

CMCS'12

**11th International Workshop on
Coalgebraic Methods in Computer Science**

CMCS 2012



Tallinn, Estonia, 31 March–1 April 2012

Short Contributions



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Institute of Cybernetics at Tallinn University of Technology

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These short contributions will not be included in the CMCS final proceedings in Springer's *Lect. Notes in Comput. Sci. (LNCS)*.

Preface

The 11th International Workshop on Coalgebraic Methods in Computer Science, CMCS 2012 was held on March 31–April 1, 2012 in Tallinn, Estonia, as a satellite event of the Joint Conference on Theory and Practice of Software, ETAPS 2012. In more than a decade of research, it has been established that a wide variety of state-based dynamical systems, like transition systems, automata (including weighted and probabilistic variants), Markov chains, and game-based systems, can be treated uniformly as coalgebras. Coalgebra has developed into a field of its own interest presenting a deep mathematical foundation, a growing field of applications, and interactions with various other fields such as reactive and interactive system theory, object-oriented and concurrent programming, formal system specification, modal and description logics, artificial intelligence, dynamical systems, control systems, category theory, algebra, analysis, etc. The aim of the CMCS workshop series is to bring together researchers with a common interest in the theory of coalgebras, their logics, and their applications.

Previous workshops have been organised in Lisbon (1998), Amsterdam (1999), Berlin (2000), Genoa (2001), Grenoble (2002) Warsaw (2003), Barcelona (2004), Vienna (2006), Budapest (2008), and Paphos (2010). Starting in 2004, CMCS has become biennial, alternating with the International Conference on Algebra and Coalgebra in Computer Science (CALCO), which, in odd-numbered years, has been formed by the union of CMCS with the International Workshop on Algebraic Development Techniques (WADT).

This volume contains the short contributions presented at CMCS 2012, complementing the proceedings volume presenting the regular papers. Short contributions describe work in progress, summarise work submitted to a conference or workshop elsewhere, or in some other way appeal to the CMCS audience. They underwent a light reviewing process. As for regular papers, contributions that describe the application of coalgebraic methods in areas that are not the central focus of the community have been particularly welcome.

March 12, 2012
London and Erlangen

Dirk Pattinson
Lutz Schröder

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Distributive Laws of Directed Containers

[Extended Abstract]

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Containers [1] are an elegant representation of a wide class of datatypes in terms of shapes and positions in shapes. In our FoSSaCS 2012 work [2], we introduced directed containers as a special case to account for the common situation where every position in a shape determines another shape, informally the subshape rooted by that position; some examples being the datatypes of nonempty lists and trees and the corresponding zipper datatypes. While containers interpret into set functors via a fully faithful monoidal functor, directed containers interpret into comonads. Further, it is also true that every comonad whose underlying functor is a container is represented by a directed container. In this paper, we develop a characterization of distributive laws between such comonads.

A container $S \triangleleft P$ is given by a set S (of shapes) and a shape-indexed family $P : S \rightarrow \mathbf{Set}$ (of positions). A morphism between containers $S \triangleleft P$ and $S' \triangleleft P'$ is a pair $t \triangleleft q$ of maps $t : S \rightarrow S'$ and $q : \Pi\{s : S\}. P'(ts) \rightarrow P s$. (We use Agda's syntax of braces for implicit arguments.) Containers form a category **Cont** carrying a monoidal structure defined by $\mathbf{ld}^c = 1 \triangleleft \lambda * . 1$ and $(S_0 \triangleleft P_0) \cdot^c (S_1 \triangleleft P_1) = \Sigma s : S_0. P_0 s \rightarrow S_1 \triangleleft \lambda (s, v). \Sigma p_0 : P_0 s. P_1 (v p_0)$ together suitable unital and associativity laws.

The interpretation of a container $S \triangleleft P$ is the set functor given by $\llbracket S \triangleleft P \rrbracket^c X = \Sigma s : S. P s \rightarrow X$, $\llbracket S \triangleleft P \rrbracket^c f (s, v) = (s, f \circ v)$. The interpretation of a container map $t \triangleleft q$ is the natural transformation $\llbracket t \triangleleft q \rrbracket^c (s, v) = (ts, v \circ q \{s\})$. $\llbracket - \rrbracket^c$ is a fully faithful monoidal functor from **Cont** to $[\mathbf{Set}, \mathbf{Set}]$.

A directed container is a container $S \triangleleft P$ together with three operations

- $\downarrow : \Pi s : S. P s \rightarrow S$ (the subshape for a position),
- $\circ : \Pi\{s : S\}. P s$ (the root),
- $\oplus : \Pi\{s : S\}. \Pi p : P s. P (s \downarrow p) \rightarrow P s$ (translation of subshape positions into positions in the global shape),

satisfying the following two shape equations and three position equations:

1. $\forall\{s\}. s \downarrow \circ = s$,
2. $\forall\{s, p, p'\}. s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$,

3. $\forall\{s, p\}. p \oplus \{s\} \circ = p,$
4. $\forall\{s, p\}. \circ\{s\} \oplus p = p,$
5. $\forall\{s, p, p', p''\}. (p \oplus \{s\} p') \oplus p'' = p \oplus (p' \oplus p'').$

Notice that, modulo the fact that the positions involved come from different sets, equations 3-5 are the equations of a monoid. Equations 1-2 make sure that equations 4-5 are well-typed. A morphism between directed containers $(S \triangleleft P, \downarrow, \circ, \oplus)$ and $(S' \triangleleft P', \downarrow', \circ', \oplus')$ is a morphism $t \triangleleft q$ between the containers $S \triangleleft P$ and $S' \triangleleft P'$ that satisfies these equations:

1. $\forall\{s, p\}. t (s \downarrow q p) = t s \downarrow' p,$
2. $\forall\{s\}. \circ\{s\} = q (\circ' \{t s\}),$
3. $\forall\{s, p, p'\}. q p \oplus \{s\} q p' = q (p \oplus' \{t s\} p').$

Here, equations 2-3 are reminiscent of the equations of a monoid morphism. Directed containers form a category **DCont**.

The interpretation $\llbracket S \triangleleft P, \downarrow, \circ, \oplus \rrbracket^{\text{dc}}$ of a directed container is the set functor $\llbracket S \triangleleft P \rrbracket^c$ together with natural transformations ε, δ where $\varepsilon(s, v) = v(\circ\{s\})$ and $\delta(s, v) = (s, \lambda p. (s \downarrow p, \lambda p'. v(p \oplus \{s\} p')))$, making a comonad. The interpretation $\llbracket t \triangleleft q \rrbracket^{\text{dc}}$ of a directed container morphism is $\llbracket t \triangleleft q \rrbracket^c$, which is a comonad morphism. $\llbracket - \rrbracket^{\text{dc}}$ is a fully-faithful functor $\mathbf{DCont} \rightarrow \mathbf{Comonads}(\mathbf{Set})$. Moreover, every comonad whose underlying functor is a container is represented by a directed container. Actually, **DCont** is isomorphic to **Comonoids(Cont)**, and that in turn is easily seen to be the pullback of $U : \mathbf{Comonads}(\mathbf{Set}) \rightarrow [\mathbf{Set}, \mathbf{Set}]$ along $\llbracket - \rrbracket^c : \mathbf{Cont} \rightarrow [\mathbf{Set}, \mathbf{Set}]$.

A sufficient condition for the composition of the underlying functors of two comonads to carry a comonad structure is that they distribute over each other. We develop the corresponding concept for directed containers and show that it is adequate.

For two directed containers $(S_0 \triangleleft P_0, \downarrow_0, \circ_0, \oplus_0)$ and $(S_1 \triangleleft P_1, \downarrow_1, \circ_1, \oplus_1)$, we define a distributive law to be given by operations

- $t_1 : \Pi s : S_0. \Pi v : P_0 s \rightarrow S_1. P_1(v(\circ_0\{s\})) \rightarrow S_0,$
- $q_0 : \Pi\{s : S_0\}. \Pi\{v : P_0 s \rightarrow S_1\}. \Pi p_1 : P_1(v(\circ_0\{s\})).$
 $P_0(t_1 s v p_1) \rightarrow P_0 s,$
- $q_1 : \Pi\{s : S_0\}. \Pi\{v : P_0 s \rightarrow S_1\}. \Pi p_1 : P_1(v(\circ_0\{s\})).$
 $\Pi p_0 : P_0(t_1 s v p_1). P_1(v(q_0\{s\}\{v\} p_1 p_0))$

satisfying the equations

1. $\forall\{s, v, p_1, p_0\}. t_1 s v p_1 \downarrow_0 p_0$
 $= t_1 (s \downarrow_0 q_0 p_1 p_0) (\lambda p'_0. v(q_0 p_1 p_0 \oplus_0 p'_0)) (q_1 p_1 p_0),$
2. $\forall\{s, v\}. t_1 s v \circ_1 = s,$
3. $\forall\{s, v, p_1, p'_1\}. t_1 s v (p_1 \oplus_1 p'_1) = t_1 (t_1 s v p_1) (\lambda p_0. v(q_0 p_1 p_0) \downarrow_1 q_1 p_1 p_0) p'_1,$
4. $\forall\{s, v, p_1\}. q_0\{s\}\{v\} p_1 \circ_0 = \circ_0,$
5. $\forall\{s, v, p_1, p_0, p'_0\}. q_0\{s\}\{v\} p_1 (p_0 \oplus_0 p'_0) = q_0 p_1 p_0 \oplus_0 q_0 (q_1 p_1 p_0) p'_0,$
6. $\forall\{s, v, p_0\}. q_0\{s\}\{v\} \circ_1 p_0 = p_0,$
7. $\forall\{s, v, p_1, p'_1, p_0\}. q_0\{s\}\{v\} (p_1 \oplus_1 p'_1) p_0 = q_0 p_1 (q_0 p'_1 p_0),$

8. $\forall\{s, v, p_1\}. q_1 \{s\} \{v\} p_1 \circ_0 = p_1,$
9. $\forall\{s, v, p_1, p_0, p'_0\}. q_1 \{s\} \{v\} p_1 (p_0 \oplus_0 p'_0) = q_1 (q_1 p_1 p_0) p'_0,$
10. $\forall\{s, v, p_0\}. q_1 \{s\} \{v\} \circ_1 p_0 = \circ_1,$
11. $\forall\{s, v, p_1, p'_1, p_0\}. q_1 \{s\} \{v\} (p_1 \oplus_1 p'_1) p_0 = q_1 p_1 (q_0 p'_1 p_0) \oplus_1 q_1 p'_1 p_0.$

If we ignore that both P_0 and P_1 are families rather than sets (i.e., confine ourselves to the special case $S_0 = S_1 = 1$), the equations 4-11 are the equations required of two monoids to have a knit or Zappa-Szép product (see [3, Lemma 3.13 (xv)]).

A distributive law as above determines a container morphism $t \triangleleft q : (S_0 \triangleleft P_0) \cdot^c (S_1 \triangleleft P_1) \rightarrow (S_1 \triangleleft P_1) \cdot^c (S_0 \triangleleft P_0)$ by $t(s, v) = (v(\circ_0 \{s\}), t_1 s v)$ and $q\{s, v\}(p_1, p_0) = (q_0 \{s\} \{v\} p_1 p_0, q_1 \{s\} \{v\} p_1 p_0)$. The interpreting natural transformation $\llbracket t \triangleleft q \rrbracket^c$ gives a distributive law θ between the comonads $\llbracket S_0 \triangleleft P_0, \downarrow_0, \circ_0, \oplus_0 \rrbracket^{\text{dc}}$ and $\llbracket S_1 \triangleleft P_1, \downarrow_1, \circ_1, \oplus_1 \rrbracket^{\text{dc}}$ by $\theta(s, v) = (\pi_0(v(\circ_0 \{s\})), \lambda p_1. (t_1 s (\pi_0 \circ v) p_1, \lambda p_0. \pi_1(v(q_0(p_1, p_0))) (q_1(p_1, p_0))))$. And conversely, any distributive law between these two comonads corresponds to a distributive law between the two directed containers. The fact that the composition of two directed containers distributing over each other is a directed container follows from the properties of $\llbracket - \rrbracket^{\text{dc}}$ (“via the semantics”), but is also easily proved directly (“syntactically”).

We see that, just as comonads whose underlying functor is the interpretation of a container have some special properties (the outer shape of the nested datastructure returned by the comultiplication is the shape of the given datastructure), so do distributive laws between such comonads have some similar properties (the outer shape of the nested datastructure returned by the distributive law is the inner shape at the outer root position of the given nested datastructure).

In the paper, we present and analyze several generic distributive laws of comonads (e.g., distributivity of any comonad over the product comonad, distributive laws for cofree comonads) in this form as well as some that are specific to comonads whose underlying functors are containers.

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Final coalgebras in categories with factorization systems

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Abstract. For a functor $T: \mathcal{C} \rightarrow \mathcal{C}$, we show that if \mathcal{C} admits a factorization system $(\mathcal{L}, \mathcal{R})$ with all arrows in \mathcal{R} monic and \mathcal{C} is \mathcal{R} -well-powered, then the final coalgebra is characterized constructively as the \mathcal{R} -union of certain sets of T -coalgebras, provided that the final sequence of T has an \mathcal{R} -arrow at some ordinal α , and T preserves \mathcal{R} -morphisms.

Final sequence: Let T be an endofunctor on a category \mathcal{C} with final object and limits of ordinal-indexed diagrams. The *final sequence* of T is a limit-preserving functor $A: \mathbf{Ord}^{op} \rightarrow \mathcal{C}$ such that, for all ordinals $\gamma \leq \beta$, $A(\beta+1) = TA(\beta)$, $A(\beta+1 \rightarrow \gamma+1) = TA(\beta \rightarrow \gamma)$, and $A(0) = 1$. Note that, since A preserves limits, for all limit ordinals β , the arrow $A(\beta) \rightarrow \lim_{\gamma < \beta} A(\gamma)$ is an isomorphism.

In [1,2], it is shown that if this sequence *stabilizes* at some α , in the sense that $f = A(\alpha+1 \rightarrow \alpha)$ is an isomorphism, then $(A(\alpha), f^{-1})$ is a final T -coalgebra. This follows since, for any T -coalgebra (X, h) and ordinal β , there exists a cone $(X, (h_\gamma)_{\gamma \in \beta^{op}})$ on $A \upharpoonright \beta^1$ uniquely determined by $A(\gamma+1 \rightarrow \gamma) \circ Th_\gamma \circ h = h_\gamma$, for all $\gamma \leq \beta$. Therefore, $h_\alpha: (X, h) \rightarrow (A(\alpha), f^{-1})$ is a morphism of T -coalgebras, which is easily seen to be unique. Note that, this method is constructive if one can determine an ordinal α at which the final sequence stabilizes.

In [3], Worrell shows that for any mono-preserving accessible endofunctor on a locally presentable category the final sequence stabilizes. However, the proof does not give any constructive bound for stabilization. If one restricts the attention to only κ -accessible endofunctors on \mathbf{Set} , $\kappa + \kappa$ steps are sufficient for the final sequence to stabilize. This bound depends heavily on the fact that in \mathbf{Set} , all monomorphisms split, which is indeed a very strong requirement.

Both the results need that the final sequence A reaches a monic arrow at some α , then stabilization follows since the category is well-powered and all $A(\gamma)$ are subobjects of $A(\alpha)$, for all $\gamma \geq \alpha$. The restriction on accessible functors on locally accessible categories ensures that these requirements hold, and that the underlying category has a (strong-epi, mono) factorization system.

A new characterization: Let $(\mathcal{L}, \mathcal{R})$ be a factorization system for \mathcal{C} , such that arrows in \mathcal{R} are monic, and let \mathcal{C} be \mathcal{R} -well-powered. Under these hypotheses, we give a characterization of a final coalgebra for a functor $T: \mathcal{C} \rightarrow \mathcal{C}$ via its final sequence. The use of the final sequence is twofold: it guarantees unicity of

¹ $A \upharpoonright \beta: \beta^{op} \rightarrow \mathcal{C}$ restricts A on the full subcategory of \mathbf{Ord}^{op} of all ordinals $\gamma \leq \beta$.

the final homomorphism, and provides a weakly final coalgebra. Notably, we do not need any bound for stabilization, still the proof is constructive.

Theorem 1. *Assume A , the final sequence of T , is such that $A(\alpha+1 \rightarrow \alpha) \in \mathcal{R}$, for some α , and that T preserves \mathcal{R} -morphisms. Then, for any T -coalgebra (X, h) and $\rho_h \circ \lambda_h \in \mathcal{L}$, $(\mathcal{L}, \mathcal{R})$ -factorization of h_α , there exists a unique arrow ϕ_h making the diagram aside commute.*

$$\begin{array}{ccccc}
 X & \xrightarrow{\lambda_h} \triangleright & F_h & \xrightarrow{\rho_h} & A(\alpha) \\
 h \downarrow & & \downarrow \phi_h & & \uparrow \\
 TX & \xrightarrow{T\lambda_h} & TF_h & \xrightarrow{T\rho_h} & A(\alpha+1)
 \end{array}$$

Moreover, if (X, h) is weakly final, then (F_h, ϕ_h) is a final T -coalgebra.

Proof. (Sketch) Since T preserves \mathcal{R} -morphisms, $T\rho_h \in \mathcal{R}$. Therefore, the outer square diagram is a lifting problem for $\lambda_h \in \mathcal{L}$ and $A(\alpha+1 \rightarrow \alpha) \circ T\rho_h \in \mathcal{R}$, and ϕ_h is solution to it. Assume (X, h) is weakly final, then (F_h, ϕ_h) is weakly final too, since λ_h is a T -homomorphism. Unicity follows by left cancellability of ρ_h , since, for any T -coalgebra (Y, k) and arrow $f: Y \rightarrow A(\alpha)$ such that $f = A(\alpha+1 \rightarrow \alpha) \circ Tf \circ k$, one proves by transfinite induction that $f = k_\alpha$. \square

Note that, $(\mathcal{L}, \mathcal{R})$ -factorizations of morphisms are not unique, hence for a given T -coalgebra (X, h) , the associated T -coalgebra (F_h, ϕ_h) is not uniquely determined. However, under the hypothesis of Theorem 1, one can fix any factorization $h_\alpha = \rho_h \circ \lambda_h$ to obtain an endofunctor F on the category of T -coalgebras, mapping objects (X, h) to (F_h, ϕ_h) , and morphisms $f: (X, h) \rightarrow (Y, k)$ to the unique solution φ_f of the lifting problem on the right. Functoriality crucially depends on the assumption that all \mathcal{R} -morphisms are monic (see the Appendix for details).

$$\begin{array}{ccccc}
 X & \xrightarrow{\lambda_h} \triangleright & F_h & \xrightarrow{\rho_h} & A(\alpha) \\
 f \downarrow & & \downarrow \varphi_f & & \parallel \\
 Y & \xrightarrow{\lambda_k} \triangleright & F_k & \xrightarrow{\rho_k} & A(\alpha)
 \end{array}$$

Finally, observe that, for all T -coalgebras (X, h) , F_h is an \mathcal{R} -subobject of $A(\alpha)$, and since \mathcal{C} is assumed to be \mathcal{R} -well-powered, there must be only a set I (up to isomorphism) of such F_h 's. Thus, if \mathcal{C} has coproducts, we are allowed to take the coproduct coalgebra $\coprod_I (F_i, \phi_i)$, which is readily seen to be weakly final, with homomorphism from any T -coalgebra (X, h) given by

$$X \xrightarrow{\lambda_h} F_h \xrightarrow{\cong} F_i \xrightarrow{in_i} \coprod_I F_i$$

where F_i is the representative of F_h in I .

Applying Theorem 1 to $\coprod_I (F_i, \phi_i)$, the final T -coalgebra is just the \mathcal{R} -union of the coalgebras in I . Notably, finality does not depend on the choice of I , which can be determined constructively by an analysis on the \mathcal{R} -subobjects of $A(\alpha)$.

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On Coalgebraic Logic over Posets

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Abstract. We relate the abstract coalgebraic logic for finitary **Set**-functors with the corresponding logic for their **Pos**-extensions.

Keywords: coalgebraic logic, duality, positive modal logic

This contribution continues the previous work of [3] and [1].

We start by recalling the following adjunction between the category of sets and the category of Boolean algebras, namely

$$T^{op} \text{Set}^{op} \begin{array}{c} \leftarrow^S \\ \perp \\ \rightarrow^P \end{array} \text{BoolAlg} \begin{array}{c} \leftarrow^L \\ \rightarrow^L \end{array} \quad (1)$$

where P maps a set to its powerset, while S maps a Boolean algebra to its set of ultrafilters. T is a (finitary) **Set**-functor coalgebraically modeling the semantics of some transition systems and L stands for the T -associated abstract Boolean logic, as in [4]. Remember that L preserves sifted colimits and coincides with $PT^{op}S$ on finitely generated free algebras.⁴

Denote by **Pos** the category of posets and monotone functions and by $D : \text{Set} \rightarrow \text{Pos}$ the functor endowing each set with the discrete order. One has a chain of adjunctions $U \vdash D \vdash C : \text{Pos} \rightarrow \text{Set}$, where U is the forgetful functor and C maps a poset to the set of its connected components. If one regards **Set** as discretely enriched over **Pos**, then D and C form an enriched adjunction, while U fails to be locally monotone.

Following [1], for a given $T : \text{Set} \rightarrow \text{Set}$, we shall call the enriched⁵ left Kan extension $\text{Lan}_D(DT)$ of DT along T the *posetification* of T . For example, it was

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⁴ Or, equivalently, $L = \text{Lan}_J(PT^{op}SJ)$, where $J : \text{BoolAlg}_\omega \rightarrow \text{BoolAlg}$ is the inclusion functor from the category BoolAlg_ω of finite Boolean algebras.

⁵ Over **Pos**, ‘enriched’ means that all functors involved, in particular $\text{Lan}_D(DT)$, are locally monotone, that is, they preserve the order on the homsets.

shown in [5] that the posetification of the finite powerset functor is the finitely generated convex powerset with convex subsets ordered by the Egli-Milner order.

We recall the (Pos-enriched) adjunction between Pos and DLat, the category of distributive lattices,

$$\text{Pos}^{op} \begin{array}{c} \xleftarrow{S'} \\ \perp \\ \xrightarrow{P'} \end{array} \text{DLat} \quad (2)$$

where P' maps a poset to the distributive lattice of its uppersets, and S' associates to each distributive lattice the poset of prime filters.

For each finitary locally monotone functor T' on Pos, one can build as in [3] a dual functor $L' : \text{DLat} \rightarrow \text{DLat}$, also locally monotone. Specifically, L' is $P'T'^{op}S'$ on finitely generated free distributive lattices and extended to all distributive lattices using sifted colimits.

Denote by W the forgetful functor $\text{BoolAlg} \rightarrow \text{DLat}$.

Theorem. *Let T be a Set-functor preserving weak pullbacks and T' its posetification. Let L and L' be the associated logic functors given by $PT^{op}S$ and $P'T'^{op}S'$ on finitely generated free algebras.*

$$\begin{array}{ccc} T^{op} \curvearrowright \text{Set}^{op} & \begin{array}{c} \xleftarrow{S} \\ \perp \\ \xrightarrow{P} \end{array} & \text{BoolAlg} \curvearrowright L \\ \downarrow D & & \downarrow W \\ T'^{op} \curvearrowright \text{Pos}^{op} & \begin{array}{c} \xleftarrow{S'} \\ \perp \\ \xrightarrow{P'} \end{array} & \text{DLat} \curvearrowright L' \end{array} \quad (3)$$

Then L' is the positive fragment of L in the sense that $L'W \cong WL$.

For the special case where T is the powerset functor, we have that L is the functor associated with Kripke's modal logic \mathbf{K} and L' the functor associated with Dunn's positive modal logic [2]. Then $L'W \cong WL$ is an abstract formulation of the following well-known facts: 1) every formula ϕ in \mathbf{K} can be written as a positive formula ϕ^+ with negation only appearing on atomic propositions. 2) ϕ and ψ are provably equivalent in \mathbf{K} iff ϕ^+ and ψ^+ are provably equivalent in positive modal logic.

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Coalgebraic Dynamic Quantum Logic

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We present our on-going work on *Coalgebraic Quantum Logic*, a new development that brings together the work on quantum (labelled) transition systems (introduced in [2]) to describe the behavior of quantum systems and coalgebraic logic [3]. The setting of our framework is inspired by Abramsky’s [1] coalgebraic representation of physical systems, which can be used to generalize the non-probabilistic setting of [2] to a quantum coalgebraic and probabilistic level.

Preliminaries We first review the basics of predicate-lifting coalgebraic logic; see, e.g., [3] for a detailed exposition. Given any set functor F , an F -coalgebra is a set S together with a function $\sigma : S \rightarrow FS$. A predicate lifting λ for F is a natural transformation $\lambda : \mathcal{Q} \rightarrow \mathcal{Q}F$, where \mathcal{Q} is the contravariant powerset functor. Fixing a set Λ of predicate liftings for F , we define a language \mathcal{L} inductively from a fixed set Prop of proposition letters, with operators \neg, \wedge, \vee and unary $[\lambda]$ for all $\lambda \in \Lambda$. Given a coalgebra (S, σ) and a valuation $V : \text{Prop} \rightarrow \mathcal{P}S$, we define the interpretation $\llbracket - \rrbracket : \mathcal{L} \rightarrow \mathcal{P}S$ inductively with the classical Boolean clauses for \neg, \wedge, \vee , and $\llbracket [\lambda]\phi \rrbracket = \sigma^{-1} \circ \lambda_S \llbracket \phi \rrbracket$.

Abramsky [1] represents physical systems with coalgebras for the following set functor F . Fixing a set Π of questions (or testable properties, i.e., closed subspaces of a given Hilbert space in the quantum case), define

$$F : X \mapsto [\{0\} + (0, 1] \times X]^\Pi.$$

Write π_1 and π_2 for the projections from $\{0\} + (0, 1] \times X$ to $[0, 1]$ and X ; π_2 is partial, with $\pi_2(0)$ undefined. Then, given a set S of states, a coalgebra (S, σ) represents that asking a question $q \in \Pi$ at a state $s \in S$ yields the answer “yes” with probability $\pi_1(\sigma(s)(q))$, bringing the system to the next state $\pi_2(\sigma(s)(q))$.

Probabilistic coalgebraic logic Using Abramsky’s coalgebras, we introduce coalgebraic logic of probabilistic dynamic modalities. Consider the family Λ of predicate liftings $\lambda^{q,p}$ for F , each labelled with $q \in \Pi$ and $p \in [0, 1]$, such that

$$\lambda_X^{q,p}(Y) := \{ \delta \in FX \mid \delta(q) \in [p, 1] \times Y \}$$

for $p > 0$, and $\lambda_X^{q,0}(Y) := \bigcup_{p>0} \lambda_X^{q,p}(Y)$. This Λ gives rise to coalgebraic logic in which $[\lambda^{q,p}]\phi$ is true at a state $s \in S$ iff asking the question q (or applying the “program” q) yields yes (or success) with probability at least p (or any probability, in the case of $p = 0$) and, after yes, ϕ is true at the next state. $[\lambda^{q,0}]\phi$ expresses the “weakest precondition” that ensures ϕ after q .

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Coalgebraic quantum semantics We further extend the framework by considering how quantum systems are characterized in terms of Abramsky's representation, thereby obtaining coalgebraic quantum semantics. Take any sets \mathcal{T} of subsets $P \subseteq S$ and \mathcal{U} of functions $U : S \rightarrow S$, and let $\Pi = \mathcal{T} \cup \mathcal{U}$ be our set of programs; our goal is to axiomatize \mathcal{T} and \mathcal{U} as the sets of testable properties and unitary transformations in quantum systems. (We write $P?$ for $P \in \mathcal{T}$ regarded as an element of Π , to indicate that it is a test rather than a proposition.)

Let us introduce the following notation. A coalgebra $\sigma : S \rightarrow FS$ associates with each $q \in \Pi$ a partial function $\bar{q} = \pi_2(\sigma(-)(q)) : S \rightarrow S$. The set of states reachable from s by some test $P?$ is $\overline{\mathcal{T}}?(s) := \{\overline{P?}(s) \mid P \in \mathcal{T}\}$. We say that t is orthogonal to s , and write $t \perp s$, if $t \notin \overline{\mathcal{T}}?(s)$. Let $\sim P := \{s \mid s \perp t \text{ for all } t \in P\}$ be the orthocomplement of $P \subseteq S$, and $P \sqcup Q := \sim(\sim P \cap \sim Q)$ the quantum join of P and Q . Now, we say that σ is a *quantum coalgebra* if it satisfies the following axioms, which are a probabilistic extension of those in [2].

1. *Closure under arbitrary conjunctions*: $\bigcap \mathcal{T}' \in \mathcal{T}$ for any $\mathcal{T}' \subseteq \mathcal{T}$.
2. *Closure under orthocomplementation*: if $P \in \mathcal{T}$, then $\sim P \in \mathcal{T}$.
3. *Atomicity*: states are testable, i.e., $\{s\} \in \mathcal{T}$ for any $s \in S$.
4. *Adequacy*: testing a true property always succeeds and does not change the state, i.e., $\sigma(s)(P?) = (1, s)$ if $s \in P \in \mathcal{T}$.
5. *Repeatability*: any testable property holds after a successful test thereof, i.e., for any $P \in \mathcal{T}$ and $s \in S$, $\overline{P?}(s) \in P$ whenever $\overline{P?}(s)$ is defined.
6. *Covering law*: if $\overline{P?}(s) \neq t \in P$, then $v \perp s$ for some $v \in \overline{\mathcal{T}}?(t) \cap P$.
7. *Self-adjointness*: $\pi_1(\sigma(\overline{P?}(s))(\{t\}?)) = \pi_1(\sigma(\overline{P?}(t))(\{s\}?))$ for any $s, t \in S$.
8. *Proper superposition*: $\overline{\mathcal{T}}?(s) \cap \overline{\mathcal{T}}?(t) \neq \emptyset$ for any $s, t \in S$.
9. For any $P_0, P_1 \in \mathcal{T}$ such that $P_0 \subseteq \sim P_1$ and for all $s \in S$, we have $\pi_1(\sigma(s)(P_0 \sqcup P_1?)) = \pi_1(\sigma(s)(P_0?)) + \pi_1(\sigma(s)(P_1?))$.
10. *Reversibility and totality*: unitary evolutions $U \in \mathcal{U}$ are deterministic (and total) bijections, i.e., for every $s \in S$ there is a $t \in S$ such that $\sigma(s)(U) = (1, t)$ and for every $t \in S$ there is an $s \in S$ such that $\sigma(s)(U) = (1, t)$.
11. *Orthogonality preservation*: $s \perp t$ iff $\overline{U}(s) \perp \overline{U}(t)$ for any $s, t \in S$ and $U \in \mathcal{U}$.
12. *Mayet's condition*: there exist $U \in \mathcal{U}, P \in \mathcal{T}$ and $t, w \in S$ such that $\{\overline{U}(s) \mid s \in P\} \subsetneq P$, $t \perp w$, and, for every $s \in \sim\sim\{t, w\}$, $\overline{U}(s) = s$.

Conclusion These structures exhibit the basic ingredients to describe the behaviour of single quantum systems. In the future we should axiomatize the coalgebraic quantum logic of these structures, as well as extend them to the multipartite case in order to construct coalgebraic models of quantum computation.

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Weak bisimulations for coalgebras over ordered functors

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Abstract. The aim of this talk is to introduce a coalgebraic setting in which it is possible to generalize and compare the two known approaches to weak bisimulation for labelled transition systems. We introduce two definitions of weak bisimulation for coalgebras over ordered functors, show their properties and give sufficient conditions for them to coincide.

The notion of a strong bisimulation plays an important role in theoretical computer science and more specifically in the theory of coalgebras (see e.g. [6]). A weak bisimulation defined for many transition systems is a relaxation of this notion by allowing silent, unobservable transitions. In the case of labelled transition systems there are two equivalent approaches to defining weak bisimulation.

Definition 1. Let Σ be a set of labels and let $\tau \in \Sigma$ be a silent transition label. Let $\langle A, \Sigma, \rightarrow \rangle$ be a labelled transition system. A symmetric relation $R \subseteq A \times A$ is a weak bisimulation if it satisfies one of the following two conditions:

Approach 1. The inclusion $(a, b) \in R$ implies that for any $\sigma \in \Sigma$ for which $\sigma \neq \tau$ if $a \xrightarrow{\sigma} a'$ then $b \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} b'$ for some $b' \in A$ and $(a', b') \in R$, and for $\sigma = \tau$ if $a \xrightarrow{\tau} a'$ then $b \xrightarrow{\tau^*} b'$ for $b' \in A$ and $(a', b') \in R$.

Approach 2. The inclusion $(a, b) \in R$ implies that for any $\sigma \in \Sigma$ for which $\sigma \neq \tau$ we have $a \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} a'$ if and only if $b \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} \circ \xrightarrow{\tau^*} b'$ for some $b' \in A$ for which $(a', b') \in R$, and for $\sigma = \tau$ we have $a \xrightarrow{\tau^*} a'$ if and only if $b \xrightarrow{\tau^*} b'$ for $b' \in A$ for which $(a', b') \in R$.

Many mathematicians and computer scientists have attempted to generalize the notion of a weak bisimulation to coalgebras (see for instance [3], [4], [5] and many other). In this talk we will present a setting in which two approaches to defining weak bisimulation will be possible. We will list their properties and give conditions for their coincidence.

Let \mathbf{Pos} be the category of all posets and monotonic mappings. Let $U : \mathbf{Pos} \rightarrow \mathbf{Set}$ be the forgetful functor. An order on a functor $S : \mathbf{Set} \rightarrow \mathbf{Set}$ (see [1] for details) is a functor $\leq : \mathbf{Set} \rightarrow \mathbf{Pos}$ for which $U \circ \leq = S$.

Note that for instance the powerset endofunctor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ is ordered by the functor $\leq : \mathbf{Set} \rightarrow \mathbf{Pos}$, which assigns to any set X the poset $(\mathcal{P}(X), \subseteq)$. Given

an order on a functor $S : \mathbf{Set} \rightarrow \mathbf{Set}$ and two mappings $f, g : X \rightarrow SY$ we write $f \leq g$ whenever $f(x) \leq g(x)$ for any $x \in X$. Let \mathbf{C} be full subcategory of the category of S -coalgebras and homomorphisms between them which is closed under taking inverse images of homomorphisms. During the talk we will introduce a notion of *coalgebraic saturator* with respect to a class \mathbf{C} as functor $\mathfrak{s} : \mathbf{C} \rightarrow \mathbf{Set}_S$ satisfying a list of very natural properties. Based on the definition of saturator we present two approaches to generalized versions of weak bisimulations.

Definition 2. *Let $S : \mathbf{Set} \rightarrow \mathbf{Set}$ be an ordered functor, let \mathbf{C} be subcategory of all S -coalgebras and homomorphisms which is closed under taking inverse images of homomorphisms and let \mathfrak{s} be a coalgebraic saturator with respect to \mathbf{C} . Let $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ be two S -coalgebras which are members of the class \mathbf{C} of S -coalgebras.*

Approach 1. *A relation $R \subseteq A \times B$ is called a weak bisimulation provided that there is a structure $\gamma_1 : R \rightarrow SR$ and a structure $\gamma_2 : R \rightarrow SR$ for which:*

- $\alpha \circ \pi_1 = S\pi_1 \circ \gamma_1$ and $S\pi_2 \circ \gamma_1 \leq \mathfrak{s}\beta \circ \pi_2$,
- $\beta \circ \pi_2 = S\pi_2 \circ \gamma_2$ and $S\pi_1 \circ \gamma_2 \leq \mathfrak{s}\beta \circ \pi_1$.

Approach 2. *A relation $R \subseteq A \times B$ is said to be a weak bisimulation provided that there is a structure $\gamma : R \rightarrow SR$ for which $\mathfrak{s}\alpha \circ \pi_1 = S\pi_1 \circ \gamma$ and $\mathfrak{s}\beta \circ \pi_2 = S\pi_2 \circ \gamma$.*

Note that the two approaches from Definition 2 coincide with those presented in Definition 1 when S is put to be $\mathcal{P}(\Sigma \times (-))$ with a natural order and an intuitively defined saturator. During the presentation we will compare the two approaches. The most notable of the results is the following.

Theorem 1. *Let $\langle A, \alpha \rangle$ be an S -coalgebra and let $S : \mathbf{Set} \rightarrow \mathbf{Set}$ weakly preserve kernel pairs. If $R \subseteq A \times A$ is an equivalence relation which is a weak bisimulation in the sense of Approach 1 then R is a weak bisimulation in the sense of Approach 2. If additionally S preserves the so called downsets then an equivalence relation $R \subseteq A \times A$ which is a weak bisimulation from Approach 2 is also a weak bisimulation in the sense of Approach 1.*

We will also show interesting examples of functors and coalgebras for which the two approaches do not coincide.

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Nondeterminism as first class citizen for Hidden Logic

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Introduction Hidden Logic provides behavioral equivalence as indistinguishability under experiments [2, 3]. In hidden logic nondeterminism is modeled by underspecification (e.g. [3, Example 9]). Underspecification results in a class of models that span the possible nondeterministic choices but in every model the choice is taken in a deterministic manner. This prevents to reason about the true nature of nondeterministic choices [5]. We extend hidden logic to model nondeterminism as first class citizen by evaluating functions to sets. Furthermore, we introduce subterm sharing to model dependency of nondeterministic choices. This allows to reason in a natural way about nondeterministic polyadic operations (e.g. stream computation, process composition).

Behavioral Specifications with Nondeterminism Let $\mathcal{S} = V \cup H$ where V is a set of *visible sorts* and H a set of *hidden sorts*. Let $\Sigma = \Sigma_{fun} \cup \Sigma_{rel} \cup \{\oplus_s \mid s \in \mathcal{S}\}$ be a signature over sorts \mathcal{S} consisting of *deterministic operations* Σ_{fun} and *nondeterministic operations* Σ_{rel} , and *nondeterministic choice operators* $\oplus_s : s \times s \rightarrow s$ for every $s \in \mathcal{S}$. As the sort of \oplus_s can be derived from the sorts of its arguments, we will omit s and write \oplus for short.

Behavioral Specifications. A *behavioral nondeterministic equation* $\ell = r$ or *behavioral deterministic equation* $\ell \doteq r$ consists of terms $\ell, r \in Ter(\Sigma, \mathcal{X})_s$ for some $s \in \mathcal{S}$. Relational symbol $=$ denotes equality of the set of nondeterministic choices, and \doteq additionally requires that the choice is unique (deterministic choice). For example, a random bit stream is specified by $\mathbf{rand} = (0 \oplus 1) : \mathbf{rand}$. A *behavioral specification* E is a set of behavioral formulas.

Behavioral Equivalence. A Σ -algebra \mathcal{A} is a tuple $\langle A, \llbracket \cdot \rrbracket \rangle$ consisting of an \mathcal{S} -sorted set A and for every $f \in \Sigma$, $f : s_1 \times \dots \times s_n \rightarrow s$ an *interpretation* $\llbracket f \rrbracket : A_{s_1} \times \dots \times A_{s_n} \rightarrow \mathcal{P}^+(A_s)$ such that $\llbracket f \rrbracket(a_1, \dots, a_n)$ is a singleton if $f \in \Sigma_{fun}$. For $f : A^n \rightarrow \mathcal{P}^+(A)$ and $B_1, \dots, B_n \subseteq A$ we let $f(B_1, \dots, B_n) = \bigcup \{f(b_1, \dots, b_n) \mid b_1 \in B_1, \dots, b_n \in B_n\}$. For $\alpha : \mathcal{X} \rightarrow A$ we define $\llbracket \cdot \rrbracket_\alpha : Ter(\Sigma, \mathcal{X}) \rightarrow \mathcal{P}^+(A)$ by: $\llbracket x \rrbracket_\alpha = \{\alpha(x)\}$, $\llbracket f(t_1, \dots, t_n) \rrbracket_\alpha = \llbracket f \rrbracket(\llbracket t_1 \rrbracket_\alpha, \dots, \llbracket t_n \rrbracket_\alpha)$, and $\llbracket s \oplus t \rrbracket_\alpha = \llbracket s \rrbracket_\alpha \cup \llbracket t \rrbracket_\alpha$. Elements $a, b \in A_s$ ($s \in \mathcal{S}$) are *behaviorally equivalent*, denoted $a \equiv b$, if $\llbracket C[* : s] \rrbracket_{* \mapsto a} = \llbracket C[* : s] \rrbracket_{* \mapsto b}$ for every context $C \in Ter(\Sigma, \{*\})_v$ with $* : s$ and $v \in V$. Due to nondeterminism, the equality here is set equality. Sets of elements $A, B \subseteq A_s$ are *behaviorally equivalent*, denoted $A \equiv B$, if $\forall a \in A. \exists b \in B. a \equiv b$ and $\forall b \in B. \exists a \in A. a \equiv b$, that is, the sets consist of the same set of behaviors. A

Σ -algebra \mathcal{A} *satisfies* a behavioral specification E , $\mathcal{A} \models E$, if for every $\alpha : \mathcal{X} \rightarrow A$ we have: $\llbracket \ell \rrbracket_\alpha \equiv \llbracket r \rrbracket_\alpha$ for every $\ell = r \in E_s$ and $\ell \doteq r \in E_\sigma$, and $|\llbracket \ell \rrbracket_\alpha| = |\llbracket r \rrbracket_\alpha| = 1$ for every $\ell \doteq r \in E_\sigma$. Observe that here \equiv is behavioral equivalence between sets of elements. Let E be a behavioral specification, and e a behavioral equation. Then e is said to be *satisfied in E* , denoted by $E \models e$, if $\mathcal{A} \models E$ implies $\mathcal{A} \models e$ for every Σ -algebra \mathcal{A} .

Behavioral Reasoning with Sharing In the presence of real nondeterminism, the *ordinary* equational reasoning is no longer sound. For instance, consider the specification:

$$\begin{aligned} \text{rand} &= (0 \oplus 1) : \text{rand} & \text{hd}(x : \sigma) &\doteq x \\ \text{dup}(\sigma) &\doteq \text{hd}(\sigma) : \text{hd}(\sigma) : \text{dup}(\text{tl}(\sigma)) & \text{tl}(x : \sigma) &\doteq \sigma \end{aligned}$$

where $\Sigma_{fun} = \{\text{hd}, \text{tl}, \text{dup}, 0, 1, :\}$ and $\Sigma_{rel} = \{\text{rand}\}$. In every model, of the specification, the interpretation $\llbracket \text{dup}(\text{rand}) \rrbracket$ is the set of all streams τ such that $\tau(2n) = \tau(2n+1)$ for all $n \in \mathbb{N}$. Let $s = \text{dup}(\text{rand})$ and $t = \text{hd}(\text{rand}) : \text{hd}(\text{rand}) : \text{dup}(\text{tl}(\text{rand}))$. Although s equates to t , the semantics of s and t do not agree. The term s contains one nondeterministic choice rand while t contains three nondeterministic choices. As a consequence, it is not guaranteed that $\tau(0) = \tau(1)$ for every $\tau \in \llbracket r \rrbracket$. The problem is caused by the lack of expressiveness of terms.

We propose a variant of term graph rewriting [1] with *sharing* to specify whether repeated occurrences of subterms are independent or represent the same nondeterministic choice. Then we get:

$$\text{dup}(\text{rand}) = \langle \text{hd}(X) : \text{hd}(X) : \text{dup}(\text{tl}(X)) \mid X = \text{rand} \rangle$$

where now X is a recursion variable used to share the occurrences of rand . That is, we introduce sharing whenever a term containing a nondeterministic symbol from Σ_{rel} is duplicated. For terms with sharing, we introduce semantics $\llbracket \cdot \rrbracket_\alpha$, equational reasoning, transformations for introduction and removal of sharing and a circular coinduction principle [4]. This interpretation of $\text{dup}(\text{rand})$ allows to deduce the equality of s and t .

Example We extend the above specification with:

$$\text{add}(\sigma, \tau) \doteq (\text{hd}(\sigma) + \text{hd}(\tau)) : \text{add}(\text{tl}(\sigma), \text{tl}(\tau)) \quad \text{zeros} \doteq 0 : \text{zeros}$$

where $\text{add}, \text{zeros} \in \Sigma_{fun}$ and $+$ the summation on the data algebra containing elements 0 and 1. We prove

$$\langle \text{add}(X, X) \mid X = \text{rand} \rangle \doteq \langle \text{zeros} \rangle$$

using circular coinduction. We have:

$$\begin{aligned} &\langle \text{hd}(\text{add}(X, X)) \mid X = \text{rand} \rangle \\ \doteq &\langle \text{hd}(\text{add}(X, X)) \mid X = (0 \oplus 1) : \text{rand} \rangle && \text{rewriting} \\ \doteq &\langle \text{hd}(\text{add}(Y : Z, Y : Z)) \mid Y = 0 \oplus 1, Z = \text{rand} \rangle && \text{unsharing } \Sigma_{fun} \\ \doteq &\langle Y + Y \mid Y = 0 \oplus 1, Z = \text{rand} \rangle && \text{rewriting} \end{aligned}$$

By a case distinction on $0 \oplus 1$ we obtain:

$$\begin{aligned} (Y = 0) \quad \dots &\doteq \langle Y + Y \mid Y = 0, Z = \text{rand} \rangle \doteq \langle 0 + 0 \mid \rangle \doteq \langle 0 \rangle, \text{ and} \\ (Y = 1) \quad \dots &\doteq \langle Y + Y \mid Y = 1, Z = \text{rand} \rangle \doteq \langle 1 + 1 \mid \rangle \doteq \langle 0 \rangle \end{aligned}$$

by unsharing both Y and Z , and then a rewrite step. Thus

$$\langle \text{hd}(\text{add}(X, X)) \mid X = \text{rand} \rangle \doteq \langle 0 \rangle \doteq \langle \text{hd}(\text{zeros}) \rangle$$

Now we use the coinduction hypothesis (CIH, [4])

$$\langle \text{freeze}(\text{add}(X, X)) \mid X = \text{rand} \rangle \doteq \langle \text{freeze}(\text{zeros}) \rangle$$

to prove that the tails are equal

$$\langle \text{freeze}(\text{tl}(\text{add}(X, X))) \mid X = \text{rand} \rangle \doteq \langle \text{freeze}(\text{tl}(\text{zeros})) \rangle$$

We have:

$$\begin{aligned} &\langle \text{freeze}(\text{tl}(\text{add}(X, X))) \mid X = \text{rand} \rangle \\ &\doteq \langle \text{freeze}(\text{tl}(\text{add}(Y : Z, Y : Z))) \mid Y = 0 \oplus 1, Z = \text{rand} \rangle && \text{as above} \\ &\doteq \langle \text{freeze}(\text{tl}((Y + Y) : \text{add}(Z, Z))) \mid Y = 0 \oplus 1, Z = \text{rand} \rangle && \text{rewriting} \\ &\doteq \langle \text{freeze}(\text{add}(Z, Z)) \mid Y = 0 \oplus 1, Z = \text{rand} \rangle && \text{rewriting} \\ &\doteq \langle \text{freeze}(\text{add}(Z, Z)) \mid Z = \text{rand} \rangle && \text{unsharing} \\ &\doteq \langle \text{freeze}(\text{zeros}) \rangle && \text{CIH} \\ &\doteq \langle \text{freeze}(\text{tl}(\text{zeros})) \rangle && \text{rewriting} \quad \square \end{aligned}$$

Conclusion We have proposed an extension of behavioral specifications with nondeterministic operations Σ_{rel} . If all symbols are deterministic ($\Sigma_{rel} = \emptyset$) then we obtain ordinary behavioral specifications, and symbols from Σ_{fun} can always be ‘unshared’, reducing equations with sharing back to usual equations without sharing.

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The Ball Monad and Its Metric Trace Semantics in Kleisli Categories

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In the semantics of programming there are two important traditions: one founded by Scott and Strachey using complete ordered sets, and one started by de Bakker and Zucker using complete metric spaces. In the ordered approach, recursion is modeled via the fixed point theorem for complete partial orders, while in the metric approach, one uses Banach's fixed point theorem. See [4] for an overview from a coalgebraic perspective.

Both traditions can also be used in the coalgebraic description of trace semantics. Trace semantics has been modeled in Kleisli categories that are enriched in directed complete partial orders (dcpos). This short paper illustrates a metric analogue of this approach, that works for a monad on **Sets**, which we call the ball monad \mathcal{B} . It describes complex probability distributions on a set.

The Ball Monad The ball monad \mathcal{B} has been introduced in [3]. On a set X one defines $\mathcal{B}(X)$ as:

$$\mathcal{B}(X) = \{\varphi: X \rightarrow \mathbb{C} \mid \sum_{x \in X} |\varphi(x)| \leq 1\},$$

where \mathbb{C} is the set of complex numbers. An element $\varphi \in \mathcal{B}(X)$ may be understood as a formal sum $\varphi = \sum_i z_i x_i$, if $X = \{x_i \mid i \in I\}$ and $z_i = \varphi(x_i) \in \mathbb{C}$.

For a function $f: X \rightarrow Y$ one defines $\mathcal{B}(f): \mathcal{B}(X) \rightarrow \mathcal{B}(Y)$ by:

$$\mathcal{B}(f)(\sum_i z_i x_i) = \sum_i z_i f(x_i).$$

We wish to describe trace semantics for coalgebras of the form $c: X \rightarrow \mathcal{B}FX$, where F is an endofunctor on **Sets**. We will do this by lifting the functor F to a functor \overline{F} on the Kleisli category $\mathcal{Kl}(\mathcal{B})$, and then viewing c as a coalgebra for \overline{F} in $\mathcal{Kl}(\mathcal{B})$. Before carrying out this procedure for the ball monad, we will show how it works for order-enriched Kleisli categories.

Trace Semantics in Kleisli Categories In this section we recall the essence of the coalgebraic description of trace semantics in Kleisli categories developed in [1,2]. This approach starts with a monad T and an endofunctor F , both on **Sets**, together with a distributive law $\lambda: FT \Rightarrow TF$ between them. This law corresponds to a lifting of F to an endofunctor $\overline{F}: \mathcal{Kl}(T) \rightarrow \mathcal{Kl}(T)$ on the Kleisli category, given by $\overline{F}(X) = F(X)$ and $\overline{F}(f) = \lambda \circ F(f)$. Under suitable additional conditions, an initial algebra $F(A) \xrightarrow{\cong} A$ in **Sets**, then yields a final coalgebra $A \xrightarrow{\cong} \overline{F}(A)$ in the Kleisli category $\mathcal{Kl}(T)$.

Theorem 1. *In the above situation, assume that:*

1. $T(0) \cong 1$, making $0 \in \mathcal{Kl}(T)$ a zero object;
2. the Kleisli category $\mathcal{Kl}(T)$ is **Dcpo**-enriched, in such a way that the zero maps are bottom element in the Kleisli homsets;
3. the lifted functor \overline{F} is locally continuous, i.e. preserves the directed joins in Kleisli homsets;
4. the functor F has an initial algebra $\alpha : F(A) \rightarrow A$.

Then $\eta \circ \alpha^{-1} : A \rightarrow \overline{F}(A)$ is a final coalgebra in $\mathcal{Kl}(T)$.

Trace Semantics for the Ball Monad Theorem 1 depends crucially on the dcpo-enrichment of the Kleisli category. It cannot be applied to the ball monad, so we will now switch from dcpos to complete metric spaces.

Let **Cms** be the category of complete metric spaces and non-expansive maps. A map $f : X \rightarrow Y$ between metric spaces is called *non-expansive* if

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$

for all $x, x' \in X$.

Proposition 1. *The category $\mathcal{Kl}(\mathcal{B})$ is **Cms**-enriched with metric on $\text{Hom}(X, Y)$ given by*

$$d(f, g) = \sup_{x \in X} \sum_{y \in Y} |f(x)(y) - g(x)(y)|.$$

We can use this metric structure on the Kleisli category $\mathcal{Kl}(\mathcal{B})$ to obtain trace semantics for the ball monad. The proof is more complicated than in the order-enriched case, since we need a contractive operator on the Kleisli homsets, as is standard in metric semantics. To achieve this, we have to modify the metrics on the homsets.

Theorem 2. *Let F be a polynomial functor on **Sets** with initial algebra $\alpha : A \xrightarrow{\cong} F(A)$, and let $\lambda : F\mathcal{B} \Rightarrow \mathcal{B}F$ be a distributive law. Then $\eta \circ \alpha^{-1} : A \rightarrow \overline{F}(A)$ is a final coalgebra for $\overline{F} : \mathcal{Kl}(\mathcal{B}) \rightarrow \mathcal{Kl}(\mathcal{B})$.*

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State-Based Simulation of Linear Course-of-Value Iteration

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The following categorial definition of course-of-value (cov) iteration as a recursion scheme is due to [2].

Definition 1 (Course-of-Value Iteration). *Let F be a **Set**-endofunctor and $F_C^\times = C \times F(-)$. Let (A, in) be an initial F -algebra. Let (Ω, out) be a final F_C^\times -coalgebra. Then every operation $\varphi : F\Omega \rightarrow C$ induces a function $\llbracket \varphi \rrbracket_F : A \rightarrow C$:*

$$\llbracket \varphi \rrbracket_F = \pi_1 \circ \text{out} \circ (\text{out}^{-1} \circ \langle \varphi, \text{id}_{F\Omega} \rangle)_F$$

The paradigmatic example is, of course, the Fibonacci function, where $C = \mathbb{N}$ and $F = 1 + (-)$, hence $A = \mathbb{N}$, $\Omega = \mathbb{N}^+ \cup \mathbb{N}^\omega$ and $F\Omega = \mathbb{N}^\infty = \mathbb{N}^* \cup \mathbb{N}^\omega$. Define $\varphi : \mathbb{N}^\infty \rightarrow \mathbb{N}$ as

$$\varphi() = 0 \qquad \varphi(a) = 1 \qquad \varphi(a, b, \dots) = a + b$$

to obtain $fib = \llbracket \varphi \rrbracket_F$. The same functor and basic pattern applies to many recursive functions that may depend on their own result for arbitrary smaller (in the sense of an initial algebra) arguments, as well as to black-box models in software engineering, and scientific models of history-dependent systems, such as the vast field of Box–Jenkins alias auto-regressive moving-average (ARMA) models [1].

Cov iteration is an elegant description for the sake of programming semantics or modelling theory. But often a more concrete presentation in terms of ordinary iteration and explicit state is preferable in either domain of application. For instance, for the Fibonacci function a well-known algorithm with linear time and constant space complexity exists that does not follow immediately from the cov-iterative presentation.

Here we give a simple definition of cov-controllable state systems, incorporating a sufficient condition that ensures correct simulation of a given cov operation. In the following, fix $F = 1 + (-)$ as above and C to an arbitrary set.

Definition 2 (State System). *A triple $(S, \sigma : C^\infty \rightarrow S, \tau : C \times S \rightarrow S)$ is called a **state system** with **state space** S , **abstraction** σ and **transition** τ . It is called an **epi-state system** if and only if σ is epi. It is said to **factor** some cov operation φ if and only if there is an arrow $\tilde{\varphi} : S \rightarrow C$ such that $\varphi = \tilde{\varphi} \circ \sigma$ and $\tau \circ \langle \tilde{\varphi}, \text{id}_S \rangle \circ \sigma = \sigma \circ \text{cons} \circ \langle \varphi, \text{id}_{C^\infty} \rangle$.*

For epi-state systems, φ determines $\tilde{\varphi}$ uniquely. The morphism specified by the second equation is abbreviated to $\delta : C^\infty \rightarrow S$.

Theorem 1. *A state system (S, σ, τ) factoring a cov operation φ simulates it.*

$$\{\!\!\{\varphi\}\!\!\}_F = \pi_1 \circ (\langle \pi_1, \tau \rangle \circ \langle \tilde{\varphi}, \text{id}_S \rangle \circ [\sigma \circ \iota_1, \pi_2])_F$$

Definition 3. *A cov operation is called k -**bounded** for $0 \leq k \leq \omega$ if and only if there is a sequence $h \in C^k$ and operation $\hat{\varphi} : C^k \rightarrow C$ such that*

$$\varphi = \hat{\varphi} \circ \text{take}(k) \circ \text{append}(h)$$

*The minimal k such that φ is k -bounded is called the **horizon** of φ . 1-bounded cov iteration coincides with **primitive** recursion, cf. [2].*

Theorem 2. *A first-in-first-out buffer of k elements of C is a state system factoring any k -bounded cov operation with codomain C , with $\tilde{\varphi} = \hat{\varphi}$.*

$$S = C^k \quad \sigma = \text{take}(k) \circ \text{append}(h) \quad \tau(c_0, (c_1, \dots, c_k)) = (c_0, \dots, c_{k-1})$$

The Fibonacci operation is well-known to have a horizon of two, with $h = (1, -1)$ and $\hat{\varphi}(a, b) = a + b$. The resulting state system specifies precisely the usual iterative algorithm. Note that this is not an epi-state system. Some relevant cov operations have infinite horizon, e.g. fractionally integrated ARMA models.

Definition 4. *A state system homomorphism between (S_1, σ_1, τ_1) and (S_2, σ_2, τ_2) , both factoring φ , is a map $h : S_1 \rightarrow S_2$ such that $h \circ \sigma_1 = \sigma_2$ and $h \circ \delta_1 = \delta_2$. Put differently, h is a morphism between two pairs of coslices under C^∞ , (S_i, σ_i) and (S_i, δ_i) , simultaneously.*

Theorem 3. *The state systems factoring a fixed cov operation φ form a category **State**(φ). The epi-state systems factoring φ form a subcategory **EpiState**(φ). Morphisms in **State**(φ) are morphisms in **EpiState**(φ) if and only if they are epi in the underlying category **Set**.*

Theorem 4. **EpiState**(φ) *has initial and final objects:*

- *The trivial state system $(C^\infty, \text{id}_{C^\infty}, \text{cons})$ is initial, with $\tilde{\varphi} = \varphi$. The corresponding unique homomorphism to any (S, σ, τ) is σ .*
- *The coimage or coequalizer of the kernel pair of φ , (S^\dagger, ψ) , gives rise to a final object $(S^\dagger, \sigma^\dagger, \tau^\dagger)$, where $\sigma^\dagger = \psi \circ \varphi$, and τ^\dagger exists as a (non-unique) solution to $\tau^\dagger \circ \langle \varphi, \sigma^\dagger \rangle = \sigma^\dagger \circ \text{cons} \circ \langle \varphi, \text{id}_{C^\infty} \rangle$. $\tilde{\varphi}^\dagger$ is the unique retraction of ψ . The corresponding unique homomorphism from any (S, σ, τ) is $\psi \circ \tilde{\varphi}$.*

The uniqueness of the model operation $\tilde{\varphi}$ for epi-state systems and the placement of a particular system along the initial-final (syntax-semantic) axis have far-reaching implications for the philosophy of black-box models in science and software engineering. Details are out of scope here and will be given in forthcoming companion papers.

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Lindenmayer Systems, Coalgebraically

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1 Introduction

Lindenmayer systems, or L-systems, are a formal model of structural growth due to the theoretical botanist Lindenmayer in 1968, in the wake of research on Chomsky grammars in computer science. They have been immensely influential as a conceptual tool for botany, as well as an application domain for computer-based modelling [2]. Like grammars, L-systems come in various classes of complexity, with optional nondeterminism, context-sensitivity or parametrization. Unlike grammars, L-systems consider only derivations where all symbols are rewritten *in parallel*, that is, they are models of decentral, vegetative growth.

Context-free grammars have been given a coalgebraic treatment [6], extending earlier work on coalgebraic representations of regular languages and automata (see e.g. [3]). In [6], grammars in Greibach normal form are first regarded as coalgebras over the functor $2 \times (\mathcal{P}(-*))^A$, which are then extended to $2 \times (-)^A$ -coalgebras using an instance of the generalized powerset construction [4]. In this framework, rules are applied sequentially, making use of leftmost derivations: this presents a fundamental difference with the case of L-systems, where we deal with parallel rewriting of all symbols present at a certain stage.

2 Coalgebraic L-Systems

The classical presentation of L-systems follows Chomsky grammars, and is strictly syntactic. Here we suggest an alternative, semantical perspective on L-systems. The key idea is to represent the single-step rules of an L-system as a finite coalgebra for a monadic functor on the category of sets. The Kleisli extension then yields a function that can be iterated to formalize multi-step derivations.

The simplest class of deterministic context-free L-systems without terminals is modelled by the list functor and its standard monadic structure. Common extensions such as terminals, nondeterminism and probabilism can be added modularly by composing the list functor with the coproduct with a constant set, the covariant finitary powerset functor and the finitely supported distribution functor, respectively. Each of these comes with a standard monadic structure. In general, the composition of monads is not a monad, but for all pairs of the functors in question, distributive laws can be given that make for a composite monadic structure consistent with the traditional semantics of L-systems.

Parametric L-systems can be incorporated using a different, but equally semantical technique. The traditional syntactical approach is to hoist parameter datatypes, guard predicates and operators onto the finite set of basic symbols. Instead, we suggest to relax the condition of finiteness on the carriers of coalgebras, and merely require that they have a finite homomorphic image. Thus, the kernel quotient of that homomorphism can be seen as a finite collection of symbols, and the internal structure of each quotient class as parametrization.

Context-sensitive L-systems are a more complicated matter. As for the Chomsky grammar case, no obvious coalgebraic presentation is available. We conjecture that a bialgebraic approach is promising, by analogy to the bialgebraic semantics of cellular automata [5].

3 Conclusion

The coalgebraic presentation of L-systems is concise, elegant and natural. Some typical problems regarding L-systems reappear as standard coalgebraic notions in disguise, while other problems even become apparent only in coalgebraic form. As an example of the former effect, consider the vast subject of botanical reasoning with L-systems, namely the classification of branching structures and organ placement on higher plants (*phyllotaxis*): it can be understood as an instance of bisimulation. As an example of the latter effect, consider the definition of probabilistic L-systems in the definitive resource [2]: productions are weighted with probabilities, and it is obviously implied that parallel rewriting steps be *stochastically independent*, but an explicit statement has simply been forgotten. Such an oversight is not possible in the coalgebraic form; stochastic independence is exactly the content of the distributive law between lists and distributions.

L-systems are very easily understood models and appeal to intuition. They showcase basic concepts of coalgebraic modelling in such a way that they could be a useful pedagogic example in the introductory teaching of coalgebra.

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Reflexive Economics and Categorical, Coalgebraic and Domain Theoretical Modelling

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Abstract. I propose the use of categorical methods of theoretical computer science as key tools for reflexive economics. Reflexive methods are deemed appropriate to model the peculiarity of social systems where the theory and the modeled system interact. This allows to approach many long standing key features not addressable by current methods. The proposed program is a far reaching approach with new and major applications for categories, coalgebras or domain theory. Reflexivity points towards a non reductionistic scientific approach of behavioral modelling of social systems and their parts beyond mechanistic analogies from physics.

Key words: economics, reflexive modelling, endogeneous control theory, biological system theory, evolutionary economics, institutional economics, coalgebra, domain theory, category theory

This short note sketches the possibilities that categorical, coalgebraic and domain theoretical structures open up for long standing modelling issues in economics. Reflexive structures, see the overviews in (Winrich 1984, Knudsen 1993, Sandri 2008), arise in economics if theory and the modelled system interact and the controller is part of the controlled and thus self-organizing system. (Morgenstern 1972), p. 707, and similarly in (Morgenstern 1928), writes

There is thus a back-coupling or feedback between the theory and the object of the theory, an interrelation which is definitely lacking in the natural sciences.

Reflexive mathematical structures are in need to model key social hyper structures like expectations as beliefs of beliefs, organizations as repair of repair, institutions as rules to change rules, markets as decentralized solving for solution concepts and other phenomena like life, value, money, culture or democracy.

Reflexivity allows a change of point of view from the outside to the inside of a system and can be thought as a generalization of the quantum mechanical situation where the observer influences the observed to the more complex situation where the observed may change the observer as well, see (Frigg 2010) for a philosophical discussion.

A key problem in economics is a model of a system, company, organization or institution, now discussed as a means for transaction cost reduction initiated in (Coase 1937, Coase 1960). Reflexive economics can address the nature of a firm or organization by a model of biological metabolism-repair systems of (Rosen 1991) which is a generalization of autopoiesis. An approach to their realization in linear systems is (Casti 1988) which we currently formalize with Samson Abramsky, by improving the lambda calculus approach in (Mossio, Longo, and Stewart 2009) which seems to involve reflexivity at the type level. By a model of a system with an internal model of itself we can model interaction. Autopoiesis and reflexivity are the core of the non mathematical sociology of (Luhmann 1998) which closely resembles the ideas for reflexive economics.

Interactive game theory is heading towards modal logic but without dwelling into reflexivity. An exception is (Heifetz and Samet 1998) who construct Hasanyi type spaces to model beliefs of beliefs in situations of incomplete information. This was modelled coalgebraically in computer science by (Moss and Viglizzo 2004). A non mathematical reflexive theory is also proposed by George Soros, one of the most successful investors, who has based his analysis of financial markets on reflexivity, the Russell paradox and open world assumptions, see (Cross and Strachan 1997, Soros 2003). Recently a first n-players Russell paradox appeared in economic game theory and has been modelled in (Abramsky and Zvesper 2010).

The only papers formally approaching reflexive structures I know about in economics are the categorical and domain theoretical papers (Vassilakis 1991, Vassilakis 1992, Vassilakis 2002a, Vassilakis 2002b). Vassilakis models games over games and rules to change rules as an approach to institutions and some other applications. These papers are largely unknown in economics which might be due to the unclear relation to usual economic models, to the lack of hints to the many applications of categories and reflexivity in economics and to the fact that there are few economists able to understand categorical mathematics. This points to the need to use the unifying nature of categories for calculus, as in (Rutten 2003) or modal logic, as in (Kurz 2006), which are the major languages currently used in economics.

(Alameddine 1990) builds on the work of Vassilakis and rules to change rules as an approach to the famous theory of justice of (Rawls 1999) who has not succeeded in treating the reflexive structure of this problem in his natural language approach. Another reflexivity arises as preferences over preferences as the core of a concept of a person, see (Frankfurt 1971) in philosophy and (Nehring 2006) in economics.

(Lescanne 2009, Lescanne and Matthieu 2010) develop the notion of rational escalation strategies in infinite coalgebraic games which should give hints how to model financial bubbles in economics. The usual economic approach to dynamics with infinite time horizons is (informally) recursive, see (Stokey, Prescott, and Lucas 1989), but should be corecursive and a clarification is an important part of dynamics in reflexive economics where the context and content coevolve. The non reflexive (expectation) dynamics in economics so far excludes bubbles at the

fundamental level but nevertheless is in need to model them, see (Santos and Woodford 1997). This is related to local and global epistemic and ontological states and behavior in economies. A proper discussion of computability, the fundamental decision problem, open world semantics and similar problems along the lines of (Lawvere 1969) has to be addressed.

As the first steps towards reflexive economics I would like to build on the beginnings of a coalgebraic research in the field of ecological management science which is rather close to economics in terms of objectives, current methods, their shortcomings and the possibilities of categorical and coalgebraic approaches. In the first paper on a model of modelling in (Hauhs and Trancon y Widemann 2010, Trancon y Widemann 2011) the usual inverse functional or state based approach inherited from physics is contrasted to the coalgebraic behavioral approach, see also (Willems 2007).

(Trancon y Widemann and Hauhs 2011a) contrast a recursion theoretical approach to path dependent dynamics with infinite histories to the reductionistic notion of markovian path independent processes. Path dependency is at the heart of evolutionary and institutional economics, see (Dopfer 2005) which is a rich source of modelling challenges in economics.

Finally (Trancon y Widemann and Hauhs 2011b) show a bialgebraic model of space and time for agent based modelling (called multi agent systems in computer science) as a solution to the usual (and naive state based) defect object oriented programming. The core problem is the need for intentional states which can not be understood in object oriented code without a formal semantics. This extends the scientific method based on states inherited from physics to the behavioral modelling of epistemic or ontological states. It allows to discuss emergent behavior of the whole system which is the explicit goal of agent based and social modelling, see (Colander 2006). Multi-agent logic and especially its compositionality as in (Abramsky 2007) is therefore a very important but largely not understood topic in economics. Interactive and decentralized models are a prerequisite to understand the fundamental economic issues of why there are markets, price systems, money or economic value, see (Hellwig 1993), who describes the challenges of monetary theory. Its underlying (to be generalized space-time) double accounting seems to be an early form of a process logic, see (Katis, Sabadini, and Walters 2008). The lack of an economic monetary theory and its need for reflexive and decentralized structures was and is my major motivation towards reflexive economics.

Challenges in network economics are summarized by (Jackson 2009, Jackson 2010) with (Kranton and Minehart 2001) being one interactive market model beyond general equilibria which are criticized by (Ackerman 2002). Placing market models as (Kranton and Minehart 2001) into the agent based modelling framework of (Trancon y Widemann and Hauhs 2011b) is a natural candidate for the first steps into reflexive economics.

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