![A \mid \Gamma]\!]_\otimes \mid \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![[\![A \mid \Gamma]\!]_\otimes \mid \mathsf{I}, \Delta]\!]_\otimes}\,= \quad \overset{\text{Lemma 2.2.3}}{[\![[\![A \mid \Gamma]\!]_\otimes \mid \mathsf{I}, \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes [\![\mathsf{I} \mid \Delta]\!]_\otimes}}{[\![[\![A \mid \Gamma]\!]_\otimes \mid \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes [\![\mathsf{I} \mid \Delta]\!]_\otimes}\,\mathsf{comp}}{[\![A \mid \Gamma, \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes [\![\mathsf{I} \mid \Delta]\!]_\otimes}\,\text{Lemma 2.2.1}}{[\![A \mid \Gamma, \Delta]\!]_\otimes \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_\otimes \otimes [\![- \mid \Delta]\!]_\otimes}\,=$$

□

Theorem 2.2.5. *The calculi* $\mathtt{SkNMILL}_{\mathsf{S}}$ *and* $\mathtt{SkNMILL}_{\mathsf{A}}$ *are equivalent, meaning that the two statements below are true:*

- *For any derivation* $f : S \mid \Gamma \vdash C$*, there exists a derivation* $\mathsf{G2A} f : [\![S \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} C$.
- *For any derivation* $f : A \vdash_{\mathsf{A}} C$*, there exists a derivation* $\mathsf{A2G} f : A \mid \ \vdash C$.

Proof. Both G2A and A2G are constructed by induction on the height of f.
We first construct G2A. For the base cases where $f = \mathsf{ax}$ or $f = \mathsf{IR}$, the corresponding $\mathtt{SkNMILL_A}$ derivations are constructed as follows:

$$\frac{}{A \mid \ \vdash A}\,\mathsf{ax} \quad\mapsto\quad \frac{}{A \vdash_{\mathsf{A}} A}\,\mathsf{id} \qquad\qquad \frac{}{- \mid \ \vdash \mathsf{I}}\,\mathsf{IR} \quad\mapsto\quad \frac{}{\mathsf{I} \vdash_{\mathsf{A}} \mathsf{I}}\,\mathsf{id}$$

Next we consider the inductive cases.
<u>Case $f = \mathsf{IL}\ f'$</u>

$$\frac{\overset{f'}{- \mid \Gamma \vdash C}}{\mathsf{I} \mid \Gamma \vdash C}\,\mathsf{IL} \quad\mapsto\quad \overset{\mathsf{G2A} f'}{[\![\mathsf{I} \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} C}$$

<u>Case $f = \otimes\mathsf{L}\ f'$</u>

$$\frac{\overset{f'}{A \mid B, \Gamma \vdash C}}{A \otimes B \mid \Gamma \vdash C}\,\otimes\mathsf{L} \quad\mapsto\quad \overset{\mathsf{G2A} f'}{[\![A \otimes B \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} C}$$

<u>Case $f = \mathsf{pass}\ f'$</u>

$$\frac{\overset{f'}{A \mid \Gamma \vdash C}}{- \mid A, \Gamma \vdash C}\,\mathsf{pass} \quad\mapsto\quad \frac{\dfrac{\dfrac{}{\mathsf{I} \otimes A \vdash_{\mathsf{A}} A}\,\lambda}{[\![\mathsf{I} \otimes A \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} [\![A \mid \Gamma]\!]_{\otimes}}\,[\![\lambda \mid \Gamma]\!]_{\otimes} \quad \overset{\mathsf{G2A} f'}{[\![A \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} C}}{[\![\mathsf{I} \otimes A \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} C}\,\mathsf{comp}$$

<u>Case $f = \multimap\mathsf{R}\ f'$</u>

$$\frac{\overset{f'}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma \vdash A \multimap B}\,\multimap\mathsf{R} \quad\mapsto\quad \frac{\dfrac{\overset{\mathsf{G2A} f'}{[\![S \mid \Gamma, A]\!]_{\otimes} \vdash_{\mathsf{A}} B}}{[\![S \mid \Gamma]\!]_{\otimes} \otimes A \vdash_{\mathsf{A}} B}}{[\![S \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} A \multimap B}\,\pi$$

<u>Case $f = \otimes\mathsf{R}(f', f'')$</u>

$$\frac{\overset{f'}{S \mid \Gamma \vdash A} \quad \overset{f''}{- \mid \Delta \vdash B}}{S \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R}$$

$$\mapsto \quad \frac{\overset{\text{Lemma 2.2.4}}{[\![S \mid \Gamma, \Delta]\!]_{\otimes} \vdash_{\mathsf{A}} [\![S \mid \Gamma]\!]_{\otimes} \otimes [\![- \mid \Delta]\!]_{\otimes}} \quad \dfrac{\overset{\mathsf{G2A} f'}{[\![S \mid \Gamma]\!]_{\otimes} \vdash_{\mathsf{A}} A} \quad \overset{\mathsf{G2A} f''}{[\![- \mid \Delta]\!]_{\otimes} \vdash_{\mathsf{A}} B}}{[\![S \mid \Gamma]\!]_{\otimes} \otimes [\![- \mid \Delta]\!]_{\otimes} \vdash_{\mathsf{A}} A \otimes B}\,\otimes}{[\![S \mid \Gamma, \Delta]\!]_{\otimes} \vdash_{\mathsf{A}} A \otimes B}\,\mathsf{comp}$$

<u>Case $f = \multimap\mathsf{L}(f', f'')$</u>

$$\dfrac{\overset{f'}{- \mid \Gamma \vdash A} \qquad \overset{f''}{B \mid \Delta \vdash C}}{A \multimap B \mid \Gamma, \Delta \vdash C}\ \multimap\mathsf{L}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{Lemma 2.2.4}}{\llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \vdash_{\mathsf{A}} (A \multimap B) \otimes \llbracket - \mid \Gamma \rrbracket_{\otimes}} \qquad \dfrac{\dfrac{\dfrac{}{A \multimap B \vdash_{\mathsf{A}} A \multimap B}\ \mathsf{id} \quad \overset{\mathsf{G2A}f'}{\llbracket - \mid \Gamma \rrbracket_{\otimes} \vdash_{\mathsf{A}} A}}{(A \multimap B) \otimes \llbracket - \mid \Gamma \rrbracket_{\otimes} \vdash_{\mathsf{A}} (A \multimap B) \otimes A}\ \otimes \quad \dfrac{\dfrac{}{A \multimap B \vdash_{\mathsf{A}} A \multimap B}\ \mathsf{id}}{(A \multimap B) \otimes A \vdash_{\mathsf{A}} B}\ \pi^{-1}}{(A \multimap B) \otimes \llbracket - \mid \Gamma \rrbracket_{\otimes} \vdash_{\mathsf{A}} B}\ \mathsf{comp}}{\llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \vdash_{\mathsf{A}} B}\ \mathsf{comp}}{\llbracket \llbracket A \multimap B \mid \Gamma \rrbracket_{\otimes} \mid \Delta \rrbracket_{\otimes} \vdash_{\mathsf{A}} \llbracket B \mid \Delta \rrbracket_{\otimes}}\ \text{Lemma 2.2.2}}{\llbracket A \multimap B \mid \Gamma, \Delta \rrbracket_{\otimes} \vdash_{\mathsf{A}} \llbracket B \mid \Delta \rrbracket_{\otimes}}\ \text{Lemma 2.2.1} \qquad \overset{\mathsf{G2A}f''}{\llbracket B \mid \Delta \rrbracket_{\otimes} \vdash_{\mathsf{A}} C}}{\llbracket A \multimap B \mid \Gamma, \Delta \rrbracket_{\otimes} \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

The construction of A2G also proceeds by induction f. For the identity rule, A2G id = ax. The structural rules λ, ρ, and α are translated to their corresponding $\mathtt{SkNMILL_S}$ derivations shown in (2.1). The comp rule is handled by the admissible scut rule. The translation of adjunction rules π and π^{-1} relies on the invertibility of $\otimes$L and $\multimap$R rules in $\mathtt{SkNMILL_S}$. The functoriality rules ($\otimes$ and $\multimap$) follow directly from $\otimes$R, $\otimes$L, $\multimap$L, and $\multimap$R. □

Remark 2.2.6. Both $\mathtt{SkNMILL_S}$ (with its equivalence $\stackrel{\circ}{=}$) and $\mathtt{SkNMILL_A}$ (with its equivalence $\doteq$) serve as presentations of the free skew monoidal closed category (FSkMCl) over At. The functions G2A and A2G can be proved to preserve these respective equivalence relations. Furthermore, G2A and A2G are inverses of each other modulo equivalence of derivations, meaning that the equivalence between $\mathtt{SkNMILL_S}$ and $\mathtt{SkNMILL_A}$ holds not just for derivability of sequents but also at the level of derivations up to equivalence. For the proof that $\mathtt{SkNMILL_S}$ with $\stackrel{\circ}{=}$ forms the FSkMCl, we refer readers to Theorem 2.3.6. For the proof of $\mathtt{SkNMILL_A}$ with $\doteq$, we refer readers to the beginning of Section 6.2.1, noting that while the construction there is for a more complex calculus, it is a conservative extension of $\mathtt{SkNMILL_A}$.

In the next section, we introduce tree sequent calculus of SkNMILL, which is also equivalent to $\mathtt{SkNMILL_S}$ and $\mathtt{SkNMILL_A}$ up to provability of sequents. In principle, they should also be equivalent up to derivation equivalences under some well-defined congruence relation on derivations in the tree calculus. However, this investigation remains as future work.

2.2.2 Tree sequent calculus

Adapted from [52], we define trees inductively by the grammar $T ::= \mathsf{Fma} \mid - \mid (T, T)$, where $-$ is an empty tree. A context is a tree with a hole defined recursively as $\mathcal{C} ::= [\cdot] \mid (\mathcal{C}, T) \mid (T, \mathcal{C})$. The substitution of a tree into a hole is defined recursively:

$$\begin{array}{rcl} \mathit{subst}([\cdot], U) & = & U \\ \mathit{subst}((T', \mathcal{C}), U) & = & (T', \mathit{subst}(\mathcal{C}, U)) \\ \mathit{subst}((\mathcal{C}, T'), U) & = & (\mathit{subst}(\mathcal{C}, U), T') \end{array}$$

We use $T[\cdot]$ to denote a context and $T[U]$ to abbreviate $\mathit{subst}(T[\cdot], U)$. Sometimes we omit parentheses for trees when it does not cause ambiguity. Sequents in

$\mathtt{SkNMILL_T}$ are in the form $T \vdash_{\mathsf{T}} A$ where T is a tree and A is a single formula. Derivations in $\mathtt{SkNMILL_T}$ are generated recursively by the following rules:

$$\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}$$

$$\dfrac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL} \qquad \dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \qquad \dfrac{T[A, B] \vdash_{\mathsf{T}} C}{T[A \otimes B] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{L} \qquad \dfrac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T, U \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}$$

$$\dfrac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap B, U] \vdash_{\mathsf{T}} C}\ {\multimap}\mathsf{L} \qquad \dfrac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap B}\ {\multimap}\mathsf{R}$$

$$\dfrac{T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc} \qquad \dfrac{T[U] \vdash_{\mathsf{T}} C}{T[-, U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL} \qquad \dfrac{T[U, -] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}$$

This calculus is similar to the ones for $\mathtt{NL}$ (non-associative Lambek calculus) [52] and $\mathtt{NL}$ with unit [17] but with semi-unital (unitL and unitR) and semi-associative (assoc) rules. The structural rule unitL, read bottom-up, removes an empty tree from the left. It helps us to correctly characterize the axiom λ in $\mathtt{SkNMILL_T}$, i.e. $\mathsf{I} \otimes A \vdash_{\mathsf{T}} A$ is derivable while $A \vdash_{\mathsf{T}} \mathsf{I} \otimes A$ is not. Analogously for the rule unitR, from a bottom-up perspective, adds an empty tree from the right, and we cannot capture ρ in $\mathtt{SkNMILL_T}$ without unitR. Similarly, assoc captures α by allowing the conversion of left-associative bracketing to right-associative bracketing among three adjacent trees:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{-, A \vdash_{\mathsf{T}} A}\ \mathsf{unitL}}{\mathsf{I}, A \vdash_{\mathsf{T}} A}\ \mathsf{IL}}{\mathsf{I} \otimes A \vdash_{\mathsf{T}} A}\ \otimes\mathsf{L} \qquad \dfrac{\dfrac{\overset{??}{X \vdash_{\mathsf{T}} \mathsf{I}} \quad \overset{??}{- \vdash_{\mathsf{T}} X}}{X, - \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \otimes\mathsf{R}}{X \vdash_{\mathsf{T}} \mathsf{I} \otimes X}\ \mathsf{unitR}$$

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR}}{A, - \vdash_{\mathsf{T}} A \otimes \mathsf{I}}\ \otimes\mathsf{R}}{A \vdash_{\mathsf{T}} A \otimes \mathsf{I}}\ \mathsf{unitR} \qquad \dfrac{\dfrac{\overset{??}{X, - \vdash_{\mathsf{T}} X}}{X, \mathsf{I} \vdash_{\mathsf{T}} X}\ \mathsf{IL}}{X \otimes \mathsf{I} \vdash_{\mathsf{T}} X}\ \otimes\mathsf{L}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax} \quad \dfrac{}{C \vdash_{\mathsf{T}} C}\ \mathsf{ax}}{B, C \vdash_{\mathsf{T}} B \otimes C}\ \otimes\mathsf{R}}{A, (B, C) \vdash_{\mathsf{T}} A \otimes (B \otimes C)}\ \otimes\mathsf{R}}{(A, B), C \vdash_{\mathsf{T}} A \otimes (B \otimes C)}\ \mathsf{assoc}}{(A \otimes B), C \vdash_{\mathsf{T}} A \otimes (B \otimes C)}\ \otimes\mathsf{L}}{(A \otimes B) \otimes C \vdash_{\mathsf{T}} A \otimes (B \otimes C)}\ \otimes\mathsf{L} \qquad \dfrac{\dfrac{\overset{??}{X, (Y, Z) \vdash_{\mathsf{T}} (X \otimes Y) \otimes Z}}{X, (Y \otimes Z) \vdash_{\mathsf{T}} (X \otimes Y) \otimes Z}\ \otimes\mathsf{L}}{X \otimes (Y \otimes Z) \vdash_{\mathsf{T}} (X \otimes Y) \otimes Z}\ \otimes\mathsf{L}$$

Remark 2.2.7. The tree sequent calculus presented here, particularly its feature of applying rules within a tree context (e.g., rule $\otimes\mathsf{L}$ acting on a tree (A, B) within a larger tree $T[A, B]$), might seem reminiscent of deep inference formalisms [33, 31, 32]. Deep inference systems are characterized by their ability to apply inference rules at arbitrary depths within general syntactic structures.

However, the motivation and technical machinery differ significantly. The tree calculus employed in this thesis serves first as an equivalent formalism to the stoup calculus, and later (in Chapter 6) as a flexible calculus to characterize certain variants of skew monoidal categories that the stoup calculus cannot characterize. The tree constructors (commas and, in Chapter 6, additionally with semicolons) define grouping of formulae in the antecedent. Deep inference formalisms, in contrast, often pursue broader proof-theoretic goals, such as symmetry in derivations, locality of rules, and normalization procedures applicable to a wide range of logics.

While the concept of "deep application" provides a point of initial comparison, the design choices, the nature of the syntactic objects (trees of formulae versus unified "structures"), and the primary applications within this thesis distinguish tree sequent calculus from deep inference. For instance, proof normalization in this thesis primarily relies on the focused stoup calculi, as the explicit structural rules of the tree calculus complicate the equational theory of derivations.

Theorem 2.2.8. $\mathtt{SkNMILL_T}$ *is cut-free, i.e. the rule*

$$\dfrac{\overset{f}{U \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

is admissible.

Proof. We perform induction on the structure of derivation f of the left premise, and if necessary, we perform subinduction on the derivation g or the complexity of the cut formula A. Cases of logical rules $\mathsf{ax}, \otimes\mathsf{L}, \otimes\mathsf{R}, \multimap\mathsf{L}$, and $\multimap\mathsf{R}$ have been discussed in [52], so we only elaborate on the new cases arising in $\mathtt{SkNMILL_T}$.

- The first new case is that $f = \mathsf{IR}$, then we inspect the structure of g.
  - If $g = \mathsf{ax} : \mathsf{I} \vdash_{\mathsf{T}} \mathsf{I}$, then we define $\mathsf{cut}(\mathsf{IR}, \mathsf{ax}) = \mathsf{IR}$.
  - If $g = \mathsf{IL}\ g'$, then there are two subcases:
    * if the I introduced by IL is the cut formula, then we define

$$\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \dfrac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad = \quad \overset{g'}{T[-] \vdash_{\mathsf{T}} C}$$

    * if the I introduced by IL is not the cut formula, then we define

$$\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \dfrac{\overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad = \quad \dfrac{\dfrac{\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{ax} \quad \overset{g'}{T[-] \vdash_{\mathsf{T}} C}}{T^{\{\mathsf{I}:=-\}}[-] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T^{\{\mathsf{I}:=-\}}[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}$$

      where $T^{\{\mathsf{I}:=-\}}[\cdot]$ means that a formula occurrence I at some fixed position in the context has been replaced by $-$.
  - If $g = \mathcal{R}\ g'$, where $\mathcal{R}$ is a one-premise rule different from IL, then $\mathsf{cut}(\mathsf{IR}, \mathcal{R}\ g') = \mathcal{R}(\mathsf{cut}(\mathsf{IR}, g'))$.
  - The cases of an arbitrary two-premise rule are similar.
- The only other new cases are IL and the structural rules, which are all one-premise left rules, where we can permute cut upwards. For example, if $f =$

unitL f', then we define

$$\dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A}}{T'[-,U] \vdash_{\mathsf{T}} A}\ \mathsf{unitL} \quad \overset{g}{T[A] \vdash_{\mathsf{T}} C}}{T[T'[-,U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut} \quad = \quad \dfrac{\dfrac{\overset{f'}{T'[U] \vdash_{\mathsf{T}} A} \quad \overset{g}{T[A] \vdash_{\mathsf{T}}}}{T[T'[U]] \vdash_{\mathsf{T}} C}\ \mathsf{cut}}{T[T'[-,U]] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}$$

The other cases are similar.

□

The proof of equivalence between $\mathtt{SkNMILL_S}$ and $\mathtt{SkNMILL_T}$ relies on the following lemmata and definitions.

Definition 2.2.9. For any tree T, T^* is the formula obtained from T by replacing commas with $\otimes$ and $-$ with I, respectively.

Lemma 2.2.10. *For any context $T[\cdot]$ and tree U, $T[U]^* = T[U^*]^*$.*

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then $[U]^* = U^*$ by the definition of substitution.
If $T[\cdot] = (T'[\cdot], T'')$, then by inductive hypothesis, we have $T'[U]^* = T'[U^*]^*$ and by definition, we have $(T'[U], T'')^* = T'[U]^* \otimes^{\mathsf{L}} T''^* = T'[U^*]^* \otimes^{\mathsf{L}} T''^* = (T'[U^*], T'')^*$.
The case $T[\cdot] = (T', T''[\cdot])$ is symmetric. □

In the remainder of the section, we will refer to uses of this lemma by double lines.

Lemma 2.2.11. *Given a context $T[\cdot]$ and a derivation $f : A \vdash_{\mathsf{A}} B$, the following rule is admissible:*

$$\dfrac{\overset{f}{A \vdash_{\mathsf{A}} B}}{T[A]^* \vdash_{\mathsf{A}} T[B]^*}\ T[f]^*$$

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then we have $T[A]^* = A$ and $T[B]^* = B$, and f is the desired derivation.
If $T[\cdot] = (T'[\cdot]; T'')$, then we construct the desired derivation as follows:

$$\dfrac{\dfrac{\dfrac{\overset{f}{A \vdash_{\mathsf{A}} B}}{T'[A]^* \vdash_{\mathsf{A}} T'[B]^*}\ T'[f]^* \quad \dfrac{}{T''^* \vdash_{\mathsf{A}} T''^*}\ \mathsf{id}}{T'[A]^* \otimes T''^* \vdash_{\mathsf{A}} T'[B]^* \otimes T''^*}\ \otimes}{(T'[A], T'')^* \vdash_{\mathsf{A}} (T'[B], T'')^*}$$

The case $T[\cdot] = (T', T''[\cdot])$ is symmetric. □

Definition 2.2.12. We define an encoding function $[\![- \mid -]\!]$ that transforms a tree and an ordered list of formulae into a tree associating to the left:

$$[\![T \mid [\,]]\!] = T$$
$$[\![T \mid B, \Gamma]\!] = [\![(T, B) \mid \Gamma]\!]$$

Lemma 2.2.13. *For any tree T and lists of formulae Γ and Δ, $[\![[\![T \mid \Gamma]\!] \mid \Delta]\!] = [\![T \mid \Gamma, \Delta]\!]$.*

Proof. The proof proceeds by induction on Δ.
If $\Delta = [\,]$, then $[\![[\![T \mid \Gamma]\!] \mid [\,]]\!] = [\![T \mid \Gamma]\!] = [\![T \mid \Gamma, [\,]]\!]$ by definition.
If $\Delta = (A, \Delta')$, then by Definition 2.2.12, inductive hypothesis, and associativity of list concatenation, we have $[\![[\![T \mid \Gamma]\!] \mid A, \Delta']\!] = [\![[\![T \mid \Gamma, A]\!] \mid \Delta']\!] \overset{\text{I.H.}}{=} [\![T \mid (\Gamma, A), \Delta']\!] = [\![T \mid \Gamma, (A, \Delta')]\!]$. □

With the above lemmata, definition, and the functions $s(S)$ that maps a stoup to a tree (i.e. $s(S) = -$ if $S = -$ or $s(S) = B$ if $S = B$), we can state and prove the equivalence between $\mathtt{SkNMILL_S}$ and $\mathtt{SkNMILL_T}$.

Theorem 2.2.14. *The calculi* $\mathtt{SkNMILL_S}$ *and* $\mathtt{SkNMILL_T}$ *are equivalent, meaning that the two statements below are true:*

- *For any derivation* $f : S \mid \Gamma \vdash C$*, there exists a derivation* $\mathsf{G2T} f : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} C$.
- *For any derivation* $f : T \vdash_{\mathsf{T}} C$*, there exists a derivation* $\mathsf{T2G} f : T^* \mid \;\vdash C$.

Proof. Both $\mathsf{G2T}$ and $\mathsf{T2G}$ are constructed by induction on height of f.

For $\mathsf{G2T}$, the interesting cases are $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$. For example, if $f = \otimes\mathsf{R}(f', f'')$, then by inductive hypothesis, we have two derivations $\mathsf{G2T}\ f' : [\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A$ and $\mathsf{G2T}\ f'' : [\![- \mid \Delta]\!] \vdash_{\mathsf{T}} B$. Our goal sequent is $[\![s(S) \mid \Gamma, \Delta]\!] \vdash_{\mathsf{T}} A\otimes$, which is constructed as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{G2T}\ f'}{[\![s(S) \mid \Gamma]\!] \vdash_{\mathsf{T}} A} \quad \dfrac{\mathsf{G2T}\ f''}{[\![- \mid \Delta]\!] \vdash_{\mathsf{T}} B}}{[\![s(S) \mid \Gamma]\!], [\![- \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \otimes\mathsf{R}}{[\![[\![s(S) \mid \Gamma]\!], - \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{assoc}^*}{[\![[\![s(S) \mid \Gamma]\!] \mid \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \mathsf{unitR}}{[\![s(S) \mid \Gamma, \Delta]\!] \vdash_{\mathsf{T}} A \otimes B}\ \text{Lemma 2.2.13}
$$

where assoc^* means multiple applications of assoc. The case of $\multimap\mathsf{L}$ is similar.

For $\mathsf{T2G}$, the proof relies on Lemma 2.2.11 heavily. For example, when $f = \mathsf{unitR}\ g$, where we have $g : T[U, -] \vdash_{\mathsf{T}} C$. By inductive hypothesis, we have $\mathsf{T2G}\ g : T[U^* \otimes \mathsf{I}]^* \mid \;\vdash C$. With Lemma 2.2.11, we construct the desired derivation as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{U^* \mid \;\vdash U^*}\ \mathsf{ax} \quad \dfrac{}{- \mid \;\vdash \mathsf{I}}\ \mathsf{IR}}{U^* \mid \;\vdash U^* \otimes \mathsf{I}}\ \otimes\mathsf{R}}{T[U^*]^* \mid \;\vdash T[U^* \otimes \mathsf{I}]^*}\ T[\otimes\mathsf{R}(\mathsf{ax}, \mathsf{IR})]^*}{T[U]^* \mid \;\vdash T[U, -]^*} \quad \dfrac{\mathsf{T2G}\ g}{T[U, -]^* \mid \;\vdash C}}{T[U]^* \mid \;\vdash C}\ \mathsf{scut}
$$

The other cases are similar. □

2.3 Categorical Semantics

Next we present a categorical semantics for $\mathtt{SkNMILL_S}$.

Definition 2.3.1. A *(left) skew monoidal closed category* $\mathbb{C}$ is a category with a unit object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ natural in B, and three natural transformations λ, ρ, α typed $\lambda_A : \mathsf{I} \otimes A \to A$, $\rho_A : A \to A \otimes \mathsf{I}$ and $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, satisfying the Mac Lane axioms in Definition 1.1.1.

Remark 2.3.2. The notion of skew monoidal closed category admits other equivalent characterizations [60, 64]. Tuples of natural transformations (λ, ρ, α) are in bijective correspondence with tuples of (extra)natural transformations (j, i, L) typed $j_A : \mathsf{I} \to A \multimap A$, $i_A : \mathsf{I} \multimap A \to A$, $L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C)$. Moreover, α and L are interdefinable with a natural transformation p typed $\mathsf{p}_{A,B,C} : (A \otimes B) \multimap C \to A \multimap (B \multimap C)$, embodying an internal version of the adjunction between $\otimes$ and $\multimap$. Additionally, j and L are interdefinable with natural transformations $\widehat{j}_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}(\mathsf{I}, A \multimap B)$ and $\widehat{L}_{A,B,C,D} : \int^E \mathbb{C}(A, E \multimap D) \times \mathbb{C}(B, C \multimap E) \to \mathbb{C}(A, B \multimap (C \multimap D))$ respectively, where $\int^E$ is a coend and $\mathbb{C}(A, B)$ means the set of morphisms from A to B. See Remark 6.0.3 for the right skew correspondents.

Example 2.3.3 (from [64]). This example explains how to turn every categorical model of $\mathtt{MILL}$ extended with a $\Box$-like modality of necessity (or something like the exponential modality ! of linear logic) into a model of $\mathtt{SkNMILL}$. Let $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ be a (possibly symmetric) monoidal closed category and let (D, ε, δ) be a comonad on $\mathbb{C}$, where $\varepsilon_A : D\,A \to A$ and $\delta_A : D\,A \to D\,(D\,A)$ are the counit and comultiplication of D. Suppose the comonad D to be *lax monoidal*, i.e., coming with a map $\mathsf{e} : \mathsf{I} \to D\mathsf{I}$ and a natural transformation m typed $\mathsf{m}_{A,B} : D\,A \otimes D\,B \to D\,(A \otimes B)$ cohering suitably with λ, ρ, α, ε and δ. Then $\mathbb{C}$ has also a skew monoidal closed structure $(\mathsf{I}, \otimes^D, {}^D{\multimap})$ given by $A \otimes^D B = A \otimes D\,B$ and $B \;{}^D{\multimap}\; C = D\,B \multimap C$. The adjunction $- \otimes D\,B \dashv D\,B \multimap -$ yields an adjunction $- \otimes^D B \dashv B \;{}^D{\multimap}\; -$. The structural laws are

$$\lambda^D_A = \mathsf{I} \otimes D\,A \xrightarrow{\mathsf{I} \otimes \varepsilon_A} \mathsf{I} \otimes A \xrightarrow{\lambda_A} A \qquad \rho^D_A = A \xrightarrow{\rho_A} A \otimes \mathsf{I} \xrightarrow{A \otimes \mathsf{e}} A \otimes D\,\mathsf{I}$$

$$\begin{aligned} \alpha^D_{A,B,C} = \; & (A \otimes D\,B) \otimes D\,C \xrightarrow{(A \otimes D\,B) \otimes \delta_C} (A \otimes D\,B) \otimes D\,(D\,C) \\ & \xrightarrow{\alpha_{A,D\,B,D\,(D\,C)}} A \otimes (D\,B \otimes D\,(D\,C)) \xrightarrow{A \otimes \mathsf{m}_{B,D\,C}} A \otimes D\,(B \otimes D\,C) \end{aligned}$$

$(\mathbb{C}, \mathsf{I}, \otimes^D, {}^D{\multimap})$ is a "genuine" skew monoidal closed category, in the sense that λ^D, ρ^D and α^D are all generally non-invertible.

Definition 2.3.4. A *(strict) skew monoidal closed functor* $F : \mathbb{C} \to \mathbb{D}$ between skew monoidal closed categories $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ and $(\mathbb{D}, \mathsf{I}', \otimes', \multimap')$ is a functor from $\mathbb{C}$ to $\mathbb{D}$ satisfying $F\mathsf{I} = \mathsf{I}'$, $F(A \otimes B) = FA \otimes' FB$ and $F(A \multimap B) = FA \multimap' FB$, also preserving the structural laws λ, ρ and α on the nose.

The formulae, derivations and the equivalence relation $\stackrel{\circ}{=}$ of the sequent calculus for $\mathtt{SkNMILL}$ determine a skew monoidal closed category $\mathsf{FSkMCl(At)}$.

Definition 2.3.5. The skew monoidal closed category $\mathsf{FSkMCl(At)}$ has as objects formulae; the operations I, $\otimes$ and $\multimap$ are the logical connectives. The set of maps between objects A and B is the set of derivations $A \mid \ \vdash B$ quotiented by the equivalence relation $\doteq$. The identity map on A is the equivalence class of ax_A, while composition is given by scut. The structural laws λ, ρ, α are given by derivations in (2.1).

This is a good definition since all equations of a skew monoidal closed category turn out to hold.

Skew monoidal closed categories with given interpretations of atoms into them constitute models of the sequent calculus of SkNMILL, in the sense specified by the following theorem.

Theorem 2.3.6. *Let $\mathbb{D}$ be a skew monoidal closed category. Given $F_{\mathsf{At}} : \mathsf{At} \to |\mathbb{D}|$ providing evaluation of atomic formulae as objects of $\mathbb{D}$, there exists a unique strict skew monoidal closed functor $F : \mathsf{FSkMCl}(\mathsf{At}) \to \mathbb{D}$.*

Proof.
Existence. Let $(\mathbb{D}, \mathsf{I}', \otimes', \multimap')$ be a skew monoidal closed category. The action F_0 on objects of the functor F is defined by induction on the input formula:

$$F_0 X = F_{\mathsf{At}} X \qquad F_0 \mathsf{I} = \mathsf{I}'$$
$$F_0(A \otimes B) = F_0 A \otimes' F_0 B \qquad F_0(A \multimap B) = F_0 A \multimap' F_0 B$$

The encoding of antecedents as formulae $[\![S \mid \Gamma]\!]_{\otimes'}$, introduced immediately after (2.2), can be replicated also in $\mathbb{D}$ by simply replacing I and $\otimes$ with I' and $\otimes'$ in the definition, where now S is an optional object and Γ is a list of objects of $\mathbb{D}$. Using this encoding, it is possible to show that each rule in $\mathtt{SkNMILL_S}$ is derivable in $\mathbb{D}$. As an illustrative case, consider the rule pass. Assume given a map $f : [\![A \mid \Gamma]\!]_{\otimes'} \to C$ in $\mathbb{D}$. Then, assuming $\Gamma = A_1, \ldots, A_n$, we can define the passivation of f typed $[\![- \mid A, \Gamma]\!]_{\otimes'} \to C$ as

$$(\ldots((\mathsf{I}' \otimes' A) \otimes' A_1)\ldots) \otimes' A_n \xrightarrow{(\ldots(\lambda'_A \otimes' A_1)\ldots)\otimes' A_n} (\ldots(A \otimes' A_1)\ldots) \otimes' A_n \xrightarrow{f} C$$

This implies the existence of a function F_1, sending each derivation $f : S \mid \Gamma \vdash A$ to a map $F_1 f : F_0([\![S \mid \Gamma]\!]_{\otimes}) \to FA$ in $\mathbb{D}$, defined by induction on the derivation f. When restricted to sequents of the form $A \mid \ \vdash B$, the function F_1 provides the action of F on maps. It is possible to show that F is a functor and strictly preserves the skew monoidal closed structure, so it is a skew monoidal closed functor.
Uniqueness. Consider another skew monoidal closed functor $F' : \mathsf{FSkMCl}(\mathsf{At}) \to \mathbb{D}$ such that $F'X = F_{\mathsf{At}} X$ for any atom X. We can verify that F' and F agree on every object and morphism in $\mathsf{FSkMCl}(\mathsf{At})$ by induction on formulae and derivations respectively. □

Thus we have proved that $\mathsf{FSkMCl}(\mathsf{At})$ is the free skew monoidal closed category.

2.4 Proof-Theoretic Semantics via Focusing

The equivalence relation $\doteq$ from Figures 2.1 and 2.2 can also be viewed as an abstract rewrite system, by orienting every equation from left to right. The resulting rewrite system is locally confluent and strongly normalizing, thus confluent with unique normal forms. Derivations in normal form thus correspond to canonical representatives of $\doteq$-equivalence classes. These representatives can be organized in a *focused sequent calculus* in the sense of [5], which describes, in a declarative fashion, a particular root-first proof search strategy for the (original, unfocused) sequent calculus.

2.4.1 A First (Naïve) Focused Sequent Calculus

As a first attempt to focusing, we naïvely merge together the rules of the focused sequent calculi of skew monoidal categories [67] and skew prounital closed categories [65]. In the resulting calculus, sequents have one of 3 possible subscript annotations, corresponding to 3 different phases of proof search: RI for "right invertible", LI for "left invertible', and F for "focusing". We will see soon that this focused sequent calculus is too permissive, in the sense that two distinct derivations in the focused system can correspond to $\stackrel{\circ}{=}$-equivalent sequent calculus derivations.

$$
\begin{array}{ll}
\text{(right invertible)} & \dfrac{S \mid \Gamma, A \vdash_{\mathsf{RI}} B}{S \mid \Gamma \vdash_{\mathsf{RI}} A \multimap B}\ {\multimap}\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash_{\mathsf{LI}} P}{S \mid \Gamma \vdash_{\mathsf{RI}} P}\ \mathsf{LI2RI} \\[2ex]
\text{(left invertible)} & \dfrac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\ {\otimes}\mathsf{L} \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{F}} P}{T \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{F2LI} \\[2ex]
\text{(focusing)} & \begin{array}{c} \dfrac{}{X \mid\ \vdash_{\mathsf{F}} X}\ \mathsf{ax} \qquad \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash_{\mathsf{F}} P}\ \mathsf{pass} \qquad \dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\ \mathsf{IR} \\[2ex] \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} A \otimes B}\ {\otimes}\mathsf{R} \qquad \dfrac{- \mid \Gamma \vdash_{\mathsf{RI}} A \quad B \mid \Delta \vdash_{\mathsf{LI}} P}{A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{F}} P}\ {\multimap}\mathsf{L} \end{array}
\end{array}
\tag{2.4}
$$

In the rules above, the metavariable P denotes a *positive* formula, i.e. $P \neq A \multimap B$, while metavariable T indicates a *negative* stoup, i.e. $T \neq \mathsf{I}$ and $T \neq A \otimes B$ (T can also be empty).

We explain the rules of the focused sequent calculus from the perspective of root-first proof search. The starting phase is 'right invertible' RI.

($\vdash_{\mathsf{RI}}$) We repeatedly apply the right invertible rule ${\multimap}\mathsf{R}$ with the goal of reducing the succedent to a positive formula P. When the succedent formula becomes positive, we move to phase LI via LI2RI.

($\vdash_{\mathsf{LI}}$) We repeatedly destruct the stoup formula via application of left invertible rules ${\otimes}\mathsf{L}$ and IL with the goal of making it negative. When this happens, we move to phase F via F2LI.

($\vdash_{\mathsf{F}}$) We apply one of the five remaining rules ax, IR, pass, ${\otimes}\mathsf{R}$ or ${\multimap}\mathsf{L}$. For the passivation rule, we move the leftmost formula A in the context to the stoup when the latter is empty. This allows us to start decomposing A using left invertible rules in phase LI. The premises of ${\otimes}\mathsf{R}$ are both in phase RI since A and B are generic formulae, in particular they could be implications. The first premise of ${\multimap}\mathsf{L}$ is in phase RI for the same reason while the second premise is in LI because the succedent formula P is positive.

The focused calculus in (2.4) is sound and complete with respect to $\mathtt{SkNMILL}_{\mathtt{S}}$ in regards to derivability, but not *equationally complete*, i.e., there exist $\stackrel{\circ}{=}$-equivalent sequent calculus derivations which have multiple distinct derivations using the rules in (2.4). In other words, the rules in (2.4) are too permissive. They facilitate two forms of non-determinism in root-first proof search that should not be there.

(i) The first premise of the $\otimes\mathsf{R}$ rule is in phase RI, since A is potentially an implication which the invertible right rule $\multimap\mathsf{R}$ could act upon. Proof search for the first premise eventually hits phase F again, when we have the possibility of applying the pass rule if the stoup is empty. This implies the existence of situations where either of the rules $\otimes\mathsf{R}$ and pass can be applied first, in both cases resulting in valid focused derivations. As an example, consider the two distinct derivations of $- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C$ under assumptions $f : X \mid \Gamma \vdash_{\mathsf{LI}} P$ and $g : - \mid \Delta \vdash_{\mathsf{RI}} C$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{f}{X \mid \Gamma \vdash_{\mathsf{LI}} P}}{X \mid \Gamma \vdash_{\mathsf{RI}} P}\,\mathsf{sw} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{X \mid \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\,\otimes\mathsf{R}}{X \mid \Gamma, \Delta \vdash_{\mathsf{LI}} P \otimes C}\,\mathsf{sw}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\,\mathsf{pass}
\qquad
\dfrac{\dfrac{\dfrac{\overset{f}{X \mid \Gamma \vdash_{\mathsf{LI}} P}}{- \mid X, \Gamma \vdash_{\mathsf{F}} P}\,\mathsf{pass}}{- \mid X, \Gamma \vdash_{\mathsf{RI}} P}\,\mathsf{sw} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\,\otimes\mathsf{R}
\tag{2.5}
$$

Here and in the rest of the thesis, the rule sw above stands for a sequence of (appropriately typed) phase switching inferences by $\mathsf{LI2RI}$ and $\mathsf{F2LI}$. The corresponding sequent calculus derivations are equated by congruence relation $\overset{\circ}{=}$ because of the first equation from Figure 2.2, i.e., the permutative conversion involving $\otimes\mathsf{R}$ and pass.

(ii) Rules $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$ appear in the same phase F, though there are situations where both rules can be applied first, which can lead to two distinct focused derivations. More precisely, there are cases when $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$ can be interchangeably applied. As an example, consider the following two valid derivations of $A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D$ under the assumption of $f : - \mid \Gamma \vdash_{\mathsf{RI}} A$, $g : X \mid \Delta \vdash_{\mathsf{LI}} P$ and $h : - \mid \Lambda \vdash_{\mathsf{RI}} D$.

$$
\begin{gathered}
\dfrac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{RI}} A} \qquad \dfrac{\dfrac{\dfrac{\overset{g}{X \mid \Delta \vdash_{\mathsf{LI}} P}}{X \mid \Delta \vdash_{\mathsf{RI}} P}\,\mathsf{sw} \qquad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{X \mid \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\,\otimes\mathsf{R}}{X \mid \Delta, \Lambda \vdash_{\mathsf{LI}} P \otimes D}\,\mathsf{sw}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\,\multimap\mathsf{L}
\\[2ex]
\dfrac{\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{RI}} A} \quad \overset{g}{X \mid \Delta \vdash_{\mathsf{LI}} P}}{A \multimap X \mid \Gamma, \Delta \vdash_{\mathsf{F}} P}\,\multimap\mathsf{L}}{A \multimap X \mid \Gamma, \Delta \vdash_{\mathsf{RI}} P}\,\mathsf{sw} \qquad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\,\otimes\mathsf{R}
\end{gathered}
\tag{2.6}
$$

The corresponding sequent calculus derivations, at the same time, are $\overset{\circ}{=}$-equivalent because of the 4th equation from Figure 2.2, the permutative conversion for $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$.

To get rid of type (i) undesired non-determinism, one might try an idea similar to the one that works in the skew monoidal non-closed case [67], namely, to prioritize pass over $\otimes\mathsf{R}$ by requiring the first premise of the latter to be a sequent in phase F. But this does not do the right thing in the skew monoidal closed case.

E.g., the sequent $- \mid Y \vdash_{\mathsf{F}} (X \multimap X) \otimes Y$ becomes underivable while its counterpart is derivable in $\mathtt{SkNMILL_S}$.

$$
\dfrac{\overset{??}{- \mid \ \vdash_{\mathsf{F}} X \multimap X} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}}{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}}{- \mid Y \vdash_{\mathsf{F}} Y}\,\mathsf{pass}}{- \mid Y \vdash_{\mathsf{RI}} Y}\,\mathsf{sw}}{- \mid Y \vdash_{\mathsf{F}} (X \multimap X) \otimes Y}\,\otimes\mathsf{R}
$$

An impulsive idea for eliminating undesired non-determinism of type (ii) is to prioritize the application of $\multimap\mathsf{L}$ over $\otimes\mathsf{R}$, e.g., by forcing the application of $\multimap\mathsf{L}$ in phase F whenever the stoup formula is an implication and restricting the application of $\otimes\mathsf{R}$ to sequents where the stoup is empty or atomic. This too leads to an unsound calculus, since the sequent $X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z$, which has a derivable correspondent in $\mathtt{SkNMILL_S}$, would not be derivable by first applying the $\multimap\mathsf{L}$ rule.

$$
\dfrac{\overset{??}{- \mid \ \vdash_{\mathsf{RI}} X} \qquad \dfrac{\dfrac{\dfrac{\overset{??}{Y \mid X \vdash_{\mathsf{RI}} Y}}{Y \mid \ \vdash_{\mathsf{RI}} X \multimap Y}\,\multimap\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Z \mid \ \vdash_{\mathsf{F}} Z}\,\mathsf{ax}}{Z \mid \ \vdash_{\mathsf{LI}} Z}\,\mathsf{sw}}{- \mid Z \vdash_{\mathsf{F}} Z}\,\mathsf{pass}}{- \mid Z \vdash_{\mathsf{RI}} Z}\,\mathsf{sw}}{Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\,\otimes\mathsf{R}}{Y \mid Z \vdash_{\mathsf{LI}} (X \multimap Y) \otimes Z}\,\mathsf{sw}}{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\,\multimap\mathsf{L}
$$

Dually, prioritizing the application of $\otimes\mathsf{R}$ over $\multimap\mathsf{L}$ leads to similar issues, e.g., the sequent $X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{F}} Y \otimes Z$ would not be derivable by first applying the $\otimes\mathsf{R}$ rule while its counterpart is derivable in $\mathtt{SkNMILL_S}$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}}{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}}{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}}{- \mid X \vdash_{\mathsf{RI}} X}\,\mathsf{sw} \qquad \dfrac{\overset{??}{Y \mid Z \vdash_{\mathsf{LI}} Y}}{Y \otimes Z \mid \ \vdash_{\mathsf{LI}} Y}\,\otimes\mathsf{L}}{X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{F}} Y}\,\multimap\mathsf{L}}{X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{RI}} Y}\,\mathsf{sw} \qquad \overset{??}{- \mid \ \vdash_{\mathsf{RI}} Z}}{X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{F}} Y \otimes Z}\,\otimes\mathsf{R}
$$

2.4.2 A Focused Sequent Calculus with Tag Annotations

In order to eliminate undesired non-determinism of type (i) between pass and $\otimes\mathsf{R}$, we need to restrict applications of pass in the derivation of the first premise of an application of $\otimes\mathsf{R}$. One way to achieve this is to force that such an application of pass is allowed only if the leftmost formula of the context is *new*, in the sense that it was not already present in the context before the $\otimes\mathsf{R}$ application. For example, with this restriction in place, the application of pass in the 2nd derivation of (2.5) would be invalid, since the formula X was already present in context before the application of $\otimes\mathsf{R}$.

Analogously, undesired non-determinism of type (ii) between $\multimap$L and $\otimes$R can be eliminated by restricting applications of $\multimap$L after an application of $\otimes$R. This can be achieved by forcing the subsequent application of $\multimap$L to split the context into two parts Γ, Δ in such a way that Γ, i.e., the context of the first premise, necessarily contains some *new* formula occurrences that were not in the context before the first $\otimes$R application. Under this restriction, the application of $\multimap$L in the 2nd derivation of (2.6) would become invalid, since all formulae in Γ are already present in context before the application of $\otimes$R.

One way to distinguish between old and new formulae occurrences in the above cases is to mark with a *tag* $\bullet$ each new formula appearing in context during the building of a focused derivation. We christen a formula occurrence "new" whenever it is moved from the succedent to the context via an application of the right implication rule $\multimap$R. In order to remember when we are building a derivation of a sequent arising as the first premise of $\otimes$R, in which the distinction between old and new formula is relevant, we mark such sequents with a tag $\bullet$ as well. More generally, we write $S \mid \Gamma \vdash^x_{ph} C$ for a sequent that can be untagged or tagged, i.e., the turnstile can be of the form $\vdash_{ph}$ or $\vdash^\bullet_{ph}$, for $ph \in \{\mathsf{RI}, \mathsf{LI}, \mathsf{F}\}$. This implies that there are a total of six sequent phases, corresponding to the possible combinations of three subscript phases with the untagged/tagged state. In tagged sequents $S \mid \Gamma \vdash^\bullet_{ph} C$, the formulae in the context Γ can be untagged or tagged, i.e., they can be of the form A or $A^\bullet$; to be precise, all untagged formulae in Γ must precede all tagged formulae (i.e., the context splits into untagged and tagged parts and, instead of possibly tagged formulae, we could alternatively work with contexts with two compartments). The formulae in the context of an untagged sequent $S \mid \Gamma \vdash_{ph} C$ must all be untagged (or, alternatively, the tagged compartment must be empty). Given a context Γ, we write Γ° for the same context where all tags have been removed from the formulae in it.

Derivations in the focused sequent calculus with tag annotations are generated by the rules

$$
\begin{array}{ll}
\text{(right invertible)} & \dfrac{S \mid \Gamma, A^x \vdash^x_{\mathsf{RI}} B}{S \mid \Gamma \vdash^x_{\mathsf{RI}} A \multimap B}\multimap\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash^x_{\mathsf{LI}} P}{S \mid \Gamma \vdash^x_{\mathsf{RI}} P}\mathsf{LI2RI} \\[2ex]
\text{(left invertible)} & \dfrac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P}\mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\otimes\mathsf{L} \qquad \dfrac{T \mid \Gamma \vdash^x_{\mathsf{F}} P}{T \mid \Gamma \vdash^x_{\mathsf{LI}} P}\mathsf{F2LI} \\[2ex]
\text{(focusing)} & \dfrac{}{X \mid \;\vdash^x_{\mathsf{F}} X}\mathsf{ax} \qquad \dfrac{A \mid \Gamma^\circ \vdash_{\mathsf{LI}} P}{- \mid A^x, \Gamma \vdash^x_{\mathsf{F}} P}\mathsf{pass} \qquad \dfrac{}{- \mid \;\vdash^x_{\mathsf{F}} \mathsf{I}}\mathsf{IR}
\end{array}
\tag{2.7}
$$

$$
\dfrac{T \mid \Gamma^\circ \vdash^\bullet_{\mathsf{RI}} A \quad - \mid \Delta^\circ \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} A \otimes B}\otimes\mathsf{R} \qquad \dfrac{- \mid \Gamma^\circ \vdash_{\mathsf{RI}} A \quad B \mid \Delta^\circ \vdash_{\mathsf{LI}} P \quad x = \bullet \supset \bullet \in \Gamma}{A \multimap B \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} P}\multimap\mathsf{L}
$$

Remember that P is a positive formula and T is a negative stoup. The side condition in rule $\multimap$L reads: if $x = \bullet$, then some formula in Γ must be tagged. For the rule pass notice that, if $x = \bullet$, it is actually forced that all formulae of Γ are tagged since the preceding context formula $A^\bullet$ is tagged. For the rules $\otimes$R and $\multimap$L similarly notice that, if some formula of Γ is tagged, then all formulae of Δ must be tagged.

The rules in (2.7), when stripped of all the tags, are equivalent to the rules in the naïve calculus (2.4). When building a derivation of an untagged sequent $S \mid \Gamma \vdash_{\mathsf{RI}} A$, the only possible way to enter a tagged phase is via an application of the $\otimes$R rule, so that sequents with turnstile marked $\vdash^\bullet_{ph}$ denote the fact that we are performing proof search for the first premise of an $\otimes$R inference (and the stoup is negative). The search for a proof of a tagged sequent $T \mid \Gamma^{\circ} \vdash^{\bullet}_{\mathsf{RI}} A$ proceeds as follows:

($\vdash^{\bullet}_{\mathsf{RI}}$) We eagerly apply the right invertible rule $\multimap\mathsf{R}$ with the goal of reducing A to a positive formula P. All formulae that get moved to the right end of the context are "new", and are therefore marked with $\bullet$. When the succedent formula becomes positive, we move to the tagged LI phase via $\mathsf{LI2RI}$.

($\vdash^{\bullet}_{\mathsf{LI}}$) Since T is a negative stoup, we can only move to the tagged F phase via $\mathsf{F2LI}$.

($\vdash^{\bullet}_{\mathsf{F}}$) The possible rules to apply (depending on the stoup and succedent formula) are ax, IR, pass, $\otimes\mathsf{R}$ or $\multimap\mathsf{L}$.

- If the stoup is empty, we have the possibility of applying the pass rule and moving the leftmost formula A in the context to the stoup, but only when this formula is marked by $\bullet$. This restriction makes it possible to remove undesired non-determinism of type (i). We then strip the context of all tags and jump to the untagged LI phase.
- If we apply $\otimes\mathsf{R}$, we remove all tags from the context Γ, Δ and move the first premise to the tagged RI phase again. The tags are removed from the context in order to reset tracking of new formulae.
- The most interesting case is $\multimap\mathsf{L}$, which can only be applied if the Γ part of the context Γ, Δ contains at least one tagged formula. This side condition implements the restriction allowing the elimination of undesired non-determinism of type (ii). All tags are removed from Γ, Δ and proof search continues in the appropriate untagged phases.

Remark 2.4.1. The focused calculus in the original work [63] includes four phases of derivations with an additional "passivation" phase, while in this thesis, we present an equivalent three-phase calculus to maintain consistency with other parts of the thesis. In particular, we remove the passivation phase by moving the pass rule to the focusing phase since it is a non-invertible rule.

The employment of tag annotations eliminates the two types of undesired non-determinism. For example, only one of the two derivations in (2.5) is valid using the rules in (2.7).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{f}{X \mid \Gamma \vdash^{\bullet}_{\mathsf{LI}} P}}{X \mid \Gamma \vdash^{\bullet}_{\mathsf{RI}} P} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{X \mid \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\otimes\mathsf{R}}{X \mid \Gamma, \Delta \vdash_{\mathsf{LI}} P \otimes C}\mathsf{sw}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\mathsf{pass}$$

(same derivation as in (2.5))

$$\dfrac{\dfrac{\overset{??}{- \mid X, \Gamma \vdash^{\bullet}_{\mathsf{F}} P}}{- \mid X, \Gamma \vdash^{\bullet}_{\mathsf{RI}} P}\mathsf{sw} \quad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} C}}{- \mid X, \Gamma, \Delta \vdash_{\mathsf{F}} P \otimes C}\otimes\mathsf{R}$$

(pass not applicable since X is not tagged)

Similarly for (2.6).

$$\dfrac{\overset{f}{- \mid \Gamma \vdash_{\mathsf{RI}} A} \quad \dfrac{\dfrac{\dfrac{\overset{g}{X \mid \Delta \vdash^{\bullet}_{\mathsf{LI}} P}}{X \mid \Delta \vdash^{\bullet}_{\mathsf{RI}} P}\mathsf{sw} \quad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{X \mid \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\otimes\mathsf{R}}{X \mid \Delta, \Lambda \vdash_{\mathsf{LI}} P \otimes D}\mathsf{sw}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\multimap\mathsf{L}$$

(same derivation as in (2.6))

$$\dfrac{\dfrac{\overset{??}{A \multimap X \mid \Gamma, \Delta \vdash^{\bullet}_{\mathsf{F}} P}}{A \multimap X \mid \Gamma, \Delta \vdash^{\bullet}_{\mathsf{RI}} P}\mathsf{sw} \quad \overset{h}{- \mid \Lambda \vdash_{\mathsf{RI}} D}}{A \multimap X \mid \Gamma, \Delta, \Lambda \vdash_{\mathsf{F}} P \otimes D}\otimes\mathsf{R}$$

($\multimap\mathsf{L}$ not applicable since Γ is tag-free)

The extra restrictions on sequents and formulae that the rules in (2.7) impose in comparison to those in (2.4) do not reduce derivability, e.g., the sequents $- \mid Y \vdash_{\mathsf{F}} (X \multimap X) \otimes Y$, whose proof requires passivation of a new formula in the derivation of the first premise of a $\otimes\mathsf{R}$ application, $X \multimap Y \mid Z \vdash_{\mathsf{RI}} (X \multimap Y) \otimes Z$, whose proof requires application of $\otimes\mathsf{R}$ before $\multimap\mathsf{L}$, and $X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{F}} Y \otimes Z$, which needs $\multimap\mathsf{L}$ invoked before $\otimes\mathsf{R}$, are all derivable.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}
}{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
}{- \mid X^{\bullet} \vdash^{\bullet}_{\mathsf{F}} X}\,\mathsf{pass}
}{- \mid X^{\bullet} \vdash^{\bullet}_{\mathsf{RI}} X}\,\mathsf{sw}
}{- \vdash \ \vdash^{\bullet}_{\mathsf{RI}} X \multimap X}\,{\multimap}\mathsf{R}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}
}{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
}{- \mid Y \vdash_{\mathsf{F}} Y}\,\mathsf{pass}
}{- \mid Y \vdash_{\mathsf{RI}} Y}\,\mathsf{sw}
}{- \mid Y \vdash_{\mathsf{F}} (X \multimap X) \otimes Y}\,\otimes\mathsf{R}
$$

$$
\dfrac{
\dfrac{
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\dfrac{
\dfrac{
\dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}
}{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
}{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}
}{- \mid X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}
\quad
\dfrac{
\dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}
}{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
}{X \multimap Y \mid X^{\bullet} \vdash^{\bullet}_{\mathsf{F}} Y}\,{\multimap}\mathsf{L}
}{X \multimap Y \mid X^{\bullet} \vdash^{\bullet}_{\mathsf{RI}} Y}\,\mathsf{sw}
}{X \multimap Y \mid \ \vdash^{\bullet}_{\mathsf{RI}} X \multimap Y}\,{\multimap}\mathsf{R}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Z \mid \ \vdash_{\mathsf{F}} Z}\,\mathsf{ax}
}{Z \mid \ \vdash_{\mathsf{LI}} Z}\,\mathsf{sw}
}{- \mid Z \vdash_{\mathsf{F}} Z}\,\mathsf{pass}
}{- \mid Z \vdash_{\mathsf{RI}} Z}\,\mathsf{sw}
}{X \multimap Y \mid Z \vdash_{\mathsf{F}} (X \multimap Y) \otimes Z}\,\otimes\mathsf{R}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}
}{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
}{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}
}{- \mid X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Y \mid \ \vdash^{\bullet}_{\mathsf{F}} Y}\,\mathsf{ax}
}{Y \mid \ \vdash^{\bullet}_{\mathsf{RI}} Y}\,\mathsf{sw}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Z \mid \ \vdash_{\mathsf{F}} Z}\,\mathsf{ax}
}{Z \mid \ \vdash_{\mathsf{LI}} Z}\,\mathsf{sw}
}{- \mid Z \vdash_{\mathsf{F}} Z}\,\mathsf{pass}
}{- \mid Z \vdash_{\mathsf{RI}} Z}\,\mathsf{sw}
}{Y \mid Z \vdash_{\mathsf{F}} Y \otimes Z}\,\otimes\mathsf{R}
}{Y \mid Z \vdash_{\mathsf{LI}} Y \otimes Z}\,\mathsf{sw}
}{Y \otimes Z \mid \ \vdash_{\mathsf{LI}} Y \otimes Z}\,\otimes\mathsf{L}
}{X \multimap (Y \otimes Z) \mid X \vdash_{\mathsf{F}} Y \otimes Z}\,{\multimap}\mathsf{L}
$$

We should point out that although the focused calculus is free of the undesired non-determinism that the naïve attempt (2.4) suffered from, it remains necessarily non-deterministic. Specifically, the focused calculus (2.7) maintains two types of *essential* non-determinism that parallel those in the focused calculus for skew monoidal categories from Section 1.2.3. While the derivations are more intricate due to the tag annotations and more additional phases and rules, the underlying principle remains the same. These types of non-determinism reflect that $\mathtt{SkNMILL_S}$ contains sequents with multiple derivations that are not $\stackrel{\circ}{=}$-equivalent. This justifies viewing the tagged focused calculus as a generalization of the focused calculus for skew monoidal categories (1.2).

1. In phase F, when the stoup is empty, there is a choice of whether to apply pass or $\otimes\mathsf{R}$ and sometimes both options lead to a derivation. For example, the sequent $X \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} X \otimes (\mathsf{I} \otimes Y)$ has two distinct derivations in the focused

system and the corresponding derivations in $\mathtt{SkNMILL_S}$ are not $\circeq$-equivalent.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{F}} X$}
\RightLabel{sw}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{RI}} X$}
\AxiomC{}
\RightLabel{IR}
\UnaryInfC{$- \mid \ \vdash^{\bullet}_{\mathsf{F}} \mathsf{I}$}
\RightLabel{sw}
\UnaryInfC{$- \mid \ \vdash^{\bullet}_{\mathsf{RI}} \mathsf{I}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$Y \mid \ \vdash_{F} Y$}
\RightLabel{sw}
\UnaryInfC{$Y \mid \ \vdash_{\mathsf{LI}} Y$}
\RightLabel{pass}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{RI}} Y$}
\RightLabel{$\otimes$R}
\BinaryInfC{$- \mid Y \vdash_{\mathsf{F}} \mathsf{I} \otimes Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{LI}} \mathsf{I} \otimes Y$}
\RightLabel{IL}
\UnaryInfC{$\mathsf{I} \mid Y \vdash_{\mathsf{LI}} \mathsf{I} \otimes Y$}
\RightLabel{$\otimes$L}
\UnaryInfC{$\mathsf{I} \otimes Y \mid \ \vdash_{\mathsf{LI}} \mathsf{I} \otimes Y$}
\RightLabel{pass}
\UnaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} \mathsf{I} \otimes Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{RI}} \mathsf{I} \otimes Y$}
\RightLabel{$\otimes$R}
\BinaryInfC{$X \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} X \otimes (\mathsf{I} \otimes Y)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{F}} X$}
\RightLabel{sw}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{RI}} X$}
\AxiomC{}
\RightLabel{IR}
\UnaryInfC{$- \mid \ \vdash^{\bullet}_{\mathsf{F}} \mathsf{I}$}
\RightLabel{sw}
\UnaryInfC{$- \mid \ \vdash^{\bullet}_{\mathsf{RI}} \mathsf{I}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$Y \mid \ \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$Y \mid \ \vdash_{\mathsf{LI}} Y$}
\RightLabel{pass}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{LI}} Y$}
\RightLabel{IL}
\UnaryInfC{$\mathsf{I} \mid Y \vdash_{\mathsf{LI}}$}
\RightLabel{$\otimes$L}
\UnaryInfC{$\mathsf{I} \otimes \mid \ \vdash_{\mathsf{LI}} Y$}
\RightLabel{sw}
\UnaryInfC{$\mathsf{I} \otimes Y \mid \ \vdash_{\mathsf{LI}} Y$}
\RightLabel{pass}
\UnaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{RI}} Y$}
\RightLabel{$\otimes$R}
\BinaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} \mathsf{I} \otimes Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid \mathsf{I} \otimes Y \vdash_{\mathsf{RI}} \mathsf{I} \otimes Y$}
\RightLabel{$\otimes$R}
\BinaryInfC{$X \mid \mathsf{I} \otimes Y \vdash_{\mathsf{F}} X \otimes (\mathsf{I} \otimes Y)$}
\end{prooftree}
$$

2. In phase F, if the succedent formula is $A \otimes B$, and the rule $\otimes$R is to be applied, the context can be split anywhere. Sometimes several of these splits can lead to a derivation. For example, the sequent $X \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (X \otimes \mathsf{I}) \otimes Y$ has two distinct derivations.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{F}} X$}
\RightLabel{sw}
\UnaryInfC{$X \mid \ \vdash^{\bullet}_{\mathsf{RI}} X$}
\AxiomC{}
\RightLabel{IR}
\UnaryInfC{$- \mid \ \vdash_{\mathsf{F}} \mathsf{I}$}
\RightLabel{sw}
\UnaryInfC{$- \mid \ \vdash_{\mathsf{LI}} \mathsf{I}$}
\RightLabel{IL}
\UnaryInfC{$\mathsf{I} \mid \ \vdash_{\mathsf{LI}} \mathsf{I}$}
\RightLabel{pass}
\UnaryInfC{$- \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}$}
\RightLabel{sw}
\UnaryInfC{$- \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}$}
\RightLabel{$\otimes$R}
\BinaryInfC{$X \mid \mathsf{I} \vdash^{\bullet}_{\mathsf{F}} X \otimes \mathsf{I}$}
\RightLabel{sw}
\UnaryInfC{$X \mid \mathsf{I} \vdash^{\bullet}_{\mathsf{RI}} X \otimes \mathsf{I}$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$Y \mid \ \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$Y \mid \ \vdash_{\mathsf{LI}} Y$}
\RightLabel{pass}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{F}} Y$}
\RightLabel{sw}
\UnaryInfC{$- \mid Y \vdash_{\mathsf{RI}} Y$}
\RightLabel{$\otimes$R}
\BinaryInfC{$X \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (X \otimes \mathsf{I}) \otimes Y$}
\end{prooftree}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{X \mid \ \vdash^{\bullet}_{\mathsf{F}} X}\,\mathsf{ax}
    }{X \mid \ \vdash^{\bullet}_{\mathsf{RI}} X}\,\mathsf{sw}
    \quad
    \dfrac{
      \dfrac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}
    }{- \mid \ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
  }{
    \dfrac{X \mid \ \vdash^{\bullet}_{\mathsf{F}} X \otimes \mathsf{I}}{X \mid \ \vdash^{\bullet}_{\mathsf{RI}} X \otimes \mathsf{I}}\,\mathsf{sw}
  }\,\otimes\mathsf{R}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}
            }{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
          }{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{pass}
        }{- \mid Y \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
      }{\mathsf{I} \mid Y \vdash_{\mathsf{LI}} Y}\,\mathsf{IL}
    }{- \mid \mathsf{I}, Y \vdash_{\mathsf{F}} Y}\,\mathsf{pass}
  }{- \mid \mathsf{I}, Y \vdash_{\mathsf{RI}} Y}\,\mathsf{sw}
}{X \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (X \otimes \mathsf{I}) \otimes Y}\,\otimes\mathsf{R}
$$

The presence of $\multimap$ adds two further types of essential nondeterminism.

3. This type is similar to type 2. In the focusing phase , if the stoup formula is $A \multimap B$, and the rule $\multimap\mathsf{L}$ is to be applied, the context can be split anywhere. Again, sometimes several of these splits lead to a derivation. For example, the sequent $\mathsf{I} \multimap (X \multimap Y) \mid \mathsf{I}, X \vdash_{\mathsf{F}} Y$ has two derivations.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}
        }{- \mid \ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{sw}
      }{\mathsf{I} \mid \ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}
    }{- \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{pass}
  }{- \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}
          }{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
        }{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}
      }{- \mid X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}
      \quad
      \dfrac{
        \dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}
      }{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
    }{X \multimap Y \mid X \vdash_{\mathsf{F}} Y}\,\multimap\mathsf{L}
  }{X \multimap Y \mid X \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
}{\mathsf{I} \multimap (X \multimap Y) \mid \mathsf{I}, X \vdash_{\mathsf{F}} Y}\,\multimap\mathsf{L}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}
  }{- \mid \ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{
                \dfrac{
                  \dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\,\mathsf{ax}
                }{X \mid \ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
              }{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}
            }{- \mid X \vdash_{\mathsf{LI}} X}\,\mathsf{sw}
          }{\mathsf{I} \mid X \vdash_{\mathsf{LI}} X}\,\mathsf{IL}
        }{- \mid \mathsf{I}, X \vdash_{\mathsf{F}} X}\,\mathsf{pass}
      }{- \mid \mathsf{I}, X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}
      \quad
      \dfrac{
        \dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}
      }{Y \mid \ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
    }{X \multimap Y \mid \mathsf{I}, X \vdash_{\mathsf{F}} Y}\,\multimap\mathsf{L}
  }{X \multimap Y \mid \mathsf{I}, X \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}
}{\mathsf{I} \multimap (X \multimap Y) \mid \mathsf{I}, X \vdash_{\mathsf{F}} Y}\,\multimap\mathsf{L}
$$

4. Finally, in phase F, if the succedent formula is $A \otimes B$ and the stoup formula is $A' \multimap B'$, then both $\otimes\mathsf{R}$ and $\multimap\mathsf{L}$ can be applied first and sometimes both options lead to a derivation. For example, the sequent $\mathsf{I} \multimap \mathsf{I} \mid Z \vdash_{\mathsf{F}} (\mathsf{I} \multimap \mathsf{I}) \otimes Z$

has two derivations.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-\mid\;\vdash_{\mathsf{F}}\mathsf{I}}\,\mathsf{IR}}{-\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{sw}}{\mathsf{I}\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{IL}}{-\mid\mathsf{I}\vdash_{\mathsf{F}}\mathsf{I}}\,\mathsf{pass}}{-\mid\mathsf{I}\vdash_{\mathsf{RI}}\mathsf{I}}\,\mathsf{sw}
\quad
\dfrac{\dfrac{\dfrac{}{-\mid\;\vdash_{\mathsf{F}}\mathsf{I}}\,\mathsf{IR}}{-\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{sw}}{\mathsf{I}\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{IL}
}{\mathsf{I}\multimap\mathsf{I}\mid\mathsf{I}^{\bullet}\vdash^{\bullet}_{\mathsf{F}}\mathsf{I}}\,{\multimap}\mathsf{L}
}{\mathsf{I}\multimap\mathsf{I}\mid\mathsf{I}^{\bullet}\vdash^{\bullet}_{\mathsf{RI}}\mathsf{I}}\,\mathsf{sw}
}{\mathsf{I}\multimap\mathsf{I}\mid\;\vdash^{\bullet}_{\mathsf{RI}}\mathsf{I}\multimap\mathsf{I}}\,{\multimap}\mathsf{R}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{Z\mid\;\vdash_{\mathsf{F}}Z}\,\mathsf{ax}}{Z\mid\;\vdash_{\mathsf{LI}}Z}\,\mathsf{sw}}{-\mid Z\vdash_{\mathsf{F}}Z}\,\mathsf{pass}}{-\mid Z\vdash_{\mathsf{LI}}Z}\,\mathsf{sw}
}{\mathsf{I}\multimap\mathsf{I}\mid Z\vdash_{\mathsf{F}}(\mathsf{I}\multimap\mathsf{I})\otimes Z}\,\otimes\mathsf{R}
$$

$$
\dfrac{
\dfrac{\dfrac{}{-\mid\;\vdash_{\mathsf{F}}\mathsf{I}}\,\mathsf{IR}}{-\mid\;\vdash_{\mathsf{RI}}\mathsf{I}}\,\mathsf{sw}
\quad
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-\mid\;\vdash_{\mathsf{F}}\mathsf{I}}\,\mathsf{IR}}{-\mid\;\vdash_{\mathsf{LI}}}\,\mathsf{sw}}{\mathsf{I}\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{IL}}{\mathsf{I}\mid\;\vdash_{\mathsf{LI}}\mathsf{I}}\,\mathsf{sw}}{-\mid\mathsf{I}^{\bullet}\vdash^{\bullet}_{\mathsf{F}}\mathsf{I}}\,\mathsf{pass}}{-\mid\mathsf{I}^{\bullet}\vdash^{\bullet}_{\mathsf{RI}}\mathsf{I}}\,\mathsf{sw}}{-\mid\;\vdash^{\bullet}_{\mathsf{RI}}\mathsf{I}\multimap\mathsf{I}}\,{\multimap}\mathsf{R}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{Z\mid\;\vdash_{\mathsf{F}}Z}\,\mathsf{ax}}{Z\mid\;\vdash_{\mathsf{LI}}Z}\,\mathsf{sw}}{-\mid Z\vdash_{\mathsf{F}}Z}\,\mathsf{pass}}{-\mid Z\vdash_{\mathsf{LI}}Z}\,\mathsf{sw}
}{-\mid Z\vdash_{\mathsf{F}}(\mathsf{I}\multimap\mathsf{I})\otimes Z}\,\otimes\mathsf{R}}{-\mid Z\vdash_{\mathsf{LI}}(\mathsf{I}\multimap\mathsf{I})\otimes Z}\,\mathsf{sw}}{\mathsf{I}\mid Z\vdash_{\mathsf{LI}}(\mathsf{I}\multimap\mathsf{I})\otimes Z}\,\mathsf{IL}
}{\mathsf{I}\multimap\mathsf{I}\mid Z\vdash_{\mathsf{F}}(\mathsf{I}\multimap\mathsf{I})\otimes Z}\,{\multimap}\mathsf{L}
$$

Note that, in the second derivation, the rule pass applies to the sequent $-\mid\mathsf{I}^{\bullet}\vdash^{\bullet}_{\mathsf{F}}\mathsf{I}$ only because the context formula $\mathsf{I}^{\bullet}$ is tagged.

Theorem 2.4.2. *The focused sequent calculus is equivalent to* $\mathtt{SkNMILL_S}$*, meaning that the two statements below are true:*

- *For any derivation* $f : S \mid \Gamma \vdash C$*, there exists a derivation* $\mathsf{focus}\ f : S \mid \Gamma \vdash_{\mathsf{RI}} C$*.*
- *For any derivation in* $f : S \mid \Gamma \vdash^{x}_{ph} C$*, there exists a derivation* $\mathsf{emb}_{ph} : S \mid \Gamma \vdash C$*, for all* $ph \in \{\mathsf{RI}, \mathsf{LI}, \mathsf{F}\}$*.*

The second statement is immediate: the emb_{ph} functions can be defined by mutual recursion where all functions erase the all phase and tag annotations. The focus functions follows from the fact that the following rules are all admissible:

$$
\begin{gathered}
\dfrac{}{A\mid\;\vdash_{\mathsf{RI}}A}\,\mathsf{ax}^{\mathsf{RI}}
\qquad
\dfrac{A\mid\Gamma\vdash_{\mathsf{RI}}C}{-\mid\Gamma\vdash_{\mathsf{RI}}C}\,\mathsf{pass}^{\mathsf{RI}}
\\
\dfrac{-\mid\Gamma\vdash_{\mathsf{RI}}A \quad B\mid\Delta\vdash_{\mathsf{RI}}C}{A\multimap B\mid\Gamma,\Delta\vdash_{\mathsf{RI}}C}\,{\multimap}\mathsf{L}^{\mathsf{RI}}
\qquad
\dfrac{-\mid\Gamma\vdash_{\mathsf{RI}}C}{\mathsf{I}\mid\Gamma\vdash_{\mathsf{RI}}C}\,\mathsf{IL}^{\mathsf{RI}}
\qquad
\dfrac{A\mid B,\Gamma\vdash_{\mathsf{RI}}C}{A\otimes B\mid\Gamma\vdash_{\mathsf{RI}}C}\,\otimes\mathsf{L}^{\mathsf{RI}}
\\
\dfrac{}{-\mid\;\vdash_{\mathsf{RI}}\mathsf{I}}\,\mathsf{IR}^{\mathsf{RI}}
\qquad
\dfrac{S\mid\Gamma,\Gamma'\vdash_{\mathsf{RI}}A \quad -\mid\Delta\vdash_{\mathsf{RI}}B}{S\mid\Gamma,\Delta\vdash_{\mathsf{RI}}[\![\Gamma'\mid A]\!]_{\multimap}\otimes B}\,\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}
\end{gathered}
\tag{2.8}
$$

The interesting one is $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}$. The tensor right rule $\otimes\mathsf{R}^{\mathsf{RI}}$, with premises and conclusion in phase RI, is an instance of the latter with empty Γ'. Without this generalization including the extra context Γ', one quickly discovers that finding a proof of $\otimes\mathsf{R}^{\mathsf{RI}}$, proceeding by induction on the structure of the derivation of the first premise, is not possible when this derivation ends with an application of $\multimap\mathsf{R}$:

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A' \vdash_{\mathsf{RI}} B'}}{S \mid \Gamma \vdash_{\mathsf{RI}} A' \multimap B'}\multimap\mathsf{R} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{S \mid \Gamma, \Delta \vdash_{\mathsf{RI}} (A' \multimap B') \otimes B}\otimes\mathsf{R}^{\mathsf{RI}} \quad = \quad ??$$

The inductive hypothesis applied to f and g would produce a derivation of the wrong sequent. The use of Γ' in the generalized rule $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}$ is there to fix precisely this issue.

To prove the admissibility of $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}$, it is essential to build a focused variant of the rule $\multimap\mathsf{R}^*$ in (2.2). We detail its construction in the upcoming proposition.

Proposition 2.4.3. *The following rule, corresponding to an iterated $\multimap$-right rule, is admissible:*

$$\dfrac{S \mid \Gamma, \Gamma'^{x} \vdash^{x}_{\mathsf{RI}} C}{S \mid \Gamma \vdash^{x}_{\mathsf{RI}} [\![\Gamma' \mid C]\!]_{\multimap}}\multimap\mathsf{R}^*$$

Proof. By structural induction on Γ':

- if Γ' is empty, then take $\multimap\mathsf{R}^*\ f = f$;
- if $\Gamma' = A, \Gamma''$, then $\multimap\mathsf{R}^*\ f = \multimap\mathsf{R}\ (\multimap\mathsf{R}^*\ f)$, i.e.

$$\dfrac{\overset{f}{S \mid \Gamma, A^x, \Gamma''^{x} \vdash^{x}_{\mathsf{RI}} C}}{S \mid \Gamma \vdash^{x}_{\mathsf{RI}} [\![A, \Gamma'' \mid C]\!]_{\multimap}}\multimap\mathsf{R}^* \quad = \quad \dfrac{\dfrac{\overset{f}{S \mid \Gamma, A^x, \Gamma''^{x} \vdash^{x}_{\mathsf{RI}} C}}{S \mid \Gamma, A^x \vdash^{x}_{\mathsf{RI}} [\![\Gamma'' \mid C]\!]_{\multimap}}\multimap\mathsf{R}^*}{S \mid \Gamma \vdash^{x}_{\mathsf{RI}} A \multimap [\![\Gamma'' \mid C]\!]_{\multimap}}\multimap\mathsf{R}$$

  where by definition $A \multimap [\![\Gamma'' \mid C]\!]_{\multimap} = [\![A, \Gamma'' \mid C]\!]_{\multimap}$.

□

Proposition 2.4.4. *The following rules, corresponding to different generalizations of the $\otimes$-right rule, are admissible:*

$$\dfrac{S \mid \Gamma, \Gamma' \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{S \mid \Gamma, \Delta \vdash_{\mathsf{RI}} [\![\Gamma' \mid A]\!]_{\multimap} \otimes B}\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'} \qquad \dfrac{S \mid \Gamma, \Gamma' \vdash_{\mathsf{LI}} P \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{S \mid \Gamma, \Delta \vdash_{\mathsf{LI}} [\![\Gamma' \mid P]\!]_{\multimap} \otimes B}\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}$$

$$\dfrac{T \mid \Gamma, \Gamma' \vdash_{\mathsf{F}} P \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} [\![\Gamma' \mid P]\!]_{\multimap} \otimes B}\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}$$

Proof. The proof proceeds by mutual induction on the first premise of each rule, which we always name f. The second premises are all named g.

Proof of $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}$:

- If $f = \multimap\mathsf{R}\ f'$, then

$$\dfrac{\dfrac{\begin{array}{c} f' \\ S \mid \Gamma, \Gamma', A' \vdash_{\mathsf{RI}} B' \end{array}}{S \mid \Gamma, \Gamma' \vdash_{\mathsf{RI}} A' \multimap B'}\ \multimap\mathsf{R} \qquad \begin{array}{c} g \\ - \mid \Delta \vdash_{\mathsf{RI}} B \end{array}}{S \mid \Gamma, \Delta \vdash_{\mathsf{RI}} [\![\Gamma' \mid A' \multimap B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}$$

$$= \dfrac{\begin{array}{c} f' \\ S \mid \Gamma, \Gamma', A' \vdash_{\mathsf{RI}} B' \end{array} \qquad \begin{array}{c} g \\ - \mid \Delta \vdash_{\mathsf{RI}} B \end{array}}{S \mid \Gamma, \Delta \vdash_{\mathsf{RI}} [\![\Gamma', A' \mid B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma', A'}$$

and $[\![\Gamma' \mid A \multimap B]\!]_{\multimap} = [\![\Gamma', A \mid B]\!]_{\multimap}$.

- If $f = \mathsf{LI2RI}\ f'$ then $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'}\ (\mathsf{LI2RI}\ f', g) = \mathsf{LI2RI}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (f', g))$.
(We do not explicitly show derivation trees in cases like this one, when rules are simply permuted.)

Proof of $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}$:

- If $f = \mathsf{IL}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (\mathsf{IL}\ f', g) = \mathsf{IL}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (f', g))$.
- If $f = \otimes\mathsf{L}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (\otimes\mathsf{L}\ f', g) = \otimes\mathsf{L}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (f', g))$.
- If $f = \mathsf{F2LI}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (\mathsf{F2LI}\ f', g) = \mathsf{F2LI}\ (\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}\ (f', g))$.

Proof of $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}$:

- If $f = \mathsf{ax}$, then Γ' is empty and $\otimes\mathsf{R}^{\mathsf{F}}_{(\,)}\ (\mathsf{ax}, g) = \otimes\mathsf{R}\ (\mathsf{ax}, g)$.
- If $f = \mathsf{IR}$, then Γ' is empty and $\otimes\mathsf{R}^{\mathsf{F}}_{(\,)}\ (\mathsf{IR}, g) = \otimes\mathsf{R}\ (\mathsf{IR}, g)$.
- If $f = \mathsf{pass}\ f'$, the passivated formula (i.e. the formula that is moved to the context from the stoup) can either belong to Γ or belong to Γ'. In the first case, $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}$ is permuted with pass, i.e. $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}\ (\mathsf{pass}\ f, g) = \mathsf{pass}\ (\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}\ (f, g))$. In the second case, Γ is empty, $\Gamma' = A', \Gamma''$ and we define:

$$\dfrac{\dfrac{\begin{array}{c} f' \\ A' \mid \Gamma'' \vdash_{\mathsf{LI}} P \end{array}}{- \mid A', \Gamma'' \vdash_{\mathsf{F}} P}\ \mathsf{pass} \qquad \begin{array}{c} g \\ - \mid \Delta \vdash_{\mathsf{RI}} B \end{array}}{- \mid \Delta \vdash_{\mathsf{F}} [\![A', \Gamma'' \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{F}}_{A', \Gamma''}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} f' \\ A' \mid \Gamma'' \vdash_{\mathsf{LI}} P \end{array}}{- \mid A'^{\bullet}, \Gamma''^{\bullet} \vdash^{\bullet}_{\mathsf{F}} P}\ \mathsf{pass}}{- \mid A'^{\bullet}, \Gamma''^{\bullet} \vdash^{\bullet}_{\mathsf{RI}} P}\ \mathsf{sw}}{- \mid \ \vdash^{\bullet}_{\mathsf{RI}} [\![A', \Gamma'' \mid P]\!]_{\multimap}}\ \multimap\mathsf{R}^{*} \qquad \begin{array}{c} g \\ - \mid \Delta \vdash_{\mathsf{RI}} B \end{array}}{- \mid \Delta \vdash_{\mathsf{F}} [\![A', \Gamma'' \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}$$

- If $f = \otimes\mathsf{R}\ (f', f'')$, there are two possibilities, depending on whether the $\otimes\mathsf{R}$ rule splits the context in Γ (or between Γ and Γ') or properly in Γ'. The two cases are analogous. We only show the second one, where $\Gamma' = \Gamma'_0, \Gamma'_1$.

$$\dfrac{\dfrac{\overset{f'}{T \mid \Gamma, \Gamma'_0 \vdash^{\bullet}_{\mathsf{RI}} A'} \quad \overset{f''}{- \mid \Gamma'_1 \vdash_{\mathsf{RI}} B'}}{T \mid \Gamma, \Gamma'_0, \Gamma'_1 \vdash_{\mathsf{F}} A' \otimes B'}\ \otimes\mathsf{R} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_0, \Gamma'_1 \mid A' \otimes B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'_0,\Gamma'_1}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\overset{f'}{T \mid \Gamma, \Gamma'_0 \vdash^{\bullet}_{\mathsf{RI}} A'} \quad \overset{f''}{- \mid \Gamma'_1 \vdash_{\mathsf{RI}} B'}}{T \mid \Gamma, \Gamma'^{\bullet}_0, \Gamma'^{\bullet}_1 \vdash^{\bullet}_{\mathsf{F}} A' \otimes B'}\ \otimes\mathsf{R}}{T \mid \Gamma, \Gamma'^{\bullet}_0, \Gamma'^{\bullet}_1 \vdash^{\bullet}_{\mathsf{RI}} A' \otimes B'}\ \mathsf{sw}}{T \mid \Gamma \vdash^{\bullet}_{\mathsf{RI}} [\![\Gamma'_0, \Gamma'_1 \mid A' \otimes B']\!]_{\multimap}}\ {\multimap}\mathsf{R}^* \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_0, \Gamma'_1 \mid A' \otimes B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}$$

- If $f = {\multimap}\mathsf{L}\ (f', f'')$, there are two possibilities, depending on whether the ${\multimap}\mathsf{L}$ rule splits the context in Γ or Γ', i.e. whether at least one formula from Γ' is moved to the first premise f'. In the first case, ${\multimap}\mathsf{L}$ can be applied first, i.e. $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'}\ ({\multimap}\mathsf{L}\ (f', f''), g) = {\multimap}\mathsf{L}\ (f', \otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'}\ (f'', g))$. In the second case, $\Gamma' = \Gamma'_0, \Gamma'_1$ with non-empty Γ'_0 and

$$\dfrac{\dfrac{\overset{f'}{- \mid \Gamma, \Gamma'_0 \vdash_{\mathsf{RI}} A'} \quad \overset{f''}{B' \mid \Gamma'_1 \vdash_{\mathsf{LI}} P}}{A' \multimap B' \mid \Gamma, \Gamma'_0, \Gamma'_1 \vdash_{\mathsf{F}} P}\ {\multimap}\mathsf{L} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{A' \multimap B' \mid \Gamma, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_0, \Gamma'_1 \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'_0,\Gamma'_1}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\overset{f'}{- \mid \Gamma, \Gamma'_0 \vdash_{\mathsf{RI}} A'} \quad \overset{f''}{B' \mid \Gamma'_1 \vdash_{\mathsf{LI}} P}}{A' \multimap B' \mid \Gamma, \Gamma'^{\bullet}_0, \Gamma'^{\bullet}_1 \vdash^{\bullet}_{\mathsf{F}} P}\ {\multimap}\mathsf{L}}{A' \multimap B' \mid \Gamma, \Gamma'^{\bullet}_0, \Gamma'^{\bullet}_1 \vdash^{\bullet}_{\mathsf{RI}} P}\ \mathsf{sw}}{A' \multimap B' \mid \Gamma \vdash^{\bullet}_{\mathsf{RI}} [\![\Gamma'_0, \Gamma'_1 \mid P]\!]_{\multimap}}\ {\multimap}\mathsf{R}^* \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{A' \multimap B' \mid \Gamma, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_0, \Gamma'_1 \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}$$

  The last application of ${\multimap}\mathsf{L}$ is justified since Γ'_0 is non-empty. □

The admissibility of the rules in (2.8) allows the construction of the function $\mathsf{focus} : S \mid \Gamma \vdash A \to S \mid \Gamma \vdash_{\mathsf{RI}} A$, replacing applications of each rule in $\mathtt{SkNMILL_S}$ with inferences by the corresponding admissible focused rule in phase RI.

Theorem 2.4.5. *The functions* $\mathsf{emb}_{\mathsf{RI}}$ *and* focus *define a bijective correspondence between the set of derivations of* $S \mid \Gamma \vdash A$ *quotiented by the equivalence relation* $\doteq$ *and the set of derivations of* $S \mid \Gamma \vdash_{\mathsf{RI}} A$*:*

- *For all* $f, g : S \mid \Gamma \vdash A$*, if* $f \doteq g$ *then* $\mathsf{focus}\ f = \mathsf{focus}\ g$.
- *For all* $f : S \mid \Gamma \vdash A$*,* $\mathsf{emb}_{\mathsf{RI}}\ (\mathsf{focus}\ f) \doteq f$.
- *For all* $f : S \mid \Gamma \vdash_{\mathsf{RI}} A$*,* $\mathsf{focus}\ (\mathsf{emb}_{\mathsf{RI}}\ f) = f$.

Proof. The proof of each statement proceeds by induction on the appropriate structure, respectively. We refer the interested reader to consult Veltri's Agda formalization, `https://github.com/niccoloveltri/code-skewmonclosed`. However, the focused calculus in the formalization includes four phases of derivations, since in [63] we considered an extra "passivation" phase. In this thesis we only have 3 phases, to keep the story consistent with the other parts of the thesis. □

The focused sequent calculus solves the *coherence problem* for skew monoidal closed categories. As proved in Theorem 2.3.6, the sequent calculus for $\mathtt{SkNMILL}$ is a presentation of the free skew monoidal closed category $\mathsf{FSkMCl}(\mathsf{At})$ on the set At. The coherence problem is the problem of deciding whether two parallel maps in $\mathsf{FSkMCl}(\mathsf{At})$ are equal. This is equivalent to deciding whether two sequent calculus derivations $f, g : A \mid \ \vdash B$ are in the same $\overset{\circ}{=}$-equivalence class. But that in turn is the same as deciding whether $\mathsf{focus}\ f = \mathsf{focus}\ g$ in the focused sequent calculus, and deciding syntactic equality of focused derivations is straightforward. $\mathtt{SkNMILL}_{\mathtt{A}}$ is a direct presentation of $\mathsf{FSkMCl}(\mathsf{At})$, but thanks to the bijection (up to $\doteq$ resp. $\overset{\circ}{=}$) between $\mathtt{SkNMILL}_{\mathtt{A}}$ and $\mathtt{SkNMILL}_{\mathtt{S}}$ derivations, we can also decide if two $\mathtt{SkNMILL}_{\mathtt{A}}$ derivations $f, g : A \vdash_{\mathsf{A}} B$ are in the same $\doteq$-equivalence class.

Chapter 3

Craig Interpolation for SkNMILL

After having established the sequent and focused calculi for SkNMILL, we now turn to discuss another logical property, Craig interpolation.

In this chapter, we will prove Craig interpolation for SkNMILL via $\mathtt{SkNMILL_S}$ by adapting Maehara's method. Moreover, we will prove proof-relevant interpolation for $\mathtt{SkNMILL_S}$, which means that our interpolation procedure not only finds an interpolant formula but also produces proofs well-behaved with respect to the scut rule.

To prove Craig interpolation, we need to modify the statement of Maehara interpolation. This modification is required due to issues similar to those encountered in the product-free Lambek calculus [54] and the implicational fragment of intuitionistic logic [38], where Maehara interpolation fails. The main result of this chapter is the following:

Theorem 3.2.1. *In $\mathtt{SkNMILL_S}$, the following two interpolation properties hold:*

(sMIP) *Given a derivation $f : S \mid \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1 \rangle$ of Γ, there exist*

- *an interpolant formula D,*
- *a derivation $g : S \mid \Gamma_0 \vdash D$,*
- *a derivation $h : D \mid \Gamma_1 \vdash C$ such that*
- *$\sigma_X(D) \leq \sigma_X(S, \Gamma_0)$ and $\sigma_X(D) \leq \sigma_X(\Gamma_1, C)$ for all atomic formulae X.*

(cMMIP) *Given a derivation $f : S \mid \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exist*

- *a partition $\langle \Delta_1, \dots, \Delta_n \rangle$ of Γ_1,*
- *a list of interpolant formulae $D_1, \dots, D_n$,*
- *$g_i : - \mid \Delta_i \vdash D_i$ for all $i \in \{1, \dots, n\}$,*
- *$h : S \mid \Gamma_0, D_1, \dots, D_n, \Gamma_2 \vdash C$ such that*
- *$\sigma_X(D_1, \dots, D_n) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D_1, \dots, D_n) \leq \sigma_X(S, \Gamma_0, \Gamma_2, C)$ for all atomic formulae X.*

In $\texttt{SkNMILL}_\texttt{S}$, the first property sMIP, which stands for *stoup Maehara interpolation*, resembles Maehara interpolation for the full Lambek calculus. Whereas cMMIP, which stands for *context Maehara multi-interpolation*, is similar to Maehara multi-interpolation for the product-free Lambek calculus, a fragment of the Lambek calculus that excludes the product (multiplicative conjunction) and thus only uses types formed with residuations (implications) $\diagdown$ and $\diagup$. This restriction results in a situation where a single formula may no longer suffice as an interpolant, sometimes requiring a sequence of formulae instead.

Motivated by the categorical interpretation of $\texttt{SkNMILL}_\texttt{S}$, we take one more step and investigate the interplay between the admissible cut rules (called scut and ccut) and the derivations produced by the interpolation algorithm of Theorem 3.2.1. In Section 2.1, we introduced an equivalence relation on derivations ($\stackrel{\circ}{=}$) that captures η-conversions and permutative conversions, and is both sound and complete with respect to the categorical semantics. We show that the sMIP and cMMIP procedures of Theorem 3.2.1 are right inverses of the admissible rules scut and ccut, respectively. Formally, we prove the following theorem:

Theorem 3.4.1.

(i) Let $g : S \mid \Gamma_0 \vdash D$ and $h : D \mid \Gamma_1 \vdash C$ be the derivations obtained by applying the sMIP *procedure on a derivation $f : S \mid \Gamma \vdash C$ with the partition $\langle \Gamma_0, \Gamma_1 \rangle$. Then* $\mathsf{scut}(g, h) \stackrel{\circ}{=} f$.

(ii) Let $g : S \mid \Gamma_0, D_1, \ldots, D_n, \Gamma_2 \vdash C$ and $h_i : - \mid \Delta_i \vdash D_i$ for $i \in [1, \ldots, n]$ be derivations obtained by applying the cMMIP *procedure on a derivation $f : S \mid \Gamma \vdash C$ with the partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$. Then* $\mathsf{ccut}^*([g_i], h) \stackrel{\circ}{=} f$.

In the above statement, ccut* denotes multiple applications of the admissible ccut rule, one for each derivation h_i. Theorems 3.2.1 and 3.4.1 together show that $\texttt{SkNMILL}_\texttt{S}$ satisfies a *proof-relevant* form of Craig interpolation, in the sense formulated in the early 1990s by Čubrić [68] in the setting of intuitionistic propositional logic and recently discussed also by Saurin [58] for (extensions of) classical linear logic.

We introduce a few admissible rules that will be employed later.

First, given a list of formulae $\Delta = A_1, \ldots, A_n$ and a list of derivations $f_i : - \mid \Gamma_i \vdash A_i$ for $i \in [1, \ldots, n]$, we define an iterated version of $\multimap$L, consisting on n applications of $\multimap$L, one for each derivation f_i:

$$\dfrac{\overset{[f_i]}{[- \mid \Gamma_i \vdash A_i]_i} \quad \overset{g}{B \mid \Lambda \vdash C}}{[\![\Delta \mid B]\!]_{\multimap} \mid \Gamma_1, \ldots, \Gamma_n, \Lambda \vdash C} \multimap\mathsf{L}^*$$

$$\begin{array}{c} \dfrac{\overset{f_n}{- \mid \Gamma_n \vdash A_n} \quad \overset{g}{B \mid \Lambda \vdash C}}{A_n \multimap B \mid \Gamma_n, \Lambda \vdash C} \multimap\mathsf{L} \\ \vdots \\ = \quad \dfrac{\dfrac{\overset{f_1}{- \mid \Gamma_1 \vdash A_1} \quad A_2 \multimap (\ldots (A_n \multimap B) \ldots) \mid \Gamma_2, \ldots, \Gamma_n, \Lambda \vdash C}{A_1 \multimap (A_2 \multimap (\ldots (A_n \multimap B) \ldots)) \mid \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \Lambda \vdash C} \multimap\mathsf{L}}{[\![\Delta \mid B]\!]_{\multimap} \mid \Gamma_1, \ldots, \Gamma_n, \Lambda \vdash C} \end{array} \tag{3.1}$$

The rule $\multimap\mathsf{L}^*$ has $n+1$ premises, the first n are collected in the list of sequents $[- \mid \Gamma_i \vdash A_i]_i$. If $n = 0$, then $\multimap\mathsf{L}^*([\,], g) = g$.

Second, given a list of derivations $f_i : - \mid \Delta_i \vdash A_i$ for $i \in [1, \ldots, n]$, we define an iterated version of ccut, consisting on n applications of ccut, one for each derivation f_i:

$$\frac{\overset{[f_i]}{[- \mid \Delta_i \vdash A_i]_i} \quad \overset{g}{S \mid \Gamma_0, A_1, \ldots, A_n, \Gamma_1 \vdash C}}{S \mid \Gamma_0, \Delta_1, \Delta_2, \ldots, \Delta_n, \Gamma_1 \vdash C}\ \mathsf{ccut}^*$$

$$= \frac{\overset{f_1}{- \mid \Delta_1 \vdash A_1} \quad \dfrac{\dfrac{\overset{f_n}{- \mid \Delta_n \vdash A_n} \quad \overset{g}{S \mid \Gamma_0, A_1, A_2, \ldots, A_n, \Gamma_1 \vdash C}}{S \mid \Gamma_0, A_1, A_2, \ldots, \Delta_n, \Gamma_1 \vdash C}\ \mathsf{ccut}}{\vdots \atop S \mid \Gamma_0, A_1, \Delta_2, \ldots, \Delta_n, \Gamma_1 \vdash C}}{S \mid \Gamma_0, \Delta_1, \Delta_2, \ldots, \Delta_n, \Gamma_1 \vdash C}\ \mathsf{ccut} \tag{3.2}$$

If $n = 0$, then $\mathsf{ccut}^*([\,], g) = g$.

3.1 Failure of Maehara Interpolation

The goal of this chapter is proving that the logic $\mathtt{SkNMILL_S}$ satisfies the Craig interpolation property. But, as already mentioned in the beginning of this chapter, we cannot follow the same proof strategy used in the (associative or non-associative) Lambek calculus, where Craig interpolation follows as a corollary to Maehara interpolation. This is because the sequent calculus of $\mathtt{SkNMILL_S}$ does not enjoy Maehara interpolation. Let us see why.

First, in analogy with the presence of two admissible cut rules (scut and ccut), there are also two different forms of interpolation. This is because the subsequence of the antecedent formulae for which we wish to find an interpolant can either contain the stoup or it can be fully included in the context. More explicitly, given an antecedent $S \mid \Gamma$, we can either: (*i*) split the context $\Gamma = \Gamma_1, \Gamma_2$ in two parts and look for an interpolant of the sub-antecedent $S \mid \Gamma_1$, or (*ii*) split the context $\Gamma = \Gamma_0, \Gamma_1, \Gamma_2$ in three parts and look for an interpolant of the sub-context Γ_1. The Maehara interpolation property in $\mathtt{SkNMILL_S}$ would then consist of two statements, a *stoup Maehara interpolation* (sMIP) and a *context Maehara interpolation* (cMIP):

(sMIP) Given $f : S \mid \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1 \rangle$ of Γ, there exists

- an interpolant formula D,
- a derivation $g : S \mid \Gamma_0 \vdash D$,
- a derivation $h : D \mid \Gamma_1 \vdash C$ such that
- $\sigma_X(D) \leq \sigma_X(S, \Gamma_0)$ and $\sigma_X(D) \leq \sigma_X(\Gamma_1, C)$ for all atomic formulae X.

(cMIP) Given $f : S \mid \Gamma \vdash C$ and a partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ, there exists

- an interpolant formula D,
- a derivation $g : - \mid \Gamma_1 \vdash D$,

– a derivation $h : S \mid \Gamma_0, D, \Gamma_2 \vdash C$ such that
– $\sigma_X(D) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D) \leq \sigma_X(S, \Gamma_0, \Gamma_2, C)$.

However, this property is not provable in $\mathtt{SkNMILL_S}$. The problem lies in the validity of the second statement cMIP. An attempt to prove this would proceed by induction on the height of the derivation $f : S \mid \Gamma \vdash C$ and then inspecting what is the last rule applied in f. The rules $\otimes$R and $\multimap$L split the context, so one should be careful to consider all possible ways in which these splittings relate to the given partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ.

The critical case is $f = \otimes\mathsf{R}(f', f'')$ with the partition $\langle \Gamma_0, (\Gamma_1', \Gamma_1''), \Gamma_2 \rangle$ and the two derivations $f' : S \mid \Gamma_0, \Gamma_1' \vdash A$ and $f'' : - \mid \Gamma_1'', \Gamma_2 \vdash B$. So this is the case when the $\otimes$R rule splits Γ_1 in two parts Γ_1', Γ_1''. By induction on f' and the partition $\langle \Gamma_0, \Gamma_1', [\,] \rangle$, we would be given a formula D and derivations $g' : - \mid \Gamma_1' \vdash D$, and $h' : S \mid \Gamma_0, D \vdash A$. By applying the inductive hypothesis on f'' and the partition $\langle [\,], \Gamma_1'', \Gamma_2 \rangle$, we would be given a formula E and derivations $g'' : - \mid \Gamma_1'' \vdash E$ and $h'' : - \mid E, \Gamma_2 \vdash B$. We obtain $\otimes\mathsf{R}(g', g'') : - \mid \Gamma_1', \Gamma_1'' \vdash D \otimes E$, but we are unable to construct the other desired proof of sequent $S \mid \Gamma_0, D \otimes E, \Gamma_1 \vdash A \otimes B$. We get very close via $\otimes\mathsf{R}(h', h'') : S \mid \Gamma_0, D, E, \Gamma_1 \vdash A \otimes B$, but we are unable to merge D and E into $D \otimes E$, since $\mathtt{SkNMILL_S}$ the $\otimes$L cannot be applied on formulae in context.

For a simple concrete counterexample, consider the derivation

$$\dfrac{\dfrac{}{X \mid \;\vdash X}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{Y \mid \;\vdash Y}\,\mathsf{ax}}{- \mid Y \vdash Y}\,\mathsf{pass}}{\dfrac{X \mid Y \vdash X \otimes Y}{X \mid Y, Z \vdash (X \otimes Y) \otimes Z}\;\; \dfrac{\dfrac{}{Z \mid \;\vdash Z}\,\mathsf{ax}}{- \mid Z \vdash Z}\,\mathsf{pass}\;\otimes\mathsf{R}}\,\otimes\mathsf{R}$$

and the partition $\langle [\,], (Y, Z), [\,] \rangle$. Suppose by contradiction that Maehara interpolation holds, so we would have a formula D and the two derivations $g : - \mid Y, Z \vdash D$ and $h : X \mid D \vdash (X \otimes Y) \otimes Z$. The variable condition of Maehara interpolation and the existence of the derivation g ensure that D does not contain atomic formulae other than Y and Z, and the latter must have a unique occurrence in D. Nevertheless, the existence of derivation h is absurd. Since X is atomic, h can only be of the form: (i) $f = \otimes\mathsf{R}(f_1, f_2)$ for some derivations $f_1 : X \mid D \vdash X \otimes Y$ and $f_2 : - \mid \;\vdash Z$, or (ii) $f = \otimes\mathsf{R}(f_1', f_2')$ for some derivations $f_1' : X \mid \;\vdash X \otimes Y$ and $f_2' : - \mid D \vdash Z$. Case (i) is impossible since there is no such f_2, while case (ii) is impossible since there is no such f_1'.

This situation is reminiscent of proving interpolation in the product-free Lambek calculus [54] and in the implicational fragment of intuitionistic logic [38]. In both these cases, Maehara interpolation fails because none of the multiplicative ($\otimes$) and additive ($\wedge$) conjunction is present. A concrete counterexample in the product-free Lambek calculus (adapted from Kanazawa [38]) is given by the derivable sequent $W, W\backslash Y, W, W\backslash X, X\backslash(Y\backslash Z) \vdash Z$ with the partition $\langle [\,], [W, W\backslash Y, W, W\backslash X], [X\backslash(Y\backslash Z)] \rangle$. This can be shown to not satisfy Maehara interpolation property. In the presence of $\otimes$, Maehara's method would produce the interpolant formula $X \otimes Y$. The situation of $\mathtt{SkNMILL_S}$ is somewhere inbetween: we have a multiplicative conjunction $\otimes$ but we cannot do much with it if a formula $A \otimes B$ is in context instead of the stoup position, since the rule $\otimes$L cannot be applied arbitrarily in the antecedent. The counterexample to MIP in product-free Lambek calculus can, when appropriately modified, also work as a counterexample to cMIP in $\mathtt{SkNMILL_S}$: consider the derivable sequent $X \multimap (Y \multimap Z) \mid W \multimap X, W, W \multimap Y, W \vdash Z$ with the partition $\langle [\,], [W \multimap X, W, W \multimap Y, W], [\,] \rangle$.

3.2 Craig Interpolation for SkNMILL

In this section, we show that $\mathtt{SkNMILL_S}$ enjoys Craig interpolation, even though it does not generally enjoy Maehara interpolation. This is again in analogy with the product-free Lambek calculus without empty antecedents.

We showed in the previous section that $\mathtt{SkNMILL_S}$ does not satisfy the context Maehara interpolation property (cMIP). We prove now that instead it satisfies a *context Maehara multi-interpolation property* (cMMIP). And the stoup Maehara interpolation property (sMIP) also holds.

Theorem 3.2.1. *In* $\mathtt{SkNMILL_S}$*, the following two interpolation properties hold:*

(sMIP) *Given a derivation* $f : S \mid \Gamma \vdash C$ *and a partition* $\langle \Gamma_0, \Gamma_1 \rangle$ *of* Γ*, there exist*

- *an interpolant formula* D*,*
- *a derivation* $g : S \mid \Gamma_0 \vdash D$*,*
- *a derivation* $h : D \mid \Gamma_1 \vdash C$ *such that*
- $\sigma_X(D) \leq \sigma_X(S, \Gamma_0)$ *and* $\sigma_X(D) \leq \sigma_X(\Gamma_1, C)$ *for all atomic formulae* X*.*

(cMMIP) *Given a derivation* $f : S \mid \Gamma \vdash C$ *and a partition* $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ *of* Γ*, there exist*

- *a partition* $\langle \Delta_1, \ldots, \Delta_n \rangle$ *of* Γ_1*,*
- *a list of interpolant formulae* $D_1, \ldots, D_n$*,*
- $g_i : - \mid \Delta_i \vdash D_i$ *for all* $i \in \{1, \ldots, n\}$*,*
- $h : S \mid \Gamma_0, D_1, \ldots, D_n, \Gamma_2 \vdash C$ *such that*
- $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_1)$ *and* $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(S, \Gamma_0, \Gamma_2, C)$ *for all atomic formulae* X*.*

These two statements of the theorem are proved mutually by structural induction on derivations. We separate the proofs for readability.

Proof of sMIP. We proceed by induction on the structure of f.
Case $f = \mathsf{ax}$. Suppose $f = \mathsf{ax} : A \mid\ \vdash A$, which forces $\Gamma_0 = \Gamma_1 = [\,]$. In this case, the interpolant formula is A and $g = h = \mathsf{ax} : A \mid\ \vdash A$, where the variable multiplicity condition is automatically satisfied.
Case $f = \mathsf{IR}$. Since $f : - \mid\ \vdash \mathsf{I}$, this forces again Γ_0 and Γ_1 to be empty lists. In this case, the interpolant formula is I and $g = \mathsf{IR} : - \mid\ \vdash \mathsf{I}$ and $h = \mathsf{IL}(\mathsf{IR}) : \mathsf{I} \mid\ \vdash \mathsf{I}$, where the variable multiplicity condition is vacuously satisfied.
Case $f = \mathsf{IL}\ f'$. Given a derivation $f' : - \mid \Gamma \vdash C$, by induction on f' with the same partition $\langle \Gamma_0, \Gamma_1 \rangle$ of Γ we obtain

- a formula D,
- a derivation $g' : - \mid \Gamma_0 \vdash D$,

– a derivation $h' : D \mid \Gamma_1 \vdash C$ such that

– $\sigma_X(D) \leq \sigma_X(\Gamma_0)$ and $\sigma_X(\Gamma_1, C)$ for all atomic formulae X.

In this case, the interpolant formula for f is D and the the two desired derivations are $g = \mathsf{IL}\ g'$ and $h = h'$. The variable multiplicity condition is automatically satisfied.

Cases $f = \otimes\mathsf{L}\ f'$ and $f = \multimap\mathsf{R}\ f'$. Analogous to the previous case.

Case $f = \mathsf{pass}\ f'$. Let $f' : A \mid \Gamma' \vdash C$ and $\Gamma = A, \Gamma'$. There are two subcases determined by the partition $\langle \Gamma_0, \Gamma_1 \rangle$ of Γ. Specifically, either Γ_0 is an empty list or not.

- If $\Gamma_0 = [\,]$, then the interpolant is I and the two desired derivations are IR and $\mathsf{IL}(\mathsf{pass}\ f')$. The variable multiplicity condition is satisfied because $\mathsf{var}(\mathsf{I}) = \emptyset$.
- If $\Gamma_0 = A, \Gamma_0'$, then by induction on f' with the partition $\langle \Gamma_0', \Gamma_1 \rangle$ we obtain
  - a formula D,
  - a derivation $g' : A \mid \Gamma_0' \vdash D$,
  - a derivation $h' : D \mid \Gamma_1 \vdash C$ such that
  - $\sigma_X(D) \leq \sigma_X(A, \Gamma_0')$ and $\sigma_X(D) \leq \sigma_X(\Gamma_1, C)$ for all atomic formulae X.

  In this case, the interpolant formula for f is D, and the two desired derivations are $g = \mathsf{pass}\ g'$ and $h = h'$. The variable multiplicity condition follows directly from the inductive hypothesis.

Case $f = \otimes\mathsf{R}(f', f'')$. Let $f' : S \mid \Lambda \vdash A$ and $f'' : - \mid \Omega \vdash B$, so that $\Gamma = \Lambda, \Omega$. We need to check how the latter splitting of Γ compares to the given partition $\langle \Gamma_0, \Gamma_1 \rangle$. There are two possibilities:

- Γ_0 is fully contained in Λ. This means that $\Lambda = \Gamma_0, \Gamma_1'$ and $\Gamma_1 = \Gamma_1', \Omega$. Then $f' : S \mid \Gamma_0, \Gamma_1' \vdash A$ and $f'' : - \mid \Omega \vdash B$. In this case, by induction on f' with the partition $\langle \Gamma_0, \Gamma_1' \rangle$ we obtain
  - a formula D,
  - a derivation $g' : S \mid \Gamma_0 \vdash D$,
  - a derivation $h' : D \mid \Gamma_1' \vdash A$ such that
  - $\sigma_X(D) \leq \sigma_X(S, \Gamma_0)$ and $\sigma_X(D) \leq \sigma_X(\Gamma_1', A)$ for all atomic formulae X.

  The desired interpolant formula is D and the desired derivations are $g = g'$ and $h = \otimes\mathsf{R}(h', f'') : D \mid \Gamma_1', \Omega \vdash A \otimes B$. The variable multiplicity condition is satisfied because $\sigma_X(D) \leq \sigma_X(\Gamma_1', A) \leq \sigma_X(\Gamma_1', \Omega, A \otimes B)$ for all atomic formulae X.
- Γ_0 splits between Λ and Ω. This means that $\Gamma_0 = \Lambda, \Gamma_0'$ and $\Omega = \Gamma_0', \Gamma_1$, and Γ_0' is non-empty. Then $f' : S \mid \Lambda \vdash A$ and $f'' : - \mid \Gamma_0', \Gamma_1 \vdash B$. In this case, by induction on f' with the partition $\langle \Lambda, [\,] \rangle$ and on f'' with the partition $\langle \Gamma_0', \Gamma_1 \rangle$, respectively, we obtain
  - formulae E and F,
  - derivations $g' : S \mid \Lambda \vdash E$ and $g'' : - \mid \Gamma_0' \vdash F$,

- derivations $h' : E \mid \ \vdash A$ and $h'' : F \mid \Gamma_1 \vdash B$ such that
- $\sigma_X(E) \leq \sigma_X(S, \Lambda)$ and $\sigma_X(E) \leq \sigma_X(A)$, and
- $\sigma_X(F) \leq \sigma_X(\Gamma_0')$ and $\sigma_X(F) \leq \sigma_X(\Gamma_1, B)$ for all atomic formulae X.

The desired interpolant formula is $D = E \otimes F$ and the desired derivations are

$$g = \dfrac{\dfrac{g'}{S \mid \Lambda \vdash E} \quad \dfrac{g''}{- \mid \Gamma_0' \vdash F}}{S \mid \Lambda, \Gamma_0' \vdash E \otimes F} \otimes\mathsf{R} \qquad h = \dfrac{\dfrac{\dfrac{h'}{E \mid \ \vdash A} \quad \dfrac{\dfrac{h''}{F \mid \Gamma_1 \vdash B}}{- \mid F, \Gamma_1 \vdash B}\ \mathsf{pass}}{E \mid F, \Gamma_1 \vdash A \otimes B} \otimes\mathsf{R}}{E \otimes F \mid \Gamma_1 \vdash A \otimes B} \otimes\mathsf{L}$$

The variable multiplicity condition is satisfied because $\sigma_X(E \otimes F) = \sigma_X(E) + \sigma_X(F) \leq \sigma_X(S, \Lambda) + \sigma_X(\Gamma_0') = \sigma_X(S, \Lambda, \Gamma_0')$ and $\sigma_X(E \otimes F) = \sigma_X(E) + \sigma_X(F) \leq \sigma_X(A) + \sigma_X(\Gamma_1, B) = \sigma_X(A, \Gamma_1, B) = \sigma_X(\Gamma_1, A \otimes B)$.

Case $f = \multimap\mathsf{L}(f', f'')$. Let $f' : - \mid \Lambda \vdash A$ and $f'' : B \mid \Omega \vdash C$, so that $\Gamma = \Lambda, \Omega$. Again we check how the latter splitting of Γ compares to the given partition $\langle \Gamma_0, \Gamma_1 \rangle$. There are two possibilities:

- Γ_1 is fully contained in Ω. This means that $\Gamma_0 = \Lambda, \Gamma_0'$ and $\Omega = \Gamma_0', \Gamma_1$. Then $f' : - \mid \Lambda \vdash A$ and $f'' : B \mid \Gamma_0', \Gamma_1 \vdash C$. In this case, by induction on f'' with the partition $\langle \Gamma_0', \Gamma_1 \rangle$ we obtain

  - a formula D,
  - a derivation $g'' : B \mid \Gamma_0' \vdash D$,
  - a derivation $h'' : D \mid \Gamma_1 \vdash C$ such that
  - $\sigma_X(D) \leq \sigma_X(B, \Gamma_0')$ and $\sigma_X(D) \leq \sigma_X(\Gamma_1, C)$ for all atomic formulae X.

  The desired interpolant formula is D and the desired derivations are $g = \multimap\mathsf{L}(f', g'') : A \multimap B \mid \Lambda, \Gamma_0' \vdash D$ and $h = h''$. The variable multiplicity condition is satisfied because $\sigma_X(D) \leq \sigma_X(B, \Gamma_0') \leq \sigma_X(A \multimap B, \Lambda, \Gamma_0')$.

- Γ_1 splits between Λ and Ω. This means that $\Lambda = \Gamma_0, \Gamma_1'$ and $\Gamma_1 = \Gamma_1', \Omega$, and Γ_1' is non-empty. Then $f' : - \mid \Gamma_0, \Gamma_1' \vdash A$ and $f'' : B \mid \Omega \vdash C$. Our goal is to find a formula D and derivations $g : A \multimap B \mid \Gamma_0 \vdash D$ and $h : D \mid \Gamma_1', \Omega \vdash C$. By induction on f'' with the partition $\langle [\,], \Omega \rangle$ we obtain

  - a formula E,
  - a derivation $g'' : B \mid \ \vdash E$,
  - a derivation $h'' : E \mid \Omega \vdash C$ such that
  - $\sigma_X(E) \leq \sigma_X(B)$ and $\sigma_X(E) \leq \sigma_X(\Omega, C)$ for all atomic formulae X.

  We also apply the cMMIP procedure (which, remember, is proved by mutual induction with sMIP) on the derivation f' with the partition $\langle \Gamma_0, \Gamma_1', [\,] \rangle$ and obtain

  - a partition $\langle \Delta_1, \ldots, \Delta_n \rangle$ of Γ_1',
  - a list of formulae $D_1, \ldots, D_n$,

- a list of derivations $g'_i : - \mid \Delta_i \vdash D_i$, for $i \in [1, \ldots, n]$,
- a derivation $h' : - \mid \Gamma_0, D_1, \ldots, D_n \vdash A$ such that
- $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma'_1)$ and $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_0, A)$ for all atomic formulae X.

The desired interpolant formula is $D = D_1 \multimap (D_2 \multimap (\ldots(D_n \multimap E)\ldots))$. The desired derivations g and h are constructed as follows:

$$g \quad = \quad \dfrac{\dfrac{\overset{h'}{- \mid \Gamma_0, D_1, \ldots, D_n \vdash A} \quad \overset{g''}{B \mid \ \vdash E}}{A \multimap B \mid \Gamma_0, D_1, \ldots, D_n \vdash E} \multimap\mathsf{L}}{A \multimap B \mid \Gamma_0 \vdash D_1 \multimap (\ldots(D_n \multimap E)\ldots)} \multimap\mathsf{R}^*$$

$$h \quad = \quad \dfrac{\overset{[g'_i]}{[- \mid \Delta_i \vdash D_i]_i} \quad \overset{h''}{E \mid \Omega \vdash C}}{D_1 \multimap (\ldots(D_n \multimap E)\ldots) \mid \Delta_1, \ldots, \Delta_n, \Omega \vdash C} \multimap\mathsf{L}^*$$

Notice that $\Gamma'_1 = \Delta_1, \ldots, \Delta_n$, so the variable multiplicity condition is easy to check.

□

Proof of cMMIP. We proceed by induction on the structure of f.
Case $f = \mathsf{ax}$. Suppose $f = \mathsf{ax} : A \mid \ \vdash A$, which means that $\Gamma_0 = \Gamma_1 = \Gamma_2 = [\,]$. In this case, the desired partition of Γ_1 is the empty one, i.e. $n = 0$. The desired lists of formulae D_i and of derivations h_i are also empty. The desired derivation g is ax.
Case $f = \mathsf{IR}$. Similar to the previous one.
Case $f = \mathsf{IL}\ f'$. Given a derivation $f' : - \mid \Gamma \vdash C$, by induction on f' with the same partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ we obtain

- a partition $\langle \Delta_0, \ldots, \Delta_n \rangle$ of Γ_1,
- a list of interpolant formulae $D_1, \ldots, D_n$,
- derivations $g'_i : - \mid \Delta_i \vdash D_i$, for $i \in [1, \ldots, n]$,
- a derivation $h' : - \mid \Gamma_0, D_1, \ldots, D_n, \Gamma_2 \vdash C$ such that
- $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_0, \Gamma_2, C)$ for all X.

The desired partition of Γ_1 is $\langle \Delta_0, \ldots, \Delta_n \rangle$, the desired list of interpolant formulae is $D_1, \ldots, D_n$. The desired derivations are $g_i = g'_i$ for $i \in [1, \ldots, n]$ and $h = \mathsf{IL}\ h'$. The variable multiplicity condition is automatically satisfied.
Cases $f = \otimes\mathsf{L}\ f'$ and $f = \multimap\mathsf{R}\ f'$. Analogous to the previous case.
Case $f = \mathsf{pass}\ f'$. Let $f' : A \mid \Gamma' \vdash C$ and $\Gamma = A, \Gamma'$. There are subcases determined by the partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$ of Γ. The most interesting case is the one where $\Gamma_0 = [\,]$ and $\Gamma_1 = A, \Gamma'_1$, so that $\Gamma' = \Gamma'_1, \Gamma_2$. The other possible cases are handled similarly to the IL case discussed above. We apply the sMIP procedure (which, remember, is proved by mutual induction with cMMIP) on the derivation f' and the partition $\langle \Gamma'_1, \Gamma_2 \rangle$, which gives us

- a formula D,

- a derivation $g' : A \mid \Gamma'_1 \vdash D$,
- a derivation $h' : D \mid \Gamma_2 \vdash C$ such that
- $\sigma_X(D) \leq \sigma_X(A, \Gamma'_1)$ and $\sigma_X(D) \leq \sigma_X(\Gamma_2, C)$ for any X.

The desired partition of A, Γ'_1 is the singleton context $[A, \Gamma'_1]$, i.e. $n = 1$. The desired list of interpolant formulae is the singleton $[D]$. The desired list of derivations g_i is the singleton list consisting only of pass $g' : - \mid A, \Gamma'_1 \vdash D$ and the desired derivation h is pass $h' : - \mid D, \Gamma_2 \vdash C$. The variable multiplicity condition follows from the inductive hypothesis.

Case $f = \otimes\mathsf{R}(f', f'')$. Let $f' : S \mid \Lambda \vdash A$ and $f'' : - \mid \Omega \vdash B$, so that $\Gamma = \Lambda, \Omega$. We need to check how the latter splitting of Γ compares to the given partition $\langle \Gamma_0, \Gamma_1, \Gamma_2 \rangle$. There are three possibilities:

- Γ_1 is fully contained in Ω. This means that $\Gamma_0 = \Lambda, \Gamma'_0$ and $\Omega = \Gamma'_0, \Gamma_1, \Gamma_2$. Then $f' : S \mid \Lambda \vdash A$ and $f'' : - \mid \Gamma'_0, \Gamma_1, \Gamma_2 \vdash B$. By inductive hypothesis on f'' with partition $\langle \Gamma'_0, \Gamma_1, \Gamma_2 \rangle$ we obtain

  - a partition $\langle \Delta_0, \ldots, \Delta_n \rangle$ of Γ_1,
  - a list of interpolant formulae $D_1, \ldots, D_n$,
  - derivations $g''_i : - \mid \Delta_i \vdash D_i$, for $i \in [1, \ldots, n]$,
  - a derivation $h'' : - \mid \Gamma'_0, D_1, \ldots, D_n, \Gamma_2 \vdash B$ such that
  - $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma_1)$ and $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma'_0, \Gamma_2, B)$ for any X.

  The desired partition of Γ_1 is $\langle \Delta_0, \ldots, \Delta_n \rangle$. The desired list of interpolant formulae is $D_1, \ldots, D_n$. The desired derivation g_i is g''_i for $i \in [1, \ldots, n]$ and the desired derivation h is $\otimes\mathsf{R}(f', h'')$. The variable multiplicity condition is satisfied because $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma'_0, \Gamma_2, B) \leq \sigma_X(S, \Lambda, \Gamma'_0, \Gamma_2, A \otimes B)$ for any X.

- Γ_1 is fully contained in Λ. This case is analogous to the one above, but now we have to use the inductive hypothesis on the derivation f' instead of f''.

- Γ_1 splits between Λ and Ω. This means that $\Gamma_1 = \Gamma'_1, \Gamma''_1$ and $\Lambda = \Gamma_0, \Gamma'_1$ and $\Omega = \Gamma''_1, \Gamma_2$, and Γ''_1 is non-empty. Then $f' : S \mid \Gamma_0, \Gamma'_1 \vdash A$ and $f'' : - \mid \Gamma''_1, \Gamma_2 \vdash B$. By inductive hypothesis on f' with the partition $\langle \Gamma_0, \Gamma'_1, [\,] \rangle$ and on f'' with the partition $\langle [\,], \Gamma''_1, \Gamma_2 \rangle$, respectively, we obtain

  - a partition $\langle \Delta_0, \ldots, \Delta_n \rangle$ of Γ'_1 and a partition $\langle \Delta_{n+1}, \ldots, \Delta_m \rangle$ of Γ''_1,
  - two lists of interpolant formulae $D_1, \ldots, D_n$ and $D_{n+1}, \ldots, D_m$,
  - derivations $g'_i : - \mid \Delta_i \vdash D_i$, for $i \in [1, \ldots, n]$, and derivations $g'_j : - \mid \Delta_j \vdash D_j$, for $j \in [n+1, \ldots, m]$,
  - derivations $h' : S \mid \Gamma_0, D_1, \ldots, D_n \vdash A$ and $h'' : - \mid D_{n+1}, \ldots, D_m, \Gamma_2 \vdash B$ such that
  - $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(\Gamma'_1)$ and $\sigma_X(D_1, \ldots, D_n) \leq \sigma_X(S, \Gamma_0, A)$, and
  - $\sigma_X(D_{n+1}, \ldots, D_m) \leq \sigma_X(\Gamma''_1)$ and $\sigma_X(D_{n+1}, \ldots, D_m) \leq \sigma_X(\Gamma_2, B)$ for any X.

The desired partition of Γ_1', Γ_1'' is $\langle \Delta_0, \ldots, \Delta_n, \Delta_{n+1}, \ldots, \Delta_m \rangle$. The desired list of interpolant formulae is $D_1, \ldots, D_n, D_{n+1}, \ldots, D_m$. The desired derivation g is

$$\dfrac{\dfrac{g'}{S \mid \Gamma_0, D_1, \ldots, D_n \vdash A} \quad \dfrac{g''}{- \mid D_{n+1}, \ldots, D_m, \Gamma_2 \vdash B}}{S \mid \Gamma_0, D_1, \ldots, D_n, D_{n+1}, \ldots, D_m, \Gamma_2 \vdash A \otimes B} \otimes\mathsf{R}$$

while the desired derivation h_i is h_i' for $i \in [1, \ldots, m]$. For the variable multiplicity condition, we have $\sigma_X(D_1, \ldots, D_m) \leq \sigma_X(S, \Gamma_0, A, \Gamma_2, B) = \sigma_X(S, \Gamma_0, \Gamma_2, A \otimes B)$ for any X.

Case $f = \multimap\mathsf{L}(f', f'')$. Analogous to the case of $\otimes\mathsf{R}$ above. □

Notice that cMMIP is invoked in the proof of sMIP, in the case $f = \multimap\mathsf{L}(f', f'')$. Conversely, sMIP is invoked in the proof of cMMIP, in the case $f = \mathsf{pass}\ f'$. The proof of Theorem 3.2.1 describes an effective procedure for building interpolant formulae and derivations. This procedure is terminating, since each recursive call happens on a derivation with height strictly smaller than the one of the derivation in input. This behavior is further confirmed in Veltri's Agda formalization, `https://github.com/niccoloveltri/code-skewmonclosed/tree/interpolation`, where the inductive proof of sMIP/cMMIP is accepted by the proof assistant as terminating.

Example 3.2.2. Let us illustrate the interpolation procedure on a simple example. We compute the stoup Maehara interpolant of the end-sequent in the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \vdash X}\mathsf{ax}}{- \mid X \vdash X}\mathsf{pass} \quad \dfrac{\dfrac{}{Y \mid\ \vdash Y}\mathsf{ax} \quad \dfrac{\dfrac{}{W \mid\ \vdash W}\mathsf{ax}}{- \mid W \vdash W}\mathsf{pass}}{Y \mid W \vdash Y \otimes W}\otimes\mathsf{R}}{\dfrac{X \multimap Y \mid X, W \vdash Y \otimes W}{- \mid X \multimap Y, X, W \vdash Y \otimes W}\mathsf{pass}}\multimap\mathsf{L} \quad \dfrac{}{Z \mid\ \vdash Z}\mathsf{ax}}{(Y \otimes W) \multimap Z \mid X \multimap Y, X, W \vdash Z}\multimap\mathsf{L} \tag{3.3}$$

with the partition $\langle [X \multimap Y], [X, W] \rangle$.
Following the procedure in the proof of Theorem 3.2.1, we are in the case when the last rule is $\multimap\mathsf{L}$ and both lists in the partition $\langle [X \multimap Y], [X, W] \rangle$ move to the context of the left premise. This means that we need to apply the cMMIP procedure to the derivation $\mathsf{pass}(\multimap\mathsf{L}(\mathsf{pass\ ax}, \otimes\mathsf{R}(\mathsf{ax}, \mathsf{pass\ ax}))) : - \mid X \multimap Y, X, W \vdash Y \otimes W$ (witnessing the left premise of $\multimap\mathsf{L}$) with the partition $\langle [X \multimap Y], [X, W], [\,] \rangle$. This produces

- a partition $\langle [X], [W] \rangle$ of $[X, W]$,
- a list of interpolant formulae $[X, W]$, and
- derivations $g_1' = \mathsf{pass\ ax} : - \mid X \vdash X$, and $g_2' = \mathsf{pass\ ax} : - \mid W \vdash W$, and $h' = \mathsf{pass}(\multimap\mathsf{L}(\mathsf{pass\ ax}, \otimes\mathsf{R}(\mathsf{ax}, \mathsf{pass\ ax}))) : - \mid X \multimap Y, X, W \vdash Y \otimes W$

satisfying the variable multiplicity condition. Next, we need to apply the sMIP procedure on the derivation $\mathsf{ax} : Z \mid\ \vdash Z$ with the partition $\langle [\,], [\,] \rangle$ which produces two derivations $\mathsf{ax} : Z \mid \ \vdash Z$ and $\mathsf{ax} : Z \mid \ \vdash Z$. Then we obtain the desired interpolant formula $X \multimap (W \multimap Z)$ and the desired derivations

$$g \quad = \quad \dfrac{\dfrac{\dfrac{\overset{h'}{- \mid X \multimap Y, X, W \vdash Y \otimes W} \quad \dfrac{}{Z \mid \ \vdash Z}\,\mathsf{ax}}{(Y \otimes W) \multimap Z \mid X \multimap Y, X, W \vdash Z}\,{\multimap}\mathsf{L}}{(Y \otimes W) \multimap Z \mid X \multimap Y, X \vdash W \multimap Z}\,{\multimap}\mathsf{R}}{(Y \otimes W) \multimap Z \mid X \multimap Y \vdash X \multimap (W \multimap Z)}\,{\multimap}\mathsf{R}$$

$$h \quad = \quad \dfrac{\overset{g_1'}{- \mid X \vdash X} \quad \dfrac{\overset{g_2'}{- \mid W \vdash W} \quad \dfrac{}{Z \mid \ \vdash Z}\,\mathsf{ax}}{W \multimap Z \mid W \vdash Z}\,{\multimap}\mathsf{L}}{X \multimap (W \multimap Z) \mid X, W \vdash Z}\,{\multimap}\mathsf{L}$$

Notice that this is crucially different from the result that Maehara's method would produce on the corresponding derivation in the associative Lambek calculus (with $\otimes$). The translation of derivation (3.3) in the associative Lambek calculus is

$$\dfrac{\dfrac{\dfrac{}{X \vdash X}\,\mathsf{ax} \quad \dfrac{\dfrac{}{Y \vdash Y}\,\mathsf{ax} \quad \dfrac{}{W \vdash W}\,\mathsf{ax}}{Y, W \vdash Y \otimes W}\,\otimes\mathsf{R}}{Y \mathbin{/} X, X, W \vdash Y \otimes W}\,{/}\mathsf{L} \quad \dfrac{}{Z \vdash Z}\,\mathsf{ax}}{Z \mathbin{/} (Y \otimes W), Y \mathbin{/} X, X, W \vdash Z}\,{/}\mathsf{L}$$

Using the Maehara interpolation procedure defined in [52], the resulting interpolant formula would be $Z \mathbin{/} (X \otimes W)$. Again, X and W can be tensored in the latter formula since the Lambek calculus admits a general left rule for $\otimes$.

We conclude this section showing how Craig interpolation follows from stoup Maehara interpolation.

Theorem 3.2.3. *For any formulae A and C, if $A \multimap C$ is provable in* SkNMILL*, then there exists a formula D such that both $A \multimap D$ and $D \multimap C$ are provable, and $\mathsf{var}(D) \subseteq \mathsf{var}(A) \cap \mathsf{var}(C)$.*

Proof. $A \multimap C$ being provable means that there is a derivation $f : - \mid \ \vdash A \multimap C$. By invertibility of the rule ${\multimap}\mathsf{R}$, we obtain a derivation $f' : - \mid A \vdash C$. Then by running the sMIP procedure on f' with the partition $\langle [A], [\,] \rangle$, we get

- a formula D,
- $g' : - \mid A \vdash D$,
- $h' : D \mid \ \vdash C$, and
- $\sigma_X(D) \leq \sigma_X(A)$ and $\sigma_X(D) \leq \sigma_X(C)$.

The formulae $A \multimap D$ and $D \multimap C$ are proved by the derivations ${\multimap}\mathsf{R}\ g' : - \mid \ \vdash A \multimap D$ and ${\multimap}\mathsf{R}(\mathsf{pass}\ h') : - \mid \ \vdash D \multimap C$, respectively. The variable condition is implied by the variable multiplicity condition. ∎

Remark 3.2.4. The interpolation property (sMIP) for $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$, the sequent calculus for skew monoidal categories, is straightforwardly proved by observing the sequents. In particular, for any derivation $f : S \mid \Gamma \vdash C$ with the partition $\langle \Gamma_0, \Gamma_1 \rangle$, we take the interpolant formula to be $[\![S \mid \Gamma_0]\!]_\otimes$ and the two desired derivations are $\mathsf{ax} : [\![S \mid \Gamma_0]\!]_\otimes \mid \;\; \vdash [\![S \mid \Gamma_0]\!]_\otimes$ and $f' : [\![S \mid \Gamma_0]\!]_\otimes \mid \Gamma_1 \vdash C$ obtained by applying $\otimes\mathsf{L}$ on f multiple times. The variable condition is satisfied since there is no linear implication, meaning that the variables of any formula in the stoup is necessarily contained in the set of variables of the succedent formula.

3.3 More Admissible Equivalences of Derivations

In this section we introduce an equation and two equivalences, that will be employed later in Section 3.4. In the construction of the scut admissibility procedure (Section 2.1), the case when the first premise is of the form $\multimap\mathsf{R}\ f$ and the second premise of the form $\multimap\mathsf{L}\ (g, h)$ (i.e. a principal cut when the cut formula is an implication) is defined as follows:

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma \vdash A \multimap B}\multimap\mathsf{R} \quad \dfrac{\overset{g}{- \mid \Delta \vdash A} \quad \overset{h}{B \mid \Lambda \vdash C}}{A \multimap B \mid \Delta, \Lambda \vdash C}\multimap\mathsf{L}}{S \mid \Gamma, \Delta, \Lambda \vdash C}\mathsf{scut}$$

$$= \dfrac{\dfrac{\overset{g}{- \mid \Delta \vdash A} \quad \overset{f}{S \mid \Gamma, A \vdash B}}{S \mid \Gamma, \Delta \vdash B}\mathsf{ccut} \quad \overset{h}{B \mid \Lambda \vdash C}}{S \mid \Gamma, \Delta, \Lambda \vdash C}\mathsf{scut}$$

This equation can be generalized to one where $\multimap\mathsf{R}$, $\multimap\mathsf{L}$ and ccut are replaced by their iterated versions $\multimap\mathsf{R}^*$, $\multimap\mathsf{L}^*$ and ccut^* introduced in Equations (2.2), (3.1) and (3.2).

Proposition 3.3.1. *Given a list of formulae* $\Lambda = A_1, \ldots, A_n$*, a derivation* $f : S \mid \Gamma_0, \Lambda \vdash B$ *and a list of derivations* $g_i : - \mid \Delta_i \vdash A_i$ *for* $i \in [1, \ldots, n]$*, the following equation is derivable:*

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma_0, \Lambda \vdash B}}{S \mid \Gamma_0 \vdash \Lambda \multimap^* B}\multimap\mathsf{R}^* \quad \dfrac{\overset{[g_i]}{[- \mid \Delta_i \vdash A_i]_i} \quad \overset{h}{B \mid \Gamma_1 \vdash C}}{\Lambda \multimap^* B \mid \Delta_1, \ldots, \Delta_n, \Gamma_1 \vdash C}\multimap\mathsf{L}^*}{S \mid \Gamma_0, \Delta_1, \ldots, \Delta_n, \Gamma_1 \vdash C}\mathsf{scut}$$

$$= \dfrac{\dfrac{\overset{[g_i]}{[- \mid \Delta_i \vdash A_i]_i} \quad \overset{f}{S \mid \Gamma_0, \Delta_1, \ldots, \Delta_n \vdash B}}{S \mid \Gamma_0, \Delta_1, \ldots, \Delta_n \vdash B}\mathsf{ccut}^* \quad \overset{h}{B \mid \Gamma_1 \vdash C}}{S \mid \Gamma_0, \Delta_1, \ldots, \Delta_n, \Gamma_1 \vdash C}\mathsf{scut}$$

Proof. Proving the validity of the equation requires various applications of the associativity equations in Proposition 2.1.2. □

In Section 2.1, the admissibility of rule scut is proved by structural recursion on the derivation of the left premise. This implies that "scut commutes with left rules in first premise", i.e. $\mathsf{scut}(\odot\mathsf{L}\ f, g) = \odot\mathsf{L}(\mathsf{scut}(f, g))$ for any one-premise left rule $\odot\mathsf{L}$ among IL, $\otimes\mathsf{L}$ and pass, and also $\mathsf{scut}(\multimap\mathsf{L}(f,f'),g) = \multimap\mathsf{L}(f,\mathsf{scut}(f',g))$. It is possible to also show that "scut commutes with right rules in second premise", but only up to equivalence $\doteq$.

Proposition 3.3.2. *The following equivalences of derivations involving* scut*,* $\otimes\mathsf{R}$*, and* $\multimap\mathsf{R}$ *are admissible in* $\mathtt{SkNMILL}_{\mathsf{S}}$*:*

$$\dfrac{\overset{f}{S \mid \Gamma \vdash A} \quad \dfrac{\overset{g}{A \mid \Delta \vdash B} \quad \overset{h}{- \mid \Lambda \vdash C}}{A \mid \Delta,\Lambda \vdash B \otimes C}\otimes\mathsf{R}}{S \mid \Gamma,\Delta,\Lambda \vdash B\otimes C}\mathsf{scut} \doteq \dfrac{\dfrac{\overset{f}{S\mid\Gamma\vdash A} \quad \overset{g}{A\mid\Delta\vdash B}}{S\mid\Gamma,\Delta\vdash B}\mathsf{scut} \quad \overset{h}{-\mid\Lambda\vdash C}}{S\mid\Gamma,\Delta,\Lambda\vdash B\otimes C}\otimes\mathsf{R}$$

$$\dfrac{\overset{f}{S \mid \Gamma \vdash B} \quad \dfrac{\overset{g}{A \mid \Delta, B \vdash C}}{A \mid \Delta \vdash B \multimap C}\multimap\mathsf{R}}{S \mid \Gamma,\Delta \vdash B\multimap C}\mathsf{scut} \doteq \dfrac{\dfrac{\overset{f}{S\mid\Gamma\vdash A} \quad \overset{g}{A\mid\Delta,B\vdash C}}{S\mid\Gamma,\Delta,B\vdash C}\mathsf{scut}}{S\mid\Gamma,\Delta\vdash B\multimap C}\multimap\mathsf{R}$$

Proof. Both equivalences are proved by structural induction on the derivation f. ☐

3.4 Proof-Relevant Interpolation

So far we have established a procedure sMIP for effectively splitting a derivation $f : S \mid \Gamma_1,\Gamma_2 \vdash C$ in two derivations $g : S \mid \Gamma_1 \vdash D$ and $h : - \mid \Gamma_2 \vdash C$, with D being "minimal" in the sense of satisfying an appropriate variable condition. A natural question arises: what happens when we compose derivations g and h using the admissible scut rule? Intuition suggests that we should get back the original derivation f, at least modulo η-conversions and permutative conversions. This in fact what happens, and this section is dedicated to proving this result.

Analogously, the cMMIP procedure splits a derivation $f : S \mid \Gamma_0,\Gamma_1,\Gamma_2 \vdash C$ in a tuple of derivations $[h_i : - \mid \Delta_i \vdash D_i]_i$ and $g : S \mid \Gamma_0, D_1,\dots,D_n,\Gamma_2 \vdash C$, with $D_1,\dots,D_n$ satisfying an appropriate variable condition. If we compose $[h_i]$ and g using the admissible ccut^* rule, we get back the original derivation modulo $\doteq$.

Similar questions have been considered by Čubrić [68] in the setting of intuitionistic propositional logic and by Saurin [58] for (extensions) of classical linear logic. They call *proof-relevant interpolation* the study of interpolation procedures in relationship to cut rules and equivalence of proofs, like our $\doteq$. In particular, Čubrić and Saurin show that interpolation procedures are in a way "right inverses" of cut rules. Here we show the same for $\mathtt{SkNMILL}_{\mathsf{S}}$: the sMIP procedure is a right inverse of scut, while the cMMIP procedure is a right inverse of ccut^*.

Theorem 3.4.1.

(i) Let $g : S \mid \Gamma_0 \vdash D$ and $h : D \mid \Gamma_1 \vdash C$ be the derivations obtained by applying the sMIP *procedure on a derivation $f : S \mid \Gamma \vdash C$ with the partition $\langle\Gamma_0,\Gamma_1\rangle$. Then* $\mathsf{scut}(g,h) \doteq f$*.*

(ii) Let $g : S \mid \Gamma_0, D_1,\dots,D_n,\Gamma_2 \vdash C$ and $h_i : - \mid \Delta_i \vdash D_i$ for $i \in [1,\dots,n]$ be derivations obtained by applying the cMMIP *procedure on a derivation $f : S \mid \Gamma \vdash C$ with the partition $\langle\Gamma_0,\Gamma_1,\Gamma_2\rangle$. Then* $\mathsf{ccut}^*([g_i],h) \doteq f$*.*

Proof. Similar to the proof of Theorem 3.2.1, statements (i) and (ii) are proved by mutual induction on the structure of derivations. We focus on the proof of statement (i), since (ii) is proved in a similar manner. We refer the interested reader to Veltri's Agda formalization, `https://github.com/niccoloveltri/code-skewmonclosed/tree/interpolation`, for all the technical details.

The proof relies on the computational behavior of the admissible rules scut and ccut defined in [67] and the proof of Theorem 2.1.1.

Case $f = \mathsf{ax}$. The goal reduces to $\mathsf{scut}(\mathsf{ax}, \mathsf{ax}) \doteq \mathsf{ax}$, which holds by definition of scut.

Case $f = \mathsf{IR}$. The goal reduces to $\mathsf{scut}(\mathsf{IR}, \mathsf{IL}\ \mathsf{IR}) \doteq \mathsf{IR}$, which holds by definition of scut.

Case $f = \mathsf{IL}\ f'$. The goal reduces to $\mathsf{scut}(\mathsf{IL}\ g', h') \doteq \mathsf{IL}\ f'$[1]. By definition of scut we have $\mathsf{scut}(\mathsf{IL}\ g', h') = \mathsf{IL}\ (\mathsf{scut}(g', h'))$. By inductive hypothesis on f', we have $\mathsf{scut}(g', h') \doteq f'$ and then by congruence of $\doteq$, we obtain $\mathsf{IL}(\mathsf{scut}(g', h')) \doteq \mathsf{IL}\ f'$, as desired.

Cases $f = \otimes\mathsf{L}\ f'$ and $f = \multimap\mathsf{R}\ f'$. Analogous to the previous case. Though the case of $\multimap\mathsf{R}$ requires an additional application of Proposition 3.3.2.

Case $f = \mathsf{pass}\ f'$. Two cases determined by whether Γ_0 is empty or not.

- In the first case, the goal reduces to $\mathsf{scut}(\mathsf{IR}, \mathsf{IL}(\mathsf{pass}\ f')) \doteq \mathsf{pass}\ f'$, which holds by definition of scut.
- In the second case, the goal reduces to $\mathsf{scut}(\mathsf{pass}\ g', h') \doteq \mathsf{pass}\ f'$. By definition of scut we have $\mathsf{scut}(\mathsf{pass}\ g', h') = \mathsf{pass}(\mathsf{scut}(g', h'))$. By inductive hypothesis on f' and congruence, the latter is $\doteq$-related to $\mathsf{pass}\ f'$.

Case $f = \otimes\mathsf{R}(f', f'')$. Two cases determined by whether Γ_0 is fully contained in the context of the left premise or not.

- In the first case, the goal reduces to $\mathsf{scut}(g', \otimes\mathsf{R}(h', f'')) \doteq \otimes\mathsf{R}(f', f'')$. By Proposition 3.3.2, we have $\mathsf{scut}(g', \otimes\mathsf{R}(h', f'')) \doteq \otimes\mathsf{R}(\mathsf{scut}(g', h'), f'')$. By inductive hypothesis on f' and congruence, the latter is $\doteq$-related to $\otimes\mathsf{R}(f', f'')$.
- In the second case, the goal reduces to showing that the derivation $\mathsf{scut}(\otimes\mathsf{R}(g, h'), \otimes\mathsf{L}\ (\otimes\mathsf{R}(h', \mathsf{pass}`h'')))$ is $\doteq$-related to $\otimes\mathsf{R}(f', f'')$. This is witnessed by the following sequence of equivalences:

$$
\begin{aligned}
&\mathsf{scut}(\otimes\mathsf{R}(g, h'), \otimes\mathsf{L}(\otimes\mathsf{R}(h', \mathsf{pass}`h''))) \\
&\quad = \mathsf{scut}(g', \otimes\mathsf{R}(h', \mathsf{scut}(g'', h''))) && \text{(by definition of scut)} \\
&\quad \doteq \otimes\mathsf{R}(\mathsf{scut}(g', h'), \mathsf{scut}(g'', h'')) && \text{(by Proposition 3.3.2)} \\
&\quad \doteq \otimes\mathsf{R}(f', f'') && \text{(by ind. hyp. on } f' \text{ and } f'' \text{ and congruence)}
\end{aligned}
$$

Case $f = \multimap\mathsf{L}(f', f'')$. Two cases determined by whether Γ_1 is fully contained in the context of the right premise or not.

- In the first case, the goal reduces to $\mathsf{scut}(\multimap\mathsf{L}(f', g''), h'') \doteq \multimap\mathsf{L}(f', f'')$. By definition of scut, we have $\mathsf{scut}(\multimap\mathsf{L}(f', g''), h'') = \multimap\mathsf{L}(f', (\mathsf{scut}(g'', h''))$. By inductive hypothesis on f'' and congruence, the latter is $\doteq$-related to $\multimap\mathsf{L}(f', f'')$.

[1] Here g' and h' are as in the proof of Theorem 3.2.1. We follow the same convention for the forthcoming cases too, where name of derivations will match the ones in the proof of Theorem 3.2.1

- In the second case, the goal reduces to showing that the derivation $\mathsf{scut}(\multimap\mathsf{R}^*(\multimap\mathsf{L}(h', g'')), \multimap\mathsf{L}^*([g'_i], h''))$ is $\stackrel{\circ}{=}$-related to $\multimap\mathsf{L}(f', f'')$. This is witnessed by the following sequence of equivalences:

$$
\begin{array}{lr}
\mathsf{scut}(\multimap\mathsf{R}^*(\multimap\mathsf{L}(h', g'')), \multimap\mathsf{L}^*([g'_i], h'')) & \\
\quad \stackrel{\circ}{=} \mathsf{scut}(\mathsf{ccut}^*([g'_i], \multimap\mathsf{L}(h', g'')), h'') & \text{(by Proposition 3.3.1)} \\
\quad = \multimap\mathsf{L}(\mathsf{ccut}^*([g'_i], h'), \mathsf{scut}(h'', g'')) & \text{(by definition of } \mathsf{scut} \text{ and } \mathsf{ccut}^*\text{)} \\
\quad \stackrel{\circ}{=} \multimap\mathsf{L}(f', f'') & \text{(by ind. hyp. on } f' \text{ and } f'' \\
& \text{and congruence)}
\end{array}
$$

  The final step employs the "inductive hypothesis" on f', which in this case means the validity of statement (*ii*) for derivation f' (remember that statements (*i*) and (*ii*) are proved simultaneously by structural induction on derivations). □

Having established the core theory of SkNMILL, including its sequent calculus, categorical semantics, focused system, and interpolation properties, we now turn to explore various extensions of this base system. The following two chapters demonstrate the modularity of our framework by showing how SkNMILL can be enriched with additional structure while preserving its essential skew character and proof-theoretic properties.

Chapter 4

A Commutative Extension of SkNMILL

We begin by investigating a commutative extension of SkNMILL that incorporates a restricted form of exchange (skew exchange), leading to a language of symmetric skew monoidal closed categories. For this extension, we develop not only the basic sequent calculus but also extend the focused calculus in Section 2.4.2 to handle skew exchange.

Veltri has recently investigated the proof theory of *symmetric* left skew monoidal categories and *symmetric* left skew closed categories [69, 70]. These are variants of Mac Lane's symmetric monoidal categories and de Schippers' symmetric closed categories [23] which are originally introduced by Bourke and Lack [14] where the natural isomorphism representing symmetry involves *three* objects rather than two. Following the ideas in Veltri's study, we discuss a commutative extension of SkNMILL, that we call SkMILL (dropping the N for "non-commutative"), and its connection to symmetric skew monoidal closed categories.

While Veltri's studies addressed symmetric skew monoidal and symmetric skew closed structures separately [69, 70], this chapter distinctively concentrates on the integrated *symmetric skew monoidal closed* case. The core of this added complexity lies in how the exchange (ex) rule affects the behavior of tagged formulae within the focused calculus. In Chapter 2, formulae newly introduced into the context by the right implication rule ($\multimap$R) maintain predictable positions when the proof search moves to the focusing phase. In SkMILL, however, the ex rule allows these new formulae to be permuted and then to appear throughout the context. This behavior demands a more complex tracking mechanism than what sufficed previously. This enhanced tracking is essential for proving the admissibility of the tensor right rule ($\otimes$R) within each phase. As we have seen in Chapter 2, this is crucial for the correctness of the focusing procedure.

4.1 Sequent Calculus

A sequent calculus for SkMILL ($\mathtt{SkMILL}_\mathtt{S}$) is obtained by adding the following *restricted exchange* rule to the collection of inference rules in $\mathtt{SkNMILL}_\mathtt{S}$:

$$\frac{S \mid \Gamma, A, B, \Delta \vdash C}{S \mid \Gamma, B, A, \Delta \vdash C}\ \mathsf{ex}_{A,B}$$

An application of rule $\mathsf{ex}_{A,B}$ results in the swapping of two adjacent formulae A and B in the context. But notice that the exchange of the formula in the stoup (when such formula exists) and a formula in context is not allowed. This implies that a derivation corresponding to the structural law s of skew symmetric monoidal closed categories (see Definition 4.2.1) exists, while a derivation corresponding to the usual symmetry of (non-skew) symmetric monoidal categories does not:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \mid\ \vdash A}\mathsf{ax} \quad \dfrac{\dfrac{}{C \mid\ \vdash C}\mathsf{ax}}{- \mid C \vdash C}\mathsf{pass}}{A \mid C \vdash A \otimes C}\otimes\mathsf{R} \quad \dfrac{\dfrac{}{B \mid\ \vdash B}\mathsf{ax}}{- \mid B \vdash B}\mathsf{pass}}{\dfrac{A \mid C, B \vdash (A \otimes C) \otimes B}{A \mid B, C \vdash (A \otimes C) \otimes B}\mathsf{ex}_{C,B}}\otimes\mathsf{R}}{A \otimes B \mid C \vdash (A \otimes C) \otimes B}\otimes\mathsf{L}}{(A \otimes B) \otimes C \mid\ \vdash (A \otimes C) \otimes B}\otimes\mathsf{L} \tag{4.1}$$

$$\dfrac{\dfrac{\overset{??}{X \mid\ \vdash Y} \quad \overset{??}{- \mid Y \vdash X}}{X \mid Y \vdash Y \otimes X}\otimes\mathsf{R}}{X \otimes Y \mid\ \vdash Y \otimes X}\otimes\mathsf{L} \qquad \dfrac{\dfrac{\overset{??}{X \mid Y \vdash Y} \quad \overset{??}{- \mid\ \vdash X}}{X \mid Y \vdash Y \otimes X}\otimes\mathsf{R}}{X \otimes Y \mid\ \vdash Y \otimes X}\otimes\mathsf{L}$$

The presence of the new ex rule requires the extension of the congruence relation $\overset{\circ}{=}$ in Figures 2.1 and 2.2 with new generating equations in Figures 4.1 and 4.2.

The first equation states that swapping the same two formulae twice is the same as doing nothing. The second equation is a form of Yang-Baxter equation, stating that the two ways in which 3 adjacent formulae A, B, C can be turned to C, B, A are the same. The remaining equations are permutative conversions. The congruence $\overset{\circ}{=}$ serves as the proof-theoretic counterpart of the equational theory of symmetric skew monoidal closed categories, introduced in Definition 4.2.1. The subsystem of equations involving only $(\mathsf{I}, \otimes)$ comes from [69].

The rule ex allows us to swap the position of two adjacent formulae in the context. But more general forms of exchange are admissible, where a formula is swapped with a list of formulae.

Proposition 4.1.1. *The following generalized exchange rules are admissible:*

$$\dfrac{S \mid \Gamma, A, \Lambda, \Delta \vdash C}{S \mid \Gamma, \Lambda, A, \Delta \vdash C}\mathsf{exs}_{A,\Lambda} \qquad \dfrac{S \mid \Gamma, \Lambda, A, \Delta \vdash C}{S \mid \Gamma, A, \Lambda, \Delta \vdash C}\mathsf{exs}_{\Lambda,A} \tag{4.2}$$

Proof. We only show the proof of $\mathsf{exs}_{A,\Lambda}$, the proof of $\mathsf{exs}_{\Lambda,A}$ is similar. The proof proceeds by induction on Λ. If Λ is empty, then $\mathsf{exs}_{A,(\,)}$ simply returns the input derivation. If $\Lambda = \Lambda', B$, we first swap A and B and then apply the inductive hypothesis:

$$\dfrac{\overset{f}{S \mid \Gamma, A, \Lambda', B, \Delta \vdash C}}{S \mid \Gamma, \Lambda', B, A, \Delta \vdash C}\mathsf{exs}_{A,(\Lambda',B)} \quad = \quad \dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, \Lambda', B, \Delta \vdash C}}{S \mid \Gamma, \Lambda', A, B, \Delta \vdash C}\mathsf{exs}_{A,\Lambda'}}{S \mid \Gamma, \Lambda', B, A, \Delta \vdash C}\mathsf{ex}_{A,B}$$

□

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta \vdash C}}{S \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}}{S \mid \Gamma, A, B, \Delta \vdash C}\,\mathsf{ex}_{B,A} \circeq \overset{f}{S \mid \Gamma, A, B, \Delta \vdash C}$$

$$\dfrac{\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, D, \Delta \vdash C}}{S \mid \Gamma, A, D, B, \Delta \vdash C}\,\mathsf{ex}_{B,D}}{S \mid \Gamma, D, A, B, \Delta \vdash C}\,\mathsf{ex}_{A,D}}{S \mid \Gamma, D, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B} \circeq \dfrac{\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, D, \Delta \vdash C}}{S \mid \Gamma, B, A, D, \Delta \vdash C}\,\mathsf{ex}_{A,B}}{S \mid \Gamma, B, D, A, \Delta \vdash C}\,\mathsf{ex}_{A,D}}{S \mid \Gamma, D, B, A, \Delta \vdash C}\,\mathsf{ex}_{B,D}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma, A, B, \Delta \vdash C}}{- \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}}{\mathsf{I} \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{IL} \circeq \dfrac{\dfrac{\overset{f}{- \mid \Gamma, A, B, \Delta \vdash C}}{\mathsf{I} \mid \Gamma, A, B, \Delta \vdash C}\,\mathsf{IL}}{\mathsf{I} \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A, B, \Delta \vdash C}}{A' \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}}{- \mid A', \Gamma, B, A, \Delta \vdash C}\,\mathsf{pass} \circeq \dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A, B, \Delta \vdash C}}{- \mid A', \Gamma, A, B, \Delta \vdash C}\,\mathsf{pass}}{- \mid A', \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma, A, B, \Delta \vdash C}}{A' \mid B', \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}}{A' \otimes B' \mid \Gamma, B, A, \Delta \vdash C}\,\otimes\mathsf{L} \circeq \dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma, A, B, \Delta \vdash C}}{A' \otimes B' \mid \Gamma, A, B, \Delta \vdash C}\,\otimes\mathsf{L}}{A' \otimes B' \mid \Gamma, B, A, \Delta \vdash C}\,\mathsf{ex}_{A,B}$$

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma_0, A, B, \Gamma_1 \vdash A'}}{S \mid \Gamma_0, B, A, \Gamma_1 \vdash A'}\,\mathsf{ex}_{A,B} \quad \overset{g}{- \mid \Delta \vdash B'}}{S \mid \Gamma_0, B, A, \Gamma_1, \Delta \vdash A' \otimes B'}\,\otimes\mathsf{R} \circeq \dfrac{\dfrac{\overset{f}{S \mid \Gamma_0, A, B, \Gamma_1 \vdash A'} \quad \overset{g}{- \mid \Delta \vdash B'}}{S \mid \Gamma_0, A, B, \Gamma_1, \Delta \vdash A' \otimes B'}\,\otimes\mathsf{R}}{S \mid \Gamma_0, B, A, \Gamma_1, \Delta \vdash A' \otimes B'}\,\mathsf{ex}A, B$$

$$\dfrac{\overset{f}{S \mid \Gamma \vdash A'} \quad \dfrac{\overset{g}{- \mid \Delta_0, A, B, \Delta_1 \vdash B'}}{- \mid \Delta_0, B, A, \Delta_1 \vdash B'}\,\mathsf{ex}_{A,B}}{S \mid \Gamma, \Delta_0, B, A, \Delta_1 \vdash A' \otimes B'}\,\otimes\mathsf{R} \circeq \dfrac{\dfrac{\overset{f}{S \mid \Gamma \vdash A'} \quad \overset{g}{- \mid \Delta_0, A, B, \Delta_1 \vdash B'}}{S \mid \Gamma, \Delta_0, A, B, \Delta_1 \vdash A' \otimes B'}\,\otimes\mathsf{R}}{S \mid \Gamma, \Delta_0, B, A, \Delta_1 \vdash A' \otimes B'}\,\mathsf{ex}A, B$$

Figure 4.1: Additional equations of derivations in SkMILL$_{\mathsf{S}}$

$$\dfrac{\dfrac{\overset{f}{-\mid \Gamma_0, A, B, \Gamma_1 \vdash A'}}{-\mid \Gamma_0, B, A, \Gamma_1 \vdash A'}\,\mathsf{ex}_{A,B} \qquad \overset{g}{B' \mid \Delta \vdash C}}{A' \multimap B' \mid \Gamma_0, B, A, \Gamma_1, \Delta \vdash C}\,{\multimap}\mathsf{L} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{-\mid \Gamma_0, A, B, \Gamma_1 \vdash A'} \quad \overset{g}{B' \mid \Delta \vdash C}}{A' \multimap B' \mid \Gamma_0, A, B, \Gamma_1, \Delta \vdash C}\,{\multimap}\mathsf{L}}{A' \multimap B' \mid \Gamma_0, B, A, \Gamma_1, \Delta \vdash C}\,\mathsf{ex}A, B$$

$$\dfrac{\overset{f}{-\mid \Gamma \vdash A'} \qquad \dfrac{\overset{g}{B' \mid \Delta_0, A, B, \Delta_1 \vdash C}}{B' \mid \Delta_0, B, A, \Delta_1 \vdash C}\,\mathsf{ex}_{A,B}}{A' \multimap B' \mid \Gamma, \Delta_0, B, A, \Delta_1 \vdash C}\,{\multimap}\mathsf{L} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{-\mid \Gamma \vdash A'} \quad \overset{g}{B' \mid \Delta_0, A, B, \Delta_1 \vdash C}}{A' \multimap B' \mid \Gamma, \Delta_0, A, B, \Delta_1 \vdash C}\,{\multimap}\mathsf{L}}{A' \multimap B' \mid \Gamma, \Delta_0, B, A, \Delta_1 \vdash C}\,\mathsf{ex}A, B$$

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta, A' \vdash B'}}{S \mid \Gamma, B, A, \Delta, A' \vdash B'}\,\mathsf{ex}_{A,B}}{S \mid \Gamma, B, A, \Delta \vdash A' \multimap B'}\,{\multimap}\mathsf{R} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta, A' \vdash B'}}{S \mid \Gamma, A, B, \Delta \vdash A' \multimap B'}\,{\multimap}\mathsf{R}}{S \mid \Gamma, B, A, \Delta \vdash A' \multimap B'}\,\mathsf{ex}_{A,B}$$

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta, A', B', \Lambda \vdash C}}{S \mid \Gamma, A, B, \Delta, B', A', \Lambda \vdash C}\,\mathsf{ex}_{A',B'}}{S \mid \Gamma, B, A, \Delta, B', A', \Lambda \vdash C}\,\mathsf{ex}_{A,B} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta, A', B', \Lambda \vdash C}}{S \mid \Gamma, A, B, \Delta, B', A', \Lambda \vdash C}\,\mathsf{ex}_{A,B}}{S \mid \Gamma, B, A, \Delta, B', A', \Lambda \vdash C}\,\mathsf{ex}_{A',B'}$$

Figure 4.2: Additional equations of derivations in `SkMILLs`*, continued*

The two forms of cut rules are admissible also in `SkMILL`. The admissibility of ccut is proved with the help of the general exchange rules in (4.2). For instance,

$$\dfrac{\overset{f}{-\mid \Gamma \vdash A} \qquad \dfrac{\overset{g}{S \mid \Delta_0, A, B, \Delta_1 \vdash C}}{S \mid \Delta_0, B, A, \Delta_1 \vdash C}\,\mathsf{ex}_{A,B}}{S \mid \Delta_0, B, \Gamma, \Delta_1 \vdash C}\,\mathsf{ccut} = \dfrac{\dfrac{\overset{f}{-\mid \Gamma \vdash A} \quad \overset{g}{S \mid \Delta_0, A, B, \Delta_1 \vdash C}}{S \mid \Delta_0, \Gamma, B, \Delta_1 \vdash C}\,\mathsf{ccut}}{S \mid \Delta_0, B, \Gamma, \Delta_1 \vdash C}\,\mathsf{exs}_{\Gamma,B}$$

$$\dfrac{\overset{f}{-\mid \Gamma \vdash B} \qquad \dfrac{\overset{g}{S \mid \Delta_0, A, B, \Delta_1 \vdash C}}{S \mid \Delta_0, B, A, \Delta_1 \vdash C}\,\mathsf{ex}_{A,B}}{S \mid \Delta_0, \Gamma, A, \Delta_1 \vdash C}\,\mathsf{ccut} = \dfrac{\dfrac{\overset{f}{-\mid \Gamma \vdash B} \quad \overset{g}{S \mid \Delta_0, A, \Gamma, \Delta_1 \vdash C}}{S \mid \Delta_0, A, \Gamma, \Delta_1 \vdash C}\,\mathsf{ccut}}{S \mid \Delta_0, \Gamma, A, \Delta_1 \vdash C}\,\mathsf{exs}_{A,\Gamma}$$

`SkMILL` can be seen as a logic of resources, akin to its non-commutative variant `SkNMILL`, for which such an interpretation is provided in the end of Section 2. The antecedent of a sequent contains the resources at hand, while the stoup position, when it is non-empty, contains the resource that is immediately usable. Now the resources in the context are not ordered anymore, but they are still ordered with respect to the resource in the stoup. This implies that, when the resource in the stoup has been used, it is possible to choose which resource in the context can be spent next. It is also possible to rearrange the position of resources in the context before splitting it when applying rules ${\multimap}\mathsf{L}$ and $\otimes\mathsf{R}$.

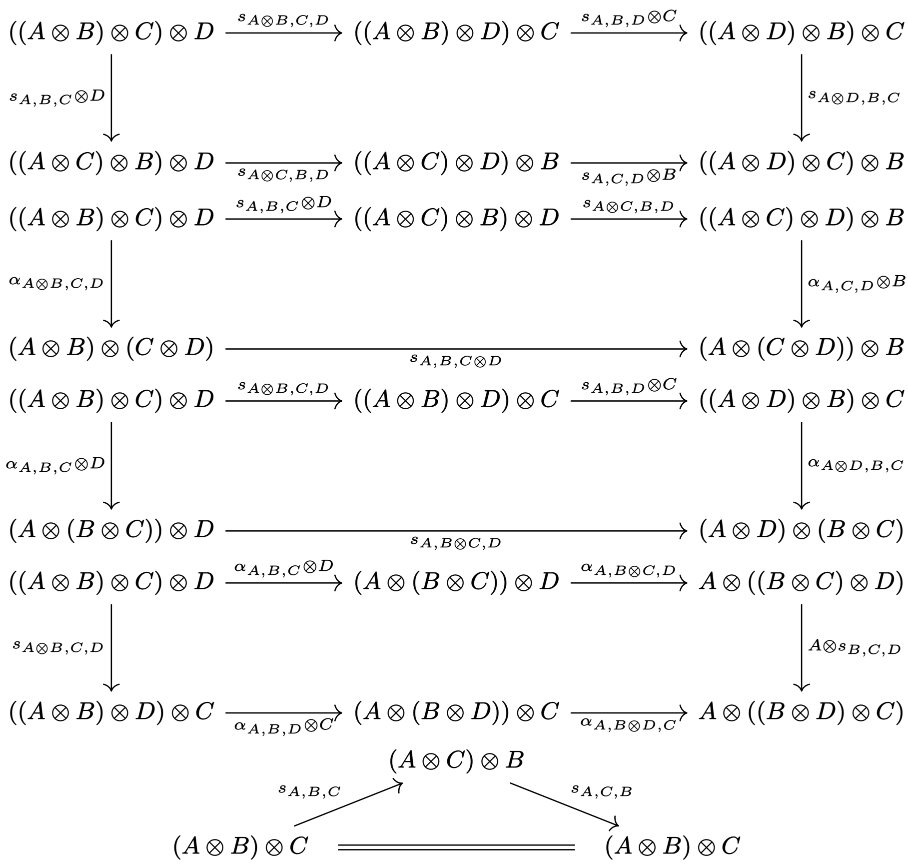

Figure 4.3: Equations related to the structural law s

4.2 Categorical Semantics

The categorical semantics of $\texttt{SkMILL}_\texttt{S}$ is given in terms of a symmetric extension of skew monoidal closed categories.

Definition 4.2.1. A *symmetric skew monoidal closed category* [14] is a skew monoidal closed category (in the sense of Definition 2.3.1) with a natural isomorphism s typed $s_{A,B,C} : (A \otimes B) \otimes C \to (A \otimes C) \otimes B$ satisfying the equations in Figure 4.3.

A *(strict) symmetric skew monoidal closed functor* is a (strict) skew monoidal closed functor $F : \mathbb{C} \to \mathbb{D}$ between symmetric skew monoidal closed categories that preserves the structural law s on the nose.

The natural isomorphism s acts on three objects A, B, C: it fixes A and swaps B and C. Therefore, there is a map between objects $(\dots((A \otimes B_1) \otimes B_2 \dots)) \otimes B_n$ and $(\dots((A \otimes B_{i_1}) \otimes B_{i_2} \dots))B_{i_n}$ for each permutation of indices i. Notice the difference with the usual structural law σ of symmetry of Mac Lane, which is typed $\sigma_{B,C} : B \otimes C \to C \otimes B$. In a symmetric skew monoidal closed category there is generally no map with such type (we will have a further discussion in Section

6.4). Clearly every symmetric monoidal closed category is also skew symmetric:

$$\begin{array}{rl} s'_{A,B,C} = & (A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C) \\ & \xrightarrow{A \otimes \sigma_{B,C}} A \otimes (C \otimes B) \xrightarrow{\alpha^{-1}_{A,C,B}} (A \otimes C) \otimes B \end{array} \tag{4.3}$$

Example 4.2.2 (Example 2.3.3 ctd)**.** Let $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ be a symmetric monoidal closed category equipped with a symmetric lax monoidal comonad (D, ε, δ). Let σ be its (non-skew) symmetry typed $\sigma_{B,C} : B \otimes C \to C \otimes B$. Then the skew monoidal closed structure $(\mathsf{I}, \otimes^D, {}^D\!\multimap)$ of Example 2.3.3 is also symmetric with $s^D_{A,B,C} : (A \otimes D\,B) \otimes D\,C \to (A \otimes D\,C) \otimes D\,B$ defined as $s'_{A,D\,B,D\,C}$ in (4.3).

Remark 4.2.3. Similar to left skew monoidal closed categories, symmetric left skew monoidal closed categories admit an equivalent characterization, i.e. the natural isomorphisms s are bijective with the natural isomorphisms $s' : B \multimap (A \multimap C) \to A \multimap (B \multimap C)$ [14]. In other words, s' correctly characterizes symmetry in a symmetric left skew *non-monoidal* closed category.

Similarly to the non-commutative case investigated in Section 2.3, the formulae, derivations and the equivalence relation $\doteq$ of the sequent calculus for `SkMILL` determine a symmetric skew monoidal closed category $\mathsf{FSymSkMCl(At)}$. This is defined analogously to the skew monoidal closed category $\mathsf{FSkMCl(At)}$ of Definition 2.3.5 with the addition of the structural law s given by the left derivation in (4.1).

Theorem 4.2.4. *Let $\mathbb{D}$ be a symmetric skew monoidal closed category. Each function $F_{\mathsf{At}} : \mathsf{At} \to |\mathbb{D}|$ extends to a unique symmetric skew monoidal closed functor $F : \mathsf{FSymSkMCl(At)} \to \mathbb{D}$.*

Proof.
Existence. Let $(\mathbb{D}, \mathsf{I}', \otimes', \multimap')$ be a skew monoidal closed category. The construction of F is analogous to the one given in the proof of Theorem 2.3.6. We only show how to derive the exchange rule $\mathsf{ex}_{A,B}$ in $\mathbb{D}$:

$$\begin{array}{c} ((((\ldots((S \otimes' A_1)\ldots) \otimes' B) \otimes' A) \otimes' A_j)\ldots) \otimes' A_n \xrightarrow{((s_{((S \otimes' A_1)\ldots),B,A} \otimes' A_j)\ldots) \otimes' A_n} \\ ((((\ldots((S \otimes' A_1)\ldots) \otimes' A) \otimes' B) \otimes' A_j)\ldots) \otimes' A_n \xrightarrow{f} C \end{array}$$

(Here S is an object of $\mathbb{D}$, not an optional formula.)
Uniqueness. Consider another symmetric skew monoidal closed functor $F' : \mathsf{FSymSkMCl(At)} \to \mathbb{D}$ such that $F'X = F_{\mathsf{At}}X$ for any atom X. We can verify that F' and F agree on every object and morphism in $\mathsf{FSymSkMCl(At)}$ by induction on formulae and derivations respectively. □

As discussed in the end of Section 2.3, this can alternatively be shown by constructing a Hilbert-style deductive system directly presenting the free symmetric skew monoidal closed category on At and appropriately proving that it is equivalent to the sequent calculus. The Hilbert-style system looks like the one for the non-symmetric case in (2.2.1), with the addition of a rule for symmetry:

$$\frac{}{(A \otimes B) \otimes C \Rightarrow (A \otimes C) \otimes B}\ s$$

The congruence relation $\doteq$ also needs to be extended with all the equations involving symmetry in Definition 4.2.1. There exists a bijection between the set of derivations of the SkMILL sequent $A \mid\ \vdash B$ modulo $\stackrel{\circ}{=}$ and the set of derivations of the sequent $A \Rightarrow B$ modulo $\doteq$.

4.3 A Focused Sequent Calculus with Tag Annotations

SkMILL also has a tagged focused subsystem of derivations corresponding to normal forms for the equivalence relation $\stackrel{\circ}{=}$ in Figures 2.1, 2.2, 4.1 and 4.2.

$$
\begin{array}{lll}
\text{(context)} & \dfrac{S \mid \Gamma \,\vdots\, \Delta, A^x, \Lambda \vdash^x_{\mathsf{C}} C}{S \mid \Gamma, A^x \,\vdots\, \Delta, \Lambda \vdash^x_{\mathsf{C}} C}\ \mathsf{ex} & \dfrac{S \mid \Gamma \vdash^x_{\mathsf{RI}} C}{S \mid\ \,\vdots\, \Gamma \vdash^x_{\mathsf{C}} C}\ \mathsf{RI2C} \\[2ex]
\text{(right invertible)} & \dfrac{S \mid A^x \,\vdots\, \Gamma \vdash^x_{\mathsf{C}} B}{S \mid \Gamma \vdash^x_{\mathsf{RI}} A \multimap B}\ {\multimap}\mathsf{R} & \dfrac{S \mid \Gamma \vdash^x_{\mathsf{LI}} P}{S \mid \Gamma \vdash^x_{\mathsf{RI}} P}\ \mathsf{LI2RI} \\[2ex]
\text{(left invertible)} & \dfrac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{IL} \quad \dfrac{A \mid B \,\vdots\, \Gamma \vdash_{\mathsf{C}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\ {\otimes}\mathsf{L} & \dfrac{T \mid \Gamma \vdash^x_{\mathsf{F}} P}{T \mid \Gamma \vdash^x_{\mathsf{LI}} P}\ \mathsf{F2LI}
\end{array}
\tag{4.4}
$$

$$
\text{(focusing)} \quad \dfrac{}{X \mid\ \vdash^x_{\mathsf{F}} X}\ \mathsf{ax} \quad \dfrac{A \mid \Gamma^\circ \vdash_{\mathsf{LI}} P}{- \mid A^x, \Gamma \vdash^x_{\mathsf{F}} P}\ \mathsf{pass} \quad \dfrac{}{- \mid\ \vdash^x_{\mathsf{F}} \mathsf{I}}\ \mathsf{IR}
$$

$$
\dfrac{T \mid \Gamma^\circ \vdash^\bullet_{\mathsf{RI}} A \quad - \mid \Delta^\circ \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} A \otimes B}\ {\otimes}\mathsf{R} \quad \dfrac{- \mid \Gamma^\circ \vdash_{\mathsf{RI}} A \quad B \mid \Delta^\circ \vdash_{\mathsf{LI}} P \quad x = \bullet \supset \bullet \in \Gamma}{A \multimap B \mid \Gamma, \Delta \vdash^x_{\mathsf{F}} P}\ {\multimap}\mathsf{L}
$$

There are two main differences with the non-commutative focused calculus in (2.7):

- Root-first proof search now starts in a new phase C, where formulae in context are permuted. The search in phase C starts from a sequent $S \mid \Gamma \,\vdots\ \vdash_{\mathsf{C}} C$ and ends in a sequent $S \mid\ \,\vdots\, \Gamma' \vdash_{\mathsf{C}} C$, where Γ' is a permutation of Γ. During this process, the context is split into two parts $\Gamma \,\vdots\, \Delta$, where formulae in Γ are ready to be moved and formulae in Δ have been already moved to their correct position. The rule ex moves one formula at a time (the rightmost one in the left side of the context Γ) until every formula is moved somewhere to the right of the separator $\vdots$. When this happens, proof search continues in phase RI. Notice that the sequent and formulae in context may also be tagged and tags are preserved during the permutation phase. An important difference with the non-commutative focused calculus is the fact that now tagged and untagged formulae in context are mixed, while in (2.7) all tagged formulae necessarily appear at the right end of the context.

- The other difference is in rules ${\multimap}\mathsf{R}$ and ${\otimes}\mathsf{L}$, which, under the bottom-up reading of the rules, are the only two rules where new formulae are added to the context. The premises of these rules are now required to go back to phase C in order to move the new formulae A to its correct position in the context via an application of the rule ex.

Theorem 4.3.1. *The focused sequent calculus in (4.4) is sound and complete with respect to the sequent calculus of Section 4.1: there is a one-to-one correspondence between the set of derivations of $S \mid \Gamma \vdash A$ quotiented by $\stackrel{\circ}{=}$ and the set of derivations of $S \mid \Gamma \,\vdots\ \vdash_{\mathsf{C}} A$.*

Soundness is obtained by defining embedding functions $\mathsf{emb_C} : S \mid \Gamma \mathbin{\vdots} \Delta \vdash^x_{\mathsf{C}} C \to S \mid \Gamma, \Delta \vdash A$ and $\mathsf{emb}_{ph} : S \mid \Gamma \vdash^x_{ph} A \to S \mid \Gamma \vdash A$, for all $ph \in \{\mathsf{C}, \mathsf{RI}, \mathsf{LI}, \mathsf{F}\}$, which erase all phase and tag annotations. Completeness is justified by proving that the following rules are admissible in phase C:

$$\frac{}{A \mid \ \vdots \ \vdash_{\mathsf{C}} A}\ \mathsf{ax^C} \qquad \frac{A \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}{- \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}\ \mathsf{pass^C} \qquad \frac{S \mid \Gamma_0, A, B, \Gamma_1 \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}{S \mid \Gamma_0, B, A, \Gamma_1 \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}\ \mathsf{ex^C}$$

$$\frac{- \mid \Gamma \mathbin{\vdots}\ \vdash_{\mathsf{C}} A \quad B \mid \Delta \mathbin{\vdots} \Lambda \vdash_{\mathsf{C}} C}{A \multimap B \mid \Gamma, \Delta \mathbin{\vdots} \Lambda \vdash_{\mathsf{C}} C}\ {\multimap}\mathsf{L^C} \qquad \frac{- \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}{\mathsf{I} \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}\ \mathsf{IL^C} \qquad \frac{A \mid B, \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}{A \otimes B \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} C}\ {\otimes}\mathsf{L^C}$$

$$\frac{S \mid \Gamma \mathbin{\vdots}\ \vdash_{\mathsf{C}} A \quad - \mid \Gamma' \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} B}{S \mid \Gamma, \Gamma' \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} A \otimes B}\ {\otimes}\mathsf{R^C} \qquad \frac{}{- \mid \ \vdots \ \vdash_{\mathsf{C}} \mathsf{I}}\ \mathsf{IR^C} \qquad \frac{S \mid \Gamma, A \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} B}{S \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} A \multimap B}\ {\multimap}\mathsf{R^C}$$

From the admissibility of these rules, we can define a focusing function $\mathsf{focus} : S \mid \Gamma \vdash A \to S \mid \Gamma \mathbin{\vdots}\ \vdash_{\mathsf{C}} A$ by induction on the input derivation. For example, $\mathsf{focus}\ ({\multimap}\mathsf{R}\ f) = {\multimap}\mathsf{R^C}\ (\mathsf{focus}\ f)$, which are both focused derivations of the sequent $S \mid \Gamma \mathbin{\vdots}\ \vdash_{\mathsf{C}} A \multimap B$ (notice that the rule ${\multimap}\mathsf{R^C}$ moves A to the end of Γ, not to the end of Δ, a difference that would make the rule inadmissible). Theorem 4.3.1 is then proved by showing the following three properties, proving that the set of derivations of $S \mid \Gamma \vdash A$ quotiented by $\doteq$ is in bijection with the set of derivations of $S \mid \Gamma \mathbin{\vdots}\ \vdash_{\mathsf{C}} A$:

- For all $f, g : S \mid \Gamma \vdash A$, if $f \doteq g$ then $\mathsf{focus}\ f = \mathsf{focus}\ g$.
- Given any $f : S \mid \Gamma \vdash A$, $\mathsf{emb_C}\ (\mathsf{focus}\ f) \doteq f$.
- Given any $f : S \mid \Gamma \mathbin{\vdots} \Delta \vdash_{\mathsf{C}} A$, $\mathsf{focus}\ (\mathsf{emb_C}\ f) = f$.

The focused sequent calculus can be used to decide whether two morphisms in the free symmetric skew monoidal closed category are equal, in the sense of being related by the equivalence relation generated by the equational theory of symmetric skew monoidal closed categories. In the non-closed case, Veltri [69] showed that the coherence problem has a more interesting solution than the one for non-skew symmetric monoidal categories. In fact, there exist pairs of non-equal morphisms which have the same underlying permutation of atomic formulae, a phenomenon that is peculiar to the skew case and is not present in the usual non-skew case. Veltri illustrates this by showing two distinct focused derivations of the sequent $- \mid X, \mathsf{I} \otimes Y \mathbin{\vdots}\ \vdash_{\mathsf{C}} X \otimes Y$. Clearly, these derivations can also be replicated in the focused sequent calculus (4.4) for `SkMILL`.

Similarly to the non-commutative case, the main challenge in the construction of the focus function lies in proving the admissibility of ${\otimes}\mathsf{R^C}$, the right $\otimes$-rule in phase C. We show how to adapt the admissibility of rule ${\multimap}\mathsf{R}^*$ from Proposition 2.4.3 and the $\otimes$-right rules from Proposition 2.4.4 in the commutative focused calculus and then show how this leads to the admissibility of ${\otimes}\mathsf{R^C}$.

To this end, we introduce some notation. We write $\Gamma \leftrightsquigarrow \Gamma'$ when Γ is a permutation of Γ'. We write $\Gamma_0 ; \Gamma_1 \bowtie \Gamma$ if the context Γ is equal to an interleaving of Γ_0 and Γ_1, which can be formally described as the following inductive relation:

$$\frac{}{\Gamma_0 ;\ \bowtie \Gamma_0} \qquad \frac{}{; \Gamma_1 \bowtie \Gamma_1} \qquad \frac{\Gamma_0 ; \Gamma_1 \bowtie \Gamma}{A, \Gamma_0 ; \Gamma_1 \bowtie A, \Gamma} \qquad \frac{\Gamma_0 ; \Gamma_1 \bowtie \Gamma}{\Gamma_0 ; A, \Gamma_1 \bowtie A, \Gamma}$$

Assume $\Gamma_0; \Gamma_1 \bowtie \Gamma$. If Γ is empty then both Γ_0 and Γ_1 are forced to be empty. If $\Gamma = A, \Gamma'$ then A is either the leftmost formula of Γ_0 or the leftmost formula of Γ_1. Crucially, the relative position of formulae in Γ_0 is preserved in Γ and the same is true for Γ_1. When $\Gamma_0; \Gamma_1 \bowtie \Gamma$, we write $\Gamma^{\Gamma_1^x}$ to indicate the context Γ in which all formulae in Γ_1 have been assigned tag x and all formulae in Γ_0 are kept untagged.

Proposition 4.3.2. *The following rule, corresponding to an iterated* $\multimap$*-right rule, is admissible:*

$$\frac{S \mid \Gamma^{\Gamma_1^x} \vdash^x_{\mathsf{RI}} C \quad \Gamma_0; \Gamma_1 \bowtie \Gamma \quad \Gamma_1 \leftrightsquigarrow \Gamma_1'}{S \mid \Gamma_0 \vdash^x_{\mathsf{RI}} [\![\Gamma_1' \mid C]\!]_{\multimap}} \multimap\mathsf{R}^*$$

Proof. By structural induction on Γ_1':

- If Γ_1' is empty, then Γ_1 is also empty and $\Gamma = \Gamma_0$. Therefore we can take $\multimap\mathsf{R}^*\ f = f$;
- If $\Gamma_1' = A, \Gamma_1''$, then Γ_1 is forced to be of the form $\Gamma_{10}, A, \Gamma_{11}$ such that $\Gamma_{10}, \Gamma_{11} \leftrightsquigarrow \Gamma_1''$. Since $\Gamma_0; \Gamma_1 \bowtie \Gamma$, then $\Gamma_0 = \Gamma_{00}, \Gamma_{01}$ and $\Gamma = \Gamma', A, \Gamma''$ such that $\Gamma_{00}; \Gamma_{10} \bowtie \Gamma'$ and $\Gamma_{01}; \Gamma_{11} \bowtie \Gamma''$. We then define $\multimap\mathsf{R}^*\ f = \multimap\mathsf{R}\ (\mathsf{ex}\ (\mathsf{RI2C}\ (\multimap\mathsf{R}^* f)))$, i.e.

$$\frac{\overset{f}{S \mid (\Gamma')^{\Gamma_{10}^x}, A^x, (\Gamma'')^{\Gamma_{11}^x} \vdash^x_{\mathsf{RI}} C}}{S \mid \Gamma_{00}, \Gamma_{01} \vdash^x_{\mathsf{RI}} [\![A, \Gamma_1'' \mid C]\!]_{\multimap}} \multimap\mathsf{R}^* \quad = \quad \cfrac{\cfrac{\cfrac{\cfrac{\overset{f}{S \mid (\Gamma')^{\Gamma_{10}^x}, A^x, (\Gamma'')^{\Gamma_{11}^x} \vdash^x_{\mathsf{RI}} C}}{S \mid \Gamma_{00}, A^x, \Gamma_{01} \vdash^x_{\mathsf{RI}} [\![\Gamma_1'' \mid C]\!]_{\multimap}} \multimap\mathsf{R}^*}{S \mid \ \vdots \Gamma_{00}, A^x, \Gamma_{01} \vdash^x_{\mathsf{C}} [\![\Gamma_1'' \mid C]\!]_{\multimap}} \mathsf{RI2C}}{S \mid A^x \vdots \Gamma_{00}, \Gamma_{01} \vdash^x_{\mathsf{C}} [\![\Gamma_1'' \mid C]\!]_{\multimap}} \mathsf{ex}}{S \mid \Gamma_{00}, \Gamma_{01} \vdash^x_{\mathsf{RI}} [\![A, \Gamma_1'' \mid C]\!]_{\multimap}} \multimap\mathsf{R}$$

□

Proposition 4.3.3. *The following rules, corresponding to different generalizations of the* $\otimes$*-right rule, are admissible:*

$$\frac{S \mid \Gamma \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{S \mid \Gamma_0, \Delta \vdash_{\mathsf{RI}} [\![\Gamma_1' \mid A]\!]_{\multimap} \otimes B} \otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1'} \qquad \frac{S \mid \Gamma \vdash_{\mathsf{LI}} P \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{S \mid \Gamma_0, \Delta \vdash_{\mathsf{LI}} [\![\Gamma_1' \mid P]\!]_{\multimap} \otimes B} \otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}$$

$$\frac{T \mid \Gamma \vdash_{\mathsf{F}} P \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma_0, \Delta \vdash_{\mathsf{F}} [\![\Gamma_1' \mid P]\!]_{\multimap} \otimes B} \otimes\mathsf{R}^{\mathsf{F}}_{\Gamma_1'}$$

where all rules include side conditions $\Gamma_0; \Gamma_1 \bowtie \Gamma$ *and* $\Gamma_1 \leftrightsquigarrow \Gamma_1'$.

Proof. The proof proceeds by mutual induction on the left premise of each rule, which we always name f. Second premises are all named g.

Proof of $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1'}$:

- If $f = {\multimap}\mathsf{R}\ (\mathsf{ex}\ (\mathsf{RI2C}\ f'))$, then $\Gamma = \Gamma', \Gamma''$ and

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{f'}{S \mid \Gamma', A', \Gamma'' \vdash_{\mathsf{RI}} B'}}{S \mid\ \vdots\, \Gamma', A', \Gamma'' \vdash_{\mathsf{C}} B'}\ \mathsf{RI2C}}{S \mid A' \vdots\, \Gamma', \Gamma'' \vdash_{\mathsf{C}} B'}\ \mathsf{ex}}{S \mid \Gamma', \Gamma'' \vdash_{\mathsf{RI}} A' \multimap B'}\ {\multimap}\mathsf{R} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{S \mid \Gamma_0, \Delta \vdash_{\mathsf{RI}} [\![\Gamma_1' \mid A' \multimap B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1'}$$

$$= \dfrac{\overset{f'}{S \mid \Gamma', A', \Gamma'' \vdash_{\mathsf{RI}} B'} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{S \mid \Gamma_0, \Delta \vdash_{\mathsf{RI}} [\![\Gamma_1', A' \mid B']\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1', A'}$$

  The application of $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1', A'}$ can be justified as follows. Since Γ is an interleaving of Γ_0 and Γ_1, this implies that $\Gamma_0 = \Gamma_{00}, \Gamma_{01}$ and $\Gamma_1 = \Gamma_{10}, \Gamma_{11}$ so that $\Gamma_{00}; \Gamma_{10} \bowtie \Gamma'$ and $\Gamma_{01}; \Gamma_{11} \bowtie \Gamma''$. Then also $\Gamma_{00}, \Gamma_{01}; \Gamma_{10}, A', \Gamma_{11} \bowtie \Gamma', A', \Gamma''$. Moreover, since $\Gamma_{10}, \Gamma_{11} \leftrightsquigarrow \Gamma_1'$, then also $\Gamma_{10}, A', \Gamma_{11} \leftrightsquigarrow \Gamma_1', A'$.

- If $f = \mathsf{LI2RI}\ f'$, then $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma_1'}\ (\mathsf{LI2RI}\ f', g) = \mathsf{LI2RI}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (f', g))$.
  (We omit derivation trees in cases where rules are simply permuted.)

Proof of $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}$:

- If $f = \mathsf{IL}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (\mathsf{IL}\ f', g) = \mathsf{IL}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (f', g))$.
- If $f = \otimes\mathsf{L}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (\otimes\mathsf{L}\ f', g) = \otimes\mathsf{L}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (f', g))$.
- If $f = \mathsf{F2LI}\ f'$, then $\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (\mathsf{F2LI}\ f', g) = \mathsf{F2LI}\ (\otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma_1'}\ (f', g))$.

Proof of $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma_1'}$:

- If $f = \mathsf{ax}$, then Γ_1' is empty and $\otimes\mathsf{R}^{\mathsf{F}}_{(\,)}\ (\mathsf{ax}, g) = \otimes\mathsf{R}\ (\mathsf{ax}, g)$.
- If $f = \mathsf{IR}$, then Γ_1' is empty and $\otimes\mathsf{R}^{\mathsf{F}}_{(\,)}\ (\mathsf{IR}, g) = \otimes\mathsf{R}\ (\mathsf{IR}, g)$.
- If $f = \mathsf{pass}\ f'$, the passivated formula A' can either belong to Γ_0 or to Γ_1, which follows from the assumption $\Gamma_0; \Gamma_1 \bowtie A', \Gamma$. In the first case $\Gamma_0 = A', \Gamma_0'$ and pass can be applied first, i.e. $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma_1'}\ (\mathsf{pass}\ f', g) = \mathsf{pass}\ (\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma_1'}\ (f', g))$. In the second case A' is the leftmost formula of Γ_1 and consequently it belongs to Γ_1' as well. Then

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash_{\mathsf{LI}} P}}{- \mid A', \Gamma \vdash_{\mathsf{F}} A}\ \mathsf{pass} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{- \mid \Gamma_0, \Delta \vdash_{\mathsf{F}} [\![\Gamma_1' \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}^{\mathsf{F}}_{\Gamma_1'}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash_{\mathsf{LI}} P}}{- \mid A'^{\bullet}, \Gamma^{\Gamma_1^{\bullet}} \vdash^{\bullet}_{\mathsf{F}} P}\ \mathsf{pass}}{- \mid A'^{\bullet}, \Gamma^{\Gamma_1^{\bullet}} \vdash^{\bullet}_{\mathsf{RI}} P}\ \mathsf{sw}}{- \mid \Gamma_0 \vdash^{\bullet}_{\mathsf{RI}} [\![\Gamma_1' \mid P]\!]_{\multimap}}\ {\multimap}\mathsf{R}^{*} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{- \mid \Gamma_0, \Delta \vdash_{\mathsf{F}} [\![\Gamma_1' \mid P]\!]_{\multimap} \otimes B}\ \otimes\mathsf{R}$$

- If $f = \otimes\mathsf{R}\ (f', f'')$, then $\Gamma = \Gamma', \Gamma''$. This implies that $\Gamma_0 = \Gamma_{00}, \Gamma_{01}$ and $\Gamma_1 = \Gamma_{10}, \Gamma_{11}$ so that $\Gamma_{00}; \Gamma_{10} \bowtie \Gamma'$ and $\Gamma_{01}; \Gamma_{11} \bowtie \Gamma''$. Then

$$\dfrac{\dfrac{\overset{f'}{T \mid \Gamma' \vdash^{\bullet}_{\mathsf{RI}} A'} \quad \overset{f''}{- \mid \Gamma'' \vdash_{\mathsf{RI}} B'}}{T \mid \Gamma', \Gamma'' \vdash_{\mathsf{F}} A' \otimes B'}\,\otimes\mathsf{R} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{T \mid \Gamma_{00}, \Gamma_{01}, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_1 \mid A' \otimes B']\!]_{\multimap} \otimes B}\,\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'_1}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\overset{f'}{T \mid \Gamma' \vdash^{\bullet}_{\mathsf{RI}} A'} \quad \overset{f''}{- \mid \Gamma'' \vdash_{\mathsf{RI}} B'}}{T \mid (\Gamma')^{\Gamma^{\bullet}_{10}}, (\Gamma'')^{\Gamma^{\bullet}_{11}} \vdash^{\bullet}_{\mathsf{F}} A' \otimes B'}\,\otimes\mathsf{R}}{T \mid (\Gamma')^{\Gamma^{\bullet}_{10}}, (\Gamma'')^{\Gamma^{\bullet}_{11}} \vdash^{\bullet}_{\mathsf{RI}} A' \otimes B'}\,\mathsf{sw}}{T \mid \Gamma_{00}, \Gamma_{01} \vdash^{\bullet}_{\mathsf{RI}} [\![\Gamma'_1 \mid A' \otimes B']\!]_{\multimap}}\,\multimap\mathsf{R}^{*} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{T \mid \Gamma_{00}, \Gamma_{01}, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_1 \mid A' \otimes B']\!]_{\multimap} \otimes B}\,\otimes\mathsf{R}$$

- If $f = \multimap\mathsf{L}\ (f', f'')$, then $\Gamma = \Gamma', \Gamma''$. This implies that $\Gamma_0 = \Gamma_{00}, \Gamma_{01}$ and $\Gamma_1 = \Gamma_{10}, \Gamma_{11}$ so that $\Gamma_{00}; \Gamma_{10} \bowtie \Gamma'$ and $\Gamma_{01}; \Gamma_{11} \bowtie \Gamma''$. We do further induction on Γ_{10} to check if there is any formula in Γ'_1 coming from the left premise f'. If Γ_{10} is empty, then the $\multimap\mathsf{L}$ can be applied first, i.e. $\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'_1}\ (\multimap\mathsf{L}\ (f', f''), g) = \multimap\mathsf{L}\ (f', \otimes\mathsf{R}^{\mathsf{LI}}_{\Gamma'_1}\ (f'', g))$. If Γ_{10} is non-empty, then $\otimes\mathsf{R}$ must be applied before $\multimap\mathsf{L}$, i.e.

$$\dfrac{\dfrac{\overset{f'}{- \mid \Gamma' \vdash_{\mathsf{RI}} A'} \quad \overset{f''}{B' \mid \Gamma'' \vdash_{\mathsf{LI}} P}}{A' \multimap B' \mid \Gamma', \Gamma'' \vdash_{\mathsf{F}} P}\,\multimap\mathsf{L} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{A' \multimap B' \mid \Gamma_{00}, \Gamma_{01}, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_1 \mid P]\!]_{\multimap} \otimes B}\,\otimes\mathsf{R}^{\mathsf{F}}_{\Gamma'_1}$$

$$= \dfrac{\dfrac{\dfrac{\dfrac{\overset{f'}{- \mid \Gamma' \vdash_{\mathsf{RI}} A'} \quad \overset{f''}{B' \mid \Gamma'' \vdash_{\mathsf{LI}} P}}{A' \multimap B' \mid (\Gamma')^{\Gamma^{\bullet}_{10}}, (\Gamma'')^{\Gamma^{\bullet}_{11}} \vdash^{\bullet}_{\mathsf{F}} P}\,\multimap\mathsf{L}}{A' \multimap B' \mid (\Gamma')^{\Gamma^{\bullet}_{10}}, (\Gamma'')^{\Gamma^{\bullet}_{11}} \vdash^{\bullet}_{\mathsf{RI}} P}\,\mathsf{sw}}{A' \multimap B' \mid \Gamma_{00}, \Gamma_{01} \vdash^{\bullet}_{\mathsf{RI}} [\![\Gamma'_1 \mid P]\!]_{\multimap}}\,\multimap\mathsf{R}^{*} \qquad \overset{g}{- \mid \Delta \vdash_{\mathsf{RI}} B}}{A' \multimap B' \mid \Gamma_{00}, \Gamma_{01}, \Delta \vdash_{\mathsf{F}} [\![\Gamma'_1 \mid P]\!]_{\multimap} \otimes B}\,\otimes\mathsf{R}$$

  The last application of $\multimap\mathsf{L}$ is justified since Γ_{10} is non-empty. □

Proposition 4.3.4. *The following rules are admissible:*

$$\dfrac{S \mid \Gamma \,\vdots\, \vdash_{\mathsf{C}} A \quad - \mid \Gamma' \,\vdots\, \Delta \vdash_{\mathsf{C}} B}{S \mid \Gamma, \Gamma' \,\vdots\, \Delta \vdash_{\mathsf{C}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{C}} \qquad \dfrac{S \mid \Gamma \,\vdots\, \Delta \vdash_{\mathsf{C}} A \quad - \mid \Delta' \vdash_{\mathsf{RI}} B}{S \mid \Gamma \,\vdots\, \Delta, \Delta' \vdash_{\mathsf{C}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{C\text{-}RI}}$$

Proof. The admissibility of $\otimes\mathsf{R}^{\mathsf{C}}$ is proved by induction on the structure of the right premise.

- If $g = \mathsf{ex}\ g'$, then $\otimes\mathsf{R}^{\mathsf{C}}\ (f, \mathsf{ex}\ g') = \mathsf{ex}\ (\otimes\mathsf{R}^{\mathsf{C}}\ (f, g'))$.
- If $g = \mathsf{RI2C}\ g'$, then $\otimes\mathsf{R}^{\mathsf{C}}\ (f, \mathsf{RI2C}\ g') = \otimes\mathsf{R}^{\mathsf{C\text{-}RI}}\ (f, g')$.

The admissibility of $\otimes\mathsf{R}^{\mathsf{C\text{-}RI}}$ is proved by induction on the left premise.

- If $f = \mathsf{ex}\ f'$, then $\otimes\mathsf{R}^{\mathsf{C\text{-}RI}}\ (\mathsf{ex}\ f', g) = \mathsf{ex}\ (\otimes\mathsf{R}^{\mathsf{C\text{-}RI}}\ (f', g))$.

- If $f = \mathsf{RI2C}\ f'$, then $\otimes\mathsf{R}^{\mathsf{C\text{-}RI}}\ (\mathsf{RI2C}\ f', g) = \mathsf{RI2C}\ (\otimes\mathsf{R}^{\mathsf{RI}}_{(\,)}\ (f', g))$, where $\otimes\mathsf{R}^{\mathsf{RI}}_{(\,)}$ is the first admissible rule $\otimes\mathsf{R}^{\mathsf{RI}}_{\Gamma'_1}$ of Proposition 4.3.3 instantiated with empty Γ'_1. The side conditions of this rule are automatically satisfied since Γ'_1 is empty.

□

We conclude the section by introducing the focusing completeness theorem of the focused calculus (4.4).

Theorem 4.3.5. *The functions* $\mathsf{emb}_{\mathsf{RI}}$ *and* focus *define a bijective correspondence between the set of derivations of $S \mid \Gamma \vdash A$ quotiented by the equivalence relation $\overset{\circ}{=}$ and the set of derivations of $S \mid \Gamma \vdash_{\mathsf{RI}} A$:*

- *For all $f, g : S \mid \Gamma \vdash A$, if $f \overset{\circ}{=} g$ then* $\mathsf{focus}\ f = \mathsf{focus}\ g$.
- *For all $f : S \mid \Gamma \vdash A$,* $\mathsf{emb}_{\mathsf{RI}}\ (\mathsf{focus}\ f) \overset{\circ}{=} f$.
- *For all $f : S \mid \Gamma \vdash_{\mathsf{RI}} A$,* $\mathsf{focus}\ (\mathsf{emb}_{\mathsf{RI}}\ f) = f$.

Proof. The proof is similar to the proof of Theorem 2.4.5. We refer the interested reader to consult the associated Agda formalization, `https://github.com/cswphilo/code-PhD-thesis/tree/main/sym-skew-mon-clo`. However, the focused calculus in the formalization includes four phases of derivations, while in this thesis, we only have three phases due to the consistency of the other parts of the thesis. □

Chapter 5

Additive Extensions

In this chapter, we first take a step back by studying the calculus $\mathtt{SkNMILLA_S}$, which extends $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ by incorporating both additive conjunction and disjunction. Starting with this simpler fragment allows us to demonstrate the modularity of our frameworks, both sequent and focused calculi. We then integrate $\mathtt{SkNMILLA_S}$ with skew exchange and linear implication to derive the sequent and focused calculi, respectively.

5.1 Sequent Calculus

We introduce a sequent calculus $\mathtt{SkNMILLA_S}$ with additives by enriching $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$ with additive conjunction and disjunction. Formulae are inductively generated by the grammar $A, B ::= X \mid A \otimes B \mid \mathsf{I} \mid A \wedge B \mid A \vee B$, where X comes from a set At of atomic formulae. We use $\wedge$ and $\vee$ to denote additive conjunction and additive disjunction, respectively, while traditionally they are named & and $\oplus$ in linear logic literature.

Derivations are inductively generated by the following rules and the rules in $\mathtt{SkNMILL_S}$ except those related to $\multimap$:

$$\frac{A \mid \Gamma \vdash C}{A \wedge B \mid \Gamma \vdash C}\ \wedge\mathsf{L}_1 \qquad \frac{B \mid \Gamma \vdash C}{A \wedge B \mid \Gamma \vdash C}\ \wedge\mathsf{L}_2 \qquad \frac{S \mid \Gamma \vdash A \quad S \mid \Gamma \vdash B}{S \mid \Gamma \vdash A \wedge B}\ \wedge\mathsf{R}$$

$$\frac{A \mid \Gamma \vdash C \quad B \mid \Gamma \vdash C}{A \vee B \mid \Gamma \vdash C}\ \vee\mathsf{L} \qquad \frac{S \mid \Gamma \vdash A}{S \mid \Gamma \vdash A \vee B}\ \vee\mathsf{R}_1 \qquad \frac{S \mid \Gamma \vdash B}{S \mid \Gamma \vdash A \vee B}\ \vee\mathsf{R}_2$$

Notice that, similar to restrictions in $\mathtt{SkNMILL}^{\mathsf{I},\otimes}$, left rules for $\wedge$ and $\vee$ can only be applied on the formula in the stoup position.

Theorem 5.1.1. $\mathtt{SkNMILLA_S}$ *enjoys cut admissibility: the following two cut rules are admissible*

$$\frac{S \mid \Gamma \vdash A \quad A \mid \Delta \vdash C}{S \mid \Gamma, \Delta \vdash C}\ \mathsf{scut} \qquad \frac{- \mid \Gamma \vdash A \quad S \mid \Delta_0, A, \Delta_1 \vdash C}{S \mid \Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{ccut}$$

Proof. Proof proceeds similarly to the proof of Theorem 2.1.1. □

$$\dfrac{}{A \wedge B \mid \ \vdash A \wedge B}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax}}{A \wedge B \mid \ \vdash A}\ \wedge\mathsf{L}_1 \quad \dfrac{\dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{A \wedge B \mid \ \vdash B}\ \wedge\mathsf{L}_2}{A \wedge B \mid \ \vdash A \wedge B}\ \wedge\mathsf{R}$$

$$\dfrac{}{A \vee B \mid \ \vdash A \vee B}\ \mathsf{ax} \quad \doteq \quad \dfrac{\dfrac{\dfrac{}{A \mid \ \vdash A}\ \mathsf{ax}}{A \mid \ \vdash A \vee B}\ \vee\mathsf{R}_1 \quad \dfrac{\dfrac{}{B \mid \ \vdash B}\ \mathsf{ax}}{B \mid \ \vdash A \vee B}\ \vee\mathsf{R}_2}{A \vee B \mid \ \vdash A \vee B}\ \vee\mathsf{L}$$

Figure 5.1: Equivalence of derivations in $\mathtt{SkNMILLA_S}$*: η-conversions*

While the left $\wedge$-rules only act on the formula in stoup position (as all the other left logical rules), other $\wedge$-rules $\wedge\mathsf{L}_i^{\mathsf{C}}$ acting on formulae in context are admissible, with this admissibility proved by induction on the height of derivations.

$$\dfrac{S \mid \Gamma, A, \Delta \vdash C}{S \mid \Gamma, A \wedge B, \Delta \vdash C}\ \wedge\mathsf{L}_1^{\mathsf{C}} \qquad \dfrac{S \mid \Gamma, B, \Delta \vdash C}{S \mid \Gamma, A \wedge B, \Delta \vdash C}\ \wedge\mathsf{L}_2^{\mathsf{C}}$$

However, this is not the case for the other left logical rules. For example, there is no way of constructing a general left $\vee$-rule $\vee\mathsf{L}^{\mathsf{C}}$ acting on a disjunction in context. This rule should be forbidden since it would make some inadmissible sequents provable in the sequent calculus. For example, the sequent $X \wedge Y \mid Y \vee X \vdash (X \otimes Y) \vee (Y \otimes X)$ is not admissible (this can be proved using the normalization procedure of Section 5.3) but a proof could be found using $\vee\mathsf{L}^{\mathsf{C}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid \ \vdash X}\ \mathsf{ax} \quad \dfrac{\dfrac{}{Y \mid \ \vdash Y}\ \mathsf{ax}}{- \mid Y \vdash Y}\ \mathsf{pass}}{X \mid Y \vdash X \otimes Y}\ \otimes\mathsf{R}}{X \wedge Y \mid Y \vdash X \otimes Y}\ \wedge\mathsf{L}_1}{X \wedge Y \mid Y \vdash (X \otimes Y) \vee (Y \otimes X)}\ \vee\mathsf{R}_1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid \ \vdash Y}\ \mathsf{ax} \quad \dfrac{\dfrac{}{X \mid \ \vdash X}\ \mathsf{ax}}{- \mid X \vdash X}\ \mathsf{pass}}{Y \mid X \vdash Y \otimes X}\ \otimes\mathsf{R}}{X \wedge Y \mid X \vdash Y \otimes X}\ \wedge\mathsf{L}_2}{X \wedge Y \mid X \vdash (X \otimes Y) \vee (Y \otimes X)}\ \vee\mathsf{R}_2}{X \wedge Y \mid Y \vee X \vdash (X \otimes Y) \vee (Y \otimes X)}\ \vee\mathsf{L}^{\mathsf{C}}$$

The equations in Figures 5.1, 5.2, and 5.3 in addition to the equations in Figures 2.1, 2.2, excluding derivations related to $\multimap$, define a congruence relation $\doteq$.

The equations in Figure 5.1 are η-conversions of $\wedge$ and $\vee$. The remaining equations are permutative conversions. Similarly to those chosen in Figures 2.1 and 2.2, the congruence $\doteq$ is meant to serve as the proof-theoretic counterpart of the equational theory of certain categories with skew structure.

5.2 Categorical Semantics

Definition 5.2.1. A *distributive skew monoidal category with binary products* is a skew monoidal category (Definition 1.2.1) with binary products and coproducts that is left-distributive, i.e., where the canonical morphism $(A \otimes C) + (B \otimes C) \to (A + B) \otimes C$ has an inverse $l : (A + B) \otimes C \to (A \otimes C) + (B \otimes C)$.

Notice that we do not require right distributivity in this definition.

$$\dfrac{\dfrac{\overset{f}{A_i \mid \Gamma \vdash A}}{A_1 \wedge A_2 \mid \Gamma \vdash A}\,\wedge\mathsf{L}_i \quad \overset{g}{- \mid \Delta \vdash B}}{A_1 \wedge A_2 \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R} \circeq \dfrac{\dfrac{\overset{f}{A_i \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A_i \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R}}{A_1 \wedge A_2 \mid \Gamma, \Delta \vdash A \otimes B}\,\wedge\mathsf{L}_i$$

$$\dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A} \quad \overset{f_2}{A' \mid \Gamma \vdash B}}{A' \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}}{- \mid A', \Gamma \vdash A \wedge B}\,\mathsf{pass} \circeq \dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A}}{- \mid A', \Gamma \vdash A}\,\mathsf{pass} \quad \dfrac{\overset{f_2}{A' \mid \Gamma \vdash B}}{- \mid A', \Gamma \vdash B}\,\mathsf{pass}}{- \mid A', \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f_1}{- \mid \Gamma \vdash A} \quad \overset{f_2}{- \mid \Gamma \vdash B}}{- \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}}{\mathsf{I} \mid \Gamma \vdash A \wedge B}\,\mathsf{IL} \circeq \dfrac{\dfrac{\overset{f_1}{- \mid \Gamma \vdash A}}{\mathsf{I} \mid \Gamma \vdash A}\,\mathsf{IL} \quad \dfrac{\overset{f_2}{- \mid \Gamma \vdash B}}{\mathsf{I} \mid \Gamma \vdash B}\,\mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f_1}{A' \mid B', \Gamma \vdash A} \quad \overset{f_2}{A' \mid B', \Gamma \vdash B}}{A' \mid B', \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}}{A' \otimes B' \mid \Gamma \vdash A \wedge B}\,\otimes\mathsf{L} \circeq \dfrac{\dfrac{\overset{f_1}{A' \mid B', \Gamma \vdash A}}{A' \otimes B' \mid \Gamma \vdash A}\,\otimes\mathsf{L} \quad \dfrac{\overset{f_2}{A' \mid B', \Gamma \vdash B}}{A' \otimes B' \mid \Gamma \vdash B}\,\otimes\mathsf{L}}{A' \otimes B' \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f_1}{A_i \mid \Gamma \vdash A} \quad \overset{f_2}{A_i \mid \Gamma \vdash B}}{A_i \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}}{A_1 \wedge A_2 \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{L}_i \circeq \dfrac{\dfrac{\overset{f_1}{A_i \mid \Gamma \vdash A}}{A_1 \wedge A_2 \mid \Gamma \vdash A}\,\wedge\mathsf{L}_i \quad \dfrac{\overset{f_2}{A_i \mid \Gamma \vdash B}}{A_1 \wedge A_2 \mid \Gamma \vdash B}\,\wedge\mathsf{L}_i}{A_1 \wedge A_2 \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f}{A \mid \Gamma \vdash A_i}}{- \mid A, \Gamma \vdash A_i}\,\mathsf{pass}}{- \mid A, \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \circeq \dfrac{\dfrac{\overset{f}{A \mid \Gamma \vdash A_i}}{A \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i}{- \mid A, \Gamma \vdash A_1 \vee A_2}\,\mathsf{pass}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A_i}}{\mathsf{I} \mid \Gamma \vdash A_i}\,\mathsf{IL}}{\mathsf{I} \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \circeq \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A_i}}{- \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i}{\mathsf{I} \mid \Gamma \vdash A_1 \vee A_2}\,\mathsf{IL}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A_i}}{A' \otimes B' \mid \Gamma \vdash A_i}\,\otimes\mathsf{L}}{A' \otimes B' \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \circeq \dfrac{\dfrac{\overset{f}{A' \mid B', \Gamma \vdash A_i}}{A' \mid B', \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i}{A' \otimes B' \mid \Gamma \vdash A_1 \vee A_2}\,\otimes\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{A'_i \mid \Gamma \vdash A_i}}{A'_1 \wedge A'_2 \mid \Gamma \vdash A_i}\,\wedge\mathsf{L}_i}{A'_1 \wedge A'_2 \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \circeq \dfrac{\dfrac{\overset{f}{A'_i \mid \Gamma \vdash A_i}}{A'_i \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i}{A'_1 \wedge A'_2 \mid \Gamma \vdash A_1 \vee A_2}\,\wedge\mathsf{L}_i$$

Figure 5.2: Equivalence of derivations in $\texttt{SkNMILLA}_{\mathsf{S}}$*: permutative conversions*

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A_i} \quad \overset{g}{B' \mid \Gamma \vdash A_i}}{A' \vee B' \mid \Gamma \vdash A_i}\,\vee\mathsf{L}}{A' \vee B' \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \;\doteq\; \dfrac{\dfrac{\overset{f}{A' \mid \Gamma \vdash A_i}}{A' \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i \quad \dfrac{\overset{f}{B' \mid \Gamma \vdash A_i}}{B' \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{R}_i}{A' \vee B' \mid \Gamma \vdash A_1 \vee A_2}\,\vee\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A} \quad \overset{f_2}{B' \mid \Gamma \vdash B}}{A' \vee B' \mid \Gamma \vdash A}\,\vee\mathsf{L} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \vee B' \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R}$$

$$\doteq\; \dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{A' \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R} \quad \dfrac{\overset{f_2}{B' \mid \Gamma \vdash A} \quad \overset{g}{- \mid \Delta \vdash B}}{B' \mid \Gamma, \Delta \vdash A \otimes B}\,\otimes\mathsf{R}}{A' \vee B' \mid \Gamma, \Delta \vdash A \otimes B}\,\vee\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A} \quad \overset{g_1}{A' \mid \Gamma \vdash B}}{A' \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R} \quad \dfrac{\overset{f_2}{B' \mid \Gamma \vdash A} \quad \overset{g_2}{B' \mid \Gamma \vdash B}}{B' \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}}{A' \vee B' \mid \Gamma \vdash A \wedge B}\,\vee\mathsf{L}$$

$$\doteq\; \dfrac{\dfrac{\overset{f_1}{A' \mid \Gamma \vdash A} \quad \overset{f_2}{B' \mid \Gamma \vdash A}}{A' \vee B' \mid \Gamma \vdash A}\,\vee\mathsf{L} \quad \dfrac{\overset{g_1}{A' \mid \Gamma \vdash B} \quad \overset{g_2}{B' \mid \Gamma \vdash B}}{A' \vee B' \mid \Gamma \vdash B}\,\vee\mathsf{L}}{A' \vee B' \mid \Gamma \vdash A \wedge B}\,\wedge\mathsf{R}$$

Figure 5.3: Equivalence of derivations in $\mathtt{SkNMILLA_S}$*: permutative conversions, continued*

Definition 5.2.2. A *(strict) skew monoidal functor* $F : \mathbb{C} \to \mathbb{D}$ between skew monoidal categories $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ and $(\mathbb{D}, \mathsf{I}', \otimes', \multimap')$ is a functor from $\mathbb{C}$ to $\mathbb{D}$ satisfying $F\mathsf{I} = \mathsf{I}'$ and $F(A \otimes B) = FA \otimes' FB$, also preserving the structural laws λ, ρ and α on the nose.

Definition 5.2.3. A skew monoidal functor is *distributive* if it also strictly preserves products, coproducts and l.

The formulae, derivations and the equivalence relation $\stackrel{\circ}{=}$ of the sequent calculus determine a *syntactic* distributive skew monoidal category with binary products $\mathsf{FDSkM(At)}$ (an acronym for *F*ree *D*istributive *Sk*ew *M*onoidal category with binary products on the set At). Its objects are formulae. The operations I and $\otimes$ are the logical connectives. The set of maps between objects A and B is the set of derivations $A \mid \ \vdash B$ quotiented by the equivalence relation $\stackrel{\circ}{=}$. The identity map on A is the equivalence class of ax_A, while composition is given by scut. The structural laws λ, ρ, α are all admissible. Products and coproducts are the additive connectives $\wedge$ and $\vee$. Left-distributivity follows from the logical rules of $\vee$ and $\otimes$. Distributive skew monoidal categories with binary products form models of our sequent calculus. Moreover the sequent calculus, as a presentation of a distributive skew monoidal category with binary products, is the *initial* one among these models. Equivalently, $\mathsf{FDSkM(At)}$ is the *free* such category on the set At.

Theorem 5.2.4. *Let $\mathbb{D}$ be a distributive skew monoidal category with binary products. Given a function $F_{\mathsf{At}} : \mathsf{At} \to |\mathbb{D}|$ evaluating atomic formulae as objects of $\mathbb{D}$, there exists a unique distributive skew monoidal functor $F : \mathsf{FDSkM(At)} \to \mathbb{D}$ for which $FX = F_{\mathsf{At}}X$, for any atom X.*

The construction of the functor F and the proof of uniqueness proceed similarly to the proof of Theorem 2.3.6.

5.3 A Focused Sequent Calculus with Tag Annotations

When oriented from left-to-right, the equations in Figures 5.1, 5.2, and 5.3 become a rewrite system, which is locally confluent and strongly normalizing, thus confluent with unique normal forms. Here we provide an explicit description of the normal forms of $\mathtt{SkNMILLA_S}$ with respect to this rewrite system.

For any sequent $S \mid \Gamma \vdash A$, a root-first proof search procedure can be defined as follows. First apply right invertible rules on the sequent until the principal connective of the succedent is non-negative, then apply left invertible rules until the stoup becomes either empty or non-positive. At this point, if we do not insist on focusing on a particular formula (either in the stoup or succedent, since no rule acts on formulae in context) as in Andreoli's focusing procedure [5], we obtain a sequent calculus with a reduced proof search space, that looks like this:

$$
\begin{array}{ll}
\text{(right invertible)} & \dfrac{S \mid \Gamma \vdash_{\mathsf{RI}} A \quad S \mid \Gamma \vdash_{\mathsf{RI}} B}{S \mid \Gamma \vdash_{\mathsf{RI}} A \wedge B}\ \wedge\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash_{\mathsf{LI}} P}{S \mid \Gamma \vdash_{\mathsf{RI}} P}\ \mathsf{LI2RI} \\[2ex]
\text{(left invertible)} & \dfrac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\ \otimes\mathsf{L} \\
& \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P \quad B \mid \Gamma \vdash_{\mathsf{LI}} P}{A \vee B \mid \Gamma \vdash_{\mathsf{LI}} P}\ \vee\mathsf{L} \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{F}} P}{T \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{F2LI} \\[2ex]
\text{(focusing)} & \dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\ \mathsf{ax} \qquad \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash_{\mathsf{F}} P}\ \mathsf{pass} \qquad \dfrac{}{- \mid \ \vdash_{\mathsf{F}} \mathsf{I}}\ \mathsf{IR}
\end{array}
\tag{5.1}
$$

$$
\dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}} A \otimes B}\ \otimes\mathsf{R} \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} A}{T \mid \Gamma \vdash_{\mathsf{F}} A \vee B}\ \vee\mathsf{R}_1 \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}} B}{T \mid \Gamma \vdash_{\mathsf{F}} A \vee B}\ \vee\mathsf{R}_2
$$

$$
\dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash_{\mathsf{F}} P}\ \wedge\mathsf{L}_1 \qquad \dfrac{B \mid \Gamma \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash_{\mathsf{F}} P}\ \wedge\mathsf{L}_2
$$

In the rules above, P is a positive formula, i.e. its principal connective is not $\wedge$, and T is a negative stoup, i.e. it is not I and its principal connective is neither $\otimes$ nor $\vee$.

This calculus is too permissive. The same sequent $S \mid \Gamma \vdash_{\mathsf{RI}} A$ may have multiple derivations which correspond to $\stackrel{\circ}{=}$-related derivations in the original sequent calculus. This happens since certain sequents in phase $\vdash_{\mathsf{F}}$ can be alternatively proved by an application of a left non-invertible rule (pass, $\wedge\mathsf{L}_1$ and $\wedge\mathsf{L}_2$) or an application of a right non-invertible rule ($\otimes\mathsf{R}$, $\vee\mathsf{R}_1$ and $\vee\mathsf{R}_2$). As concrete examples, both sequents $- \mid X, Y \vdash_{\mathsf{F}} X \otimes Y$ and $X \wedge Y \mid \ \vdash_{\mathsf{F}} X \vee Y$ have multiple distinct proofs in this calculus, but their corresponding proofs in the original calculus are $\stackrel{\circ}{=}$-related.

In phase $\vdash_{\mathsf{F}}$, only non-invertible rules can be applied, so the question is: how to arrange the order between non-invertible rules without causing undesired nondeterminism and losing completeness with respect to $\mathtt{SkNMILLA_S}$ and its equivalence relation $\stackrel{\circ}{=}$? Similarly to the focused calculus in previous sections, our strategy is to prioritize left non-invertible rules over right ones, unless this does not lead to a valid derivation and the other way around is necessary. For example, consider the sequent $X \wedge Y \mid \ \vdash_{\mathsf{F}} (X \wedge Y) \vee Z$. Proof search fails if we apply $\wedge\mathsf{L}_i$ before $\vee\mathsf{R}_1$. A valid proof is obtained only when applying $\vee\mathsf{R}_1$ before $\wedge\mathsf{L}_i$. Rule sw is an abbreviation for the application of multiple consecutive phase switching rules.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid \ \vdash_{\mathsf{F}} X}\ \mathsf{ax}}{X \mid \ \vdash_{\mathsf{LI}} X}\ \mathsf{F2LI}}{X \wedge Y \mid \ \vdash_{\mathsf{F}} X}\ \wedge\mathsf{L}_1}{X \wedge Y \mid \ \vdash_{\mathsf{RI}} X}\ \mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid \ \vdash_{\mathsf{F}} Y}\ \mathsf{ax}}{Y \mid \ \vdash_{\mathsf{LI}} Y}\ \mathsf{F2LI}}{X \wedge Y \mid \ \vdash_{\mathsf{F}} Y}\ \wedge\mathsf{L}_2}{X \wedge Y \mid \ \vdash_{\mathsf{RI}} Y}\ \mathsf{sw}}{X \wedge Y \mid \ \vdash_{\mathsf{RI}} X \wedge Y}\ \wedge\mathsf{R}}{X \wedge Y \mid \ \vdash_{\mathsf{F}} (X \wedge Y) \vee Z}\ \vee\mathsf{R}_1
\tag{5.2}
$$

In this example it was possible to first apply $\vee\mathsf{R}_1$ since, after the application of $\wedge\mathsf{R}$, different left $\wedge$-rules are applied in different branches of the proof tree. If we would have applied the same rule $\wedge\mathsf{L}_1$ to both premises (imagine that $X = Y$ for this to be possible), then we could have obtained a $\stackrel{\circ}{=}$-equivalent derivation by moving the application of $\wedge\mathsf{L}_1$ to the bottom of the proof tree.

It is also possible that the two premises of $\wedge\mathsf{R}$ correspond to $\overset{\circ}{=}$-inequivalent derivations. For example, consider the following proof of sequent $- \mid \mathsf{I} \otimes X \vdash_{\mathsf{F}} ((I \otimes X) \wedge (\mathsf{I} \otimes X)) \vee Y$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{F}} X}\,\mathsf{ax}}{X \mid\ \vdash_{\mathsf{LI}} X}\,\mathsf{F2LI}}{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}}{- \mid X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}}{- \mid X \vdash_{\mathsf{F}} \mathsf{I} \otimes X}\,\otimes\mathsf{R}}{- \mid X \vdash_{\mathsf{LI}} \mathsf{I} \otimes X}\,\mathsf{F2LI}}{\mathsf{I} \mid X \vdash_{\mathsf{LI}} \mathsf{I} \otimes X}\,\mathsf{IL}}{\mathsf{I} \otimes X \mid\ \vdash_{\mathsf{LI}} \mathsf{I} \otimes X}\,\otimes\mathsf{L}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{F}} \mathsf{I} \otimes X}\,\mathsf{pass}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{RI}} \mathsf{I} \otimes X}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{F}} X}\,\mathsf{ax}}{X \mid\ \vdash_{\mathsf{LI}} X}\,\mathsf{F2LI}}{- \mid X \vdash_{\mathsf{F}} X}\,\mathsf{pass}}{- \mid X \vdash_{\mathsf{LI}} X}\,\mathsf{F2LI}}{\mathsf{I} \mid X \vdash_{\mathsf{LI}} X}\,\mathsf{IL}}{\mathsf{I} \otimes X \mid\ \vdash_{\mathsf{LI}} X}\,\otimes\mathsf{L}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{F}} X}\,\mathsf{pass}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{RI}} X}\,\mathsf{sw}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{F}} \mathsf{I} \otimes X}\,\otimes\mathsf{R}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{RI}} \mathsf{I} \otimes X}\,\mathsf{sw}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{RI}} (\mathsf{I} \otimes X) \wedge (\mathsf{I} \otimes X)}\,\wedge\mathsf{R}}{- \mid \mathsf{I} \otimes X \vdash_{\mathsf{F}} ((\mathsf{I} \otimes X) \wedge (\mathsf{I} \otimes X)) \vee Y}\,\vee\mathsf{R}_1$$

In this case, the right non-invertible rule $\vee\mathsf{R}_1$ must be applied before the left non-invertible rule pass. This is because pass is used in the proof of the first premise of $\wedge\mathsf{R}$, but it is not used in the proof of the second premise, $\otimes\mathsf{R}$ is used instead. If both proofs of the premises would have used pass (for example, they could have been the same exact proof), then it would have been possible to apply pass before $\vee\mathsf{R}_1$.

In general, a right non-invertible rule should be applied before a left non-invertible one if, after the possible application of some $\wedge\mathsf{R}$ rules, either: (*i*) a right non-invertible rule or the ax rule is applied to one of the premises; (*ii*) $\wedge\mathsf{L}_1$ and $\wedge\mathsf{L}_2$ are applied to different premises. Therefore, we have to make sure that in the focused sequent calculus, after the application of a right non-invertible rule, not all premises use the same left non-invertible rule, because in this case the latter rule could be applied first.

In order to keep track of this, we use a system of *tag annotations* and we introduce new phases of proof search where sequents are annotated by *list of tags*. There are four tags: $\mathbb{P}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{R}$. Intuitively, they respectively correspond to rules pass, $\wedge\mathsf{L}_1, \wedge\mathsf{L}_2$ and all the remaining non-invertible rules in phase $\vdash_{\mathsf{F}}$. A list of tags l is called *valid* if it is non-empty and either (*i*) $\mathbb{R} \in l$ or (*ii*) both $\mathbb{C}_1 \in l$ and $\mathbb{C}_2 \in l$.

Derivations in the focused sequent calculus with tag annotations are generated by the rules:

$$
\begin{array}{ll}
\text{(right invertible)} & \dfrac{S \mid \Gamma \vdash_{\mathsf{RI}}^{l_1?} A \quad S \mid \Gamma \vdash_{\mathsf{RI}}^{l_2?} B}{S \mid \Gamma \vdash_{\mathsf{RI}}^{l_1?,l_2?} A \wedge B} \wedge\mathsf{R} \qquad \dfrac{S \mid \Gamma \vdash_{\mathsf{LI}}^{t?} P}{S \mid \Gamma \vdash_{\mathsf{RI}}^{t?} P} \mathsf{LI2RI} \\[3ex]
\text{(left invertible)} & \begin{array}{c} \dfrac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P} \mathsf{IL} \qquad \dfrac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P} \otimes\mathsf{L} \\ \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P \quad B \mid \Gamma \vdash_{\mathsf{LI}} P}{A \vee B \mid \Gamma \vdash_{\mathsf{LI}} P} \vee\mathsf{L} \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{F}}^{t?} P}{T \mid \Gamma \vdash_{\mathsf{LI}}^{t?} P} \mathsf{F2LI} \end{array} \\[3ex]
\text{(focusing)} & \begin{array}{c} \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{- \mid A, \Gamma \vdash_{\mathsf{F}}^{\mathbb{P}?} P} \mathsf{pass} \qquad \dfrac{}{X \mid \ \vdash_{\mathsf{F}}^{\mathbb{R}?} X} \mathsf{ax} \\ \dfrac{}{- \mid \ \vdash_{\mathsf{F}}^{\mathbb{R}?} \mathsf{I}} \mathsf{IR} \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B \quad l \text{ valid}}{T \mid \Gamma, \Delta \vdash_{\mathsf{F}}^{\mathbb{R}?} A \otimes B} \otimes\mathsf{R} \\ \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A \quad l \text{ valid}}{T \mid \Gamma \vdash_{\mathsf{F}}^{\mathbb{R}?} A \vee B} \vee\mathsf{R}_1 \qquad \dfrac{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} B \quad l \text{ valid}}{T \mid \Gamma \vdash_{\mathsf{F}}^{\mathbb{R}?} A \vee B} \vee\mathsf{R}_2 \\ \dfrac{A \mid \Gamma \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash_{\mathsf{F}}^{\mathbb{C}_1?} P} \wedge\mathsf{L}_1 \qquad \dfrac{B \mid \Gamma \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash_{\mathsf{F}}^{\mathbb{C}_2?} P} \wedge\mathsf{L}_2 \end{array}
\end{array}
\tag{5.3}
$$

We use l for lists of tags and t for single tags. The notation $l?$ indicates that the sequent is either untagged or assigned the list of tags l. Similarly for notation $t?$. We discuss the proof search procedures of untagged and tagged sequents separately. The proof search of a sequent $S \mid \Gamma \vdash_{\mathsf{RI}} A$ proceeds as follows:

($\vdash_{\mathsf{RI}}$) We apply the right invertible rule $\wedge\mathsf{R}$ eagerly to decompose the succedent until its principal connective is not $\wedge$, then we move to the left invertible phase $\vdash_{\mathsf{LI}}$ with an application of LI2RI.

($\vdash_{\mathsf{LI}}$) We apply left invertible rules until the stoup becomes irreducible, then move to the focusing phase $\vdash_{\mathsf{F}}$ with an application of F2LI.

($\vdash_{\mathsf{F}}$) We apply one of the remaining rules. Since the sequents are not marked by tags at this point, rules pass, ax, IR and $\wedge\mathsf{L}_i$ can be directly applied when stoups, contexts and succedents are of the appropriate form. If we decide to apply a right non-invertible rule, we need to come up with a valid list of tags l and subsequently continue proof search in tagged right invertible phase $\vdash_{\mathsf{RI}}^{l}$, which is described below. Notice that only the first premise of $\otimes\mathsf{R}$ is tagged, the second premise is not, i.e. its proof search continues in phase $\vdash_{\mathsf{RI}}$.

The proof search of a sequent $T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A$ proceeds as follows (notice that at this point in proof search the stoup T is necessarily irreducible):

($\vdash_{\mathsf{RI}}^{l}$) We apply the $\wedge\mathsf{R}$ rule to decompose the succedent and split the list of tags carefully until the succedent becomes non-negative and the list of tags becomes a singleton t, then we move to phase $\vdash_{\mathsf{LI}}^{t}$ with an application of LI2RI.

($\vdash_{\mathsf{LI}}^{t}$) Since the stoup is either empty or a negative formula, we immediately switch to phase $\vdash_{\mathsf{F}}$ with an application of F2LI. This motivates why sequents in rules IL, $\otimes$L, $\vee$L are not tagged.

($\vdash^t_{\mathsf{F}}$) If $t = \mathbb{R}$ we can apply either ax, IR or another right non-invertible rule. Again, when applying right non-invertible rules we need to come up with a new valid list of tags. Left non-invertible rules can be applied only when the tag is correct, i.e. pass with tag $\mathbb{P}$, $\wedge\mathsf{L}_1$ with tag $\mathbb{C}_1$, and $\wedge\mathsf{L}_2$ with tag $\mathbb{C}_2$.

The derivation in (5.2) can be reconstructed in the focused calculus with tag annotations.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \vdash_{\mathsf{F}} X}\,\mathsf{ax}}{X \mid\ \vdash_{\mathsf{LI}} X}\,\mathsf{sw}}{X \wedge Y \mid\ \vdash^{\mathbb{C}_1}_{\mathsf{F}} X}\,\wedge\mathsf{L}_1}{X \wedge Y \mid\ \vdash^{\mathbb{C}_1}_{\mathsf{RI}} X}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid\ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}}{Y \mid\ \vdash_{\mathsf{LI}} Y}\,\mathsf{sw}}{X \wedge Y \mid\ \vdash^{\mathbb{C}_2}_{\mathsf{F}} Y}\,\wedge\mathsf{L}_2}{X \wedge Y \mid\ \vdash^{\mathbb{C}_2}_{\mathsf{RI}} Y}\,\mathsf{sw}}{\dfrac{X \wedge Y \mid\ \vdash^{\mathbb{C}_1,\mathbb{C}_2}_{\mathsf{RI}} X \wedge Y}{X \wedge Y \mid\ \vdash_{\mathsf{F}} (X \wedge Y) \vee Z}\,\vee\mathsf{R}_1}\,\wedge\mathsf{R} \tag{5.4}$$

The list of tags $[\mathbb{C}_1, \mathbb{C}_2]$ is valid since it contains both $\mathbb{C}_1$ and $\mathbb{C}_2$.

Notice that the list of tags is not predetermined when a right non-invertible rule is applied, we have to come up with one ourselves. Practically, the list l can be computed by continuing proof search until, in each branch, we hit the first application of a rule in phase $\vdash_{\mathsf{F}}$, each with its own (necessarily uniquely determined) single tag t. Take l as the concatenation of the resulting ts and check whether it is valid. If it is not, backtrack and apply a left non-invertible rule instead.

Theorem 5.3.1. *The focused sequent calculus with tag annotations in (5.3) is sound and complete with respect to the sequent calculus in* `SkNMILLA`$_{\mathtt{S}}$.

Soundness is immediate because there exist functions $\mathsf{emb}_{ph} : S \mid \Gamma \vdash^{l?}_{ph} A \to S \mid \Gamma \vdash A$, for all $ph \in \{\mathsf{RI}, \mathsf{LI}, \mathsf{F}\}$, which erase all phase and tag annotations. Completeness follows from the fact that the following rules are all admissible:

$$\dfrac{- \mid \Gamma \vdash_{\mathsf{RI}} C}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{RI}} C}\,\mathsf{IL}^{\mathsf{RI}} \quad \dfrac{A \mid B, \Gamma \vdash_{\mathsf{RI}} C}{A \otimes B \mid \Gamma \vdash_{\mathsf{RI}} C}\,\otimes\mathsf{L}^{\mathsf{RI}}$$

$$\dfrac{A \mid \Gamma \vdash_{\mathsf{RI}} C}{- \mid A, \Gamma \vdash_{\mathsf{RI}} C}\,\mathsf{pass}^{\mathsf{RI}} \quad \dfrac{}{A \mid\ \vdash_{\mathsf{RI}} A}\,\mathsf{ax}^{\mathsf{RI}} \quad \dfrac{}{- \mid\ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{IR}^{\mathsf{RI}}$$

$$\dfrac{A \mid \Gamma \vdash_{\mathsf{RI}} C \quad B \mid \Gamma \vdash_{\mathsf{RI}} C}{A \vee B \mid \Gamma \vdash_{\mathsf{RI}} C}\,\vee\mathsf{L}^{\mathsf{RI}} \quad \dfrac{S \mid \Gamma \vdash_{\mathsf{RI}} A \quad - \mid \Delta \vdash_{\mathsf{RI}} B}{S \mid \Gamma, \Delta \vdash_{\mathsf{RI}} A \otimes B}\,\otimes\mathsf{R}^{\mathsf{RI}} \tag{5.5}$$

$$\dfrac{A \mid \Gamma \vdash_{\mathsf{RI}} C}{A \wedge B \mid \Gamma \vdash_{\mathsf{RI}} C}\,\wedge\mathsf{L}^{\mathsf{RI}}_1 \quad \dfrac{B \mid \Gamma \vdash_{\mathsf{RI}} C}{A \wedge B \mid \Gamma \vdash_{\mathsf{RI}} C}\,\wedge\mathsf{L}^{\mathsf{RI}}_2$$

$$\dfrac{S \mid \Gamma \vdash_{\mathsf{RI}} A}{S \mid \Gamma \vdash_{\mathsf{RI}} A \vee B}\,\vee\mathsf{R}^{\mathsf{RI}}_1 \quad \dfrac{S \mid \Gamma \vdash_{\mathsf{RI}} B}{S \mid \Gamma \vdash_{\mathsf{RI}} A \vee B}\,\vee\mathsf{R}^{\mathsf{RI}}_2$$

The admissibility of the rules in (5.5), apart from the right non-invertible ones, is proved by structural induction on derivations. The same strategy cannot be applied to right non-invertible rules. For example, if the premise of $\vee\mathsf{R}^{\mathsf{RI}}_1$ ends with an application of $\wedge\mathsf{R}$, we get immediately stuck:

$$\dfrac{\dfrac{\overset{f}{S \mid \Gamma \vdash_{\mathsf{RI}} A'} \quad \overset{g}{S \mid \Gamma \vdash_{\mathsf{RI}} B'}}{S \mid \Gamma \vdash_{\mathsf{RI}} A' \wedge B'}\wedge\mathsf{R}}{S \mid \Gamma \vdash_{\mathsf{RI}} (A' \wedge B') \vee B}\vee\mathsf{R}_1^{\mathsf{RI}} \qquad = \quad ??$$

The inductive hypothesis applied to f and g would produce wrong sequents for the target conclusion. This is fixed by proving the admissibility of more general rules. In order to state and prove this, we need to first introduce a few lemmata. The first one shows that applying several $\wedge\mathsf{R}$ rules in one step is admissible.

Let $\mathsf{conj}(A)$ be the list of formulae obtained by decomposing additive conjunctions $\wedge$ in the formula A. Concretely, $\mathsf{conj}(A) = \mathsf{conj}(A'), \mathsf{conj}(B')$ if $A = A' \wedge B'$ and $\mathsf{conj}(A) = A$ otherwise.

Lemma 5.3.2. *The following rules*

$$\dfrac{\overset{fs}{[T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[1,\dots,n]}}}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A}\wedge\mathsf{R}_t^* \qquad \dfrac{\overset{fs}{[S \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\dots,n]}}}{S \mid \Gamma \vdash_{\mathsf{RI}} A}\wedge\mathsf{R}^*$$

are admissible, where $\mathsf{conj}(A) = [P_1, \dots, P_n]$ *and* $l = [t_1, \dots, t_n]$.

Proof. We show the case of $\wedge\mathsf{R}_t^*$, the other one is similar. Let $fs : [T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_i$ be a list of derivations. The proof proceeds by induction on A.

- If $A \neq A' \wedge B'$, then fs consists of a single derivation f. Define $\wedge\mathsf{R}_t^*\ fs = \mathsf{F2LI}\ (\mathsf{LI2RI}\ f)$.
- If $A = A' \wedge B'$, then there exist lists of derivations $fs_1 : [T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[1,\dots,m]}$ and $fs_2 : [T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[m+1,\dots,n]}$, and lists of tags $l_1 = t_1, \dots, t_m$ and $l_2 = t_{m+1}, \dots, t_n$, so that fs is the concatenation of fs_1 and fs_2 and l is the concatenation of l_1 and l_2. Apply $\wedge\mathsf{R}$ at the bottom, then proceed recursively:

$$\dfrac{\overset{fs}{[T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[1,\dots,n]}}}{T \mid \Gamma \vdash^{l} A' \wedge B'}\wedge\mathsf{R}_t^*$$

$$= \dfrac{\dfrac{\overset{fs_1}{[T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[1,\dots,m]}}}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l_1} A'}\wedge\mathsf{R}_t^* \quad \dfrac{\overset{fs_2}{[T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[m+1,\dots,n]}}}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l_2} B'}\wedge\mathsf{R}_t^*}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l_1,l_2} A' \wedge B'}\wedge\mathsf{R}$$

∎

The second lemma corresponds to the invertibility of phase $\vdash_{\mathsf{RI}}$.

Lemma 5.3.3. *Given* $f : S \mid \Gamma \vdash_{\mathsf{RI}} A$*, there is a list of derivations* $fs : [S \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\dots,n]}$ *with* $f = \wedge\mathsf{R}^* fs$.

Proof. The proof proceeds by structural induction on $f : S \mid \Gamma \vdash_{\mathsf{RI}} A$.

- If $f = \mathsf{LI2RI}\ f_1$, then A is non-negative. Take fs as the singleton list consisting exclusively of f_1.

- If $f = \wedge\mathsf{R}\ (f_1, f_2)$, then by inductive hypothesis we have $fs_1 : [S \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}$ and $fs_2 : [S \mid \Gamma \vdash_{\mathsf{LI}} P_i']_{i\in[1,\ldots,m]}$. Take fs as the concatenation of fs_1 and fs_2. □

Proposition 5.3.4. *The following rules*

$$\frac{\begin{array}{c} fs \\ [S \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}\end{array}}{S \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}_1^{\mathsf{LI}} \qquad \frac{\begin{array}{c} fs \\ [S \mid \Gamma \vdash_{\mathsf{LI}} Q_i]_{i\in[1,\ldots,m]}\end{array}}{S \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}_2^{\mathsf{LI}}$$

$$\frac{\begin{array}{c} fs \\ [S \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}\end{array} \quad - \mid \Delta \vdash_{\mathsf{RI}} B'}{S \mid \Gamma, \Delta \vdash_{\mathsf{LI}} A \otimes B'}\ \otimes\mathsf{R}^{\mathsf{LI}}$$

are admissible, where $\mathsf{conj}(A) = [P_1, \ldots, P_n]$ *and* $\mathsf{conj}(B) = [Q_1, \ldots, Q_m]$.

Proof. The list of derivations fs is non-empty, so we let $fs = [f_1, fs']$. We proceed by induction on f_1. We only present the proof for $\vee\mathsf{R}_1^{\mathsf{LI}}$, the admissibility of $\vee\mathsf{R}_2^{\mathsf{LI}}$ and $\otimes\mathsf{R}^{\mathsf{LI}}$ is proved similarly.

If f_1 ends with the application of a left invertible rule, then all the derivations in fs' necessarily end with the same rule as well. Therefore, we permute this rule with $\vee\mathsf{R}_1^{\mathsf{LI}}$ and apply the inductive hypothesis.

If $f_1 = \mathsf{F2LI}\ f_1'$, then all the derivations in fs' necessarily end with $\mathsf{F2LI}$ as well. We generate a list of tags l by examining the shape of each derivation in fs: we add $\mathbb{P}$ for each pass, $\mathbb{C}_1$ for each $\wedge\mathsf{L}_1$, $\mathbb{C}_2$ for each $\wedge\mathsf{L}_2$ and $\mathbb{R}$ for the remaining rules. There are two possibilities:

- The resulting list l is valid. We switch to phase $\vdash_{\mathsf{F}}$ and apply $\vee\mathsf{R}_1$ followed by $\wedge\mathsf{R}_t^*$:

$$\dfrac{\dfrac{\begin{array}{c} fs^* \\ [T \mid \Gamma \vdash_{\mathsf{F}} P_i]_{i\in[1,\ldots,n]}\end{array}}{[T \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}}\ [\mathsf{F2LI}]}{T \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}_1^{\mathsf{LI}} \quad = \quad \dfrac{\dfrac{\dfrac{\begin{array}{c} fs^{*\prime} \\ [T \mid \Gamma \vdash_{\mathsf{F}}^{t_i} P_i]_{i\in[1,\ldots,n]}\end{array}}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A}\ \wedge\mathsf{R}_t^*}{T \mid \Gamma \vdash_{\mathsf{F}} A \vee B}\ \vee\mathsf{R}_1}{T \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \mathsf{F2LI}$$

  A rule wrapped in square brackets, like $[\mathsf{F2LI}]$ above, denotes the application of the rule to the conclusion of each derivation in the list. The list of derivations fs^* is obtained from fs by applying $[\mathsf{F2LI}]$, i.e. $fs = [\mathsf{F2LI}]\, fs^*$, while $fs^{*\prime}$ is a list of derivations whose conclusions are tagged version of those in fs^*, which can be easily constructed from fs^*.

- The list l is invalid. In this case, all elements in fs end with the same left non-invertible rule, so we permute the rule down with $\vee\mathsf{R}_1^{\mathsf{LI}}$ and continue recursively. Here is an example where all derivations in fs end with an application of pass, i.e. $fs = [\mathsf{F2LI}]\ ([\mathsf{pass}]\, fs^*)$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} fs^* \\ [A' \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}\end{array}}{[- \mid A', \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}}\ [\mathsf{pass}]}{[- \mid A', \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}}\ [\mathsf{F2LI}]}{- \mid A', \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}_1^{\mathsf{LI}} \quad = \quad \dfrac{\dfrac{\dfrac{\begin{array}{c} fs^* \\ [A' \mid \Gamma \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}\end{array}}{A' \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}_1^{\mathsf{LI}}}{- \mid A', \Gamma \vdash_{\mathsf{F}} A \vee B}\ \mathsf{pass}}{- \mid A', \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \mathsf{F2LI}$$

□

Finally, a right non-invertible rule in (5.5) is defined as follows: first invert its premises (for $\otimes\mathsf{R}^{\mathsf{RI}}$, only the left premise) using Lemma 5.3.3. Then apply the corresponding generalized rule in Proposition 5.3.4.

We can construct a function $\mathsf{focus} : S \mid \Gamma \vdash A \to S \mid \Gamma \vdash_{\mathsf{RI}} A$ by structural recursion on the input derivation. Each inference rule in $\mathtt{SkNMILLA}_\mathsf{S}$ is sent to the corresponding admissible rule in (5.5). For example, $\mathsf{focus}\ (\vee\mathsf{R}_1\ f) = \vee\mathsf{R}_1^{\mathsf{RI}}\ (\mathsf{focus}\ f)$. Furthermore, it can be shown that $\mathsf{emb}_{\mathsf{RI}}$ and focus are each other's inverses, in the sense made precise by the following theorem.

Theorem 5.3.5. *The functions* $\mathsf{emb}_{\mathsf{RI}}$ *and* focus *define a bijective correspondence between the set of derivations of* $S \mid \Gamma \vdash A$ *quotiented by the equivalence relation* $\doteq$ *and the set of derivations of* $S \mid \Gamma \vdash_{\mathsf{RI}} A$*:*

- *For all* $f, g : S \mid \Gamma \vdash A$*, if* $f \doteq g$ *then* $\mathsf{focus}\ f = \mathsf{focus}\ g$*.*
- *For all* $f : S \mid \Gamma \vdash A$*,* $\mathsf{emb}_{\mathsf{RI}}\ (\mathsf{focus}\ f) \doteq f$*.*
- *For all* $f : S \mid \Gamma \vdash_{\mathsf{RI}} A$*,* $\mathsf{focus}\ (\mathsf{emb}_{\mathsf{RI}}\ f) = f$*.*

Proof. The first bullet is proved by structural induction on the given equality proof $e : f \doteq g$. The other bullets are proved by structural induction on f. See the associated Agda formalization, `https://github.com/cswphilo/code-PhD-thesis/tree/main/skew-mon-conjunction-disjunction`, for details. □

5.4 Skew Exchange

We consider a "skew" commutative extension of the sequent calculus in $\mathtt{SkNMILLA}_\mathsf{S}$ obtained by adding the skew exchange rule that has been introduced in Section 4.1:

$$\frac{S \mid \Gamma, A, B, \Delta \vdash C}{S \mid \Gamma, B, A, \Delta \vdash C}\ \mathsf{ex}$$

Note that exchanging the formula in the stoup, whenever the latter is non-empty, with a formula in context is not allowed. The new rule ex comes with additional permutative equations for the congruence relation $\doteq$ (Figure 5.4).

The resulting sequent calculus enjoys categorical semantics in distributive symmetric skew monoidal categories with binary products, that has a natural isomorphism $s_{A,B,C} : A \otimes (B \otimes C) \to A \otimes (C \otimes B)$ representing a form of "skew symmetry" involving three objects instead of two (see Section 4).

The focused sequent calculus is extended with a new phase $\vdash_{\mathsf{C}}$ (for "context") where the exchange rule can be applied. Rule $\otimes\mathsf{L}$ has to be modified, since we need to give the possibility to move the formula B to a different position in the context.

$$\frac{S \mid \Gamma \,\vdots\, \Delta, A, \Lambda \vdash_{\mathsf{C}} C}{S \mid \Gamma, A \,\vdots\, \Delta, \Lambda \vdash_{\mathsf{C}} C}\ \mathsf{ex} \qquad \frac{S \mid \Gamma \vdash_{\mathsf{RI}} C}{S \mid \quad \vdots\, \Gamma \vdash_{\mathsf{C}} C}\ \mathsf{RI2C} \qquad \frac{A \mid B \,\vdots\, \Gamma \vdash_{\mathsf{C}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\ \otimes\mathsf{L}$$

Root-first proof search now begins in the new phase $\vdash_{\mathsf{C}}$, where formulae in context are permuted. We start with a sequent $S \mid \Gamma \,\vdots\, \quad \vdash_{\mathsf{C}} C$ and end with a sequent $S \mid \quad \vdots\, \Gamma' \vdash_{\mathsf{C}} C$ where Γ' is a permutation of Γ. In the process, the context is divided into two parts $\Gamma \,\vdots\, \Delta$, where the formulae in Γ are ready to be moved while

$$
\dfrac{\dfrac{\overset{f}{A_i \mid \Gamma, A, B, \Delta \vdash C}}{A_i \mid \Gamma, B, A, \Delta \vdash C}\;\mathsf{ex}_{A,B}}{A_1 \wedge A_2 \mid \Gamma, B, A, \Delta \vdash C}\;\wedge\mathsf{L}_i
\quad \doteq \quad
\dfrac{\dfrac{\overset{f}{A_i \mid \Gamma, A, B, \Delta \vdash C}}{A_1 \wedge A_2 \mid \Gamma, A, B, \Delta \vdash C}\;\wedge\mathsf{L}_i}{A_1 \wedge A_2 \mid \Gamma, B, A, \Delta \vdash C}\;\mathsf{ex}_{A,B}
$$

$$
\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta \vdash A'}}{S \mid \Gamma, B, A, \Delta \vdash A'}\;\mathsf{ex}_{A,B} \quad \dfrac{\overset{g}{S \mid \Gamma, A, B, \Delta \vdash B'}}{S \mid \Gamma, B, A, \Delta \vdash B'}\;\mathsf{ex}_{A,B}}{S \mid \Gamma, B, A, \Delta \vdash A' \wedge B'}\;\wedge\mathsf{R}
$$

$$
\doteq \quad
\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta \vdash A'} \quad \overset{g}{S \mid \Gamma, A, B, \Delta \vdash B'}}{S \mid \Gamma, A, B, \Delta \vdash A' \wedge B'}\;\wedge\mathsf{R}}{S \mid \Gamma, B, A, \Delta \vdash A' \wedge B'}\;\mathsf{ex}_{A,B}
$$

$$
\dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A, B, \Delta \vdash C}}{S \mid \Gamma, B, A, \Delta \vdash A'}\;\mathsf{ex}_{A,B} \quad \dfrac{\overset{g}{B' \mid \Gamma, A, B, \Delta \vdash C}}{S \mid \Gamma, B, A, \Delta \vdash B'}\;\mathsf{ex}_{A,B}}{A' \vee B' \mid \Gamma, B, A, \Delta \vdash C}\;\vee\mathsf{L}
$$

$$
\doteq \quad
\dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A, B, \Delta \vdash C} \quad \overset{g}{B' \mid \Gamma, A, B, \Delta \vdash C}}{A' \vee B' \mid \Gamma, A, B, \Delta \vdash C}\;\vee\mathsf{L}}{A' \vee B' \mid \Gamma, B, A, \Delta \vdash C}\;\mathsf{ex}_{A,B}
$$

$$
\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta \vdash A_i}}{S \mid \Gamma, B, A, \Delta \vdash A_i}\;\mathsf{ex}_{A,B}}{S \mid \Gamma, B, A, \Delta \vdash A_1 \vee A_2}\;\vee\mathsf{R}_i
\quad \doteq \quad
\dfrac{\dfrac{\overset{f}{S \mid \Gamma, A, B, \Delta \vdash A_i}}{A_1 \wedge A_2 \mid \Gamma, A, B, \Delta \vdash C}\;\vee\mathsf{R}_i}{S \mid \Gamma, B, A, \Delta \vdash A_1 \vee A_2}\;\mathsf{ex}_{A,B}
$$

Figure 5.4: Additional equations in **SkNMILLA$_\mathsf{S}$** *with exchange*

those in Δ have already been placed in their final position. Once all formulae in Γ have been moved, we switch to phase $\vdash_{\mathsf{RI}}$ with an application of rule RI2C. Note that sequents in phase $\vdash_{\mathsf{C}}$ are not marked by list of tags, since after the application of right non-invertible rules there is no need to further permute formulae in context. Moreover, no new formulae can appear in context via applications of rule $\otimes$L, since the stoup is irreducible at this point.

As already mentioned, rule $\otimes$L has been modified. Its premise is now a sequent in phase $\vdash_{\mathsf{C}}$, which allows a further application of ex for the relocation of the formula B to a different position in the context.

5.5 Linear Implication

Finally, we consider a sequent calculus of `SkNMILL` enriched by $\wedge$ and $\vee$. This is obtained by extending the sequent calculus $\mathtt{SkNMILLA}_{\mathtt{S}}$ with a linear implication $\multimap$ and two introduction rules (alternatively, it can be considered as an extension of $\mathtt{SkNMILL}_{\mathtt{S}}$ with four introduction rules for $\wedge$ and $\vee$):

$$\frac{- \mid \Gamma \vdash A \quad B \mid \Delta \vdash C}{A \multimap B \mid \Gamma, \Delta \vdash C} \multimap\mathsf{L} \qquad \frac{S \mid \Gamma, A \vdash B}{S \mid \Gamma \vdash A \multimap B} \multimap\mathsf{R}$$

The presence of $\multimap$ requires the extension of the congruence relation $\circeq$ with additional permutative equations (Figure 5.5).

The sequent calculus enjoys categorical semantics in distributive skew monoidal categories with binary products, which moreover are endowed with a *closed structure*, i.e. a functor $\multimap$: $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ natural in B (see Definition 2.3.1). There is no need to require left-distributivity, since this can now be proved using the adjunction and the universal property of coproducts.

We now discuss the extension of the focused sequent calculus. This is more complicated than the extension considered in Section 5.4. In order to understand the increased complexity, let us include the two new rules $\multimap$R and $\multimap$L in the "naive" focused sequent calculus in (5.1). The right $\multimap$-rule is invertible, so it belongs to phase $\vdash_{\mathsf{RI}}$, while the left rule is not, so it goes in phase $\vdash_{\mathsf{F}}$.

$$\frac{- \mid \Gamma \vdash_{\mathsf{RI}} A \quad B \mid \Delta \vdash_{\mathsf{LI}} P}{A \multimap B \mid \Gamma, \Delta \vdash_{\mathsf{F}} P} \multimap\mathsf{L} \qquad \frac{S \mid \Gamma, A \vdash_{\mathsf{RI}} B}{S \mid \Gamma \vdash_{\mathsf{RI}} A \multimap B} \multimap\mathsf{R}$$

As we know, this calculus is too permissive, and the inclusion of the above rules increases the non-deterministic choices in proof search even further. As a strategy for taming non-determinism, as before we decide to prioritize left non-invertible rules over right non-invertible ones. So we need to think of all possible situations when a right non-invertible rule must be applied before a left non-invertible one. The presence of $\multimap$ creates two new possibilities: (*i*) $\multimap$L splits the context differently in different premises, or (*ii*) left non-invertible rules manipulate formulae that have been moved to the context by applications of $\multimap$R, i.e. the two types of counterexamples (2.5) and (2.6) mentioned in Section 2.4.1. To understand the former better, let us look at one example.

$$\dfrac{\dfrac{\overset{f}{A_i \mid \Gamma, A \vdash B}}{A_i \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R}}{A_1 \wedge A_2 \mid \Gamma \vdash A \multimap B}\ \wedge\mathsf{L}_i \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A_i \mid \Gamma, A \vdash B}}{A_1 \wedge A_2 \mid \Gamma, A \vdash B}\ \wedge\mathsf{L}_i}{A_1 \wedge A_2 \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R}$$

$$\dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{g}{B \mid \Delta \vdash A_i}}{A \multimap B \mid \Gamma, \Delta \vdash A_i}\ {\multimap}\mathsf{L}}{A \multimap B \mid \Gamma, \Delta \vdash A_1 \vee A_2}\ \vee\mathsf{R}_i \quad \circeq \quad \dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \dfrac{\overset{g}{B \mid \Delta \vdash A_i}}{B \mid \Delta \vdash A_1 \vee A_2}\ \vee\mathsf{R}_i}{A \multimap B \mid \Gamma, \Delta \vdash A_1 \vee A_2}\ {\multimap}\mathsf{L}$$

$$\dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A \vdash B}}{A' \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R} \quad \dfrac{\overset{g}{B' \mid \Gamma, A \vdash B}}{B' \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R}}{A' \vee B' \mid \Gamma \vdash A \multimap B}\ \vee\mathsf{L} \quad \circeq \quad \dfrac{\dfrac{\overset{f}{A' \mid \Gamma, A \vdash B} \quad \overset{g}{B' \mid \Gamma, A \vdash B}}{A' \vee B' \mid \Gamma, A \vdash B}\ \vee\mathsf{L}}{A' \vee B' \mid \Gamma \vdash A \multimap B}\ {\multimap}\mathsf{R}$$

$$\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \dfrac{\overset{g}{B \mid \Delta \vdash A'} \quad \overset{h}{B \mid \Delta \vdash B'}}{B \mid \Delta \vdash A' \wedge B'}\ \wedge\mathsf{R}}{A \multimap B \mid \Gamma, \Delta \vdash A' \wedge B'}\ {\multimap}\mathsf{L}$$

$$\circeq \quad \dfrac{\dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{g}{B \mid \Delta \vdash A'}}{A \multimap B \mid \Gamma, \Delta \vdash A'}\ {\multimap}\mathsf{L} \quad \dfrac{\overset{f}{- \mid \Gamma \vdash A} \quad \overset{h}{B \mid \Delta \vdash B'}}{A \multimap B \mid \Gamma, \Delta \vdash B'}\ {\multimap}\mathsf{L}}{A \multimap B \mid \Gamma, \Delta \vdash A' \wedge B'}\ \wedge\mathsf{R}$$

Figure 5.5: Additional equations in $\mathtt{SkNMILLA}_{\mathsf{S}}$ *with linear implication*

Consider the sequent $\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (\mathsf{I} \wedge \mathsf{I}) \otimes Y$ and the following proof:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I} \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{- \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{pass}}{- \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
\quad
\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I} \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}}\,\multimap\mathsf{L}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{- \mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{- \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I} \mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{- \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{pass}}{- \mid \mathsf{I} \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{F}} \mathsf{I}}\,\multimap\mathsf{L}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I} \wedge \mathsf{I}}\,\wedge\mathsf{R}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{Y \mid\ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}}{Y \mid\ \vdash_{\mathsf{LI}} Y}\,\mathsf{F2LI}}{- \mid Y \vdash_{\mathsf{F}} Y}\,\mathsf{pass}}{- \mid Y \vdash_{\mathsf{RI}} Y}\,\mathsf{sw}
}{\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I}, Y \vdash_{\mathsf{F}} (\mathsf{I} \wedge \mathsf{I}) \otimes Y}\,\otimes\mathsf{R}
\tag{5.6}
$$

Here $\otimes\mathsf{R}$ must be applied first, before $\multimap\mathsf{L}$. In the proofs of the two premises of $\wedge\mathsf{R}$, which prove the same sequent $\mathsf{I} \multimap \mathsf{I} \mid \mathsf{I} \vdash_{\mathsf{RI}} \mathsf{I}$, rule $\multimap\mathsf{L}$ splits the context in different ways: in the left branch the unit I in context is sent to the left premise, while in the right branch it goes to the right premise. If the application of the rule $\multimap\mathsf{L}$ would have split the context in the same way, then we could have applied $\multimap\mathsf{L}$ before $\otimes\mathsf{R}$.

This motivates the addition of two new tags, corresponding to the two situations previously discussed: on top of $\mathbb{P}, \mathbb{C}_1, \mathbb{C}_2$ and $\mathbb{R}$, a tag could either be of the form Γ, for each context Γ, or of the form $\bullet$. The validity condition for list of tags needs to be updated. A list of tags l is now valid if it is non-empty and either (i) $\mathbb{R} \in l$, (ii) both $\mathbb{C}_1 \in l$ and $\mathbb{C}_2 \in l$, (iii) there exist contexts Γ, Γ' such that $\Gamma \in l$, $\Gamma' \in l$ and $\Gamma \neq \Gamma'$, or (iv) $\bullet \in l$. Following the strategy in Section 2.4.2, on top of tag annotations for sequents, we also require tag annotations for formulae in context. There is only one tag $\bullet$ for formulae. The tag on the formula $A^{\bullet}$ means that A has been previously moved to the context by an application of $\multimap\mathsf{R}$ in phase $\vdash^{l}_{\mathsf{RI}}$.

Here are the inference rules of the focused sequent calculus with linear implication:

(right invertible)

$$\frac{S \mid \Gamma \vdash^{l_1?}_{\mathsf{RI}} A \quad S \mid \Gamma \vdash^{l_2?}_{\mathsf{RI}} B}{S \mid \Gamma \vdash^{l_1?,l_2?}_{\mathsf{RI}} A \wedge B}\ \wedge\mathsf{R} \qquad \frac{S \mid \Gamma, A^{\bullet ?} \vdash^{l?}_{\mathsf{RI}} B}{S \mid \Gamma \vdash^{l?}_{\mathsf{RI}} A \multimap B}\ \multimap\mathsf{R}$$

$$\frac{S \mid \Gamma \vdash^{t?}_{\mathsf{LI}} P}{S \mid \Gamma \vdash^{t?}_{\mathsf{RI}} P}\ \mathsf{LI2RI}$$

(left invertible)

$$\frac{- \mid \Gamma \vdash_{\mathsf{LI}} P}{\mathsf{I} \mid \Gamma \vdash_{\mathsf{LI}} P}\ \mathsf{IL} \qquad \frac{A \mid B, \Gamma \vdash_{\mathsf{LI}} P}{A \otimes B \mid \Gamma \vdash_{\mathsf{LI}} P}\ \otimes\mathsf{L}$$

$$\frac{A \mid \Gamma \vdash_{\mathsf{LI}} P \quad B \mid \Gamma \vdash_{\mathsf{LI}} P}{A \vee B \mid \Gamma \vdash_{\mathsf{LI}} P}\ \vee\mathsf{L} \qquad \frac{T \mid \Gamma \vdash^{t?}_{\mathsf{F}} P}{T \mid \Gamma \vdash^{t?}_{\mathsf{LI}} P}\ \mathsf{F2LI}$$

(focusing)

$$\frac{}{X \mid \ \vdash^{\mathbb{R}?}_{\mathsf{F}} X}\ \mathsf{ax} \qquad \frac{}{- \mid \ \vdash^{\mathbb{R}?}_{\mathsf{F}} \mathsf{I}}\ \mathsf{IR}$$

$$\frac{A \mid \Gamma^{\circ} \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash^{\mathbb{C}_1?}_{\mathsf{F}} P}\ \wedge\mathsf{L}_1 \qquad \frac{B \mid \Gamma^{\circ} \vdash_{\mathsf{LI}} P}{A \wedge B \mid \Gamma \vdash^{\mathbb{C}_2?}_{\mathsf{F}} P}\ \wedge\mathsf{L}_2 \tag{5.7}$$

$$\frac{T \mid \Gamma^{\circ} \vdash^{l}_{\mathsf{RI}} A \quad - \mid \Delta^{\circ} \vdash_{\mathsf{RI}} B \quad l \text{ valid}}{T \mid \Gamma, \Delta \vdash^{\mathbb{R}?}_{\mathsf{F}} A \otimes B}\ \otimes\mathsf{R}$$

$$\frac{T \mid \Gamma^{\circ} \vdash^{l}_{\mathsf{RI}} A \quad l \text{ valid}}{T \mid \Gamma \vdash^{\mathbb{R}?}_{\mathsf{F}} A \vee B}\ \vee\mathsf{R}_1 \qquad \frac{T \mid \Gamma^{\circ} \vdash^{l}_{\mathsf{RI}} B \quad l \text{ valid}}{T \mid \Gamma \vdash^{\mathbb{R}?}_{\mathsf{F}} A \vee B}\ \vee\mathsf{R}_2$$

$$\frac{A \mid \Gamma^{\circ} \vdash_{\mathsf{LI}} P \quad \begin{array}{l}\text{if } A^{\bullet ?} = A \\ \text{then } t \text{ does not exist or } t = \mathbb{P} \\ \text{else } t = \bullet\end{array}}{- \mid A^{\bullet ?}, \Gamma \vdash^{t?}_{\mathsf{F}} P}\ \mathsf{pass}$$

$$\frac{- \mid \Gamma, \Delta^{\circ} \vdash_{\mathsf{RI}} A \quad B \mid \Lambda^{\circ} \vdash_{\mathsf{LI}} P \quad \begin{array}{l}\text{if } \Delta^{\bullet} \text{ is empty} \\ \text{then } t \text{ does not exist or } t = \Gamma \\ \text{else } t = \bullet\end{array}}{A \multimap B \mid \Gamma, \Delta^{\bullet}, \Lambda \vdash^{t?}_{\mathsf{F}} P}\ \multimap\mathsf{L}$$

Again P indicates a non-negative formula, which now means that its principal connective is neither $\wedge$ nor $\multimap$. The notation $\Gamma^{\bullet}$ means that all the formulae in Γ are tagged, while Γ° indicates that all the tags on formulae in Γ have been erased. We write $A^{\bullet ?}$ to denote A if the formula appears in an untagged sequent and $A^{\bullet}$ if it appears in a sequent marked with a list of tags l or a single tag t.

Tags of the form $t = \Gamma$ are used to record different splitting of context in applications of $\multimap$L, while tag $t = \bullet$ marks when rule $\multimap$L sends tagged formulae to the left premise and when rule pass moves a tagged formula to the stoup.

Rule $\multimap$R moves a formula A from the succedent to the right end of the context. If its conclusion is marked by a list of tags l, then A is also tagged with $\bullet$.

The side condition in rule $\multimap$L should be read as follows. The tagged context $\Delta^{\bullet}$ starts with the leftmost tagged formula in the sequent. If $\Delta^{\bullet}$ is empty, then the sequent is either untagged (so there is no t) or the tag t is equal to Γ. If $\Delta^{\bullet}$ is non-empty, then $t = \bullet$. In particular, $\Delta^{\bullet}$ contains at least one tagged formula, which must have appeared in context from an application of $\multimap$R. If $\Delta^{\bullet}$ is empty and $t = \Gamma$, no new (meaning: tagged with $\bullet$) formula is moved to the left premise. If $t = \Gamma$ then we are performing proof search inside the premise of a right non-invertible rule and t belongs to some valid list of tags l. List l could be valid because of a different branch in the proof tree where $\multimap$L is also applied but the context has been split differently (so its tag would be Γ' for some $\Gamma \neq \Gamma'$).

Rule pass has a similar side condition to $\multimap\mathsf{L}$. If A does not have a tag, then the sequent is also untagged or the tag t is equal to $\mathbb{P}$. If A has tag $\bullet$, then t must also be $\bullet$. In other words, if $t = \bullet$ then the formula that pass moves to the stoup must also be tagged with $\bullet$, i.e. must have been added to the context by an application of $\multimap\mathsf{R}$.

We can reconstruct the derivation in (5.6) within the focused sequent calculus with tags in (5.7).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-\mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{-\mid\ \vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-\mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{-\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I}\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{-\mid \mathsf{I}\vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{pass}}{-\mid \mathsf{I}\vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{F}}^{[\,]} \mathsf{I}}\,\multimap\mathsf{L}}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{RI}}^{[\,]} \mathsf{I}}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-\mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{-\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I}\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{-\mid \mathsf{I}\vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{pass}}{-\mid \mathsf{I}\vdash_{\mathsf{RI}} \mathsf{I}}\,\mathsf{sw} \quad \dfrac{\dfrac{\dfrac{}{-\mid\ \vdash_{\mathsf{F}} \mathsf{I}}\,\mathsf{IR}}{-\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{F2LI}}{\mathsf{I}\mid\ \vdash_{\mathsf{LI}} \mathsf{I}}\,\mathsf{IL}}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{F}}^{[\mathsf{I}]} \mathsf{I}}\,\multimap\mathsf{L}}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{RI}}^{[\mathsf{I}]} \mathsf{I}}\,\mathsf{sw}}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I}\vdash_{\mathsf{RI}}^{[\,],[\mathsf{I}]} \mathsf{I}\wedge\mathsf{I}}\,\wedge\mathsf{R} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Y\mid\ \vdash_{\mathsf{F}} Y}\,\mathsf{ax}}{Y\mid\ \vdash_{\mathsf{LI}} Y}\,\mathsf{F2LI}}{-\mid Y\vdash_{\mathsf{F}} Y}\,\mathsf{pass}}{-\mid Y\vdash_{\mathsf{RI}} Y}\,\mathsf{sw}}{\dfrac{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I},Y\vdash_{\mathsf{F}} (\mathsf{I}\wedge\mathsf{I})\otimes Y}{\mathsf{I}\multimap\mathsf{I}\mid \mathsf{I},Y\vdash_{\mathsf{RI}} (\mathsf{I}\wedge\mathsf{I})\otimes Y}\,\mathsf{sw}}\,\otimes\mathsf{R}$$

The list of tags $[[\,],[\mathsf{I}]]$ is valid since it contains distinct tags $[\,]$ and $[\mathsf{I}]$, indicating that there exist two applications of $\multimap\mathsf{L}$ that split the context differently.

Proving completeness of the extended focused sequent calculus is more involved than in the absence of implication. Concretely, the complication resides in stating and proving the analog of Proposition 5.3.4. First, define an operation $\mathsf{impconj}(A)$ which produces a list of pairs of lists of formulae and formulae as follows:

$$\begin{array}{lll}
\mathsf{impconj}(A) & = \mathsf{impconj}(A'), \mathsf{impconj}(B') & \text{when } A = A' \wedge B' \\
\mathsf{impconj}(A) & = ((A', \Gamma'_1), B'_1), \ldots, ((A', \Gamma'_n), B'_n) & \text{when } A = A' \multimap B' \text{ and} \\
 & & \mathsf{impconj}(B') \\
 & & = ([(\Gamma'_1, B'_1), \ldots, (\Gamma'_n, B'_n)]) \\
\mathsf{impconj}(A) & = ([\,], A) & \text{otherwise}
\end{array}$$

For example,

$$\begin{array}{l}
\mathsf{impconj}(A \multimap (B \multimap (X \wedge (C \vee D) \wedge (Y \multimap Z)))) \\
\qquad\qquad = [([A,B], X), ([A,B], C \vee D), ([A,B,Y], Z)]
\end{array}$$

Then we prove the adjusted Lemma 5.3.2 for the focused sequent calculus in (5.7):

Lemma 5.5.1. *The following rules*

$$\dfrac{\begin{array}{c} fs \\ [T \mid \Gamma, \Gamma'_i \vdash_{\mathsf{F}}^{t_i} P_i]_{i \in [1,\ldots,n]} \end{array}}{T \mid \Gamma \vdash_{\mathsf{RI}}^{l} A}\,\mathsf{RI}^*_t \qquad \dfrac{\begin{array}{c} fs \\ [S \mid \Gamma, \Gamma'_i \vdash_{\mathsf{LI}} P_i]_{i \in [1,\ldots,n]} \end{array}}{S \mid \Gamma \vdash_{\mathsf{RI}} A}\,\mathsf{RI}^*$$

are admissible, where $\mathsf{impconj}(A) = [(\Gamma'_1, P_1), \ldots, (\Gamma'_n, P_n)]$ *and* $l = [t_1, \ldots, t_n]$.

Proof. The proof follows a similar structure to that of Lemma 5.3.2, with one key difference: the additional case where $A = A' \multimap B'$. We have a collection of derivations $fs' : [T \mid \Gamma, A'^{\bullet}, \Gamma''_i \vdash P_i]_{i\in[1,\ldots,n]}$ and $\mathsf{impconj}(B') = [(\Gamma''_1, P_1), \ldots, (\Gamma''_n, P_n)]$, where each Γ'_i is formed by adding $A'^{\bullet}$ to the front of Γ''_i for all i from 1 to n. We apply $\multimap\mathsf{R}$ at bottom, then proceed recursively:

$$\dfrac{\overset{fs}{[T \mid \Gamma, \Gamma'_i \vdash^{t_i}_{\mathsf{F}} P_i]_{i\in[1,\ldots,n]}}}{T \mid \Gamma \vdash^{l}_{\mathsf{RI}} A' \multimap B'}\ \mathsf{RI}^*_t \quad = \quad \dfrac{\dfrac{\overset{fs'}{[T \mid \Gamma, A'^{\bullet}, \Gamma''_i \vdash^{t_i}_{\mathsf{F}} P_i]_{i\in[1,\ldots,n]}}}{T \mid \Gamma, A'^{\bullet} \vdash^{l}_{\mathsf{RI}} B'}\ \mathsf{RI}^*_t}{T \mid \Gamma \vdash^{l}_{\mathsf{RI}} A' \multimap B'}\ \multimap\mathsf{R}$$

□

The statement of Proposition 5.3.4 for the focused sequent calculus in (5.7) then becomes:

Proposition 5.5.2. *The following rules*

$$\dfrac{\overset{fs}{[S \mid \Gamma, \Gamma'_i \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}}}{S \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}^{\mathsf{LI}}_1 \qquad \dfrac{\overset{fs}{[S \mid \Gamma, \Gamma''_i \vdash_{\mathsf{LI}} Q_i]_{i\in[1,\ldots,m]}}}{S \mid \Gamma \vdash_{\mathsf{LI}} A \vee B}\ \vee\mathsf{R}^{\mathsf{LI}}_2$$

$$\dfrac{\overset{fs}{[S \mid \Gamma, \Gamma'_i \vdash_{\mathsf{LI}} P_i]_{i\in[1,\ldots,n]}} \qquad - \mid \Delta \vdash_{\mathsf{RI}} B'}{S \mid \Gamma, \Delta \vdash_{\mathsf{LI}} A \otimes B'}\ \otimes\mathsf{R}^{\mathsf{LI}}$$

are admissible, where $\mathsf{impconj}(A) = [(\Gamma'_1, P_1), \ldots, (\Gamma'_n, P_n)]$ *and* $\mathsf{impconj}(B) = [(\Gamma''_1, Q_1), \ldots, (\Gamma''_m, Q_m)]$.

Proof. The proof proceeds similarly to that of Proposition 5.3.4. The key case is $f_1 = \mathsf{F2LI}\, f'_1$. In this case, every derivation in fs must also end with $\mathsf{F2LI}$. Our task is to generate a list of tags l that records the structure of derivations in fs. We do this by examining each derivation in fs and assigning tags based on specific rules we encounter.

– For any $f' = \mathsf{pass}\, f''$, we check the passivated formula A. If it is tagged, then we assign $\bullet$ to the derivation, otherwise we assign $\mathbb{P}$.

– For conjunction rules, we add $\mathbb{C}_1$ when we see $\wedge\mathsf{L}_1$, and $\mathbb{C}_2$ when we see $\wedge\mathsf{L}_2$.

– For any $f' = \multimap\mathsf{L}(f''_1, f''_2)$, if there is any tagged formula sent to f''_1, then we assign $\bullet$ to the derivation, otherwise we assign the context sent to f''_1 as the tag.

– We assign $\mathbb{R}$ for the remaining rules.

The remaining procedure is similar to that in the proof of Proposition 5.3.4. □

Chapter 6

Semi-Substructural Logics Beyond Stoup

Having established the core theory of SkNMILL and explored several extensions, we now turn to examining variants where the stoup sequent calculus approach fails to capture the underlying structure. While the calculi discussed so far, including their additive and symmetric extensions, work well with a stoup sequent calculus, some semi-substructural logics do not fit this formalism.

Before we get into an example of such logics, we first distinguish left and right-closed structures. The definitions of variants of skew monoidal closed categories we have introduced in Chapters 2–5 are all *right-closed*. Logically speaking, they all correspond to *right residuation*. The left-closed structure is similar to the right-closed one but adjoint to left tensoring rather than right tensoring.[1] For example, left skew monoidal left-closed categories are defined as follows.

Definition 6.0.1. A left skew monoidal *left-closed* category $\mathbb{C}$ is a category with a unit object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimapinv\ : \mathbb{C} \times \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ forming an adjunction $B \otimes - \dashv - \multimapinv B$ natural in B,

Notice that the position of B at the left-hand side of the adjunction is now at the left argument instead of the right compared to the definition of left skew monoidal right-closed categories.

Left skew monoidal left-closed categories are not very well-behaved. In particular, we currently do not know how to develop a stoup sequent calculus for them.

Left-closedness can be considered redundant in the sense that it reduces to right-closedness for the reversed tensor. However, the price is that we need to introduce right skewness. Given a left skew monoidal structure $(\mathsf{I}, \otimes)$ on a category $\mathbb{C}$, we can define a reverse tensor $\otimes^{\mathsf{rev}}$ that switches the two arguments of a tensor, i.e. $X \otimes^{\mathsf{rev}} Y = Y \otimes X$. The structure $(\mathsf{I}, \otimes^{\mathsf{rev}})$ on $\mathbb{C}$ is not left skew but right skew monoidal. This duality generalizes to closed structures, in the sense that if $(\mathsf{I}, \otimes, \multimapinv)$ is left skew monoidal left-closed, the structure $(\mathsf{I}, \otimes^{\mathsf{rev}}, \multimapinv)$ is right skew monoidal right-closed. In a right skew monoidal category, the structural laws are

[1] We use the words "left-closed" and "right-closed" in the way Lambek [45] did. Many authors, e.g. [36], use them the other way around.

directed in the opposite direction compared to their left skew counterparts. The formal definition of right skew monoidal right-closed categories is the following.

Definition 6.0.2. A *right skew monoidal right-closed category* $\mathbb{C}$ is a category with a unit object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ forming an adjunction $- \otimes B \dashv B \multimap -$ natural in B, and three natural transformations typed $\lambda^{\mathsf{R}}_A : A \to \mathsf{I} \otimes A$, $\rho^{\mathsf{R}}_A : A \otimes \mathsf{I} \to A$ and $\alpha^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$. The equations on morphisms are analogous to those in Definition 1.2.1 but modified to fit the right skew monoidal structure.

Remark 6.0.3. Similar to Remark 2.3.2, natural transformations $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{\mathsf{R}})$ are in bijective correspondence with natural transformations $(j^{\mathsf{R}}, i^{\mathsf{R}}, L^{\mathsf{R}})$, typed as $j^{\mathsf{R}}_{A,B} : \mathbb{C}(\mathsf{I}, A \multimap B) \to \mathbb{C}(A, B)$, $i^{\mathsf{R}}_A : A \to \mathsf{I} \multimap A$ and $L^{\mathsf{R}}_{A,B,C,D} : \mathbb{C}(A, B \multimap (C \multimap D)) \to \int^E \mathbb{C}(A, E \multimap D) \times \mathbb{C}(B, C \multimap E)$. In parts of the next sections, where we only work with thin categories (for any two objects A and B, $\mathbb{C}(A, B)$ is either empty or a singleton set), it is safe to replace $\int^E$ with an existential quantifier.

In the rest of the thesis, we mainly consider right-closed structures, therefore, when we mention “closed”, we always mean right-closed unless specified.

While constructing an axiomatic calculus is straightforward by reversing the axioms λ, ρ, and α in $\mathtt{SkNMILL_A}$, developing a sequent calculus with stoup is challenging. The natural approach would be to create a dual version of $\mathtt{SkNMILL_S}$, where sequents take the form $\Gamma \mid S \vdash A$, swapping the positions of stoup and context. The antecedents would form right-associative trees structured as $(A_n, (\dots, (A_1, A_0)) \dots)$. Nevertheless, $\multimap$ in right skew monoidal categories, by definition, is again a right residuation, implying that $\multimap\mathsf{L}$ and $\multimap\mathsf{R}$ should resemble those in $\mathtt{SkNMILL_S}$. This requirement then necessitates contexts to appear on the right-hand side of the stoup.

To resolve this problem, we adapt the formalism from [52], using sequent calculus with tree as antecedents to characterize these categories.

In subsequent sections, we first introduce skew monoidal bi-closed categories, which could be thought of as categories generated by connecting isolated left $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}})$ and right $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}}, \multimap^{\mathsf{R}})$ skew monoidal closed structures with the isomorphism $\gamma : A \otimes^{\mathsf{L}} B \to B \otimes^{\mathsf{R}} A$ that allows us jumping back and forth between two tensors. Next, we develop equivalent axiomatic and tree calculi to characterize them. We then discuss sound and complete relational models of axiomatic calculi and establishing correspondence theorems between frame conditions and structural laws. In the last section, we extend the analysis to their symmetric extensions.

6.1 Skew Monoidal Bi-closed Categories

Definition 6.1.1. A skew monoidal bi-closed (SkBiC) is a category $\mathbb{C}$ that consists of the following data: a left skew monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}})$ and two functors $\otimes^{\mathsf{R}}$ and $\multimap^{\mathsf{R}}$ forming an adjunction $- \otimes^{\mathsf{R}} B \dashv B \multimap^{\mathsf{R}} -$ natural in B, and a natural isomorphism $\gamma : A \otimes^{\mathsf{L}} B \to B \otimes^{\mathsf{R}} A$.

It follows that $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}}, \multimap^{\mathsf{R}})$ forms a right skew monoidal closed category, where the right skew structural rules are dictated by the left skew ones via γ, i.e. $\lambda^{\mathsf{R}} = \gamma \circ \rho$,

$\rho^{\mathsf{R}} = \gamma^{-1} \circ \lambda$, and $\alpha^{\mathsf{R}} = \gamma \otimes^{\mathsf{R}} C \circ \gamma \circ \alpha \circ \gamma^{-1} \circ A \otimes^{\mathsf{R}} \gamma^{-1}$ diagrammatically:

$$
\begin{array}{ccc}
A & \xrightarrow{\lambda^{\mathsf{R}}} & \mathsf{I} \otimes^{\mathsf{R}} A \\
\| & & \uparrow{\scriptstyle \gamma} \\
A & \xrightarrow[\rho]{} & A \otimes^{\mathsf{L}} \mathsf{I}
\end{array}
\qquad
\begin{array}{ccc}
A \otimes^{\mathsf{R}} \mathsf{I} & \xrightarrow{\rho^{\mathsf{R}}} & A \\
{\scriptstyle \gamma^{-1}}\downarrow & & \| \\
\mathsf{I} \otimes^{\mathsf{L}} A & \xrightarrow[\lambda]{} & A
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C) & \xrightarrow{\alpha^{\mathsf{R}}} & (A \otimes^{\mathsf{R}} B) \otimes^{\mathsf{R}} C \\
{\scriptstyle A \otimes^{\mathsf{R}} \gamma^{-1}}\downarrow & & \uparrow{\scriptstyle \gamma \otimes^{\mathsf{R}} C} \\
A \otimes^{\mathsf{R}} (C \otimes^{\mathsf{L}} B) & & (B \otimes^{\mathsf{L}} A) \otimes^{\mathsf{R}} C \\
{\scriptstyle \gamma^{-1}}\downarrow & & \uparrow{\scriptstyle \gamma} \\
(C \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} A & \xrightarrow[\alpha]{} & C \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} A)
\end{array}
$$

Notice that Definition 6.1.1 combines concepts from skew bi-monoidal and skew bi-closed categories as introduced in [64].

In contrast to the categorical model of associative Lambek calculus, monoidal bi-closed category, we do not explicitly define both left ($\diagdown$) and right residuation ($\diagup$). Instead, we have two right residuations corresponding to different tensor products. However, it is worth noting that a left residuation for one tensor can be derived from the right residuation of the other tensor. Specifically, if we take $A \otimes^{\mathsf{R}} B = B \otimes^{\mathsf{L}} A$, then $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{R}})$ forms a left skew monoidal left-closed category. In here we are more akin to the main reference [64], with the natural isomorphism γ, and selecting a specific tensor, we can simulate both left and right residuations.

In the rest of the section, we usually omit subscripts of natural transformations.

6.2 Calculi for SkBiC

6.2.1 Axiomatic Calculus

By defining new formulae and adding rules in $\mathtt{SkNMILL_A}$, we can have an axiomatic calculus $\mathtt{SkNMBiC_A}$, where formulae (Fma) are inductively generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \otimes^{\mathsf{L}} B \mid A \multimap^{\mathsf{L}} B \mid A \otimes^{\mathsf{R}} B \mid A \multimap^{\mathsf{R}} B$. $\otimes^{\mathsf{L}}$ and $\multimap^{\mathsf{L}}$ ($\otimes^{\mathsf{R}}$ and $\multimap^{\mathsf{R}}$) represent left (right) skew multiplicative conjunction and implication, respectively.

Derivations in $\mathtt{SkNMBiC_A}$ are inductively generated by the following rules:

$$\frac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id} \qquad \frac{A \vdash_{\mathsf{A}} B \quad B \vdash_{\mathsf{A}} C}{A \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

$$\frac{A \vdash_{\mathsf{A}} C \quad B \vdash_{\mathsf{A}} D}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}} \qquad \frac{C \vdash_{\mathsf{A}} A \quad B \vdash_{\mathsf{A}} D}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{A}} C \multimap^{\mathsf{L}} D}\ \multimap^{\mathsf{L}} \qquad \frac{C \vdash_{\mathsf{A}} A \quad B \vdash_{\mathsf{A}} D}{A \multimap^{\mathsf{R}} B \vdash_{\mathsf{A}} C \multimap^{\mathsf{R}} D}\ \multimap^{\mathsf{R}}$$

$$\frac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} A}\ \lambda \qquad \frac{}{A \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \rho \qquad \frac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \alpha$$

$$\frac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{R}} A}\ \gamma \qquad \frac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{L}} A}\ \gamma^{-1}$$

$$\frac{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} C}{\overline{A \vdash_{\mathsf{A}} B \multimap^{\mathsf{L}} C}}\ \pi \qquad \frac{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} C}{\overline{A \vdash_{\mathsf{A}} B \multimap^{\mathsf{R}} C}}\ \pi^{\mathsf{R}}$$

For any $f : A \vdash_{\mathsf{A}} B$ and $g : C \vdash_{\mathsf{A}} D$, we define $f \otimes^{\mathsf{R}} g$ as $\gamma \circ (g \otimes^{\mathsf{L}} f) \circ \gamma^{-1}$. λ^{R}, ρ^{R}, and α^{R} are also derivable.
Similar to the constructions in [67, 66, 65, 69, 63], $\mathtt{SkNMBiC_A}$ generates the free skew monoidal bi-closed category ($\mathsf{FSkMBiC(At)}$) over a set At in the following way:

- Objects of $\mathsf{FSkMBiC(At)}$ are formulae (Fma).

- Morphisms between formulae A and B consist of derivations of sequents $A \vdash B$. These morphisms are considered equivalent when they are related by the congruence relation $\doteq$, which is defined similarly to Figure 2.5 but includes additional equations where γ is an isomorphism:

$$\frac{\dfrac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{L}} A}\ \gamma^{-1} \quad \dfrac{}{B \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} A \otimes^{\mathsf{R}} B}\ \gamma}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} A \otimes^{\mathsf{R}} B}\ \mathsf{comp} \doteq \frac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} A \otimes^{\mathsf{R}} B}\ \mathsf{id}$$

$$\frac{\dfrac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{R}} A}\ \gamma \quad \dfrac{}{B \otimes^{\mathsf{R}} A \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B}\ \gamma^{-1}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B}\ \mathsf{comp} \doteq \frac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B}\ \mathsf{id}$$

Notice that by the definition of $f \otimes^{\mathsf{R}} g$ and γ being an isomorphism, γ and γ^{-1} are natural transformations. For example, $\gamma \circ f \otimes^{\mathsf{L}} g \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \mathsf{id} \doteq \gamma \circ f \otimes^{\mathsf{L}} g \circ \gamma^{-1} \circ \gamma = g \otimes^{\mathsf{R}} f \circ \gamma$. Similarly, naturality of $(\lambda^{\mathsf{R}}, \rho^{\mathsf{R}}, \alpha^{R})$ and the Mac Lane axioms corresponding to them hold as well.

Theorem 6.2.1. *Let $\mathbb{D}$ be a skew monoidal bi-closed category. Given $F_{\mathsf{At}} : \mathsf{At} \to |\mathbb{D}|$ providing evaluation of atomic formulae as objects of $\mathbb{D}$, there exists a unique skew monoidal bi-closed functor $F : \mathsf{FSkMBiC(At)} \to \mathbb{D}$.*

Proof.
Existence. Let $(\mathbb{D}, \mathsf{I}', \otimes'^{\mathsf{L}}, \multimap'^{\mathsf{L}}, \otimes'^{\mathsf{R}}, \multimap'^{\mathsf{R}})$ be a skew monoidal bi-closed category. We define a function $F_0 : \mathsf{FSkMBiC(At)} \to |\mathbb{D}|$ by induction on the input formula:

$$F_0 X = F_{\mathsf{At}} X \qquad F_0 \mathsf{I} = \mathsf{I}'$$
$$F_0(A \otimes^{\mathsf{L}} B) = F_0 A \otimes'^{\mathsf{L}} F_0 B \qquad F_0(A \multimap^{\mathsf{L}} B) = F_0 A \multimap'^{\mathsf{L}} F_0 B$$
$$F_0(A \otimes^{\mathsf{R}} B) = F_0 A \otimes'^{\mathsf{R}} F_0 B \qquad F_0(A \multimap^{\mathsf{R}} B) = F_0 A \multimap'^{\mathsf{R}} F_0 B$$

There exists a function F_1 that maps every morphism in $\mathsf{FSkMBiC(At)}$ by replacing I with I', $\otimes^{\mathsf{L}}$ with $\otimes'^{\mathsf{L}}$, and similarly for all other connectives. This means that given any derivation $f : A \vdash_{\mathsf{A}} B$ in $\mathsf{FSkMBiC(At)}$, F_1 sends f to $F_1 f : \mathbb{D}(F_0 A, F_0 B)$, which is defined by induction on f. The functor $F : \mathsf{FSkMBiC(At)} \to \mathbb{D}$ is then defined by putting F_0 and F_1 together, which forms a skew monoidal bi-closed functor. It is possible to show that F preserves the skew monoidal bi-closed structure, so it is a skew monoidal bi-closed functor.
Uniqueness. Consider another skew monoidal closed functor $F' : \mathsf{FSkMBiC(At)} \to \mathbb{D}$ such that $F'X = F_{\mathsf{At}}X$ for any atom X. We can verify that F' and F agree on every object and morphism in $\mathsf{FSkMBiC(At)}$ by induction on formulae and derivations respectively. □

6.2.2 Tree Sequent Calculus

Trees in $\mathtt{SkNMBiC_T}$ are inductively defined by the grammar $T ::= \mathsf{Fma} \mid - \mid (T,T) \mid (T;T)$. What we have defined are trees with two different ways of linking nodes: through the use of commas and semicolons, corresponding to $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. Contexts and substitution are defined analogously to those of $\mathtt{SkNMILL_T}$. Sequents are in the form $T \vdash_{\mathsf{T}} A$ analogous to those in Section 2.2.2.
Derivations in $\mathtt{SkNMBiC_T}$ are generated recursively by the following rules:

$$\frac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \qquad \frac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{T[-] \vdash_{\mathsf{T}} C}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL}$$

$$\frac{T[A,B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T,U \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{L}} B, U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{L} \qquad \frac{T, A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{L}} B}\ \multimap^{\mathsf{L}}\mathsf{R}$$

$$\frac{T[U_0,(U_1,U_2)] \vdash_{\mathsf{T}} C}{T[(U_0,U_1),U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}^{\mathsf{L}} \qquad \frac{T[U] \vdash_{\mathsf{T}} C}{T[-,U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}^{\mathsf{L}} \qquad \frac{T[U,-] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}^{\mathsf{L}}$$

$$\frac{T[U_0,U_1] \vdash_{\mathsf{T}} C}{\overline{T[U_1;U_0] \vdash_{\mathsf{T}} C}}\ \otimes\mathsf{comm}$$

$$\frac{T[A;B] \vdash_{\mathsf{T}} C}{T[A \otimes^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{R}}\mathsf{L} \qquad \frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\ \otimes^{\mathsf{R}}\mathsf{R}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{R}}\mathsf{L} \qquad \frac{T; A \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B}\ \multimap^{\mathsf{R}}\mathsf{R}$$

$$\frac{T[(U_0;U_1);U_2] \vdash_{\mathsf{T}} C}{T[U_0;(U_1;U_2)] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}^{\mathsf{R}} \qquad \frac{T[U] \vdash_{\mathsf{T}} C}{T[U;-] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}^{\mathsf{R}} \qquad \frac{T[-;U] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR}^{\mathsf{R}}$$

We can think of these rules as originating from two separate calculi: $\mathtt{SkNMILL_T}$ (the red part plus ax, IR, and IL) and another for right skew monoidal closed categories (the blue part plus ax, IR, and IL), linked by $\otimes\mathsf{comm}$, in other words, we can mimic all the blue rules in the style of $\mathtt{SkNMILL_T}$ (only commas appear in antecedents) and conversely, the red rules can be expressed using the blue rules. For example, we can express $\otimes^{\mathsf{R}}\mathsf{L}$, $\otimes^{\mathsf{R}}\mathsf{R}$ and $\multimap^{\mathsf{R}}\mathsf{L}$ in the style of $\texttt{SkNMILL}_{\mathsf{T}}$:

$$\frac{T[A,B] \vdash_{\mathsf{T}} C}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{R}}\mathsf{L}' \quad \mapsto \quad \dfrac{\dfrac{T[A,B] \vdash_{\mathsf{T}} C}{T[B;A] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{comm}}{T[B \otimes^{\mathsf{R}} A] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{R}}\mathsf{L}$$

$$\frac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\ \otimes^{\mathsf{R}}\mathsf{R}' \quad \mapsto \quad \dfrac{\dfrac{T \vdash_{\mathsf{T}} A \quad U \vdash_{\mathsf{T}} B}{T;U \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\ \otimes^{\mathsf{R}}\mathsf{L}}{U,T \vdash_{\mathsf{T}} A \otimes^{\mathsf{R}} B}\ \otimes\mathsf{comm}$$

$$\frac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{R}}\mathsf{L}' \quad \mapsto \quad \dfrac{\dfrac{U \vdash_{\mathsf{T}} A \quad T[B] \vdash_{\mathsf{T}} C}{T[A \multimap^{\mathsf{R}} B; U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{R}}\mathsf{L}}{T[U, A \multimap^{\mathsf{R}} B] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{comm}$$

$$\frac{A,T \vdash_{\mathsf{T}} B}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B}\ \multimap^{\mathsf{R}}\mathsf{R}' \quad \mapsto \quad \dfrac{\dfrac{A,T \vdash_{\mathsf{T}} B}{T;A \vdash_{\mathsf{T}} B}\ \otimes\mathsf{comm}}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{R}} B}\ \multimap^{\mathsf{R}}\mathsf{R}$$

It is also possible to define a categorical model that models the simplified calculus by explicitly defining $A \otimes^{\mathsf{R}} B = B \otimes^{\mathsf{L}} A$. Then in the categorical model, we immediately get two residuations, and we do not have to discuss γ, so that the semantics is simplified. However, we keep the complex form of syntax and semantics to follow the main reference [64] and display the symmetry of the syntax.

Theorem 6.2.2. *Similar to* $\texttt{SkNMILL}_{\mathsf{T}}$*,* cut *is admissible in* $\texttt{SkNMBiC}_{\mathsf{T}}$*.*

$$\frac{U \vdash_{\mathsf{T}} A \quad T[A] \vdash_{\mathsf{T}} C}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

Proof. The proof proceeds similarly to that of Theorem 2.2.8. For the new logical rules in blue, the proofs follow the same pattern as their red counterparts. Since $\otimes\mathsf{comm}$ and all the logical and structural rules in blue are one-premise left rules, we can permute cut upwards. □

The equivalence between $\texttt{SkNMBiC}_{\mathtt{A}}$ and $\texttt{SkNMBiC}_{\mathsf{T}}$ can be proved by induction on the height of derivations with the following admissible rules, definition, and lemmata:

$$\frac{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{A}} C}{T[A,B] \vdash_{\mathsf{A}} C}\ \otimes^{\mathsf{L}}\mathsf{L}^{-1} \qquad \frac{T \vdash_{\mathsf{A}} A \multimap^{\mathsf{L}} B}{T,A \vdash_{\mathsf{A}} B}\ \multimap^{\mathsf{L}}\mathsf{R}^{-1}$$

$$\frac{T[A \otimes^{\mathsf{R}} B] \vdash_{\mathsf{A}} C}{T[A;B] \vdash_{\mathsf{A}} C}\ \otimes^{\mathsf{R}}\mathsf{L}^{-1} \qquad \frac{T \vdash_{\mathsf{A}} A \multimap^{\mathsf{R}} B}{T;A \vdash_{\mathsf{A}} B}\ \multimap^{\mathsf{R}}\mathsf{R}^{-1}$$

Definition 6.2.3. For any tree T, $T^{\#}$ is the formula obtained from T by replacing commas with $\otimes^{\mathsf{L}}$ and semicolons with $\otimes^{\mathsf{R}}$, and $-$ with I, respectively.

Lemma 6.2.4. *For any context* $T[\cdot]$ *and tree* U*,* $T[U]^{\#} = T[U^{\#}]^{\#}$*.*

Proof. The proof proceeds similarly to the one of Lemma 2.2.10. □

In the remainder of the chapter, we will refer to uses of this lemma by double lines. For example, given a derivation $f : T'[A]^{\#} \otimes^{\mathsf{L}} T''^{\#} \vdash_{\mathsf{A}} C$, we can rewrite the antecedent of the conclusion sequent as

$$\dfrac{\dfrac{f}{T'[A]^{\#} \otimes^{\mathsf{L}} T''^{\#} \vdash_{\mathsf{A}} C}}{(T'[A], T'')^{\#} \vdash_{\mathsf{A}} C}$$

Lemma 6.2.5. *Given a context $T[\cdot]$ and a derivation $f : A \vdash_{\mathsf{A}} B$, the following rule is admissible:*

$$\dfrac{\dfrac{f}{A \vdash_{\mathsf{A}} B}}{T[A]^{\#} \vdash_{\mathsf{A}} T[B]^{\#}}\ T[f]^{\#}$$

Proof. The proof proceeds by induction on the structure of $T[\cdot]$.
If $T[\cdot] = [\cdot]$, then we have $T[A]^{\#} = A$ and $T[B]^{\#} = B$, and f is the desired derivation.
If $T[\cdot] = (T'[\cdot]; T'')$, then we construct the desired derivation as follows:

$$\dfrac{\dfrac{\dfrac{f}{T'[A]^{\#} \vdash_{\mathsf{A}} T'[B]^{\#}} \quad \dfrac{}{T''^{\#} \vdash_{\mathsf{A}} T''^{\#}}\ \mathsf{id}}{T'[A]^{\#} \otimes^{\mathsf{R}} T''^{\#} \vdash_{\mathsf{A}} T'[B]^{\#} \otimes^{\mathsf{R}} T''^{\#}}\ \otimes^{\mathsf{R}}}{(T'[A]; T'')^{\#} \vdash_{\mathsf{A}} (T'[B]; T'')^{\#}}$$

The case $T[\cdot] = (T'; T''[\cdot])$ is symmetric, while other cases are covered in the proof of Lemma 2.2.11. ☐

Theorem 6.2.6. $\mathtt{SkNMBiC_T}$ *is equivalent to* $\mathtt{SkNMBiC_A}$*, meaning that the following two statements are true:*

1. *For any derivation $f : A \vdash_{\mathsf{A}} C$, there exists a derivation $\mathsf{A2T}f : A \vdash_{\mathsf{T}} C$.*
2. *For any derivation $f : T \vdash_{\mathsf{T}} C$, there exists a derivation $\mathsf{T2A}f : T^{\#} \vdash_{\mathsf{A}} C$.*

Proof. We first construct A2T by structural induction on the derivation f.
Case $f = \mathsf{id}$.

$$\dfrac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id} \quad \mapsto \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}$$

Case $f = \mathsf{comp}(f', f'')$.

$$\dfrac{\dfrac{f'}{A \vdash_{\mathsf{A}} B} \quad \dfrac{f''}{B \vdash_{\mathsf{A}} C}}{A \vdash_{\mathsf{A}} C}\ \mathsf{comp} \quad \mapsto \quad \dfrac{\dfrac{\mathsf{A2T}f'}{A \vdash_{\mathsf{T}} B} \quad \dfrac{\mathsf{A2T}f''}{B \vdash_{\mathsf{T}} C}}{A \vdash_{\mathsf{T}} C}\ \mathsf{cut}$$

Case $f = \otimes^{\mathsf{L}}(f', f'')$.

$$\dfrac{\dfrac{f'}{A \vdash_{\mathsf{A}} C} \quad \dfrac{f''}{B \vdash_{\mathsf{A}} D}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\mathsf{A2T}f'}{A \vdash_{\mathsf{T}} C} \quad \dfrac{\mathsf{A2T}f''}{B \vdash_{\mathsf{T}} D}}{A, B \vdash_{\mathsf{T}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}}\mathsf{R}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C \otimes^{\mathsf{L}} D}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = {\multimap^{\mathsf{L}}} (f', f'')$.

$$\frac{\begin{array}{c} f' \\ C \vdash_{\mathsf{A}} A \end{array} \quad \begin{array}{c} f'' \\ B \vdash_{\mathsf{A}} D \end{array}}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{A}} C \multimap^{\mathsf{L}} D}\ {\multimap^{\mathsf{L}}} \quad \mapsto \quad \dfrac{\dfrac{\begin{array}{c} \mathsf{A2T}\, f' \\ C \vdash_{\mathsf{T}} A \end{array} \quad \begin{array}{c} \mathsf{A2T}\, f'' \\ B \vdash_{\mathsf{T}} D \end{array}}{A \multimap^{\mathsf{L}} B, C \vdash_{\mathsf{T}} D}\ {\multimap^{\mathsf{L}}}\mathsf{L}}{A \multimap^{\mathsf{L}} B \vdash_{\mathsf{T}} C \multimap^{\mathsf{L}} D}\ {\multimap^{\mathsf{L}}}\mathsf{R}$$

Case $f = \lambda$.

$$\frac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} A}\ \lambda \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{-, A \vdash_{\mathsf{T}} A}\ \mathsf{unitL}^{\mathsf{L}}}{\mathsf{I}, A \vdash_{\mathsf{T}} A}\ \mathsf{IL}}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{T}} A}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \rho$.

$$\frac{}{A \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \rho \quad \mapsto \quad \dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR}}{A, - \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \otimes^{\mathsf{R}}\mathsf{R}}{A \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} \mathsf{I}}\ \mathsf{unitR}^{\mathsf{L}}$$

Case $f = \alpha$.

$$\frac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \alpha \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax} \quad \dfrac{}{C \vdash_{\mathsf{T}} C}\ \mathsf{ax}}{B, C \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} C}\ \otimes^{\mathsf{L}}\mathsf{R}}{A, (B, C) \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \otimes^{\mathsf{L}}\mathsf{R}}{(A, B), C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \mathsf{assoc}^{\mathsf{L}}}{(A \otimes^{\mathsf{L}} B), C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \otimes^{\mathsf{L}}\mathsf{L}}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \gamma$.

$$\frac{}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{R}} A}\ \gamma \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax} \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{B; A \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\ \otimes^{\mathsf{R}}\mathsf{R}}{A, B \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\ \otimes\mathsf{comm}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} B \otimes^{\mathsf{R}} A}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \gamma^{-1}$.

$$\frac{}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{L}} A}\ \gamma^{-1} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax} \quad \dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax}}{B, A \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\ \otimes^{\mathsf{L}}\mathsf{R}}{A; B \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\ \otimes\mathsf{comm}^{-1}}{A \otimes^{\mathsf{R}} B \vdash_{\mathsf{T}} B \otimes^{\mathsf{L}} A}\ \otimes^{\mathsf{R}}\mathsf{L}$$

Case $f = \pi\ f'$.

$$\dfrac{\dfrac{f'}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} C}}{A \vdash_{\mathsf{A}} B \multimap^{\mathsf{L}} C}\ \pi \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\mathsf{A2T} f'}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C}}{A, B \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L}^{-1}}{A \vdash_{\mathsf{T}} B \multimap^{\mathsf{L}} C}\ \multimap^{\mathsf{L}}\mathsf{R}$$

Case $f = \pi^{-1}\ f'$.

$$\dfrac{\dfrac{f'}{A \vdash_{\mathsf{A}} B \multimap^{\mathsf{L}} C}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} C}\ \pi^{-1} \quad \mapsto \quad \dfrac{\dfrac{\dfrac{\mathsf{A2T} f'}{A \vdash_{\mathsf{T}} B \multimap^{\mathsf{L}} C}}{A, B \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{R}^{-1}}{A \otimes^{\mathsf{L}} B \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Other cases for $\multimap^{\mathsf{R}}$ and π^{R} are similar.
We construct $\mathsf{T2A}$ by structural induction on f as well.
Case $f = \mathsf{ax}$.

$$\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \mapsto \quad \dfrac{}{A \vdash_{\mathsf{A}} A}\ \mathsf{id}$$

Case $f = \mathsf{IR}$.

$$\dfrac{}{- \vdash_{\mathsf{T}} \mathsf{I}}\ \mathsf{IR} \quad \mapsto \quad \dfrac{}{\mathsf{I} \vdash_{\mathsf{A}} \mathsf{I}}\ \mathsf{id}$$

Case $f = \mathsf{IL}\ f'$.

$$\dfrac{\dfrac{f'}{T[-] \vdash_{\mathsf{T}} C}}{T[\mathsf{I}] \vdash_{\mathsf{T}} C}\ \mathsf{IL} \quad \mapsto \quad \dfrac{\dfrac{\mathsf{T2A} f'}{T[-]^{\#} \vdash_{\mathsf{A}} C}}{T[\mathsf{I}]^{\#} \vdash_{\mathsf{A}} C}$$

Case $f = \otimes\mathsf{comm}\ f'$

$$\dfrac{\dfrac{f'}{T[U_0, U_1] \vdash_{\mathsf{T}} C}}{T[U_1; U_0] \vdash_{\mathsf{T}} C}\ \otimes\mathsf{comm}$$

$$\mapsto \dfrac{\dfrac{\dfrac{\dfrac{}{U_1^{\#} \otimes^{\mathsf{R}} U_0^{\#} \vdash_{\mathsf{A}} U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}}\ \gamma^{-1}}{T[U_1^{\#} \otimes^{\mathsf{R}} U_0^{\#}]^{\#} \vdash_{\mathsf{A}} T[U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}]^{\#}}\ T[\gamma^{-1}]^{\#}}{T[U_1; U_0]^{\#} \vdash_{\mathsf{A}} T[U_0, U_1]^{\#}} \qquad \dfrac{\mathsf{T2Af'}}{T[U_0, U_1]^{\#} \vdash_{\mathsf{A}} C}}{T[U_1; U_0]^{\#} \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

Case $f = \otimes\mathsf{L}\ f'$

$$\dfrac{\dfrac{f'}{T[A, B] \vdash_{\mathsf{T}} C}}{T[A \otimes^{\mathsf{L}} B] \vdash_{\mathsf{T}} C}\ \otimes^{\mathsf{L}}\mathsf{L} \quad \mapsto \quad \dfrac{\dfrac{\mathsf{T2A} f'}{T[A, B]^{\#} \vdash_{\mathsf{A}} C}}{T[A \otimes^{\mathsf{L}} B]^{\#} \vdash_{\mathsf{A}} C}$$

Case $f = \otimes^{\mathsf{L}}\mathsf{R}(f', f'')$.

$$\dfrac{\overset{f'}{T \vdash_{\mathsf{T}} A} \quad \overset{f''}{U \vdash_{\mathsf{T}} B}}{T, U \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R} \quad \mapsto \quad \dfrac{\dfrac{\overset{\mathsf{T2A}f'}{T^\# \vdash_{\mathsf{A}} A} \quad \overset{\mathsf{T2A}f''}{U^\# \vdash_{\mathsf{A}} B}}{T^\# \otimes^{\mathsf{L}} U^\# \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}}{(T,U)^\# \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B}$$

Case $f = \multimap^{\mathsf{L}}\mathsf{L}$.

$$\dfrac{\overset{f'}{U \vdash_{\mathsf{T}} A} \quad \overset{f''}{T[B] \vdash_{\mathsf{T}} C}}{T[A \multimap^{\mathsf{L}} B, U] \vdash_{\mathsf{T}} C}\ \multimap^{\mathsf{L}}\mathsf{L}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{g}{(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^\# \vdash_{\mathsf{A}} (A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A}}{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^\#]^\# \vdash_{\mathsf{A}} T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A]^\#}\ T[g]^\# \quad \dfrac{\overset{h}{(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} B}}{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} A]^\# \vdash_{\mathsf{A}} T[B]^\#}\ T[h]^\#}{T[(A \multimap^{\mathsf{L}} B) \otimes^{\mathsf{L}} U^\#]^\# \vdash_{\mathsf{A}} T[B]^\#}\ \mathsf{comp}}{T[(A \multimap^{\mathsf{L}} B), U]^\# \vdash_{\mathsf{A}} T[B]^\#} \quad \overset{\mathsf{T2A}f''}{T[B]^\# \vdash_{\mathsf{A}} C}}{T[(A \multimap^{\mathsf{L}} B), U^\#]^\# \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

where $g = \otimes^{\mathsf{L}}(\mathsf{id}, \mathsf{T2A}f')$ and $h = \pi^{-1}(\mathsf{id})$.

Case $f = \multimap^{\mathsf{L}}\mathsf{R}\ f'$

$$\dfrac{\overset{f'}{T, A \vdash_{\mathsf{T}} B}}{T \vdash_{\mathsf{T}} A \multimap^{\mathsf{L}} B}\ \multimap^{\mathsf{L}}\mathsf{R} \quad \mapsto \quad \dfrac{\overset{\mathsf{T2A}f'}{T^\# \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} B}}{T^\# \vdash_{\mathsf{A}} A \multimap^{\mathsf{L}} B}\ \pi$$

Case $f = \mathsf{assoc}^{\mathsf{L}}\ f'$

$$\dfrac{\overset{f'}{T[U_0, (U_1, U_2)] \vdash_{\mathsf{T}} C}}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}\ \mathsf{assoc}^{\mathsf{L}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(U_0^\# \otimes^{\mathsf{L}} U_1^\#) \otimes^{\mathsf{L}} U_2^\# \vdash_{\mathsf{A}} U_0^\# \otimes^{\mathsf{L}} (U_1^\# \otimes^{\mathsf{L}} U_2^\#)}\ \alpha}{T[(U_0^\# \otimes^{\mathsf{L}} U_1^\#) \otimes^{\mathsf{L}} U_2^\#]^\# \vdash_{\mathsf{A}} T[U_0^\# \otimes^{\mathsf{L}} (U_1^\# \otimes^{\mathsf{L}} U_2^\#)]^\#}\ T[\alpha]^\#}{T[(U_0, U_1), U_2]^\# \vdash_{\mathsf{A}} T[U_0, (U_1, U_2)]^\#} \quad \overset{\mathsf{T2A}f'}{T[U_0, (U_1, U_2)]^\# \vdash_{\mathsf{A}} C}}{T[(U_0, U_1), U_2]^\# \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

Case $f = \mathsf{unitL}^{\mathsf{L}}\ f'$

$$\dfrac{\overset{f'}{T[U] \vdash_{\mathsf{T}} C}}{T[-, U] \vdash_{\mathsf{T}} C}\ \mathsf{unitL}^{\mathsf{L}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} U^\# \vdash_{\mathsf{A}} U^\#}\ \lambda}{T[\mathsf{I} \otimes^{\mathsf{L}} U^\#]^\# \vdash_{\mathsf{A}} T[U^\#]^\#}\ T[\lambda]^\#}{T[-, U]^\# \vdash_{\mathsf{A}} T[U]^\#} \quad \overset{\mathsf{T2A}f'}{T[U]^\# \vdash_{\mathsf{T}} C}}{T[-, U]^\# \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

Case $f = \mathsf{unitR^L}\ f'$

$$\dfrac{\begin{array}{c} f' \\ T[U,-] \vdash_{\mathsf{T}} C \end{array}}{T[U] \vdash_{\mathsf{T}} C}\ \mathsf{unitR^L}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{U^{\#} \vdash_{\mathsf{A}} U^{\#} \otimes^{\mathsf{L}} \mathsf{I}}\ \rho}{T[U^{\#}]^{\#} \vdash_{\mathsf{A}} T[U^{\#} \otimes^{\mathsf{L}} \mathsf{I}]^{\#}}\ T[\rho]^{\#}}{T[U]^{\#} \vdash_{\mathsf{A}} T[U,-]^{\#}} \qquad \begin{array}{c} \mathsf{T2A}f' \\ T[U,-]^{\#} \vdash_{\mathsf{T}} C \end{array}}{T[U]^{\#} \vdash_{\mathsf{A}} C}\ \mathsf{comp}$$

Other cases for right skew rules are similar. □

6.3 Relational Semantics of $\mathtt{SkNMBiC_A}$

In this section, we present the relational semantics of $\mathtt{SkNMBiC_A}$, which is characterized modularly, allowing us to construct models for semi-substructural logics step by step by incorporating additional structural conditions into the frame. Moreover, by adapting the definition from relational monoid [56], we extend the scope of ternary relational semantics to include frame conditions on multiplicative unit. The modularity of ternary frame semantics allows us to provide a proof for the poset version of the main theorems concerning the interdefinability of a series of skew structured categories discussed in [64].

A preordered ternary frame with a special subset is $\langle W, \leq, \mathbb{I}, \mathbb{L} \rangle$, where W is a set, $\leq$ is a preorder relation on W, $\mathbb{I}$ is a downwards closed subset of W, and $\mathbb{L}$ is an arbitrary ternary relation on W, where $\mathbb{L}$ is upwards closed in the first two arguments and downwards closed in the last argument with respect to $\leq$. For example, given $\mathbb{L}abc$, if we have $a \leq a'$, $b \leq b'$, and $c' \leq c$, then $\mathbb{L}a'b'c'$.

Definition 6.3.1. We list properties of ternary relations which we will focus on.

Left Skew Associativity (LSA)	$\forall a, b, c, d, x \in W, \mathbb{L}abx\ \&\ \mathbb{L}xcd$ $\longrightarrow \exists y \in W$ such that $\mathbb{L}bcy\ \&\ \mathbb{L}ayd$.
Left Skew Left Unitality (LSLU)	$\forall a, b \in W, e \in \mathbb{I}, \mathbb{L}eab \longrightarrow b \leq a$.
Left Skew Right Unitality (LSRU)	$\forall a \in W, \exists e \in \mathbb{I}$ such that $\mathbb{L}aea$.
Right Skew Associativity (RSA)	$\forall a, b, c, d, x \in W, \mathbb{L}bcx\ \&\ \mathbb{L}axd$ $\longrightarrow \exists y \in W$ such that $\mathbb{L}aby\ \&\ \mathbb{L}ycd$.
Right Skew Left Unitality (RSLU)	$\forall a \in W, \exists e \in \mathbb{I}$ such that $\mathbb{L}eaa$.
Right Skew Right Unitality (RSRU)	$\forall a, b \in W, e \in \mathbb{I}, \mathbb{L}aeb \longrightarrow b \leq a$.

Given another ternary relation $\mathbb{R}$, we define

$$\mathbb{LR}\text{-reverse} \quad \forall a, b, c \in W, \mathbb{L}abc \longleftrightarrow \mathbb{R}bac.$$

The associativity and unitality conditions are adapted from the theory of relational monoids [56] and relational semantics for Lambek calculus [24].

A $\mathtt{SkNMBiC_A}$ frame is a quintuple $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$, where $\mathbb{LR}$-reverse is satisfied, $\mathbb{L}$ satisfies LSA, LSLU, LSRU, and $\mathbb{R}$ automatically satisfies RSA, RSLU, RSRU because of $\mathbb{LR}$-reverse.

Unlike studies in NL e.g. [24, 50, 52], where two associativity conditions simultaneously hold for a relation or not, we explore two relations where one satisfies LSA and the other satisfies RSA. Another distinction from the existing studies on semantics for NL with unit [17] (or non-commutative linear logic [1]) is that while W is commonly assumed to be a unital magma (or monoid in the case of linear logic), here, we should consider that the unit behaves differently for different relations.

We denote the set of downwards closed subsets of W as $\mathcal{P}_{\downarrow}(W)$.

Definition 6.3.2. A function $v : \mathsf{Fma} \to \mathcal{P}_{\downarrow}(W)$ on a $\mathtt{SkNMBiC_A}$ frame is a valuation if it satisfies:

$$\begin{array}{ll} v(\mathsf{I}) & = \mathbb{I} \\ v(A \otimes^{\mathsf{L}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{L}abc\} \\ v(A \multimap^{\mathsf{L}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{L}cab \Rightarrow b \in v(B)\} \\ v(A \otimes^{\mathsf{R}} B) & = \{c : \exists a \in v(A), b \in v(B),\ \mathbb{R}abc\} \\ v(A \multimap^{\mathsf{R}} B) & = \{c : \forall a \in v(A), b \in W,\ \mathbb{R}cab \Rightarrow b \in v(B)\} \end{array}$$

We define a $\mathtt{SkNMBiC_A}$ model to be a $\mathtt{SkNMBiC_A}$ frame with a valuation function, i.e. $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$. A sequent $A \vdash_{\mathsf{A}} B$ is valid in a model $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ if $v(A) \subseteq v(B)$ and is valid in a frame if for any v for that frame, $v(A) \subseteq v(B)$.

Theorem 6.3.3 (Soundness)**.** *If a sequent $A \vdash_{\mathsf{A}} B$ is provable in* $\mathtt{SkNMBiC_A}$ *then it is valid in any* $\mathtt{SkNMBiC_A}$ *model.*

Proof. The proof is adapted from [24, 52], where the cases of α and α^{R} have been discussed. Therefore, we only elaborate on new cases arising in $\mathtt{SkNMBiC_A}$.

- If the derivation is the axiom $\lambda : \mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} A$, then for any $\mathtt{SkNMBiC_A}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(\mathsf{I} \otimes^{\mathsf{L}} A)$, there exist $e \in \mathbb{I}$, $a' \in v(A)$, and $\mathbb{L}ea'a$. By LSLU, we know that $a \leq a'$, and then $a \in v(A)$.

- If the derivation is the axiom $\rho : A \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} \mathsf{I}$, then for any $\mathtt{SkNMBiC_A}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $a \in v(A)$, by LSRU, there exists $e \in \mathbb{I}$ such that $\mathbb{L}aea$, which means that $a \in v(A \otimes^{\mathsf{L}} \mathsf{I})$.

- If the derivation is the axiom $\gamma : A \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} B \otimes^{\mathsf{R}} A$, then for any $\mathtt{SkNMBiC_A}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $c \in v(A \otimes^{\mathsf{L}} B)$, there exist $a \in v(A)$ and $b \in v(B)$ such that $\mathbb{L}abc$. By $\mathbb{L}\mathbb{R}$-reverse, we have $\mathbb{R}bac$, therefore $c \in v(B \otimes^{\mathsf{R}} A)$.

- The case of γ^{-1} is similar. □

Definition 6.3.4. The canonical model of $\mathtt{SkNMBiC_A}$ is $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ where

- $W = \mathsf{Fma}$ and $A \leq B$ if and only if $A \vdash_{\mathsf{A}} B$,
- $\mathbb{I} = v(\mathsf{I})$,
- $\mathbb{L}ABC$ if and only if $C \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B$,
- $\mathbb{R}ABC$ if and only if $C \vdash_{\mathsf{A}} A \otimes^{\mathsf{R}} B$, and
- $v(A) = \{B \mid B \vdash_{\mathsf{A}} A$ is provable in $\mathtt{SkNMBiC_A}\}$.

Lemma 6.3.5. *The canonical model is a* $\mathtt{SkNMBiC_A}$ *model.*

Proof.

– The set $(\mathsf{Fma}, \vdash_{\mathsf{A}})$ is a preorder because of the rules id and comp, and the set $\mathbb{I}$ is downwards closed because of comp. The relations $\mathbb{L}$ and $\mathbb{R}$ are downwards closed in their last argument because of the rule comp. They are upwards closed in their first two arguments due to the rules $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$, respectively. These facts ensure that $\langle \mathsf{Fma}, \vdash_{\mathsf{A}}, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$ is a ternary frame.

– We show two cases (LSRU and LSRU) of the proof that $\mathbb{L}, \mathbb{R}$ satisfy their corresponding conditions, while other cases are similar.

(LSLU) Given any two formulae A and B, and $J \in \mathbb{I}$ with $\mathbb{L}JAB$, we have $J \vdash_{\mathsf{A}} \mathsf{I}$, and $B \vdash_{\mathsf{A}} J \otimes^{\mathsf{L}} A$, then we can construct $B \vdash_{\mathsf{A}} A$ as follows:

$$\dfrac{\dfrac{B \vdash_{\mathsf{A}} J \otimes^{\mathsf{L}} A \quad \dfrac{J \vdash_{\mathsf{A}} \mathsf{I} \quad \dfrac{}{A \vdash_{\mathsf{A}} A}\,\mathsf{id}}{J \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\otimes^{\mathsf{L}}}{B \vdash_{\mathsf{A}} \mathsf{I} \otimes^{\mathsf{L}} A}\,\mathsf{comp} \quad \dfrac{}{\mathsf{I} \otimes^{\mathsf{L}} A \vdash_{\mathsf{A}} A}\,\lambda}{B \vdash_{\mathsf{A}} A}\,\mathsf{comp}$$

(LSRU) By the axiom ρ, for any formula A, we have $A \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} \mathsf{I}$, i.e. $\mathbb{L}AIA$.

– The valuation v is downwards closed because of the rule comp. The other conditions on connectives are satisfied by definition.

Therefore, $\langle \mathsf{Fma}, \vdash_{\mathsf{A}}, \mathbb{I}, \mathbb{L}, \mathbb{R}, v \rangle$ is a $\mathtt{SkNMBiC_A}$ model. □

Theorem 6.3.6 (Completeness). *If $A \vdash_{\mathsf{A}} B$ is valid in any $\mathtt{SkNMBiC_A}$ model, then it is provable in $\mathtt{SkNMBiC_A}$.*

Proof. If $A \vdash_{\mathsf{A}} B$ is valid in any $\mathtt{SkNMBiC_A}$ model, then it is valid in the canonical model, i.e. $v(A) \subseteq v(B)$ in the canonical model. From $A \vdash_{\mathsf{A}} A$, by definition of v, we have $A \in v(A)$, and because $v(A) \subseteq v(B)$, we know that $A \in v(B)$, therefore $A \vdash_{\mathsf{A}} B$. □

We show a correspondence between frame conditions and the validity of structural laws in frames.

Theorem 6.3.7. *For any ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R} \rangle$,*

	$\mathbb{LR}$-reverse holds		$\longleftrightarrow$		*γ and γ^{-1} valid*	
$\alpha^{(\mathsf{R})}$ valid	$\longleftrightarrow$		*LSA (RSA) holds*		$\longleftrightarrow$	*$L^{(\mathsf{R})}$ valid*
$\lambda^{(\mathsf{R})}$ valid	$\longleftrightarrow$		*LSLU (RSLU) holds*		$\longleftrightarrow$	*$j^{(\mathsf{R})}$ valid*
$\rho^{(\mathsf{R})}$ valid	$\longleftrightarrow$		*LSRU (RSRU) holds*		$\longleftrightarrow$	*$i^{(\mathsf{R})}$ valid*

Proof. The first case is that $\mathbb{LR}$-reverse holds if and only if γ and γ^{-1} are valid, i.e. $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$.

($\longrightarrow$) For any $x \in v(A \otimes^{\mathsf{L}} B) \subseteq W$, there exists $a \in v(A), b \in v(B)$ and $\mathbb{L}abx$. By $\mathbb{LR}$-reverse, we have $\mathbb{R}bax$ meaning that $x \in v(B \otimes^{\mathsf{R}} A)$. The other way around is similar.

($\longleftarrow$) Suppose that for any v, A, B, we have $v(A \otimes^{\mathsf{L}} B) = v(B \otimes^{\mathsf{R}} A)$. Consider any $a, b, x \in W$ such that $\mathbb{L}abx$. We take $v(A) = a{\downarrow}$ and $v(B) = b{\downarrow}$ for some $A, B \in \mathsf{At}$. By the definition of v and assumption, x belongs to $v(A \otimes^{\mathsf{L}} B)$ which is equal to $v(B \otimes^{\mathsf{R}} A)$, therefore $\mathbb{R}bax$. The other direction is similar.

λ : LSLU holds if and only if λ is valid.

($\longrightarrow$) This is similar to case of λ in the proof of Theorem 6.3.3.

($\longleftarrow$) Suppose that λ is valid, i.e. for any A and v, we have $v(\mathsf{I} \otimes^{\mathsf{L}} A) \subseteq v(A)$. Consider any $a, b \in W$, $e \in \mathbb{I}$ such that $\mathbb{L}eab$. We take $v(A) = a{\downarrow}$ for some $A \in \mathsf{At}$. By $\mathbb{L}eab$ and the assumption, we know that $b \in v(A)$, which means that $b \leq a$.

ρ : LSRU holds if and only if ρ is valid.

($\longrightarrow$) This is similar to case of ρ in the proof of Theorem 6.3.3.

($\longleftarrow$) Suppose ρ is valid, i.e. for any A and v, $v(A) \subseteq v(A \otimes^{\mathsf{L}} \mathsf{I})$. Consider any $a \in W$. We take $v(A) = a{\downarrow}$ for some $A \in \mathsf{At}$. By the assumption, there exist $a' \in v(A)$ and $e \in \mathbb{I}$ such that $\mathbb{L}a'ea$. Because $\mathbb{L}$ is upwards closed in its first argument, we know that $\mathbb{L}aea$.

α : LSA holds if and only if α is valid.

($\longrightarrow$) For any $s \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$, there exists $a \in v(A), b \in v(B), x \in v(A \otimes^{\mathsf{L}} B), c \in v(C), \mathbb{L}abx$, and $\mathbb{L}xcs$. By LSA, there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ays$, then by definition of v, $y \in v(B \otimes^{\mathsf{L}} C)$ and $s \in v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$.

($\longleftarrow$) Suppose that α is valid, i.e. for any A, B, C, v, we have $v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C) \subseteq v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$. Consider any $a, b, x, c, d \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(A \otimes^{\mathsf{L}} B)$ and $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$. By the assumption, d belongs to $v(A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C))$ as well, which means that there exist $a', b', y, c' \in W$ such that $\mathbb{L}b'c'y$ and $\mathbb{L}a'yd$. Because $\mathbb{L}$ is upwards closed in its first and second arguments, we have $\mathbb{L}bcy$ and $\mathbb{L}ayd$ as desired.

L : LSA holds if and only if for any A, B, C and v, $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$.

($\longrightarrow$) For any $s \in v(B \multimap^{\mathsf{L}} C)$, we show $s \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. By definition, from assumptions $x \in v(A \multimap^{\mathsf{L}} B)$, $\mathbb{L}sxy$, $y \in v(A \multimap^{\mathsf{L}} C)$, $a \in A$, $c \in W$, and $\mathbb{L}yac$, we have to prove that $c \in C$. By LSA, there exists $x' \in W$ such that $\mathbb{L}xax'$ and $\mathbb{L}sx'c$. We get $x' \in B$ due to $x \in v(A \multimap^{\mathsf{L}} B)$. Thus, we have $c \in C$ because $s \in v(B \multimap^{\mathsf{L}} C)$.

($\longleftarrow$) Suppose that for any A, B, C and v, we have $v(B \multimap^{\mathsf{L}} C) \subseteq v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. Consider $a, b, x, c, d \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. Take $v(A) = c{\downarrow}$, $v(B) = \{y \mid \mathbb{L}bcy\}$, and $v(C) = \{d' \mid \exists y \in v(B), \mathbb{L}ayd'\}$ for some $A, B, C \in \mathsf{At}$. Given any $y \in v(B)$ and any $d' \in W$, if $\mathbb{L}ayd'$, then by definition of $v(C)$, $d' \in v(C)$, therefore $a \in v(B \multimap^{\mathsf{L}} C)$. By assumption, $a \in v((A \multimap^{\mathsf{L}} B) \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$ as well, which means that, for any $b' \in v(A \multimap^{\mathsf{L}} B)$, $x' \in W$, $c' \in v(A)$ and $d' \in W$, if $\mathbb{L}ab'x'$, then $x' \in v(A \multimap^{\mathsf{L}} C)$, and if $\mathbb{L}x'c'd'$, then $d' \in C$. By the definition of $v(B)$ and assumptions $\mathbb{L}abx$ and $\mathbb{L}xcd$, we have $b \in v(A \multimap^{\mathsf{L}} B)$, $x \in v(A \multimap^{\mathsf{L}} C)$, therefore $d \in v(C)$, which means that there exists $y \in W$ such that $\mathbb{L}bcy$ and $\mathbb{L}ayd$.

j^{R} : RSLU holds if and only if for any A, B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$.

($\longrightarrow$) By RSLU, for all $a \in v(A)$, there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$, then we have $a \in v(B)$ because $e \in v(A \multimap^{\mathsf{R}} B)$.

($\longleftarrow$) Suppose that for any A, B and v, if $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$, then $v(A) \subseteq v(B)$. Consider any $a \in W$. We take $v(A) = a\downarrow$ and $v(B) = \{b \mid \exists e \in \mathbb{I}, \mathbb{R}eab\}$ for some $A, B \in \mathsf{At}$. For any $e' \in \mathbb{I}$, $a' \in v(A)$, and $b' \in W$, if $\mathbb{R}e'a'b'$, then because $\mathbb{R}$ is upwards closed in its second argument, we have $b' \in v(B)$, which means $e' \in v(A \multimap^{\mathsf{R}} B)$. Therefore $\mathbb{I} \subseteq v(A \multimap^{\mathsf{R}} B)$. From the assumption, we can now conclude that $v(A) \subseteq v(B)$. In particular, $a \in v(B)$, which means that there exists $e \in \mathbb{I}$ such that $\mathbb{R}eaa$.

L^{R} : RSA holds if and only if for any A, B, C, D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$ then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$.

($\longrightarrow$) We expand the assumption.
For any A, B, C, D, $a \in v(A)$, and $b, z \in W$, if $b \in v(B)$ and $\mathbb{R}abz$ then $z \in v(C \multimap^{\mathsf{R}} D)$ and for all $z \in v(C \multimap^{\mathsf{R}} D)$, for all $c, d \in W$ if $c \in v(C)$ and $\mathbb{R}zcd$, then $d \in v(D)$. In other words, for any $z, d \in W$, if there are $a \in v(A)$, $b \in v(B)$, $c \in v(C)$, $\mathbb{R}abz$, and $\mathbb{R}zcd$, then $d \in v(D)$.
We take $E = B \otimes^{\mathsf{R}} C$ and show it satisfies the two following statements:

- For any $a \in v(A)$, we show that $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$. For any $x \in v(B \otimes^{\mathsf{R}} C)$ and $d \in W$, if $\mathbb{R}axd$, then by definition of $\otimes^{\mathsf{R}}$, we have $\mathbb{R}bcx$, where $b \in v(B)$ and $c \in v(C)$. By RSA, there exists $z \in W$ such that $\mathbb{R}abz$, and $\mathbb{R}zcd$. By the expanded assumption, $d \in v(D)$. Therefore $a \in v((B \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$.
- For any $b \in v(B)$, $c \in v(C)$, and $x \in W$, suppose $\mathbb{R}bcx$, then $x \in v(B \otimes^{\mathsf{R}} C)$ by definition of $\otimes^{\mathsf{R}}$. Therefore $b \in v(C \multimap^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$.

($\longleftarrow$) Assume that, for any A, B, C, D and v, if $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$, then there exists X such that $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$. Suppose that we have $a, b, c, d, x \in W$ such that $\mathbb{R}axd$ and $\mathbb{R}bcx$, then we take $v(A) = a\downarrow$, $v(B) = b\downarrow$, $v(C) = c\downarrow$, and $v(D) = \{d' \mid \exists y, \mathbb{R}aby \& \mathbb{R}ycd'\}$ for some $A, B, C, D \in \mathsf{At}$. For any $a' \in v(A)$, given any $b' \in v(B)$, $x' \in W$, $c' \in v(C)$, $d' \in W$ such that $\mathbb{R}a'b'x'$ and $\mathbb{R}x'c'd'$. Because $\mathbb{R}$ is upwards closed in its first and second arguments, by the definition of $v(D)$, we have $d' \in v(D)$, which means $v(A) \subseteq v(B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D))$. By the assumption, there exists X such that

(1) $v(A) \subseteq v(X \multimap^{\mathsf{R}} D)$, which means that for any $a' \in v(A)$, given any $x' \in X$, $d' \in W$, if $\mathbb{R}a'x'd'$, then $d' \in v(D)$, and

(2) $v(B) \subseteq v(C \multimap^{\mathsf{R}} X)$, which means that for any $b' \in v(B)$, given any $c' \in v(C)$ and $x' \in W$, if $\mathbb{R}b'c'x'$, then $x' \in v(X)$.

By $\mathbb{R}bcx$, and (2), we know that $x \in v(X)$. By $\mathbb{R}axd$, and (1), we know that $d \in v(D)$, which means that there exists $y \in W$ such that $\mathbb{R}aby$ and $\mathbb{R}ycd$.

The other cases are similar to the arguments above. □

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left (right) skew associative if $\mathbb{L}$ satisfies LSA (RSA). For other conditions, the naming is similar. If $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ satisfies LSA, LSLU, and LSRU (respectively RSA, RSLU, RSRU), then it is a left (respectively right) skew frame.

We can think of a $\mathtt{SkNMBiC_A}$ frame $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}\rangle$ as a combination of two ternary frames $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ (left skew frame) and $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ (right skew frame) sharing the same set of possible worlds, where the ternary relations are interdefinable by $\mathbb{L}\mathbb{R}$-reverse. Whenever $\mathbb{L}\mathbb{R}$-reverse holds, then $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is left skew if and only if $\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ is right skew. In fact, we have:

$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew associative	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew associative
$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew left unital	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew right unital
$\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ left skew right unital	$\longleftrightarrow$	$\langle W, \leq, \mathbb{I}, \mathbb{R}\rangle$ right skew left unital

If we state the structural laws semantically rather than syntactically, as in the sequent calculus $\mathtt{SkNMBiC_A}$, we can reformulate Theorem 6.3.7 without referring to sequents and valuations. For example, we can define $\otimes^{\mathsf{L}}$ on downwards closed sets of worlds as $A \otimes^{\mathsf{L}} B = \{c : \exists a \in A \;\&\; b \in B \;\&\; \mathbb{L}abc\}$ and express α as $(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \subseteq A \otimes^{\mathsf{L}} (B \otimes^{\mathsf{L}} C)$. It is the case that α holds in a frame if and only if it satisfies LSA.

We construct a thin SkBiC from the frame $\langle W, \leq, \mathsf{I}, \mathbb{L}, \mathbb{R}\rangle$ and provide algebraic proofs for the main theorems in [64]. The objects in the category are downwards closed subsets of W and for A, B, we have a map $A \to B$ if and only if $A \subseteq B$.

Corollary 6.3.8. *The category $(\mathcal{P}_{\downarrow}(W), \subseteq)$ generated from any* $\mathtt{SkNMBiC_A}$ *frame is a thin* SkBiC.

A frame $\langle W, \leq, \mathbb{I}, \mathbb{L}\rangle$ is associative normal if it satisfies LSA and RSA simultaneously, and left (right) unital normal if LSLU and RSLU (LSRU and RSRU) are satisfied. Analogously, we define normality conditions on skew monoidal closed categories.

Definition 6.3.9. A left skew monoidal closed category is

- *associative normal* if α is a natural isomorphism;
- *left unital normal* if λ is a natural isomorphism;
- *right unital normal* if ρ is a natural isomorphism.

Each normality condition can be expressed equivalently using j, i, and L. The normality conditions for right skew monoidal closed categories follow the same pattern, but with α^R, λ^R, and ρ^R instead of α, λ, and ρ.

By Theorem 6.3.7, we have a thin version of the main results in [64].

Corollary 6.3.10. *Given any frame, for the category $(\mathcal{P}_{\downarrow}(W), \subseteq)$ generated from the frame we have:*

$(\mathbb{I}, \otimes^{\mathsf{L}})$ *left skew monoidal*	$\longleftrightarrow$	$(\mathbb{I}, \multimap^{\mathsf{L}})$ *left skew closed*
$(\mathbb{I}, \otimes^{\mathsf{R}})$ *right skew monoidal*	$\longleftrightarrow$	$(\mathbb{I}, \multimap^{\mathsf{R}})$ *right skew closed*

Moreover, if the frame satisfies $\mathbb{LR}$*-reverse then:*

$$
\begin{array}{lcl}
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left skew monoidal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right skew monoidal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left skew closed} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right skew closed} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \otimes^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \otimes^{\mathsf{R}}) \textit{ left unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ associative normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ associative normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ left unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ right unital normal} \\
(\mathbb{I}, \multimap^{\mathsf{L}}) \textit{ right unital normal} & \longleftrightarrow & (\mathbb{I}, \multimap^{\mathsf{R}}) \textit{ left unital normal}
\end{array}
$$

6.4 $\mathtt{SkNMBiC}_{\mathtt{A}}$ with Symmetry

An exchange rule can be added to both associative and non-associative Lambek calculus to allow permutation of formulae in context [52]. It is well-known that two implications $\setminus$ and $/$ collapse into one in commutative Lambek calculus, i.e. for any formulae A and B, $A \setminus B$ is logically equivalent to B / A. In particular, consider an axiomatic presentation of non-associative Lambek calculus (1.3) with exchange $\mathsf{ex} : A \otimes B \vdash_{\mathsf{A}} B \otimes A$, both $A \setminus B \vdash_{\mathsf{A}} B / A$ and $B / A \vdash_{\mathsf{A}} A \setminus B$ are provable.

$$
\dfrac{\dfrac{\dfrac{}{(A \setminus B) \otimes A \vdash_{\mathsf{A}} A \otimes (A \setminus B)}\,\mathsf{ex} \quad \dfrac{\dfrac{}{A \setminus B \vdash_{\mathsf{A}} A \setminus B}\,\mathsf{id}}{A \otimes (A \setminus B) \vdash_{\mathsf{A}} B}\,\setminus^{-1}_{\mathsf{RES}}}{(A \setminus B) \otimes A \vdash_{\mathsf{A}} B}\,\mathsf{comp}}{A \setminus B \vdash_{\mathsf{A}} B / A}\,/_{\mathsf{RES}}
$$

$$
\dfrac{\dfrac{\dfrac{}{A \otimes (B / A) \vdash_{\mathsf{A}} (B / A) \otimes A}\,\mathsf{ex} \quad \dfrac{\dfrac{}{(B / A) \vdash_{\mathsf{A}} B / A}\,\mathsf{id}}{(B / A) \otimes A \vdash_{\mathsf{A}} B}\,/^{-1}_{\mathsf{RES}}}{A \otimes (B / A) \vdash_{\mathsf{A}} B}\,\mathsf{comp}}{B / A \vdash_{\mathsf{A}} A \setminus B}\,\setminus_{\mathsf{RES}}
$$

This leads to a natural question: can (and if so, how) $\mathtt{SkNMILL}_{\mathtt{A}}$ be extended with exchange? An immediate idea is to add the the following axiom to $\mathtt{SkNMILL}_{\mathtt{A}}$:

$$
\dfrac{}{A \otimes B \vdash_{\mathsf{A}} B \otimes A}\,\mathsf{ex}
$$

Following this axiom, we can define a derivable rule exr that swaps any two adjacent formulae in the antecedent. This rule is defined through combinations of the axioms ex and id and the rules comp and $\otimes$. For example, given a derivation $f : (A \otimes B) \otimes C \vdash_{\mathsf{A}} D$ and the goal sequent $(B \otimes A) \otimes C \vdash_{\mathsf{A}} D$, we can use the derivable rule:

$$
\dfrac{\dfrac{f}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} D}}{(B \otimes A) \otimes C \vdash_{\mathsf{A}} D}\,\mathsf{exr}
= \dfrac{\dfrac{\dfrac{}{B \otimes A \vdash_{\mathsf{A}} A \otimes B}\,\mathsf{ex} \quad \dfrac{}{C \vdash_{\mathsf{A}} C}\,\mathsf{id}}{(B \otimes A) \otimes C \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\otimes \quad \dfrac{f}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} D}}{(B \otimes A) \otimes C \vdash_{\mathsf{A}} D}\,\mathsf{comp} \tag{6.1}
$$

However, as observed by Bourke and Lack [14], the axiom ex makes the calculus fully normal, i.e. λ^{-1}, ρ^{-1}, and α^{-1} are provable.

$$\lambda^{-1} = \qquad \frac{\dfrac{}{A \otimes \mathsf{I} \vdash_{\mathsf{A}} \mathsf{I} \otimes A}\,\mathsf{ex} \qquad \dfrac{}{\mathsf{I} \otimes A \vdash_{\mathsf{A}} A}\,\lambda}{A \otimes \mathsf{I} \vdash_{\mathsf{A}} A}\,\mathsf{comp}$$

$$\rho^{-1} = \qquad \frac{\dfrac{}{A \vdash_{\mathsf{A}} A \otimes \mathsf{I}}\,\rho \qquad \dfrac{}{A \otimes \mathsf{I} \vdash_{\mathsf{A}} \mathsf{I} \otimes A}\,\mathsf{ex}}{A \vdash_{\mathsf{A}} \mathsf{I} \otimes A}\,\mathsf{comp}$$

$$\alpha^{-1} = \frac{\dfrac{\dfrac{}{(C \otimes B) \otimes A \vdash_{\mathsf{A}} C \otimes (B \otimes A)}\,\alpha \qquad \dfrac{\dfrac{\dfrac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\mathsf{id}}{(B \otimes A) \otimes C \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\mathsf{exr}}{C \otimes (B \otimes A) \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\mathsf{exr}}{\dfrac{(C \otimes B) \otimes A \vdash_{\mathsf{A}} (A \otimes B) \otimes C}{(B \otimes C) \otimes A \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\mathsf{exr}}\,\mathsf{comp}}{A \otimes (B \otimes C) \vdash_{\mathsf{A}} (A \otimes B) \otimes C}\,\mathsf{exr}$$

Therefore, we follow the design of axiomatic calculus (called Hilbert-style calculus in the original papers) in Veltri's studies [69, 70], where symmetry is represented by the following axioms (notations are modified to fit our discussion):

$$\frac{}{(A \otimes B) \otimes C \vdash_{\mathsf{A}} (A \otimes C) \otimes B}\,s \qquad \frac{}{B \multimap (A \multimap C) \vdash_{\mathsf{A}} A \multimap (B \multimap C)}\,s'$$

The axiom s is introduced for the axiomatic calculus of symmetric left skew monoidal categories where $\multimap$ is not present, while s' is the dual case for symmetric left skew closed categories.

These axioms only take care of symmetric left skew categories. In the remainder of the section, we first extend the proof-theoretical analysis to symmetric right skew and symmetric skew monoidal bi-closed categories. We will first introduce the definition of symmetric right skew monoidal closed categories, then prove the equivalence of the axioms of symmetry proof-theoretically. After that we introduce the commutative extension of $\mathtt{SkNMBiC_A}$ ($\mathtt{SkNMBiC_T}$), called $\mathtt{SkMBiC_A}$ ($\mathtt{SkMBiC_T}$) and prove the equivalence of the axiomatic and tree calculi. Finally, we prove that $\mathtt{SkMBiC_A}$ is sound and complete with respect to the preordered ternary relation model and extend Theorem 6.3.7 with the structural laws of symmetry.

Definition 6.4.1. A *symmetric right skew monoidal closed category* $\mathbb{C}$ is a right skew monoidal closed category equipped with a natural isomorphism $s^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to B \otimes (A \otimes C)$ satisfying the equations in Figure 6.1, which are similar to the ones in Figure 4.3 with modified bracketing.

Similar to what we saw in Remark 4.2.3, there exists a bijective correspondence with natural isomorphisms $\mathsf{s'^R} : \int^F \mathbb{C}(B, F \multimap D) \times \mathbb{C}(A, C \multimap F) \to \int^E \mathbb{C}(A, E \multimap D) \times \mathbb{C}(B, C \multimap E)$ in a symmetric right skew *non-monoidal* closed category. We prove the bijective correspondence between s and s^{R} and s' and s'^{R} proof-theoretically. For a smoother discussion, we define derivable rules sr, $\mathsf{s'r}$, and $\mathsf{s^Rr}$ similarly to the definition of the rule exr in (6.1).

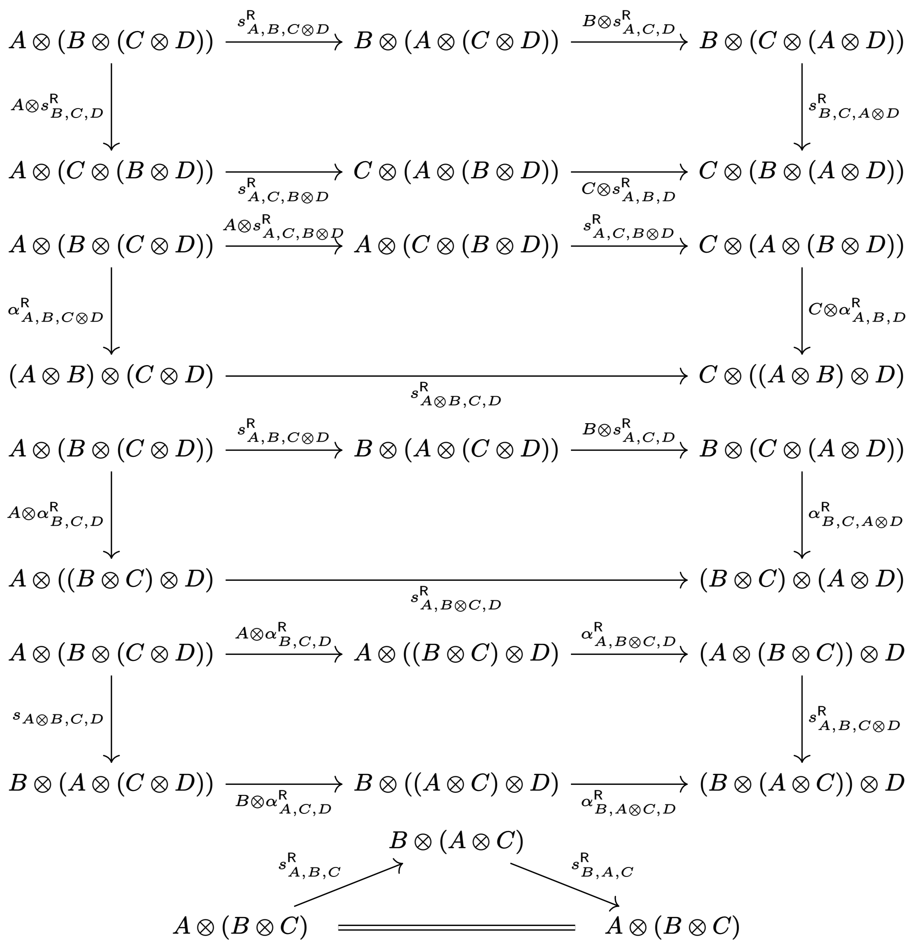

Figure 6.1: Equations of morphisms in symmetric right skew monoidal closed category.

Theorem 6.4.2. *In an extension of* $\mathtt{SkNMILL_A}$*, if*

$$s : (A \otimes B) \otimes C \vdash_{\mathsf{A}} (A \otimes C) \otimes B$$

is derivable in the calculus, then s' is derivable and vice versa.

Proof. <u>From s to s'.</u>

$$\begin{array}{cl}
\hline
B \multimap (A \multimap C) \vdash_{\mathsf{A}} B \multimap (A \multimap C) & \mathsf{ax} \\
\hline
B \multimap (A \multimap C) \otimes B \vdash_{\mathsf{A}} A \multimap C & \pi^{-1} \\
\hline
((B \multimap (A \multimap C)) \otimes B) \otimes A \vdash_{\mathsf{A}} C & \pi^{-1} \\
\hline
((B \multimap (A \multimap C)) \otimes A) \otimes B \vdash_{\mathsf{A}} C & \mathsf{sr} \\
\hline
(B \multimap (A \multimap C)) \otimes A \vdash_{\mathsf{A}} B \multimap C & \pi \\
\hline
B \multimap (A \multimap C) \vdash_{\mathsf{A}} A \multimap (B \multimap C) & \pi
\end{array}$$

<u>From s' to s.</u>

$$\begin{array}{cl}
\hline
(A \otimes C) \otimes B \vdash_{\mathsf{A}} (A \otimes C) \otimes B & \mathsf{ax} \\
\hline
A \otimes C \vdash_{\mathsf{A}} B \multimap ((A \otimes C) \otimes B) & \pi \\
\hline
A \vdash_{\mathsf{A}} C \multimap (B \multimap ((A \otimes C) \otimes B)) & \pi \\
\hline
A \vdash_{\mathsf{A}} B \multimap (C \multimap ((A \otimes C) \otimes B)) & \mathsf{s'r} \\
\hline
A \otimes B \vdash_{\mathsf{A}} C \multimap ((A \otimes C) \otimes B) & \pi^{-1} \\
\hline
(A \otimes B) \otimes C \vdash_{\mathsf{A}} (A \otimes C) \otimes B & \pi^{-1}
\end{array}$$

□

Theorem 6.4.3. *In an extension of* $\mathtt{SkNMILL_A}$*, if*

$$s^{\mathsf{R}} : A \otimes (B \otimes C) \vdash_{\mathsf{A}} B \otimes (A \otimes C)$$

is derivable, then the statement

> s'^{R} : *If there exists a formula F such that two sequents $B \vdash_{\mathsf{A}} F \multimap D$ and $A \vdash_{\mathsf{A}} C \multimap F$ are derivable, then there exists a formula E such that two sequents $A \vdash_{\mathsf{A}} E \multimap D$ and $B \vdash_{\mathsf{A}} C \multimap E$ are derivable.*

holds.
Conversely, if s'^{R} holds in the calculus, then s^{R} is derivable.

Proof. <u>From s^{R} to s'^{R}.</u> Suppose that there exists a formula F such that two sequents $B \vdash_{\mathsf{A}} F \multimap D$ and $A \vdash_{\mathsf{A}} C \multimap F$ derivable, then we take $E = B \otimes C$ and construct desired derivations of $A \vdash_{\mathsf{A}} (B \otimes C) \multimap D$ and $B \vdash_{\mathsf{A}} C \multimap (B \otimes C)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash_{\mathsf{A}} B}\ \mathsf{id} \quad \dfrac{\stackrel{\text{Assumption}}{A \vdash_{\mathsf{A}} C \multimap F}}{A \otimes C \vdash_{\mathsf{A}} F}\ \pi^{-1}}{B \otimes (A \otimes C) \vdash_{\mathsf{A}} B \otimes F}\ \otimes \qquad \dfrac{\stackrel{\text{Assumption}}{B \vdash_{\mathsf{A}} F \multimap D}}{B \otimes F \vdash_{\mathsf{A}} D}\ \pi^{-1}}{B \otimes (A \otimes C) \vdash_{\mathsf{A}} D}\ \mathsf{comp}}{A \otimes (B \otimes C) \vdash_{\mathsf{A}} D}\ \mathsf{s^R r}}{A \vdash_{\mathsf{A}} (B \otimes C) \multimap D}\ \pi$$

$$\dfrac{\dfrac{}{B \otimes C \vdash_{\mathsf{A}} B \otimes C}\ \mathsf{id}}{B \vdash_{\mathsf{A}} C \multimap (B \otimes C)}\ \pi$$

<u>From s'^{R} to s^{R}.</u> To prove the sequent $A \otimes (B \otimes C) \vdash_{\mathsf{A}} B \otimes (A \otimes C)$, we start from the following two axiom sequents $\mathsf{id} : B \otimes (A \otimes C) \vdash_{\mathsf{A}} B \otimes (A \otimes C)$ and $\mathsf{id} : A \otimes C \vdash_{\mathsf{A}} A \otimes C$. By applying π on both sequents, we obtain $\pi\ \mathsf{id} : B \vdash_{\mathsf{A}} (A \otimes C) \multimap (B \otimes (A \otimes C))$ and $\pi\ \mathsf{id} : A \vdash_{\mathsf{A}} C \multimap (A \otimes C)$. We take $A \otimes C = F$ to apply s'^{R}, then there exists a formula E such that two sequents $A \vdash_{\mathsf{A}} E \multimap (B \otimes (A \otimes C))$ and $B \vdash_{\mathsf{A}} C \multimap E$ hold. The desired derivation is constructed as follows:

$$\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{A}} A}\,\mathsf{id} \qquad \dfrac{\begin{array}{c}\text{By } s'^{\mathsf{R}} \\ B \vdash_{\mathsf{A}} C \multimap E\end{array}}{B \otimes C \vdash_{\mathsf{A}} E}\,\pi^{-1}}{A \otimes (B \otimes C) \vdash_{\mathsf{A}} A \otimes E}\,\otimes \qquad \dfrac{\begin{array}{c}\text{By } s'^{\mathsf{R}} \\ A \vdash_{\mathsf{A}} E \multimap (B \otimes (A \otimes C))\end{array}}{A \otimes E \vdash_{\mathsf{A}} B \otimes (A \otimes C)}\,\pi^{-1}}{A \otimes (B \otimes C) \vdash_{\mathsf{A}} B \otimes (A \otimes C)}\,\mathsf{comp}$$

□

Definition 6.4.4. A *symmetric skew monoidal bi-closed category* (SymSkMBiC) is a skew monoidal bi-closed category with the natural isomorphism s as in Definition 4.2.1. s^{R} is defined as $B \otimes^{\mathsf{L}} \gamma \circ \gamma \circ s \circ \gamma^{-1} \circ A \otimes^{\mathsf{R}} \gamma^{-1}$, diagrammatically:

$$\begin{array}{ccccc}
A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C) & \xrightarrow{A \otimes^{\mathsf{R}} \gamma^{-1}} & A \otimes^{\mathsf{R}} (C \otimes^{\mathsf{L}} B) & \xrightarrow{\gamma^{-1}} & (C \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} A \\
\downarrow {\scriptstyle s^{\mathsf{R}}} & & & & \downarrow {\scriptstyle s} \\
B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C) & \xleftarrow[B \otimes^{\mathsf{R}} \gamma]{} & B \otimes^{\mathsf{R}} (C \otimes^{\mathsf{L}} A) & \xleftarrow[\gamma]{} & (C \otimes^{\mathsf{L}} A) \otimes^{\mathsf{L}} B
\end{array}$$

The axiomatic calculus that is sound and complete with respect to SymSkMBiC is $\mathtt{SkMBiC}_{\mathtt{A}}$ which is extended from $\mathtt{SkNMBiC}_{\mathtt{A}}$ by adding the axiom:

$$\dfrac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\, s$$

The axiom s^{R} is defined by transforming the diagram in Definition 6.4.4 into a proof in $\mathtt{SkMBiC}_{\mathtt{A}}$, and then by Theorems 6.4.2 and 6.4.3, s' and s'^{R} are derivable in $\mathtt{SkMBiC}_{\mathtt{A}}$.

Moreover, we can construct the free SymSkMBiC (FSymSkMBiC(At)) over a set At by a similar construction of FSkMBiC(At) in Section 6.2.1.

On the other hand, the commutative extension of $\mathtt{SkNMBiC}_{\mathtt{T}}$ ($\mathtt{SkMBiC}_{\mathtt{T}}$) is defined by adding the following two rules:

$$\dfrac{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}{T[(U_0, U_2), U_1] \vdash_{\mathsf{T}} C}\,\mathsf{ex}^{\mathsf{L}} \qquad \dfrac{T[U_0; (U_1; U_2)] \vdash_{\mathsf{T}} C}{T[U_1; (U_0; U_2)] \vdash_{\mathsf{T}} C}\,\mathsf{ex}^{\mathsf{R}}$$

A result similar to Theorems 6.4.2 and 6.4.3 can also be proved in $\mathtt{SkMBiC}_{\mathtt{T}}$. We adopt a symmetric presentation to emphasize that $\mathtt{SkMBiC}_{\mathtt{T}}$ should be viewed as a combination of two distinct calculi, connected through the rule $\otimes\mathsf{comm}$.

Moreover, $\mathtt{SkMBiC}_{\mathtt{A}}$ and $\mathtt{SkMBiC}_{\mathtt{T}}$ are equivalent.

Theorem 6.4.5. SkMBiC$_{\mathsf{A}}$ *is equivalent to* SkMBiC$_{\mathsf{T}}$*, meaning that the following two statements are true:*

- *For any derivation $f : A \vdash_{\mathsf{A}} C$, there exists a derivation $\mathsf{A2T}f : A \vdash_{\mathsf{T}} C$.*
- *For any derivation $f : T \vdash_{\mathsf{T}} C$, there exists a derivation $\mathsf{T2A}f : T^{\#} \vdash_{\mathsf{A}} C$, where $T^{\#}$ transforms a tree into a formula by replacing commas with $\otimes^{\mathsf{L}}$ and semicolons with $\otimes^{\mathsf{R}}$, and $-$ with I, respectively.*

Proof. We extend the proof of Theorem 6.2.6 by examining the additional cases of s (for A2T) and ex^{L} and ex^{R} (for T2A).

Case $f = s$

$$\dfrac{}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ s$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash_{\mathsf{T}} A}\ \mathsf{ax} \quad \dfrac{}{C \vdash_{\mathsf{T}} C}\ \mathsf{ax}}{A, C \vdash_{\mathsf{T}} A \otimes^{\mathsf{L}} C}\ \otimes^{\mathsf{L}}\mathsf{R} \quad \dfrac{}{B \vdash_{\mathsf{T}} B}\ \mathsf{ax}}{(A, C), B \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{R}}{(A, B), C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \mathsf{ex}^{\mathsf{L}}}{(A \otimes^{\mathsf{L}} B), C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{L}}{(A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{T}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\ \otimes^{\mathsf{L}}\mathsf{L}$$

Case $f = \mathsf{ex}^{\mathsf{L}}\ f'$

$$\dfrac{\overset{f'}{T[(U_0, U_1), U_2] \vdash_{\mathsf{T}} C}}{T[(U_0, U_2), U_1] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{L}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{(U_0^{\#} \otimes^{\mathsf{L}} U_2^{\#}) \otimes^{\mathsf{L}} U_1^{\#} \vdash_{\mathsf{T}} (U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#}}\ s}{T[(U_0^{\#} \otimes^{\mathsf{L}} U_2^{\#}) \otimes^{\mathsf{L}} U_1^{\#}]^{\#} \vdash_{\mathsf{T}} T[(U_0^{\#} \otimes^{\mathsf{L}} U_1^{\#}) \otimes^{\mathsf{L}} U_2^{\#}]^{\#}}\ T[\mathsf{s}]^{\#}}{T[(U_0, U_2), U_1]^{\#} \vdash_{\mathsf{A}} T[(U_0, U_1), U_2]^{\#}} \quad \overset{\mathsf{T2A}f'}{T[(U_0, U_1), U_2]^{\#} \vdash_{\mathsf{T}} C}}{T[(U_0, U_2), U_1]^{\#} \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

Case $f = \mathsf{ex}^{\mathsf{R}}\ f'$

$$\dfrac{\overset{f'}{T[U_0; (U_1; U_2)] \vdash_{\mathsf{T}} C}}{T[U_1; (U_0; U_2)] \vdash_{\mathsf{T}} C}\ \mathsf{ex}^{\mathsf{R}}$$

$$\mapsto \quad \dfrac{\dfrac{\dfrac{\dfrac{}{U_1^{\#} (\otimes^{\mathsf{R}} U_0^{\#} \otimes^{\mathsf{R}} U_2^{\#}) \vdash_{\mathsf{T}} U_0^{\#} \otimes^{\mathsf{R}} (U_1^{\#} \otimes^{\mathsf{R}} U_2^{\#})}\ s^{\mathsf{R}}}{T[U_1^{\#} \otimes^{\mathsf{R}} (U_0^{\#} \otimes^{\mathsf{R}} U_2^{\#})]^{\#} \vdash_{\mathsf{T}} T[U_0^{\#} \otimes^{\mathsf{R}} (U_1^{\#} \otimes^{\mathsf{R}} U_2^{\#})]^{\#}}\ T[\mathsf{s}^{\mathsf{R}}]^{\#}}{T[U_1; (U_0; U_2)]^{\#} \vdash_{\mathsf{A}} T[U_0; (U_1; U_2)]^{\#}} \quad \overset{\mathsf{T2A}f'}{T[U_0; (U_1; U_2)]^{\#} \vdash_{\mathsf{T}} C}}{T[U_1; (U_0; U_2)]^{\#} \vdash_{\mathsf{T}} C}\ \mathsf{comp}$$

□

Recall that in commutative Lambek calculus (both associative and non-associative), the two implications collapse into one. However, this is not the case in either $\mathtt{SkMBiC_A}$ or $\mathtt{SkMBiC_T}$. Specifically, for any formulae A and B, neither of the sequents $A \multimap^{\mathsf{L}} B \vdash_i A \multimap^{\mathsf{R}} B$ nor $A \multimap^{\mathsf{R}} B \vdash_i A \multimap^{\mathsf{L}} B$ $(i \in \{\mathsf{A}, \mathsf{T}\})$ is provable. We demonstrate this non-provability by taking A and B as atomic formulae.

$$\dfrac{\dfrac{??}{(X \multimap^{\mathsf{L}} Y) \otimes^{\mathsf{R}} X \vdash_{\mathsf{A}} Y}}{X \multimap^{\mathsf{L}} Y \vdash_{\mathsf{A}} X \multimap^{\mathsf{R}} Y}\ \pi^{\mathsf{R}} \qquad \dfrac{\dfrac{??}{(X \multimap^{\mathsf{R}} Y) \otimes^{\mathsf{L}} X \vdash_{\mathsf{A}} Y}}{X \multimap^{\mathsf{R}} Y \vdash_{\mathsf{A}} X \multimap^{\mathsf{L}} Y}\ \pi$$

$$\dfrac{\dfrac{\dfrac{??}{(X \multimap^{\mathsf{L}} Y) ; X \vdash_{\mathsf{T}} Y}}{(X \multimap^{\mathsf{L}} Y) \otimes^{\mathsf{R}} X \vdash_{\mathsf{T}} Y}\ \otimes^{\mathsf{R}}\mathsf{L}}{X \multimap^{\mathsf{L}} Y \vdash_{\mathsf{T}} X \multimap^{\mathsf{R}} Y}\ \multimap^{\mathsf{R}}\mathsf{R} \qquad \dfrac{\dfrac{\dfrac{??}{(X \multimap^{\mathsf{R}} Y) , X \vdash_{\mathsf{T}} Y}}{(X \multimap^{\mathsf{R}} Y) \otimes^{\mathsf{L}} X \vdash_{\mathsf{T}} Y}\ \otimes^{\mathsf{L}}\mathsf{L}}{X \multimap^{\mathsf{R}} Y \vdash_{\mathsf{T}} X \multimap^{\mathsf{L}} Y}\ \multimap^{\mathsf{R}}\mathsf{R}$$

Lastly, we can analyze skew symmetry through the lens of ternary relational semantics and obtain a sound and complete model of $\mathtt{SkMBiC_A}$. Furthermore, we obtain the correspondence theorem of ternary frame conditions and validity of structural laws.

Definition 6.4.6. We list the frame conditions properties of skew commutativity:

Left Skew Commutativity (LSC)	$\forall a, b, c, d, x \in W, \mathbb{L}abx \ \&\ \mathbb{L}xcd \longrightarrow \exists y \in W \text{ s.t. } \mathbb{L}acy \ \&\ \mathbb{L}ybd.$
Right Skew Commutativity (RSC)	$\forall a, b, c, d, x \in W, \mathbb{L}bcx \ \&\ \mathbb{L}axd \longrightarrow \exists y \in W \text{ s.t. } \mathbb{L}acy \ \&\ \mathbb{L}byd.$

A $\mathtt{SkMBiC_A}$ frame is a $\mathtt{SkNMBiC_A}$ frame where $\mathbb{L}$ and $\mathbb{R}$ additionally satisfy LSC and RSC, respectively. A $\mathtt{SkMBiC_A}$ model is a $\mathtt{SkMBiC_A}$ frame with a valuation function.

Theorem 6.4.7 (Soundness)**.** *If a sequent $A \vdash_{\mathsf{A}} B$ is provable in $\mathtt{SkMBiC_A}$ then it is valid in any $\mathtt{SkMBiC_A}$ model.*

Proof. The proof is extended from the proof of Theorem 6.3.3 by examining one additional case, $f = s : (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B$. For any $\mathtt{SkMBiC_A}$ model $\langle W, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ and any $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$, there exist $x \in v(A \otimes^{\mathsf{L}} B)$ and $c \in v(C)$ such that $\mathbb{L}xcd$. Moreover, there exist $a \in v(A)$ and $b \in v(B)$ such that $\mathbb{L}abx$. By LSC, we know that there exist $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}ybd$, which means that $d \in v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$. □

Definition 6.4.8. The canonical model of $\mathtt{SkMBiC_A}$ is $\langle W, \leq, \mathbb{I}, \mathbb{L}, \mathbb{R}, v\rangle$ where

- $W = \mathsf{Fma}$ and $A \leq B$ if and only if $A \vdash_{\mathsf{A}} B$,
- $\mathbb{I} = v(\mathsf{I})$,
- $\mathbb{L}ABC$ if and only if $C \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B$,
- $\mathbb{R}ABC$ if and only if $C \vdash_{\mathsf{A}} A \otimes^{\mathsf{R}} B$, and
- $v(A) = \{B \mid B \vdash_{\mathsf{A}} A \text{ is provable in } \mathtt{SkMBiC_A}\}$.

Lemma 6.4.9. *The canonical model is a $\mathtt{SkMBiC_A}$ model.*

Proof. The proof proceeds similarly to the proof of Lemma 6.3.5 but with one additional case showing that LSC is satisfied.
Given five formulae A, B, C, C', D and two derivations $f : C' \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B$ and $g : D \vdash_{\mathsf{A}} C' \otimes^{\mathsf{L}} C$, then we take $A \otimes^{\mathsf{L}} C$ as the desired formula. The first desired sequent $A \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} C$ is derivable and the other desired sequent $D \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B$ is constructed as follows:

$$\dfrac{\overset{g}{D \vdash_{\mathsf{A}} C' \otimes^{\mathsf{L}} C} \qquad \dfrac{\dfrac{\overset{f}{C' \vdash_{\mathsf{A}} A \otimes^{\mathsf{L}} B} \qquad \dfrac{}{C \vdash_{\mathsf{A}} C}\,\mathsf{ax}}{C' \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C}\,\otimes^{\mathsf{L}} \qquad \dfrac{}{(A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C}\,s}{C' \otimes^{\mathsf{L}} C \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\,\mathsf{comp}}{D \vdash_{\mathsf{A}} (A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B}\,\mathsf{comp}$$

□

Following the same argument in the proof of Theorem 6.3.6, we have:

Theorem 6.4.10 (Completeness)**.** *If $A \vdash_{\mathsf{A}} B$ is valid in any $\mathtt{SkMBiC_A}$ model, then it is provable in $\mathtt{SkMBiC_A}$.*

Finally, we extend the correspondence between frame conditions and validity of structural laws to the symmetric case.

Theorem 6.4.11. *For any ternary frame $\langle W, \leq, \mathbb{I}, \mathbb{L} \rangle$,*

$$s \textit{ valid} \quad \longleftrightarrow \quad \textit{LSC holds} \quad \longleftrightarrow \quad s' \textit{ valid}$$

$$s^{\mathsf{R}} \textit{ valid} \quad \longleftrightarrow \quad \textit{RSC holds} \quad \longleftrightarrow \quad s'^{\mathsf{R}} \textit{ valid}$$

Proof. s : LSC holds if and only if s is valid.

- ($\longrightarrow$) This is similar to the case of s in the proof of Theorem 6.4.7.
- ($\longleftarrow$) Suppose that s is valid, i.e. for any A, B, C, $v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C) \subseteq v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(A \otimes^{\mathsf{L}} B)$ and $d \in v((A \otimes^{\mathsf{L}} B) \otimes^{\mathsf{L}} C)$. By the assumption, $d \in v((A \otimes^{\mathsf{L}} C) \otimes^{\mathsf{L}} B)$ as well, which means that there exist $a', b', y, c' \in W$ such that $\mathbb{L}a'c'y$ and $\mathbb{L}yb'd$. Because $\mathbb{L}$ is upward closed in its first and second argument, we have $\mathbb{L}acy$ and $\mathbb{L}ybd$ as desired.

s' : LSC holds if and only if s' is valid.

- ($\longrightarrow$) Suppose that LSC holds, we show that for any A, B, C, $v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C)) \subseteq v(A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{L}} C))$. Consider any $d \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. Assume that there exists $a \in v(A)$, $b \in v(B)$, and $x, c \in W$ such that $\mathbb{L}dax$ and $\mathbb{L}xbc$. Our goal is to prove that $c \in v(C)$. By LSC, there exists $y \in W$ such that $\mathbb{L}dby$ and $\mathbb{L}yac$, then by the assumption $d \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$, we know that $c \in v(C)$.
- ($\longleftarrow$) Suppose that s' is valid, i.e. for any A, B, C, $v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C)) \subseteq v(A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{L}} C))$. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}abx$ and $\mathbb{L}xcd$. Take $v(A) = b{\downarrow}$, $v(B) = c{\downarrow}$, and $v(C) = \{d' \mid \exists y.\mathbb{L}acy \& \mathbb{L}ybd\}$ for some $A, B, C \in \mathsf{At}$. Consider any $c' \in v(B)$, $b' \in v(A)$, $y', d' \in W$,

$\mathbb{L}ac'y'$ and $\mathbb{L}y'b'd'$. Because $\mathbb{L}$ is upwards closed in its second argument, we have $\mathbb{L}acy'$ and $\mathbb{L}y'bd'$, which means that $y' \in v(A \multimap^{\mathsf{L}} C)$ and $d' \in v(C)$, therefore $a \in v(B \multimap^{\mathsf{L}} (A \multimap^{\mathsf{L}} C))$. By validity of s', $\mathbb{L}abx$, and $\mathbb{L}xcd$, we know that $d \in v(C)$, i.e. there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}ybd$.

s^{R} : RSC holds if and only if s^{R} is valid.

($\longrightarrow$) Suppose that RSC holds, we show that for any A, B, C, $v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C)) \subseteq v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$. Consider any $d \in v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. By definition, there exists $a \in v(A)$, $b \in v(B)$, $c \in v(C)$, $x \in v(B \otimes^{\mathsf{R}} C)$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. By RSC, there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, then by definition, we know that $y \in v(A \otimes^{\mathsf{R}} C)$ and therefore $d \in v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$.

($\longleftarrow$) Suppose that s^{R} is valid. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. We take $v(A) = a{\downarrow}, v(B) = b{\downarrow}, v(C) = c{\downarrow}$ for some $A, B, C \in \mathsf{At}$, then we know that $x \in v(B \otimes^{\mathsf{R}} C)$ and $d \in v(A \otimes^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. By the assumption, $d \in v(B \otimes^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$ as well, which means that that there exist $a', b', y, c' \in W$ such that $\mathbb{L}a'c'y$ and $\mathbb{L}b'yd$. Because $\mathbb{L}$ is upwards closed in its first and second argument, we have $\mathbb{L}acy$ and $\mathbb{L}byd$ as desired.

s'^{R} : RSC holds if and only if s'^{R} is valid.

($\longrightarrow$) Suppose that RSC holds, we show that for any formulae $A, B, C,\ D$, if there exists a formula F such that $v(B) \subseteq v(F \multimap^{\mathsf{R}} D)$ and $v(A) \subseteq v(C \multimap^{\mathsf{R}} F)$ then there exists a formula E such that $v(A) \subseteq v(E \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} E)$. Take $E = B \otimes^{\mathsf{R}} C$, then clearly $v(B) \subseteq v(C \multimap^{\mathsf{R}} (B \otimes^{\mathsf{R}} C))$. For any $a \in v(A)$, if there exist $x \in v(B \multimap^{\mathsf{R}} C)$ and $d \in W$ such that $\mathbb{L}axd$, then by definition, there exist $b \in v(B)$ and $c \in v(C)$ such that $\mathbb{L}bcx$. By RSC, there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, then by $v(B) \subseteq v(F \multimap^{\mathsf{R}} D)$, $d \in v(D)$, therefore $a \in v(E \multimap^{\mathsf{R}} D)$.

($\longleftarrow$) Suppose that s'^{R} is valid. Consider any $a, b, c, d, x \in W$ such that $\mathbb{L}bcx$ and $\mathbb{L}axd$. Take $v(A) = a{\downarrow}$, $v(B) = b{\downarrow}$, $v(C) = c{\downarrow}$, and $v(D) = \{d' \mid \exists y. \mathbb{L}acy \& \mathbb{L}byd\}$ for some $A, B, C, D \in \mathsf{At}$. Clearly, $v(A)$ is a subset of $v(C \multimap^{\mathsf{R}} (A \otimes^{\mathsf{R}} C))$. For any $b' \in v(B)$, if there exist $y' \in v(A \otimes^{\mathsf{R}} C)$ and $d' \in W$ and $\mathbb{L}b'y'd'$, then by definition, there exist $a' \in v(A)$ and $c' \in v(C)$ such that $\mathbb{L}a'c'y'$. Because $\mathbb{L}$ is upwards closed in its first and second argument, we have $\mathbb{L}acy'$ and $\mathbb{L}by'd'$, which means that $d' \in v(D)$ and therefore $v(B) \in v((A \otimes^{\mathsf{R}} C) \multimap^{\mathsf{R}} D)$. Take $F = A \otimes^{\mathsf{R}} C$, then by s'^{R}, there exists a formula E such that $v(A) \subseteq v(E \multimap^{\mathsf{R}} D)$ and $v(B) \subseteq v(C \multimap^{\mathsf{R}} E)$. By $b \in v(C \multimap^{\mathsf{R}} E)$ and $\mathbb{L}bcx$, we have $x \in v(E)$. By $a \in v(E \multimap^{\mathsf{R}} D)$ and $\mathbb{L}axd$, we have $d \in v(D)$, which means that there exists $y \in W$ such that $\mathbb{L}acy$ and $\mathbb{L}byd$, as desired. □

Chapter 7

Conclusion

This thesis investigated semi-substructural logics, which arise naturally from the study of skew monoidal categories, following the project initiated by Uustalu,Veltri, and Zeilberger [67]. We extended their analysis to various variants of skew monoidal categories, solving their coherence problems following the approach in [67], i.e. through the development of appropriate focused proof systems. These investigations led us to identify a broader class of logics that we term semi-substructural logics.

This perspective is particularly evidenced by our results on Craig interpolation for SkNMILL. We show that while Craig interpolation holds for this logic, it cannot be established through traditional Maehara's method. Instead, we developed a modified version of Maehara's method that simultaneously establishes two forms of interpolation: stoup Maehara interpolation (sMIP) and context Maehara multi-interpolation (cMMIP). Furthermore, similar to Čubrić [68] and Saurin's [58] work on proof-relevant interpolation for intuitionistic and linear logic, we prove that our interpolation procedures are right inverses of the admissible cut rules. This demonstrates that semi-substructural logics can exhibit rich proof-theoretic properties that deserve study in their own right.

The development of relational semantics for semi-substructural logics and the establishment of correspondence theorems further reinforces this view. These results show that semi-substructural logics have well-behaved semantic interpretations and satisfy important meta-logical properties. The correspondence theorems in particular demonstrate that the structural rules of these logics correspond naturally to frame conditions in the relational semantics, much like in modal and substructural logics.

Taken together, these results establish semi-substructural logics as a new interesting class of logical systems. While their origin in the study of skew categorical structures provided the initial motivation, our work shows that they possess rich logical properties that make them interesting objects of study in their own right. The proof theory, model theory, and categorical semantics of these logics form a coherent picture that suggests their fundamental importance in logic and theoretical computer science.

Looking forward, this work opens several promising directions for future research.

1. One direction is to investigate whether tag annotations can scale to non-skew

classical and intuitionistic linear logic. The starting point is the multiplicative additive linear logic (MALL). The non-skew setting presents additional complexity compared to the skew case, since multiple formulae can be under focus simultaneously during the synchronous phase (our focusing phase). Our calculus aims to provide an alternative presentation of the maximally multi-focused proofs described by Chaudhuri et al. [19], where normal forms are intrinsic to the system itself rather than identified through external criteria.

2. Another direction is to understand the universal property of interpolant triples (D, g, h) produced by Maehara's method in (semi-)substructural logics. This involves exploring different notions of equality between interpolants from a category-theoretic perspective, from strict equality and identical derivations, to more relaxed notions based on isomorphisms or triples being determined uniquely up to zigzag. We aim to characterize interpolants and derivations through universal properties and determine whether Maehara's interpolation procedures are not just right but also left inverses of cut rules.

3. Beyond the category-theoretical aspect, we will investigate extending these results to other semi-substructural logics, particularly the fully additive calculi where uniform interpolation property [3] may arise.

4. A deeper exploration of symmetric right skew closed categories remains as future work, particularly focusing on their coherence conditions without relying on monoidal structures. This investigation builds upon the foundational work of Day and Laplaza [21], who explored a hierarchy of closed categories, from symmetric monoidal closed through symmetric closed and closed, to non-associative closed categories. Their research provided concrete examples where the Day convolution version of structural laws fails to be bijective. This approach will extend the framework by studying symmetric skew closed categories. Unlike symmetric closed categories which are inherently associative, we have seen that symmetric skew closed categories are not automatically associative, providing an opportunity for more fine-grained work on this topic.

5. In Section 6.4, we have established results for the special case of poset categories, where there is at most one morphism between each pair of objects. The natural next step is to extend these results to non-poset categories, which requires finding appropriate coherence conditions for symmetric right skew closed categories. This extension will extend the Eilenberg-Kelly theorem [25, 64] to symmetric skew monoidal closed categories.

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Acknowledgements

I sincerely thank my supervisors, Tarmo Uustalu and Niccolò Veltri. They not only gave me the invaluable opportunity to pursue a PhD in Tallinn but also showed faith in a novice initially without a computer science background, meticulously cultivating me to be a qualified researcher. Their enthusiasm, academic integrity, and curiosity for logic guided me throughout this challenging yet rewarding PhD career.

Many thanks to my colleagues in the Logic and Semantics group and the Laboratory for Compositional Systems and Methods. The time we shared in discussions was invaluable.

I would also like to thank everyone who participated in the review process for this thesis. Thanks to their diligent efforts, numerous typos have been corrected, and several parts of the thesis have been improved. The thesis is significantly better as a result of their contributions.

My heartfelt thanks to my parents, Chi-Hsiung and Hui-Chen, for their consistent encouragement.

Finally, special thanks to my wife, Wan-Chen, for her unwavering love and support.

My research was supported by the ESF funded Estonian IT Academy research measure (project 2014-2020.4.05.19-0001) and the Estonian Research Council grant PSG749. My participation in many conferences and workshops was generously supported by the EU COST Actions CA19135 (CERCIRAS) and CA20111 (EuroProofNet), the student travel awards of the Association for Symbolic Logic, and the MSCA Action "Philosophical, Logical, and Experimental Routes to Substructurality" (PLEXUS).

Abstract

Proof Theory of Semi-Substructural Logics

In this thesis, we investigate the proof theory of semi-substructural logics. These emerge as the logical counterparts of skew monoidal categories and their variants. Unlike traditional monoidal categories, skew monoidal categories relax the associativity and unitality laws from natural isomorphisms to natural transformations with specific orientations, making these logics intermediate between non-commutative intuitionistic linear logic or Lambek calculus and their non-associative variants.

We begin with a proof-theoretical analysis of skew monoidal closed categories via their corresponding logic, skew non-commutative multiplicative intuitionistic linear logic (SkNMILL) with the sequent calculus with stoup ($\mathtt{SkNMILL}_{\mathtt{S}}$). We prove that $\mathtt{SkNMILL}_{\mathtt{S}}$ enjoys cut-elimination, Craig interpolation, variations of Maehara interpolation and proof-relevant interpolation. An interesting problem with variants of skew monoidal categories is their coherence problem, i.e. can we have an effective method to determine whether there is a canonical morphism between two objects and whether there can be at most one, or how to enumerate them or how to establish equality of two morphisms if there can be multiple morphisms. The case of skew monoidal categories is more delicate than that of monoidal categories because the relaxation allows for multiple canonical morphisms for some pairs of objects. To solve the coherence problem, we develop a focused sequent calculus with tag annotations that derives just one representative for every equivalence class of derivations in $\mathtt{SkNMILL}_{\mathtt{S}}$.

Subsequently, we extend both $\mathtt{SkNMILL}_{\mathtt{S}}$ and its focused calculus to several variants: a commutative extension ($\mathtt{SkMILL}_{\mathtt{S}}$) with restricted exchange involving three objects rather than two, an additive extension ($\mathtt{SkNMILLA}_{\mathtt{S}}$) with conjunction and disjunction, and their combinations, each of which corresponds to a specialization of skew monoidal categories.

For semi-substructural logics that cannot be characterized well by the stoup sequent calculus approach, e.g., skew monoidal bi-closed categories, we develop axiomatic and tree sequent calculi inspired from non-associative Lambek calculus and prove their equivalence. Additionally, we prove soundness and completeness with respect to ternary relational semantics for these logics and the correspondence theorem between structural laws and frame conditions.

These investigations demonstrate, we believe, that semi-substructural logics are interesting and enjoy rich syntactic and semantic properties, beyond their motivation as logical counterparts of variants of skew monoidal categories.

Some of the key results, in particular, cut-elimination and normalization for $\mathtt{SkMILL}_{\mathtt{S}}$ and $\mathtt{SkNMILLA}_{\mathtt{S}}$, were formalized in the Agda proof assistant in the process of writing this thesis.

Kokkuvõte

Pool-allstruktuursete loogikate tõestusteooria

Käesolevas doktoritöös uurime pool-allstruktuursete loogikate tõestusteooriat. Need loogikasüsteemid on kiivmonoidiliste kategooriate ja nende mitmesuguste variantide loogikalisteks vasteteks. Erinevalt normaalsetest monoidilistest kategooriatest on assotsiatiivsuse ja ühikuseadused kiivmonoidilistes kategooriates loomulike isomorfismide asemel spetsiifilise suunaga loomulikud teisendused. Nõnda paigutuvad need loogikad mittekommutatiivse intuitsionistliku loogika ja Lambeki arvutuse ning nende mitteassotsiatiivsete nõrgenduste vahele.

Me alustame kiivmonoidilistele suletud kategooriatele vastava loogika ehk kiiva mittekommutatiivse intuitsionistliku lineaarloogika ($\mathtt{SkNMILL}$) eripositsiooniga sekventsiarvutuse ($\mathtt{SkNMILL}_{\mathtt{S}}$) tõestusteoreetilise analüüsiga. Me näitame, arvutusel $\mathtt{SkNMILL}_{\mathtt{S}}$ on lõikereegli lubatavuse ja Craigi interpoleeritavuse omadused, nimelt Maehara interpoleeritavus teatud kujul ja ka tuletusi arvestav interpolatsioon. Huvitavaks küsimuseks kiivmonoidiliste kategooriate variantide puhul on nende koherentsiprobleem ehk et kas leidub efektiivne meetod määramaks, kas kahe objekti vahel leidub kanooniline morfism ja kas neid on ülimalt üks või kuidas neid loetleda või kuidas teha kindlaks teha kahe morfismi võrdsus, kui neid saab olla mitu. Kiivmonoidiliste kategooriate juhtum on keerulisem kui monoidiliste kategooriate oma—kahe objekti vahel võib leiduda mitu kanoonilist morfismi. Koherentsiprobleemi lahendamiseks arendame me arvutusest $\mathtt{SkNMILL}_{\mathtt{S}}$ märgendeid kasutava fokuseeritud versiooni, milles arvutuse $\mathtt{SkNMILL}_{\mathtt{S}}$ ekvivalentsed tuletused on esindatud üheainsa tuletusega.

Edasi käsitleme mitut sekventsiarvutuse $\mathtt{SkNMILL}_{\mathtt{S}}$ laiendust koos fokuseeritud versioonidega: kommutatiivset laiendust ($\mathtt{SkMILL}_{\mathtt{S}}$), mida iseloomustab piiratud vahetusreegel, aditiivset laiendust ($\mathtt{SkNMILLA}_{\mathtt{S}}$), mis toetab konjunktsiooni ja disjunktsiooni konnektiive, ja nende kombinatsiooni. Igaüks neist laiendustest vastab kiivmonoidiliste kategooriate teatud spetsialisatsioonile.

Poolstruktuursete loogikate jaoks, mida ei saa õigesti kirjeldada eripositsiooniga sekventsiarvutuse abil, nagu nt kiivmonoidiliste bi-suletud kategooriate loogika, ammutame inspiratsiooni mitteassotsiatiivse Lambeki arvutuse jaoks tehtud töödest. Me esitame nende jaoks aksiomaatilised ja puusekventside arvutused ning näitame, et nad on ekvivalentsed. Lisaks näitame nende korrektsuse ja täielikkuse ternaarse relatsioonilise semantika suhtes ning vastavusteoreemi struktuursete seaduste ja raamitingimuste vahel.

Meie hinnangul näitavad need leiud, et pool-allstruktuursed loogikad on deduktiivsete ja semantiliste omaduste poolest rikkad ning pakuvad loogikasüsteemidena enamat huvi kui et on kiivmonoidiliste kategooriate variantide loogikalised vasted.

Mitu võtmetulemust, sh lõikereeglite lubatavuse ja tõestuste normaliseeritavuse arvutuste $\mathtt{SkMILL}_{\mathtt{S}}$ ja $\mathtt{SkNMILLA}_{\mathtt{S}}$ juures, formaliseerisime töö käigus tõestusassistendiga Agda.

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3. C.-S. Wan. Semi-substructural logics à la Lambek. In A. Indrzejczak and M. Zawidzki, editors, *Proceedings of 11th International Conference on Non-classical Logics: Theory and Applications, NCL 2024*, volume 415 of *Electronic Proceedings in Theoretical Computer Science*, pages 195–213. Open Publishing Association, 2024

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Teadustegevus

Publikatsioonid

1. T. Uustalu, N. Veltri, and C.-S. Wan. Proof theory of skew non-commutative MILL. In A. Indrzejczak and M. Zawidzki, editors, *Proceedings of 10th International Conference on Non-classical Logics: Theory and Applications, NCL 2022*, volume 358 of *Electronic Proceedings in Theoretical Computer Science*, pages 118–135. Open Publishing Association, 2022

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Konverentsiettekanded

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  - Ülemaailmse loogikapäeva 2022 Logic in Estonia töötuba, Tallinn.

ISSN 2585-6901 (PDF)
ISBN 978-9916-80-325-7 (PDF)