TALLINNA TEHNIKAINSTITUUDI TOIMETUSED
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1. **The Pareto Constants.** Discussing the observational data from a number of European countries, Pareto has shown the income-distribution to be fitted by the formula

$$z = c x^{-m}, \tag{1}$$

where z denotes the number of inhabitants with an annual income exceeding x, while c and m are positive constants determined by local conditions. Equation (1) gives a straight line on logarithmic scale, and the statisticians generally use the Method of Least Squares for the computation of the Pareto constants c and m of this line. Now there is no reason to expect that the least sum of square deviations in the logarithms warrants also the least sum of square deviations in the numbers z as well. Besides, this method of computation is, practically, rather laborious. Taking into consideration that formula (1) is evidently not applicable to very small values of x [1]), we can interpret the Pareto Law as a method for interpolating the variable z in a certain interval $f < x < \infty$ of the variable x. The values of the parameters c and m must then be determined by two arbitrary conditions. Such a conception permits us to compute c and m in a simpler manner.

Let N be the total number of tax-payers, that is to say the number of inhabitants with an income exceeding any free of duty limit $x = f$. As the first arbitrary condition for using (1) we can require N to be the value of z in (1) if $x = f$, which means

$$N = c f^{-m}. \tag{2}$$

Let further ξ be the average income of the taxed population. We can also require ξ to be the average income in the

1) See e. g. A. Timpe, *Einführung in die Finanz- und Wirtschaftsmathematik*, Berlin 1934, p. 111.

interval $f < x < \infty$ according to formula (1). This gives as second arbitrary condition

$$\xi N = \int_0^N x\, dz, \tag{3}$$

where the integral is geometrically represented in common coordinates by the area within the x-axis, curve (1), the straight line $z = N$, and the z-axis. Since, according to (1)

$$dz = -c m x^{-m-1}\, dx,$$

this integral becomes

$$\int_f^\infty c m x^{-m}\, dx.$$

The integral converges if $m > 1$. We put now

$$a m = 1, \quad m > 1 \tag{4}$$

and find by referring to (2):

$$\int_0^N x\, dz = \frac{c m f^{1-m}}{m-1} = \frac{fN}{1-a}.$$

Condition (3) thus gives

$$\xi = \frac{f}{1-a}, \quad m = \frac{\xi}{\xi - f}. \tag{5}$$

Having computed m we finally obtain from (2):

$$c = N f^m. \tag{6}$$

The Pareto constants are thus expressed by f and ξ.

In the following developments the relation

$$\xi - f = \xi a, \tag{7}$$

resulting from (5), will be employed several times.

The quantities f and ξ are exactly determined within the body of tax-payers. The average value ξ especially characterizes to a certain extent the whole taxed community; the computation of c and m from f and ξ is therefore well justified.

The N tax-payers form into two categories, the „impecunious“ class with $x \leqq \xi$, and the „well-off“ class with $x > \xi$. According to the Pareto Law the number W of the „well-off“ amounts to

$$W = c\xi^{-m} = cf^{-m}(1-a)^m = N(1-a)^m, \tag{8}$$

while the number I of the „impecunious“ is

$$I = N - W = N[1-(1-a)^m]. \tag{9}$$

For purposes of numerical discussion we choose arbitrarily

$$N = 347600, \qquad f = 600, \qquad \xi = 1461,$$

i. e., values which may be said to approximate the actual conditions in some Baltic state. Our formulae give in this example:

$$\begin{aligned} 1-a &= 0{,}4107 \\ a &= 0{,}5893 \\ m &= 1{,}697 \\ c &= 1801 \cdot 10^7 \\ (1-a)^m &= 0{,}2209 \\ W &= 76800 \\ I &= 270\,800 \\ fN &= 2086 \cdot 10^5 \\ W\xi &= 1122 \cdot 10^5 \end{aligned}$$

2. **Intensity of Assessment.** Starting with the Pareto Law we now proceed to establish a rational progressive income-tax tariff.

Let Y be the tax imposed on a total income x. The ratio

$$\frac{Y}{x} = y \tag{10}$$

represents the r a t e of assessment. In the case of a continuous progressive tariff y ought to be a monotonously increasing continuous function $y(x)$ of the variable x in the interval $f < x < \infty$. We can choose y in such a manner that the derivate $\frac{dy}{dx}$ also represents a continuous function in our interval. Evidently y is positive in the whole interval and not permitted to exceed the value 1. But it would be a mistake to think that all these conditions together are sufficient for the real progressiveness of the respective deduced tariff-values Y. Indeed, interpreting Y as

$$Y = \int_f^x \frac{dY}{dx}\, dx\,,$$

we notice that the derivate

$$\frac{dY}{dx} = x\frac{dy}{dx} + y \tag{11}$$

does not necessarily increase monotonously if the above mentioned conditions are fulfilled by y. By way of example in the case of an exponential tariff

$$y = \mu\,(1 - e^{-kx})$$

there exists a maximum for $\frac{dY}{dx}$ at the point $kx = 2$.

Now $\frac{dY}{dx}$ shows the intensity of assessment at the point x, i. e. the tax due for the surplus money unit just exceeding x. A real progressiveness of the tariff is only present if $\frac{dY}{dx}$ increases monotonously, in other words if the sum due for each excess-profits unit is perpetually growing.

In construing a real progressive tariff it is therefore necessary to start not from y as usual[2]), but from a monotonously increasing derivate function $\frac{dY}{dx}$. The monotony of y is then warranted, for in such a case

$$\frac{d^2Y}{dx^2} = x\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx}$$

being positive, y cannot possess a maximum, where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \leqq 0$. It stands to reason that $\frac{dY}{dx}$ is not permitted to exceed the value 1. For a certain positive quantity $\mu \leqq 1$ we have therefore

$$\lim_{x \to \infty} \frac{dY}{dx} = \mu\,. \tag{12}$$

2) See A. Timpe, l. c., p. 109—110.

On the other hand a continuous function $\frac{dY}{dx}$ necessarily vanishes at the point $x = f$, for incomes less than f are exempt from taxation. The curve $\frac{dY}{dx}$ therefore begins at $x = f$ with the value 0 and increases asymptotically up to μ. There may be points of inflexion; at least one such point of inflexion appears if the second derivate of Y varies continuously and the third derivate at the beginning $x = f$ is positive. We are trying to establish a law with just only one point of inflexion.

3. **The Tariff.** We can adjust the point of inflexion of the curve $\frac{dY}{dx}$ to the distribution law (1). It is an obvious suggestion to take this point at the place where the tax-payers fall into two classes, viz. at $x = \xi$. The „well-off" class bears by far the most important part of the total taxation. We shall therefore consider first the tariff for this class.

An obvious assumption in this case would be

$$\frac{dY}{dx} = \mu - \frac{K}{x}, \quad 0 < K < \mu\xi, \quad x \geqq \xi, \tag{13}$$

where the constant K will be definitely fixed later. Let λ designate the rate of assessment corresponding to $x = \xi$. Then we can write

$$Y = \lambda\xi + \int_{\xi}^{x} \left(\mu - \frac{K}{x}\right) dx =$$

$$= \lambda\xi + \mu(x - \xi) - K \ln \frac{x}{\xi} \qquad (x \geqq \xi),$$

and putting

$$\mu - \lambda = \nu, \tag{14}$$

we obtain

$$\left.\begin{aligned} Y &= \mu x - \left(\nu\xi + K \ln \frac{x}{\xi}\right) \\ y &= \mu - \frac{1}{x}\left(\nu\xi + K \ln \frac{x}{\xi}\right). \end{aligned}\right\} \quad (x \geqq \xi) \tag{15}$$

The curve $\frac{dY}{dx}$ in the „impecunious" part must fit closely to (13) at the point $x = \xi$, i. e., there must exist a common tangent at both parts of the curve in this point. Since in the „well-off" direction at this point

$$\frac{dY}{dx} = \mu - \frac{K}{\xi}, \quad \frac{d^2Y}{dx^2} = \frac{K}{\xi^2}, \tag{16}$$

the same has to be true in the „impecunious" direction. Now $\frac{dY}{dx}$ vanishes at $x = f$, as we have mentioned above. The analytically simplest possibility to satisfy all these conditions in the „impecunious" part of the curve is given by the assumption

$$\frac{dY}{dx} = p\,(x - f) + q\,(x - f)^2 \quad (f \leqq x \leqq \xi) \tag{17}$$

where p and q are convenient constants. Taking into consideration that

$$Y(\xi) = \lambda\xi = \int_f^\xi \frac{dY}{dx}\,dx = \frac{p}{2}\,(\xi - f)^2 + \frac{q}{3}\,(\xi - f)^3,$$

we get:

$$\begin{aligned} p\,(\xi - f) + q\,(\xi - f)^2 &= \mu - \frac{K}{\xi} \\ p + 2\,q\,(\xi - f) &= \frac{K}{\xi^2} \\ \frac{p}{2}\,(\xi - f)^2 + \frac{q}{3}\,(\xi - f)^3 &= \lambda\xi. \end{aligned} \tag{18}$$

Relation (7) permits to substitute ξa for $\xi - f$ in these simultaneous linear equations. The solution p, q, K, of (18) is found to be

$$\begin{aligned} p &= \frac{2}{\xi\,(4 + a)}\,[3\,\lambda\,m\,(2\,m + 1) - \mu] \\ q &= \frac{3}{a\,\xi^2\,(4 + a)}\,[\mu - 2\,\lambda\,m\,(m + 1)] \\ K &= \frac{2\,\xi}{4 + a}\,[2\,\mu - 3\,\lambda\,m], \end{aligned} \tag{19}$$

as the reader may verify by substituting these values in (18).

In this way we have finally determined not only p and q in (17) but also K in (13). The „impecunious" class tariff runs

$$\left.\begin{aligned} Y &= (x-f)^2\left[\frac{p}{2}+\frac{q}{3}\,(x-f)\right] \\ y &= \frac{(x-f)^2}{x}\left[\frac{p}{2}+\frac{q}{3}\,(x-f)\right] \end{aligned}\right\} \quad f \leqq x \leqq \xi\,. \tag{20}$$

This tariff (20) is fit for use only if $p > 0$; if q is negative there exists no point of inflexion at $x = \xi$. We demand therefore additionally $p > 0$, $q > 0$. This gives the restrictions

$$2\,\lambda m\,(m+1) < \mu < 3\,\lambda m\,(2\,m+1)\,. \tag{21}$$

The choice of μ is thus limited by the chosen value of λ and the constant m. Under these restrictions (21) we have always $K > 0$, because $2\,\mu > 3\,\lambda m$; the other condition $K < \mu\xi$ is satisfied as well.

In our imaginary example of § 1 we have from (21)

$$9{,}16\,\lambda < \mu < 22{,}3\,\lambda\,.$$

Taking for purposes of numerical discussion $\lambda = 0{,}01$, $\mu = 0{,}20$ we find these restrictions satisfied. The values, in this case, are:

$$p = 7{,}0698 \cdot 10^{-6} \qquad \frac{K}{\xi} = 0{,}15213$$

$$q = 5{,}6382 \cdot 10^{-8}$$

$$K = 222{,}24 \qquad \frac{K}{\xi^2} = 1{,}0413 \cdot 10^{-4}.$$

Table I shows the values of $100\,y$, Y, $100\,\frac{dY}{dx}$ and $10^6\,\frac{dy}{dx}$ corresponding to these data.

Generally the point of inflexion of the y-curve lies not at $x = \xi$, but more on the right. The abscissa of this point, say x_0, is determined by

$$\ln\frac{x_0}{\xi} = \frac{3\,K - 2\,\nu\,\xi}{2\,K}\,. \tag{22}$$

In our example $x_0 = 1878$.

Table I

x	$100\,y$	Y	$100\,\frac{dY}{dx}$	$10^6\,\frac{dy}{dx}$
600	0	0	0	0
800	0,04	0,29	0,37	4,12
1000	0,18	1,77	1,18	10,00
1200	0,44	5,33	2,45	16,75
1400	0,85	11,88	4,17	23,71
1461	1,00	14,61	4,79	25,94
1600	1,39	22,24	6,11	29,50
1800	2,00	36,00	7,65	31,39
2000	2,63	52,60	8,89	31,30
2500	4,12	103,00	11,11	27,96
3000	5,42	162,51	12,59	23,92
3500	6,52	228,20	13,65	20,37
4000	7,46	298,40	14,44	17,45
4500	8,27	372,15	15,06	15,09
5000	8,98	449,00	15,55	13,14
6000	10,14	608,40	16,30	10,27
7000	11,06	774,20	16,82	8,23
8000	11,81	944,80	17,22	6,76
9000	12,43	1118,70	17,53	5,67
10000	12,95	1295	17,78	4,83
20000	15,70	3140	18,89	1,60
30000	16,84	5051	19,26	0,81
40000	17,47	6988	19,44	0,49
50000	17,87	8935	19,56	0,34
60000	18,161	10897	19,63	0,25
70000	18,375	12862	19,68	0,19
80000	18,541	14833	19,72	0,15
90000	18,674	16807	19,75	0,12
100000	18,783	18783	19,78	0,10
200000	19,314	38628	19,89	0,03
300000	19,513	58539	19,93	0,01
400000	19,619	78476	19,94	0,01
500000	19,685	98425	19,96	0,01
600000	19,731	118386	19,96	0,00
700000	19,764	138348	19,97	0,00
800000	19,790	158320	19,97	0,00
900000	19,810	178290	19,98	0,00
1000000	19,827	198270	19,98	0,00

Fig. 1 shows the Pareto Law (heavy line), the curves y, Y and $\frac{dY}{dx}$ for the chosen numerical example on bi-logarithmic scale. Fig. 2 shows the same on ordinary scale but only for

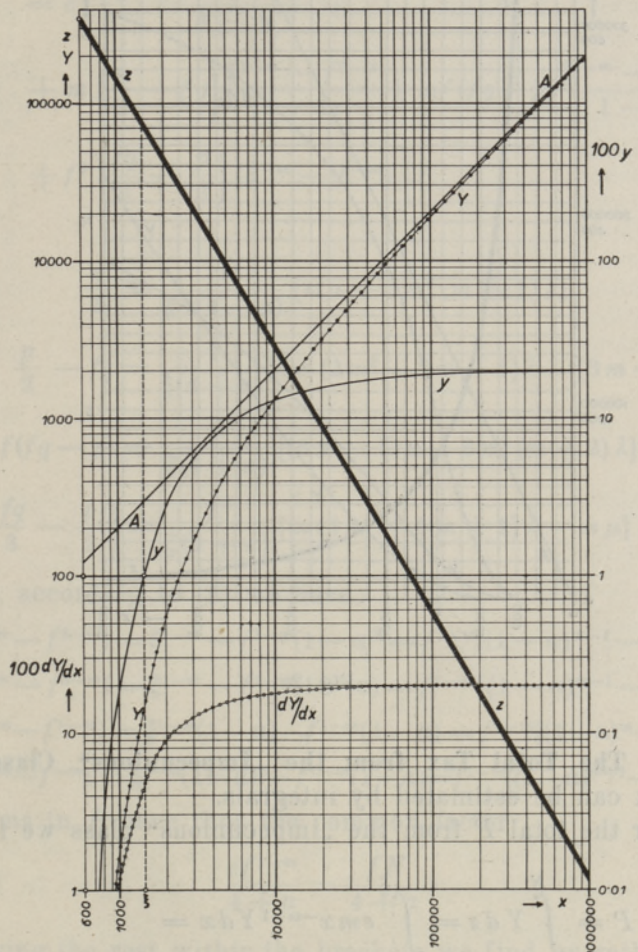

Fig. 1.

the narrow interval $0 < x < 5000$. In Fig. 1 we see an asymptote AA to the Y-curve; in Fig. 2 the corresponding straight line AA points out only an asymptotical direction. In this connection it should be noted that the existence of an asymptotical direction is already a sufficient argument for the usefulness of any progressive tariff; the existence of the asymptote itself is not required.

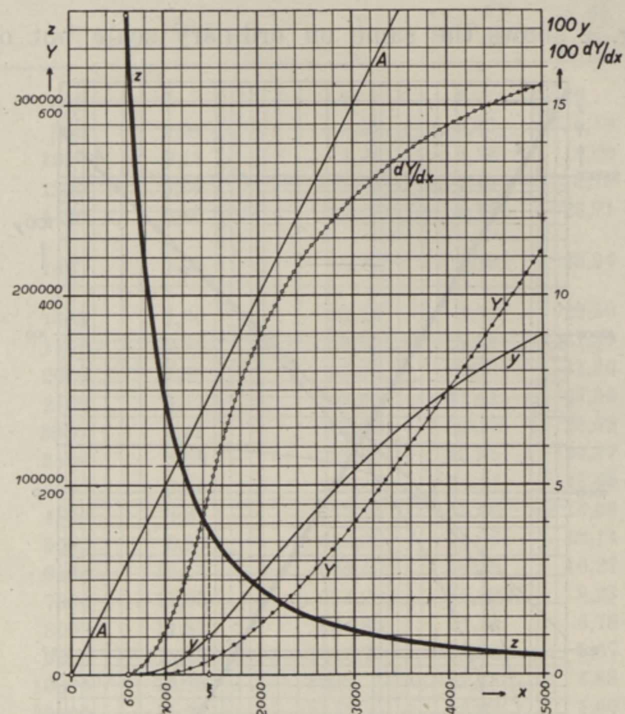

Fig. 2.

4. **The Total Tax from the „Impecunious“ Class.** The total tax can be estimated by integrals.

For the total P from the „impecunious“ class we find:

$$P = \int_W^N Y\,dz = \int_f^\xi c\,m\,x^{-m-1}\,Y\,dx =$$

$$= c\int_f^\xi m\,x^{-m-1}\left[\frac{p}{2}(x-f)^2 + \frac{q}{3}(x-f)^3\right]dx =$$

$$= c\int_f^\xi m\left[\frac{q}{3}x^{2-m} + \left(\frac{p}{2} - fq\right)x^{1-m}\right]dx +$$

$$+ c \int_f^{\xi} m \left[f(fq - p)\, x^{-m} + f^2 \left(\frac{p}{2} - \frac{fq}{3} \right) x^{-m-1} \right] dx =$$

$$= c \left[\frac{mq}{3} \cdot \frac{\xi^{3-m} - f^{3-m}}{3-m} + \right.$$

$$+ m \left(\frac{p}{2} - fq \right) \frac{\xi^{2-m} - f^{2-m}}{2-m} + mf(fq-p) \frac{\xi^{1-m} - f^{1-m}}{1-m} +$$

$$\left. + f^2 \left(\frac{fq}{3} - \frac{p}{2} \right) (\xi^{-m} - f^{-m}) \right].$$

Now

$$\frac{q}{3} = \frac{1}{\xi^2 (4+a)} [m\mu - 2m^2(m+1)\lambda]$$

$$\frac{p}{2} - fq = \frac{1}{\xi(4+a)} [3m(2m^2 + 2m - 1)\lambda - (3m-2)\mu]$$

$$f(fq - p) = \frac{f}{\xi(4+a)} [(3m-1)\mu - 6m^2(m+2)\lambda]$$

$$f^2 \left(\frac{fq}{3} - \frac{p}{2} \right) = \frac{f^2}{\xi(4+a)} [m(2m^2 + 6m + 1)\lambda - m\mu]$$

Further, according to (5) we have:

$$\xi^{-2}(\xi^{3-m} - f^{3-m}) = \xi^{1-m} - f^{1-m}(1-a)^2 = f^{1-m}[(1-a)^{m-1} - (1-a)^2]$$

$$\xi^{-1}(\xi^{2-m} - f^{2-m}) = \xi^{1-m} - f^{1-m}(1-a) = f^{1-m}[(1-a)^{m-1} - (1-a)]$$

$$f\xi^{-1}(\xi^{1-m} - f^{1-m}) = \xi^{1-m}(1-a) - f^{1-m}(1-a) = f^{1-m}[(1-a)^m - (1-a)]$$

$$f^2\xi^{-1}(\xi^{-m} - f^{-m}) = \xi^{1-m}(1-a)^2 - f^{1-m}(1-a) = f^{1-m}[(1-a)^{m+1} - (1-a)].$$

The items in P have thus the common factor

$$\frac{cf^{1-m}}{4+a} = \frac{fN}{4+a}.$$

Simplifying the rest within the brackets we find by rather laborious arithmetical work:

$$P = \frac{fN}{(4+a)(3-m)(2-m)} \Big\{ [(1-a)^{m-1}(2m^2 + 73m - 19 - 6a) +$$

$$-6(5m+1)]\lambda - [(1-a)^{m-1}(11 - 6a) - 4]\mu \Big\} =$$

$$= fN(G\lambda - H\mu) \tag{23}$$

In this expression (23) G and H become indefinite if $m = 2$ or $m = 3$. Indeed in integrating we supposed $m \neq 2$ and $m \neq 3$. Reference to the limits gives in these cases:

$$P(2) = \frac{fN}{9}\left[(132 \ln 2 - 88{,}5)\lambda + (5{,}5 - 8 \ln 2)\mu\right]$$

$$P(3) = \frac{fN}{13}\left[\left(\frac{1082}{9} - 288 \ln 1{,}5\right)\lambda + \left(12 \ln 1{,}5 - \frac{44}{9}\right)\mu\right],$$

i. e.

$$\begin{aligned} P(2) &= fN(0{,}3328\,\lambda - 0{,}005020\,\mu) \\ P(3) &= fN(0{,}2652\,\lambda - 0{,}001793\,\mu) \end{aligned} \tag{24}$$

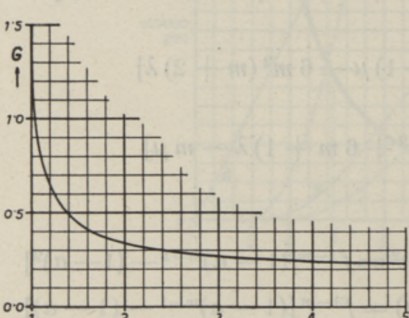

Fig. 3.

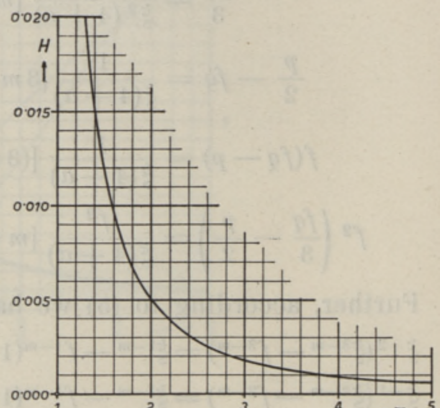

Fig. 4.

Fig. 3 shows the values of G, Fig. 4 those of H, depending upon m. Some values of G and H are given in Table II.

In our numerical example

$$m = 1{,}697, \quad fN = 2086 \cdot 10^5, \quad \lambda = 0{,}01, \quad \mu = 0{,}20$$

we get

$$G = 0{,}3832, \quad H = 0{,}008031$$

and

$$P = fN(0{,}3832\,\lambda - 0{,}008031\,\mu) = 799 \cdot 10^5\,\lambda - 168 \cdot 10^4\,\mu = 463000.$$

Table II

m	G	H	m	G	H
1,0	1,4000	0,100000	2,3	0,3031	0,003464
1,1	0,8152	0,043258	2,4	0,2957	0,003105
1,2	0,6405	0,027725	2,5	0,2891	0,002801
1,3	0,5447	0,019793	2,6	0,2833	0,002540
1,4	0,4830	0,015008	2,7	0,2781	0,002314
1,5	0,4397	0,011843	2,8	0,2734	0,002117
1,6	0,4074	0,009620	2,9	0,2691	0,001945
1,7	0,3825	0,007988	3,0	0,2652	0,001793
1,8	0,3626	0,006750	3,1	0,2617	0,001658
1,9	0,3464	0,005787	3,2	0,2585	0,001538
2,0	0,3328	0,005020	3,3	0,2555	0,001431
2,1	0,3214	0,004399	3,4	0,2528	0,001335
2,2	0,3116	0,003888	3,5	0,2502	0,001248

Fig. 5 shows a sequence of straight lines

$$799000 \cdot 100\,\lambda - 16800 \cdot 100\,\mu = P$$

on rectangular coordinates $100\,\lambda$ and $100\,\mu$. According to (21)

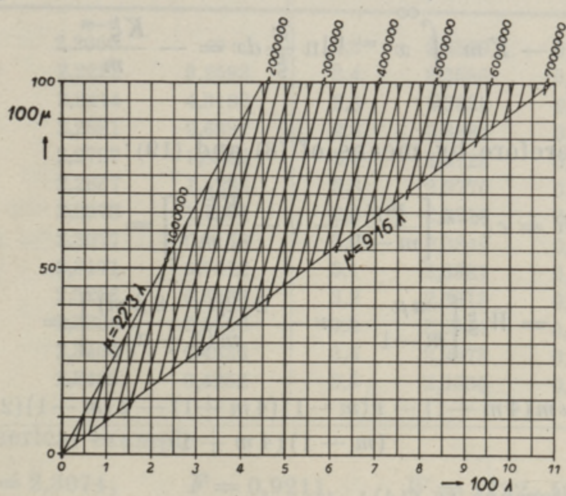

Fig. 5.

in this nomographical representation of P we need to consider only the triangle between the straight lines

$$\mu = 9{,}16\,\lambda, \qquad \mu = 1, \qquad \mu = 22{,}3\,\lambda.$$

There is no difficulty in plotting such a nomogram for any given values of m, f and N.

5. **The Total Tax from the „Well-off" Class.** The total tax R from the tax-payers belonging to the „well-off" class can be expressed by the integral

$$R = \int_0^W Y\,dz = \int_\xi^\infty c\,m\,x^{-m-1} Y\,dx =$$

$$= c\int_\xi^\infty m x^{-m-1}\left[\mu x - \nu\xi - K\ln\frac{x}{\xi}\right]dx =$$

$$= c\left[\frac{m\,\xi^{1-m}\,\mu}{m-1} - \nu\xi^{1-m} - K\,m\int_\xi^\infty x^{-m-1}\ln\frac{x}{\xi}\,dx\right].$$

Integration by parts gives

$$-K\,m\int_\xi^\infty x^{-m-1}\ln\frac{x}{\xi}\,dx = -\frac{K\xi^{-m}}{m}\,.$$

We get therefore by means of (8) and (19):

$$R = c\,\xi^{1-m}\left[\frac{m\mu}{m-1} - \nu - \frac{K\xi^{-1}}{m}\right] =$$

$$= W\xi\left[\frac{m\mu}{m-1} - \nu - \frac{2\,(2\mu - 3\,\lambda\,m)}{m\,(4+a)}\right] =$$

$$= W\xi\cdot\frac{\mu\,m\,(4\,m+1) - \nu\,(m-1)\,(4\,m+1) - 2\,(m-1)(2\,\mu - 3\,\lambda m)}{(m-1)\,(4\,m+1)},$$

or, with reference to (14):

$$R = W\xi\left[\frac{10\,m+1}{4\,m+1}\,\lambda + \frac{5}{(m-1)\,(4\,m+1)}\,\mu\right] =$$

$$= W\xi\,(E\,\lambda + F\mu)\,. \tag{25}$$

Fig. 6 shows E and F as functions of m. Some numerical values of E and F are given in Table III.

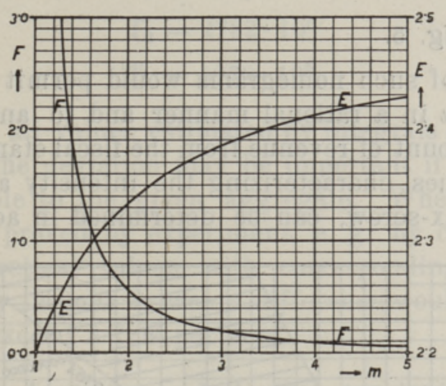

Fig. 6.

Table III

m	E	F	m	E	F
1,0	2,2000	∞	2,3	2,3529	0,3771
1,1	2,2222	9,2593	2,4	2,3585	0,3369
1,2	2,2414	4,3103	2,5	2,3636	0,3030
1,3	2,2581	2,6882	2,6	2,3684	0,2741
1,4	2,2727	1,8939	2,7	2,3729	0,2493
1,5	2,2857	1,4286	2,8	2,3770	0,2277
1,6	2,2973	1,1261	2,9	2,3810	0,2089
1,7	2,3077	0,9158	3,0	2,3846	0,1923
1,8	2,3171	0,7622	3,1	2,3881	0,1777
1,9	2,3256	0,6460	3,2	2,3913	0,1647
2,0	2,3333	0,5556	3,3	2,3944	0,1531
2,1	2,3404	0,4836	3,4	2,3973	0,1427
2,2	2,3469	0,4252	3,5	2,4000	0,1333

In our numerical example

$$E = 2{,}3074, \qquad F = 0{,}9211, \qquad W = 1122 \cdot 10^5,$$

and therefore

$$\begin{aligned} R &= 1122 \cdot 10^5 \cdot [2{,}3074\,\lambda + 0{,}9211\,\mu] = \\ &= 2660 \cdot 10^5\,\lambda + 1033 \cdot 10^5 \mu = \\ &= 23320000\,. \end{aligned}$$

Fig. 7 represents the nomogram of

$$2660000 \cdot 100\,\lambda + 1033000 \cdot 100\,\mu = R$$

similarly to Fig. 5.

The use of such nomograms would permit the legislator to choose λ and μ in a rational manner and to anticipate the corresponding amount of revenue from the fiscal standpoint. A series of average values, characterizing the intensity and the distribution of the tax-screw, can be determined in advance by using

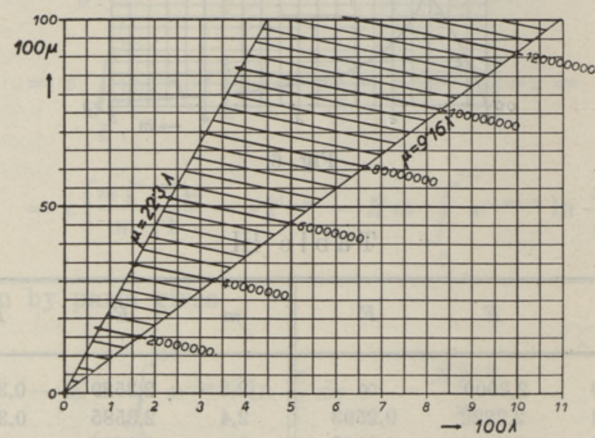

Fig. 7.

the formulae developed above. For this purpose one has to take into account that the total income T of the entire taxed population is given by

$$T = N\xi = \frac{Nf}{1-a}, \tag{26}$$

while the total income T_2 of the „well-off“ class is

$$T_2 = \int_{\xi}^{\infty} c\,m\,x^{-m}\,dx = Nf\,(1-a)^{m-2} = \frac{W\xi}{1-a}, \tag{27}$$

and therefore the total income T_1 of the „impecunious“ class

$$T_1 = T - T_2 = \frac{Nf - W\xi}{1-a}. \tag{28}$$

In the numerical example we have

$$T_1 = 2347 \cdot 10^5$$
$$T_2 = 2732 \cdot 10^5$$
$$T = 5079 \cdot 10^5 .$$

Of course, all values estimated by integrals are but approximately true. The deviation may be important if the Pareto Law is not applicable to the given aggregate. Whether this is the case must be proved by experiment, e. g. by comparing some computed theoretical values with corresponding observational data. Generally in such a case a difference would be admissible if it does not exceed a few per cent.

Summary. Assuming $m > 1$, this constant of the Pareto Law (1) can be determined by (5) from the average income ξ and the free of duty limit f. Then the other constant c is determined by (6), where N is the number of tax-payers. The „intensity of assessment“ $\frac{dY}{dx}$ of the tax Y must steadily increase with the income x if the tariff is required to be really progressive. A tariff taking into account the distribution of the income is given by formulae (15) and (20), where y denotes the rate of assessment $\frac{Y}{x}$. This tariff differs analytically for the „impecunious“ and the „well-off“ class, i. e. for incomes respectively less and more than ξ. According to the Pareto Law the total duties of both classes are found to be linear functions (23) and (25) of λ and μ, with coefficients depending upon the Pareto constants; here λ is the rate of assessment at $x = \xi$, and μ designates the asymptotical value of this rate for an infinitely increasing x. A numerical example based on arbitrarily chosen data illustrates the theoretical considerations.

In the numerical example we have

$$T_1 = 2297 \cdot 10^5,$$

$$T_2 = 2782 \cdot 10^5,$$

$$T = 5079 \cdot 10^5.$$

Of course, all values estimated by integrals are but approximately true. The deviation may be important if the Pareto Law is not applicable to the given aggregate. Whether this is the case must be proved by experiment, e. g. by comparing some computed theoretical values with corresponding observational data. Generally in such a case a difference would be admissible if it does not exceed a few per cent.

Summary. Assuming $n > 1$, this constant of the Pareto Law (1) can be determined by (15) from the average income ξ and the free of duty limit l. Then the other constant c is determined by (6), where N is the number of tax-payers. The „intensity of assessment" $\frac{dY}{dx}$ of the tax Y must steadily increase with the income x if the tariff is required to be really progressive. A tariff taking into account the distribution of the income is given by formulae (15) and (20), where y denotes the rate of assessment $\frac{Y}{x}$. This tariff differs analytically for the „impecunious" and the „well-off" class, i. e. for incomes respectively less and more than ξ. According to the Pareto Law the total duties of both classes are found to be linear functions (23) and (25) of λ and μ, with coefficients depending upon the Pareto constants; here λ is the rate of assessment at $x = \xi$, and μ designates the asymptotical value of this rate for an infinitely increasing x. A numerical example based on arbitrarily chosen data illustrates the theoretical considerations.