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Partial and Relational Algebraic Theories

Chad Mitchell Nester

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

Chad Nester

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signature

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CHAD MITCHELL NESTER

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Contents

List of Publications

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3. C. Nester. The Structure of Concurrent Process Histories. In *International Conference on Coordination Models and Languages*, pages 209-224, 2021.
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6. G. Boisseau, C. Nester, and M. Román. Cornering Optics. In *Proceedings of the Fifth International Conference on Applied Category Theory*, pages 97-110, 2023.
7. C. Nester. Concurrent Process Histories and Resource Transducers. In *Logical Methods in Computer Science*, volume 19, issue 1, 2023.
8. C. Nester and N. Voorneveld. Protocol Choice and Iteration for the Free Cornering. In *Journal of Logical and Algebraic Methods in Programming*, 100942, 2023.

Author's Contribution to the Publications

1. I was the sole author. The results of the paper are my own, I wrote the paper, and I presented the paper at the corresponding conference. This paper is unrelated to the monograph contained herein.

2. I contributed the main result characterising varieties of partial algebraic theories, wrote the corresponding part of the paper, and presented the paper at the corresponding conference.

3. I was the sole author. The results of the paper are my own, I wrote the paper, and I presented the paper at the corresponding conference. This paper is unrelated to the monograph contained herein.

4. I was the sole author. The results of the paper are my own, I wrote the paper, and presented the paper at the corresponding conference. This paper is unrelated to the monograph contained herein.

5. I was the sole author. The results of the paper are my own, and I presented the paper at the corresponding conference.

6. I attribute the main result characterising categories of optics equally to myself, Guillaume Boisseau, and Mario Román. I wrote the paper, and presented it at the corresponding conference. This paper is unrelated to the monograph contained herein.

7. I was the sole author. The results of the paper are my own, and I wrote the paper. This paper is unrelated to the monograph contained herein.

8. I attribute the results concerning protocol choice and iteration in the free cornering equally to myself and Niels Voorneveld. We wrote the paper together. This paper is unrelated to the monograph contained herein.

Abbreviations

DCR Category	Discrete Cartesian Restriction Category
CW Category	Carboni-Walters Category

Chapter 1

Introduction

Universal algebra studies kinds of algebraic structure, such as groups and rings, by algebraic means. The central idea of universal algebra is that of a *theory*, which serves to describe a kind of structure. The *models* of a given theory are the concrete instances of that kind of structure. Initially, universal algebra was concerned only with what are called *algebraic theories*. To the category theorist, an algebraic theory $\mathbb{X}$ is simply a small category with finite products, with a model of $\mathbb{X}$ being a functor $\mathbb{X} \to \mathsf{Set}$ that preserves the finite product structure.

Contrast this to the algebraist, who typically works with algebraic theories in the form of a *classical presentation*. Classical presentations (Σ, E) begin with a set Σ, called the *signature*, which has elements $f_{/n}$ consisting of a function symbol f together with an arity $n \in \mathbb{N}$. The set of *terms* over Σ is constructed as follows:

$$\frac{i \in \mathbb{N}}{x_i \quad \text{term}} \qquad\qquad \frac{f_{/n} \in \Sigma \qquad t_1 \quad \text{term} \qquad \cdots \qquad t_n \quad \text{term}}{f(t_1, \ldots, t_n) \quad \text{term}}$$

That is, we assume a countably infinite supply of variables x_i, each of which is a term, with more complex terms constructed by applying a function symbol of the signature to a number of terms specified by its arity. In addition to a signature Σ, each classical presentation comes with a set E of *equations* over Σ, which has elements $t_1 = t_2$ of pairs of terms over Σ.

An *interpretation* of signature Σ consists of a set X, called the *carrier*, together with a function $[\![f]\!] : X^n \to X$ for each $f_{/n} \in \Sigma$. Given an interpretation of Σ, each assignment of variables x_i to elements $[\![x_i]\!] \in X$, extends uniquely to an assignment of terms over Σ to elements of $\mathbb{X}$ as in $[\![f(t_1, \ldots, t_n)]\!] = [\![f]\!]([\![t_1]\!], \ldots, [\![t_n]\!]) \in X$. Now an interpretation of Σ is said to *satisfy* an equation $t_1 = t_2$ over Σ in case $[\![t_1]\!] = [\![t_2]\!]$ for all possible variable assignments. A *model* of a classical presentation (Σ, E) is an interpretation of Σ that satisfies all of the equations in E.

For example, one classical presentation of the theory of monoids consists of the signature $\Sigma_{\mathsf{Mon}} = \{m_{/2}, e_{/0}\}$ and the set E_{Mon} of equations containing:

$$m(x_1, m(x_2, x_3)) = m(m(x_1, x_2), x_3) \qquad m(e(), x_1) = x_1 \qquad m(x_1, e()) = x_1$$

Then an interpretation of Σ_{Mon} consists of a set X together with a binary operation $[\![m]\!] : X \times X \to X$ and a distinguished element $[\![e]\!] : 1 \to X$. For such an interpretation to satisfy the equations of E_{Mon} is for the binary operation to be associative, and to have the distinguished element as a left and right unit. That is, models of the classical presentation $(\Sigma_{\mathsf{Mon}}, E_{\mathsf{Mon}})$ are precisely monoids.

Upon encountering a classical presentation (Σ, E) the category theorist performs an act of mental substitution. Any signature Σ defines a category with finite products: the objects are natural numbers $n, m \in \mathbb{N}$, and the morphisms $\langle t_1, \ldots, t_m \rangle : n \to m$ are m-tuples of terms t_i over Σ restricted to the variables $x_1, \ldots, x_n$. Composition of $\langle s_1, \ldots, s_m \rangle : n \to m$ and $\langle t_1, \ldots, t_k \rangle : m \to k$ is defined by parallel substitution as in:

$$\langle t_1[^{s_1,\ldots,s_m}/_{x_1,\ldots,x_m}], \ldots, t_k[^{s_1,\ldots,s_m}/_{x_1,\ldots,x_m}]\rangle : n \to k$$

Now the product of n and m is $n+m$ with projections $\langle x_1, \ldots, x_n \rangle : n+m \to n$ and $\langle x_{n+1}, \ldots, x_{n+m} \rangle : n+m \to m$, and the pairing of $\langle f_1, \ldots, f_n \rangle : k \to n$ and $\langle g_1, \ldots, g_m \rangle : k \to m$ is given by $\langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle : k \to n+m$. Next, the equations of E determine equations between tuples of terms, and we quotient our category by these equations. For the category theorist this category — call it $\mathsf{T}(\Sigma, E)$ — is the content of the classical presentation (Σ, E). Validating this perspective is the fact that to specify a model of (Σ, E) is precisely to specify a functor $\mathsf{T}(\Sigma, E) \to \mathsf{Set}$ that preserves the finite product structure.

The perspectives of the category theorist and the algebraist complement each other. It is the work of the algebraist concerning individual algebraic theories that motivates the category theorist's interest in the notion of theory itself. Conversely the algebraist ought to be encouraged by the categorical perspective, which gives a robust mathematical account of the basic machinery underlying algebraic theories. A sensible notion of theory must support both perspectives, providing a perspicuous syntax for working with individual theories as well as a solid categorical foundation.

In this thesis I propose two notions of theory. First, *partial algebraic theories* allow the specification of algebraic structures involving partial functions — which are like functions save that they need not be defined on their entire domain. Next, *relational algebraic theories* allow the specification of algebraic structures involving binary relations. Partial algebraic theories are strictly more expressive than the classical notion of algebraic theory, and relational algebraic theories are strictly more expressive than partial algebraic theories. Each of these notions of theory admits both intuitive syntactic presentation to please the algebraist, and robust categorical semantics to please the category theorist.

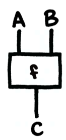
The syntax of both partial and relational algebraic theories is based on string diagrams for symmetric monoidal categories. Suppose that we represent any given morphism $f : A \to B$ of some category as a box labeled f which has wires labeled A and B protruding from its top and bottom respectively. It is then natural to represent the composite $fg : A \to C$ of $f : A \to B$ and $g : B \to C$ as two such boxes, one above the other, with the two wires labeled B merged into a single wire connecting the boxes. Identity morphisms are simply represented as labeled wires.

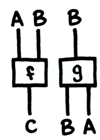
$f \rightsquigarrow$ 

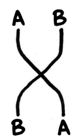
$fg \rightsquigarrow$ 

$1_A \rightsquigarrow$ 

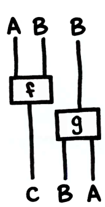
One way to understand the axioms of a category is to notice that they allow us to interpret these diagrams unambiguously. A *monoidal category* is a category equipped with *tensor products*. It is easiest to explain what this means in terms of our box-and-wire scheme for representing morphisms. First, in a monoidal category the boxes corresponding to morphisms may have *zero or more* wires protruding from the top and bottom, each wire labeled with an object of the category. Multiple wires indicates that the domain/codomain is a tensor product of objects $A \otimes B$, while zero wires indicates the unit I of the tensor product operation. Next, for any two morphisms f and g we may construct their tensor product $f \otimes g$, which we represent by placing the diagrams representing f and g beside one another. Composition is represented by connecting wires as before. A *symmetric* monoidal category also has *braiding morphisms* $\sigma_{A,B}$, which allow wires to cross.

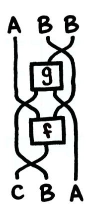
$f \rightsquigarrow$ 

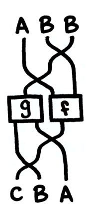
$f \otimes g \rightsquigarrow$ 

$\sigma_{A,B} \rightsquigarrow$ 

The axioms of a symmetric monoidal category can be summarized by saying that they allow us to interpret these diagrams unambiguously. For example, the following diagrams would all represent the same morphism:

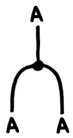






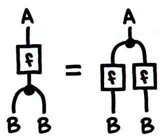
While the details have been elided here, these *string diagrams* are a precise and rigorous method of representing morphisms of a symmetric monoidal category.

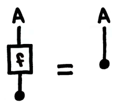
It so happens that categories with finite products are symmetric monoidal categories in which each object A is equipped with a *copying* morphism $\delta_A : A \to A \otimes A$ and *discarding* morphism $\varepsilon_A : A \to I$. We represent δ_A and ε_A with the following string diagrams:

$\delta_A \quad \rightsquigarrow$ 

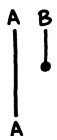
$\varepsilon_A \quad \rightsquigarrow$ 

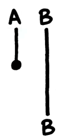
The copying and discarding morphisms must be *natural*, in the sense that for any morphism $f : A \to B$ we have $f\delta_B = \delta_A(f \otimes f)$ and $f\varepsilon_B = \varepsilon_A$. Intuitively, the naturality axiom for the copying morphisms tells us that performing f and then copying the output is the same as copying the input and then performing f on each copy. The naturality axiom for the discarding morphisms tells us that performing f and then discarding the output is the same thing as discarding the input.

$f\delta_B = \delta_A(f \otimes f) \quad \rightsquigarrow$ 

$f\varepsilon_B = \varepsilon_A \quad \rightsquigarrow$ 

Now finite products can be expressed in terms of the copying and discarding morphisms. The terminal object is the unit I of the tensor product operation, with $\varepsilon_A : A \to I$ the unique such morphism. The product of A and B is $A \otimes B$, with projections $1_A \otimes \varepsilon_B : A \otimes B \to A$ and $\varepsilon_A \otimes 1_B : A \otimes B \to B$ and with the pairing map for $f : C \to A$ and $g : C \to B$ given by $\delta_C(f \otimes g) : C \to A \otimes B$.

$1_A \otimes \varepsilon_B \quad \rightsquigarrow$ 

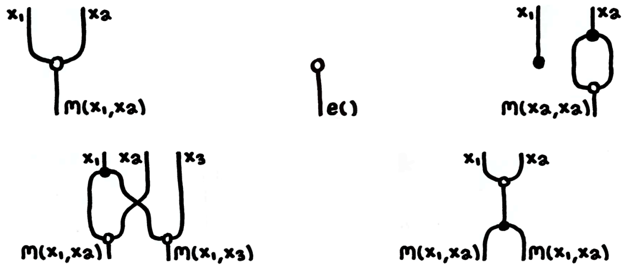
$\varepsilon_A \otimes 1_B \quad \rightsquigarrow$ 

$\delta_C(f \otimes g) \quad \rightsquigarrow$ 

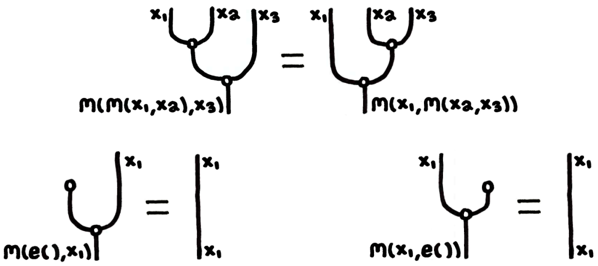
These string diagrams give an alternative method of presenting algebraic theories. Given a signature Σ we represent $f_{/n} \in \Sigma$ as a diagram with n input wires and one output wire. We omit the wire labels, as in this case they would all be the same. Then string diagrams built from these component diagrams together with the copying and discarding morphisms correspond to tuples of terms over Σ. Each input wire corresponds to a variable x_i, and each output wire corresponds to a term. For example, recall the signature $\Sigma_{\mathsf{Mon}} = \{m_{/2}, e_{/0}\}$ from our classical presentation of the theory of monoids. Let us represent m and e as follows:

$m_{/2}$ ⇝  $e_{/0}$ ⇝ 

Then for example the following string diagrams correspond to the tuples of terms that label the output wires, in the variables that label the input wires:



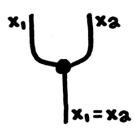
Now the equations E_{Mon} may be expressed string-diagrammatically as follows:



I will call this sort of thing a *Cartesian monoidal presentation.* We construct models of Cartesian monoidal presentations much as we construct models of classical presentations. We must choose a set X to be the carrier, and for each $f_{/n} \in \Sigma$ we must specify a function $X^n \to X$. Then each string diagram over Σ defines a function $X^n \to X^m$, and we have a *model* in case the equations are satisfied. As with classical presentations, models of Cartesian monoidal presentations are equivalently functors into Set that preserve the finite product structure.

The syntax of partial and relational algebraic theories is obtained by modifying the notion of Cartesian monoidal presentation. For partial algebraic theories, we assume an additional string-diagrammatic component $\mu_A : A \otimes A \to A$ which is to be understood as a *partial equality test.* That is, an abstract version of the partial function which, given x and y, returns x if $x = y$ and is otherwise undefined.

μ_A ⇝ 



While for partial algebraic theories we retain the assumption that the copying morphisms are natural in the sense that $f\delta_B = \delta_A(f \otimes f)$, we no longer ask that the discarding morphisms are natural. To obtain the corresponding notion of *model* we interpret our string diagrams in the category of sets and *partial* functions, proceeding much as before. For relational algebraic theories we assume yet another component $\eta_A : I \to A$ which we can think of as *existential quantification.*

$\eta_A \quad \rightsquigarrow$ 

For relational algebraic theories, the naturality axioms are replaced by the assumption that $\delta_A(f \otimes f)\mu_B = f$ for all $f : A \to B$. To obtain the corresponding notion of *model* we interpret our string diagrams in the category of sets and *relations.*

In the same way that algebraic theories correspond to categories with finite products, partial algebraic theories correspond to *discrete Cartesian restriction (DCR) categories*, and relational algebraic theories correspond to *Carboni-Walters (CW) categories.* I further characterise the *varieties* of partial and relational algebraic theories, being those categories that arise as the category of models of a given theory. For partial algebraic theories, the varieties turn out to be the *locally finitely presentable (LFP)* categories, and for relational algebraic theories one obtains the *definable* categories. This leads to a result concerning *Morita equivalence* of theories, being the situation in which two theories present the same variety. We obtain that for all of classical, partial, and relational algebraic theories, two theories are Morita equivalent if and only if they have equivalent idempotent splitting completions.

From a technical perspective, the primary novelty in all of this consists of two strict 2-equivalences of 2-categories. First, a 2-category of DCR categories and structure-preserving functors is shown to be equivalent to the usual 2-category of categories with finite limits. Second, a 2-category of CW categories and structure-preserving functors is shown to be equivalent to the usual 2-category of regular categories. Crucially, to obtain these equivalences one must take *monoidal lax transformations* as 2-cells both in the 2-category of DCR categories and the 2-category of CW categories. The development also contains significant results concerning DCR and CW categories in and of themselves.

1.0.1 Contributions

The notion of partial algebraic theory presented in this thesis is novel, although it has certainly been influenced by the work of Bonchi et al. [7], in which the notion of relational algebraic theory presented in this thesis originates. The central technical contributions of the thesis are Theorem 3.5.12 and Theorem 4.5.12. Significant corollaries of these results include Theorem 3.6.5, Theorem 4.6.14, Theorem 3.6.6, and Theorem 4.6.15, which characterise varieties and Morita equivalence in the setting of partial and relational algebraic theories.

Other significant contributions include the string-diagrammatic characterisation of DCR categories in Theorem 3.2.4, The alternative axiom scheme for CW categories in Lemma 4.1.6, the characterisation of partial monics in DCR categories in Lemma 3.2.10, and the technical result in Lemma 4.3.9, which is useful for reasoning about relations in regular categories.. The various idempotent splitting biadjunctions, being Lemma 2.3.6, Lemma 2.3.9, Lemma 2.4.27, Lemma 3.2.15 and Lemma 4.1.26 might together be seen to constitute a minor contribution.

Section 4.5 recapitulates a number of known results concerning CW categories and regular categories as necessary ingredients for Theorem 4.5.12. In this I have loosely followed the path taken by Freyd and Scedrov [32], but have avoided the concept of *tabulation*, instead preferring to work with split coreflexives in CW categories directly. The result is that many of the proofs in Section 4.5 are different than those in the literature, which may also constitute a minor contribution.

1.0.2 Related Work

CW categories were introduced as "cartesian bicategories of relations" by Carboni and Walters [14]. Much of the theory of CW categories originates in the work of Freyd and Scedrov on *allegories*, in which the CW categories are present as the "unitary pre-tabular allegories". Bonchi, Pavlovic and Sobocinski were the first to frame CW categories as relational algebraic theories [7].

Restriction categories were developed by Cockett and Lack [21, 22, 24]. DCR categories arise in the work of Cockett, Hofstra and Guo on *range categories* [17, 18]. The theory of restriction categories builds on a long history of approaches to categorifying partial functions, including [25, 46] and in particular [11], which anticipates the development of DCR categories and their connection to categories with finite limits.

The original variety theorem in universal algebra is due to Birkhoff [6]. That categories with finite products can be seen as algebraic theories is due to Lawvere [41], and the associated syntax-semantics adjunction (categorical variety theorem) is due to Adámek, Lawvere and Rosickỳ [2]. The standard reference for LFP categories and Gabriel-Ulmer duality is the monograph of Adámek and Rosickỳ [3]. Definable categories and their analogue of Gabriel-Ulmer duality were introduced by Kuber and Rosickỳ [38], although here I follow the presentation of Lack and Tendas [40].

String-diagrammatic presentation of algebraic theories is made possible by a result of Fox [30]. A contribution of this thesis is an analogous result (Theorem 3.2.4) enabling string-diagrammatic presentation of DCR categories, which is inspired by the work of Giles on *discrete inverse categories* [33]. The string-diagrammatic presentation of CW categories — used here to present relational algebraic theories — is due to Carboni and Walters [14].

Partial algebraic theories are equivalent in expressive power to a number of existing notions of theory, including essentially algebraic theories [31, 1], partial Horn theories (which include quasi-equational theories) [45], and finite limit sketches [3]. In each case theories correspond to categories with finite limits. Relational algebraic theories will be equivalent in expressive power to any notion of theory in which theories correspond to regular categories.

The relationship between CW categories and regular categories has also been treated at the level of 2-categories in a recent preprint of Fong and Spivak [28]. Therein, the authors claim to show that the 2-category of regular categories, regular functors, and regular transformations is equivalent to the 2-category of CW categories, CW functors, and natural transformations whose components are maps. Although it is not too far from the truth, this claim is incorrect. To obtain an equivalence of 2-categories one must work with CW categories, CW functors, and *lax* transformations whose components are maps, as in Theorem 4.5.12.

The central results of this thesis have been published separately. Specifically the results concerning partial algebraic theories appear in [26], and the results concerning relational algebraic theories appear in [42].

1.0.3 Organisation

The thesis is divided into three chapters following this introduction. In Chapter 2 certain background material necessary for the remainder of the thesis is recapitulated. This material includes: a brief introduction to string diagrams for symmetric monoidal categories with a particular emphasis on symmetric monoidal categories constructed from a monoidal signature (Section 2.1); a treatment of the idempotent splitting completion of a category and some of its elementary properties (Section 2.2); a very brief account of certain 2-categorical ideas (Section 2.3); a treatment of Fox's theorem characterising categories with finite products (Section 2.4.1); a treatment of the classical method of presenting algebraic theories (Section 2.4.2); an exposition of the alternative method of presenting algebraic theories by string diagrams (Section 2.4.3); and finally a review of the variety theorem for algebraic theories and the associated characterisation of Morita equivalence (Section 2.4.4).

Chapter 3 is concerned with the development of partial algebraic theories. It begins by introducing the notions of cartesian restriction (CR) category (Section 3.1) and discrete cartesian restriction (DCR) category (Section 3.2) together with some of their properties. Of particular importance is Theorem 3.2.4, which is a new result that allows the string-diagrammatic presentation of DCR categories. Section 3.3 introduces partial algebraic theories, gives a method of presenting them, and illustrates the idea with a number of examples. The remainder of the chapter builds towards the variety theorem for partial algebraic theories with the intermediate goal of Theorem 3.5.12, which establishes a correspondence between DCR categories and categories with finite limits. More precisely, Section 3.4 sets up the necessary results about finite limits and categories of partial maps, Section 3.5 contains the proof of Theorem 3.5.12, and Section 3.6 contains the variety theorem for partial algebraic theories along with the associated characterisation of Morita equivalence.

Chapter 4 is concerned with the development of relational algebraic theories. It begins with Section 4.1, which introduces Carboni-Walters (CW) categories and establishes a number of elementary results concerning such categories. This is followed by Section 4.2, which contains the notion of relational algebraic theory, a method of presenting relational algebraic theories, and a number of examples. The remainder of the chapter builds towards the variety theorem for relational algebraic theories with the intermediate goal of Theorem 4.5.12, which establishes a correspondence between CW categories and regular categories. More precisely, Section 4.3 and Section 4.4 contain necessary results about regular categories and categories of relations, Section 4.5 contains the proof of Theorem 4.5.12, and Section 4.6 contains the variety theorem for relational algebraic theories along with the associated characterisation of Morita equivalence.

Chapter 2

Background Material

Before we begin, let us briefly discuss notation and conventions. All categories in the thesis are assumed to be locally small. Composition of arrows $f : A \to B$ and $g : B \to C$ is written $fg : A \to C$. We may also write $g \circ f : A \to C$, but will *never* write $gf : A \to C$. Moreover, we will work only with strict monoidal categories as it greatly simplifies the presentation. In this thesis "monoidal category" means "strict monoidal category". In keeping with this assumption we work with strictly associative finite product structure. The category Set — along with the closely related categories Par and Rel — will be an exception to this rule. The cartesian monoidal structure on Set is not strict. Nonetheless, we will pretend that it is. There is little risk in doing so, given that every monoidal category is equivalent to a strict one. Moreover, the results of this thesis also hold for monoidal categories in general. It is hoped that the assumption of strictness will make the development herein more approachable. In this section we make some small effort to distinguish arrows of a given monoidal category from the associated string diagrams, but in general we treat the two interchangeably. For the uninitiated, a good reference on string diagrams for monoidal categories is [47].

2.1 Symmetric Monoidal Presentations

In this section we review the basic concepts involved in presenting a symmetric monoidal category by generators and equations. Write X^* for the free monoid on a set X. $X_1 \otimes \cdots \otimes X_n \in X^*$ are sequences of elements $X_1, \ldots, X_n \in X$. The unit $I \in X^*$ denotes the empty such sequence, and the monoid operation is given by concatenation: $(X_1 \otimes \cdots \otimes X_n) \otimes (Y_1 \otimes \cdots \otimes Y_m) = X_1 \otimes \cdots \otimes X_n \otimes Y_1 \otimes \cdots \otimes Y_m$. We may now begin with the appropriate notion of signature:

Definition 2.1.1. A *monoidal signature* Γ consists of a set $\mathfrak{s}(\Gamma)$ of *sorts*, a set $|\Gamma|$

of *generators*, and two functions $\delta_0, \delta_1 : |\Gamma| \to \mathfrak{s}(\Gamma)^*$. We call $\delta_0(\gamma)$ the *arity* of γ and $\delta_1(\gamma)$ the *coarity* of γ. We write $\gamma : X \to Y \in \Gamma$ to mean that $\gamma \in |\Gamma|$ with $\delta_0(\gamma) = X$ and $\delta_1(\gamma) = Y$.

Given such a signature, we can construct a symmetric monoidal category as follows:

Definition 2.1.2. Let Γ be a monoidal signature. The symmetric monoidal category $\mathsf{S}(\Gamma)$ of *symmetric monoidal terms over* Γ is constructed as follows:

objects are elements of $\mathfrak{s}(\Gamma)^*$.

arrows are generated by:

$$\frac{\gamma : A \to B \in \Gamma}{\gamma : A \to B} \qquad \frac{f : A \to B \qquad g : B \to C}{fg : A \to C} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{1_A : A \to A}$$

$$\frac{f : A \to B \qquad g : C \to D}{f \otimes g : A \otimes C \to B \otimes D} \qquad \frac{A, B \in \mathfrak{s}(\Gamma)^*}{\sigma_{A,B} : A \otimes B \to B \otimes A} \qquad \frac{}{\square : I \to I}$$

These arrows are subject to a number of equations. First, equations concerning coherence:

$$1_I = \square \qquad 1_{A \otimes B} = 1_A \otimes 1_B \qquad \sigma_{I,A} = 1_A \qquad \sigma_{A,I} = 1_A$$

$$\sigma_{A \otimes B, C \otimes D} = (1_A \otimes \sigma_{B,C} \otimes 1_D)(\sigma_{A,C} \otimes \sigma_{B,D})(1_C \otimes \sigma_{A,D} \otimes 1_B)$$

With the remaining equations being:

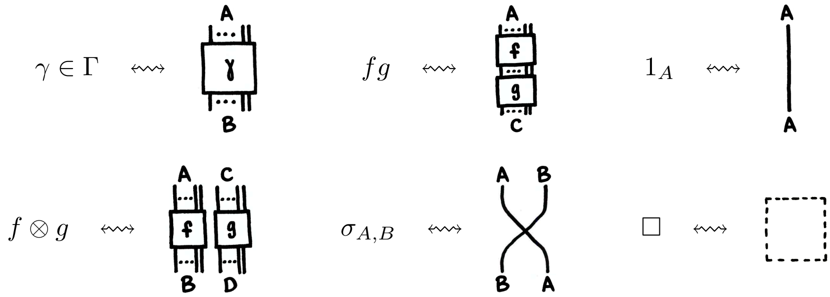
$$1_A f = f \qquad f 1_B = f \qquad \square \otimes f = f \qquad f \otimes \square = f$$

$$(fg)h = f(gh) \qquad (f \otimes g) \otimes h = f \otimes (g \otimes h) \qquad (f \otimes g)(h \otimes k) = fh \otimes gk$$

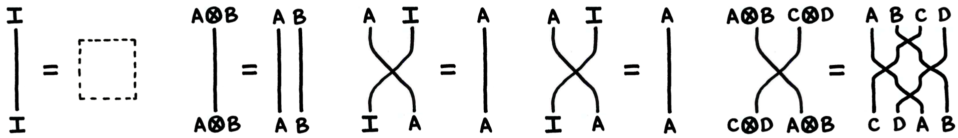
$$\sigma_{A,B}\sigma_{B,A} = 1_{A \otimes B} \qquad \sigma_{A,A'}(g \otimes f) = (f \otimes g)\sigma_{B,B'}$$

This defines a symmetric monoidal category with the evident composition, identities, and symmetric monoidal structure.

It is convenient to specify symmetric monoidal presentations using string diagrams. We recall the diagrammatic conventions:



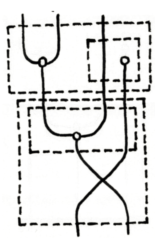
Then the equations concerning coherence become:



Of the remaining equations, the first seven ensure that our diagrams are not ambiguous, and the final two equations become:

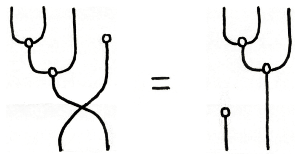
For example, consider the monoidal signature Γ_{Mon} with sorts $\mathfrak{s}(\Gamma_{\mathsf{Mon}}) = \{X\}$ and generators $m : X \otimes X \to X$ and $e : I \to X \in \Gamma$. We adopt the following diagrammatic notation for m and e:

Then the arrow $(m \otimes (1 \otimes e))\,;((m \otimes 1)\,;\sigma) : X \otimes X \otimes X \to X \otimes X$ of $\mathsf{S}(\Gamma)$ is depicted as:



where the dashed line boxes correspond to parentheses, serving to illustrate the connection between parts of the expression and parts of the string diagram.

Two parallel arrows of $\mathsf{S}(\Gamma)$ are equal if and only if it is possible to continuously deform the corresponding string diagrams into each other [37]. For example in $\mathsf{S}(\Gamma_{\mathsf{Mon}})$ we know that $(m \otimes e)(m \otimes 1_X)(\sigma_{X,X}) = (m \otimes 1_X)m(e \otimes 1_X)$ as in:



The notion of equation between string diagrams over a monoidal signature works the same way that and equations of terms does classically:

Definition 2.1.3. Let Γ be a monoidal signature. A *symmetric monoidal equation over* Γ is a pair (f, g) where $f, g : A \to B$ in $\mathsf{S}(\Gamma)$. We often write $f = g$ instead of (f, g).

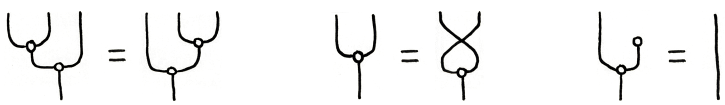
When packaged together, we call a signature together with equations a *presentation*:

Definition 2.1.4. A *symmetric monoidal presentation* (Γ, E) consists of a monoidal signature Γ together with a set E of symmetric monoidal equations over Γ.

We consider a presentation to present the category of terms over the associated signature, modulo the associated equations:

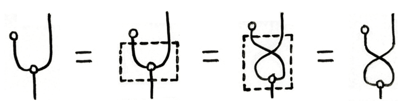
Definition 2.1.5. Let (Γ, E) be a symmetric monoidal presentation. Write $\mathsf{S}(\Gamma, E)$ for the symmetric monoidal category obtained by quotienting $\mathsf{S}(\Gamma)$ by the equations of E. We say that $\mathsf{S}(\Gamma, E)$ is *presented by* (Γ, E), and similarly we say that (Γ, E) *presents* $\mathsf{S}(\Gamma, E)$.

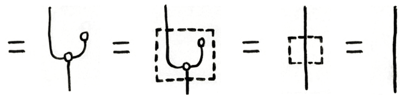
For example, consider the symmetric monoidal presentation $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ where E_{CMon} consists of the following equations:



which in turn express the associativity, commutativity, and unitality of m. Then $\mathsf{S}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ is the prop of commutative monoids [39].

String diagrams are susceptible to equational reasoning, sometimes called *diagrammatic reasoning* in this context. If $f = g$ in $\mathsf{S}(\Gamma, E)$ then substituting the corresponding string diagrams for each other is sound in any context. For example in E_{CMon} we only ask for the right unit law. As an illustration of diagrammatic reasoning we show that the left unit law holds in $\mathsf{S}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ as well:





Observe that it is possible for different symmetric monoidal presentations to present the same symmetric monoidal category. For example, if $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}'})$ is such that $E_{\mathsf{CMon}'}$ contains the associativity, commutativity, and *left* unitality axioms then $\mathsf{S}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}}) \cong \mathsf{S}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}'})$.

Structure preserving functors between monoidal categories are given as follows:

Definition 2.1.6. Let $(\mathbb{X}, \otimes, I)$ and $(\mathbb{Y}, \otimes, I)$ be monoidal categories. A *monoidal functor* $F : (\mathbb{X}, \otimes, I) \to (\mathbb{Y}, \otimes, I)$ is a functor $F : \mathbb{X} \to \mathbb{Y}$ together with a family of natural isomorphisms $\phi^F_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ and an isomorphism $\phi^F_I : I \to FI$ such that for all objects X, Y, Z of $\mathbb{X}$ we have:

(i) $(1_{FX} \otimes \phi^F_{Y,Z})\phi^F_{X,Y\otimes Z} = (\phi^F_{X,Y} \otimes 1_{FZ})\phi^F_{X\otimes Y,Z}$

(ii) $(1_{FX} \otimes \phi^F_I)\phi^F_{X,I} = 1_{FX}$

(iii) $(\phi^F_I \otimes 1_{FX})\phi^F_{I,X} = 1_{FX}$

where naturality of the $\phi^F_{X,Y}$ simply means that for all $f : X \to X'$ and $g : Y \to Y'$ of $\mathbb{X}$ we have $\phi^F_{X,Y} F(f \otimes g) = (F(f) \otimes F(g))\phi^F_{X',Y'}$.

Definition 2.1.7. Let $(\mathbb{X}, \otimes, I)$ and $(\mathbb{Y}, \otimes, I)$ be symmetric monoidal categories. A monoidal functor $F : (\mathbb{X}, \otimes, I) \to (\mathbb{Y}, \otimes, I)$ is called *symmetric* in case $\phi^F_{X,Y} F(\sigma_{X,Y}) = \sigma_{FX,FY}\phi^F_{Y,X}$.

Definition 2.1.8. Let $(\mathbb{X}, \otimes, I)$ and $(\mathbb{Y}, \otimes, I)$ be symmetric monoidal categories, and suppose that $F, G : (\mathbb{X}, \otimes, I) \to (\mathbb{Y}, \otimes, I)$ are symmetric monoidal functors. Then a *monoidal natural transformation* $\alpha : F \to G$ is a natural transformation such that $\phi^F_{X,Y}\alpha_{X\otimes Y} = (\alpha_X \otimes \alpha_Y)\phi^G_{X,Y}$ and $\phi^F_I \alpha_I = \phi^G_I$.

In this thesis we will systematically omit the coherence isomorphisms $\phi^F_{X,Y}$ and ϕ^F_I for the sake of readability. That is, we will behave as though for monoidal functors F we have $FX \otimes FY = F(X \otimes Y)$ and $FI = I$ with $\phi^F_{X,Y}$ and ϕ^F_I identity arrows, and as though for monoidal natural transformations α we have $\alpha_X \otimes \alpha_Y = \alpha_{X\otimes Y}$. Strictly speaking, the coherence isomorphisms should be included. However, we will not be doing anything particularly technical with monoidal functors in this thesis, and omitting the coherences does not lead us astray.

2.2 Splitting Idempotents

In this section we recapitulate a few details surrounding split idempotents and the idempotent splitting completion, which will be used heavily in our development. In particular Lemma 2.3.6 will later be specialized in a number of different directions, and plays a key role in the characterisation of varieties and Morita equivalence for the theories we consider. Those familiar with the idempotent splitting completion may safely skip ahead to the next section, referring back to Lemma 2.3.6 as needed.

We begin by recalling the basic definitions:

Definition 2.2.1. An arrow $f : A \to A$ in a category is said to be *idempotent* in case $ff = f$.

Definition 2.2.2. An idempotent $f : A \to A$ in a category $\mathbb{C}$ *splits* in case there exist arrows $s : X \to A$ and $r : A \to X$ of $\mathbb{C}$ such that:

$$\begin{tikzcd}
& X \arrow[r, "s"] & A \\
X \arrow[ur, equal] \arrow[r, "s"'] & A \arrow[u, "r"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

Notice that in this case s is a section, and is therefore monic. Dually, r is a retraction.

Notice that splittings of a given idempotent are unique up to isomorphism:

Lemma 2.2.3. *Suppose that an idempotent $f : A \to A$ splits in two different ways:*

$$\begin{tikzcd}
& X \arrow[r, "s_X"] & A \\
X \arrow[ur, equal] \arrow[r, "s_X"'] & A \arrow[u, "r_X"'] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& Y \arrow[r, "s_Y"] & A \\
Y \arrow[ur, equal] \arrow[r, "s_Y"'] & A \arrow[u, "r_Y"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

Then $s_X r_Y : X \to Y$ and $s_Y r_X : Y \to X$ define an isomorphism $X \cong Y$.

Proof. We have:

$$\begin{tikzcd}
X \arrow[r, "s_X"] \arrow[dr, equal] & A \arrow[r, "r_Y"] \arrow[d, "r_X"'] \arrow[dr, "f"] & Y \arrow[d, "s_Y"] \\
& X \arrow[r, "s_X"'] \arrow[dr, equal] & A \arrow[d, "r_X"] \\
& & X
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
Y \arrow[r, "s_Y"] \arrow[dr, equal] & A \arrow[r, "r_X"] \arrow[d, "r_Y"'] \arrow[dr, "f"] & X \arrow[d, "s_X"] \\
& Y \arrow[r, "s_Y"'] \arrow[dr, equal] & A \arrow[d, "r_Y"] \\
& & Y
\end{tikzcd}$$

as required. □

In any category we may formally split any collection of idempotents to obtain a new category in which those idempotents split. The procedure of formally splitting the class of all idempotents is also known as the *Cauchy completion* [8] and *Karoubi envelope* of a category. While any collection of idempotents can be split, the construction is better-behaved when that collection of idempotents contains all the identity morphisms of the category in question. We will build this assumption into our terminology:

Definition 2.2.4. Let $\mathbb{X}$ be a category. A *species of idempotent* in $\mathbb{X}$ is a set $\mathcal{E}$ of idempotents in $\mathbb{X}$ such that for all objects A of $\mathbb{X}$, $1_A \in \mathcal{E}$.

We may formally split a species of idempotent in a given category as follows:

Definition 2.2.5. Let $\mathbb{X}$ be a category, and let $\mathcal{E}$ be a species of idempotent in $\mathbb{X}$. Define the category $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ as follows:

objects are pairs (X, a) where X is an object of $\mathbb{X}$ and $a : X \to X$ is in $\mathcal{E}$.

arrows $f : (X, a) \to (Y, b)$ are arrows $f : X \to Y$ of $\mathbb{X}$ such that $afb = f$.

composition is given by composition in $\mathbb{X}$.

The **identity** on (X, a) is given by $a : X \to X$.

In case $\mathcal{E}$ is the species of *all* idempotents in $\mathbb{X}$, we write $\mathsf{Split}(\mathbb{X}) = \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$. First, we observe that $\mathbb{X}$ embeds into $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$:

Lemma 2.2.6. *Suppose $\mathcal{E}$ is a species of idempotent in $\mathbb{X}$. Then there is an embedding $[\![-]\!] : \mathbb{X} \to \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ defined by $[\![A]\!] = (A, 1_A)$ on objects and by $[\![f]\!] = f : (A, 1_A) \to (B, 1_B)$ on arrows $f : A \to B$.*

Proof. $[\![-]\!]$ preserves composition since composition in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is given by composition in $\mathbb{X}$, and preserves identities since 1_A is the identity on $(A, 1_A)$. Thus, $[\![-]\!]$ is a functor. It remains to show that $[\![-]\!]$ is an embedding, i.e., is faithful and injective on objects. This is straightforward: if $[\![f]\!] = [\![g]\!]$ then we have $f = [\![f]\!] = [\![g]\!] = g$ as required, as morphisms in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ are given by morphisms of $\mathbb{X}$. It follows that $[\![-]\!]$ is faithful. For injectivity on objects, if we have $[\![A]\!] = [\![B]\!]$ then immediately we have $(A, 1_A) = [\![A]\!] = [\![B]\!] = (B, 1_B)$, and so $A = B$ as required. Thus, $[\![-]\!] : \mathbb{X} \to \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is an embedding. $\square$

Now, the key property of the idempotent splitting completion $\mathsf{Split}_{\mathcal{E}}$ is that in it, the idempotents of $\mathcal{E}$ split:

Lemma 2.2.7. *Let $\mathcal{E}$ be a species of idempotent in a category $\mathbb{X}$, and let $e : A \to A \in \mathcal{E}$. Then $[\![e]\!] : (A, 1_A) \to (A, 1_A)$ splits in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ via $e : (A, e) \to (A, 1_A)$ (the section) and $e : (A, 1_A) \to (A, e)$ (the retraction).*

Proof. $e : (A, e) \to (A, 1_A)$ and $e : (A, 1_A) \to (A, e)$ are well-defined since $ee1_A = e = 1_A ee$. Recall that e is the identity on $(A, e) \to (A, e)$, and then since $ee = e : (A, e) \to (A, e)$ we have that $e : (A, 1_A) \to (A, 1_A)$ splits as required. $\square$

Lemma 2.2.6 specialises to symmetric monoidal categories:

Lemma 2.2.8. *Let $\mathbb{X}$ be a symmetric monoidal category and $\mathcal{E}$ be species of idempotent in $\mathbb{X}$. Then:*

(i) $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ *is a symmetric monoidal category.*

(ii) There is an embedding

$$[\![-]\!] : \mathbb{X} \hookrightarrow \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$$

defined by $[\![X]\!] = (X, 1_X)$ on objects $[\![f]\!] = f$ on arrows $f : X \to Y$. Further, $[\![-]\!]$ is a symmetric monoidal functor.

Proof. (i) The tensor product is defined on objects as in $(X, A) \otimes (Y, b) = (X \otimes Y, a \otimes b)$ with unit $I = (I, 1_I)$, and the tensor product of arrows in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is given by their tensor product in $\mathbb{X}$. The braiding maps are given by $\sigma_{(X,a),(Y,b)} = (a \otimes b)\sigma_{X,Y}(b \otimes a)$. Then for the inverse property of the braiding we have $\sigma_{(X,a),(Y,b)}\sigma_{(Y,b),(X,a)} = (a \otimes b)\sigma_{X,Y}(b \otimes a)(b \otimes a)\sigma_{Y,X}(a \otimes b) = (aa \otimes bb)\sigma_{X,Y}\sigma_{Y,X}(aa \otimes bb) = (a \otimes b) = 1_{(X,a)\otimes(Y,b)}$. For naturality, if $f : (X, a) \to (X', a')$ and $g : (Y, b) \to (Y', b')$ we have $(f \otimes g)\sigma_{(X',a'),(Y',b')} = (f \otimes g)(a' \otimes b')\sigma_{X',Y'}(b' \otimes a') = (a \otimes b)\sigma_{X,Y}(g \otimes f)(b' \otimes a') = (a \otimes b)\sigma_{X,Y}(b \otimes a)(g \otimes f) = \sigma_{(X,a),(Y,b)}(g \otimes f)$ as required. The claim follows.

(ii) We have that $[\![-]\!]$ is an embedding by Lemma 2.2.6. That it is symmetric monoidal is immediate.

$\square$

This concludes our discussion of idempotent splitting for now, although we will shortly return to these ideas when we split idempotents at the 2-categorical level to obtain a biadjunction. First, we take a brief detour into 2-category theory.

2.3 Equivalence and Adjunction of 2-categories

The core technical results of this thesis consist of a number of adjunctions and equivalences between strict 2-categories. While a full treatment of the theory of 2-categories is out of scope, in this section I briefly recall the relevant notions of equivalence and adjunction. One need not necessarily know anything about 2-categories to appreciate the notions of partial and relational algebraic theory, but a surface-level understanding of the basic concepts is necessary if one is to engage with the related technical content. A good reference that covers the basics of 2-category theory is the recent monograph of Johnson and Yau [36]. Less basic but more relevant to the subject at hand is the work of Gurski on biequivalence [35].

What does it mean for two mathematical objects to be the same? For sets a good answer is that two sets are the same in case there is a bijection between them. Then anything we can do with one set can be transported to the other by working across the bijection. More generally it is usually reasonable to consider objects A and B of a category to be the same when they are *isomorphic*, which is to say that there are morphisms $f : A \to B$ and $g : B \to A$ with $fg = 1_A$ and $gf = 1_B$. We write $A \cong B$ to indicate the existence of an isomorphism. This kind of sameness is sensitive to the ambient category. For example, it is a very different thing for two groups to be isomorphic in the category of groups and group homomorphisms than for them to be isomorphic in the category of sets and functions. The important thing to remember is that isomorphism is a notion of sameness relative to some ambient category.

If we instead work relative to an ambient 2-category the situation is more complicated. The source of this complexity is of course the 2-cells. In particular it is now possible for *morphisms* (i.e., 1-cells) f and g to be isomorphic to each other via 2-cells $\alpha : f \to g$ and $\beta : g \to f$ such that $\alpha\beta = 1_f$ and $\beta\alpha = 1_g$. This leads to the notion of an *equivalence* between two objects (i.e., 0-cells) of a 2-category, which can be understood as "isomorphism up to isomorphism". Explicitly, two 0-cells A and B are *equivalent* in case there are 1-cells $f : A \to B$ and $g : B \to A$ such that $fg \cong 1_A$ and $gf \cong 1_B$. The point being that the composites are now isomorphic to identities instead of being literally equal. We write $A \simeq B$ to indicate the existence of an equivalence. Equivalence turns out to be a good notion of sameness in 2-categories. For example the prototypical 2-category is Cat, which has categories as 0-cells, functors as 1-cells, and natural transformations as 2-cells. The resulting notion of equivalence of categories is the standard one.

In fact this notion of equivalence makes sense in any *bicategory*, which is a slightly weaker sort of 2-category. The pattern outlined above repeats, and to find a good notion of sameness for bicategories (and therefore 2-categories) one must work in a *tricategory*. The resulting notion of sameness is called *biequivalence* and has been considered in detail by Gurski [35]. A lot of knowledge surrounding biequivalence, and bicategories (including 2-categories) more generally seems to be mathematical folklore. The main idea is that we want to work up to isomorphism as far as possible. At some point between bicategories and tricategories it becomes impressively tedious and error-prone to keep track of the details involved in this, but for our purposes all we need is the resulting notion of sameness between 2-categories that results from the process. For our purposes we need only work explicitly with what I will call a *strict 2-equivalence* of strict 2-categories. This is an instance of the more general notion of biequivalence of bicategories, in which many of the coherence isomorphisms are identities.

A brief aside: what we will call a 2-functor is often called a strict 2-functor, in keeping with our earlier terminological choices. We give an elementary definition:

Definition 2.3.1. Let $\mathbb{C}$ and $\mathbb{D}$ be 2-categories. A *2-functor* $F : \mathbb{C} \to \mathbb{D}$ consists of mappings F_0 sending 0-cells of $\mathbb{C}$ to 0-cells of $\mathbb{D}$, F_1 sending 1-cells of $\mathbb{C}$ to 1-cells of $\mathbb{D}$, and F_2 sending 2-cells of $\mathbb{C}$ to 2-cells of $\mathbb{D}$ such that composition and identities are preserved at the level of both 1-cells and 2-cells:

$$F_1(f)F_1(g) = F_1(fg) \qquad\qquad F_1(1_A) = 1_{F_0(A)}$$

$$F_2(\alpha)F_2(\beta) = F_2(\alpha\beta) \qquad\qquad F_2(1_f) = 1_{F_1(f)}$$

and further that horizontal composition of 2-cells is preserved:

$$F_2(\alpha \star \beta) = F_2(\alpha) \star F_2(\beta)$$

We drop the subscripts when they are clear in context. As with functors, we may also sometimes omit the parentheses, as in $F(A) = FA$. Similarly, what we will call a 2-natural transformation what is often called a *strict 2-natural transformation*

Definition 2.3.2. A *2-natural transformation* $\lambda : F \to G$ for 2-functors $F, G : \mathbb{C} \to \mathbb{D}$ consists of a family $\lambda_A : FA \to GA$ of 1-cells in $\mathbb{D}$ such that for any 1-cell $f : A \to B$ of $\mathbb{C}$ we have $\lambda_A G(f) = F(f)\lambda_B$ and, moreover, such that for any 1-cells $f, g : A \to B$ and 2-cell $\beta : f \to g$ of $\mathbb{C}$, we have $F(\beta) \star 1_{\lambda_B} = 1_{\lambda_A} \star G(\beta)$. A 2-natural transformation is said to be *invertible* in case its components λ_A are invertible.

Now by strict 2-equivalence of 2-categories we mean:

Definition 2.3.3. Suppose that $\mathbb{C}$ and $\mathbb{D}$ are 2-categories. A *strict 2-equivalence* $\mathbb{C} \simeq \mathbb{D}$ consists of 2-functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ together with invertible 2-natural transformations $\eta : 1_{\mathbb{C}} \to FG$ and $\varepsilon : GF \to 1_{\mathbb{D}}$.

While we will only be constructing strict 2-equivalences in our development, the related notion of strict 2-adjunction is not general enough for our purposes. Instead we must use the more general notion of *biadjunction*, which is to biequivalence of 2-categories as adjunctions are to equivalence of categories. We will use the following formulation of biadjunction, found in [43], seemingly adapted from Gurski [35]:

Definition 2.3.4. A pair of 2-functors $L : \mathbb{C} \to \mathbb{D}$ and $R : \mathbb{D} \to \mathbb{C}$ form a *biadjunction* $L \dashv R$ in case there is an equivalence of categories $\mathbb{D}(L(C), D) \simeq \mathbb{C}(C, R(D))$ which is natural[1] in both C and D.

2.3.1 Idempotent Splitting Biadjunctions

Let Cat_s be the full sub 2-category of Cat on the 0-cells $\mathbb{X}$ in which the idempotents of $\mathbb{X}$ split. Splitting idemoptents yields a 2-functor mapping each category to its idempotent splitting completion:

[1] In fact one only needs to ask for a *pseudo*-natural equivalence of hom-categories, but for the purposes of this thesis we can get away with insisting that the equivalence be strictly natural.

Lemma 2.3.5. *Splitting idempotents defines a 2-functor* $\mathsf{Split} : \mathsf{Cat} \to \mathsf{Cat}_s$.

Proof. On 1-cells $F : \mathbb{X} \to \mathbb{Y}$ of Cat we take $\mathsf{Split}(F) : \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{Y})$ to be the functor defined on objects by $\mathsf{Split}(F)(X,a) = (FX, F(a))$ and defined on arrows $f : (X,a) \to (Y,b)$ of $\mathsf{Split}(\mathbb{X})$ by $\mathsf{Split}(F)(f) = F(f)$. We have that $\mathsf{Split}(F)$ is well defined as in $F(a)F(f)F(b) = F(afb) = F(f)$, and it preserves composition and identities because F does. Clearly Split preserves composition and identities at the level of 1-cells. On 2-cells $\alpha : F \to G$ of Cat we define $\mathsf{Split}(\alpha) : \mathsf{Split}(F) \to \mathsf{Split}(G)$ by $\mathsf{Split}(\alpha)_{(X,a)} = F(a)\alpha_X G(a) : (FX, F(a)) \to (GX, G(a))$. The components are well-defined as in $F(a)\mathsf{Split}(\alpha)_{(X,a)}G(a) = F(a)F(a)\alpha_X G(a)G(a) = F(a)\alpha_X G(a) = \mathsf{Split}(\alpha)_{(X,a)}$, and $\mathsf{Split}(\alpha)$ is a natural transformation as follows: for each arrow $f : (X,a) \to (Y,b)$ of $\mathsf{Split}(\mathbb{X})$ we have $\mathsf{Split}(F)(f)\mathsf{Split}(\alpha)_{(X,a)} = F(f)F(b)\alpha_Y G(b) = F(a)F(f)\alpha_Y G(b) = F(a)\alpha_X G(f)G(b) = F(a)\alpha_X G(a)G(f)$ as in:

$$\begin{tikzcd}
\mathsf{Split}(F)(X,a) \arrow[r, "\mathsf{Split}(\alpha)_{(X,a)}"] \arrow[d, "\mathsf{Split}(F)(f)"'] & \mathsf{Split}(G)(X,a) \arrow[d, "\mathsf{Split}(G)(f)"] \\
\mathsf{Split}(F)(Y,b) \arrow[r, "\mathsf{Split}(\alpha)_{(Y,b)}"'] & \mathsf{Split}(G)(Y,b)
\end{tikzcd}$$

Now $\mathsf{Split}(1_F)_{(X,a)} = F(a)F(a) = F(a) = 1_{F(X,a)}$ and for $\alpha : F \to G$ and $\beta : G \to H$ we have $(\mathsf{Split}(\alpha)\mathsf{Split}(\beta))_{(X,a)} = F(a)\alpha_X G(a)G(a)\beta_X H(a)$
$= F(a)\alpha_X \beta_X H(a)H(a) = F(a)(\alpha\beta)_X H(a) = \mathsf{Split}(\alpha\beta)_{(X,a)}$. It follows that Split preserves composition and identities at the level of 2-cells. For horizontal composition of 2-cells, suppose $\alpha : F \to F'$ and $\beta : F \to F'$ are 2-cells of Cat. Then $\mathsf{Split}(\alpha \star \beta)_{(X,a)} = G(F(a))\beta_{FX} G'(F(a))G'(F(a)\alpha_X F'(a)) = (\mathsf{Split}(\alpha) \star \mathsf{Split}(\beta))_{(X,a)}$, and it follows that Split is a 2-functor. □

Moreover, this 2-functor is part of a biadjunction:

Lemma 2.3.6. *There is a biadjunction*

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r, shift left, "\mathsf{Split}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_s \arrow[l, hook, shift left]
\end{tikzcd}$$

where the right adjoint $\mathsf{Cat}_s \hookrightarrow \mathsf{Cat}$ *is the evident inclusion.*

Proof. We must exhibit a natural equivalence of categories

$$\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \simeq \mathsf{Cat}(\mathbb{X}, \mathbb{C})$$

for any category $\mathbb{X}$ and category with split idempotents $\mathbb{C}$. To that end, we define a functor $(-)^\sharp : \mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \to \mathsf{Cat}(\mathbb{X}, \mathbb{C})$ as follows: on objects $F : \mathsf{Split}(\mathbb{X}) \to \mathbb{C}$ of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ let $F^\sharp : \mathbb{X} \to \mathbb{C}$ be defined by $F^\sharp(X) = F(X, 1_X)$ and on arrows by $F^\sharp(f) = F(f)$. Clearly $F^\sharp$ is a functor, so the object mapping of $(-)^\sharp$ is well-defined. For arrows $\alpha : F \to G$ of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ let $\alpha^\sharp : F^\sharp \to G^\sharp$ be the natural transformation with components $\alpha^\sharp_X = \alpha_{(X,1_X)}$. This is natural as in $F^\sharp(f)\alpha^\sharp_Y = F(f)\alpha_{(Y,1_Y)} = \alpha_{(X,1_X)}G(f) = \alpha^\sharp_X G^\sharp(f)$. We have that $(-)^\sharp$ is a functor as in $(\alpha^\sharp\beta^\sharp)_X = \alpha^\sharp_X\beta^\sharp_X = \alpha_{(X,1_X)}\beta_{(X,1_X)} = (\alpha\beta)_{(X,1_X)} = (\alpha\beta)^\sharp_X$ and $(1_F)^\sharp_X = (1_F)_{(X,1_X)} = 1_{F(X,1_X)} = 1_{F^\sharp(X)} = (1_{F^\sharp})_X$.

We proceed to show that $(-)^\sharp$ is full, faithful, and essentially surjective, beginning with the latter. To that end, suppose $F : \mathbb{X} \to \mathbb{C}$ is an object of $\mathsf{Cat}(\mathbb{X}, \mathbb{C})$. Define $\widehat{F} : \mathsf{Split}(\mathbb{X}) \to \mathbb{C}$ to be the functor that maps objects (X, a) of $\mathsf{Split}(\mathbb{X})$ to the carrier[2] of the splitting of $F(a) : FX \to FX$ in $\mathbb{C}$, as in:

$$\begin{tikzcd}
 & \widehat{F}(X,a) \arrow[r, "s_a"] & FX \\
\widehat{F}(X,a) \arrow[ur, equal] \arrow[r, "s_a"'] & FX \arrow[u, "r_a"'] \arrow[ur, "F(a)"'] &
\end{tikzcd}$$

with $\widehat{F}$ defined on arrows $f : (X, a) \to (Y, b)$ of $\mathsf{Split}(\mathbb{X})$ by $\widehat{F}(f) = s_aF(f)r_b$. Now $\widehat{F}$ is a functor: for identities we have $\widehat{F}(1_{(X,a)}) = \widehat{F}(F(a)) = s_aF(a)r_a = s_ar_as_ar_a = 1_{\widehat{F}(X,a)}$ and for composition we have $\widehat{F}(f)\widehat{F}(g) = s_aF(f)r_bs_bF(g)r_c = s_aF(f)F(b)F(g)r_c = s_aF(fg)r_c = \widehat{F}(fg)$ for any $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (Z, c)$ in $\mathsf{Split}(\mathbb{X})$. In particular, this means that $\widehat{F}$ is an object of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$. Now, consider $(\widehat{F})^\sharp$. On objects we have $(\widehat{F})^\sharp(X) = \widehat{F}(X, 1_X)$ is the carrier of the splitting of 1_{FX}, as in:

$$\begin{tikzcd}
 & \widehat{F}(X,1_X) \arrow[r, "s_{1_X}"] & FX \\
\widehat{F}(X,1_X) \arrow[ur, equal] \arrow[r, "s_{1_X}"'] & FX \arrow[u, "r_{1_X}"'] \arrow[ur, "F(1_X)=1_{FX}"'] &
\end{tikzcd}$$

Notice in particular that this means $s_{1_X} : \widehat{F}(X, 1_X) \to FX$ is an isomorphism with inverse $s^{-1}_{1_X} = r_{1_X}$. These sections give the components $\phi_X = s_{1_X}$ of a natural isomorphism $\phi : (\widehat{F})^\sharp \to F$. For naturality we have $\widehat{F}^\sharp(f)\phi_Y = s_{1_X}F(f)r_{1_Y}s_{1_Y} = s_{1_X}F(f) = \phi_XF(f)$ for any $f : X \to Y$ in $\mathbb{X}$, as required. Thus $F \cong (\widehat{F})^\sharp$ in Cat. It follows that $(-)^\sharp$ is essentially surjective.

Next, we show that $(-)^\sharp$ is full. To that end, suppose that F, G are objects of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ and that $\alpha : F^\sharp \to G^\sharp$ is an arrow of $\mathsf{Cat}(\mathbb{X}, \mathbb{C})$. Define $\widehat{\alpha} : F \to G$ by $\widehat{\alpha}_{(X,a)} = F(a)\alpha_XG(a)$. Now $\widehat{\alpha}$ is natural as in $F(f)\widehat{\alpha}_{(X,a)} = F(f)F(b)\alpha_YG(b) = F(a)F(f)\alpha_YG(b) = F(a)\alpha_XG(f)G(b) = F(a)\alpha_XG(a)G(f) =$

[2]Here we are using the axiom of choice to choose a splitting of each idempotent in $\mathbb{C}$. Another approach is to work with categories in which idempotents have *distinguished* splittings, in which case the axiom of choice is not needed.

$\widehat{\alpha}_{(Y,b)}G(f)$ for any $f : (X,a) \to (Y,b)$ of $\mathsf{Split}(\mathbb{X})$, and so in particular $\widehat{\alpha}$ is an arrow of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$. Consider $(\widehat{\alpha})^\sharp : F^\sharp \to G^\sharp$. We have $(\widehat{\alpha})^\sharp_X = \widehat{\alpha}_{(X,1_X)} = F(1_X)\alpha_X G(1_X) = \alpha_X$, and so $\widehat{\alpha}^\sharp = \alpha$ and $(-)^\sharp$ is full.

Finally, we show that $(-)^\sharp$ is faithful. To that end, let $\alpha, \beta : F \to G$ be arrows of $\mathsf{Cat}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ and suppose that $\alpha^\sharp = \beta^\sharp$. Notice that we have the following idempotent splitting in $\mathbb{C}$:

$$\begin{tikzcd}
& G(X,a) \arrow[r, "G(a)"] & G(X,1_X) \\
G(X,a) \arrow[ur, equal] \arrow[r, "G(a)"'] & G(X,1_X) \arrow[u, "G(a)"'] \arrow[ur, "G(a)"'] &
\end{tikzcd}$$

so in particular we have that $G(a) : G(X,a) \to G(X,1_X)$ is monic. Now we have $\alpha_{(X,a)}G(a) = F(a)\alpha_{(X,1_X)} = F(a)\alpha^\sharp_X = F(a)\beta^\sharp_X = F(a)\beta_{(X,1_X)} = \beta_{(X,a)}G(a)$, and then since $G(a)$ is monic we have $\alpha_{(X,a)} = \beta_{(X,a)}$ which gives $\alpha = \beta$. Thus, $(-)^\sharp$ is faithful. It follows that $(-)^\sharp$ is an equivalence of categories. Clearly this equivalence is natural in $\mathbb{X}$ and $\mathbb{C}$, and the claim follows. $\square$

A version of the above argument also holds for symmetric monoidal categories.

Definition 2.3.7. The 2-category SMC has 0-cells symmetric monoidal categories, 1-cells symmetric monoidal functors, and 2-cells monoidal natural transformations.

Our idempotent splitting 2-functor extends to the symmetric monoidal setting:

Lemma 2.3.8. *Let SMC_s be the full sub 2-category of SMC on the 0-cells in which the idempotents split. Then splitting idempotents yields a 2-functor $\mathsf{Split} : \mathsf{SMC} \to \mathsf{SMC}_s$.*

Proof. The 2-functor Split is defined the same way it is in Lemma 2.3.5. We must show that for any symmetric monoidal functor $F : \mathbb{X} \to \mathbb{Y}$ the functor $\mathsf{Split}(F) : \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{Y})$ is symmetric monoidal, and that for any monoidal natural transformation $\alpha : F \to G$ the natural transformation $\mathsf{Split}(\alpha) : \mathsf{Split}(F) \to \mathsf{Split}(G)$ is monoidal. For the former, we have $\mathsf{Split}(F)((X,a) \otimes (Y,b)) =$ $\mathsf{Split}(F)(X \otimes Y, a \otimes b) = (F(X \otimes Y), F(a \otimes b)) = (FX \otimes FY, F(a) \otimes F(b)) = (FX, F(a)) \otimes (FY, F(b)) = \mathsf{Split}(F)(X,a) \otimes \mathsf{Split}(F)(Y,b)$ as required. For the latter, we have $\mathsf{Split}(\alpha)_{(X,a)\otimes(Y,b)} = \mathsf{Split}(\alpha)_{(X\otimes Y, a\otimes b)} = F(a \otimes b)\alpha_{X\otimes Y}G(a \otimes b) = (F(a) \otimes F(b))(\alpha_X \otimes \alpha_Y)(G(a) \otimes G(b)) = \mathsf{Split}(\alpha)_{(X,a)} \otimes \mathsf{Split}(\alpha)_{(Y,b)}$ and $\mathsf{Split}(\alpha)_{(I,1_I)} = F(1_I)\alpha_I G(1_I) = 1_I$. The claim follows. $\square$

This 2-functor is also left biadjoint to the evident forgetful functor:

Lemma 2.3.9. *There is a biadjunction:*

$$\begin{tikzcd}
\mathsf{SMC} \arrow[r, bend left, "\mathsf{Split}"] \arrow[r, phantom, "\perp"] & \mathsf{SMC}_s \arrow[l, hook, bend left]
\end{tikzcd}$$

where the right adjoint $\mathsf{SMC}_s \hookrightarrow \mathsf{SMC}$ *is the evident inclusion.*

Proof. The proof is largely identical to the proof of Lemma 2.3.6, but we must deal with a number of proof obligations arising from the monoidal structure. As before, we define a functor $(-)^\sharp : \mathsf{SMC}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \to \mathsf{SMC}(\mathbb{X}, \mathbb{C})$ and show that it gives a natural equivalence of categories. To show that $(-)^\sharp$ is essentially surjective we assume an object F of $\mathsf{SMC}(\mathbb{X}, \mathbb{C})$ and define an object $\widehat{F}$ of $\mathsf{SMC}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$. On objects (X, a) of $\mathsf{Split}(\mathbb{X})$ we again take $\widehat{F}(X, a)$ to be the carrier of the splitting of $F(a) : FX \to FX$ in $\mathbb{C}$, as in:

$$\begin{tikzcd}
& \widehat{F}(X,a) \arrow[r, "s_a"] & FX \\
\widehat{F}(X,a) \arrow[ur, equal] \arrow[r, "s_a"'] & FX \arrow[u, "r_a"'] \arrow[ur, "F(a)"'] &
\end{tikzcd}$$

and on arrows $f : (X, a) \to (Y, b)$ we again define $\widehat{F}(f) = s_a F(f) r_b$. In doing so we have used the axiom of choice to choose a splitting for each idempotent $F(a)$ in $\mathbb{C}$. While this is enough to yield a functor $\widehat{F}$, we now require $\widehat{F}$ to be symmetric monoidal. To obtain this we must impose additional conditions on our choice of splittings[3]. Specifically, we ask that our choice of splittings be coherent with respect to the monoidal category structure in the sense that $\widehat{F}((X, a) \otimes (Y, b)) = \widehat{F}(X, a) \otimes \widehat{F}(Y, b)$ with $s_{a\otimes b} = s_a \otimes s_b$ and $r_{a\otimes b} = r_a \otimes r_b$ and further $\widehat{F}(I, 1_I) = I$ with $s_{1_I} = r_{1_I} = 1_I$. Clearly a coherent choice of splittings always exists. Then $\widehat{F}$ preserves the monoid structure on objects by assumption. For $f : (X, a) \to (X', a')$ and $g : (Y, b) \to (Y', b')$ we have $\widehat{F}(f \otimes g) = s_{a\otimes b} F(f \otimes g) r_{a'\otimes b'} = (s_a \otimes s_b)(F(f) \otimes F(g))(r_{a'} \otimes r_{b'}) = s_a F(f) r_{a'} \otimes s_b F(g) r_{b'} = \widehat{F}(f) \otimes \widehat{F}(g)$, and we have $\widehat{F}(\sigma_{(X,a),(Y,b)}) = s_{a\otimes b} F(\sigma_{X,Y}) r_{b\otimes a} = (s_a \otimes s_b)\sigma_{FX,FY}(r_b \otimes r_a) = (s_a r_a \otimes s_b r_b)\sigma_{\widehat{F}(X,a),\widehat{F}(Y,b)} = \sigma_{\widehat{F}(X,a),\widehat{F}(Y,b)}$, and it follows that $\widehat{F}$ is a symmetric monoidal functor.

The only other difference from the proof of Lemma 2.3.6 appears in the proof that $(-)^\sharp$ is full. Again we assume an arrow $\alpha : F^\sharp \to G^\sharp$ of $\mathsf{SMC}(\mathbb{X}, \mathbb{C})$ and define an arrow $\widehat{\alpha} : F \to G$ of $\mathsf{SMC}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ by $\widehat{\alpha}_{(X,a)} = F(a)\alpha_X G(a)$. We have already seen that this defines a natural transformation, but must now show that it defines a *monoidal* natural transformation. As required, we have $\widehat{\alpha}_{(X,a)\otimes(Y,b)} = \widehat{\alpha}_{(X\otimes Y, a\otimes b)} = F(a\otimes b)\alpha_{X\otimes Y} G(a\otimes b) = F(a)\alpha_X G(a) \otimes F(b)\alpha_Y G(b) = \widehat{\alpha}_{(X,a)} \otimes \widehat{\alpha}_{(Y,b)}$ and $\widehat{\alpha}_{(I,1_I)} = F(1_I)\alpha_I G(1_I) = 1_I$. The claim follows. $\square$

[3] Again, we can avoid the axiom of choice by working with categories in which idempotents have *distinguished* splittings, in which case we can impose our coherence axioms directly.

2.4 Algebraic Theories

In this section we recapitulate the basic definitions surrounding the notion of algebraic theory and state a number of related results. We follow the modern approach of Adamek et al. [4]. We pay an unusual amount of attention to presentations of algebraic theories via the usual term syntax, which we will call *classical presentations* in order to contrast them with the *Cartesian monoidal presentations* of the next section.

2.4.1 Finite Products and Fox's Theorem

To begin, let us recall the notion of binary product in a category:

Definition 2.4.1. Let $\mathbb{X}$ be a category, and let A, B be objects of $\mathbb{X}$. A *binary product* of A and B in $\mathbb{X}$ is a diagram:

$$A \xleftarrow{\pi_0^{A,B}} A \times B \xrightarrow{\pi_1^{A,B}} B$$

with the property that for any arrows $f : C \to A$ and $g : C \to B$ of $\mathbb{X}$ there exists a unique arrow $\langle f, g \rangle : C \to A \times B$ such that:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_0^{A,B}"] \arrow[r, "\pi_1^{A,B}"'] & B \end{tikzcd}$$

We refer to $\pi_0^{A,B}$ and $\pi_1^{A,B}$ as *projections* and refer to $\langle f, g \rangle$ as the *pairing* of f and g. We write $\pi_0^{A,B} = \pi_0$ and $\pi_1^{A,B} = \pi_1$ when confusion is unlikely. We write $\Delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$ for the diagonal map.

For example, in the category Set of sets and functions the binary product of sets A and B is given by their Cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$ with the evident projections and pairings. Another elementary example arises in the theory of posets, viewed as skeletal categories in which the hom-sets contain at most one element. Here a binary product of x and y is precisely their *meet* $x \cap y$, with the projections giving $x \cap y \leq x$ and $x \cap y \leq y$ and pairing giving that when $z \leq x$ and $z \leq y$ we have $z \leq x \cap y$.

Closely related to the notion of binary product is that of a terminal object:

Definition 2.4.2. Let $\mathbb{X}$ be a category. A *terminal object* in $\mathbb{X}$ is an object 1 such that for each object A of $\mathbb{X}$ there exists a unique arrow $!_A : A \to 1$ in $\mathbb{X}$.

For example, in Set any singleton set $\{*\}$ is a terminal object. In a poset, a terminal object is the greatest element of the poset, with the unique arrow giving

$x \leq 1$ for all x.

Definition 2.4.3. A category $\mathbb{X}$ is said to have *finite products* in case $\mathbb{X}$ has a terminal object and $\mathbb{X}$ contains a binary product of all pairs A, B of objects in $\mathbb{X}$.

For example, Set has finite products in the manner discussed above. Posets with finite products, being those posets with a greatest element that admit all binary meets, are called *bounded meet-semilattices.*

Catgories with finite products may be characterised in terms of commutative comonoid structure:

Definition 2.4.4. Let $\mathbb{X}$ be a symmetric monoidal category. A *commutative comonoid* in $\mathbb{X}$ consists of a triple $(X, \delta_X, \varepsilon_X)$ where X is an object of $\mathbb{X}$, $\delta_X : X \to X \otimes X$, and $\varepsilon_X : X \to I$ such that:

$$\delta_X(1_X \otimes \delta_X) = \delta_X(\delta_X \otimes 1_X) \qquad \delta_X \sigma_{X,X} = \delta_X \qquad \delta_X(1_X \otimes \varepsilon_X) = 1_X$$

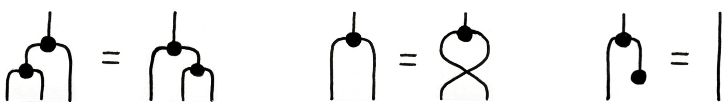
We say that X is the *carrier* of $(X, \delta_X, \varepsilon_X)$.

We adopt the following string-diagrammatic notation for such commutative comonoid structure $(X, \delta_X, \varepsilon_X)$:

$\delta_X \rightsquigarrow$ 

$\varepsilon_X \rightsquigarrow$ 

Then for example the equations of a commutative comonoid become:



The connection between commutative comonoids and categories with finite products is known as Fox's Theorem, given as follows:

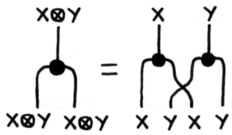
Theorem 2.4.5 ([30])**.** *A category with finite products is the same thing as a symmetric monoidal category* $(\mathbb{X}, \otimes, I)$ *such that:*

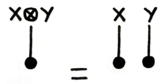
1. *Each object* X *of* $\mathbb{X}$ *is the carrier of a commutative comonoid* $(X, \delta_X, \varepsilon_X)$*.*

2. *The commutative comonoid structure is* coherent*. That is,* $(X \otimes Y, \delta_{X \otimes Y}, \varepsilon_{X \otimes Y})$ *is determined by* $(X, \delta_X, \varepsilon_X)$ *and* $(Y, \delta_Y, \varepsilon_Y)$ *in the sense that:*

$$\delta_{X \otimes Y} = (\delta_X \otimes \delta_Y)(1_X \otimes \sigma_{X,Y} \otimes 1_Y) \qquad \varepsilon_{X \otimes Y} = \varepsilon_X \otimes \varepsilon_Y$$

Diagrammatically:



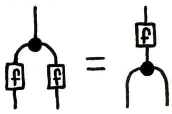


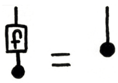
and further $\varepsilon_I = 1_I = \delta_I$.

3. *The commutative comonoid structure is* natural. *That is, for each arrow* $f : X \to Y$ *of* $\mathbb{X}$*, we have:*

$$f\delta_Y = \delta_X(f \otimes f) \qquad\qquad f\varepsilon_Y = \varepsilon_X$$

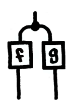
Diagrammatically:



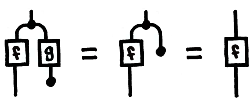


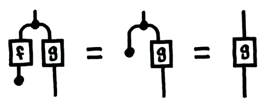
Put another way, we require every morphism $f : X \to Y$ *to be a comonoid homomorphism from* $(X, \delta_X, \varepsilon_X)$ *to* $(Y, \delta_Y, \varepsilon_Y)$.

Proof. If $\mathbb{X}$ has finite products then $(X, \Delta_X = \langle 1_X, 1_X\rangle, !_X)$ defines coherent and natural commutative comonoid structure on $\mathbb{X}$. Conversely, if $\mathbb{X}$ is a symmetric monoidal category with coherent and natural commutative comonoid structure, then $A \xleftarrow{1_A\otimes\varepsilon_B} A \otimes B \xrightarrow{\varepsilon_A\otimes 1_B} B$ is a product: for $f : C \to A$ and $g : C \to B$ the arrow $\langle f, g\rangle : C \to A \otimes B$ is given by $\delta_C(f \otimes g)$, as in:

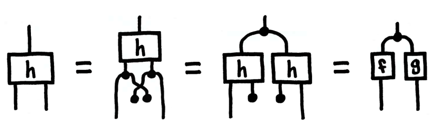


We have $\langle f, g\rangle\pi_0 = f$ and $\langle f, g\rangle\pi_1 = g$ as in:





For uniqueness, let $h : C \to A \otimes B$. Then $h = \langle f, g\rangle$ as in:



Further, I is a terminal object with $!_A = \varepsilon_A$. If $f : A \to I$ then $f = f\varepsilon_I = \varepsilon_A$, and so $!_A$ is the unique arrow $A \to I$. The claim follows. Moreover, the construction of commutative comonoid structure from finite product structure and construction of finite product structure from commutative comonoid structure are mutually inverse. $\square$

In order to emphasize this perspective we will sometimes call such categories *cartesian monoidal.* That is:

Definition 2.4.6. A *cartesian monoidal category* is a symmetric monoidal category with coherent and natural commutative comonoid structure. That is, a category with finite products.

The appropriate notion of structure-preserving functor between Cartesian monoidal categories (that is, functor that preserpves finite product structure) is as follows:

Definition 2.4.7. Let $\mathbb{X}, \mathbb{Y}$ be Cartesian monoidal categories. A *Cartesian monoidal functor* $F : \mathbb{X} \to \mathbb{Y}$ is a symmetric monoidal functor that preserves the commutative comonoid structure in the sense that $F(\delta_A) = \delta_{FA}\phi^F_{A,A}$ and $F(\varepsilon_A) = \varepsilon_{FA}\phi^F_I$.

When working with cartesian monoidal functors we will continue to omit the coherence isomorphisms, behaving as though $F(\delta_A) = \delta_{FA}$ and $F(\varepsilon_A) = \varepsilon_{FA}$.

Conveniently, natural transformations between Cartesian monoidal functors are automatically monoidal:

Lemma 2.4.8. *Let $\mathbb{X}, \mathbb{Y}$ be Cartesian monoidal categories, let $F, G : \mathbb{X} \to \mathbb{Y}$ be Cartesian monoidal functors, and let $\alpha : F \to G$ be a natural transformation. Then α is monoidal.*

Proof. Since α is natural we have:

$$\begin{tikzcd}
F(A \otimes B) \arrow[r, "\alpha_{A\otimes B}"] \arrow[d, "F(\pi_0) = \pi_0"'] & G(A \otimes B) \arrow[d, "G(\pi_0) = \pi_0"] \\
FA \arrow[r, "\alpha_A"'] & GA
\end{tikzcd}$$

Then $(\alpha_A \otimes \alpha_B)\pi_0 = \langle \pi_0\alpha_A, \pi_1\alpha_B\rangle\pi_0 = \pi_0\alpha_A = \alpha_{A\otimes B}\pi_0$. Similarly we have $(\alpha_A \otimes \alpha_B)\pi_1 = \alpha_{A\otimes B}\pi_1$, and so we have $\alpha_A \otimes \alpha_B = \alpha_{A\otimes B}$ by the universal property of binary products.

Next, we have $\alpha_I = !_I = 1_I$ because $!_I : I \to I$ is the unique arrow of $I \to I$. Thus, α is a monoidal natural transformation. $\square$

We assemble the above notions into a 2-category:

Definition 2.4.9. CM is the 2-category with small Cartesian monoidal categories as 0-cells, Cartesian monoidal functors as 1-cells, and (necessarily monoidal) natural transformations as 2-cells.

2.4.2 Algebraic Theories and Classical Presentations

We begin with the definition of algebraic theory:

Definition 2.4.10. An *Algebraic Theory* is a small cartesian monoidal category.

Models of a theory are given by structure-preserving functors:

Definition 2.4.11. A *model* of an algebraic theory $\mathbb{X}$ is a cartesian monoidal functor $\mathbb{X} \to \mathsf{Set}$.

And model morphisms correspond to natural transformations:

Definition 2.4.12. Let $\mathbb{X}$ be an algebraic theory and let $F, G : \mathbb{X} \to \mathsf{Set}$ be models of $\mathbb{X}$. Then a *model morphism* $\alpha : F \to G$ is a natural transformation.

While it has many advantages, this notion of algebraic theory is rather more abstract than the one typically encountered. The systems of generators and equations that one usually uses to present a theory are what we call *classical presentations*. We proceed to develop this notion now. Following this, we will give an alternative method of presenting algebraic theories in terms of string diagrams, building on the contents of Section 2.1. We begin with the appropriate notion of signature:

Definition 2.4.13. A *classical signature* Σ consists of a set $\mathfrak{s}(\Sigma)$ of *sorts*, a set $|\Sigma|$ of *generators*, and functions $\delta_0 : |\Sigma| \to \mathfrak{s}(\Sigma)^*$ and $\delta_1 : |\Sigma| \to \mathfrak{s}(\Sigma)$. For $\gamma \in |\Sigma|$ we call $\delta_0(\gamma)$ the *arity* of γ and call $\delta_1(\gamma)$ the *coarity* of γ. Notice that in a classical signature the coarity is an element of $\mathfrak{s}(\Sigma)$, not of $\mathfrak{s}(\Sigma)^*$. We write $\gamma : X \to Y \in \Sigma$ to mean that $\gamma \in |\Sigma|$ with $\delta_0(\gamma) = X$ and $\delta_1(\gamma) = Y$.

The classical notion of term over a signature will be familiar from universal algebra:

Definition 2.4.14. Let Σ be a classical signature. The category $\mathsf{T}(\Sigma)$ of *classical terms over* Σ is given as follows:

objects are elements of $\mathfrak{s}(\Sigma)^*$.

arrows are tuples of terms. *Terms* are given as follows:

$$\frac{A \in \mathfrak{s}(\Sigma) \qquad n \in \mathbb{N}}{x_n^A : A}$$

$$\frac{\gamma \in |\Sigma| \qquad \delta_0(\gamma) = A_1 \otimes \cdots \otimes A_n \qquad \delta_1(\gamma) = A \qquad t_1 : A_1, \ldots, t_n : A_n}{\gamma(t_1, \ldots, t_n) : A}$$

Arrows $\langle f_1, \ldots, f_m \rangle : A_1 \otimes \cdots \otimes A_n \to B_1 \otimes \cdots \otimes B_m$ are m-tuples of terms in variables $x_1^{A_1}, \ldots, x_n^{A_n}$. We write $x_n = x_n^A$ when the typing is clear from context.

composition is given by substitution. Given $\langle f_1, \ldots, f_h \rangle : A_1 \otimes \cdots \otimes A_n \to C_1 \otimes \cdots \otimes C_h$ and $\langle g_1, \ldots, g_m \rangle : C_1 \otimes \cdots \otimes C_h \to B_1 \otimes \cdots \otimes B_m$ where $f_1, \ldots f_h$ are defined in variables $x_1^{A_1}, \ldots, x_n^{A_n}$ and $g_1, \ldots, g_m$ are defined in variables $x_1^{C_1}, \ldots, x_h^{C_h}$, we define the composite as follows:

$$\langle f_1, \ldots, f_h \rangle \langle g_1, \ldots, g_m \rangle \triangleq \langle g_1[^{f_1, \ldots, f_h}/_{x_1, \ldots, x_h}], \ldots, g_m[^{f_1, \ldots, f_h}/_{x_1, \ldots, x_h}] \rangle$$

the **identity** on $A_1 \otimes \cdots \otimes A_n$ is given by:

$$\langle x_1^{A_1}, \ldots, x_n^{A_n} \rangle : A_1 \otimes \cdots \otimes A_n \to A_1 \otimes \cdots \otimes A_n$$

It is straightforward to verify that $\mathsf{T}(\Sigma)$ does indeed form a small category. Perhaps the most important property of $\mathsf{T}(\Sigma)$ is that it has finite products:

Proposition 2.4.15. *Let Σ be a classical signature. Then $\mathsf{T}(\Sigma)$ has finite products.*

Proof. We show that $\mathsf{T}(\Sigma)$ has binary products and a terminal object. The terminal object is I, the unit of the object monoid $\mathfrak{s}(\Sigma)^*$, where the unique arrow $!_A = \langle\rangle : A \to I$ is given by the empty tuple for all objects A of $\mathsf{T}(\Sigma)$. For any $A = A_1 \otimes \cdots \otimes A_n$ and $B = B_1 \otimes \cdots \otimes B_m$ with $A_1, \ldots, B_m \in \mathfrak{s}(\Sigma)$ there is a product diagram:

$$A \xleftarrow{\pi_1 = \langle x_1^{A_1}, \ldots, x_n^{A_n} \rangle} A \otimes B \xrightarrow{\pi_2 = \langle x_{n+1}^{B_1}, \ldots, x_{n+m}^{B_m} \rangle} B$$

Given $C = C_1 \otimes \cdots \otimes C_h$ and arrows $f = \langle f_1, \ldots, f_n \rangle : C \to A$ and $g = \langle g_1, \ldots, g_m \rangle$ of $\mathsf{T}(\Sigma)$ we define $\langle f, g \rangle = \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle : C \to A \otimes B$. Then we have:

$$\begin{aligned}
&\langle f, g \rangle \pi_1 = \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle \langle x_1^{A_1}, \ldots, x_n^{A_n} \rangle \\
&= \langle x_1[^{f_1, \ldots, f_n, g_1, \ldots, g_m}/_{x_1, \ldots, x_{n+m}}], \ldots, x_n[^{f_1, \ldots, f_n, g_1, \ldots, g_m}/_{x_1, \ldots, x_{n+m}}] \rangle \\
&= \langle f_1, \ldots, f_n \rangle = f
\end{aligned}$$

and similarly $\langle f, g \rangle \pi_2 = g$. For any $h = \langle h_1, \ldots, h_{n+m} \rangle : C \to A \otimes B$ satisfying $h\pi_1 = f$ and $h\pi_2 = g$ we have:

$$\begin{aligned}
&\langle f_1, \ldots, f_n \rangle = f = h\pi_1 = \langle h_1, \ldots, h_{n+m} \rangle \langle x_1, \ldots, x_n \rangle \\
&= \langle x_1[^{h_1, \ldots h_{n+m}}/_{x_1, \ldots x_{n+m}}], \ldots x_n[^{h_1, \ldots h_{n+m}}/_{x_1, \ldots x_{n+m}}] \rangle \\
&= \langle h_1, \ldots, h_n \rangle
\end{aligned}$$

and similarly we have $\langle g_1, \ldots, g_m \rangle = \langle h_{n+1}, \ldots, h_{n+m} \rangle$. It follows that

$$\langle f, g \rangle = \langle f_1, \ldots, f_n, g_1, \ldots, g_m \rangle = \langle h_1, \ldots, h_{n+m} \rangle = h$$

as required. We conclude that $\mathsf{T}(\Sigma)$ has finite products. $\square$

An equation is again a pair of terms:

Definition 2.4.16. Let Σ be a signature. Then a *classical equation* over Σ is a pair (f, g) where $f, g : A \to B$ in $\mathsf{T}(\Sigma)$. We often write $f = g$ instead of (f, g).

And a presentation is again given by a signature together with equations:

Definition 2.4.17. A *classical presentation* (Σ, E) consists of a classical signature Σ together with a set E of classical equations over Σ.

Finally, a presentation is again taken to present the category of terms over the associated signature, modulo the associated equations:

Definition 2.4.18. Let (Σ, E) be a classical presentation. Write $\mathsf{T}(\Sigma, E)$ for the category with finite products obtained by quotienting $\mathsf{T}(\Sigma)$ by the equations of E. We call $\mathsf{T}(\Sigma, E)$ the *algebraic theory presented by* (Σ, E), and similarly we say that (Σ, E) *presents* $\mathsf{T}(\Sigma, E)$.

Example 2.4.19. Consider the classical presentation $(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ with a single sort $\mathfrak{s}(\Sigma_{\mathsf{Mon}}) = \{X\}$, with generators $|\Sigma_{\mathsf{Mon}}| = \{m : X \otimes X \to X, e : I \to X\}$, and with equations $E_{\mathsf{CMon}} = \{m(x_1, m(x_2, x_3)) = m(m(x_1, x_2), x_3), m(x_1, x_2) = m(x_2, x_1), m(x_1, e()) = m(x_1)\}$. Now, consider the algebraic theory $\mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ presented by $(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$.

Recall that a model of $\mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ is precisely a functor $F : \mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}}) \to \mathsf{Set}$ that preserves finite products. Observe that since F preserves finite products the object mapping is determined by a set FX and the arrow mapping is determined by functions $F(m) : FX \times FX \to FX$ and $F(e) : 1 \to FX$. For F to be well-defined we must also have that terms made equal by E_{CMon} are equal in the image of F. That is, a model of $\mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ is precisely a commutative monoid $(FX, F(m), F(e))$.

Recall further that for $F, G : \mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}}) \to \mathsf{Set}$ a model morphism $\alpha : F \to G$ is a natural transformation. Now $\mathfrak{s}(\Sigma_{\mathsf{Mon}}) = \{X\}$ and since F, G are Cartesian, we know that α is necessarily monoidal. It follows that any $\alpha : F \to G$ is determined by the component $\alpha_X : FX \to GX$. Thus, to give a model morphism $\alpha : F \to G$ is precisely to give a function $\alpha_X : FX \to GX$ satisfying

$$
\begin{tikzcd}
FX \times FX \arrow[r, "\alpha_X \times \alpha_X"] \arrow[d, "F(m)"'] & GX \times GX \arrow[d, "G(m)"] \\
FX \arrow[r, "\alpha_X"'] & GX
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "1_1"] \arrow[d, "F(e)"'] & 1 \arrow[d, "G(e)"] \\
FX \arrow[r, "\alpha_X"'] & GX
\end{tikzcd}
$$

This is precisely to say that $\alpha_X : (FX, F(m), F(e)) \to (GX, G(m), G(e))$ is a monoid homomorphism. Thus, the category of models and model morphisms of $\mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ is the category of commutative monoids and monoid homomorphisms. That is, $(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ presents the theory of commutative monoids.

2.4.3 Cartesian Monoidal Presentations

Theorem 2.4.5 gives us a way to present algebraic theories (i.e., Cartesian monoidal categories) string-diagrammatically, by adapting the approach of Section 2.1. The appropriate notion of signature is again a monoidal signature, and for terms we define:

Definition 2.4.20. Let Γ be a monoidal signature. The small Cartesian monoidal category $\mathsf{C}(\Gamma)$ of *Cartesian monoidal terms over* Γ is constructed the same way $\mathsf{S}(\Gamma)$ is, but with additional generating arrows:

$$\frac{A \in \mathfrak{s}(\Gamma)^*}{\delta_A : A \to A \otimes A} \qquad\qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\varepsilon_A : A \to I}$$

Additional equations concerning coherence:

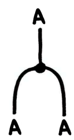
$$\delta_I = \square \qquad \varepsilon_I = \square \qquad \delta_{A\otimes B} = (\delta_A \otimes \delta_B)(1_A \otimes \sigma_{A,B} \otimes 1_B) \qquad \varepsilon_{A\otimes B} = \varepsilon_A \otimes \varepsilon_B$$

And a few remaining additional equations:

$$\delta_A(\delta_A \otimes 1_A) = \delta_A(1_A \otimes \delta_A) \qquad \delta_A(1_A \otimes \varepsilon_A) = 1_A \qquad \delta_A \sigma_{A,A} = \delta_A$$

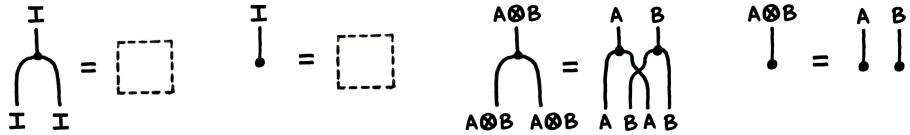
$$f\delta_B = \delta_A(f \otimes f) \qquad\qquad f\varepsilon_B = \varepsilon_A$$

It is convenient to specify cartesian monoidal presentations using string diagrams. We recall the diagrammatic convention for cartesian monoidal categories:

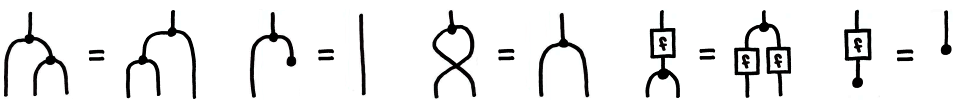
$\delta_A \quad \leftrightsquigarrow$ 

$\varepsilon_A \quad \leftrightsquigarrow$ 

Then the equations concerning coherence become:



and the remaining additional equations become:



Equations are again pairs of terms:

Definition 2.4.21. Let Γ be a monoidal signature. A *Cartesian monoidal equation over* Γ consists of a pair (f, g) where $f, g : A \to B \in \mathsf{C}(\Gamma)$ are Cartesian monoidal terms over Γ. We typically write $f = g$ instead of (f, g).

And a presentation is again a signature together with equations:

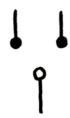
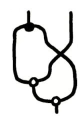
Definition 2.4.22. A *Cartesian monoidal presentation* (Γ, E) consists of a monoidal signature Γ together with a set E of Cartesian monoidal equations over Γ.

As before, a presentation is taken to present the category of terms over the associated signature, modulo the associated equations:

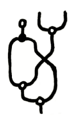
Definition 2.4.23. Let (Γ, E) be a Cartesian monoidal presentation. Write $\mathsf{C}(\Gamma, E)$ for the small Cartesian monoidal category obtained by quotienting $\mathsf{C}(\Gamma)$ by the equations of E. We say that $\mathsf{C}(\Gamma, E)$ is the *algebraic theory presented by* (Γ, E).

Example 2.4.24. Recall the symmetric monoidal presentation $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$, and notice that it is also a Cartesian monoidal presentation via the evident inclusion $\mathsf{S}(\Gamma) \to \mathsf{C}(\Gamma)$ for any monoidal signature Γ. Then $\mathsf{C}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}}) \cong \mathsf{T}(\Sigma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ of Example 2.4.19. That is, $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{CMon}})$ presents the algebraic theory of commutative monoids.

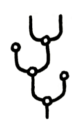
Classical and Cartesian monoidal presentations of algebraic theories are equivalently expressive. Intuitively, the input (top) wires of a term (string diagram) over a monoidal signature correspond to variables, and each output wire describes a term in those variables. For example, terms in the classical presentation of the theory of monoids and the Cartesian monoidal presentation correspond to the following terms in variables x_1 and x_2:

$m(x_1, x_2) \rightsquigarrow$  $e \rightsquigarrow$  $m(m(x_1, x_2), x_1) \rightsquigarrow$ 

The variables of the term in question correspond to the input wires, and the term itself to the output wire. Substitution of terms for variables is replaced by composition. For example:

$m(m(x_1, x_2), x_1)[{}^{e, m(x_1, x_2)}/x_1, x_2] \rightsquigarrow$ 

where the result of performing the substitution is equivalently the result of diagrammatic manipulation:

$m(m(e, m(x_1, x_2)), e) \rightsquigarrow$ 

A difference between Cartesian monoidal and classical presentations is that in the former terms may have arbitrary co-arity, while in the latter the co-arity must be singleton. This difference is a superficial one, since the Cartesian structure allows us to represent generators $\gamma : X \to B_0 \otimes B_1$ as $\gamma_0 = \gamma\pi_0 : X \to B_0$ and $\gamma_1 = \gamma\pi_1 : X \to B_1$, with γ being constructible from γ_0 and γ_1 as in $\gamma = \langle \gamma_0, \gamma_1 \rangle$.

2.4.4 Varieties and Morita Equivalence

In this section we recall the *variety theorem* for algebraic theories. The variety theorem concerns categories which arise as the models and model morphisms of some algebraic theory $\mathbb{X}$, which are called *varieties*. The theorem takes the form of a biadjunction between the 2-category CM of algebraic theories and a 2-category $\mathsf{Var}^{\mathsf{op}}$ of varieties. One way to think of algebraic theories is as a kind of syntax, with the associated varieties being the corresponding semantics. Because of this the biadjunction of the variety theorem is sometimes called a *syntax-semantics duality*. We also recall a related characterisation of *Morita equivalence* for algebraic theories, which is what it is called when two theories present the same variety. Specifically, two algebraic theories present the same variety (i.e., are Morita equivalent) if and only if they have equivalent idempotent splitting completions.

We begin by specialising Lemma 2.2.8 to the cartesian monoidal case:

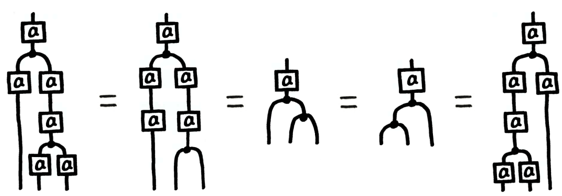
Lemma 2.4.25. *Let $\mathbb{X}$ be a Cartesian monoidal category. Then*

(i) $\mathsf{Split}(\mathbb{X})$ *is a Cartesian monoidal category.*

(ii) The embedding $[\![-]\!] : \mathbb{X} \hookrightarrow \mathsf{Split}(\mathbb{X})$ *preserves the Cartesian monoidal structure.*

Proof. (i) We know from Lemma 2.2.8 that $\mathsf{Split}(\mathbb{X})$ is symmetric monoidal. The comultiplication $\delta_{(X,a)}$ and counit $\varepsilon_{(X,a)}$ of the Cartesian monoidal structure in $\mathsf{Split}(\mathbb{X})$ are, respectively:

and it is straightforward to verify that this satisfies the axioms of a Cartesian monoidal category. For example, we have that $\delta_{(X,a)}$ is coassociative as follows:



(ii) Immediate.

□

The 2-functor of Lemma 2.3.8 specialises as well:

Lemma 2.4.26. *Let* CM_s *be the full sub 2-category of* CM *on the 0-cells* $\mathbb{X}$ *in which the idempotents split. Then splitting idempotents defines a 2-functor* $\mathsf{Split} : \mathsf{CM} \to \mathsf{CM}_s$*.*

Proof. As in Lemma 2.3.5, on 1-cells $F : \mathbb{X} \to \mathbb{Y}$ define $\mathsf{Split}(F) : \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{Y})$ on objects by $\mathsf{Split}(F)(X, a) = (FX, F(a))$ and on arrows by $\mathsf{Split}(F)(f) = F(f)$. On 2-cells define $\mathsf{Split}(\alpha)_{(X,a)} = F(a)\alpha_X G(a)$. The proof that this defines a 2-functor $\mathsf{Split} : \mathsf{CM} \to \mathsf{CM}_s$ is largely the same as the proof of Lemma 2.3.8. We need only show that if $F : \mathbb{X} \to \mathbb{Y}$ is a cartesian monoidal functor then $\mathsf{Split}(F) : \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{Y})$ is also cartesian monoidal. We have $\mathsf{Split}(F)(\delta_{(X,a)})$ $= \mathsf{Split}(F)(a\delta_X(a \otimes a)) = F(a)F(\delta_X)(F(a) \otimes F(a)) = F(a)\delta_{FX}(F(a) \otimes F(a)) = \delta_{(FX,F(a))} = \delta_{\mathsf{Split}(F)(X,a)}$ and $\mathsf{Split}(F)(\varepsilon_{(X,a)}) = \mathsf{Split}(F)(a\varepsilon_X) = F(a)F(\varepsilon_X) = F(a)\varepsilon_{FX} = \varepsilon_{(FX,F(a))} = \varepsilon_{\mathsf{Split}(F)(X,a)}$, as required. □

And finally the biadjunction from Lemma 2.3.9 also specialises to the cartesian monoidal case:

Lemma 2.4.27. *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{CM} \arrow[r, shift left, "\mathsf{Split}"] \arrow[r, phantom, "\perp"] & \mathsf{CM}_s \arrow[l, shift left, hook] \end{tikzcd}$$

where the right adjoint $\mathsf{CM}_s \hookrightarrow \mathsf{CM}$ *is the evident inclusion.*

Proof. The proof is similar to the proof of Lemma 2.3.9. As before, we define $(-)^\sharp : \mathsf{CM}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \to \mathsf{CM}(\mathbb{X}, \mathbb{C})$ which is natural in $\mathbb{X}$ and $\mathbb{C}$, and show that it is full, faithful, and essentially surjective.

The proof that $(-)^\sharp$ is essentially surjective is almost identical to the corresponding part of Lemma 2.3.9. The only difference is that now we must also show that our $\widehat{F}$ is always a cartesian monoidal functor. We have already seen that it is symmetric monoidal. For the cartesian monoidal structure, we have $\widehat{F}(\delta_{(X,a)}) = s_a F(a)F(\delta_X)(F(a) \otimes F(a))(r_a \otimes r_a) = s_a \delta_{FX}(r_a \otimes r_a) = s_a r_a \delta_{\widehat{F}(X,a)} = \delta_{\widehat{F}(X,a)}$ and $\widehat{F}(\varepsilon_{(X,a)}) = s_a F(a)F(\varepsilon_X) = s_a \varepsilon_{FX} = \varepsilon_{\widehat{F}(X,a)}$. It now follows by corresponding argument from Lemma 2.3.9 that $(-)^\sharp$ is essentially surjective. The argument that $(-)^\sharp$ is full and faithful in Lemma 2.3.9 gives that it is full and faithful here as well. □

We will compose this biadjunction with a biequivalence due to Adamek, Rosický and Lawvere [2] in order to obtain the variety theorem. First, let Var be the 2-

category with 0-cells the varieties[4] $\mathsf{CM}(\mathbb{X}, \mathsf{Set})$ for some algebraic theory $\mathbb{X}$, with 1-cells given by functors that both admit a left adjoint and commute with sifted colimits (see e.g., [4]), and with 2-cells given by natural transformations. The relationship between algebraic theories and varieties is as follows:

Theorem 2.4.28 ([2])**.** *There is a biequivalence:*

$$\begin{tikzcd} \mathsf{CM}_s \arrow[r, shift left, "\mathsf{CM}(-{,}\mathsf{Set})"] \arrow[r, phantom, "\simeq"] & \mathsf{Var}^{\mathsf{op}} \arrow[l, shift left, "\mathsf{Var}(-{,}\mathsf{Set})"] \end{tikzcd}$$

where CM_s *is the 2-category of small Cartesian monoidal categories in which all idempotents split, Cartesian monoidal functors, and natural transformations.*

Now we compose our idempotent splitting biadjunction with the above biequivalence to obtain a biadjunction relating arbitrary algebraic theories to the associated variety:

Theorem 2.4.29 ([2])**.** *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{CM} \arrow[r, shift left, "\mathsf{Mod}"] \arrow[r, phantom, "\perp"] & \mathsf{Var}^{\mathsf{op}} \arrow[l, shift left, "\mathsf{Th}"] \end{tikzcd}$$

Proof. Combining Lemma 2.4.27 and Theorem 2.4.28 gives:

$$\begin{tikzcd} \mathsf{CM} \arrow[r, shift left, "\mathsf{Split}"] \arrow[r, phantom, "\perp"] & \mathsf{CM}_s \arrow[l, shift left, hook] \arrow[r, shift left, "\mathsf{CM}(-{,}\mathsf{Set})"] \arrow[r, phantom, "\simeq"] & \mathsf{Var}^{\mathsf{op}} \arrow[l, shift left, "\mathsf{Var}(-{,}\mathsf{Set})"] \end{tikzcd}$$

□

Considered alone the biequivalence $\mathsf{CM}_s \simeq \mathsf{Var}^{\mathsf{op}}$ gives:

Theorem 2.4.30 ([2])**.** *Two algebraic theories* $\mathbb{X}$ *and* $\mathbb{Y}$ *present equivalent varieties* $\mathsf{CM}(\mathbb{X}, \mathsf{Set}) \simeq \mathsf{CM}(\mathbb{Y}, \mathsf{Set})$ *if and only if* $\mathsf{Split}(\mathbb{X}) \simeq \mathsf{Split}(\mathbb{Y})$.

That is, two algebraic theories are Morita equivalent if and only if splitting the idempotents yields equivalent categories.

[4] Strictly speaking it does not make sense to write $\mathsf{CM}(\mathbb{X}, \mathsf{Set})$, since the 0-cells of CM are the *small* cartesian monoidal categories and Set is not small. Nonetheless, there is a category of cartesian monoidal functors $\mathbb{X} \to \mathsf{Set}$ and natural transformations between them, which we denote $\mathsf{CM}(\mathbb{X}, \mathsf{Set})$.

Chapter 3

Partial Algebraic Theories

The aim of this chapter is to develop a notion of *partial* algebraic theory analogous to the algebraic theories treated in the previous section, but with the important difference that the operations are interpreted as *partial functions*, as opposed to total functions. More precisely, while models of algebraic theories are valued in the category Set of sets and (total) functions, models of partial algebraic theories will be valued in the category Par of sets and partial functions. To begin, recall:

Definition 3.0.1. The category Par of *sets and partial functions* is given as follows:

objects are sets.

arrows $f : X \to Y$ are partial functions. That is, pairs $(\mathsf{dom}(f), \mathsf{def}(f))$ where $\mathsf{dom}(f) \subseteq X$ is the *domain of definition* of f and $\mathsf{def}(f) : \mathsf{dom}(f) \to Y$ is a (total) function. Given a partial function $f : X \to Y$ and some $X' \subseteq X$ we write $f|_{X'}$ for the partial function $(\mathsf{dom}(f) \cap X',\, f')$ where $f' : \mathsf{dom}(f) \cap X' \to Y$ is $\mathsf{def}(f)$ restricted to the (potentially smaller) domain of definition $\mathsf{dom}(f) \cap X'$. Similarly, given $Y' \subseteq Y$, write $f^{-1}(Y') = \{x \in \mathsf{dom}(f) \mid \mathsf{def}(f)(x) \in Y'\}$.

composition of $f : X \to Y$ and $g : Y \to Z$ is given by

$$fg = (f^{-1}(\mathsf{dom}(g)),\, (\mathsf{def}(f)|_{f^{-1}(\mathsf{dom}(g))}\, \mathsf{def}(g))$$

The **identity** on X is $(X, 1_X)$.

It is straightforward to verify that this data makes Par a category. Further, there is a natural partial order on the hom-sets $\mathsf{Par}(X, Y)$ given by:

$$f \leq g \quad \Leftrightarrow \quad \mathsf{dom}(f) \subseteq \mathsf{dom}(g) \;\wedge\; g|_{\mathsf{dom}(f)} = f.$$

and in fact Par is enriched in the category of posets and monotone functions.

Algebraic theories correspond to Cartesian monoidal categories, which describe the behavior of the Cartesian product of sets in Set. Analogously, partial algebraic theories will describe the behavior of the Cartesian product of sets in Par. We proceed to develop the categorical structure that will play this role.

3.1 Cartesian Restriction Categories

Restriction categories are abstract categories of partial maps. Instead of working with sets and functions, we will impose axioms on a category to the effect that it admits a notion of partiality. Let us begin with a brief introduction to the theory of restriction categories. More details can be found in [21, 22, 24, 17]. The immediate goal is the notion of a *Cartesian restriction category* (CR category), which captures part of the behavior of the Cartesian product of sets in the category Par. We begin as follows:

Definition 3.1.1 ([21, 2.1.1])**.** A *restriction category* is a category in which every arrow $f : A \to B$ admits a *domain of definition* $\overline{f} : A \to A$ satisfying:

[R.1] $\overline{f}f = f$

[R.2] $\overline{f}\,\overline{g} = \overline{g}\overline{f}$ where $g : A \to C$

[R.3] $\overline{\overline{f}g} = \overline{f}\overline{g}$ where $g : A \to C$

[R.4] $f\overline{g} = \overline{fg}f$ where $g : B \to C$

Intuitively, $\overline{f}$ is a sort of partial identity map which represents the domain of definition of f, in that $\overline{f}$ is defined precisely when f is defined, in which case it acts as the identity. Arrows of the form $\overline{f}$ are called *restriction idempotents*. Arrows $f : A \to B$ with $\overline{f} = 1_A$ are called *total*, and form a subcategory $\mathsf{Total}(\mathbb{C})$.

Example 3.1.2. Par is a restriction category where for $f = (\mathsf{dom}(f), \mathsf{def}(f)) : X \to Y$ we define $\overline{f} = (\mathsf{dom}(f), \mathsf{dom}(f)) : X \to X$. The subcategory $\mathsf{Total}(\mathsf{Par})$ of total maps in Par is precisely Set. Here we have written $\mathsf{dom}(f) : \mathsf{dom}(f) \rightarrowtail X$ to indicate the subobject witnessing the inclusion of sets $\mathsf{dom}(f) \subseteq X$.

We recapitulate a few elementary properties of restriction idempotents:

Lemma 3.1.3 ([21, 2.1])**.** *Let $\mathbb{X}$ be a restriction category. Then:*

(i) $\overline{f}\,\overline{f} = \overline{f}$ *where* $f : A \to B$.

(ii) $\overline{fg}\,\overline{f} = \overline{fg}$ *where* $f : A \to B$, $g : B \to C$.

(iii) $\overline{f\overline{g}} = \overline{fg}$ *where* $f : A \to B, g : B \to C$.

(iv) $\overline{\overline{f}} = \overline{f}$ *where* $f : A \to B$.

(v) $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}$ *where* $f : A \to B$, $g : A \to C$.

(vi) If $f : A \to B$ *is monic, then* $\overline{f} = 1_A$ *and so* f *is total.*

Proof. (i) By [**R.3**] and [**R.1**] we have $\overline{f}\,\overline{f} = \overline{\overline{f}f} = \overline{f}$.

(ii) By [**R.2**], [**R.3**], and [**R.1**] we have $\overline{fg}\,\overline{f} = \overline{f}\,\overline{fg} = \overline{\overline{f}fg} = \overline{fg}$.

(iii) By [**R.4**], [**R.3**], and (ii) we have $\overline{f\overline{g}} = \overline{\overline{fg}f} = \overline{fg}\,\overline{f} = \overline{fg}$.

(iv) Using (iii) we have $\overline{f} = \overline{1f} = \overline{1\overline{f}} = \overline{\overline{f}}$.

(v) Using [**R.3**] and (iv) we have $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{\overline{g}} = \overline{f}\,\overline{g}$.

(vi) If f is monic then $1f = f = \overline{f}f$ which gives $\overline{f} = 1$.

□

An analogue of the poset-enrichment of Par is present in any restriction category:

Lemma 3.1.4 ([21, 2.1.4])**.** *Any restriction category is enriched in the category of posets and monotone functions, with the ordering on hom-sets given by:*

$$f \leq g \Leftrightarrow \overline{f}g = f$$

Proof. The ordering is reflexive in case $\overline{f}f = f$ for all f, which follows from [**R.1**]. For transitivity, if $\overline{f}g = f$ and $\overline{g}h = g$ then we have $\overline{f}h = \overline{\overline{f}g}h = \overline{f}\,\overline{g}h = \overline{f}g = f$ as required. For antisymmetry, if $\overline{f}g = f$ and $\overline{g}f = g$ then $f = \overline{f}g = \overline{f}\,\overline{g}f = \overline{g}\,\overline{f}f = \overline{g}f = g$ as required. It follows that in a restriction category $\mathbb{X}$ each hom-set $\mathbb{X}(A, B)$ is a poset. It remains to show that composition is monotone. If $\overline{f}g = f$ and $\overline{h}k = h$ then we have $\overline{fh}gk = \overline{f\overline{h}}gk = \overline{\overline{fh}\,\overline{f}}gk = \overline{fh}\,\overline{f}gk = \overline{fh}fk = f\overline{h}k = fg$, and the claim follows. □

There is an evident notion of structure-preserving functor between restriction categories:

Definition 3.1.5 ([21, 2.2.1])**.** Let $\mathbb{X}, \mathbb{Y}$ be restriction categories. A *restriction functor* $F : \mathbb{X} \to \mathbb{Y}$ is a functor that preserves domains of definition, in the sense that $F\overline{f} = \overline{Ff}$.

Restriction categories, restriction functors (which are automatically poset-enriched), and lax natural transformations (Definition 3.1.15) form a 2-category RCat. Any category is a restriction category with $\overline{f} = 1_X$ for all f, which we call the *trivial* restriction structure on that category. The usual notion of limit diagram has unfortunate consequences when considered in a restriction category. For example in a restriction category with finite products where the projection maps are total, we have:

$$\overline{f} = \overline{\langle f, 1\rangle \pi_0} = \overline{\langle f, 1\rangle \overline{\pi_0}} = \overline{\langle f, 1\rangle \overline{\pi_1}} = \overline{\langle f, 1\rangle \pi_1} = \overline{1} = 1$$

meaning that the restriction structure is the trivial one. Instead, one works with *restriction limits*, which are formal limits in RCat [24]. Following Cockett and Hofstra [19], an explicit definition of the resulting notion of terminal object is:

Definition 3.1.6 ([19, 2.2])**.** A *restriction terminal object* in a restriction category is an object 1 such that for any object A there is a unique total map $!_A : A \to 1$ and, moreover, for any $f : A \to B$ the inequation $f!_B \leq !_A$ holds.

and an explicit definition of the resulting notion of binary product is:

Definition 3.1.7 ([19, 2.2])**.** In a restriction category, a diagram of the form

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

in which π_0 and π_1 are total is called a *restriction product* in case for any two arrows $f : C \to A$ and $g : C \to B$, there is a unique arrow $\langle f, g\rangle : C \to A \times B$ such that $\langle f, g\rangle \pi_0 = \overline{g} f$ and $\langle f, g\rangle \pi_1 = \overline{f} g$.

A consequence of this definition is that $\langle f, g\rangle \pi_0 \leq f$ and $\langle f, g\rangle \pi_1 \leq g$. One obtains a lax version of the usual diagram characterizing products:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
A \arrow[r, phantom, "\geq"] & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] \arrow[r, phantom, "\leq"] & B
\end{tikzcd}$$

Categories with finite products are sometimes called "Cartesian categories". This motivates the following terminology for categories with finite restriction limits:

Definition 3.1.8 ([19, 2.2])**.** A restriction category with a restriction terminal object and a restriction product for every pair of objects therein in called a *Cartesian restriction category* (*CR category*).

Example 3.1.9. Par is a CR category. Any singleton set is a restriction terminal object, and the restriction product is given by the Cartesian product of sets.

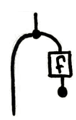
An important difference between Cartesian monoidal categories and CR categories is that the same symmetric monoidal category may be a CR category in more than one way, but may be a Cartesian monoidal category in at most one way. For example, Par is a restriction category both with the trivial restriction structure and the one discussed in Example 3.1.2. However, once a restriction structure is chosen on a category the resulting restriction category can be a CR category in at most one way [21]. A potential point of confusion is that while Par does have binary products, they do not correspond to the Cartesian product of sets. The categorical product of A and B in Par is $(A+\{\star\})\times(B+\{\star\})-\{(\star,\star)\}$. This can be seen via the equivalence $1/\mathsf{Set}\simeq\mathsf{Par}$. Limits in the coslice category $1/\mathsf{Set}$ are calculated pointwise, and the functor $1/\mathsf{Set}\to\mathsf{Par}$ removes the point. For more on this see [23].

CR categories have appeared in the literature under a variety of different names, including *p-category with a one-element object* [46] and *partially Cartesian category* [25]. Our development rests on the fact that CR categories admit a monoidal presentation analogous to the presentation of categories with finite products given by Theorem 2.4.5. Specifically, we have:

Theorem 3.1.10 ([24, Theorem 5.2]). *A CR category is the same thing as a symmetric monoidal category where every object is equipped with a commutative comonoid structure that is coherent and has natural comultiplication. That is, for any $f : A \to B$ we have $f\delta_B = \delta_A(f\otimes f)$.*

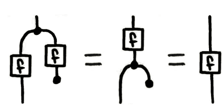
Proof. If $\mathbb{X}$ is a Cartesian restriction category, then it is straightforward to show that $(A, \Delta_A = \langle 1_A, 1_A\rangle, !_A)$ is a cocommutative comonoid, and that the resulting structure is coherent. Naturality holds by $f\Delta = f\langle 1,1\rangle = \langle f,f\rangle = \Delta(f\times f)$.

For the converse, suppose $\mathbb{C}$ is a symmetric monoidal category equipped with a coherent commutative comonoid structure in which the comultiplication is natural. We must show that $\mathbb{C}$ is a Cartesian restriction category. Define $\overline{f}$ as follows:

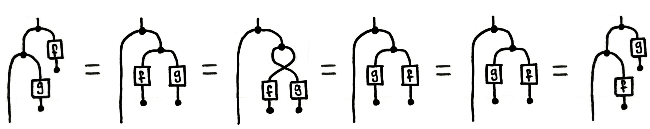


This satisfies the restriction axioms as follows:

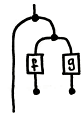
We have $\overline{f}f = f$ as in:



We have $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ as in:

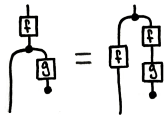


For $\overline{\overline{f}g} = \overline{f}\,\overline{g}$ notice that $\overline{\overline{f}g}$ is:

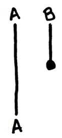


which is equal to $\overline{f}\,\overline{g}$ as in 2.

We have $f\overline{g} = \overline{fg}f$ as in:



So $\mathbb{C}$ is a restriction category. Next, define the restriction product of A and B to be $A \otimes B$ with projection maps:



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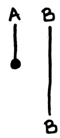
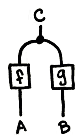
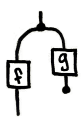


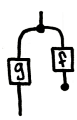
Now, for any $f : C \to A$ and $g : C \to B$, define the pairing map $\langle f, g \rangle$ by:



Then we have $\langle f, g \rangle \pi_0 = \overline{g}f$ and $\langle f, g \rangle \pi_1 = \overline{f}g$ immediately, each denoting the same string diagrams as in:

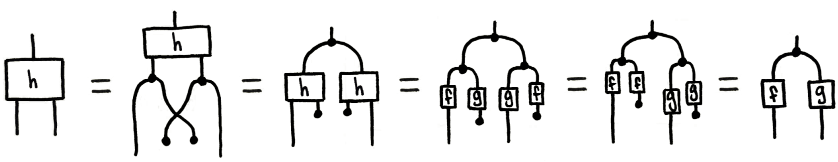


and

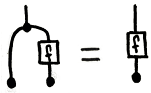


as required.

In order to show that the pairing operation is unique, suppose that $h : C \to A \otimes B$ is such that $h\pi_0 = \overline{g}f$ and $h\pi_1 = \overline{f}g$. Then we have $h = \langle f, g \rangle$ as follows:



We have shown that $\mathbb{C}$ has restriction products. The restriction terminal object of $\mathbb{C}$ is I, with $!_A = \varepsilon_A$. For any $f : A \to B$ we have $\overline{f!_B}!_A = \delta_A(f!_B \otimes !_A) = f!_B$ as in:

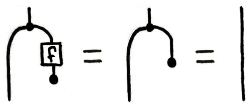


It follows that $f!_B \leq !_A$ as required. Thus, $\mathbb{C}$ is a CR category. Moreover, the construction of CR category structure from commutative comonoid structure and the construction of commutative comonoid structure from CR category structure are mutually inverse. $\square$

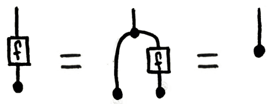
From this perspective a CR category is very similar to a Cartesian monoidal category. The only difference is that the counit of the comonoid does not need to be natural. In fact, the total maps of a CR category are precisely those which are natural with respect to the counit:

Lemma 3.1.11. *Let $\mathbb{C}$ be a Cartesian restriction category, and let $f : A \to B$ in $\mathbb{C}$. Then f is total (in the sense that $\overline{f} = 1_A$) if and only if $f\varepsilon_B = \varepsilon_A$.*

Proof. If $f\varepsilon_B = \varepsilon_A$ then $\overline{f} = 1_A$ as in:



Conversely, if $\overline{f} = 1_A$ then $f\varepsilon_B = \varepsilon_A$ as in:



$\square$

For example, the comultiplication morphisms δ_A are necessarily total. Notice that as a consequence of Theorem 2.4.5 we obtain:

Corollary 3.1.12. *Let $\mathbb{C}$ be a Cartesian restriction category. Then* $\mathsf{Total}(\mathbb{C})$ *is Cartesian monoidal.*

This presentation of the Cartesian restriction category structure suggests a notion of structure-preserving functor:

Definition 3.1.13. A *CR functor* between two CR categories $F : \mathbb{X} \to \mathbb{Y}$ is a symmetric monoidal functor that preserves the CR category structure in the sense that $F(\delta_A) = \delta_{FA}\phi^F_{A,A}$ and $F(\varepsilon_A) = \varepsilon_{FA}\phi^F_I$.

As with the other sorts of monoidal functors considered in this thesis, we omit the coherence isomorphisms, behaving as though $F(\delta_A) = \delta_{FA}$ and $F(\varepsilon_A) = \varepsilon_{FA}$. CR functors are restriction functors:

Lemma 3.1.14. *If $\mathbb{X}, \mathbb{Y}$ are CR categories and $F : \mathbb{X} \to \mathbb{Y}$ is a CR functor then F is a restriction functor. That is, $F(\overline{f}) = \overline{F(f)}$ for all $f : A \to B$ of $\mathbb{X}$.*

Proof. We have $F(\overline{f}) = F(\delta_A(1_A \otimes f\varepsilon_B)) = F(\delta_A)(F(1_A) \otimes F(f)F(\varepsilon_B)) = \delta_{FA}(1_{FA} \otimes F(f)\varepsilon_{FB}) = \overline{F(f)}$. □

While there are many interesting notions of transformation between restriction functors (see e.g., [20]), the one that is relevant here is that of *monoidal lax transformation*. Monoidal lax transformations make sense at the more general level of poset-enriched categories, where in the case of restriction categories the poset enrichment is the one from Lemma 3.1.4.

Definition 3.1.15. Let $\mathbb{X}, \mathbb{Y}$ be poset-enriched categories and let $F, G : \mathbb{X} \to \mathbb{Y}$ be poset-enriched functors, meaning that $f \le g \Rightarrow Ff \le Fg$ and $Gf \le Gg$. Then a *lax transformation* $\alpha : F \to G$ consists of a morphism $\alpha_X : FX \to GX$ of $\mathbb{Y}$ for each object X of $\mathbb{X}$ such that for all arrows $f : X \to Y$ of $\mathbb{X}$ we have $F(f)\alpha_Y \le \alpha_X G(f)$. That is, such that the usual naturality square commutes up to $\le$ as in:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\le"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

Definition 3.1.16. Let $\mathbb{X}, \mathbb{Y}$ be poset-enriched monoidal categories, and let $F, G : \mathbb{X} \to \mathbb{Y}$ be poset-enriched monoidal functors. A lax transformation $\alpha : F \to G$ is called *monoidal* in case $(\alpha_X \otimes \alpha_Y)\phi^G_{X,Y} = \phi^F_{X,Y}\alpha_{X\otimes Y}$ and $\phi^F_I \alpha_I = \phi^G_I$.

As with monoidal natural transformations, we will systematically omit the coherence isomorphisms when working with monoidal lax transformations. That is, we behave as though $\alpha_X \otimes \alpha_Y = \alpha_{X\otimes Y}$ and $\alpha_I = 1_I$.

Curiously, the components of a monoidal lax transformation between CR functors are always total:

Lemma 3.1.17. *Let $F, G : \mathbb{X} \to \mathbb{Y}$ be CR functors and let $\alpha : F \to G$ be a monoidal lax transformation. Then the components of α are necessarily total in the sense that $\overline{\alpha_X} = 1_{FX}$ for all objects X of $\mathbb{X}$.*

Proof. We have $\varepsilon_{FX} = F(\varepsilon_X)\alpha_I \le \alpha_X G(\varepsilon_X) = \alpha_X\varepsilon_{GX}$. We have $\alpha_X\varepsilon_{GX} \le \varepsilon_{FX}$ because I is a restriction terminal object, so $\varepsilon_{FX} = \alpha_A\varepsilon_{GX}$. Now by Lemma 3.1.11 the components of α_A of α are total. □

Conversely, lax transformations of CR functors with total components are always monoidal:

Lemma 3.1.18. *Let $\alpha : F \to G$ be a lax transformation of CR functors whose components are total in the sense that $\overline{\alpha_X} = 1_{FX}$ for all X of $\mathbb{X}$. Then α is necessarily monoidal.*

Proof. It follows from lax naturality of α that $\pi_0\alpha_X \leq \alpha_{X\otimes Y}\pi_0$, but both sides of this inequation are total so we have $\pi_0\alpha_X = \alpha_{X\otimes Y}\pi_0$. Similarly we have $\pi_1\alpha_Y = \alpha_{X\otimes Y}\pi_1$. It follows by the universal property of the restriction product that $\alpha_X \otimes \alpha_Y = \langle \pi_0\alpha_X, \pi_1\alpha_Y\rangle = \alpha_{X\otimes Y}$. The fact that $\alpha_I : I \to I$ is total means that it must be the identity. □

Monoidal lax transformations are the more generally applicable notion, but we will frequently use the fact that components of such transformations between CR functors are necessarily total.

3.2 DCR Categories

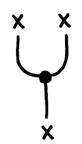
Cartesian restriction categories do not capture all the behavior of the Cartesian product of sets in Par that is involved in our notion of partial algebraic theory. In particular, we will require CR categories with the following extra structure:

Definition 3.2.1 ([17, 2.18])**.** A *discrete Cartesian restriction category* (DCR category) is a CR category in which for each object A there is an arrow $\mu_A : A\otimes A \to A$ that is *partial inverse* to δ_A. That is, $\delta_A\mu_A = \overline{\delta_A} = 1_A$ and $\mu_A\delta_A = \overline{\mu_A}$.

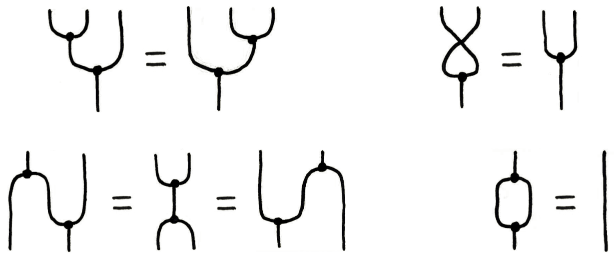
Example 3.2.2. Par is a DCR category. For any set A, $\mu_A : A \otimes A \to A$ is the partial function that maps (x, y) to x in case $x = y$, and is otherwise undefined.

DCR categories admit presentation in terms of their symmetric monoidal structure, extending Theorem 3.1.10. An important component of this result is a notion of commutative special Frobenius algebra in which the monoid does not have a unit, which we call a *non-unital Frobenius algebra*. More precisely:

Definition 3.2.3. A *non-unital Frobenius algebra* $(X, \delta_X, \mu_X, \varepsilon_X)$ in a symmetric monoidal category consists of a commutative comonoid $(X, \delta_X, \varepsilon_X)$ and a commutative semigroup (X, μ_X) such that (X, δ_X, μ_X) is a semi-Frobenius algebra. Diagrammatically, this is the comonoid structure we have already seen together with μ_X, which is depicted in string diagrams as:



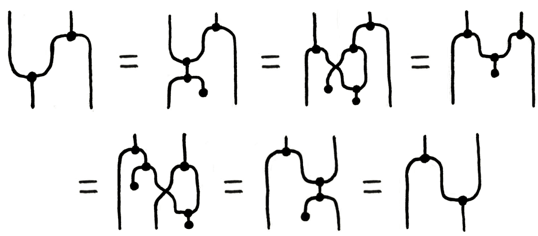
subject to the following additional equations:



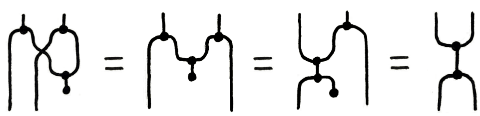
Note that there is some redundancy in the equational presentation above, as discussed in [12]. We are now ready to extend Theorem 3.1.10 to DCR categories:

Theorem 3.2.4. *A DCR category is the same thing as a symmetric monoidal category where every object A is equipped with a coherent non-unital Frobenius algebra structure $(A, \delta_A, \varepsilon_A, \mu_A)$ with natural comultiplication. That is, for any $f : A \to B$ we have $f\delta_B = \delta_A(f \otimes f)$.*

Proof. Suppose $\mathbb{X}$ is a CR category in which each $\delta_A : A \to A \otimes A$ has a partial inverse $\mu_A : A \otimes A \to A$. Then it is straightforward to show that μ is coherent with respect to the monoidal structure, and that μ_A is always associative and commutative. The special equation holds because $\delta_A \mu_A = \overline{\delta_A} = 1_A$, and for the Frobenius equations we use that $\overline{\mu_A} = \mu_A \delta_A$ to obtain:



From which both Frobenius identities follow. For the converse, the special equation gives that $\delta_A \mu_A = 1 = \overline{\delta_A}$, and further we have $\mu_A \delta_A = \overline{\mu_A}$ as in:



meaning that μ_A is a partial inverse for δ_A. □

Note that Theorem 3.2.4 owes much to the work of Giles [33] on discrete inverse categories. There the relevant notion is that of a *commutative special semi-Frobenius algebra*, which is obtained by removing both the unit and counit from a commutative special Frobenius algebra. The difference, of course, is that for DCR categories we retain the counit.

Theorem 3.2.4 suggests that a functor of DCR categories should be a CR functor with the additional property that $F(\mu_A) = \mu_{FA}$. This is true for every such functor, since partial inverses are unique and are preserved by restriction functors [21]. As such, we will take CR functors as our notion of morphism of DCR category. Taking monoidal lax transformations as 2-cells gives a 2-category:

Definition 3.2.5. DCR is the 2-category of small DCR categories, CR functors, and monoidal lax transformations.

This 2-category plays an important role in our development of partial theories, analogous to the role of the 2-category CM in the context of algebraic theories.

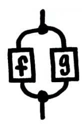
The difference between CR categories and DCR categories is succinctly captured by the useful notion of meets in a restriction category:

Definition 3.2.6 ([17, 2.8])**.** A restriction category $\mathbb{C}$ is said to have *meets* in case each hom-set $\mathbb{C}(X, Y)$ has meets with respect to the ordering of Lemma 3.1.4, which are further preserved by precomposition in the sense that $h(f \cap g) = hf \cap hg$ for any $f, g : X \to Y$ and $h : Z \to X$.

In particular notice that we do not ask for meets to be unital, or for meets to be preserved by postcomposition. We then have:

Lemma 3.2.7 ([17, 2.20])**.** *Let $\mathbb{C}$ be a CR category. Then $\mathbb{C}$ is a DCR category if and only if $\mathbb{C}$ has meets.*

Proof. If $\mathbb{C}$ is a DCR category then the meet of $f, g : X \to Y$ is given by $f \cap g = \delta_X(f \otimes g)\mu_Y$. In string diagrams:

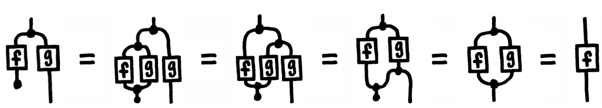


Conversely if $\mathbb{C}$ has meets them $\mu_X : X \times X \to X$ is given by $\pi_0^{X,X} \cap \pi_1^{X,X}$. $\square$

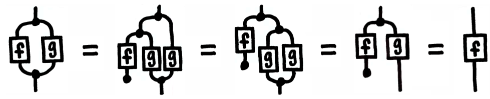
The ordering on hom-sets in a DCR category is expressible via the meet:

Lemma 3.2.8. *Let $\mathbb{X}$ be a DCR category and let $f, g : X \to Y$ in $\mathbb{X}$. Then $f \leq g$ if and only if $f \cap g = f$.*

Proof. Suppose that $f \cap g = f$. Then we have $\overline{f}g = f$ as in:



For the converse, suppose that $\overline{f}g = f$. Then we have $f \cap g = \overline{f}g = f$ as in:



$\square$

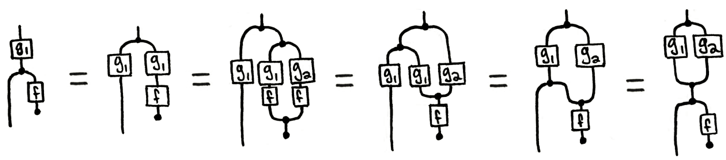
In a DCR category all arrows are natural with respect to δ, and we have already seen that arrows that are natural with respect to ε are the total ones (Lemma 3.1.11). Theorem 3.2.4 raises the question of what property the arrows that are natural with respect to μ have. The answer is that they are precisely the *partial monics*, as in:

Definition 3.2.9 ([17, 2.1])**.** Let $\mathbb{C}$ be a restriction category, and let $f : A \to B$ in $\mathbb{C}$. We say that f is *partial monic* in case for all $g_1, g_2 : C \to A$ in $\mathbb{C}$, $g_1 f = g_2 f$ implies $g_1 \overline{f} = g_2 \overline{f}$.

For example, the partial monics in Par are precisely the partial injective functions. Partial monics play an important role in the development of *range categories* [17]. Notice also that f is partial monic and total if and only if f is monic. Now, as promised, we have:

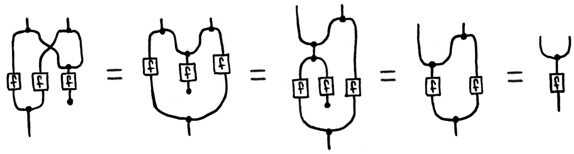
Lemma 3.2.10. *Let $\mathbb{C}$ be a DCR category, and let $f : A \to B$ in $\mathbb{C}$. Then f is partial monic if and only if $\mu_A f = (f \otimes f)\mu_B$.*

Proof. Suppose $(f \otimes f)\mu_B = \mu_A f$, and suppose $g_1 f = g_2 f$ for some $g_1, g_2 : C \to A$. Then we have $g_1 \overline{f} = (g_1 \cap g_2)\overline{f}$ as in:

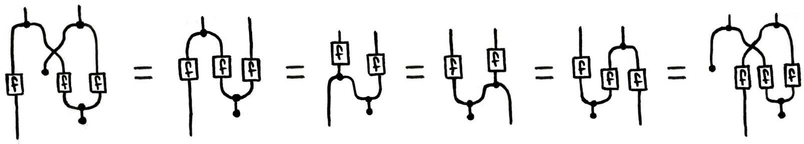


Similarly, we have $g_2 \overline{f} = (g_1 \cap g_2)\overline{f}$, and so $g_1 \overline{f} = g_2 \overline{f}$ and f is partial monic.

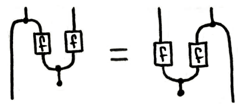
For the converse, suppose that f is partial monic. For *any* $f : A \to B$ we have $\overline{\mu_A f}(f \otimes f)\mu_B = \mu_A f$ as in:



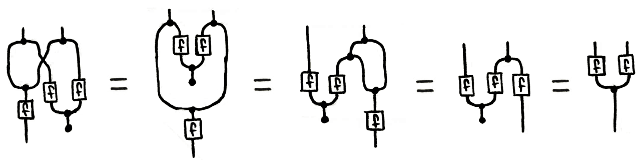
and so $\mu_A f \leq (f \otimes f)\mu_B$. Thus it suffices to show that $(f \otimes f)\mu_B \leq \mu_A f$. To do this, first notice that $\overline{(f\otimes f)\mu_B}(1_A\otimes\varepsilon_A)f = \overline{(f\otimes f)\mu_B}(\varepsilon_A\otimes 1_A)f$ as in:



Since f is partial monic this gives $\overline{(f\otimes f)\mu_B}(1_A\otimes\varepsilon_A)\overline{f} = \overline{(f\otimes f)\mu_B}(\varepsilon_A\otimes 1_A)\overline{f}$. Reducing both sides of this equation yields:



It follows that $\overline{(f\otimes f)\mu_B}\mu_A f = (f\otimes f)\mu_B$ as in:



and so $(f\otimes f)\mu_B = \mu_A f$, as required. $\square$

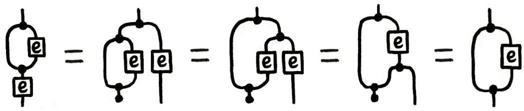
Since our attention is already focused on DCR categories, let us take the opportunity to discuss DCR categories with split restriction idempotents. Recall that *restriction idempotents* in a restriction category are those morphisms $e : X \to X$ which occur as the domain of definition of some arrow $f : X \to Y$ in the sense that $e = \overline{f}$. Equivalently, e is a restriction idempotent in case we have $e \leq 1_X$ or, also equivalently, in case $e = \overline{e}$. Split restriction idempotents are important to the theory of restriction categories more broadly, and in the specific case of DCR categories enjoy a remarkable property:

Lemma 3.2.11 ([16, Lemma 5.32]). *Let $\mathbb{X}$ be a DCR category. Then restriction idempotents split in $\mathbb{X}$ if and only if all idempotents split in $\mathbb{X}$.*

Proof. If all idempotents split then the restriction idempotents do. For the converse, suppose that all restriction idempotents split, and let $e : A \to A$ be an arbitrary idempotent. Then $(1\cap e) = \overline{(1\cap e)}$ is a restriction idempotent, so we know that it splits as in:

$$\begin{tikzcd} & X \arrow[r, "s"] & A \\ X \arrow[ur, equal] \arrow[r, "s"'] & A \arrow[u, "r"'] \arrow[ur, "1\cap e"'] & \end{tikzcd}$$

for some r, s. Now we have $(1 \cap e)e = (1 \cap e)$ as in:



which in turn gives that re and s split e as in:

$$ser = srser = s(1 \cap e)er = s(1 \cap e)r = srsr = 1_X$$

and

$$ers = e(1 \cap e) = (e \cap ee) = (e \cap e) = e$$

as required. □

Splitting the restriction idempotents in a DCR category is well-behaved:

Lemma 3.2.12 ([17, 2.12])**.** *Let $\mathbb{X}$ be a DCR category and $\mathcal{R}$ be the collection of restriction idempotents $e = \overline{e}$ in $\mathbb{X}$. Then:*

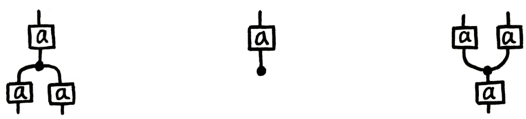
(i) $\mathsf{Split}_{\mathcal{R}}(\mathbb{X})$ *is a DCR category.*

(ii) $\mathbb{X}$ *is a sub-DCR category of* $\mathsf{Split}_{\mathcal{R}}(\mathbb{X})$ *in the sense that there is an embedding:*

$$[\![-]\!] : \mathbb{X} \hookrightarrow \mathsf{Split}_{\mathcal{R}}(\mathbb{X})$$

that preserves the Cartesian restriction structure.

Proof. (i) The symmetric monoidal structure on $\mathsf{Split}_{\mathcal{R}}(\mathbb{X})$ is given as in Lemma 2.2.8. For the DCR category structure we define $\delta_{(X,a)}$, $\varepsilon_{(X,a)}$ and $\mu_{(X,a)}$ as, respectively:



It is straightforward to verify that this structure satisfies the axioms of a DCR category. The commutative comonoid identities hold the same way they do for Lemma 2.4.25. For the remaining identities notice that any restriction idempotent $a = \overline{a} : X \to X$ is trivially partial monic since $fa = ga$ implies $f\overline{a} = fa = ga = g\overline{a}$. Then by Lemma 3.2.10 we have $\mu_X a = (a \otimes a)\mu_X$, and the remaining identities of a DCR category follow easily.

(ii) As in Lemma 2.2.6, The inclusion is defined by $[\![X]\!] = (X, 1_X)$ on objects and sends $f : X \to Y$ to $f : (X, 1_X) \to (Y, 1_Y)$. Clearly this preserves the DCR structure.

$\square$

A restriction category is called *split* in case every restriction idempotent therein splits. Note that as a consequence of Lemma 3.2.11 we have that $\mathsf{Split}_{\mathcal{R}}(\mathbb{X})$ is split, and moreover we may drop the subscript $\mathcal{R}$ when working with DCR categories:

Corollary 3.2.13. *Let $\mathbb{X}$ be a DCR category. Then $\mathsf{Split}_{\mathcal{R}}(\mathbb{X}) \simeq \mathsf{Split}(\mathbb{X})$.*

Let DCR_s be the full sub 2-category of DCR on the 0-cells with split idempotents. Splitting idempotents in DCR categories extends to a 2-functor:

Lemma 3.2.14. *There is a 2-functor $\mathsf{Split} : \mathsf{DCR} \to \mathsf{DCR}_s$ given as follows:*

On 0-cells Split sends $\mathbb{X}$ to the idempotent splitting completion $\mathsf{Split}(\mathbb{X})$.

On 1-cells $F : \mathbb{X} \to \mathbb{Y}$ we define $\mathsf{Split}(F) : \mathsf{Split}(\mathbb{X}) \to \mathsf{Split}(\mathbb{Y})$ by $\mathsf{Split}(F)(X, a) = (FX, F(a))$ on objects and by $\mathsf{Split}(F)(f) = F(f)$ on arrows.

On 2-cells $\alpha : F \to G$ the components of $\mathsf{Split}(\alpha) : \mathsf{Split}(F) \to \mathsf{Split}(G)$ are given by $\mathsf{Split}(\alpha)_{(X,a)} = F(a)\alpha_X G(a)$.

Proof. We must show that $\mathsf{Split}(F)$ is a DCR functor, and that $\mathsf{Split}(\alpha)$ is a monoidal lax transformation. For the former, $\mathsf{Split}(F)$ preserves composition as in $\mathsf{Split}(F)(f)\mathsf{Split}(F)(g) = F(f)F(g) = F(fg) = \mathsf{Split}(F)(fg)$ and identities as in $\mathsf{Split}(F)(1_{(X,a)}) = \mathsf{Split}(F)(a) = F(a) = 1_{(FX,Fa)} = 1_{\mathsf{Split}(F)(X,a)}$. That $\mathsf{Split}(F)$ preserves the DCR structure is immediate. Next, $\mathsf{Split}(\alpha)$ is lax natural as in $\mathsf{Split}(f)\mathsf{Split}(\alpha)_{(Y,b)} = F(f)F(b)\alpha_Y G(b) = F(a)F(f)\alpha_Y G(b) \leq F(a)\alpha_X G(f)G(b) = F(a)\alpha_X G(a)G(f) = \mathsf{Split}(\alpha)_{(G,y)}\mathsf{Split}(G)(f)$ for any $f : (X, a) \to (Y, b)$ of $\mathsf{Split}(\mathbb{X})$. To show that $\mathsf{Split}(\alpha)$ is monoidal, it suffices to show that it has total components. We have $\overline{\mathsf{Split}(\alpha_X)} = \overline{F(a)\alpha_X G(a)} \geq \overline{F(a)F(a)\alpha_X} = \overline{F(a)\overline{\alpha_X}} = \overline{F(a)} = F(a) = 1_{\mathsf{Split}(F)(X,a)}$ as required, and the claim follows. $\square$

Using Lemma 2.3.9 we obtain:

Lemma 3.2.15. *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{DCR} \arrow[r, shift left, "\mathsf{Split}"] \arrow[r, phantom, "\perp"] & \mathsf{DCR}_s \arrow[l, shift left, hook] \end{tikzcd}$$

where the right adjoint $\mathsf{DCR}_s \hookrightarrow \mathsf{DCR}$ is the evident inclusion.

Proof. The proof is similar to the proof of Lemma 2.3.9. As before, we define $(-)^\sharp : \mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \to \mathsf{DCR}(\mathbb{X}, \mathbb{C})$ which is natural in $\mathbb{X}$ and $\mathbb{C}$, and show that it is full, faithful, and essentially surjective.

The proof that $(-)^\sharp$ is essentially surjective is almost identical to the corresponding part of Lemma 2.3.9. The only difference is that now we must also show that our $\widehat{F}$ is always a DCR functor. We have already seen that it is symmetric monoidal. For the DCR structure, we have $\widehat{F}(\delta_{(X,a)}) = s_a F(a) F(\delta_X)(F(a) \otimes F(a))(r_a \otimes r_a) = s_a \delta_{FX}(r_a \otimes r_a) = s_a r_a \delta_{\widehat{F}(X,a)} = \delta_{\widehat{F}(X,a)}$ and $\widehat{F}(\varepsilon_{(X,a)}) = s_a F(a) F(\varepsilon_X) = s_a \varepsilon_{FX} = \varepsilon_{\widehat{F}(X,a)}$. It now follows by corresponding argument from Lemma 2.3.9 that $(-)^\sharp$ is essentially surjective.

We proceed to show that $(-)^\sharp$ is full. To that end suppose that F, G are objects of $\mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$ and that $\alpha : F^\sharp \to G^\sharp$ is an arrow of $\mathsf{DCR}(\mathbb{X}, \mathbb{C})$. Define $\widehat{\alpha} : F \to G$ by $\widehat{\alpha}_{(X,a)} = F(a)\alpha_X G(a)$. Now $\widehat{\alpha}$ is lax natural as in $F(f)\widehat{\alpha}_{(Y,b)} = F(f)F(b)\alpha_Y G(b) = F(a)F(f)\alpha_Y G(b) \leq F(a)\alpha_X G(f) G(b) = F(a)\alpha_X G(a) G(f) = \widehat{\alpha}_{(X,a)} G(f)$ for $f : (X,a) \to (Y,b)$ of $\mathsf{Split}(\mathbb{X})$. We have that the components of $\widehat{\alpha}$ are total as in $\overline{\widehat{\alpha}_{(X,a)}} = \overline{F(a)\alpha_X G(a)} \geq \overline{F(a)\overline{\alpha_X}} = F(a)$,and so $\widehat{\alpha} : F \to G$ is monoidal, and is thus a morphism of $\mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$. Consider $\widehat{\alpha}^\sharp : F^\sharp \to G^\sharp$. In particular, $\widehat{\alpha}^\sharp_X = \widehat{\alpha}_{(X,1_X)} = F(1_X)\alpha_X G(1_X) = \alpha_X$ and so we have $\widehat{\alpha}^\sharp = \alpha$. It follows that $(-)^\sharp$ is full.

Finally, we must show that $(-)^\sharp$ is faithful. To that end suppose that $\alpha, \beta : F \to G$ are morphisms of $\mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C})$, and that $\alpha^\sharp = \beta^\sharp$ in $\mathsf{DCR}(\mathbb{X}, \mathbb{C})$. Observe that this means $\alpha_{(X,1_X)} = \alpha^\sharp_X = \beta^\sharp_X = \beta_{(X,1_X)}$. Now, notice that $F(a) : F(X,1_X) \to F(X,1_X)$ and $G(a) : G(X,1_X) \to G(X,1_X)$ split in $\mathbb{C}$ as:

$$\begin{tikzcd}
 & F(X,a) \arrow[r, "F(a)"] & F(X,1_X) \\
F(X,a) \arrow[ur, equal, "F(a)=1_{F(X,a)}"] \arrow[r, "F(a)"'] & F(X,1_X) \arrow[u, "F(a)"'] \arrow[ur, "F(a)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & G(X,a) \arrow[r, "G(a)"] & G(X,1_X) \\
G(X,a) \arrow[ur, equal, "G(a)=1_{G(X,a)}"] \arrow[r, "G(a)"'] & G(X,1_X) \arrow[u, "G(a)"'] \arrow[ur, "G(a)"'] &
\end{tikzcd}$$

and so in particular $F(a) : F(X,a) \to F(X,1_X)$ and $G(a) : G(X,a) \to G(X,1_X)$ are both monic because they are sections. Next, lax naturality of α gives $F(a)\alpha_{(X,1_X)} \leq \alpha_{(X,a)} G(a)$ as in:

$$\begin{tikzcd}
F(X,a) \arrow[r, "\alpha_{(X,a)}"] \arrow[d, "F(a)"'] \arrow[dr, phantom, "\leq"] & G(X,a) \arrow[d, "G(a)"] \\
F(X,1_X) \arrow[r, "\alpha_{(X,1_X)}"'] & G(X,1_X)
\end{tikzcd}$$

In a restriction category every monic is total (Lemma 3.1.3), and the components of a monoidal lax transformation of CR functors are necessarily total (Lemma 3.1.17), and so we have $F(a)\alpha_{(X,1_X)} = \alpha_{(X,a)}G(a)$ since total morphisms related via $\leq$ are equal. Since $G(a) : G(X,a) \to G(X,1_X)$ is monic this gives $\alpha_{(X,a)} = F(a)\alpha_{(X,1_X)}G(a) : F(X,a) \to G(X,a)$ via $F(a)\alpha_{(X,1_X)}G(a)G(a) = \alpha_{(X,a)}G(a)G(a)G(a) = \alpha_{(X,a)}G(a)$. A similar argument gives $\beta_{(X,a)} = F(a)\beta_{(X,1_X)}G(a)$, and so $\alpha_{(X,a)} = F(a)\alpha_{(X,1_X)}G(a) = F(a)\beta_{(X,1_X)}G(a) = \beta_{(X,a)}$. Thus $\alpha = \beta$, and it follows that $(-)^\sharp$ is faithful.

It follows that $(-)^\sharp : \mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathbb{C}) \to \mathsf{DCR}(\mathbb{X}, \mathbb{C})$ is an equivalence of categories. Clearly this is natural in $\mathbb{X}$ and $\mathbb{C}$, which is enough to establish the promised biadjunction. $\square$

3.3 Partial Algebraic Theories

This section introduces partial algebraic theories, gives a method of presenting them, and gives a number of examples. In presenting the notion of partial algebraic theory we will echo the abstract approach to classical algebraic theories of Section 2.4.2. Just as small categories with finite products play the role of classical algebraic theory, small DCR categories will play the role of partial algebraic theory. Following this we will develop *partial term presentations* of partial algebraic theories, analogous to the Cartesian monoidal presentations of classical algebraic theories developed in Section 2.4.3. Note in particular that this method of presenting partial algebraic theories by means of equations on a monoidal signature is made possible by Theorem 3.2.4. Following this, we will illustrate partial algebraic theories and partial term presentations thereof with a number of examples. Let us begin with the notion of partial algebraic theory:

Definition 3.3.1. A *partial algebraic theory* is a small DCR category.

In the same way that models of a classical algebraic theory are given by structure-preserving functors into Set, models of a partial algebraic theory are given by structure-preserving functors into Par. Explicitly:

Definition 3.3.2. A *model* of a partial algebraic theory $\mathbb{X}$ is a CR functor $F : \mathbb{X} \to \mathsf{Par}$.

Model morphisms are monoidal lax transformations between the underlying functors:

Definition 3.3.3. Let $F, G : \mathbb{X} \to \mathsf{Par}$ be models of a partial algebraic theory $\mathbb{X}$. A *model morphism* $\alpha : F \to G$ is a monoidal lax transformation.

Thus, the 2-category DCR (Definition 3.2.5) occupies the same position in relation to partial algebraic theories that the 2-category CM (Definition 2.4.9) occupies in relation to classical algebraic theories.

In our presentations of partial algebraic theories, arrows of the free DCR category over a given monoidal signature will play the role of terms. Explicitly:

Definition 3.3.4. Let Γ be a monoidal signature. The small DCR category $\mathsf{P}(\Gamma)$ of *partial terms over* Γ is constructed the same way $\mathsf{S}(\Gamma)$ is, but with additional generating arrows:

$$\frac{A \in \mathfrak{s}(\Gamma)^*}{\delta_A : A \to A \otimes A} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\mu_A : A \otimes A \to A} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\varepsilon_A : A \to I}$$

additional equations concerning coherence:

$$\delta_I = \square \qquad \mu_I = \square \qquad \varepsilon_I = \square \qquad \delta_{A \otimes B} = (\delta_A \otimes \delta_B)(1_A \otimes \sigma_{A,B} \otimes 1_B)$$

$$\mu_{A \otimes B} = (1_A \otimes \sigma_{A,B} \otimes 1_B)(\mu_A \otimes \mu_B) \qquad \varepsilon_{A \otimes B} = \varepsilon_A \otimes \varepsilon_B$$

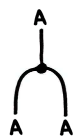
and remaining additional equations:

$$\delta_A(\delta_A \otimes 1_A) = \delta_A(1_A \otimes \delta_A) \qquad \delta_A(1_A \otimes \varepsilon_A) = 1_A \qquad \delta_A \sigma_{A,A} = \delta_A$$

$$(1_A \otimes \mu_A)\mu_A = (\mu_A \otimes 1_A)\mu_A \qquad \sigma_{A,A}\mu_A = \mu_A \qquad (\delta_A \otimes 1_A)(1_A \otimes \mu_A) = \mu_A \delta_A$$

$$(1_A \otimes \delta_A)(\mu_A \otimes 1_A) = \mu_A \delta_A \qquad \delta_A \mu_A = 1_A \qquad f\delta_B = \delta_A(f \otimes f)$$

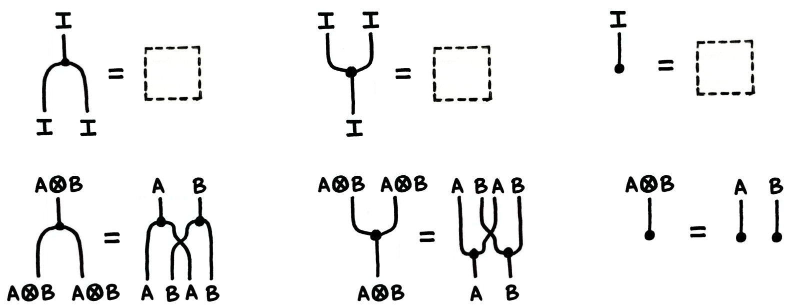
It is convenient to specify partial term presentations using string diagrams. We recall the diagrammatic convention for DCR categories:

$\delta_A \leftrightsquigarrow$ 

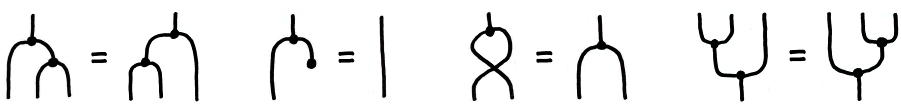
$\varepsilon_A \leftrightsquigarrow$ 

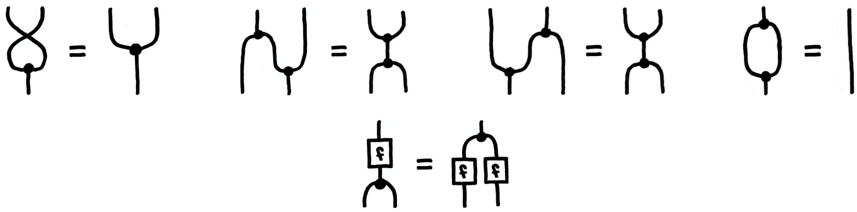
$\mu_A \leftrightsquigarrow$ 

Then the additional equations concerning coherence become:



and the remaining additional equations become:





Equations over a signature are pairs of terms:

Definition 3.3.5. Let Γ be a monoidal signature. A *partial term equation over* Γ is a pair (f, g) where $f, g : A \to B$ in $\mathsf{P}(\Gamma)$. We often write $f = g$ instead of (f, g).

A presentation of a partial theory is a signature together with a collection of equations between terms over that signature:

Definition 3.3.6. A *partial term presentation* (Γ, E) consists of a monoidal signature Γ together with a set E of partial term equations over Γ.

To construct the theory presented by some presentation, we quotient the category of terms by the equations:

Definition 3.3.7. Let (Γ, E) be a partial term presentation. Write $\mathsf{P}(\Gamma, E)$ for the small DCR category obtained by quotienting $P(\Gamma)$ by the equations of E. We say that $\mathsf{P}(\Gamma, E)$ is *presented by* (Γ, E), and similarly we say that (Γ, E) *presents* $\mathsf{P}(\Gamma, E)$.

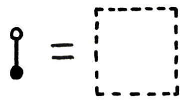
The rest of this section is dedicated to a series of examples of partial algebraic theories. We begin with a partial version of the theory of pointed sets, which serves to illustrate a difference between partial algebraic theories and their classical counterpart:

Example 3.3.8 (Possibly Pointed Sets)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Point}}, \emptyset)$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{Point}}) = \{X\}$, a single generator $e : I \to X$ written as in:

and with no generating equations. Models $F : \mathsf{P}(\Gamma_{\mathsf{Point}}, \emptyset) \to \mathsf{Par}$ of the associated partial algebraic theory consist of a partial function $F(e) : I \to FX$ from the one-element set $I = \{*\}$ into the carrier FX. Of course there will be a model $I \to FX$ for each element of FX, but it is important to note that the partial function $I \to FX$ which is undefined on $*$ is also a model. Intuitively, models of this theory consist of a set together with an *optional* point of that set. Let us call these *possibly pointed sets.*

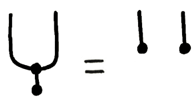
A morphism $\alpha : F \to G$ of models $F, G : \mathsf{P}(\Gamma_{\mathsf{Point}}, \emptyset) \to \mathsf{Par}$ consists of a total function $\alpha_X : FX \to GX$ with the property that $F(e)\alpha_X \leq G(e)$. That is, if $F(e)(*)$ is defined then so is $G(e)(*)$ and moreover $F(e)(*) = G(e)(*)$. Thus, a morphism of possibly pointed sets is a function that preserves the point if it exists. A subtlety here is that if $F(e)(*)$ is defined then $G(e)(*)$ *must be defined*, and so for a morphism of possibly pointed sets to exist the point in the codomain must be at least as defined as the point in the domain. That is, there cannot be a morphism of possibly pointed sets whose domain is truly pointed and whose codomain is not.

If we were to add the following equation to our presentation of possibly pointed sets:



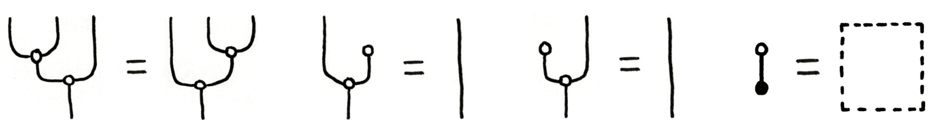
then the resulting partial algebraic theory would have pointed sets (in the usual sense) as models and functions preserving the point as model morphisms.

Example 3.3.9 (Collapsible Sets)**.** Consider the partial term presentation $(\emptyset_X, E_{\mathsf{Collapse}})$ with a single sort $\mathfrak{s}(\emptyset_X) = \{X\}$, no generators, with E_{Collapse} consisting of a single generating equation to the effect that μ_X is total:



Models $F : \mathsf{P}(\emptyset_X, E_{\mathsf{Collapse}}) \to \mathsf{Par}$ of the associated partial algebraic theory are then sets FX with the property that for all $x, y \in FX$, $x = y$. Let us call such sets *collapsible*. It is easy to see that a collapsible set is either empty or singleton. A morphism of models is simply a function between the carriers.

Example 3.3.10 (Partial Monoids)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{pMon}})$ with Γ_{Mon} as in Example 2.1 and E_{pMon} consisting of equations:

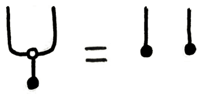


Then $\mathsf{P}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{pMon}})$ is the partial algebraic theory of *partial monoids*. A model of this theory $F : \mathsf{P}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{pMon}}) \to \mathsf{Par}$ corresponds to a set FX together with a partial function $F(m) : FX \times FX \to FX$ and a unit element $F(e) \in FX$ satisfying the equations of E_{pMon}.

These differ from models of the algebraic theory $\mathsf{C}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{Mon}})$ of monoids in that the binary operation need not be defined on all pairs of elements of the carrier.

A morphism of models $\alpha : F \to G$ of $\mathsf{P}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{pMon}})$ is a total function $\alpha_X : FX \to GX$ such that for all $a, b \in FX$, $\alpha_X(F(m)(a,b)) \leq G(m)(\alpha_X(a), \alpha_X(b))$ and $\alpha_X(F(e)) = G(e)$. Explicitly, α_X must preserve the unit, and if $F(m)(a,b)$ is defined, then so is $G(m)(\alpha_X(a), \alpha_X(b))$, and further we have $\alpha_X(F(m)(a,b)) = G(m)(\alpha_X(a), \alpha_X(b))$.

If we also impose equations to the effect that the binary operation is total:



then we obtain the partial algebraic theory of monoids. That is, models are monoids (in the usual sense), and model morphisms are monoid homomorphisms.

This method of representing an algebraic theory as a partial algebraic theory works for any cartesian monoidal presentation:

Example 3.3.11 (Cartesian Monoidal Presentations)**.** Let (Γ, E) be a cartesian monoidal presentation. Notice that every cartesian monoidal equation can also be interpreted as a partial term equation. Define E' to be the union of E with the set of equations:

$$\{f\varepsilon_B = \varepsilon_A \mid f : A \to B \in \Gamma\}$$

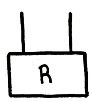
Now models and model morphisms of the partial algebraic theory $\mathsf{P}(\Gamma, E')$ coincide with models and model morphisms of the classical algebraic theory $\mathsf{C}(\Gamma, E)$. The extra equations of E' ensure that all of the operations in a model are total in spite of the fact that they inhabit Par.

There is also a presentation-independent version of this:

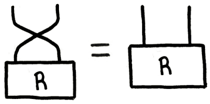
Example 3.3.12 (Classical Algebraic Theories)**.** Let $\mathbb{X}$ be a classical algebraic theory. Then $\mathsf{Par}(\mathbb{X}_{\mathsf{eq}})$ — see Definition 3.4.10 — is a partial algebraic theory, where $(-)_{\mathsf{eq}}$ is the equaliser completion of a category with finite limits [9]. The models and model morphisms of the algebraic theory $\mathbb{X}$ and the corresponding partial algebraic theory $\mathsf{Par}(\mathbb{X}_{\mathsf{eq}})$ then coincide.

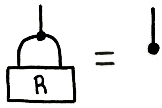
It is possible to express properties of relations, modeled as domains of definition, using partial algebraic theories. To illustrate this we give a partial algebraic theory of equivalence relations:

Example 3.3.13 (Equivalence Relations)**.** Consider the partial term presentation $(\Gamma_{\mathsf{BinRel}}, E_{\mathsf{Eq}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{BinRel}}) = \{X\}$, a single generator $R : X \otimes X \to I \in \Gamma_{\mathsf{BinRel}}$, written as in:

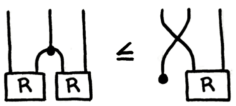


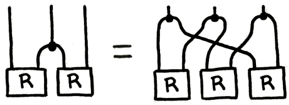
and with E_{Eq} consisting of equations expressing the symmetry and reflexivity of R:



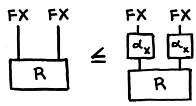


and either the equation below on the right or the inequation below on the left, each of which express the transitivity of R. Recall that *inequations* of partial terms, as in Lemma 3.1.4, may be expressed as equations by using the meet as in Lemma 3.2.8. As such, we may use them freely when specifying partial algebraic theories.





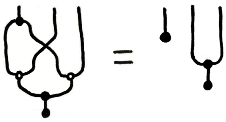
Now a model $F : \mathsf{P}(\Gamma_{\mathsf{BinRel}}, E_{\mathsf{Eq}})$ of the resulting partial algebraic theory consists of a set FX together with an equivalence relation $=_F \subseteq FX \times FX$ corresponding to the domain of definition of $F(R) : FX \otimes FX \to I$. A morphism $\alpha : F \to G$ of models of $\mathsf{P}(\Gamma_{\mathsf{BinRel}}, E_{\mathsf{Eq}})$ is a function $\alpha_X : FX \to GX$ with $a =_F b \Rightarrow \alpha_X(a) =_G \alpha_X(b)$, which arises from the requirement that α is a lax transformation:

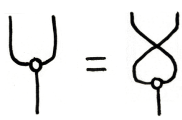


Thus, the category of models and model morphisms of $\mathsf{P}(\Gamma_{\mathsf{BinRel}}, E_{\mathsf{Eq}})$ is the category of *Bishop sets* (*setoids*) [44].

In particular we note that partial monoids show up in the computer science literature in the form of separation algebras. The partial algebraic theory of separation algebras is particularly instructive as it show how to formulate the cancellation property.

Example 3.3.14 (Separation Algebras)**.** A *separation algebra* [10] is a partial commutative monoid that is *cancellative* in the sense that if $m(a, b) = m(a, c)$ then $b = c$. We can capture separation algebras as a partial algebraic theory with the partial term presentation $(\Gamma_{\mathsf{Mon}}, E_{\mathsf{Sep}})$ where E_{Sep} contains the equations E_{pMon} of a partial monoid (Example 3.3.10) along with the following equations expressing cancellativity and commutativity:

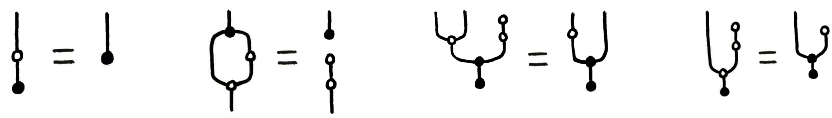




Then models of $\mathsf{P}(\Gamma_{\mathsf{Mon}}, E_{\mathsf{Sep}})$ are separation algebras and model morphisms are partial monoid homomorphisms.

Example 3.3.15 (Effect Algebras)**.** An *effect algebra* [29] is a partial commutative monoid with a unary operator $(-)^{\perp}$ satisfying the orthocomplementation axioms $m(a, a^{\perp}) = e^{\perp}$ and $m(a, b) = e^{\perp} \Leftrightarrow a^{\perp} = b$ in which the zero-one law $m(a, e^{\perp}) \downarrow \Leftrightarrow a = e$ holds ($e^{\perp}$ plays the role of "one"). We can capture effect algebras as a partial algebraic theory with the partial term presentation $(\Gamma_{\mathsf{Eff}}, E_{\mathsf{Eff}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{Eff}}) = \{X\}$, generators $\Gamma_{\mathsf{Eff}} = \Gamma_{\mathsf{Mon}} \cup \{(-)^{\perp} : X \to X\}$ with $(-)^{\perp}$ written as in:

and equations E_{Eff} consisting of the equations for partial commutative monoids E_{pMon} together the commutativity equation and:



Then models of $\mathsf{P}(\Gamma_{\mathsf{Eff}}, E_{\mathsf{Eff}})$ are effect algebras, and model morphisms $\alpha_X : FX \to GX$ are partial monoid homomorphisms with the additional property that $\alpha_X(a^{\perp}) = \alpha_X(a)^{\perp}$ for all $a \in X$.

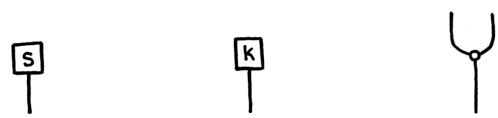
Example 3.3.16 (Partial Combinatory Algebras)**.** A *partial combinatory algebra* (PCA) is a set A with a binary partial operation $- \bullet - : A \times A \to A$, and elements $\mathsf{s}, \mathsf{k} \in A$ s.t. for any $x, y, z \in A$:

(i) $(\mathsf{k} \bullet x) \bullet y \simeq x$

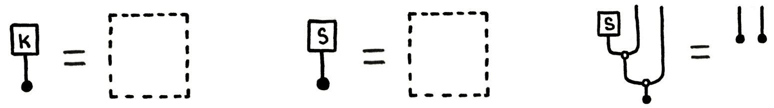
(ii) $((\mathsf{s} \bullet x) \bullet y) \bullet z \simeq (x \bullet z) \bullet (y \bullet z)$

(iii) $(\mathsf{s} \bullet x) \bullet y$ is defined

where "$\simeq$" is Kleene equality. We can capture PCAs as a partial algebraic theory with the partial term presentation $(\Gamma_{\mathsf{PCA}}, E_{\mathsf{PCA}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{PCA}}) = \{X\}$, generators as in:



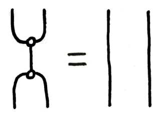
where E_{PCA} consists of equations that ensure the totality of k and s (so that they give elements of the carrier), and an equation corresponding to (iii):



as well as equations corresponding to (i) and (ii):

Then models of $\mathsf{P}(\Gamma_{\mathsf{PCA}}, E_{\mathsf{PCA}})$ are partial combinatory algebras and model morphisms are functions that preserve all of $-\bullet-$, k, and s in the sense that $\alpha(\mathsf{k}) = \mathsf{k}$, $\alpha(\mathsf{s}) = \mathsf{s}$, and if $a \bullet b$ is defined then $\alpha(a \bullet b) = \alpha(a) \bullet \alpha(b)$ for all a, b.

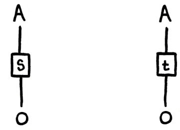
Example 3.3.17 (Pairing Functions)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Pair}}, E_{\mathsf{Pair}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{Pair}}) = \{X\}$, with two generators — pictured below left – which we think of as *pairing* and *unpairing* respectively, where E_{Pair} consists of the equation below on the right:



Then models of $\mathsf{P}(\Gamma_{\mathsf{Pair}}, E_{\mathsf{Pair}})$ are sets equipped with a surjective pairing, and model morphisms map pairs to pairs. For example, $\mathbb{N}$ and Cantor's pairing function, or Λ – the set of untyped λ-terms – with the usual pairing and projection functions. Note that our equation makes pairing a section, and so it is total. Note also that this axiomatisation allows the domain of unpairing to be larger than the range of pairing.

Thus far, our examples have mostly been single-sorted. Next we develop a progression of multi-sorted examples of partial algebraic theories, culminating in the partial algebraic theory of cartesian closed categories.

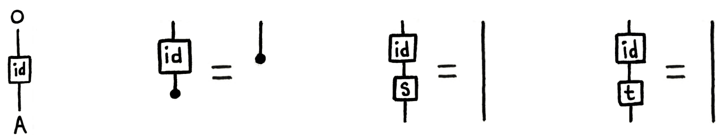
Example 3.3.18 (Directed Graphs)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Graph}}, E_{\mathsf{Graph}})$ where $\mathfrak{s}(\Gamma_{\mathsf{Graph}}) = \{O, A\}$ contains a sort O of vertices and a sort A of edges, Γ_{Graph} contains generators $s : A \to O$ and $t : A \to O$ (pictured below left), and E_{Graph} contains equations to the effect that t and s are total (pictured below right).



Then models of $\mathsf{P}(\Gamma_{\mathsf{Graph}}, E_{\mathsf{Graph}})$ are directed graphs. For such two models $F, G : \mathsf{P}(\Gamma_{\mathsf{Graph}}, E_{\mathsf{Graph}}) \to \mathsf{Par}$, a model morphism $\alpha : F \to G$ is given by functions $\alpha_O : FO \to GO$ (mapping vertices to vertices) and $\alpha_A : FA \to GA$ (mapping edges to edges) which satisfy:

and so model morphisms are directed graph homomorphisms.

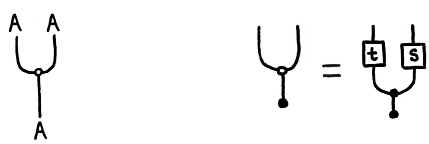
Example 3.3.19 (Reflexive Graphs)**.** Consider the partial term presentation $(\Gamma_{\mathsf{RGraph}}, E_{\mathsf{RGraph}})$ where Γ_{RGraph} is the result of adding an extra generator $id : O \to A$ (pictured below left) to Γ_{Graph} (see Example 3.3.18), and E_{RGraph} is the result of adding the equations pictured below right to E_{Graph}.



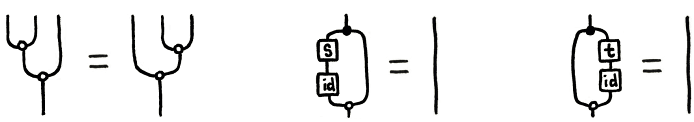
Then models of $\mathsf{P}(\Gamma_{\mathsf{RGraph}}, E_{\mathsf{RGraph}})$ are *reflexive* graphs, and model morphisms are graph homomorphisms that preserve the identity edge in the sense that $\alpha_1(id(v)) = id(\alpha_0(v))$ for all edges v.

We remark that Example 3.3.19 and Example 3.3.18 could also be presented as classical algebraic theories, since all the operations are total. We next give a partial algebraic theory of categories, in which the composition operation is *not* total, and as such cannot be presented as an algebraic theory.

Example 3.3.20 (Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Cat}}, E_{\mathsf{Cat}})$ obtained by extending Γ_{RGraph} with a generator $m : A \otimes A \to A$ representing composition of arrows (pictured below left), and extending E_{RGraph} with an equation to the effect that the composite of two arrows is defined when the target of the first arrows matches the source of the second arrow (below right):

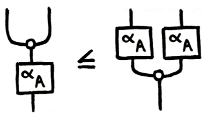


along with equations to the effect that composition is associative and unital:



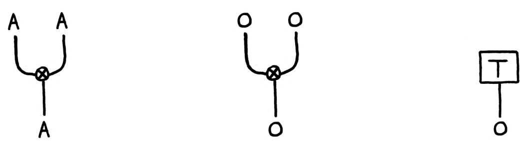
and finally equations concerning the source and target of composite arrows:

Models of $\mathsf{P}(\Gamma_{\mathsf{Cat}}, E_{\mathsf{Cat}})$ are (small) categories. For two such models $F, G : \mathsf{P}(\Gamma_{\mathsf{Cat}}, E_{\mathsf{Cat}}) \to \mathsf{Par}$ a model morphism $\alpha : F \to G$ consists of functions $\alpha_O : FO \to GO$ (mapping objects to objects) and $\alpha_A : FA \to GA$ (mapping arrows to arrows) which must satisfy the equations from Examples 3.3.18 and 3.3.19 as well as the following inequation:

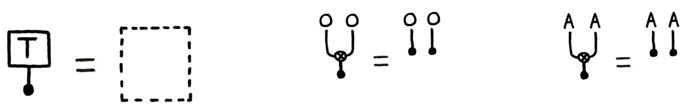


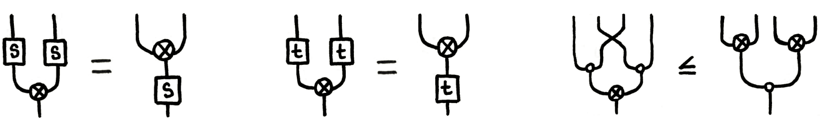
This states that if f and g are composable then so are $\alpha_A(f)$ and $\alpha_A(g)$, and further that $\alpha_A(fg) = \alpha_A(f)\alpha_A(g)$. If this were an equality, it would insist also that if $\alpha_A(f)$ and $\alpha_A(g)$ are composable, then so are f and g, which is not always the case. Thus, model morphisms are precisely functors.

Example 3.3.21 (Strict Monoidal Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{MCat}}, E_{\mathsf{MCat}})$ obtained by extending Γ_{Cat} with additional generators corresponding to the tensor product operation and unit object:

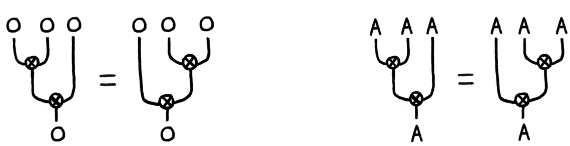


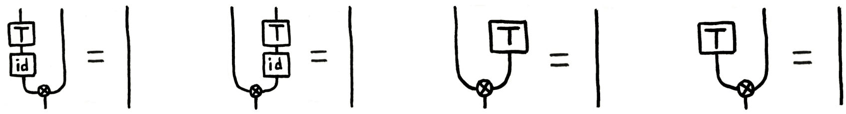
and extending E_{Cat} with additional equations as in:





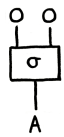
as well as equations to the effect that the tensor product is associative and unital:



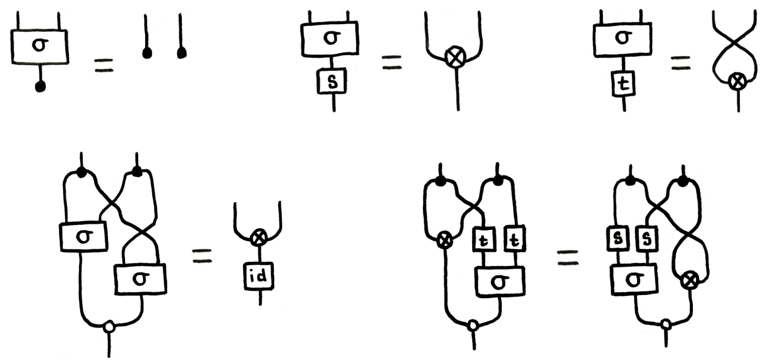


Models of $\mathsf{P}(\Gamma_{\mathsf{MCat}}, E_{\mathsf{MCat}})$ are strict monoidal categories and model morphisms are strict monoidal functors.

Example 3.3.22 (Symmetric Strict Monoidal Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{SMCat}}, E_{\mathsf{SMCat}})$ obtained by extending Γ_{MCat} with an additional generator $\sigma : O \otimes O \to A$ for the braiding:



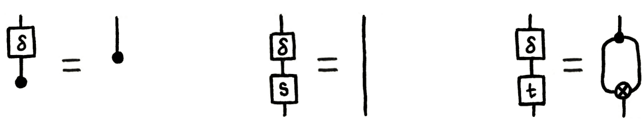
and extending E_{MCat} with additional equations as in:

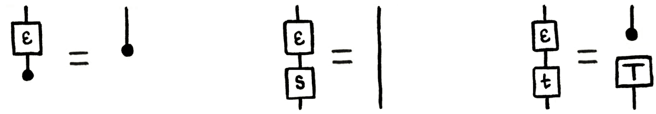


Models of $\mathsf{P}(\Gamma_{\mathsf{SMCat}}, E_{\mathsf{SMCat}})$ are symmetric strict monoidal categories, and model morphisms are symmetric strict monoidal functors.

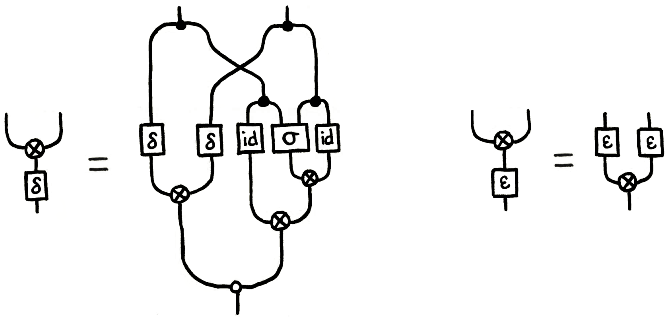
Example 3.3.23 (Cartesian Restriction Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{CRCat}}, E_{\mathsf{CRCat}})$ obtained by extending Γ_{SMCat} with generators $\delta : O \to A$ and $\varepsilon : O \to A$ for the comonoid structure:

and extending E_{SMCat} with additional equations as in:

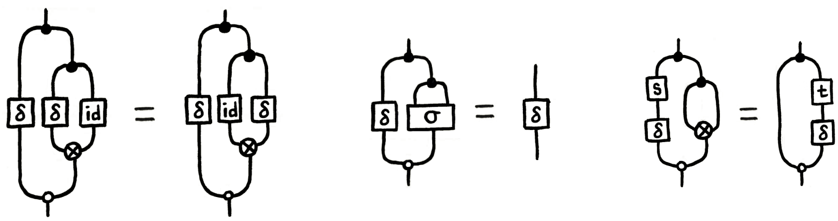


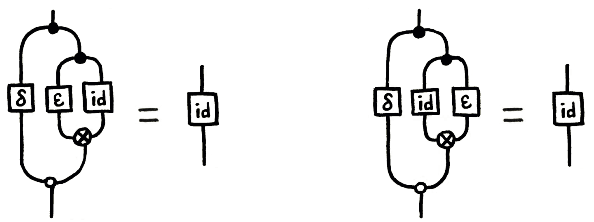


along with equations insisting that δ and ε are coherent with respect to the monoidal structure:



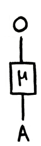
And finally equations for the commutative comonoid axioms, and naturality of δ:



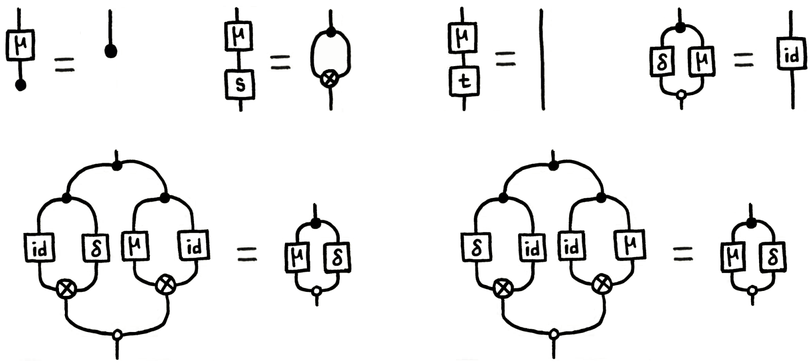


Then (using Theorem 3.1.10) we have that models of $\mathsf{P}(\Gamma_{\mathsf{CRCat}}, E_{\mathsf{CRCat}})$ are CR categories, and that model morphisms are CR functors.

Example 3.3.24 (Discrete Cartesian Restriction Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{DCRCat}}, E_{\mathsf{DCRCat}})$ obtained by extending Γ_{CRCat} with a generator $\mu : O \to A$:

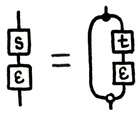


and extending E_{CRCat} with additional equations as in:



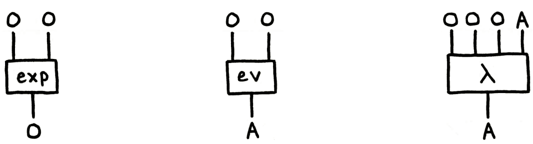
Models of $\mathsf{P}(\Gamma_{\mathsf{DCRCat}}, E_{\mathsf{DCRCat}})$ are discrete cartesian restriction categories, and model morphisms are CR functors.

Example 3.3.25 (Cartesian Monoidal Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{CRCat}}, E_{\mathsf{CCat}})$ where E_{CCat} is obtained by extending E_{CRCat} with a single equation:

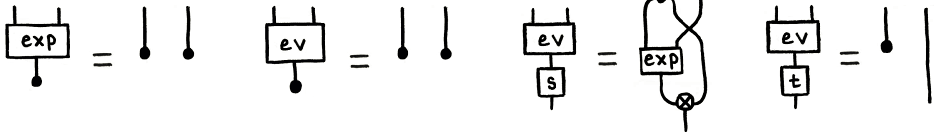


Models of $\mathsf{P}(\Gamma_{\mathsf{CRCat}}, E_{\mathsf{CCat}})$ are cartesian monoidal categories (using Theorem 2.4.5), and model morphisms are cartesian monoidal functors.

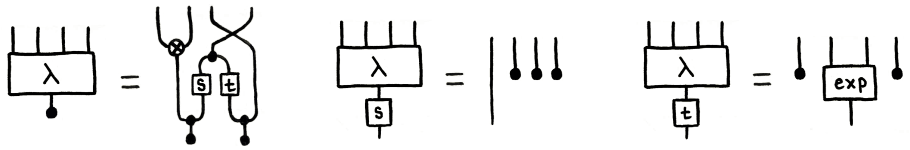
Example 3.3.26 (Cartesian Closed Categories)**.** Consider the partial term presentation $(\Gamma_{\mathsf{CCC}}, E_{\mathsf{CCC}})$ obtained by extending $(\Gamma_{\mathsf{CRCat}}, E_{\mathsf{CCat}})$ with generators $\mathsf{exp} : O \otimes O \to O$, $\mathsf{ev} : O \otimes O \to A$, and $\lambda : O \otimes O \otimes O \otimes A \to A$:



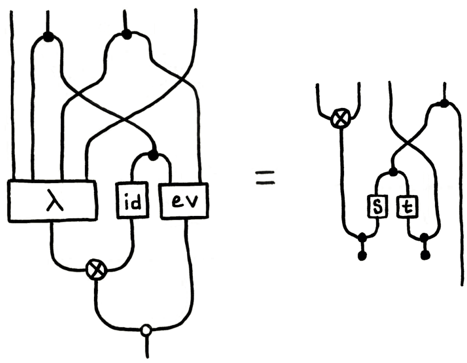
The idea is that $\mathsf{exp}(A, B)$ represents the internal hom $[A, B]$, that ev gives the evaluation maps, and that λ gives the "name" maps of the cartesian closed structure. E_{CCC} is obtained by extending E_{CCat} with the necessary equations relating the internal hom and the evaluation maps:



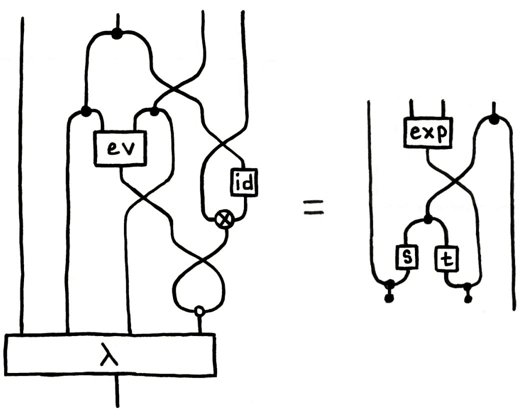
We also require equations governing the domain of definition, source, and target of our morphism names. In particular we must have that $\lambda(X, A, B, f)$ is defined precisely in case $f : X \times A \to B$, in which case it yields a morphism $\lambda(X, A, B, f) : X \to [A, B]$. Diagrammatically:



Further, we must ask that evaluating names works as intended, in the sense that $f : X \times A \to B$ then $(\lambda(X, A, B, f) \times 1)\mathsf{ev} = f$. Diagrammatically:



Finally, we ask that when $g : X \to [A, B]$ we have $\lambda(X, A, B, (g \times 1)\mathsf{ev}) = g$ holds:



Models of $\mathsf{P}(\Gamma_{\mathsf{CCC}}, E_{\mathsf{CCC}})$ are strict cartesian closed categories, and model morphisms are strict cartesian closed functors (preserving both hom-objects and names in addition to the cartesian monoidal structure).

This presentation of cartesian closed categories is due to Freyd: a version of it is given immediately after the first appearance of the notion of essentially algebraic theory in [31], albeit somewhat informally, and using very different syntax.

3.4 Finite Limits and Partial Maps

Our next goal will be to characterise the categories of models and model morphisms of partial algebraic theories (the *varieties*). In doing so we will also obtain a characterisation of *Morita equivalence* for partial algebraic theories, which is what we will call the situation in which two different partial algebraic theories determine equivalent categories of models and model morphisms (i.e., *present the same variety*).

Remark 3.4.1. In what follows we will write $f : A \rightarrowtail B$ when we wish to emphasize that the arrow $f : A \to B$ is monic.

To this end, we recall certain elementary facts concerning categories with finite limits, and in particular the construction of categories of partial maps internal to such categories. Ultimately, we will show that DCR categories and categories with finite limits are the 0-cells of equivalent 2-categories. This will allow us to connect DCR categories to the wider literature. It will be convenient for us to place equalisers at the center of our discussion. Recall:

Definition 3.4.2. Let $\mathbb{X}$ be a category, and let $f, g : A \to B$ be morphisms of $\mathbb{X}$. An *equaliser* of f and g is a diagram[1]:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "+"] & B \end{tikzcd}$$

such that for any $k : C \to A$ with $kf = kg$, there exists a unique $h : C \to E$ satisfying $he = k$, as in:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "+"] & B \\ C \arrow[u, "h"] \arrow[ur, "k"'] & & \end{tikzcd}$$

A category in which an equaliser exists for any two parallel morphisms is said to *have equalisers*. For example, Set has equalisers, with the equaliser of two functions $f, g : A \to B$ given by the set $E = \{a \in A \mid f(a) = g(a)\}$ together with the inclusion $E \to A$.

Definition 3.4.3. A Cartesian monoidal category $\mathbb{X}$ that has equalisers is said to have *finite limits*.

While finite limits admit a number of elementary characterisations, the definition above lines up nicely with the notion of DCR category. We will also work with categories of spans, which are naturally presented in terms of *pullbacks*. Recall:

[1]Here the "+" inside the diagram indicates that that sub-diagram *need not commute* — a convention we adopt from Freyd and Scedrov [32] — so that while the diagram states $ef = eg$, it does not state that $f = g$.

Definition 3.4.4. Let $f : A \to C$ and $g : B \to C$ be arrows of a category $\mathbb{X}$. A *pullback* of f and g is a diagram:

$$\begin{tikzcd}
A \times_C B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

such that for any $a : D \to A$ and $b : D \to B$ with $af = bg$ there exists a unique $h : D \to A \times_C B$ such that:

$$\begin{tikzcd}
D \arrow[dr, "h"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& A \times_C B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Say that a category *has pullbacks* in case it contains a pullback of any two morphisms $f : A \to C$ and $g : B \to C$. Given a pullback diagram labeled in the above manner, we refer to p_1 as the *pullback of f along g* and, dually, refer to p_0 as the *pullback of g along f*.

For example, in Set the pullback of two functions $f : A \to C$ and $g : B \to C$ is given by $A \times_C B = \{(a, b) \mid a \in A, b \in B, f(a) = g(b)\}$. More generally:

Lemma 3.4.5. *Let $\mathbb{X}$ be a category with finite limits. Suppose $f : A \to C, g : B \to C$ are arrows of $\mathbb{X}$ with equaliser:*

$$\begin{tikzcd}
E \arrow[r, "e"] & A \times B \arrow[r, shift left=1.5ex, "\pi_0 f"] \arrow[r, shift right=1.5ex, "\pi_1 g"'] \arrow[r, phantom, "+"] & C
\end{tikzcd}$$

Then the following diagram is a pullback:

$$\begin{tikzcd}
E \arrow[r, "e\pi_1"] \arrow[d, "e\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

So in particular, if $\mathbb{X}$ has finite limits then it has pullbacks.

Proof. Suppose $f : A \to C$ and $g : B \to C$ in $\mathbb{X}$. Let

$$\begin{tikzcd}
E \arrow[r, "e"] & A \times B \arrow[r, shift left=1.5ex, "\pi_0 f"] \arrow[r, shift right=1.5ex, "\pi_1 g"'] \arrow[r, phantom, "+"] & C
\end{tikzcd}$$

be an equaliser. Then

$$\begin{tikzcd} E \arrow[r, "e\pi_1"] \arrow[d, "e\pi_0"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is a pullback as follows: suppose we have $a : D \to A$ and $b : D \to B$ such that:

$$\begin{tikzcd} D \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Then we have:

$$\begin{tikzcd} E \arrow[r, "e"] & A \times B \arrow[r, shift left, "\pi_0 f"] \arrow[r, shift right, "\pi_1 g"'] & C \\ D \arrow[ur, "\langle a,b \rangle"'] & & \end{tikzcd}$$

and so there exists a unique $h : D \to E$ such that:

$$\begin{tikzcd} E \arrow[r, "e"] & A \times B \arrow[r, shift left, "\pi_0 f"] \arrow[r, shift right, "\pi_1 g"'] & C \\ D \arrow[u, dashed, "h"] \arrow[ur, "\langle a,b \rangle"'] & & \end{tikzcd}$$

We show that this $h : D \to E$ is also the unique arrow such that:

$$\begin{tikzcd} D \arrow[dr, "h"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & E \arrow[r, "e\pi_1"] \arrow[d, "e\pi_0"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

To that end, suppose we have $k : D \to E$ such that $ke\pi_0 = a$ and $ke\pi_1 = b$. Then $ke = \langle a, b \rangle$ by the universal property of binary products. Recall that h is the unique morphism $D \to E$ such that $he = \langle a, b \rangle$. It follows that $h = k$. Thus, our diagram is indeed a pullback. Since f and g were arbitrary, it follows that $\mathbb{X}$ has pullbacks. $\square$

In the presence of a terminal object, pullbacks can be used to construct equalisers, giving a sort of converse to the above lemma. An important fact about pullbacks is that the pullback of a monic along any arrow is again monic:

Lemma 3.4.6. *Let $m : B \to C$ be monic, and suppose the following diagram is a* *pullback:*

$$\begin{tikzcd} A \times_C B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "m"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Then p_0 is monic.

Proof. Suppose we have $a, b : D \to A \times_C B$ such that $ap_0 = bp_0$. We must show that in this case $a = b$. First, notice that we have $ap_1 m = ap_0 f = bp_0 f = bp_1 m$, and so since m is monic we have $ap_1 = bp_1$. Now, we have both of:

$$\begin{tikzcd} D \arrow[dr, "a"] \arrow[drr, bend left, "ap_1=bp_1"] \arrow[ddr, bend right, "ap_0=bp_0"'] & & \\ & A \times_C B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "m"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \qquad \begin{tikzcd} D \arrow[dr, "b"] \arrow[drr, bend left, "ap_1=bp_1"] \arrow[ddr, bend right, "ap_0=bp_0"'] & & \\ & A \times_C B \arrow[r, "p_1"] \arrow[d, "p_0"'] & B \arrow[d, "m"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

and it follows from the universal property of our assumed pullback that $a = b$. Thus, p_0 is monic. □

There is an evident notion of structure-preserving functor between categories with finite limits, and an evident 2-category of categories with finite limits:

Definition 3.4.7. A Cartesian monoidal functor $F : \mathbb{X} \to \mathbb{Y}$ between categories $\mathbb{X}, \mathbb{Y}$ with finite limits is said to *preserve finite limits* in case it maps equalisers in $\mathbb{X}$ to equalisers in $\mathbb{Y}$.

Definition 3.4.8. Lex is the 2-category of small categories with finite limits, finite limit preserving functors, and natural transformations.

In any category with finite limits, spans with a monic left leg can be understood as a kind partial morphism. If $f : M \to Y$ is any morphism and $m : M \rightarrowtail X$ is monic, then the span:

$$\begin{tikzcd} & M \arrow[dl, tail, "m"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

represents a partial morphism $(m, f) : X \to Y$. The left leg plays the role of $\mathsf{dom}(m, f)$, exhibiting A as a subobject of X, and the right leg plays the role of $\mathsf{def}(m, f)$.

Here we work with a slight modification of the usual construction, in which the left leg of our spans in required to be *regular monic*, as in:

Definition 3.4.9. A morphism $m : M \to X$ is called *regular monic* in case for some $f, g : X \to Y$ the diagram:

$$\begin{tikzcd} M \arrow[r, "m"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

is an equaliser.

Notice in particular that every regular monic is monic and that the pullback of a regular monic along any arrow is again a regular monic. Further, each identity morphism is a regular monic, and the composite of two regular monics is again a regular monic.

In any category with finite limits these partial morphisms define a category:

Definition 3.4.10. Suppose that $\mathbb{C}$ is a category with finite limits. Then the category $\mathsf{Par}(\mathbb{C})$ of *partial morphisms in* $\mathbb{C}$ is defined as follows:

objects are objects of $\mathbb{C}$.

arrows $(m, f) : X \to Y$ are equivalence classes of spans $X \xleftarrow{m} A \xrightarrow{f} Y$ where m is regular monic. Two spans (m, f) and (m', f') are equivalent if and only if there is an isomorphism α such that:

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[dl, "m"'] \arrow[drr, "f"] & A' \arrow[dll, "m'"] \arrow[dr, "f'"] & \\ X & & & Y. \end{tikzcd}$$

composition is defined by pullback. Explicitly, the composite of $(m, f) : A \to B$ and $(m', g) : B \to C$ is the outer span of the diagram below on the left

$$\begin{tikzcd} & & X \times_B X' \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\ & X \arrow[dl, "m"'] \arrow[dr, "f"] & & X' \arrow[dl, "m'"'] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd} \qquad \begin{tikzcd} X \times_B X' \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "m'"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

where the diagram above right is a pullback. Note that it doesn't matter which pullback, since any two choices will give isomorphic spans, and therefore equal morphisms.

identities are diagonal spans $(1_A, 1_A) : A \to A$.

We obtain an analogue of the partial order on hom-sets of Par by relaxing the notion of equivalence in the above definition. Specifically, given (m, f) and (m', f')

in $\mathsf{Par}(\mathbb{C})(X,Y)$ we say that $(m,f) \leq (m',f')$ in case we have:

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[dl, "m"'] \arrow[drr, "f"' near end] & A' \arrow[dll, "m'" near end] \arrow[dr, "f'"] & \\ X & & & Y. \end{tikzcd}$$

for *any* $\alpha : A \to A'$. In this way, $\mathsf{Par}(\mathbb{C})$ is also enriched in the category of posets and monotone functions.

This notion of partial morphism coincides with the usual notion of partial function when considered in the category of sets:

Observation 3.4.11. There is an isomorphism of categories $\mathsf{Par} \cong \mathsf{Par}(\mathsf{Set})$. Further, the partial orders on $\mathsf{Par}(X,Y)$ and $\mathsf{Par}(\mathsf{Set})(X,Y)$ coincide.

These categories of partial morphisms are always split DCR categories:

Lemma 3.4.12. *If $\mathbb{C}$ has finite limits then $\mathsf{Par}(\mathbb{C})$ is a DCR category. Further, idempotents in $\mathsf{Par}(\mathbb{C})$ split.*

Proof. The monoidal structure in $\mathsf{Par}(\mathbb{C})$ is given by the monoidal structure in $\mathbb{C}$, in the sense that the tensor product of two objects A, B is $A \otimes B$ and the unit is I. The tensor product of arrows is given by $(f_0, f_1) \otimes (g_0, g_1) = (f_0 \otimes g_0, f_1 \otimes g_1)$, and the braiding is given by $\sigma_{A,B} = (1_{A \otimes B}, \sigma_{A,B}) = (\sigma_{B,A}, 1_{B \otimes A})$. The DCR structure is given as in $\delta_A = (1_A, \delta_A)$, $\varepsilon_A = (1_A, !_A)$, and $\mu_A = (\delta_A, 1_A)$. The restriction idempotents in $\mathsf{Par}(\mathbb{C})$ are those arrows of the form $(m,m) : A \to A$. Any such idempotent splits as in:

$$\begin{tikzcd} & A \arrow[r, "{(1,m)}"] & A \\ A \arrow[ur, equal] \arrow[r, "{(1,m)}"'] & A \arrow[u, dashed, "{(m,1)}"] \arrow[ur, "{(m,m)}"'] & \end{tikzcd}$$

Then by Lemma 3.2.11 we have that all idempotents in $\mathsf{Par}(\mathbb{C})$ split. $\square$

3.5 Relating DCR Categories and Finite Limits

In this section we establish a strict 2-equivalence between the 2-category DCR_s of split small DCR categories and the 2-category Lex of small categories with finite limits.

We have already seen that when $\mathbb{C}$ has finite limits $\mathsf{Par}(\mathbb{C})$ is a split DCR category (Lemma 3.4.12). Clearly $\mathsf{Par}(\mathbb{C})$ is small if $\mathbb{C}$ is. We show that this extends to a 2-functor $\mathsf{Par} : \mathsf{Lex} \to \mathsf{DCR}_s$. If $\mathbb{C}$ and $\mathbb{D}$ are small categories with finite limits and $F : \mathbb{C} \to \mathbb{D}$ is a finite-limit preserving functor, then we obtain a CR functor

$\mathsf{Par}(F) : \mathsf{Par}(\mathbb{C}) \to \mathsf{Par}(\mathbb{D})$, defined on objects by $\mathsf{Par}(F)(A) = F(A)$, and on arrows by

$$\begin{tikzcd}[column sep=small] & X \arrow[dl, "m"'] \arrow[dr, "f"] & & & & FX \arrow[dl, "Fm"'] \arrow[dr, "Ff"] & \\ A & & B & \overset{\mathsf{Par}(F)}{\mapsto} & FA & & FB \end{tikzcd}$$

Since F preserves finite limits, we have that $\mathsf{Par}(F)(\delta_A) = (F1_A, F\Delta_A) = (1_{FA}, \Delta_{FA}) = \delta_{FA} = \delta_{\mathsf{Par}(F)(A)}$ and $\mathsf{Par}(F)(\varepsilon_A) = (F1_A, F!_A) = (1_{FA}, !_{FA}) = \varepsilon_{\mathsf{Par}(F)(A)}$, so $\mathsf{Par}(F)$ preserves the CR structure. This defines the action of Par on 1-cells. We present the action of Par on 2-cells as a lemma:

Lemma 3.5.1. *If $F, G : \mathbb{C} \to \mathbb{D}$ are finite limit preserving functors between categories with finite limits and $\alpha : F \to G$ is a natural transformation, define $\mathsf{Par}(\alpha) : \mathsf{Par}(F) \to \mathsf{Par}(G)$ by defining the component of $\mathsf{Par}(\alpha)$ at A in $\mathbb{C}$ to be:*

$$\begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] \arrow[d, phantom, "\mathsf{Par}(\alpha)_A"] & \\ FA & {} & GA \end{tikzcd}$$

Then $\mathsf{Par}(\alpha) : \mathsf{Par}(F) \to \mathsf{Par}(G)$ is a monoidal lax transformation.

Proof. First, we show that for any arrow (m, f) of $\mathsf{Par}(\mathbb{C})$ we have

$$\begin{tikzcd} FA \arrow[r, "\mathsf{Par}(\alpha)_A"] \arrow[d, "\mathsf{Par}(F)(m,f)"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "\mathsf{Par}(G)(m,f)"] \\ FB \arrow[r, "\mathsf{Par}(\alpha)_B"'] & GB \end{tikzcd}$$

To that end, let the following two diagrams define, respectively, $\mathsf{Par}(\alpha)_A \mathsf{Par}(G)(m, f) = (\pi_0, \pi_1 Gf)$ and $\mathsf{Par}(F)(m, f)\mathsf{Par}(\alpha)_B = (Fm, Ff\alpha_B)$:

$$\begin{tikzcd}[column sep=small] & & FA \wedge GX \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\ & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] & & GX \arrow[dl, "Gm"'] \arrow[dr, "Gf"] & \\ FA & & GA & & GB \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & FX \arrow[dl, "1_{FX}"'] \arrow[dr, "Ff"] & & \\ & FX \arrow[dl, "Fm"'] \arrow[dr, "Ff"] & & FB \arrow[dl, "1_{FB}"'] \arrow[dr, "\alpha_B"] & \\ FA & & FB & & GB \end{tikzcd}$$

From this, we obtain a morphism $h : FX \to FA \wedge GX$ by the universal property of the first pullback:

$$\begin{tikzcd} FX \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\alpha_X"] \arrow[ddr, bend right, "Fm"'] & & \\ & FA \wedge GX \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & GX \arrow[d, "Gm"] \\ & FA \arrow[r, "\alpha_A"'] & GA \end{tikzcd}$$

which gives a morphism of spans

$$\begin{tikzcd}
 & FX \arrow[r, "h"] \arrow[dl, "Fm"'] \arrow[drr, "Ff\alpha_B"] & FA \wedge GX \arrow[dll, "\pi_0"] \arrow[dr, "\pi_1 Gf"] & \\
FA & & & GB.
\end{tikzcd}$$

Thus, $\mathsf{Par}(\alpha)$ is a lax transformation. Obviously the components of $\mathsf{Par}(\alpha)$ are total, and so it is a monoidal lax transformation as required. □

It remains only to show that Par preserves composition and identities at the level of 1-cells and 2-cells, and that preserves horizontal composition of 2-cells, all of which are immediate. We therefore have:

Lemma 3.5.2. $\mathsf{Par} : \mathsf{Lex} \to \mathsf{DCR}_s$ *is a 2-functor.*

□

Recall that for any restriction category $\mathbb{X}$ the *total* maps, being those f with $\overline{f} = 1$ – equivalently with $f\varepsilon = \varepsilon$ – form a subcategory $\mathsf{Total}(\mathbb{X})$ of $\mathbb{X}$. We have:

Proposition 3.5.3 ([17, 2.14])**.** *If* $\mathbb{X}$ *is a split DCR category then* $\mathsf{Total}(\mathbb{X})$ *has finite limits.*

Proof. The comonoid structure is always total, and the total maps are precisely the comonoid homomorphisms. It follows that $\mathsf{Total}(\mathbb{X})$ has finite products. We must show that it also has equalisers. To that end suppose $f, g : A \to B$ are parallel arrows in $\mathsf{Total}(\mathbb{X})$. Let the splitting of the restriction idempotent $\overline{f \cap g} = \overline{\delta_A(f \otimes g)\mu_B} : A \to A$ in $\mathbb{X}$ be as in:

$$\begin{tikzcd}
 & E \arrow[r, "s"] & A \\
E \arrow[ur, "1_E"] \arrow[r, "s"'] & A \arrow[u, "r"'] \arrow[ur, "\overline{f\cap g}"'] &
\end{tikzcd}$$

Then the equaliser of f, g is as follows:

$$\begin{tikzcd}
E \arrow[r, "s"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

In any DCR category we have that $\overline{f \cap g} f = \overline{f \cap g} g$. It follows that s equalises f and g as in:

$$sf = srsf = s\overline{f \cap g} f = s\overline{f \cap g} g = srsg = sg$$

Next, for any total $h : X \to A$ with $hf = hg$ the arrow $hr : X \to E$ is total as in $\overline{hr} = \overline{hrs} = \overline{hf \cap hg} = \overline{hf} = 1_X$. Further, since f, g are total we have

$hrs = h\overline{f\cap g} = \overline{hf \cap hg}h = \overline{hf}h = h\overline{f} = h$. To see that h is the unique such map, suppose we have $k : X \to A$ with $ks = h$. Then $ks = h = hrs$ which means that $k = hr$ as s is monic. It follows that $\mathsf{Total}(\mathbb{X})$ has equalisers, and thus finite limits. ☐

There is an important connection between regular monics in $\mathsf{Total}(\mathbb{X})$ and the *restriction monics* of $\mathbb{X}$, as in:

Definition 3.5.4 ([21, 2.3.1])**.** An arrow $m : M \to X$ of a restriction category $\mathbb{X}$ is a *restriction monic* in case it splits a restriction idempotent. That is, in case there exists some $w : X \to M$ such that

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

for some arrow $f : X \to Y$ of $\mathbb{X}$. In this case we say that m *splits* $\overline{f}$. Notice that in this case $wm = \overline{f} = \overline{\overline{f}} = \overline{wm}$.

Specifically, since equalisers in $\mathsf{Total}(\mathbb{X})$ are constructed by splitting restriction idempotents we have:

Lemma 3.5.5 ([17, 2.16])**.** *Let $\mathbb{X}$ be a DCR category. If $m : M \to X$ is regular monic in $\mathsf{Total}(\mathbb{X})$ then $m : M \to X$ is a restriction monic in $\mathbb{X}$.*

☐

Notice that a given restriction monic splits at most one restriction idempotent, as in:

Lemma 3.5.6 ([21, 2.25])**.** *In any restriction category:*

(i) if $mw = 1$ and $mk = 1$ with $wm = \overline{wm}$ and $km = \overline{km}$ then $w = k$

(ii) if $mw = 1$ and $nw = 1$ with $wm = \overline{wm}$ and $wn = \overline{wn}$ then $n = m$

Proof. (i) $w = wmkmw = \overline{wm}\,\overline{km}w = \overline{km}\,\overline{wm}w = kmwmw = k$

(ii) $m = mwnwm = m\overline{wn}\,\overline{wm} = m\overline{wm}\,\overline{wn} = mwmwn = n$

☐

Now the assignment of $\mathbb{X}$ to $\mathsf{Total}(\mathbb{X})$ extends to a 2-functor $\mathsf{Total} : \mathsf{DCR}_s \to \mathsf{Lex}$. If $\mathbb{X}$ and $\mathbb{Y}$ are split DCR categories and $F : \mathbb{X} \to \mathbb{Y}$ is a CR functor then restricting F to the categories of total maps defines a finite limit preserving functor $\mathsf{Total}(F) : \mathsf{Total}(\mathbb{X}) \to \mathsf{Total}(\mathbb{Y})$. Similarly $F, G : \mathbb{X} \to \mathbb{Y}$ are CR functors between DCR categories $\mathbb{X}$ and $\mathbb{Y}$ and $\alpha : F \to G$ is a monoidal lax transformation then α restricts to a natural transformation $\alpha : \mathsf{Total}(\mathbb{F}) \to \mathsf{Total}(\mathbb{G})$. Naturality follows from lax naturality together with the fact that all components of α are total maps: for any $f : A \to B$ of $\mathsf{Total}(\mathbb{X})$ we know $F(f)\mathsf{Total}(\alpha)_B = F(f)\alpha_B \leq \alpha_A G(f) = \mathsf{Total}(\alpha)_A G(f)$, but since both composites are total this is an equality, as required. It is easy to see that Total preserves composition and idenitities for both 1- and 2-cells and preserves horizontal composition of 2-cells. We therefore have:

Lemma 3.5.7. $\mathsf{Total} : \mathsf{DCR}_s \to \mathsf{Lex}$ *is a 2-functor.*

Having two 2-functors $\mathsf{Total} : \mathsf{DCR}_s \to \mathsf{Lex}$ and $\mathsf{Par} : \mathsf{Lex} \to \mathsf{DCR}_s$ it remains to show that there are invertible strict 2-natural transformations $1_{\mathsf{Lex}} \to \mathsf{Total} \circ \mathsf{Par}$ and $1_{\mathsf{DCR}_s} \to \mathsf{Par} \circ \mathsf{Total}$. We begin with the former, which has components as in:

Lemma 3.5.8. *Let $\mathbb{X}$ be a small category with finite limits. Then there is an isomorphism $\phi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Total}(\mathsf{Par}(\mathbb{X}))$ defined on objects by $\phi_{\mathbb{X}}(X) = X$ and on arrows $f : X \to Y$ by $\phi_{\mathbb{X}}(f) = (1_X, f)$.*

Proof. For $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{X}$ we have $\phi_{\mathbb{X}}(f)\phi_{\mathbb{X}}(g) = (1_X, f)(1_Y, g) = (1_X, fg) = \phi_{\mathbb{X}}(fg)$. Further, we have $\phi_{\mathbb{X}}(1_X) = (1_X, 1_X) = 1_X = 1_{\phi_{\mathbb{X}}}$ and so $\phi_{\mathbb{X}}$ is a functor. Define $\phi_{\mathbb{X}}^{-1} : \mathsf{Total}(\mathsf{Par}(\mathbb{X}))$ on objects by $\phi_{\mathbb{X}}^{-1}(X) = X$ and on arrows $(m, f) : X \to Y$ by $\phi_{\mathbb{X}}^{-1}(m, f) = f$. This is well-defined because an arrow $(m, f) : X \to Y$ of $\mathsf{Par}(\mathbb{X})$ is total if and only if $(m, f) = (1_X, f)$, which also gives immediately that $\phi_{\mathbb{X}}^{-1}$ and $\phi_{\mathbb{X}}$ are inverses, as required. $\square$

Now the required strict 2-natural transformation is given as follows:

Lemma 3.5.9. *There is an invertible strict 2-natural transformation $\phi : 1_{\mathsf{Lex}} \to \mathsf{Total} \circ \mathsf{Par}$ with components $\phi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Total}(\mathsf{Par}(\mathbb{X}))$.*

Proof. We show that this defines a strict 2-natural transformation. To that end, suppose $F : \mathbb{X} \to \mathbb{Y}$ is a 1-cell of Lex. Then we have:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\phi_{\mathbb{X}}"] \arrow[d, "F"'] & \mathsf{Total}(\mathsf{Par}(\mathbb{X})) \arrow[d, "\mathsf{Total}(\mathsf{Par}(F))"] \\ \mathbb{Y} \arrow[r, "\phi_{\mathbb{Y}}"'] & \mathsf{Total}(\mathsf{Par}(\mathbb{Y})) \end{tikzcd}$$

as follows: for objects X of $\mathbb{X}$, $\mathsf{Total}(\mathsf{Par}(F))(\phi_{\mathbb{X}}(X)) = \mathsf{Total}(\mathsf{Par}(F))(X) = \mathsf{Par}(F)(X) = FX$. For arrows $f : X \to Y$, $\mathsf{Total}(\mathsf{Par}(F))(\phi_{\mathbb{X}}(f)) =$
$\mathsf{Total}(\mathsf{Par}(F))(1_X, f) = \mathsf{Par}(F)(1_X, f) = (F(1_X), F(f)) = (1_{FX}, F(f)) = \phi_{\mathbb{Y}}(F(f))$. Thus $\phi_{\mathbb{X}}\mathsf{Total}(\mathsf{Par}(F)) = F\phi_{\mathbb{Y}}$. Moreover for any 1-cells $F, G : \mathbb{X} \to \mathbb{Y}$ and 2-cell $\beta : F \to G$ of Lex we immediately have $\beta \star 1_{\phi_{\mathbb{Y}}} = 1_{\phi_{\mathbb{X}}} \star \mathsf{Total}(\mathsf{Par}(\beta))$. It follows that $\phi : 1_{\mathsf{Lex}} \to \mathsf{Total} \circ \mathsf{Par}$ is a strict 2-natural transformation. We have already seen that each component $\phi_{\mathbb{X}}$ of ϕ is an isomorphism (Lemma 3.5.8), and the claim follows. $\square$

Next, we give the components of the other strict 2-natural transformation that forms our equivalence:

Lemma 3.5.10. *Let $\mathbb{X}$ be a small DCR category. Then there is an isomorphism $\psi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Par}(\mathsf{Total}(\mathbb{X}))$ defined on objects by $\psi_{\mathbb{X}}(X) = X$ and on arrows $f : X \to Y$ by $\psi_{\mathbb{X}}(f) = (m, mf)$ where m splits $\overline{f}$.*

Proof. For $\psi_{\mathbb{X}}(f) = (m, mf)$ to be well-defined we must have that mf is total. Let m be part of the following splitting:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Then we have $\overline{mf} = \overline{m\overline{f}} = \overline{mwm} = \overline{m} = 1_M$ as required. To see that $\psi_{\mathbb{X}}$ preserves identities, let $\psi_{\mathbb{X}}(1_X) = (p, p1_X) = (p, p)$ where p splits $\overline{1_X} = 1_X$ as in:

$$\begin{tikzcd} & P \arrow[r, "p"] & X \\ P \arrow[ur, equal] \arrow[r, "p"'] & X \arrow[u, "b"'] \arrow[ur, "1_X"'] & \end{tikzcd}$$

Then $b : X \to P$ is an isomorphism of spans as in:

$$\begin{tikzcd} & X \arrow[r, "b"] \arrow[dl, "1_X"'] \arrow[drr, "1_X" near end] & P \arrow[dll, "p" near end] \arrow[dr, "p"] & \\ X & & & X \end{tikzcd}$$

and so we have $\psi_{\mathbb{X}}(1_X) = (p, p) = (1_X, 1_X) = 1_{\psi_{\mathbb{X}}(X)}$. To see that $\psi_{\mathbb{X}}$ preserves composition, suppose $f : X \to Y$, $g : Y \to Z$ in $\mathbb{X}$ and let $\psi_{\mathbb{X}}(f) = (m, mf)$ and $\psi_{\mathbb{X}}(g) = (n, ng)$ with m and n part of splittings:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "\overline{f}"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & N \arrow[r, "n"] & Y \\ N \arrow[ur, equal] \arrow[r, "n"'] & Y \arrow[u, "u"'] \arrow[ur, "\overline{g}"'] & \end{tikzcd}$$

Now, let $n' : N' \to M$ split $\overline{mfg}$ as in:

$$\begin{tikzcd} & N' \arrow[r, "n'"] & M \\ N' \arrow[ur, equal] \arrow[r, "n'"'] & M \arrow[u, "u'"'] \arrow[ur, "\overline{mfg}"'] & \end{tikzcd}$$

We show that the following square is a pullback in $\mathsf{Total}(\mathbb{X})$

$$\begin{tikzcd}
N' \arrow[r, "n'mfu"] \arrow[d, "n'"'] & N \arrow[d, "n"] \\
M \arrow[r, "mf"'] & Y
\end{tikzcd}$$

The square commutes as in $n'mfun = m'mf\overline{g} = n'\overline{mfg}mf = n'u'n'mf = n'mf$ and $n'mfu : N' \to N$ is total as in $\overline{n'mfu} = \overline{n'mf\overline{g}} = \overline{n'\overline{mfg}} = \overline{n'u'n'} = \overline{n'} = 1_{N'}$. Suppose we have arrows $k : Z \to M$ and $h : Z \to N$ of $\mathsf{Total}(\mathbb{X})$ such that $kmf = hn$. Then for $ku' : Z \to N'$ we have $kun' = k\overline{mfg} = \overline{kmfg}k = \overline{hng}k = \overline{h\overline{ng}}k = \overline{h}k = k$ and $ku'n'mfu = k\overline{mfg}mfu = kmfuhnu = h$. We show that ku' is the unique such arrow $Z \to N'$. To that end, suppose we have $\alpha : Z \to N'$ with $\alpha n' = k$ and $\alpha n'mfu = h$. Then $\alpha = \alpha n'u' = ku'$ as required. Thus, we have our pullback:

$$\begin{tikzcd}
X \arrow[dr, dashed, "ku'"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& N' \arrow[r, "n'mfu"] \arrow[d, "n'"'] & N \arrow[d, "n"] \\
& M \arrow[r, "mf"'] & Y
\end{tikzcd}$$

In particular this means that $\psi_{\mathbb{X}}(f)\psi_{\mathbb{X}}(g)$ is given by the span:

$$\begin{tikzcd}
& & N' \arrow[dl, "n'"'] \arrow[dr, "n'mfu"] & & \\
& M \arrow[dl, "m"'] \arrow[dr, "mf"] & & N \arrow[dl, "n"'] \arrow[dr, "ng"] & \\
X & & Y & & Z
\end{tikzcd}$$

and we have $n'mfung = n'mf\overline{g}g = n'mfg$, so $\psi_{\mathbb{X}}(f)\psi_{\mathbb{X}}(g) = (n'm, n'mfg)$. Now, let $\psi_{\mathbb{X}}(fg) = (m', m'fg)$ with m' part of the splitting:

$$\begin{tikzcd}
& M' \arrow[r, "m'"] & X \\
M' \arrow[ur, equal] \arrow[r, "m'"'] & X \arrow[u, "w'"'] \arrow[ur, "\overline{fg}"'] &
\end{tikzcd}$$

Notice that $n'm : N' \to X$ also splits $\overline{fg}$, as in:

$$\begin{tikzcd}
 & & N' \arrow[r, "n'"] & M \arrow[r, "m"] & X \\
 & & M \arrow[u, "u'"] \arrow[ur, "\overline{mfg}"'] & & \\
N' \arrow[r, "n'"'] \arrow[uurr, equal] & M \arrow[r, "m"'] \arrow[ur, equal] & X \arrow[u, "w"] \arrow[uurr, "\overline{fg}"'] & &
\end{tikzcd}$$

where the rightmost square commutes via $w\overline{mfg}m = wm\overline{fg} = \overline{f}\,\overline{fg} = \overline{\overline{f}fg} = \overline{fg}$. It follows that $n'mw' : N' \to M'$ and $m'wu' : M' \to N'$ define an isomorphism, which is in fact as isomorphism of spans:

$$\begin{tikzcd}
 & N' \arrow[r, "n'mw'"] \arrow[dl, "n'm"'] \arrow[drr, "n'mfg"' near end] & M' \arrow[dll, "m'" near end] \arrow[dr, "m'fg"] & \\
X & & & Z
\end{tikzcd}$$

via $n'mw'm' = n'm\overline{fg} = n'mwu'n'm = n'm$. Thus $\psi_{\mathbb{X}}(f)\psi_{\mathbb{X}}(g) = (n'm, n'mfg) = (m', m'fg) = \psi_{\mathbb{X}}(fg)$, and it follows that $\psi_{\mathbb{X}}$ is a functor.

Our functor $\psi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Par}(\mathsf{Total}(\mathbb{X}))$ is identity-on-objects, so to show that ψ_X is an isomorphism it suffices to show that it is full and faithful. We begin by showing that $\psi_{\mathbb{X}}$ is full. To that end, suppose $(m, f) : X \to Y$ in $\mathsf{Par}(\mathsf{Total}(\mathbb{X}))$. Now m is a regular monic in $\mathsf{Total}(\mathbb{X})$ and so by Lemma 3.5.5 we have that m is a restriction monic in $\mathbb{X}$. That is, for some w we have:

$$\begin{tikzcd}
 & M \arrow[r, "m"] & X \\
M \arrow[r, "m"'] \arrow[ur, equal] & X \arrow[u, "w"] \arrow[ur, "\overline{wm}"'] &
\end{tikzcd}$$

Now, let $\psi_{\mathbb{X}}(wf) = (m', m'wf)$ where m' is part of the splitting:

$$\begin{tikzcd}
 & M' \arrow[r, "m'"] & X \\
M' \arrow[r, "m'"'] \arrow[ur, equal] & X \arrow[u, "w'"] \arrow[ur, "\overline{wf} = \overline{w\overline{f}} = \overline{w} = \overline{wm}"'] &
\end{tikzcd}$$

Then since m and m' both split $\overline{wm}$ we have that $m'w : M' \to M$ is an isomorphism. In fact, it is an isomorphism of spans:

$$\begin{tikzcd}
 & M' \arrow[r, "m'w"] \arrow[dl, "m'"'] \arrow[drr, "m'wf"' near end] & M \arrow[dll, "m" near end] \arrow[dr, "f"] & \\
X & & & Y
\end{tikzcd}$$

via $m'wm = m'\overline{wm} = m'w'm' = m'$. Then we have $\psi_{\mathbb{X}}(wf) = (m', m'wf) = (m, f)$ and so $\psi_{\mathbb{X}}$ is full.

To see that $\psi_{\mathbb{X}}$ is faithful, suppose $\psi_{\mathbb{X}}(f) = (m, mf) = (n, ng) = \psi_{\mathbb{X}}(g)$ with m and n as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "\overline{f}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & N \arrow[r, "n"] & Y \\ N \arrow[ur, equal] \arrow[r, "n"'] & Y \arrow[u, "u"'] \arrow[ur, "\overline{g}"'] & \end{tikzcd}$$

Then our assumption that $(m, mf) = (n, ng)$ is the assumption that there is an isomorphism of spans $\alpha : M \to N$ as in:

$$\begin{tikzcd} & M \arrow[r, "\alpha"] \arrow[dl, "m"'] \arrow[drr, "mf"', near end] & N \arrow[dr, "ng"] \arrow[dll, "n", near end] & \\ X & & & Y \end{tikzcd}$$

Notice that this gives $mf = \alpha ng = mg$ and $ng = \alpha^{-1}mf = nf$. Then we have $f \leq g$ via $f = \overline{f}f = wmf = wmg = \overline{f}g$ and $g \leq f$ via $g = \overline{g}g = ung = unf = \overline{g}f$. Thus $f = g$, and so $\psi_{\mathbb{X}}$ is faithful. It follows that $\psi_{\mathbb{X}}$ is an isomorphism, as required. $\square$

Now the transformation itself is given as follows:

Lemma 3.5.11. *There is an invertible strict 2-natural transformation* $\psi : 1_{\mathsf{DCR}_s} \to \mathsf{Par} \circ \mathsf{Total}$ *with components* $\psi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Par}(\mathsf{Total}(\mathbb{X}))$.

Proof. We show that this is strict 2-natural. To that end, suppose $F : \mathbb{X} \to \mathbb{Y}$ is a 1-cell of DCR_s. Then we have:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi_{\mathbb{X}}"] \arrow[d, "F"'] & \mathsf{Par}(\mathsf{Total}(\mathbb{X})) \arrow[d, "\mathsf{Par}(\mathsf{Total}(F))"] \\ \mathbb{Y} \arrow[r, "\psi_{\mathbb{Y}}"'] & \mathsf{Par}(\mathsf{Total}(\mathbb{Y})) \end{tikzcd}$$

as follows: for objects X of $\mathbb{X}$ we have $\mathsf{Par}(\mathsf{Total}(F))(\psi_{\mathbb{X}}(X)) = \mathsf{Par}(\mathsf{Total}(F))(X) = \mathsf{Total}(F)(X) = FX = \psi_{\mathbb{Y}}(F(X))$. For arrows $f : \mathbb{X} \to \mathbb{Y}$ of $\mathbb{X}$, let $\psi_{\mathbb{X}}(f) = (m, mf)$ and let $\psi_{\mathbb{Y}}(F(f)) = (m', m'F(f))$ with m and m' as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "\overline{f}"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M' \arrow[r, "m'"] & FX \\ M' \arrow[ur, equal] \arrow[r, "m'"'] & FX \arrow[u, "w'"'] \arrow[ur, "\overline{F(f)}"'] & \end{tikzcd}$$

Notice that $F(m)$ also splits $\overline{F(f)}$ as in:

$$\begin{tikzcd}
 & FM \arrow[r, "F(m)"] & FX \\
FM \arrow[ur, equal] \arrow[r, "F(m)"'] & FX \arrow[u, "F(w)"'] \arrow[ur, "F(\overline{f}) = \overline{F(f)}"'] &
\end{tikzcd}$$

It follows that $m'F(w) : M' \to FX$ is an isomorphism. In fact, it is an isomorphism of spans:

$$\begin{tikzcd}
 & M' \arrow[r, "m'F(w)"] \arrow[dl, "m'"'] \arrow[drr, "m'F(f)"' near end] & FM \arrow[dll, "F(m)" near end] \arrow[dr, "F(mf)"] & \\
FX & & & FY
\end{tikzcd}$$

via $m'F(w)F(m) = m'\overline{F(f)} = m'w'm' = m'$ and $m'F(w)F(mf) = m'\overline{F(f)}F(f) = m'F(f)$. That is, we have $(m', m'F(f)) = (F(m), F(mf))$. In turn this gives $\psi_{\mathbb{Y}}(F(f)) = (m', m'F(f)) = (F(m), F(mf)) = \mathsf{Par}(F)(m, mf) = \mathsf{Par}(\mathsf{Total}(F))(\psi_{\mathbb{X}}(f))$ as required. Moreover, for any 1-cells $F, G : \mathbb{X} \to \mathbb{Y}$ and 2-cell $\beta : F \to G$ we immediately have $\beta \star 1_{\psi_{\mathbb{Y}}} = 1_{\psi_{\mathbb{X}}} \star \mathsf{Par}(\mathsf{Total}(\mathbb{Y}))$. Thus $\psi : 1_{\mathsf{DCR}_s} \to \mathsf{Par} \circ \mathsf{Total}$ is strict 2-natural. We have already seen that each component $\psi_{\mathbb{X}}$ of ψ is an isomorphism (Lemma 3.5.10), and the claim follows. □

Now the promised equivalence of 2-categories follows immediately:

Theorem 3.5.12. *There is strict 2-equivalence:*

$$\begin{tikzcd}
\mathsf{DCR}_s \arrow[r, shift left, "\mathsf{Total}"] \arrow[r, phantom, "\simeq"] & \mathsf{Lex} \arrow[l, shift left, "\mathsf{Par}"]
\end{tikzcd}$$

□

3.6 Varieties and Morita Equivalence

In this section we assemble our results to obtain a *variety theorem* for partial algebraic theories in the form of a syntax-semantics adjunction. Specifically, we briefly recall locally finitely presentable (LFP) categories and the Gabriel-Ulmer duality, which connects LFP categories to categories with finite limits. Our variety theorem then follows immediately: the categories that arise as the models and model morphisms of some partial algebraic theory are precisely the LFP categories. This gives an analogue of Theorem 2.4.28 for partial algebraic theories. We end with a discussion of Morita equivalence for partial algebraic theories, in which we find that two partial algebraic theories present the same category of models and model morphisms if and only if they have equivalent idempotent splitting completions. This gives an analogue of Theorem 2.4.30 for partial algebraic theories.

A consequence of Theorem 3.5.12 is that the categories that arise as the models and model morphisms of some partial algebraic theory correspond to those categories of the form[2] $\mathsf{Lex}(\mathbb{X}, \mathsf{Set})$. Fortunately, categories $\mathsf{Lex}(\mathbb{X}, \mathsf{Set})$ are well-studied in categorical algebra, and turn our to be precisely the *locally finitely presentable* categories [3]. The precise formulation of this correspondence is known as *Gabriel-Ulmer duality*. We briefly recall the Gabriel-Ulmer duality, beginning with a recapitulation of the machinery involved.

The objects of a category can often be equipped with a natural notion of *size*. For example: the size of a set is its cardinality, and the size of a group is the cardinality of its smallest presentation. For our purposes the important objects will be those of *finite* size. Remarkably, there is an element-free notion of finite object:

Definition 3.6.1 ([3, 1.1])**.** In a category $\mathbb{C}$, an object X is said to be *finitely presentable* if the hom-functor $\mathbb{C}(X, -)$ preserves directed colimits.

While this notion of finiteness may appear to be an obscure one, it is quite robust. For example: the finitely presentable objects of Set are precisely the finite sets, and the finitely presentable groups are those admitting a finite presentation.

A category is locally finitely presentable if it can be reconstructed from the collection of finitely presentable objects therein:

Definition 3.6.2 ([3, 1.9])**.** A *locally finitely presentable (LFP)* category $\mathcal{K}$ is a cocomplete category such that:

(i) the full subcategory of $\mathcal{K}$ on the finitely presentable objects therein is small.

(ii) every object of $\mathcal{K}$ is a directed colimit of finitely presentable objects.

LFP categories enjoy a number of satisfying categorical properties [3]. Much as the appropriate notion of morphism for varieties (Section 2.4.4) is that of right adjoints that preserve sifted colimits, the appropriate notion of morphism for LFP categories is that of right adjoints that preserve directed colimits. This leads to the following 2-category of LFP categories:

Definition 3.6.3. LFP is the 2-category of LFP categories, right adjoints that preserve directed colimits, and natural transformations.

Gabriel-Ulmer duality may now be stated as follows:

[2]Strictly speaking it does not make sense to write $\mathsf{Lex}(\mathbb{X}, \mathsf{Set})$, since the 0-cells of Lex are *small* and Set is not small. Nonetheless, there is a category of finite limit preserving functors $\mathbb{X} \to \mathsf{Set}$, which we denote $\mathsf{Lex}(\mathbb{X}, \mathsf{Set})$.

Theorem 3.6.4. *There is a biequivalence:*

$$\mathsf{Lex} \underset{\mathsf{LFP}(-,\mathsf{Set})}{\overset{\mathsf{Lex}(-,\mathsf{Set})}{\rightleftarrows}} \mathsf{LFP}^{\mathsf{op}} \quad (\simeq)$$

□

A classical reference for the proof is [15].

The promised variety theorem follows easily from the foregoing machinery:

Theorem 3.6.5. *There is a biadjunction*

$$\mathsf{DCR} \underset{\mathsf{Th}}{\overset{\mathsf{Mod}}{\rightleftarrows}} \mathsf{LFP}^{\mathsf{op}} \quad (\perp)$$

Proof. We promote the strict 2-equivalence of Theorem 3.5.12 to a biequivalence and compose it with the biadjunction of Lemma 3.2.15 on the left and with the Gabriel-Ulmer duality on the right to obtain a biadjunction:

$$\mathsf{DCR} \overset{\mathsf{Split}}{\underset{\perp}{\rightleftarrows}} \mathsf{DCR}_s \underset{\mathsf{Par}}{\overset{\mathsf{Total}}{\underset{\simeq}{\rightleftarrows}}} \mathsf{Lex} \underset{\mathsf{LFP}(-,\mathsf{Set})}{\overset{\mathsf{Lex}(-,\mathsf{Set})}{\underset{\simeq}{\rightleftarrows}}} \mathsf{LFP}^{\mathsf{op}}$$

□

It may not be immediately clear what this tells us about the category of models and model morphisms of a particular partial algebraic theory, so let us briefly discuss. Consider an arbitrary partial algebraic theory $\mathbb{X}$. Par is a split restriction category, so models of $\mathbb{X}$ and models of $\mathsf{Split}(\mathbb{X})$ coincide since the image of any restriction idempotent of $\mathbb{X}$ already splits in Par. Thus the category of models of $\mathbb{X}$ and model morphisms thereof is $\mathsf{DCR}_s(\mathsf{Split}(\mathbb{X}), \mathsf{Par})$. Transporting this across the equivalence of Theorem 3.5.12 yields $\mathsf{Lex}(\mathsf{Total}(\mathsf{Split}(\mathbb{X})), \mathsf{Set})$, which is LFP. Conversely for any LFP category $\mathbb{C}$ we know that $\mathsf{Par}(\mathsf{LFP}(\mathbb{C}, \mathsf{Set}))$ is a partial algebraic theory with $\mathbb{C}$ as its category of models. We may conclude that the categories of models of partial algebraic theories correspond to the LFP categories.

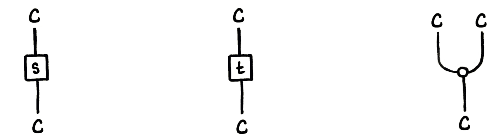
Dropping the idempotent splitting biadjunction from our syntax-semantics duality yields a biequivalence $\mathsf{DCR}_s \simeq \mathsf{LFP}^{\mathsf{op}}$, which in particular characterizes Morita equivalence of partial algebraic theories:

Theorem 3.6.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be partial algebraic theories. Then $\mathbb{X}$ and $\mathbb{Y}$ present equivalent categories of models and model morphisms if and only if $\mathsf{Split}(\mathbb{X})$ and $\mathsf{Split}(\mathbb{Y})$ are equivalent.*

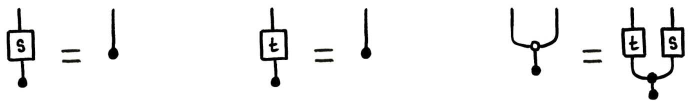
□

This is rather encouraging, in that Morita equivalence works the same way for partial algebraic theories that it does for algebraic theories. Consider for example the difference between the two-sorted theory of categories given in Example 3.3.20 and the single sorted theory of categories:

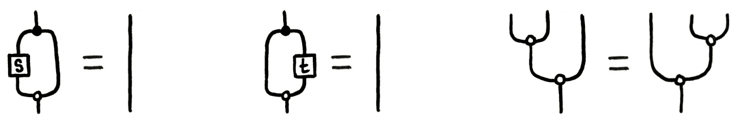
Example 3.6.7 (Categories, Again)**.** Consider the partial term presentation $(\Gamma_{\mathsf{Cat}'}, E_{\mathsf{Cat}'})$ over a single sort $\mathfrak{s}(\Gamma_{\mathsf{Cat}'}) = \{C\}$ representing the arrows of the category, and generators $\Gamma_{\mathsf{Cat}'}$ corresponding to the source, target, and composition operations:



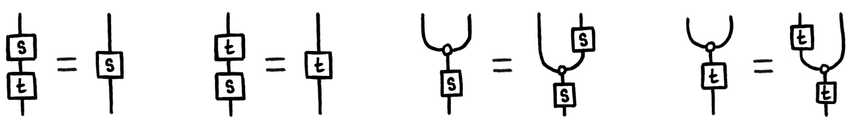
In the single-sorted presentation of categories objects are represented by their identity morphisms. The source and target operations must be total, and again we ask that the composite of two morphisms is defined if and only if the target of the first matches the source of the second. As such, we ask that $E_{\mathsf{Cat}'}$ contain the following equations:



Additionally, we require equations to the effect that the composition operation is associative and unital:



and finally we require equations to the effect that the source (target) of an identity map is the object it represents, and that the source (target) of a composite is the source (target) of the first (second) component:



It is well known that the single-sorted presentation of the theory of categories is equivalent to the two-sorted one, so we already know that splitting idempotents in the two partial algebraic theories will yield equivalent categories. Nonetheless, we feel that it is instructive to consider this in some detail. Notice that in the single-sorted theory of categories the source and target operations are idempotent as in $ss = sts = st = s$ and $tt = tst = ts = t$.

Further, there is an isomorphism $(C, s) \cong (C, t)$ in the idempotent splitting completion given by $s : (C, s) \to (C, t)$ and $t : (C, t) \to (C, s)$. Now $(C, t) \cong (C, s)$ correspond to the sort of objects in the two-sorted presentation, and $(C, 1_C)$ corresponds to the sort of arrows. The generator $id : A \to O$ that maps each object to its corresponding identity morphism in the two-sorted presentation is represented in the single-sorted presentation by $t : (C, t) \to (C, 1_C)$ and $s : (C, s) \to (C, 1_C)$. Constructing an equivalence of categories between the idempotent splittings of our two theories is now routine, giving an alternate proof that the two different presentations of the theory of categories are Morita equivalent. That is, that $\mathsf{Split}(\mathsf{P}(\Gamma_{\mathsf{Cat}'}, E_{\mathsf{Cat}'})) \simeq \mathsf{Split}(\mathsf{P}(\Gamma_{\mathsf{Cat}}, E_{\mathsf{Cat}}))$. It is intriguing that the two-sorted presentation of the theory of categories is the more canonical one in this sense.

Chapter 4

Relational Algebraic Theories

The aim of this chapter is to develop a notion of *relational* algebraic theory analogous to the algebraic theories and partial algebraic theories treated in previous sections, but with the important difference that the operations are treated as *relations*, as opposed to functions or partial functions. More precisely, while models of algebraic theories are valued in the category Set of sets and (total) functions, and models of partial algebraic theories are valued in the category Par of sets and partial functions, models of relational algebraic theories will be valued in the category Rel of sets and relations. To begin, we recall:

Definition 4.0.1. The category Rel of *sets and relations* is given as follows:

objects are sets.

arrows $f : X \to Y$ are subsets of $X \times Y$.

composition of $f : X \to Y$ and $g : Y \to Z$ is given by:

$$fg = \{(x, z) \in X \times Z \mid \exists y \in Y.(x, y) \in f \wedge (y, z) \in g\}$$

The **identity** on X is $1_X = \{(x, x) \mid x \in \mathbb{X}\}$

It is straightforward to verify that Rel is a category. Further, notice that there is a partial order on the hom-sets $\mathsf{Rel}(X, Y)$ given by subset inclusion. In fact, Rel is enriched in the category of posets and monotone functions.

In the same way that Cartesian monoidal categories (algebraic theories) describe the behavior of the Cartesian product of sets in Set and DCR categories (partial algebraic theories) describe the behavior of the Cartesian product of sets in Par, relational algebraic theories will describe the behavior of the Cartesian product of sets in Rel. We proceed to develop the categorical structure that will play this role.

4.1 CW Categories

We have seen that symmetric monoidal categories with certain additional structure capture the behavior of the Cartesian product of sets in the category Set of sets and functions (Cartesian monoidal categories), and in the category Par of sets and partial functions (DCR categories). We proceed to show how this scheme extends to the category Rel of sets and relations. Specifically, we introduce what we will call *Carboni-Walters categories* (*CW categories*), which capture the behavior of the Cartesian product of sets in Rel. We note that our CW categories are more commonly called “Cartesian bicategories of relations”, following the terminology of the paper in which they first appear [14]. In the context of modern category theory this name is misleading, and so we adopt the alternative proposed in [7].

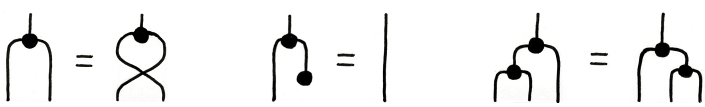
Much as Cartesian monoidal categories are characterised by commutative comonoid structure, CW categories will be characterised by commutative special Frobenius algebra structure:

Definition 4.1.1. Let $\mathbb{X}$ be a symmetric strict monoidal category. A *commutative special Frobenius algebra* in $\mathbb{X}$ is a 5-tuple $(X, \delta_X, \mu_X, \varepsilon_X, \eta_X)$, as in

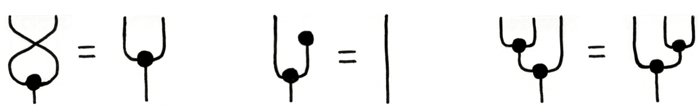
$\delta_X \leftrightsquigarrow$  $\mu_X \leftrightsquigarrow$  $\varepsilon_X \leftrightsquigarrow$  $\eta_X \leftrightsquigarrow$ 

such that

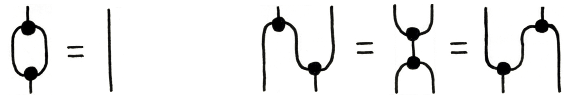
(i) $(X, \delta_X, \varepsilon_X)$ is a commutative comonoid:



(ii) (X, μ_X, η_X) is a commutative monoid:



(iii) μ_X and δ_X satisfy the special and Frobenius equations:

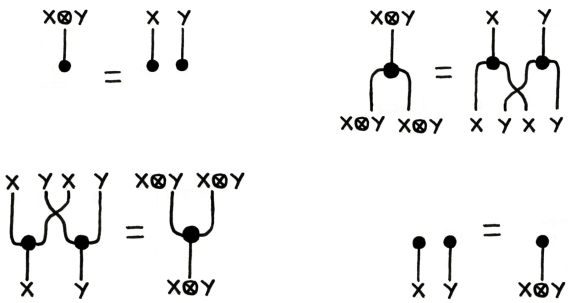


An intermediate notion is that of a hypergraph category, in which objects are coherently equipped with commutative special Frobenius algebra structure:

Definition 4.1.2 ([27, 2.12]). A symmetric monoidal category $\mathbb{X}$ is called a *hypergraph category* in case:

(i) Each object X of $\mathbb{X}$ is equipped with a commutative special Frobenius algebra.

(ii) The Frobenius algebra structure is coherent, i. e., for all X, Y we have:

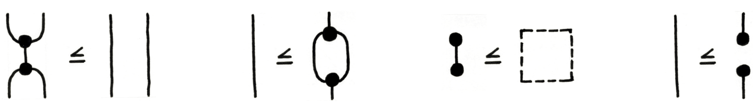


Now a CW category is a poset-enriched hypergraph category satisfying certain additional equations:

Definition 4.1.3 ([14, 2.1]). A *CW category* is a poset-enriched hypergraph category $\mathbb{X}$ such that:

(i) The comonoid structure is *lax natural*. That is, for all arrows f of $\mathbb{X}$:

(ii) Each of the Frobenius algebras satisfy:



Example 4.1.4. The category Rel is a CW category with

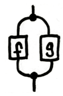
$$\delta_X = \{(x, (x, x)) \mid x \in X\} \qquad \mu_X = \{((x, x), x) \mid x \in X\}$$

$$\varepsilon_X = \{(x, *) \mid x \in X\} \qquad \eta_X = \{(*, x) \mid x \in X\}$$

where $*$ is the unique element of the singleton set $I = \{*\}$.

In a CW category every hom-set admits binary meets:

Lemma 4.1.5 ([7, 4.14])**.** *Every CW category has meets of parallel arrows, with the meet of $f, g : X \to Y$ defined by $f \cap g = \delta_X(f \otimes g)\mu_X$. As a string diagram:*

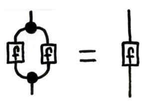


Further, the meet determines the poset-enrichment in that $f \leq g \Leftrightarrow f \cap g = f$.

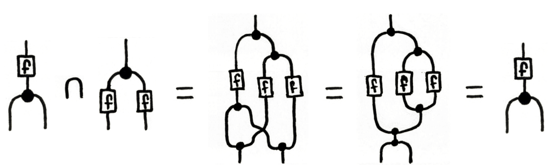
□

We have defined CW categories by taking the ordering on hom-sets as primitive and imposing axioms to govern its behavior (Definition 4.1.3). We point out that it is also possible to axiomatise the meet operator directly, deriving the ordering. This way of presenting the axioms is simpler, requiring only one family of equations beyond those of a hypergraph category. Specifically, we have:

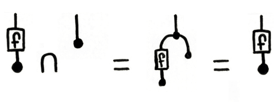
Lemma 4.1.6. *A CW category is precisely a hypergraph category in which for each arrow $f : X \to Y$ we have $\delta_X(f \otimes f)\mu_Y = f$, as in:*



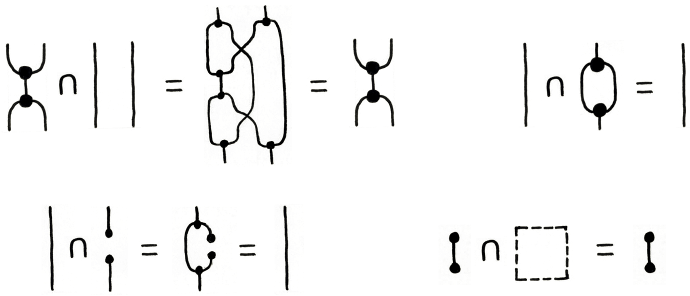
Proof. Let $\mathbb{X}$ be a hypergraph category, and suppose that for all arrows f of $\mathbb{X}$ we have $\delta_X(f \otimes f)\mu_Y = f$. Define $f \cap g$ and $f \leq g$ as in Lemma 4.1.5. It is straightforward to verify that this is a preorder-enrichment of $\mathbb{X}$. We show that the conditions of Definition 4.1.3 are satisfied, beginning with lax naturality: For any arrow f of $\mathbb{X}$ we have



and also



and so f is lax natural. Next, we show that the Frobenius algebra structure satisfies the required inequations:



Thus, $\mathbb{X}$ is a CW category. Conversely, if $\mathbb{X}$ is a CW category then $\mathbb{X}$ is a hypergraph category, and we have immediately that $f \cap f = f$, as required. $\square$

It should come as no surprise that the notion of structure-preserving functor between CW categories is the one that preserves the Frobenius algebra structure:

Definition 4.1.7. A *CW functor* $F : \mathbb{X} \to \mathbb{Y}$ of CW categories $\mathbb{X}, \mathbb{Y}$ is a symmetric monoidal functor that preserves the Frobenius algebra structure as in:

$$F(\delta_X) = \delta_{FX}\phi^F_{X,X} \qquad\qquad F(\varepsilon_X) = \varepsilon_{FX}\phi^F_I$$

$$F(\mu_X) = \phi^{F(-1)}_{X,X}\mu_{FX} \qquad\qquad F(\eta_X) = \phi^{F(-1)}_I\eta_{FX}$$

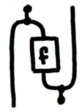
When working with CW functors we will continue to systematically omit the coherence isomorphisms. Like DCR categories, the correct notion of 2-cell between CW functors turns out to be that of a monoidal lax transformation (Definition 3.1.15 and Definition 3.1.16). Small CW categories, CW functors, and monoidal lax transformations form a 2-category CW. This 2-category will play an important role in our development of relational algebraic theories, analogous to the role of CM for algebraic theories and DCR for partial algebraic theories.

Definition 4.1.8. CW is the 2-category of small CW categories, CW functors, and monoidal lax transformations.

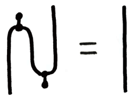
CW categories have a lot of structure. A good place to start exploring this structure is with the analogue of the relational converse. In Rel, for any $f \subseteq X \times Y$ we may define its converse $f^\circ \subseteq Y \times X$ by $f^\circ = \{(y, x) \mid (x, y) \in f\}$. What follows is an abstract version of this that makes sense in any CW category:

Lemma 4.1.9 ([14, 2.4])**.** *Every CW category $\mathbb{X}$ has an identity-on-objects contravariant involution $(-)^\circ : \mathbb{X}^{\mathsf{op}} \to \mathbb{X}$ which maps $f : X \to Y$ to $f^\circ = (\eta_X\delta_X \otimes$*

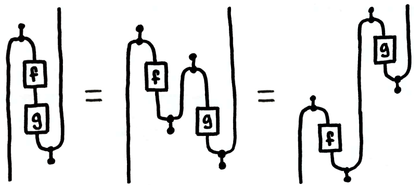
$1_Y)(1_X \otimes f \otimes 1_Y)(1_X \otimes \mu_Y \varepsilon_Y) : Y \to X$ *as in:*



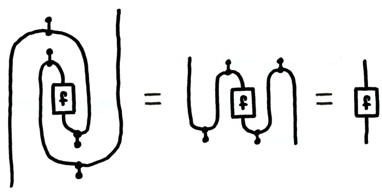
Proof. We have $1_X^\circ = 1_X$ as in:



We have $(fg)^\circ = g^\circ f^\circ$ as in:



and finally we have $f^{\circ\circ} = f$ as in:



$\square$

Notice in particular that this is indeed the relational converse when specialised to Rel. This generalised converse operation is well-behaved. For example, we have:

Lemma 4.1.10. *Let $\mathbb{X}$ be a CW category. Then:*

(i) $(f \otimes g)^\circ = f^\circ \otimes g^\circ$

(ii) $(f \cap g)^\circ = g^\circ \cap f^\circ$

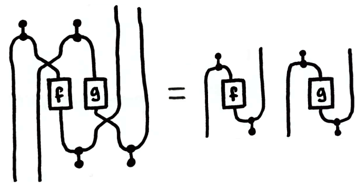
(iii) If $f \leq g$ *then* $f^\circ \leq g^\circ$

(iv) $\delta_X^\circ = \mu_X$ *and* $\varepsilon_X^\circ = \eta_X$

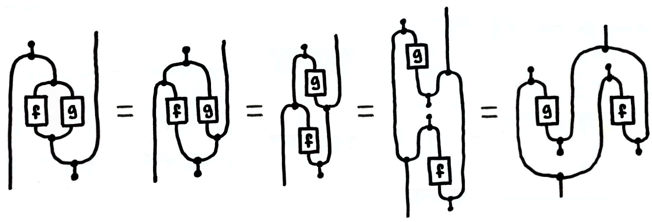
(v) $\mu_X f \leq (f \otimes f)\mu_Y$ *and* $\eta_X f \leq \eta_Y$ *for all* $f : X \to Y$

(vi) $f \leq f f^\circ f$

Proof. (i) We have $(f \otimes g)^\circ = f^\circ \otimes g^\circ$ as in:

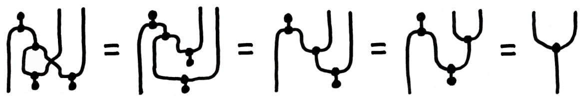


(ii) We have $(f \cap g)^\circ = g^\circ \cap f^\circ$ as in:

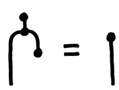


(iii) $f \leq g$ if and only if $f \cap g = f$, but then we have $f^\circ \cap g^\circ = (f \cap g)^\circ = f^\circ$ and so $f^\circ \leq g^\circ$.

(iv) We have $\delta_X^\circ = \mu_X$ as in:

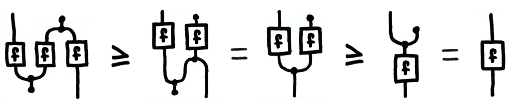


and we have $\varepsilon_X^\circ = \eta_X$ as in:



(v) The axioms of a CW category give $f\delta_Y \leq \delta_X(f \otimes f)$ and $f\varepsilon_Y \leq \varepsilon_X$ directly. Then we have $(\mu_X f) = (f^\circ \delta_X)^\circ \leq (\delta_Y(f^\circ \otimes f^\circ))^\circ = (f \otimes f)\mu_Y$ and $\eta_X f = (f^\circ \varepsilon_X)^\circ \leq \varepsilon_Y^\circ = \eta_Y$ as required.

(vi) We have $ff^\circ f \geq f$ as in:



$\square$

We pay special attention to the arrows in a CW category that are natural with respect to one or more of the generators of the Frobenius algebra structure. Specifically:

Definition 4.1.11 ([7, Section 4]). An arrow $f : X \to Y$ in a CW category is called:

(i) *simple* in case $f\delta_Y = \delta_X(f \otimes f)$ (if and only if $\delta_X(f \otimes f) \leq f\delta_Y$))

(ii) *total* in case $f\varepsilon_Y = \varepsilon_X$ (if and only if $\varepsilon_X \leq f\varepsilon_Y$)

(iii) *surjective* in case $\eta_X f = \eta_Y$ (if and only if ($\eta_Y \leq \eta_X f$)

(iv) *injective* in case $\mu_X f = (f \otimes f)\mu_Y$ (if and only if $(f \otimes f)\mu_Y \leq \mu_X f$)

Notice that in Rel an arrow $f : X \to Y$ is simple in case each $x \in X$ is related to at most one $y \in Y$. Similarly, f is total in case each $x \in X$ is related to at least one $y \in Y$, is surjective in case every $y \in Y$ is related to at least one $x \in X$, and is injective in case each $y \in Y$ is related to at most one $x \in X$. It is useful to consider the meaning of the above properties in Rel to form an intuition about their behavior in a general CW category, since that intuition will usually be correct.

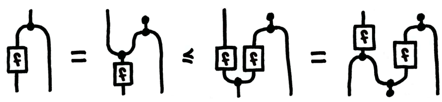
Perhaps surprisingly, the properties named in Definition 4.1.11 all admit characterisations in terms of the generalised converse operation. We first require a technical lemma:

Lemma 4.1.12 ([14, 2.4]). *Let $f : X \to Y$ be an arrow in a CW category. Then:*

(i) $\delta_X(f \otimes 1_X) \leq f\delta_Y(1_X \otimes f^\circ)$

(ii) $(f \otimes 1_Y)\mu_Y \leq (1_X \otimes f^\circ)\mu_X f$

Proof. (i) We have:



(ii) $(f \otimes 1_Y)\mu_Y = (\delta_Y(f^\circ \otimes 1_Y))^\circ \leq (f^\circ\delta_X(1_X \otimes f))^\circ = (1_X \otimes f^\circ)\mu_X f$. ∎

Technical lemma in hand, we obtain the promised characterisation of Definition 4.1.11 in terms of the generalised converse:

Lemma 4.1.13 ([7, 4.4]). *Let $f : X \to Y$ be an arrow in a CW category. Then:*

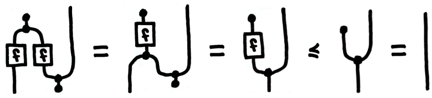
(i) f *is simple if and only if* $f^\circ f \leq 1_Y$

(ii) f *is total if and only if* $1_X \leq ff^\circ$

(iii) f *is surjective if and only if* $1_Y \leq f^\circ f$

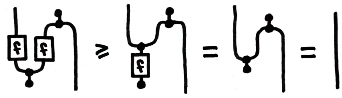
(iv) f *is injective if and only if* $ff^\circ \leq 1_X$

Proof. (i) If f is simple then $f^\circ f \leq 1_Y$ as in:



Conversely, if $f^\circ f \leq 1_Y$ then we have $\delta_X(f \otimes f) \leq f\delta_Y(1_Y \otimes f^\circ f) \leq f\delta_Y$ and so f is simple.

(ii) If f is total then $ff^\circ \geq 1_X$ as in:



Conversely, if $1_X \leq ff^\circ$ then f is total via $\varepsilon_Y \leq ff^\circ\varepsilon_Y \leq f\varepsilon_Y$.

(iii) Similar to (ii).

(iv) Similar to (iii).

□

In Rel, the arrows that are both simple and total are the *functions*. The class of simple and total arrows in a CW category will be of particular importance to our development, warranting a name:

Definition 4.1.14 ([14, 1.5])**.** An arrow $f : X \to Y$ in a CW category is called a *map* in case it is both simple and total.

A good analogy is that the maps of a CW category are like the total arrows of a restriction category (Definition 3.1.1). Like total arrows in a restriction category, the maps in a CW category form a subcategory:

Lemma 4.1.15 ([14, 1.6])**.** *The maps of a CW category* $\mathbb{X}$ *form a subcategory* $\mathsf{Map}(\mathbb{X})$*. Moreover, this category is Cartesian monoidal.*

Proof. It is clear that maps compose, and that identity morphisms are maps. That $\mathsf{Map}(\mathbb{X})$ is Cartesian monoidal then follows immediately from Theorem 2.4.5. □

For example, we have $\mathsf{Map}(\mathsf{Rel}) \cong \mathsf{Set}$ since the maps in Rel are precisely the functions.

As an aside, we acknowledge that the word "map" is commonly used alongside "morphism" and "arrow" as a generic term for a morphism in a category. We must be careful to avoid this usage when working with CW categories, since the maps enjoy a number of properties that arbitrary arrows do not. Specifically, we have:

Lemma 4.1.16 ([32, 2.1])**.** *For* f *and* g *maps in a CW category* $\mathbb{X}$*, we have:*

(i) $f = ff^\circ f$

(ii) *If* $f \leq g$ *then* $f = g$

(iii) *If* f° *is a map then* f *is an isomorphism in* $\mathsf{Map}(\mathbb{X})$ *with* $f^{-1} = f^\circ$

(iv) *A map* $f : X \to Y$ *is monic if and only if it is injective*

(v) *A map* $f : X \to Y$ *is epic if and only if it is surjective*

Proof. (i) f simple gives $ff^\circ f \leq f$ and f total gives $f \leq ff^\circ f$, so $f = ff^\circ f$ by antisymmetry.

(ii) $f \leq g$ gives $f^\circ \leq g^\circ$ and we have $g \leq f$ as in $g \leq ff^\circ g \leq fg^\circ g \leq f$, which gives $f = g$ by antisymmetry.

(iii) $f : X \to Y$ simple and total gives $f^\circ f \leq 1_Y$ and $1_X \leq ff^\circ$. Then since f° is a map so are ff° and $f^\circ f$, and so both of these inequations are in fact equations.

(iv) Suppose $f : X \to Y$ is monic. Then $f = ff^\circ f$ gives $1_X = ff^\circ$, which gives that f is injective. For the converse, suppose that f is injective and that $g_0 f = g_1 f$. Since f is total and injective we have $ff^\circ = 1_X$ which gives $g_0 = g_0 ff^\circ = g_1 ff^\circ = g_1$, and it follows that f is monic.

(v) Suppose $f : X \to Y$ is epic. Then $f = ff^\circ f$ gives $1_Y = f^\circ f$, which in turn gives that f is surjective. For the converse, suppose that f is surjective and that $fg_0 = fg_1$. Since f is surjective and simple we have $1_Y = f^\circ f$ which gives $g_0 = f^\circ f g_0 = f^\circ f g_1 = g_1$, and it follows that f is epic.

□

The components of monoidal lax transformations between CW functors are necessarily maps:

Lemma 4.1.17 ([7, 4.17]). *If* $\mathbb{X}, \mathbb{Y}$ *are CW categories,* $F, G : \mathbb{X} \to \mathbb{Y}$ *are CW functors, and* $\alpha : F \to G$ *is a monoidal lax transformation, then each component* $\alpha_X : FX \to GX$ *of* α *is necessarily a map.*

Proof. We have $\delta_{FA}(\alpha_A \otimes \alpha_A) \leq \alpha_A \delta_{GA}$ and $\varepsilon_{FA} \leq \alpha_A \varepsilon_{GA}$ as in:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "\delta_{FA} = F(\delta_A)"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "\delta_{GA} = G(\delta_A)"] \\ FA \otimes FA \arrow[r, "\alpha_A \otimes \alpha_A"'] & GA \otimes GA \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "\varepsilon_{FA} = F(\varepsilon_A)"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "\varepsilon_{GA} = G(\varepsilon_A)"] \\ I \arrow[r, "\alpha_I = 1_I"'] & I \end{tikzcd}$$

The converse of each of these inequalities holds for any arrow of a CW category, and it follows that α_A is a map. □

Much as with lax transformations of CR functors, asking lax transformations of CW functors to have components that are maps is equivalent to asking that they are monoidal:

Lemma 4.1.18. *Let $F, G : \mathbb{X} \to \mathbb{Y}$ be CW functors, and let $\alpha : F \to G$ be a lax transformation with components that are maps, in the sense that α_X is a map for each X of $\mathbb{X}$. Then α is necessarily monoidal.*

Proof. As in the proof of Lemma 3.1.18. □

Notice the similarity to Lemma 3.1.17, in which it is shown that the components of a monoidal lax transformation between CR functors are necessarily total. While it is not directly related to our development, we point out a further connection between CW categories and DCR categories:

Lemma 4.1.19. *Let $\mathbb{X}$ be a CW category, and let $\mathsf{Sim}(\mathbb{X})$ be the subcategory of simple arrows in $\mathbb{X}$. Then $\mathsf{Sim}(\mathbb{X})$ is a DCR category.*

Proof. Immediate from Theorem 3.2.4. □

In fact, it is straightforward to verify that $\mathsf{Sim}(\mathbb{X})$ is a *regular restriction category*, as defined in [18]. We will see later on how maps in a CW category are related to regular categories (without the restriction structure), pointing to a larger story about regularity that we will not pursue in this thesis.

As in the case of algebraic theories and partial algebraic theories, our results concerning relational algebraic theories involve splitting certain species of idempotent. We introduce the relevant notions now:

Definition 4.1.20 ([32, 2.1])**.** An arrow $f : X \to X$ in a CW category is called:

(i) *reflexive* if $1 \leq f$

(ii) *symmetric* if $f^\circ \leq f$

(iii) *transitive* if $ff \leq f$

(iv) *coreflexive* if $f \leq 1$

(v) a *partial equivalence relation* (*PER*) if it is symmetric and transitive

(vi) a *equivalence relation* if it is reflexive, symmetric, and transitive

These notions are connected in a straightforward way:

Lemma 4.1.21. *Let $f : X \to X$ in a CW category. Then:*

(i) If f is coreflexive then it is idempotent

(ii) If f is reflexive and transitive then f is idempotent

(iii) If f is symmetric and transitive then f is idempotent

(iv) f is symmetric and idempotent if and only if f is a PER

(v) f is coreflexive if and only if it is a simple PER

Proof. (i) We have $ff \leq f$ immediately and $ff = f(1 \cap f) = f \cap ff \leq f$, so by antisymmetry $ff = f$ as required.

(ii) $1 \leq f$ gives $1 \cap f = 1$ and $ff \leq f$ gives $ff \cap f = ff$. Then we have $f = f(1 \cap f) = f \cap ff = ff$.

(iii) $f \leq ff^\circ f \leq fff \leq ff$.

(iv) If f is a PER then it is symmetric and transitive, and so is also idempotent as shown above. Conversely, if f is a symmetric idempotent then it is trivially transitive, and is therefore a PER.

(v) If f is coreflexive then immediately $f^\circ \leq 1$ and so $f^\circ f \leq 1$ and f is simple. Further $f \leq ff^\circ f \leq f^\circ$ gives $f = f^\circ$ so f is symmetric. We have already seen that all coreflexives are idempotent, and it follows that f is a PER. Conversely, if f is a simple PER then we have $f = ff = f^\circ f \leq 1$ as required. □

Splittings of coreflexives in a CW category behave very well:

Lemma 4.1.22 ([32, 2.163])**.** *Let $a : X \to X$ be coreflexive in a CW category $\mathbb{X}$, and suppose that a splits as in:*

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "w"'] \arrow[ur, "a"'] & \end{tikzcd}$$

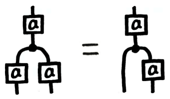
Then $w = m^\circ$, and m is an injective map.

Proof. First $a \leq 1$ gives $a^\circ \leq 1^\circ = 1$ and we have $w^\circ = mww^\circ \leq mm^\circ mww^\circ = mm^\circ w^\circ = m(wm)^\circ = ma^\circ \leq m$ and conversely $m^\circ = m^\circ mw \leq m^\circ mww^\circ w = m^\circ w^\circ w = (wm)^\circ w = a^\circ w \leq w$, and so $m = m^{\circ\circ} \leq w^\circ$ which gives $w^\circ = m$ by antisymmetry, and of course we also have $m^\circ = w$.

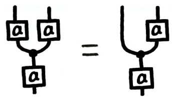
Notice that $mm^\circ = 1_M$ gives that m is total and injective. For m simple we have $m^\circ m = wm = a \leq 1_X$, and it follows that m is a map. □

PERs in a CW category have a useful property:

Lemma 4.1.23. *Let $a : X \to X$ be a PER in a CW category $\mathbb{X}$. Then we have $a\delta_X(a \otimes a) = a\delta(1_X \otimes a)$ and $(a \otimes a)\mu_X a = (1_X \otimes a)\mu_X a$ as in:*



and



Moreover we have $a\delta_X(a \otimes a) = a\delta_X(a \otimes 1_X)$ and $(a \otimes a)\mu_X a = (a \otimes 1_X)\mu_X a$.

Proof. Lemma 4.1.12 gives $a\delta_X(a \otimes a) \leq aa\delta_X(1_X \otimes a^\circ a) = a\delta_X(1_X \otimes aa) = a\delta_X(1_X \otimes a)$. Conversely we have $a\delta_X(1_X \otimes a) = aa\delta_X(1_X \otimes a) \leq a\delta_X(a \otimes aa) = a\delta_X(a \otimes a)$, and so by antisymmetry we have $a\delta_X(a \otimes a) = a\delta_X(1_X \otimes a)$. The proof that $(a \otimes a)\mu_X a = (1_X \otimes a)\mu_X a$ is similar, as is the proof that $a\delta_X(a \otimes a) = a\delta_X(a \otimes 1_X)$ and $(a \otimes a)\mu_X a = (a \otimes 1_X)\mu_X a$. $\square$

This allows us to prove a version of Lemma 2.2.8 for CW categories:

Proposition 4.1.24 ([32, 2.169])**.** *Let $\mathbb{X}$ be a CW category and $\mathcal{E}$ be a collection of PERs in $\mathbb{X}$ such that $1_X \in \mathcal{E}$ for all objects A of $\mathbb{X}$. Then:*

(i) $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ *is a CW category.*

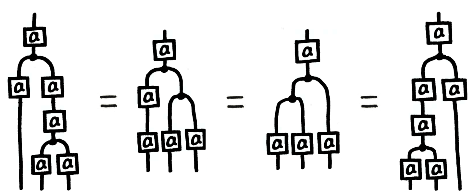
(ii) There is an embedding

$$[\![-]\!] : \mathbb{X} \hookrightarrow \mathsf{Split}_{\mathcal{E}}(\mathbb{X})$$

that preserves the CW category structure.

Proof. (i) For the CW structure, $\delta_{(X,a)}$,$\varepsilon_{(X,a)}$,$\mu_{(X,a)}$, and $\eta_{(X,a)}$ are, respectively:

and it is straightforward to verify that this structure satisfies the axioms of a CW category by using Lemma 4.1.23. For example, we have that $\delta_{(X,a)}$ is coassociative as in:



(ii) The inclusion is defined by $[\![X]\!] = (X, 1_X)$ on objects and by $[\![f]\!] = f$ on arrows. Clearly this preserves the CW structure.

$\square$

Let $\mathsf{CW}_{\mathsf{per}}$ be the full sub 2-category of CW on the 0-cells $\mathbb{X}$ in which the PERs split. In the same way that splitting idempotents extends to a 2-functor $\mathsf{Cat} \to \mathsf{Cat}_s$, splitting PERs extends to a 2-functor $\mathsf{CW} \to \mathsf{CW}_{\mathsf{per}}$ as follows:

Lemma 4.1.25. *There is a 2-functor* $\mathsf{Split}_{\mathsf{per}} : \mathsf{CW} \to \mathsf{CW}_{\mathsf{per}}$ *given as follows:*

On 0-cells $\mathsf{Split}_{\mathsf{per}}$ *sends* $\mathbb{X}$ *to* $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$.

On 1-cells $F : \mathbb{X} \to \mathbb{Y}$ *we define* $\mathsf{Split}_{\mathsf{per}}(F) : \mathsf{Split}_{\mathsf{per}}(\mathbb{X}) \to \mathsf{Split}_{\mathsf{per}}(\mathbb{Y})$ *by* $\mathsf{Split}_{\mathsf{per}}(F)(X, a) = (FX, F(a))$ *on objects and by* $\mathsf{Split}_{\mathsf{per}}(F)(f) = F(f)$ *on arrows.*

On 2-cells $\alpha : F \to G$ *the components of* $\mathsf{Split}_{\mathsf{per}}(\alpha) : \mathsf{Split}_{\mathsf{per}}(F) \to \mathsf{Split}_{\mathsf{per}}(G)$ *are given by* $\mathsf{Split}_{\mathsf{per}}(\alpha)_{(X,a)} = F(a)\alpha_X G(a)$.

Proof. That each $\mathsf{Split}_{\mathsf{per}}(F)$ is a CW functor is immediate. We must show that each $\mathsf{Split}_{\mathsf{per}}(\alpha)$ is a monoidal lax transformation. Then for any $f : (X, a) \to (Y, b)$ in $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$ we have $\mathsf{Split}_{\mathsf{per}}(F)(f)\,\mathsf{Split}_{\mathsf{per}}(\alpha)_{(Y,b)} = Fa\,Ff\,\alpha_Y\,Gb \leq Fa\,\alpha_X\,Ga\,Gf = \mathsf{Split}_{\mathsf{per}}(\alpha)_{(X,a)}\mathsf{Split}_{\mathsf{per}}(G)(f)$, and so $\mathsf{Split}_{\mathsf{per}}(\alpha)$ is lax natural. It remains to show that the components of $\mathsf{Split}_{\mathsf{per}}(\alpha)$ are maps. We have $\mathsf{Split}_{\mathsf{per}}(\alpha)(X, a)\varepsilon_{G(X,a)} = F(a)\alpha_X G(a)\varepsilon_{GX} \geq F(a)\alpha_X\varepsilon_{GX} = F(a)\varepsilon_{FX} = \varepsilon_{F(X,a)}$ and $\mathsf{Split}_{\mathsf{per}}(\alpha)_{(X,a)}\delta_{G(X,a)} = F(a)\alpha_X G(a)\delta_{GX}(G(a)\otimes G(a)) \geq F(a)\alpha_X\delta_{GX}(G(a)\otimes G(a)) = F(a)\delta_{FX}\alpha_X(G(a)\otimes G(a)) \geq F(a)\delta_{FX}(F(a)\otimes F(a))(F(a)\alpha_X G(a)\otimes F(a)\otimes G(a)) = \delta_{F(X,a)}(\mathsf{Split}_{\mathsf{per}}(\alpha)_X \otimes \mathsf{Split}_{\mathsf{per}}(\alpha)_X)$ as required. It follows that $\mathsf{Split}_{\mathsf{per}}(\alpha)$ is a monoidal lax transformation. It is straightforward to verify that $\mathsf{Split}_{\mathsf{per}}$ preserves composition and identities for 1-cells and 2-cells, and preserves horizontal composition of 2-cells. The claim follows. ∎

We also obtain a version of Lemma 2.3.9 for CW categories:

Lemma 4.1.26. *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{CW} \arrow[r, shift left, "\mathsf{Split}_{\mathsf{per}}"] \arrow[r, phantom, "\perp"] & \mathsf{CW}_{\mathsf{per}} \arrow[l, hook, shift left] \end{tikzcd}$$

where the right adjoint $\mathsf{CW}_{\mathsf{per}} \hookrightarrow \mathsf{CW}$ *is the evident forgetful functor.*

Proof. The proof is similar to the proof of Lemma 2.3.9. As before, we define $(-)^\sharp : \mathsf{CW}_{\mathsf{per}}(\mathsf{Split}_{\mathsf{per}}(\mathbb{X}), \mathbb{C}) \to \mathsf{CW}(\mathbb{X}, \mathbb{C})$ which is natural in $\mathbb{X}$ and $\mathbb{C}$, and show that it is full, faithful, and essentially surjective.

The proof that $(-)^\sharp$ is essentially surjective is almost identical to the corresponding part of Lemma 2.3.9. The only difference is that now we must also show that our $\widehat{F}$ is always a CW functor. We have already seen that it is symmetric monoidal. For the CW structure: since s_a is simple we have $\widehat{F}(\delta_{(X,a)}) = s_a F(a)F(\delta_X)(F(a)\otimes F(a))(r_a \otimes r_a) = s_a\delta_{FX}(r_a \otimes r_a) = \delta_{\widehat{F}(X,a)}(s_a r_a \otimes s_a r_a) = \delta_{\widehat{F}(X,a)}$; since s_a is total we have $\widehat{F}(\varepsilon_{(X,a)}) = s_a F(a)F(\varepsilon_X) = s_a\varepsilon_{FX} = \varepsilon_{\widehat{F}(X,a)}$;

since s_a is injective we have $\widehat{F}(\mu_{(X,a)}) = (s_a \otimes s_a)(F(a) \otimes F(a))F(\mu_X)F(a)r_a = (s_a \otimes s_a)\mu_{FX}r_a = \mu_{\widehat{F}(X,a)}s_a r_a = \mu_{\widehat{F}(X,a)}$; and finally since $r_a = s_a^\circ$ is surjective we have $\widehat{F}(\eta_{(X,a)}) = F(\eta_X)F(a)r_a = \eta_{FX}r_a = \eta_{\widehat{F}(X,a)}$. It follows that $\widehat{F}$ is a CW functor, and that $(-)^\sharp$ is essentially surjective.

To see that $(-)^\sharp$ is full, suppose F, G are objects of $\mathsf{CW}_{\mathsf{per}}(\mathsf{Split}_{\mathsf{per}}(\mathbb{X}), \mathbb{C})$ and that $\alpha : F^\sharp \to G^\sharp$ is an arrow of $\mathsf{CW}(\mathbb{X}, \mathbb{C})$ Define $\widehat{\alpha} : F \to G$ by $\widehat{\alpha}_{(X,a)} = F(a)\alpha_X G(a)$. Now $\widehat{\alpha}$ is lax natural as in $F(f)\widehat{\alpha}_{(Y,b)} = F(f)F(b)\alpha_Y G(b) = F(a)F(f)\alpha_Y G(b) \leq F(a)\alpha_X G(f)G(b) = F(a)\alpha_X G(a)G(f) = \widehat{\alpha}_{(X,a)}G(f)$ for $f : (X,a) \to (Y,b)$ of $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$. The components of $\widehat{\alpha}_{(X,a)}$ are maps as in $\widehat{\alpha}_{(X,a)}\varepsilon_{G(X,a)} =$
$F(a)\alpha_X G(a)\varepsilon_{GX} \geq F(a)\alpha_X\varepsilon_{GX} = F(a)\varepsilon_{FX} = \varepsilon_{F(X,a)}$ and $\widehat{\alpha}_{(X,a)}\delta_{F(X,a)} =$
$F(a)\alpha_X G(a)\delta_{GX}(G(a)\otimes G(a)) \geq F(a)\alpha_X\delta_{GX}(G(a)\otimes G(a)) = F(a)\delta_{FX}(\alpha_X G(a)\otimes \alpha_X G(a)) \geq F(a)\delta_{FX}(F(a) \otimes F(a))(\alpha_X \otimes \alpha_X)(G(a) \otimes G(a)) = \delta_{F(X,a)}(\widehat{\alpha}_{(X,a)} \otimes \widehat{\alpha}_{(X,a)})$. So $\widehat{\alpha} : F \to G$ is a monoidal lax transformation, and is thus a morphism of $\mathsf{CW}_{\mathsf{per}}(\mathsf{Split}_{\mathsf{per}}(\mathbb{X}), \mathbb{C})$ Consider $\widehat{\alpha}^\sharp : F^\sharp \to G^\sharp$. We have $\widehat{\alpha}^\sharp_X = \widehat{\alpha}_{(X,1_X)} = F(1_X)\alpha_X G(1_X) = \alpha_X$, and so $\widehat{\alpha}^\sharp = \alpha$. It follows that $(-)^\sharp$ is full.

Finally, to see that $(-)^\sharp$ is faithful suppose that $\alpha, \beta : F \to G$ are morphisms of $\mathsf{CW}_{\mathsf{per}}(\mathsf{Split}_{\mathsf{per}}(\mathbb{X}), \mathbb{C})$ such that $\alpha^\sharp = \beta^\sharp$ in $\mathsf{CW}(\mathbb{X}, \mathbb{C})$. Notice that in this case we have $\alpha_{(X,1_X)} = \alpha^\sharp_X = \beta^\sharp_X = \beta_{(X,1_X)}$, and further that for any object (X,a) of $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$ we have that $F(a) : F(X,1_X) \to F(X,1_X)$ and $G(a) : G(X,1_X) \to G(X,1_X)$ split as in:

$$\begin{tikzcd}
 & F(X,a) \arrow[r, "F(a)"] & F(X,1_X) \\
F(X,a) \arrow[ur, equal, "F(a)=1_{F(X,a)}"] \arrow[r, "F(a)"'] & F(X,1_X) \arrow[u, "F(a)"] \arrow[ur, "F(a)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & G(X,a) \arrow[r, "G(a)"] & G(X,1_X) \\
G(X,a) \arrow[ur, equal, "G(a)=1_{G(X,a)}"] \arrow[r, "G(a)"'] & G(X,1_X) \arrow[u, "G(a)"] \arrow[ur, "G(a)"'] &
\end{tikzcd}$$

In particular this means that $F(a) : F(X,a) \to F(X,1_X)$ and $G(a) : F(X,a) \to F(X,1_X)$ are both injective maps. Lax naturality of α gives $F(a)\alpha_{(X,1_X)} \leq \alpha_{(X,a)}G(a)$ as in:

$$\begin{tikzcd}
F(X,a) \arrow[r, "\alpha_{(X,a)}"] \arrow[d, "F(a)"'] \arrow[dr, phantom, "\leq"] & F(X,a) \arrow[d, "G(a)"] \\
F(X,1_X) \arrow[r, "\alpha_{(X,1_X)}"'] & G(X,1_X)
\end{tikzcd}$$

The components of α are necessarily maps, and so in fact we have $F(a)\alpha_{(X,1_X)} =$

$\alpha_{(X,a)}G(a)$. But then $F(a)\alpha_{(X,1_X)}G(a)G(a) = \alpha_{(X,a)}G(a)$ and so $F(a)\alpha_{(X,1_X)}G(a) = \alpha_{(X,a)}$ since $G(a) : F(X,a) \to F(X,1_X)$ is monic. A similar argument gives $F(a)\beta_{(X,1_X)}G(a) = \beta_{(X,a)}$. Then we have $\alpha_{(X,a)} = F(a)\alpha_{(X,1_X)}G(a) = F(a)\alpha^\sharp_X G(a) = F(a)\beta^\sharp_X G(a) = F(a)\beta_{(X,1_X)}G(a) = \beta_{(X,a)}$ and so $\alpha = \beta$. It follows that $(-)^\sharp$ is faithful. $\square$

This argument also works if, instead of splitting the PERs in a CW category, we split the coreflexives or the equivalence relations. Specifically, let $\mathsf{CW}_{\mathsf{cor}}$ and $\mathsf{CW}_{\mathsf{eq}}$ be the full sub 2-categories of CW on the 0-cells with in which coreflexives and, respectively, equivalence relations split. Then as above we have 2-functors $\mathsf{Split}_{\mathsf{cor}} : \mathsf{CW} \to \mathsf{CW}_{\mathsf{cor}}$ and $\mathsf{Split}_{\mathsf{eq}} : \mathsf{CW}_{\mathsf{cor}} \to \mathsf{CW}_{\mathsf{eq}}$. The argument of Lemma 4.1.26 then gives:

Corollary 4.1.27. *There are biadjunctions:*

$$\begin{tikzcd} \mathsf{CW} \arrow[r, shift left, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[r, phantom, "\perp"] & \mathsf{CW}_{\mathsf{cor}} \arrow[l, hook, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{CW} \arrow[r, shift left, "\mathsf{Split}_{\mathsf{per}}"] \arrow[r, phantom, "\perp"] & \mathsf{CW}_{\mathsf{per}} \arrow[l, hook, shift left] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{CW}_{\mathsf{cor}} \arrow[r, shift left, "\mathsf{Split}_{\mathsf{eq}}"] \arrow[r, phantom, "\perp"] & \mathsf{CW}_{\mathsf{eq}} \arrow[l, hook, shift left] \end{tikzcd}$$

Where the right biadjoints are the evident forgetful functors.

It turns out that splitting all the PERs in a CW category is equivalent to splitting all the idempotents, much in the way that splitting all the restriction idempotents in a DCR category is equivalent to splitting all of the idempotents (Lemma 3.2.11). The proof is somewhat more involved, and our next goal will be to establish this fact. We begin with a definition:

Definition 4.1.28 ([32, 2.16(11)])**.** Two idempotents $a, b : X \to X$ in a CW category $\mathbb{X}$ are said to be *neighbours* in case $a = aba$ and $b = bab$.

Next, a technical lemma:

Lemma 4.1.29 ([32, 2.16(11)])**.** *Let $a : X \to X$ be idempotent in a CW category $\mathbb{X}$. Then we have:*

(i) $a \cap a^\circ$ is a PER.

(ii) If a splits, then a and $a \cap a^\circ$ are neighbours.

(iii) If $b : X \to X$ is idempotent and a and b are neighbours, then a splits if and only if b splits.

Proof. (i) We must show that $a \cap a^\circ$ is symmetric and transitive. For symmetry we have $(e \cap e^\circ)^\circ = e^\circ \cap (e^\circ)^\circ = e^\circ \cap e = e \cap e^\circ$, and for transitivity we have $(e \cap e^\circ)(e \cap e^\circ) \leq (e \cap e^\circ)e \cap (e \cap e^\circ)e^\circ \leq ee \cap e^\circ e^\circ = e \cap e^\circ$. The claim follows.

(ii) Suppose that a splits as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "m^\circ"'] \arrow[ur, "a"'] & \end{tikzcd}$$

Then we have $(a\cap a^\circ)a(a\cap a^\circ) = (wm\cap m^\circ w^\circ)wm(wm\cap m^\circ w^\circ) \leq m^\circ(mwm\cap w^\circ)wm(wmw\cap m^\circ)w^\circ \leq m^\circ mwmwmwmww^\circ = m^\circ w^\circ = a^\circ$ and $(a\cap a^\circ) \leq a$, so by the universal property of the meet we have $(a\cap a^\circ)a(a\cap a^\circ) \leq (a\cap a^\circ)$. Conversely, we have $(a\cap a^\circ) = (a\cap a^\circ)(a\cap a^\circ)(a\cap a^\circ) \leq (a\cap a^\circ)a(a\cap a^\circ)$, and so $a\cap a^\circ = (a\cap a^\circ)a(a\cap a^\circ)$. Next, we have $a = wm = w(mwmw\cap 1_M)m \leq wm(wmw\cap m^\circ)m \leq wm(wm\cap m^\circ w^\circ)wm = a(a\cap a^\circ)a$ and conversely $a(a\cap a^\circ)a \leq aaa = a$, so we have $a = a(a\cap a^\circ)a$. It follows that a and $a\cap a^\circ$ are neighbours.

(iii) Suppose that a splits as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \arrow[u, "m^\circ"'] \arrow[ur, "a"'] & \end{tikzcd}$$

Then we have $bm^\circ mb = bab = b$ and also $mbbm^\circ = mm^\circ mbm^\circ mm^\circ = mabam^\circ = mam^\circ = mm^\circ mm^\circ = 1_M$, so b splits as in:

$$\begin{tikzcd} & M \arrow[r, "mb"] & X \\ M \arrow[ur, equal] \arrow[r, "mb"'] & X \arrow[u, "bm^\circ"'] \arrow[ur, "b"'] & \end{tikzcd}$$

Similarly, if b splits then a splits. The claim follows.

$\square$

Then, as promised, we have:

Lemma 4.1.30 ([32, 2.16(11)])**.** *Let $\mathbb{X}$ be a CW category. Then PERs in $\mathbb{X}$ split if and only if all idempotents in $\mathbb{X}$ split.*

Proof. If all idempotents in $\mathbb{X}$ split then PERs split, since PERs are idempotent. For the converse, suppose that PERs in $\mathbb{X}$ split and let $a : X \to X$ be idempotent in $\mathbb{X}$. Now $a : (X, 1_X) \to (X, 1_X)$ splits in $\mathsf{Split}(\mathbb{X})$, and so we know $a : (X, 1_X) \to (X, 1_X)$ and $a\cap a^\circ : (X, 1_X) \to (X, 1_X)$ are neighbours in $\mathsf{Split}(\mathbb{X})$. Recall that there is a faithful functor $[\![-]\!] : \mathbb{X} \to \mathsf{Split}(\mathbb{X})$ defined by $[\![X]\!] = (X, 1_X)$ on objects and by $[\![f]\!] = f$ on arrows. We have $[\![a(a\cap a^\circ)a]\!] = a(a\cap a^\circ)a = a = [\![a]\!]$ and $[\![(a\cap a^\circ)a(a\cap a^\circ)]\!] = (a\cap a^\circ)a(a\cap a^\circ) = (a\cap a^\circ) = [\![(a\cap a^\circ)]\!]$ in $\mathsf{Split}(\mathbb{X})$, but then since $[\![-]\!]$ is faithful we have that a and $a \cap a^\circ$ are neighbours in $\mathbb{X}$. Now $a \cap a^\circ$ is a PER, so by assumption it splits in $\mathbb{X}$, which means that a splits in $\mathbb{X}$. The claim follows. $\square$

Corollary 4.1.31 ([32, 2.16(11)]). *Let $\mathbb{X}$ be a CW category. Then $\mathsf{Split}_{\mathsf{per}}(\mathbb{X}) \simeq \mathsf{Split}(\mathbb{X})$.*

Another consequence of this is that our three idempotent splitting biadjunctions are related:

Lemma 4.1.32 ([32, 2.169]). *Let $\mathbb{X}$ be a CW category. Then we have:*

$$\mathsf{Split}_{\mathsf{per}}(\mathbb{X}) \simeq \mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$$

Proof. It suffices to show that all idempotents split in $\mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$, for which it suffices to show that all PERs split therein, for which it suffices in turn to show that any PER $a : X \to X$ of $\mathbb{X}$ splits in $\mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$. We have that $1_X \cap a : X \to X$ is coreflexive, so $(X, 1_X \cap a)$ is an object of $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$. Moreover, we have $(1_X \cap a)a(1_X \cap a) = (a \cap a)a(a \cap a) = aaa = a$ by Lemma 4.1.23, which means that $a : (X, 1_X \cap a) \to (X, 1_X \cap a)$ is an arrow of $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$. Now clearly $1_X \cap a \leq a$, and so $a : (X, 1_X \cap a) \to (X, 1_X \cap a)$ is in fact an equivalence relation in $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$. It follows that a splits in $\mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$, as required. $\square$

This extends to a result about our biadjunctions:

Corollary 4.1.33. *The following diagram of left biadjoints commutes:*

$$\begin{tikzcd}
\mathsf{CW} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[dr, "\mathsf{Split}_{\mathsf{per}}"'] & \mathsf{CW}_{\mathsf{cor}} \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"] \\
& \mathsf{CW}_{\mathsf{per}}
\end{tikzcd}$$

4.2 Relational Algebraic Theories

We begin with the notion of relational algebraic theory:

Definition 4.2.1 ([7, 4.1]). A *relational algebraic theory* is a small CW category.

Much as models of classical algebraic theories are given by structure-preserving functors into Set and models of partial algebraic theories are given by structure-preserving functors into Par, models of relational algebraic theories are given by structure-preserving functors into Rel:

Definition 4.2.2 ([7, 4.1]). A *model* of a relational algebraic theory $\mathbb{X}$ is a CW functor $F : \mathbb{X} \to \mathsf{Rel}$.

As with partial algebraic theories, model morphisms are monoidal lax transformations between the underlying functors:

Definition 4.2.3 ([7, 4.1])**.** Let $F, G : \mathbb{X} \to \mathsf{Rel}$ be models of a relational algebraic theory $\mathbb{X}$. A *model morphism* $\alpha : F \to G$ is a lax transformation.

Thus, the 2-category CW (Definition 4.1.8) occupies the same position with respect to relational algebraic theories that CM and DCR occupy with respect to classical and partial algebraic theories, respectively.

In our presentations of relational algebraic theories, arrows of the free CW category over a given monoidal signature will play the role of terms. Explicitly:

Definition 4.2.4. Let Γ be a monoidal signature. The small CW category $\mathsf{R}(\Gamma)$ of *relational terms over* Γ is constructed the same way $\mathsf{S}(\Gamma)$ is, but with additional generating arrows:

$$\frac{A \in \mathfrak{s}(\Gamma)^*}{\delta_A : A \to A \otimes A} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\mu_A : A \otimes A \to A} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\varepsilon_A : A \to I} \qquad \frac{A \in \mathfrak{s}(\Gamma)^*}{\eta_A : I \to A}$$

additional equations concerning coherence:

$$\delta_I = \square \qquad \mu_I = \square \qquad \varepsilon_I = \square \qquad \eta_I = \square$$

$$\delta_{A\otimes B} = (\delta_A \otimes \delta_B)(1_A \otimes \sigma_{A,B} \otimes 1_B)\mu_{A\otimes B} = (1_A \otimes \sigma_{A,B} \otimes 1_B)(\mu_A \otimes \mu_B)$$

$$\varepsilon_{A\otimes B} = \varepsilon_A \otimes \varepsilon_B \qquad \eta_{A\otimes B} = \eta_A \otimes \eta_B$$

and remaining additional equations:

$$\delta_A(\delta_A \otimes 1_A) = \delta_A(1_A \otimes \delta_A) \qquad \delta_A(1_A \otimes \varepsilon_A) = 1_A \qquad \delta_A \sigma_{A,A} = \delta_A$$

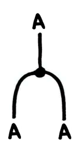
$$(1_A \otimes \mu_A)\mu_A = (\mu_A \otimes 1_A)\mu_A \qquad (\eta_A \otimes 1_A)\mu_A = 1_A \qquad \sigma_{A,A}\mu_A = \mu_A$$

$$(\delta_A \otimes 1_A)(1_A \otimes \mu_A) = \mu_A \delta_A \qquad (1_A \otimes \delta_A)(\mu_A \otimes 1_A) = \mu_A \delta_A$$

$$\delta_A(f \otimes f)\mu_B = f \quad \forall f : A \to B$$

Note in particular that the final family of equations gives $\delta_A \mu_A = 1_A$.

It is convenient to specify relational term presentations using string diagrams. We recall the diagrammatic convention for CW categories:

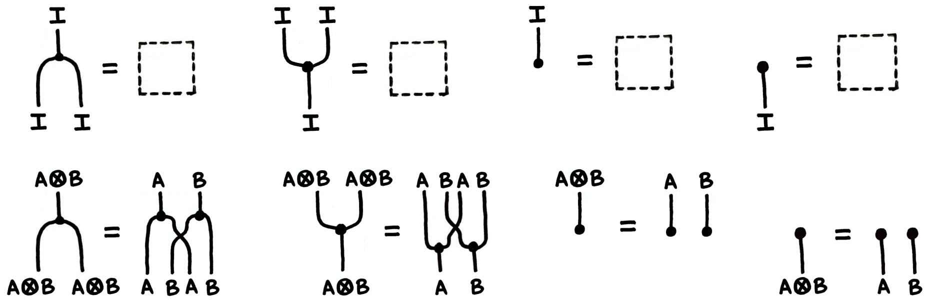
δ_A ⇝ 

ε_A ⇝ 

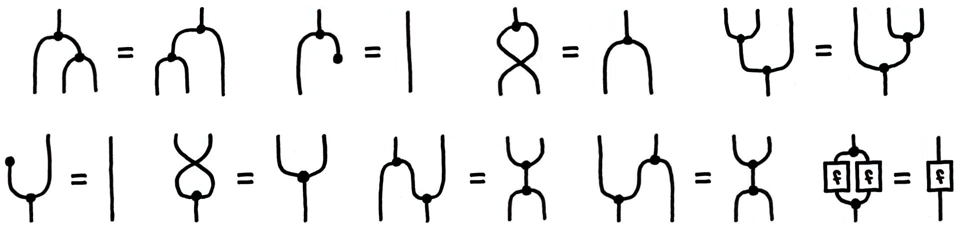
μ_A ⇝ 

η_A ⇝ 

Then the equations concerning coherence become:



and the remaining additional equations become:



Equations over a signature are again pairs of terms:

Definition 4.2.5. Let Γ be a monoidal signature. A *relational term equation over* Γ is a pair (f, g) where $f, g : A \to B \in \mathsf{R}(\Gamma)$. We may write $f = g$ instead of (f, g).

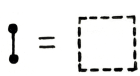
A presentation of a relational algebraic theory is a signature together with a collection of relational term equations over that signature:

Definition 4.2.6. A *relational term presentation* (Γ, E) consists of a monoidal signature Γ together with a set E of relational term equations over Γ.

To construct the theory presented by a presentation, we quotient the category of terms by the equations:

Definition 4.2.7. Let (Γ, E) be a relational term presentation. Write $\mathsf{R}(\Gamma, E)$ for the small CW category obtained by quotienting $\mathsf{R}(\Gamma)$ by the equations of E. We say that $\mathsf{R}(\Gamma, E)$ is *presented by* (Γ, E), and similarly we say that (Γ, E) *presents* $\mathsf{R}(\Gamma, E)$.

Example 4.2.8 (Nonempty Sets). Let $(\emptyset_X, E_{\mathsf{nonempty}})$ be the relational term presentation with a single sort $\mathfrak{s}(\emptyset_X) = \{X\}$ and no generating symbols, in which E_{nonempty} consists of the following equation:



Models of the associated relational algebraic theory $\mathsf{R}(\emptyset_X, E_{\mathsf{nonempty}})$ are sets X such that the generating equation is satisfied in Rel:

$$\eta_X \varepsilon_X = \{(*, *)\} = \square_I$$

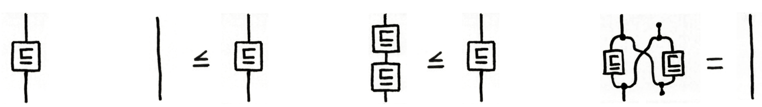
where η_X and ε_X are defined as in Definition 4.0.1. If we calculate the relational composite, we find that:

$$\eta_X \varepsilon_X = \{(*, *) \mid \exists x \in X.(*, x) \in \eta_X \wedge (x, *) \in \varepsilon_X\} = \{(*, *) \mid \exists x \in X\}$$

and so models are nonempty sets. Model morphisms are simply functions. Contrast this to the category of *pointed* sets, in which morphisms must preserve the distinguished point.

Recall that *inequations* of arrows in CW categories may be expressed as equations using the meet. We use inequations in our presentations of relational algebraic theories as syntactic sugar. That is, $f \leq g$ corresponds to the equation $f \cap g = f$. For example:

Example 4.2.9 (Posets)**.** Consider the relational term presentation $(\Gamma_{\sqsubseteq}, E_{\sqsubseteq})$ with a single sort $\mathfrak{s}(\Gamma_{\sqsubseteq}) = \{X\}$, a single generator $\sqsubseteq : X \to X$ (below left), and equations $E_{\sqsubseteq}$ to the effect that $\sqsubseteq$ is reflexive, transitive, and antisymmetric (below right).

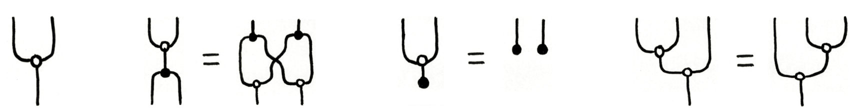


Models $F : \mathsf{R}(\Gamma_{\sqsubseteq}, E_{\sqsubseteq}) \to \mathsf{Rel}$ of the associated relational algebraic theory are precisely posets FX. Model morphisms are monotone functions.

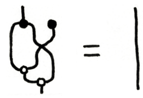
Example 4.2.10 (Regular Semigroups)**.** A *semigroup* is a set equipped with an associative binary operation, denoted by juxtaposition. A semigroup S is *regular* [34] in case:

$$\forall a \in S. \exists x \in S. axa = a$$

Consider the relational term presentation $(\Gamma_{\mathsf{rsg}}, E_{\mathsf{rsg}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{rsg}}) = \{X\}$ a single generator $\Gamma_{\mathsf{rsg}} = \{m : X \otimes X \to X\}$ (below left), in which E_{rsg} consists of equations to the effect that m is simple, total, and associative (below right)



together with an equation expressing the regularity condition:



Then models of $\mathsf{R}(\Gamma_{\mathsf{rsg}}, E_{\mathsf{rsg}})$ are regular semigroups and model morphisms are semigroup homomorphisms thereof.

Example 4.2.11 (Von Neumann Regular Rings)**.** A ring R is said to be *Von Neumann regular* [49] in case the multiplication monoid is a regular semigroup. The theory of rings is algebraic, and so adding the regularity axiom from Example 4.2.10 to the set of equations allows us to capture Von Neumann regular rings as a relational algebraic theory.

Example 4.2.12 (Effectoids)**.** An *effectoid* [48] is a set A equipped with a unary relation $\notin \mapsto \_ \subseteq A$, a binary relation $\_ \preceq \_ \subseteq A \times A$, and a ternary relation $\_ ; \_ \mapsto \_ \subseteq A \times A \times A$ satisfying:

(Identity) For all $a, a' \in A$,

$$\exists x \in A.(\notin \mapsto x) \wedge (x\,; a \mapsto a') \Leftrightarrow a \preceq a' \Leftrightarrow \exists y \in A.(\notin \mapsto y) \wedge (a\,; y \mapsto a')$$

(Associativity) For all $a, b, c, d \in A$,

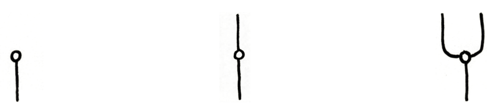
$$\exists x.(a\,; b \mapsto x) \wedge (x\,; c \mapsto d) \Leftrightarrow \exists y.(b\,; c \mapsto y) \wedge (a\,; y \mapsto d)$$

(Reflexive Congruence 1) For all $a \in A$, $a \preceq a$.

(Reflexive Congruence 2) For all $a, a' \in A$, $(\notin \mapsto a) \wedge (a \preceq a') \Rightarrow (\notin \mapsto a')$

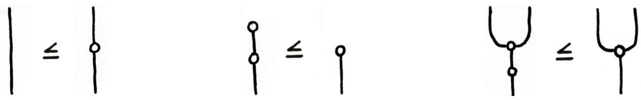
(Reflexive Congruence 3) For all $a, b, c \in A$, $\exists x.(a\,; b \mapsto x) \wedge (x \preceq c) \Rightarrow (a\,; b \preceq c)$

Effectoids admit a nice presentation as a relational algebraic theory. Consider the relational term presentation $(\Gamma_{\mathsf{effect}}, E_{\mathsf{effect}})$ with a single sort $\mathfrak{s}(\Gamma_{\mathsf{effect}}) = \{X\}$, generators Γ_{effect} corresponding respectively to the unary, binary, and ternary relations as in:



and with equations in E_{effect} corresponding to the identity and associativity axioms as in:

along with equations corresponding the the reflexive congruence axioms as in:



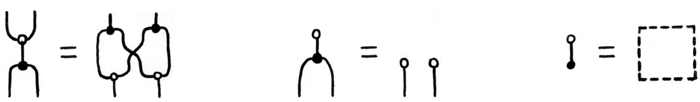
The models of $\mathsf{R}(\Gamma_{\mathsf{effect}}, E_{\mathsf{effect}})$ are precisely effectoids.

Example 4.2.13 (Generalized Separation Algebras)**.** A *generalized separation algebra* [5] is a partial monoid satisfying the left and right cancellativity axioms, which further satisfies the conjugation axiom:

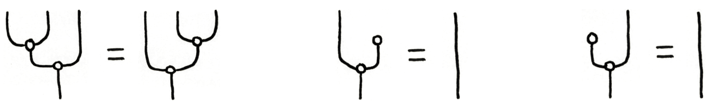
$$\forall x, y.(\exists z.x \circ z = y) \Leftrightarrow (\exists w.w \circ x = y)$$

We give a relational algebraic theory of generalized separation algebras. Consider $(\Gamma_{\mathsf{gsa}}, E_{\mathsf{gsa}})$ where $\Gamma_{\mathsf{gsa}} = \{X\}$ and Γ_{gsa} contains generators corresponding to the monoid operation and the unit as in:

Both are required to be simple, and the unit is required to be total. Thus E_{gsa} must contain equations as in:

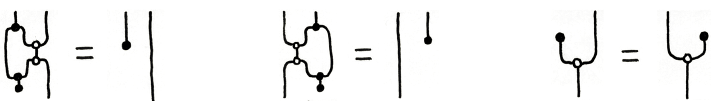


along with equations corresponding to the associativity and unitality axioms:



For the sake of convenience we define upside-down versions of the generators as in:

This allows us to state the rest of the equations in E_{gsa} more compactly, corresponding to left cancellativity, right cancellativity, and conjugation, respectively:



Then models of $\mathsf{R}(\Gamma_{\mathsf{gsa}}, E_{\mathsf{gsa}})$ are generalized separation algebras and model morphisms are partial monoid homomorphisms.

Example 4.2.14 (Algebraic Theories)**.** Let $\mathbb{X}$ be an algebraic theory, and let $(\mathbb{X}_{\mathsf{eq}})_{\mathsf{reg/lex}}$ be the regular completion of $\mathbb{X}$ [9, 13]. $\mathsf{Rel}((\mathbb{X}_{\mathsf{eq}})_{\mathsf{reg/lex}})$ — see Definition 4.4.1 — is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of $\mathbb{X}$ (as an algebraic theory). Conversely, if $\mathbb{X}$ is a relational algebraic theory, then the maps of $\mathbb{X}$ form a subcategory $\mathsf{Map}(\mathbb{X})$. $\mathsf{Map}(\mathbb{X})$ has finite products, and so defines an algebraic theory in the usual sense. Further, the notions of model and model morphism for relational algebraic theories restrict to the usual notions for algebraic theories on the category of maps.

Example 4.2.15 (Essentially Algebraic Theories)**.** An *essentially algebraic theory* [45] is (among many equivalent presentations) a category $\mathbb{X}$ with finite limits. Models are the finite-limit preserving functors $\mathbb{X} \to \mathsf{Set}$, and model morphisms are natural transformations. For $\mathbb{X}$ an essentially algebraic theory let $\mathbb{X}_{\mathsf{reg/lex}}$ be the regular completion of $\mathbb{X}$ [13]. Then $\mathsf{Rel}(\mathbb{X}_{\mathsf{reg/lex}})$ — see Definition 4.4.1 — is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of $\mathbb{X}$ (as an essentially algebraic theory).

Example 4.2.16 (Partial Algebraic Theories)**.** Let $\mathbb{X}$ be a partial algebraic theory. Then $\mathsf{Total}(\mathsf{Split}(\mathbb{X}))$ is a category with finite limits, and so $\mathsf{Rel}(\mathsf{Total}(\mathsf{Split}(\mathbb{X}))_{\mathsf{reg/lex}})$ — see Definition 4.4.1 — is a relational algebraic theory. Further, its models and model morphisms coincide with the models and model morphisms of $\mathbb{X}$ as a partial algebraic theory.

Example 4.2.17 (Cartesian Monoidal Presentations)**.** Let (Γ, E) be a cartesian monoidal presentation. Notice that every cartesian monoidal equation can also be interpreted as a relational term equation. Define E' to be the union of E with the following set of equations:

$$\{f\varepsilon_B = \varepsilon_A \mid f : A \to B \in \Gamma\} \cup \{f\delta_B = \delta_A(f \otimes f) \mid f : A \to B \in \Gamma\}$$

Now models and model morphisms of the relational algebraic theory $\mathsf{R}(\Gamma, E')$ coincide with models and model morphisms of the classical algebraic theory $\mathsf{C}(\Gamma, E)$. The extra equations of E' ensure that all of the operations are total functions in spite of the fact that they inhabit Rel.

Example 4.2.18 (Partial Term Presentations)**.** Let (Γ, E) be a partial term presentation. Notice that every partial term equation can also be interpreted as a relational term equation. Define E' to be the union of E with the following set of equations:

$$\{f\delta_B = \delta_A(f \otimes f) \mid f : A \to B \in \Gamma\}$$

Now models and model morphisms of the relational algebraic theory $\mathsf{R}(\Gamma, E')$ coincide with models and model morphisms of the partial algebraic theory $\mathsf{P}(\Gamma, E)$. The extra equations of E' ensure that all of the operations are partial functions in spite of the fact that they inhabit Rel.

4.3 Regular Categories

Our next goal will be to characterise the categories of models and model morphisms of relational algebraic theories (the *varieties*). In doing we will also obtain a characterisation of *Morita equivalence* for relational algebraic theories, which is what we will call the situation in which two different relational algebraic theories determine the same category of models and model morphisms (i.e., *present the same variety*).

To this end we recall the definition and elementary properties of regular categories. Ultimately we will show that CW categories and regular categories are the 0-cells of equivalent 2-categories. This will allow us to connect CW categories to the wider literature. We begin with the notion of subobject:

Definition 4.3.1. Let $\mathbb{X}$ be a category, and let A be an object of $\mathbb{X}$. A *subobject* of A consists of a pair (B, m) where B is an object of $\mathbb{X}$ and $m : B \to A$ is monic in $\mathbb{X}$.

Subobjects of A form a preorder: for (B, m) and (C, n) subobjects of A we say that $(B, m) \leq (C, n)$ in case there is an arrow $f : B \to C$ of $\mathbb{X}$ such that:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "m"'] & C \arrow[dl, "n"] \\ A & \end{tikzcd}$$

Notice that there can be at most one such morphism: if we have $g : B \to C$ with $gn = m$ then $fn = m = gn$ and so $f = g$ since n is monic. Consequently, if $(B, m) \leq (C, n)$ and $(C, n) \leq (B, m)$ then this is realised by the components of a (necessarily unique) isomorphism $B \cong C$. Transitivity of $\leq$ is realised by composition in $\mathbb{X}$ and reflexivity of $\leq$ is realised by the identity arrows. When we speak of subobjects of A we are often truly speaking of elements of the posetal reflection of this preorder, tacitly identifying (B, m) and (C, n) when $(B, m) \leq (C, n)$ and $(C, n) \leq (B, m)$.

Reasoning about subobjects in an arbitrary category by analogy to subsets in Set is remarkably robust. For example, in the category of groups and group homomorphisms a subobject of a group G is precisely a subgroup of G.

Central to the notion of regular category is the notion of regular epic, which is simply an arrow that arises as a coequaliser:

Definition 4.3.2. A morphism $e : Y \to E$ is called *regular epic* in case for some $f, g : X \to Y$ the diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "+"] & Y \arrow[r, "e"] & E \end{tikzcd}$$

is a coequaliser. Note in particular that every regular epic is epic. We write $e : Y \twoheadrightarrow E$ to indicate that the arrow e is regular epic.

Next, we introduce the notion of kernel pair. We will see later on that in a regular category any regular epic is in fact the coequaliser of its own kernel pair, which makes it easier to check whether or not something is regular epic.

Definition 4.3.3. The *kernel pair* of an arrow $f : X \to Y$, if it exists, consists of a pair of arrows f_0 and f_1 such that the diagram:

$$\begin{tikzcd} \mathsf{Ker}(f) \arrow[r, "f_0"] \arrow[d, "f_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback. Note in particular that in a category with pullbacks every morphism has a kernel pair.

In a regular category the regular epics interact with the monics. We recall a few facts about monics that will be required in our development:

Lemma 4.3.4. *(i) if fg is monic then f is monic*

(ii) For any arrow $f : X \to Y$ in a category with finite products, $\langle f, 1_X \rangle : X \to X \times Y$ and $\langle 1_X, f \rangle : X \to Y \times X$ are both monic.

(iii) $m : X \to Y$ is monic if and only if the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "m"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

(iv) If $m : X \to Y$ and $m_0, m_1 : \mathsf{Ker}(m) \to X$ is the kernel pair of m, then m is monic if and only if $m_0 = m_1$.

Proof. (i) If $hf = kf$ then $hfg = kfg$ which gives $h = k$ since fg is monic.

(ii) If $h\langle f, 1_X\rangle = k\langle f, 1_X\rangle$ then $h = h\langle f, 1_X\rangle\pi_1 = k\langle f, 1_X\rangle\pi_1 = k$ and so $\langle f, 1_X\rangle$ is monic. Similarly, $\langle 1_X, f\rangle$ is monic.

(iii) If m is monic and we have $hm = km$ for $h, k : Z \to X$ then $h = k$ since m is monic, which gives the unique map $Z \to X$ making the square in question a pullback. Conversely, if the square is a pullback and $fm = gm$ for $f, g : Z \to X$ then the arrow $\alpha : Z \to X$ given by the universal property of the pullback has $f = \alpha = g$, and so m is monic.

(iv) If m is monic then we have $m_0 m = m_1 m$ and so immediately $m_0 = m_1$. For the converse, suppose $m_0 = m_1$. Then for any $f, g : Z \to X$ with $fm = gm$ we have:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \mathsf{Ker}(m) \arrow[r, "m_0"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "m"] \\
& X \arrow[r, "m"'] & Y
\end{tikzcd}$$

which gives $f = \alpha m_0 = \alpha m_1 = g$, and so m is monic.

□

Now the definition of regular category is as follows:

Definition 4.3.5. A category $\mathbb{C}$ with finite limits is called *regular* in case:

(i) Coequalisers of kernel pairs exist, and

(ii) The pullback of a regular epic along any arrow is regular epic.

The regular epics in a regular category enjoy many properties, including:

Lemma 4.3.6. *In a regular category $\mathbb{C}$, we have:*

(i) Any regular epic is the coequaliser of its own kernel pair.

(ii) A regular epic that is also monic is an isomorphism.

(iii) The composite of two regular epics is regular epic.

(iv) If f and fg are regular epic, so is g.

Proof. (i) Suppose that we have a coequaliser:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "+"] & Y \arrow[r, two heads, "e"] & E
\end{tikzcd}$$

and let $e_0, e_1 : \mathsf{Ker}(e) \to Y$ be the kernel pair of $e : Y \to E$. Now let $e' : Y \twoheadrightarrow E'$ be the coequaliser of e_0 and e_1. Of course, e also equalises e_0 and e_1, so we obtain $h : E' \to E$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(e) \arrow[r, shift left=1.5, "e_0"] \arrow[r, shift right=1.5, "e_1"'] \arrow[r, phantom, "+"] & Y \arrow[r, two heads, "e'"] \arrow[dr, two heads, "e"'] & E' \arrow[d, dashed, "h"] \\
& & E
\end{tikzcd}$$

Now, let $\alpha : X \to \mathsf{Ker}(e)$ be the morphism given by the universal property of the kernel pair pullback as in:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& \mathsf{Ker}(e) \arrow[r, "e_0"] \arrow[d, "e_1"'] & Y \arrow[d, two heads, "e"] \\
& Y \arrow[r, two heads, "e"'] & E
\end{tikzcd}$$

Then we have $fe' = \alpha e_0 e' = \alpha e_1 e' = ge'$, from which we obtain $k : E \to E'$ as in:

$$\begin{tikzcd}
X \arrow[r, shift left=1.5, "f"] \arrow[r, shift right=1.5, "g"'] \arrow[r, phantom, "+"] & Y \arrow[r, two heads, "e"] \arrow[dr, two heads, "e'"'] & E \arrow[d, dashed, "k"] \\
& & E'
\end{tikzcd}$$

Then we have $e'hk = ek = e' \Rightarrow hk = 1_{E'}$ since e' is epic. Similarly $ekh = e'h = e \Rightarrow kh = 1_E$ since e is epic, and so k and h are mutually inverse and e is the coequaliser of its kernel pair e_0, e_1.

(ii) Let $e : X \to Y$ be both regular epic and monic, and let e_0, e_1 be the kernel pair of e. Then $e_0 e = e_1 e \Rightarrow e_0 = e_1$ since e is monic, and we have $h : X \to Y$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(e) \arrow[r, shift left=1.5, "e_0"] \arrow[r, shift right=1.5, "e_1"'] & X \arrow[r, "e"] \arrow[dr, equal] & Y \arrow[d, dashed, "h"] \\
& & X
\end{tikzcd}$$

So in particular we have $eh = 1_X$. We also have $ehe = e \Rightarrow he = 1_Y$ since e is epic. Thus, e is an isomorphism.

(iii) Suppose that we have regular epics $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$. Pasting pullback diagrams gives:

$$
\begin{tikzcd}
\mathsf{Ker}(fg) \arrow[r, two heads] \arrow[d, two heads] \arrow[rr, bend left, "(fg)_0"] \arrow[dd, bend right, "(fg)_1"'] \arrow[dr, phantom, "\lrcorner"] & X_0 \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, two heads, "f"] \\
X_1 \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Ker}(g) \arrow[r, two heads, "g_0"] \arrow[d, two heads, "g_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, two heads, "g"] \\
X \arrow[r, two heads, "f"'] & Y \arrow[r, two heads, "g"'] & Z
\end{tikzcd}
$$

where $(fg)_0, (fg)_1 : \mathsf{Ker}(fg) \to X$ gives the kernel pair of fg. Let $\alpha : \mathsf{Ker}(fg) \to \mathsf{Ker}(g)$ as in:

$$
\begin{tikzcd}
\mathsf{Ker}(fg) \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "(fg)_0 f"] \arrow[ddr, bend right, "(fg)_1 f"'] & & \\
& \mathsf{Ker}(g) \arrow[r, "g_0"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

Notice in particular that α is epic since it is a composite of (regular) epics $\mathsf{Ker}(fg) \twoheadrightarrow X_0 \twoheadrightarrow \mathsf{Ker}(g)$. Let $f_0, f_1 : \mathsf{Ker}(f) \to X$ be the kernel pair of f. Then we have an arrow $\beta : \mathsf{Ker}(f) \to \mathsf{Ker}(fg)$ as in:

$$
\begin{tikzcd}
\mathsf{Ker}(f) \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "f_0"] \arrow[ddr, bend right, "f_1"'] & & \\
& \mathsf{Ker}(fg) \arrow[r, "(fg)_0"] \arrow[d, "(fg)_1"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "fg"] \\
& X \arrow[r, "fg"'] & Y
\end{tikzcd}
$$

Now $(fg)_0 fg = \alpha g_0 g = \alpha g_1 g = (fg)_1 fg$ and for any $h : Z \to W$ with $(fg)_0 h = (fg)_1 h$ we have $f_0 h = \beta (fg)_0 h = \beta (fg)_1 h = f_1 h$, and so since f is the coequaliser of its kernel pair we obtain $k : Y \to W$ as in:

$$
\begin{tikzcd}
\mathsf{Ker}(f) \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, dashed, "k"] \\
& & W
\end{tikzcd}
$$

Next, we have $g_0 k = g_1 k$ since $\alpha g_0 k = (fg)_0 fk = (fg)_0 h = (fg)_1 h =$

$(fg)_1 fk = \alpha g_1 k$ and α is epic. Since g is the coequaliser of its kernel pair we obtain $w : Z \to W$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(g) \arrow[r, shift left=2, "g_0"] \arrow[r, phantom, "+"] \arrow[r, shift right=2, "g_1"'] & Y \arrow[r, "g"] \arrow[dr, "k"'] & Z \arrow[d, dashed, "w"] \\
& & W
\end{tikzcd}$$

Now $fgw = fk = h$ and we have:

$$\begin{tikzcd}
\mathsf{Ker}(fg) \arrow[r, shift left=2, "(fg)_0"] \arrow[r, phantom, "+"] \arrow[r, shift right=2, "(fg)_1"'] & X \arrow[r, "fg"] \arrow[dr, "h"'] & Z \arrow[d, dashed, "w"] \\
& & W
\end{tikzcd}$$

where w is the unique such morphism since for any $w' : Z \to W$ with $fgw' = h$ we have $w = w'$ because $fgw = h = fgw'$ and fg is epic (being the composite of two epic arrows). Thus, fg is the coequaliser of its kernel pair, and is therefore regular epic.

(iv) Suppose we have $f : X \to Y$ and $g : Y \to Z$ with f and fg regular epic. Let $g_0, g_1 : \mathsf{Ker}(g) \to Y$ be the kernel pair of g, and let $(fg)_0, (fg)_1 : \mathsf{Ker}(fg) \to X$ be the kernel pair of fg. Then as before we have:

$$\begin{tikzcd}
\mathsf{Ker}(fg) \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "(fg)_0 f"] \arrow[ddr, bend right, "(fg)_1 f"'] & & \\
& \mathsf{Ker}(g) \arrow[r, "g_0"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Now, suppose we have $h : Y \to W$ such that $g_0 h = g_1 h$. Then $(fg)_0 fh = \alpha g_0 h = \alpha g_1 h = (fg)_1 fh$ and we obtain $w : Z \to W$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(fg) \arrow[r, shift left=2, "(fg)_0"] \arrow[r, phantom, "+"] \arrow[r, shift right=2, "(fg)_1"'] & X \arrow[r, "fg"] \arrow[dr, "fh"'] & Z \arrow[d, dashed, "w"] \\
& & W
\end{tikzcd}$$

Then we have $gw = h$ since f is epic, which gives:

$$\begin{tikzcd} \mathsf{Ker}(g) \arrow[r, shift left, "g_0"] \arrow[r, shift right, "g_1"'] \arrow[r, phantom, "+"] & Y \arrow[r, "g"] \arrow[dr, "h"'] & Z \arrow[d, dashed, "w"] \\ & & W \end{tikzcd}$$

where w is the unique such arrow because for any $w' : Z \to W$ with $gw' = h$ we have $fgw' = fh = fgw$ which gives $w = w'$ since fg is epic. Thus g is the coequaliser of its kernel pair, and is therefore regular epic.

□

One of the most important properties of regular categories is that in them, the regular epics and monics form a stable factorisation system.

Proposition 4.3.7. *In a regular category $\mathbb{C}$ each arrow factors as a regular epic followed by a monic. Moreover, for each commutative diagram:*

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r, tail, "m"'] & W \end{tikzcd}$$

in $\mathbb{C}$ there exists a unique arrow $t : Y \to Z$ such that:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "t"'] \\ Z \arrow[r, tail, "m"'] & W \end{tikzcd}$$

That is, every regular category has a stable factorisation system (L, R) with L the class of regular epics and M the class of monics.

Proof. We first show that each arrow of $\mathbb{C}$ factors as a regular epic followed by a monic. To that end, suppose $f : X \to Y$ in $\mathbb{C}$. Let $f_0, f_1 : \mathsf{Ker}(f) \to X$ be the kernel pair of f, and let $e : X \twoheadrightarrow E$ be the coequaliser of f_0 and f_1. Now we have $m : E \to Y$ as in:

$$\begin{tikzcd} \mathsf{Ker}(f) \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[r, phantom, "+"] & X \arrow[r, two heads, "e"] \arrow[dr, "f"'] & E \arrow[d, dashed, "m"] \\ & & Y \end{tikzcd}$$

It remains to show that m is monic. Let $m_0, m_1 : \mathsf{Ker}(m) \to E$ be the kernel pair of m. Pasting pullback diagrams gives:

$$\begin{tikzcd}
\mathsf{Ker}(f) \arrow[r, two heads] \arrow[d, two heads] \arrow[rr, bend left, "f_0"] \arrow[dd, bend right, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "e"] \arrow[dd, bend left, "f"] \\
Y_1 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Ker}(m) \arrow[r, "m_0"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\
Y \arrow[r, two heads, "e"'] \arrow[rr, bend right, "f"'] & E \arrow[r, "m"'] & Y
\end{tikzcd}$$

It follows that $\alpha : \mathsf{Ker}(f) \to \mathsf{Ker}(m)$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(f) \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f_0 e"] \arrow[ddr, bend right, "f_1 e"'] & & \\
& \mathsf{Ker}(m) \arrow[r, "m_0"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "m"] \\
& E \arrow[r, "m"'] & Y
\end{tikzcd}$$

is epic, since it is a composite of (regular) epics $\mathsf{Ker}(f) \twoheadrightarrow Y_0 \twoheadrightarrow \mathsf{Ker}(m)$. This gives $m_0 = m_1$ via $\alpha m_0 = f_0 e = f_1 e = \alpha m_1$. Then by Lemma 4.3.4 we have that m is monic, as required.

Next, suppose that we have a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
Z \arrow[r, tail, "m"'] & W
\end{tikzcd}$$

Let $e_0, e_1 : \mathsf{Ker}(e) \to X$ be the kernel pair of e. Then $e_0 f m = e_0 e g = e_1 e g = e_1 f m$ and so $e_0 f = e_1 f$ since m is monic. We obtain $t : Y \to Z$ as in:

$$\begin{tikzcd}
\mathsf{Ker}(e) \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "e"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "t"] \\
& & Z
\end{tikzcd}$$

Further, $etm = fm = eg$ and so $tm = g$ since e is epic. Thus we have:

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "t"'] \\ Z \arrow[r, tail, "m"'] & W \end{tikzcd}$$

where the universal property of the coequaliser $e : X \twoheadrightarrow Y$ ensures that t is the unique such arrow. The claim follows. $\square$

The monic part of the factorisation of $f : X \to Y$ in a regular category defines a subobject of Y, which we call the *image* of f.

We require a few more minor facts about regular categories before moving on. First, we have:

Lemma 4.3.8. *In a regular category any regular epic is* extremal. *That is, if $e = gm$ for some g and some monic m, then m is an isomorphism.*

Proof. We have seen that in every regular category the regular epics and monics form an orthogonal factorisation system. In particular this means that any regular epic is left orthogonal to all monics. Suppose $e : Y \twoheadrightarrow E$ is regular epic, that $m : M \rightarrowtail E$ is monic, and that $e = gm$ for some $g : Y \to M$. Then we have a unique diagonal filler $h : E \to M$ for the following square:

$$\begin{tikzcd} Y \arrow[r, two heads, "e"] \arrow[d, "g"'] & E \arrow[d, equal] \arrow[dl, dashed, "h"] \\ M \arrow[r, tail, "m"'] & E \end{tikzcd}$$

Notice that we immediately have $hm = 1_E$. But then we have $mhm = m = 1_M m$ which gives $mh = 1_M$ since m is monic. It follows that m is an isomorphism. $\square$

While the following result may at first seem to be overly specific, it will be useful in the sequel for reasoning about internal relations in a regular category.

Lemma 4.3.9. *In a regular category two spans $\langle f, g\rangle : R \to A \times B$ and $\langle h, k\rangle : S \to A \times B$ are such that the image of $\langle f, g\rangle$ is a subobject of the image of $\langle h, k\rangle$ if and only if there exists a regular epic $a : X \to R$ along with any morphism $b : X \to S$ such that*

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & X \arrow[u, two heads, "a"'] \arrow[d, "b"] & B \\ & S \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

Proof. Suppose we have a commutative diagram of the appropriate form. We must show that the image I_R of $\langle f, g\rangle : R \to A \times B$ is isomorphic to $I_R \cap I_S$. Begin by constructing the meet, then we have:

$$
\begin{tikzcd}
X \arrow[rr, bend left, "b"] \arrow[dr, dotted, "x"] \arrow[dd, bend right, two heads, "a"'] & & S \arrow[d, two heads] \arrow[dd, bend left, "\langle h,k\rangle"] \\
& R \cap S \arrow[r, tail] \arrow[d, tail, "m"'] \arrow[dr, phantom, very near start, "\lrcorner"] & I_S \arrow[d, tail] \\
R \arrow[r, two heads] \arrow[rr, bend right, "\langle f,g\rangle"'] & I_R \arrow[r, tail] & A \times B
\end{tikzcd}
$$

And since regular epics are left-orthogonal to monics we have:

$$
\begin{tikzcd}
X \arrow[r, two heads, "a"] \arrow[d, "x"'] & R \arrow[d, two heads] \arrow[dl, dotted] \\
R \cap S \arrow[r, tail, "m"'] & I_R
\end{tikzcd}
$$

But then since $R \twoheadrightarrow I_R$ is extremal epic we have that $m : R \cap S \xrightarrow{\sim} R$ is an isomorphism, as required.

Conversely, suppose that there is an isomorphism $\alpha : I_R \xrightarrow{\sim} R \cap S$. Then the following diagram is a pullback:

$$
\begin{tikzcd}
I_R \arrow[r, "\alpha", "\sim"'] \arrow[d, two heads, "1"', "\sim"] \arrow[dr, phantom, very near start, "\lrcorner"] & R \cap S \arrow[r, tail] & I_S \arrow[d, tail] \\
I_R \arrow[rr, tail] & & A \times B
\end{tikzcd}
$$

Now, define a, b by repeated pullback as in:

$$
\begin{tikzcd}
X \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[rrr, bend left, "b"] \arrow[dd, bend right, two heads, "a"'] & Q \arrow[rr, tail] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & S \arrow[d, two heads] \arrow[dd, bend left, "\langle h,k\rangle"] \\
P \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & I_R \arrow[r, "\alpha", "\sim"'] \arrow[d, two heads, "1"', "\sim"] \arrow[dr, phantom, very near start, "\lrcorner"] & R \cap S \arrow[r, tail] & I_S \arrow[d, tail] \\
R \arrow[r, two heads] \arrow[rrr, bend right, "\langle f,g\rangle"'] & I_R \arrow[rr, tail] & & A \times B
\end{tikzcd}
$$

and the outer commutative square gives the required diamond shape. □

The notion of functor between regular categories is straightforward:

Definition 4.3.10. A functor $F : \mathbb{X} \to \mathbb{Y}$ between regular categories $\mathbb{X}$ and $\mathbb{Y}$ is itself called *regular* if it preserves finite limits and regular epics.

Finally, we define a 2-category of regular categories:

Definition 4.3.11. REG is the 2-category of small regular categories, regular functors, and natural transformations.

4.4 Categories of Relations

To begin, we recall the category Rel of sets and relations, which will serve as the universe of models for relational theories in the same way that the category Set of sets and functions is the universe of models for classical algebraic theories.

In any regular category we can construct an abstract analogue of Definition 4.0.1. Instead of sub*sets* $R \subseteq A \times B$, we represent relations as sub*objects* $R \rightarrowtail A \times B$. This approach to categorifying the theory of relations has a relatively long history [32], and integrates well with standard categorical logic due to the ubiquity of regular categories there.

Definition 4.4.1. Let $\mathbb{C}$ be a regular category. The associated category of *internal relations*, $\mathsf{Rel}(\mathbb{C})$, is defined as follows:

objects are objects of $\mathbb{C}$

arrows $f : X \to Y$ of $\mathsf{Rel}(\mathbb{C})$ are monics $\langle f_0, f_1 \rangle : F \rightarrowtail X \times Y$ in $\mathbb{C}$ modulo equivalence as subobjects of $X \times Y$. We write $m \looparrowright f : X \to Y$ to indicate that $m : R \rightarrowtail X \times Y$ in $\mathbb{C}$ gives $f : X \to Y$ in $\mathsf{Rel}(\mathbb{C})$.

composition of two arrows $\langle f_0, f_1 \rangle \looparrowright f : X \to Y$ and $\langle g_0, g_1 \rangle \looparrowright g : Y \to Z$ of $\mathsf{Rel}(\mathbb{C})$ is defined by first constructing the pullback below on the left. This defines an arrow $\langle g_0' f_0, f_1' g_1 \rangle$, and the composite $fg : X \to Z$ is given by the monic part of the image factorization of this arrow, as pictured below right.

$$\begin{tikzcd}
F \times_Y G \arrow[r, "f_1'"] \arrow[d, "g_0'"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "g_0"] \\
F \arrow[r, "f_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
R \times_B S \arrow[rr, "\langle g_0' f_0, f_1' g_1 \rangle"] \arrow[dr, two heads] & & X \times Z \\
& FG \arrow[ur, tail, "\langle (fg)_0, (fg)_1 \rangle"'] &
\end{tikzcd}$$

identities are as in $\Delta_X = \langle 1_X, 1_X \rangle \looparrowright 1_X : X \to X$.

Remark 4.4.2. While the above definition is the standard one, it introduces a novel piece of notation that will be used heavily in the sequel. Specifically, the expression:

$$m \looparrowright f : X \to Y$$

indicates that the arrow $f : X \to Y$ of $\mathsf{Rel}(\mathbb{C})$ is given by the subobject $m : R \rightarrowtail X \times Y$ in $\mathbb{C}$. It is hoped that reiterating this explicitly will help to head off any notational confusion going forward.

That internal relations in a regular category are themselves a category is non-trivial:

Lemma 4.4.3. *If $\mathbb{C}$ is a regular category then $\mathsf{Rel}(\mathbb{C})$ is a category.*

Proof. We must show that composition is associative and unital. We begin with associativity. Suppose that we have arrows $\langle f_0, f_1 \rangle \looparrowright f : X \to Y$, $\langle g_0, g_1 \rangle \looparrowright g : Y \to Z$, and $\langle h_0, h_1 \rangle \looparrowright h : Z \to W$ of $\mathsf{Rel}(\mathbb{C})$. We will show that both $(fg)h$ and $f(gh)$ are given by the monic part of the image factorisation of the arrow $\langle m_0 p_0 f_0, m_1 q_1 h_1 \rangle : M \rightarrowtail X \times W$ defined by repeated pullback as in:

$$\begin{tikzcd}
 & & & M \arrow[dl, "m_0"'] \arrow[dr, "m_1"] & & & \\
 & & P \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & & Q \arrow[dl, "q_0"'] \arrow[dr, "q_1"] & & \\
 & F \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & & G \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & & H \arrow[dl, "h_0"'] \arrow[dr, "h_1"] & \\
X & & Y & & Z & & W
\end{tikzcd}$$

To begin, recall that $\langle (fg)_0, (fg)_1 \rangle \looparrowright fg : X \to Z$ as in the image factorisation:

$$\begin{tikzcd}
P \arrow[rr, "\langle p_0 f_0, p_1 g_1 \rangle"] \arrow[dr, two heads, "e_0"'] & & X \times Z \\
 & FG \arrow[ur, tail, "\langle (fg)_0, (fg)_1 \rangle"'] &
\end{tikzcd}$$

Now we construct $\langle ((fg)h)_0, ((fg)h)_1 \rangle \looparrowright (fg)h : X \to W$ by constructing the pullback of h_0 along $(fg)_1$ (below left) and taking the the image of the resulting arrow $N \to X \times W$ (below right):

$$
\begin{tikzcd}
N \arrow[r, "n_1"] \arrow[d, "n_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[r, "h_1"] \arrow[d, "h_0"] & W \\
FG \arrow[r, "(fg)_1"'] \arrow[d, "(fg)_0"'] & Z \\
X
\end{tikzcd}
\qquad
\begin{tikzcd}
N \arrow[rr, "\langle n_0(fg)_0, n_1h_1 \rangle"] \arrow[dr, two heads, "e_1"'] & & X \times X \\
& (FG)H \arrow[ur, tail, "\langle ((fg)h)_0, ((fg)h)_1 \rangle"'] &
\end{tikzcd}
$$

Then we obtain $k : M \to N$ as in:

$$
\begin{tikzcd}
M \arrow[d, "m_0"'] \arrow[dr, dashed, "k"] \arrow[drr, bend left, "m_1q_1"] & & \\
P \arrow[dr, bend right, two heads, "e_0"'] & N \arrow[r, "n_1"] \arrow[d, "n_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "h_0"] \\
& FG \arrow[r, "(fg)_1"'] & Z
\end{tikzcd}
$$

Now, we have:

$$
\begin{tikzcd}
M \arrow[r, "k"] \arrow[d, "m_0"'] & N \arrow[r, "n_1"] \arrow[d, "n_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "h_0"] \\
P \arrow[r, two heads, "e_0"'] & FG \arrow[r, "(fg)_1"'] & Z
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M \arrow[r, "m_1"] \arrow[rr, bend left, "kn_1"] \arrow[d, "m_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q \arrow[r, "q_1"] \arrow[d, "q_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d, "h_0"] \\
P \arrow[r, "p_1"'] \arrow[rr, bend right, "e_0(fg)_1"'] & G \arrow[r, "g_1"'] & Z
\end{tikzcd}
$$

And so the left-hand square in the diagram above left is a pullback (by the pullback pasting lemma). In particular, this means that $k : M \twoheadrightarrow N$ is regular epic. Now let $\langle (fgh)_0, (fgh)_1 \rangle \looparrowright fgh : X \to W$ be the arrow given by the image of $\langle m_0p_0f_0, m_1q_1h_1 \rangle : M \to X \times W$, as in:

$$
\begin{tikzcd}
M \arrow[rr, "\langle m_0p_0f_0, m_1q_1h_1 \rangle"] \arrow[dr, two heads, "e_2"'] & & X \times W \\
& FGH \arrow[ur, tail, "\langle (fgh)_0, (fgh)_1 \rangle"'] &
\end{tikzcd}
$$

Now, we have:

$$\begin{tikzcd}
M \arrow[rrrr, "\langle m_0p_0f_0, m_1q_1h_1\rangle"] \arrow[d, equal] \arrow[dr, two heads, "k"] & & & & X \times W \arrow[d, equal] \\
M \arrow[ddrr, two heads, "e_2"'] & N \arrow[rr, "\langle n_0(fg)_0, n_1h_1\rangle"] \arrow[dr, two heads, "e_1"'] & & X \times W \arrow[ur, equal] & X \times W \\
& & (FG)H \arrow[ur, tail, "\langle ((fg)h)_0, ((fg)h)_1\rangle"] \arrow[d, dashed, "\alpha", "\cong"'] & & \\
& & FGH \arrow[uurr, tail, "\langle (fgh)_0, (fgh)_1\rangle"'] & &
\end{tikzcd}$$

That is, we have that $\langle((fg)h)_0, ((fg)h)_1\rangle : (FG)H \rightarrowtail X \times W$ and $\langle(fgh)_0, (fgh)_1\rangle : FGH \rightarrowtail X \times W$ are both the image of $\langle m_0p_0f_0, m_1q_1h_1\rangle : M \to X \times W$. It follows that they are equal as subobjects (via the induced isomorphism α in the above diagram), which gives $fgh = (fg)h : X \to W$ in $\mathsf{Rel}(\mathbb{C})$. A similar argument gives that $fgh = f(gh)$, from which we conclude that composition in $\mathsf{Rel}(\mathbb{C})$ is associative.

For unitality, observe that for any $\langle f_0, f_1\rangle \looparrowright f : X \to Y$ the composite $1_X f$ is given by the image of the outer span of:

$$\begin{tikzcd}
& & F \arrow[dl, equal] \arrow[dr, "f_1"] & & \\
& F \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & & Y \arrow[dl, equal] \arrow[dr, equal] & \\
X & & Y & & Y
\end{tikzcd}$$

since the operation of pulling back along an identity arrow is itself the identity. That is, $1_X f$ is given by the monic part of the image of $\langle f_0, f_1\rangle$, but this is already monic and is therefore its own image. Thus $1_X f = f$. Similarly, we have $f = f1_Y$. It follows that $\mathsf{Rel}(\mathbb{C})$ is a category. $\square$

Example 4.4.4. Set is a regular category, and the category of internal relations in $\mathsf{Rel}(\mathsf{Set})$ is precisely the usual category of sets and relations Rel.

Lemma 4.4.5 ([14, 1.4])**.** *Let $\mathbb{C}$ be a regular category. Then $\mathsf{Rel}(\mathbb{C})$ is a CW category with split coreflexives.*

Proof. $\mathsf{Rel}(\mathbb{C})$ is a CW category with the Frobenius algebra structure given by:

$$\langle 1_X, \Delta_X\rangle \looparrowright \delta_X : X \rightarrowtail X \times X \qquad \langle \Delta_X, 1_X\rangle \looparrowright \mu_X : X \times X \to X$$

$$\langle 1_X, !_X\rangle \looparrowright \varepsilon_X : X \to I \qquad \langle !_X, 1_X\rangle \looparrowright \eta_X : I \to X$$

Suppose $\langle f_0, f_1\rangle \looparrowright f : X \to X$ is coreflexive. Then $f \leq 1_X$ gives a morphism of spans $\alpha : F \to X$ as in:

$$\begin{tikzcd}
& F \arrow[r, "\alpha"] \arrow[dl, "f_0"'] \arrow[drr, "f_1"] & X \arrow[dll, equal] \arrow[dr, equal] & \\
X & & & X
\end{tikzcd}$$

In particular, this gives $f_0 = \alpha = f_1$, which means $\langle \alpha, \alpha \rangle \looparrowright f$. Now $\alpha : F \to X$ is monic since $\langle f_0, f_1 \rangle = \langle \alpha, \alpha \rangle = \alpha \Delta_X$ is monic. Let $\langle 1_F, \alpha \rangle \looparrowright m : F \to X$ and $\langle \alpha, 1_F \rangle \looparrowright m^\circ : X \to F$ in $\mathsf{Rel}(\mathbb{C})$. Since α is monic we have $\langle 1_F, 1_F \rangle \looparrowright mm^\circ = 1_F$ and $\langle \alpha, \alpha \rangle \looparrowright m^\circ m = f$ in $\mathsf{Rel}(\mathbb{C})$. Thus, f splits in $\mathsf{Rel}(\mathbb{C})$ and the claim follows. $\square$

4.5 Relating CW Categories and Regular Categories

In this section we establish a strict 2-equivalence between the 2-category $\mathsf{CW_{cor}}$ of small CW categories with split coreflexives and the 2-category Reg of small regular categories.

We have already seen that if $\mathbb{C}$ is a regular category then $\mathsf{Rel}(\mathbb{C})$ is a CW category with split coreflexives (Lemma 4.4.5). Clearly $\mathsf{Rel}(\mathbb{C})$ is small if $\mathbb{C}$ is. We show that this extends to a 2-functor $\mathsf{Rel} : \mathsf{Reg} \to \mathsf{CW_{cor}}$. If $\mathbb{C}$ and $\mathbb{D}$ are regular categories and $F : \mathbb{C} \to \mathbb{D}$ is a regular functor, then we obtain a CW functor $\mathsf{Rel}(F) : \mathsf{Rel}(\mathbb{C}) \to \mathsf{Rel}(\mathbb{D})$. This functor is defined on objects by $\mathsf{Rel}(F)(X) = FX$, and sends arrows $m \looparrowright f : X \to Y$ in $\mathsf{Rel}(\mathbb{C})$ to $Fm \looparrowright \mathsf{Rel}(F)(f) : FX \to FY$ in $\mathsf{Rel}(\mathbb{D})$. We have that $\mathsf{Rel}(F)$ is a functor since F is a regular functor. In particular, $\mathsf{Rel}(F)$ preserves composition because F preserves images. Further, $\mathsf{Rel}(F)$ is a CW functor because F preserves the Cartesian monoidal structure of $\mathbb{C}$. For 2-cells $\alpha : F \to G : \mathbb{C} \to \mathbb{D}$ of Reg, define $\mathsf{Rel}(\alpha) : \mathsf{Rel}(F) \to \mathsf{Rel}(G)$ by $\langle 1_{FA}, \alpha_A \rangle \looparrowright \mathsf{Rel}(\alpha)_A$. We must show that $\mathsf{Rel}(\alpha)$ is a monoidal lax transformation. To that end, let $\langle f_0, f_1 \rangle \looparrowright f : X \to Y$ in $\mathsf{Rel}(\mathbb{C})$ and consider that:

$$\mathsf{Rel}(\alpha)_X \mathsf{Rel}(G)(f) \qquad \text{and} \qquad \mathsf{Rel}(F)(f)\mathsf{Rel}(\alpha)_Y$$

are defined by first constructing pullbacks

$$\begin{tikzcd}
P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & GR \arrow[d, "Gf_0"] \\
FX \arrow[r, "\alpha_X"'] & GX
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
Q \arrow[r, "q_1"] \arrow[d, "q_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & FY \arrow[d, "1_{FB}"] \\
FR \arrow[r, "Ff_1"'] & FY
\end{tikzcd}$$

and then taking images as in

$$
\begin{tikzcd}
P \arrow[rr, "\langle p_0, p_1 G(f_1)\rangle"] \arrow[dr, two heads] & & FX \times GY \\
& I_P \arrow[ur, tail] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Q \arrow[rr, "\langle q_0 F(f_0), q_1 \alpha_Y\rangle"] \arrow[dr, two heads] & & FX \times GY \\
& I_Q \arrow[ur, tail] &
\end{tikzcd}
$$

Thus, it suffices to show that I_Q is a subobject of I_P. Since α is natural we have $q_0 \alpha_R G(f_0) = q_0 F(f_0) \alpha_X$, which induces an arrow $h : Q \to P$ as in:

$$
\begin{tikzcd}
Q \arrow[r, "q_0"] \arrow[d, "q_0"'] \arrow[dr, dotted, "h"] & FR \arrow[dr, "\alpha_R"] & \\
FR \arrow[dr, "Ff"'] & P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & GR \arrow[d, "Gf"] \\
& FX \arrow[r, "\alpha_X"'] & GX
\end{tikzcd}
$$

and then we have $h\langle p_0, p_1 Gg\rangle = \langle q_0 Ff, q_1 \alpha_B\rangle$ immediately in the first component, and for the second component as in:

$$
\begin{tikzcd}
& P \arrow[r, "p_1"] & GR \arrow[dr, "Gg"] & \\
Q \arrow[ur, "h"] \arrow[r, "q_0"] \arrow[dr, "q_1"'] & FR \arrow[ur, "\alpha_R"'] \arrow[dr, "Fg"] & & GB \\
& FB \arrow[r, "1"'] & FB \arrow[ur, "\alpha_B"'] &
\end{tikzcd}
$$

and then we have

$$
\begin{tikzcd}
& I_Q \arrow[dl] \arrow[dr] & \\
FA & Q \arrow[u, two heads] \arrow[d, "\twoheadrightarrow \circ h"] & QB \\
& I_P \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

so by Lemma 4.3.9 I_Q is a subobject of I_P, as required, and we may conclude that $\mathsf{Rel}(\alpha) : \mathsf{Rel}(F) \to \mathsf{Rel}(G)$ is a lax transformation. To see that $\mathsf{Rel}(\alpha)$ is monoidal it suffices to show that each component is a map, which is straightforward. Thus $\mathsf{Rel}(\alpha)$ is a monoidal lax transformation. Clearly Rel preserves composition and identities at the level of 1- and 2-cells as well as horizontal composition of 2-cells. We record:

Lemma 4.5.1. $\mathsf{Rel} : \mathsf{Reg} \to \mathsf{CW}_{\mathsf{cor}}$ *is a 2-functor.*

□

This 2-functor is the first half of our 2-equivalence. Our next goal will be to construct the second half of our 2-equivalence, which will be a 2-functor $\mathsf{Map} : \mathsf{CW}_{\mathsf{cor}} \to \mathsf{Reg}$. This will take some doing. We begin by introducing a notion of domain of definition for CW categories, which will play an important role.

Definition 4.5.2 ([32, 2.122])**.** For $f : X \to Y$ in a CW category define the *domain of definition* of f to be $\mathsf{dom}(f) = (1 \cap ff^\circ) : X \to X$. Note that the domain of definition of any arrow is coreflexive.

We establish some elementary properties of this domain of definition:

Lemma 4.5.3. *(i)* $\mathsf{dom}(f) = 1_X$ *if and only if* $f : X \to Y$ *is total*

(ii) $\mathsf{dom}(f)f = f$

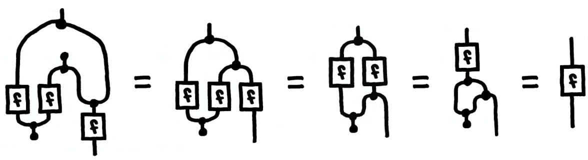
(iii) If f *is simple then* $\mathsf{dom}(f^\circ) = f^\circ f$

(iv) If f *is simple then* $\mathsf{dom}(f \cap g)f = f \cap g$

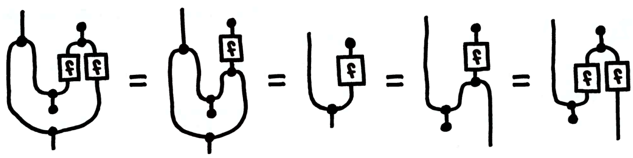
(v) If f *and* g *are simple then* $\mathsf{dom}(f \cap g)f = \mathsf{dom}(f \cap g)g$

Proof. (i) Note that we always have $\mathsf{dom}(f) = 1 \cap ff^\circ \leq 1_X$. If f is total then $1_X \leq ff^\circ$ and we have $\mathsf{dom}(f) = 1_X \cap ff^\circ \geq 1_X \cap 1_X = 1_X$ and so by antisymmetry $\mathsf{dom}(f) = 1_X$. Conversely, if $\mathsf{dom}(f) = 1_X$ then $1_X \cap ff^\circ = \mathsf{dom}(f) = 1_X$ and so $1_X \leq ff^\circ$, which means that f is total.

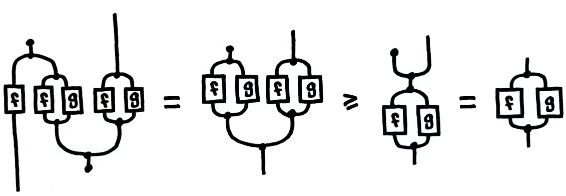
(ii) We have $\mathsf{dom}(f)f = (1 \cap ff^\circ)f = f$ as in:



(iii) We have $\mathsf{dom}(f^\circ) = 1 \cap f^\circ f = f^\circ f$ as in:



(iv) First, we have $\mathsf{dom}(f \cap g)f = (1_X \cap (f \cap g)(f \cap g)^\circ)f \geq f \cap g$ as in:



Conversely $\mathsf{dom}(f \cap g) = (1 \cap (f \cap g)(f \cap g)^\circ)f \leq (f \cap g)(f \cap g)^\circ f = (f \cap g)(f^\circ \cap g^\circ)f \leq (f \cap g)f^\circ f \leq (f \cap g)$, and so we have $f \cap g = \mathsf{dom}(f \cap g)f$ by antisymmetry.

(v) We have $\mathsf{dom}(f \cap g)f = f \cap g = g \cap f = \mathsf{dom}(g \cap f)g = \mathsf{dom}(f \cap g)g$.

□

Our 2-functor Map will send 0-cells of $\mathsf{CW}_{\mathsf{cor}}$ to their category of maps. We show that this results in a category with finite limits, and characterise its regular epics.

Lemma 4.5.4 ([32, 2.147])**.** *Let $\mathbb{X}$ be a CW category with split coreflexives. Then:*

(i) $\mathsf{Map}(\mathbb{X})$ *has finite limits*

(ii) A map $f : X \to Y$ in $\mathbb{X}$ is surjective if and only if it is regular epic in $\mathsf{Map}(\mathbb{X})$

(iii) Regular epics are stable under pullback in $\mathsf{Map}(\mathbb{X})$

(iv) Coequalisers of kernel pairs exist in $\mathsf{Map}(\mathbb{X})$

Proof. (i) We have already seen that $\mathsf{Map}(\mathbb{X})$ has finite products (Lemma 4.1.15). We show that it also has equalisers. To that end, suppose $f, g : X \to Y$ are arrows of $\mathsf{Map}(\mathbb{X})$, and let $\mathsf{dom}(f \cap g)$ split as in:

$$\begin{tikzcd} & E \arrow[r, "e"] & X \\ E \arrow[ur, equal] \arrow[r, "e"'] & X \arrow[u, "e^\circ"'] \arrow[ur, "\mathsf{dom}(f \cap g)"'] & \end{tikzcd}$$

Then Lemma 4.1.22 gives that e is a map. We show that the following diagram is an equaliser in $\mathsf{Map}(\mathbb{X})$:

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[r, phantom, "+"] & Y \end{tikzcd}$$

If we have $h : Z \to X$ with $hf = hg$ then $he^\circ : Z \to E$ is such that $he^\circ e = h\mathsf{dom}(f \cap g) = h(1 \cap (f \cap g)(f \cap g)^\circ) = h \cap h(f \cap g)(f \cap g)^\circ = (hf \cap hg)(f^\circ \cap g^\circ) = hff^\circ hgg^\circ \geq h \cap h = h$ and also $he^\circ e = h\mathsf{dom}(f \cap g) \leq h$, so by antisymmetry $he^\circ e = h$. We have that he° is simple as in $(he^\circ)^\circ he^\circ = eh^\circ he^\circ \leq ee^\circ = 1_E$ and total as in $he^\circ(he^\circ)^\circ = he^\circ eh^\circ = hh^\circ \geq 1_X$, and so he° is a map.

We must show that for any map $k : Z \to E$ with $ke = h$ we have $k = he^\circ$. In this case we have $ke = h = he^\circ e$ but then $k = he^\circ$ because e is monic. Thus $e : E \to X$ is the equaliser of f and g, from which it follows that $\mathsf{Map}(\mathbb{X})$ has finite limits.

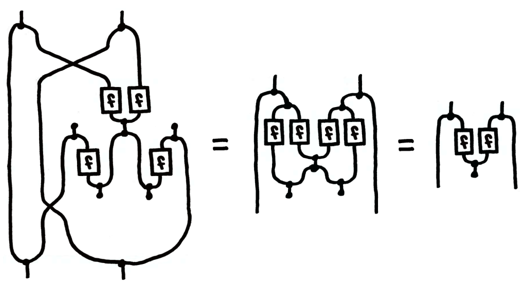
(ii) Suppose that $f : X \to Y$ is a surjective map in $\mathbb{X}$. Let $e : E \to X \times X$ be the equaliser of $\pi_0 f, \pi_1 f : X \times X \to Y$ in $\mathsf{Map}(\mathbb{X})$. In particular this means that e is the monic part of the splitting of $\mathsf{dom}(\pi_0 f \cap \pi_1 f)$ (below left), and that the square below right is a pullback in $\mathsf{Map}(\mathbb{X})$ (see Lemma 3.4.5):

$$
\begin{tikzcd}
 & E \arrow[r, "e"] & X \times X \\
E \arrow[ur, equal] \arrow[r, "e"'] & X \times X \arrow[u, "e^\circ"'] \arrow[ur, "\mathsf{dom}(\pi_0 f \cap \pi_1 f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "e\pi_0"] \arrow[d, "e\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

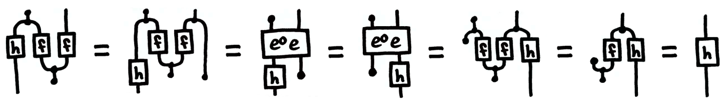
Thus the kernel pair of f is $e\pi_0, e\pi_1$. We must show that f is the coequaliser of $e\pi_0$ and $e\pi_1$. To that end, suppose we have $h : X \to Z$ with $e\pi_0 h = e\pi_1 h$. Then we have $f^\circ h : Y \to Z$ as in:

$$
\begin{tikzcd}
E \arrow[r, shift left, "e\pi_0"] \arrow[r, shift right, "e\pi_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, dashed, "f^\circ h"] \\
 & & Z
\end{tikzcd}
$$

To see that $ff^\circ h = h$, first notice that $\mathsf{dom}(\pi_0 f \cap \pi_1 f) = \mathsf{dom}((f \otimes f)\mu_Y) = (1 \cap (f \otimes f)\mu_Y \delta_Y (f^\circ \otimes f^\circ))$ may be written as follows:



Then using that $e^\circ e(h \otimes \varepsilon_X) = e^\circ e\pi_0 h = e^\circ e\pi_1 h = e^\circ e(\varepsilon_X \otimes h)$ we have $ff^\circ h = h$ as in:

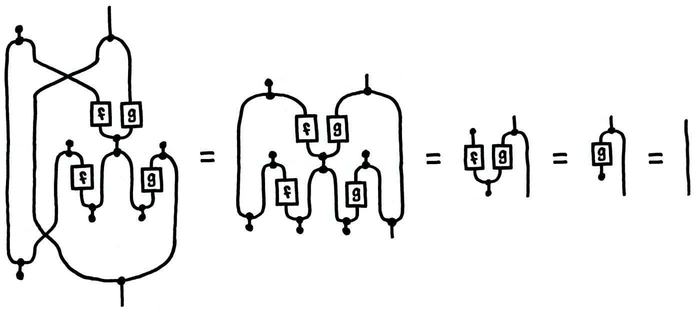


It follows that $f^\circ h$ is simple via $(f^\circ h)^\circ f^\circ h = h^\circ f f^\circ h = h^\circ h \leq 1_Z$ and that $f^\circ h$ is total via $f^\circ h(f^\circ h)^\circ = f^\circ h h^\circ f \geq f^\circ f = 1_Y$, so it is an arrow of $\mathsf{Map}(\mathbb{X})$. For any other map $k : Y \to Z$ with $fk = h$ we have $k = f^\circ h$ since f is epic (because it is surjective) and $fk = h = ff^\circ h$. Thus, f coequalises its kernel pair, and is regular epic. For the converse, if f is regular epic in $\mathsf{Map}(\mathbb{X})$ then f is epic, and so f is surjective.

(iii) Let $f : X \twoheadrightarrow Y$ be regular epic in $\mathsf{Map}(\mathbb{X})$, and let $g : Z \to Y$ be another map. Then Lemma 3.4.5 tells us that the pullback of f along g given by $m\pi_0$ and $m\pi_1$ (below left), where $m : M \to X \times Z$ is the monic part of the splitting of $\mathsf{dom}(\pi_0 f \cap \pi_1 g)$ (below right):

$$\begin{tikzcd} M \arrow[r, "m\pi_0"] \arrow[d, "m\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[r, "m"] & X \times Z \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \times Z \arrow[u, "m^\circ"'] \arrow[ur, "\mathsf{dom}(\pi_0 f \cap \pi_1 g)"'] & \end{tikzcd}$$

So our goal is to show that $m\pi_1 : M \to Z$ is regular epic. It suffices to show that $m\pi_1$ is surjective. Using that f is surjective and that f and g are maps, we have $(m\pi_1)^\circ m\pi_1 = \pi_1^\circ m^\circ m\pi_1 = \pi_1^\circ \mathsf{dom}(\pi_0 f \cap \pi_1 g)\pi_1 = \pi_1^\circ \mathsf{dom}((f \otimes g)\mu_Y)\pi_1 = \pi_1^\circ(1 \cap (f \otimes g)\mu_Y \delta_Y (f^\circ \otimes g^\circ))\pi_1 = 1_Z$ as in:



Thus $m\pi_1$ is surjective, and it follows that regular epics are stable under pullback in $\mathsf{Map}(\mathbb{X})$.

(iv) Suppose $f : X \to Y$ in $\mathsf{Map}(\mathbb{X})$, and let $e : E \to X \times X$ be the equaliser of $\pi_0 f, \pi_1 f : X \times X \to Y$. In particular this means that e splits $\mathsf{dom}(\pi_0 f \cap \pi_1 f)$ (below left) and that the square below on the right is a pullback in $\mathsf{Map}(\mathbb{X})$:

$$\begin{tikzcd} & E \arrow[r, "e"] & X \times X \\ E \arrow[ur, equal] \arrow[r, "e"'] & X \times X \arrow[u, "e^\circ"'] \arrow[ur, "\mathsf{dom}(\pi_0 f \cap \pi_1 f)"'] & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "e\pi_0"] \arrow[d, "e\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Thus $e\pi_0, e\pi_1$ is the kernel pair of f. Let $\mathsf{dom}(f^\circ)$ split as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & Y \\ M \arrow[ur, equal] \arrow[r, "m"'] & Y \arrow[u, "m^\circ"'] \arrow[ur, "\mathsf{dom}(f^\circ)"'] & \end{tikzcd}$$

We will show that the following diagram is a coequaliser in $\mathsf{Map}(\mathbb{X})$:

$$\begin{tikzcd} E \arrow[r, shift left=2, "e\pi_0"] \arrow[r, shift right=2, "e\pi_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "fm^\circ"] & M \end{tikzcd}$$

First, that f is total gives that fm° is total as in $fm^\circ(fm^\circ)^\circ = fm^\circ mf^\circ = f\mathsf{dom}(f^\circ)f^\circ = ff^\circ \geq 1_X$. Moreover, since f is simple we have that fm° is simple and surjective as in $1_M = mm^\circ mm^\circ = m\mathsf{dom}(f^\circ)m^\circ = mf^\circ fm^\circ = (fm^\circ)^\circ fm^\circ$. In particular, we have shown that fm° is regular epic in $\mathsf{Map}(\mathbb{X})$. Now, suppose we have $h : X \to Z$ with $e\pi_0 h = e\pi_1 h$ as in:

$$\begin{tikzcd} E \arrow[r, shift left=2, "e\pi_0"] \arrow[r, shift right=2, "e\pi_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "fm^\circ"] \arrow[dr, "h"'] & M \\ & & Z \end{tikzcd}$$

Notice that $\mathsf{dom}(\pi_0 f \cap \pi_1 f)\pi_0 h = e^\circ e\pi_0 h = e^\circ e\pi_1 h = \mathsf{dom}(\pi_0 f \cap \pi_1 f)\pi_1 h$, which gives $ff^\circ h$ as in part (ii) of this lemma. Then we have $fm^\circ mf^\circ h = f\mathsf{dom}(f^\circ)f^\circ h = ff^\circ h = h$ as in:

$$\begin{tikzcd} E \arrow[r, shift left=2, "e\pi_0"] \arrow[r, shift right=2, "e\pi_1"'] \arrow[r, phantom, "+"] & X \arrow[r, "fm^\circ"] \arrow[dr, "h"'] & M \arrow[d, dashed, "mf^\circ h"] \\ & & Z \end{tikzcd}$$

We show that $mf^\circ h : M \to Z$ is the unique such map. To that end, suppose that for some $k : M \to Z$ in $\mathsf{Map}(\mathbb{X})$ we have $fm^\circ k = h$. Then $fm^\circ k = h = ff^\circ h = f\mathsf{dom}(f^\circ)f^\circ h = fm^\circ mf^\circ h$, which gives $k = mf^\circ h$ since fm° is regular epic. It follows that our diagram is indeed a coequaliser, and we conclude that coequalisers of kernel pairs exist in $\mathsf{Map}(\mathbb{X})$.

$\square$

As a consequence of our lemma we have:

Theorem 4.5.5 ([32, 2.147])**.** *Let $\mathbb{X}$ be a CW category with split coreflexives. Then $\mathsf{Map}(\mathbb{X})$ is regular.*

$\square$

We show that the assignment of $\mathbb{X}$ to $\mathsf{Map}(\mathbb{X})$ extends to a 2-functor $\mathsf{Map} : \mathsf{CW}_{\mathsf{cor}} \to \mathsf{Reg}$. On 1-cells $F : \mathbb{X} \to \mathbb{Y}$ we define $\mathsf{Map}(F) : \mathsf{Map}(\mathbb{X}) \to \mathsf{Map}(\mathbb{Y})$ on objects by $\mathsf{Map}(F)(X) = FX$ and on arrows $f : X \to Y$ by $\mathsf{Map}(F)(f) = F(f) : FX \to FY$. That $\mathsf{Map}(F)$ preserves finite limits and regular epics is immediate, and so it is a regular functor. For 2-cells $\alpha : F \to G$, $\mathsf{Map}(\alpha)$ is defined by $\mathsf{Map}(\alpha_X) = \alpha_X : FX \to GX$. Now if $f : X \to Y$ in $\mathsf{Map}(\mathbb{X})$, we use the fact that

α is a monoidal lax transformation to obtain

$$\begin{aligned} \mathsf{Map}(F)(f)\mathsf{Map}(\alpha)_Y &= Ff\,\alpha_Y \\ &\leq \alpha_X\, Gf = \mathsf{Map}(\alpha)_X \mathsf{Map}(G)(f) \end{aligned}$$

By Lemma 4.1.17 we have that both sides of this inequation are maps, so Lemma 4.1.16 tells us that it is in fact an equation. Then we have that $\mathsf{Map}(\alpha)$ is a natural transformation. Clearly Map preserves composition and identities at the level of 1- and 2-cells as well as horizontal composition of 2-cells. We record:

Lemma 4.5.6. $\mathsf{Map} : \mathsf{CW}_{\mathsf{cor}} \to \mathsf{Reg}$ *is a 2-functor.*

□

This will be the second half of our strict 2-equivalence. To construct the required natural isomorphisms we require an intermediate result:

Lemma 4.5.7. *Let* $\mathbb{C}$ *be a regular category, and let* $\langle f_0, f_1\rangle \looparrowright f : X \to Y$ *in* $\mathsf{Rel}(\mathbb{C})$*. Then* f *is a map in* $\mathsf{Rel}(\mathbb{C})$ *if and only if* f_0 *is an isomorphism in* $\mathbb{C}$*.*

Proof. Let the kernel pairs of f_0 and f_1 be $a_0, a_1 : \mathsf{Ker}(f_0) \to F$ and $b_0, b_1 : \mathsf{Ker}(f_1) \to F$ respectively, as in:

$$\begin{tikzcd} \mathsf{Ker}(f_0) \arrow[r, "a_0"] \arrow[d, "a_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "f_0"] \\ F \arrow[r, "f_0"'] & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{Ker}(f_1) \arrow[r, "b_0"] \arrow[d, "b_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, "f_1"] \\ F \arrow[r, "f_1"'] & Y \end{tikzcd}$$

So in particular $f^\circ f$ is given by image of $\langle a_0 f_1, a_1 f_1\rangle : \mathsf{Ker}(f_0) \to Y \times Y$ and ff° is given by the image of $\langle b_0 f_0, b_1 f_0\rangle : \mathsf{Ker}(f_1) \to X \times X$.

Suppose f is a map. Then f is simple and total, and so we have $f^\circ f \leq 1_Y$ and $1_X \leq ff^\circ$. Then Lemma 4.3.9 gives:

$$\begin{tikzcd} & \mathsf{Ker}(f_0) \arrow[dl, "a_0 f_1"'] \arrow[dr, "a_1 f_1"] & \\ Y \arrow[dr, equal] & N \arrow[u, two heads, "n"'] \arrow[d, "u"] & Y \arrow[dl, equal] \\ & Y & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & M \arrow[u, two heads, "m"'] \arrow[d, "w"] & X \\ & \mathsf{Ker}(f_1) \arrow[ul, "b_0 f_0"] \arrow[ur, "b_1 f_0"'] & \end{tikzcd}$$

Now $a_0 f_1 = a_1 f_1$ since $n a_0 f_1 = u = n a_1 f_1$ and n is epic. Then we have $a_0\langle f_0, f_1\rangle = \langle a_0 f_0, a_0 f_1\rangle = \langle a_1 f_0, a_1 f_1\rangle = a_1\langle f_0, f_1\rangle$, and so $a_0 = a_1$ since $\langle f_0, f_1\rangle$ is monic. Then by Lemma 4.3.4 we have that f_0 is monic. Now since m is regular epic it is extremal epic (Lemma 4.3.8), which in particular gives that f_0 is an isomorphism since $m = w b_0 f_0$ and f_0 is monic.

For the converse, suppose that f_0 is an isomorphism. Then since f_0 is monic it follows that $\langle f_1, f_1\rangle \looparrowright f^\circ f$. We have:

$$\begin{tikzcd}
& F \arrow[dl, "f_1"'] \arrow[d, equal] \arrow[dr, "f_1"] & \\
Y \arrow[dr, equal] & F \arrow[d, "f_1"] & Y \arrow[dl, equal] \\
& Y &
\end{tikzcd}$$

and then by Lemma 4.3.9 we have $f^\circ f \leq 1_Y$, and so f is simple. To see that f is total, recall that $\langle 1_X, !_X\rangle \looparrowright \varepsilon_X$ in $\mathsf{Rel}(\mathbb{C})$. Then $f\varepsilon_Y$ in $\mathsf{Rel}(\mathbb{C})$ is given by the image of $\langle f_0, !_F\rangle$, and we have:

$$\begin{tikzcd}
& X \arrow[dl, equal] \arrow[dr, "!_X"] & \\
X & F \arrow[u, two heads, "f_0"'] & 1 \\
& F \arrow[ul, two heads, "f_0"] \arrow[u, equal] \arrow[ur, "!_F"'] &
\end{tikzcd}$$

where f_0 is regular epic because it is an isomorphism. Then Lemma 4.3.9 gives that $\varepsilon_X \leq f\varepsilon_Y$. Thus f is total, and so f is a map. The claim follows. $\square$

Now for the components of the first required natural isomorphism we have:

Lemma 4.5.8. *Let $\mathbb{C}$ be a regular category. Then there is an isomorphism $\phi_{\mathbb{C}} : \mathbb{C} \to \mathsf{Map}(\mathsf{Rel}(\mathbb{C}))$ defined to be the identity on objects, and defined on arrows $f : X \to Y$ of $\mathbb{C}$ by $\langle 1_X, f\rangle \looparrowright \phi_{\mathbb{C}}(f) : X \to Y$ in $\mathsf{Map}(\mathsf{Rel}(\mathbb{C}))$.*

Proof. It is immediate that $\phi_{\mathbb{C}}$ is a functor. Since $\phi_{\mathbb{C}}$ is identity-on-objects it suffices to show that it is full and faithful. To see that $\phi_{\mathbb{C}}$ is full, suppose $\langle f_0, f_1\rangle \looparrowright f : X \to Y$ in $\mathsf{Map}(\mathsf{Rel}(\mathbb{C}))$. Then since f is a map we know that f_0 is an isomorphism and then $\langle f_0, f_1\rangle$ and $\langle 1_X, f_0^{-1}f_1\rangle$ are equal as subobjects of $X \times Y$, so $f = \phi_{\mathbb{C}}(f_0^{-1}f_1)$. It follows that $\phi_{\mathbb{C}}$ is full. To see that $\phi_{\mathbb{C}}$ is faithful suppose that $\phi_{\mathbb{C}}(f) = \phi_{\mathbb{C}}(g)$ for $f, g : X \to Y$ in $\mathbb{C}$. Then $\langle 1_X, f\rangle$ and $\langle 1_X, g\rangle$ are equal as subobjects of $X \times Y$, which means there is an isomorphism $\alpha : X \to X$ with $1_X = \alpha 1_X$ and $f = \alpha g$. Then $g = 1_X g = \alpha g = f$, and it follows that $\phi_{\mathbb{C}}$ is faithful. Thus, $\phi_{\mathbb{C}}$ is an isomorphism. $\square$

With the natural isomorphism itself given as in:

Lemma 4.5.9. *There is an invertible strict 2-natural transformation $\phi : 1_{\mathsf{REG}} \to \mathsf{Map} \circ \mathsf{Rel}$ with components $\phi_{\mathbb{C}} : \mathbb{C} \to \mathsf{Map}(\mathsf{Rel}(\mathbb{C}))$.*

Proof. We show that this defines a strict 2-natural transformation. To that end, suppose that $F : \mathbb{C} \to \mathbb{D}$ is a 1-cell of Reg. Then we have:

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\phi_{\mathbb{C}}"] \arrow[d, "F"'] & \mathsf{Map}(\mathsf{Rel}(\mathbb{C})) \arrow[d, "\mathsf{Map}(\mathsf{Rel})(F))"] \\
\mathbb{D} \arrow[r, "\phi_{\mathbb{D}}"'] & \mathsf{Map}(\mathsf{Rel}(\mathbb{D}))
\end{tikzcd}
$$

as follows: both composites send X in $\mathbb{C}$ to FX in $\mathbb{D}$. On arrows $f : X \to Y$ of $\mathbb{C}$, $\mathsf{Map}(\mathsf{Rel}(F))(\phi_{\mathbb{C}}(f)) = \mathsf{Map}(\mathsf{Rel}(F))(\langle 1_X, f\rangle \looparrowright \phi_{\mathbb{C}}(f)) = \mathsf{Rel}(F)(\langle 1_X, f\rangle \looparrowright \phi_{\mathbb{C}}(f)) = \langle 1_{FX}, F(f)\rangle \looparrowright F(\phi_{\mathbb{C}}(f)) = \phi_{\mathbb{D}}(F(f))$. For any 1-cells $F, G : \mathbb{C} \to \mathbb{D}$ and 2-cell $\beta : F \to G$ it is straightforward to verify that $\beta \star 1_{\phi_{\mathbb{D}}} = 1_{\phi_{\mathbb{C}}} \star \mathsf{Map}(\mathsf{Rel}(\beta))$. The claim follows. $\square$

The components of the second required natural isomorphism are given by:

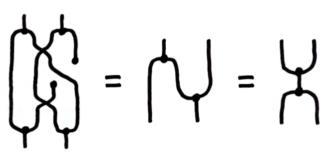
Lemma 4.5.10. *Let $\mathbb{C}$ be a CW category with split coreflexives. Then there is an isomorphism $\psi_{\mathbb{C}} : \mathbb{C} \to \mathsf{Rel}(\mathsf{Map}(\mathbb{C}))$ which maps $f : X \to Y$ in $\mathbb{C}$ to $m \looparrowright \psi_{\mathbb{C}}(f) : X \to Y$ in $\mathsf{Rel}(\mathsf{Map}(\mathbb{C}))$ where $m : M \rightarrowtail X \times Y$ splits $1 \cap \pi_0 f \pi_1^\circ$ in $\mathbb{C}$, as in:*

$$
\begin{tikzcd}
 & M \arrow[r, "m"] & X \times Y \\
M \arrow[ur, equal] \arrow[r, "m"'] & X \times Y \arrow[u, "m^\circ"'] \arrow[ur, "1\cap\pi_0 f\pi_1^\circ"'] &
\end{tikzcd}
$$

Proof. We first show that $\psi_{\mathbb{C}}$ is a functor. We begin by showing that $\psi_{\mathbb{C}}$ preserves identities. Consider $\psi_{\mathbb{C}}(1_X) : X \to X$ given by the monic part $m : M \rightarrowtail X \times X$ of the splitting of $1 \cap \pi_0\pi_1^\circ$, as in:

$$
\begin{tikzcd}
 & M \arrow[r, "m"] & X \times X \\
M \arrow[ur, equal] \arrow[r, "m"'] & X \times X \arrow[u, "m^\circ"'] \arrow[ur, "1\cap\pi_0\pi_1^\circ"'] &
\end{tikzcd}
$$

Notice that $1 \cap \pi_0\pi_1^\circ = \mu_X\delta_X$ as in:



This means that δ_X also splits $1 \cap \pi_0\pi_1^\circ$, as in:

$$\begin{tikzcd}
 & X \arrow[r, "\delta_X"] & X \times X \\
X \arrow[ur, equal] \arrow[r, "\delta_X"'] & X \times X \arrow[u, "\mu_X"] \arrow[ur, "1\cap\pi_0\pi_1^\circ = \mu_X\delta_X"'] &
\end{tikzcd}$$

And so $\delta_X : X \rightarrowtail X \times X$ and $m : M \rightarrowtail X \times X$ equal as subobjects of $X \times X$. Then $\langle 1_X, 1_X \rangle \looparrowright 1_X = \psi_{\mathbb{C}}(1_X)$ in $\mathsf{Rel}(\mathsf{Map}(\mathbb{C}))$, as required.

Next, we show that $\psi_{\mathbb{C}}$ preserves composition. To that end, suppose we have arrows $f : X \to Y$ and $g : Y \to Z$ of $\mathbb{C}$. Suppose further that $m \looparrowright \psi_{\mathbb{C}}(f)$ such that m and n split $1 \cap \pi_0 f \pi_1^\circ$ and $1 \cap \pi_0 g \pi_1^\circ$ respectively, as in:

$$\begin{tikzcd}
 & M \arrow[r, "m"] & X \times Y \\
M \arrow[ur, equal] \arrow[r, "m"'] & X \times Y \arrow[u, "m^\circ"] \arrow[ur, "1\cap\pi_0 f\pi_1^\circ"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N \arrow[r, "n"] & Y \times Z \\
N \arrow[ur, equal] \arrow[r, "n"'] & Y \times Z \arrow[u, "n^\circ"] \arrow[ur, "1\cap\pi_0 g\pi_1^\circ"'] &
\end{tikzcd}$$

and let $p \looparrowright \psi_{\mathbb{C}}(fg)$ such that p splits $1 \cap \pi_0 f g \pi_1^\circ$ as in:

$$\begin{tikzcd}
 & P \arrow[r, "p"] & X \times Z \\
P \arrow[ur, equal] \arrow[r, "p"'] & X \times Z \arrow[u, "p^\circ"] \arrow[ur, "1\cap\pi_0 fg\pi_1^\circ"'] &
\end{tikzcd}$$

Now, let $e : E \rightarrowtail M \times N$ split $\mathsf{dom}(\pi_0 m \pi_1 \cap \pi_1 n \pi_0)$ (below left). It follows that the square below right is a pullback:

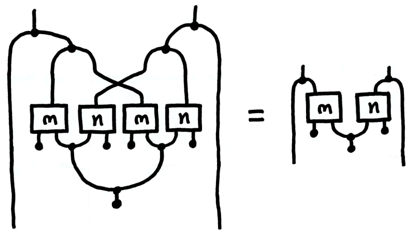
$$\begin{tikzcd}
 & E \arrow[r, "e"] & M \times N \\
E \arrow[ur, equal] \arrow[r, "e"'] & M \times N \arrow[u, "e^\circ"] \arrow[ur, "\mathsf{dom}(\pi_0 m\pi_1\cap\pi_1 n\pi_0)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "e\pi_1"] \arrow[d, "e\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, "n\pi_0"] \\
M \arrow[r, "m\pi_1"'] & Y
\end{tikzcd}$$

Then we have $q \looparrowright \psi_{\mathbb{C}}(f)\psi_{\mathbb{C}}(g)$ where $q : Q \rightarrowtail X \times Z$ is the image of $\langle e\pi_0 m\pi_0, e\pi_1 n\pi_1 \rangle = e(m\pi_0 \otimes n\pi_1) : E \to X \times Z$, as in:

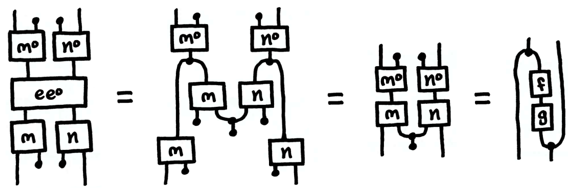
$$\begin{tikzcd}
E \arrow[rr, "e(m\pi_0\otimes n\pi_1)"] \arrow[dr, two heads, "i"'] & & X \times Z \\
 & Q \arrow[ur, tail, "q"'] &
\end{tikzcd}$$

Now, notice that $e^\circ e = \mathsf{dom}(\pi_0 m\pi_1 \cap \pi_1 n\pi_0) = 1 \cap (\pi_0 m\pi_1 \cap \pi_1 n\pi_0)(\pi_0 m\pi_1 \cap \pi_1 n\pi_0)^\circ$

may be written as follows:



It follows that q splits $1\cap\pi_0 fg\pi_1^\circ$: we have $qq^\circ = 1_Q$ because q is total and injective (being a monic map), and then since i is a surjective map we have $i^\circ i = 1_E$, which gives $q^\circ q = q^\circ i^\circ iq = (iq)^\circ iq = (e(m\pi_0 \otimes n\pi_1))^\circ e(m\pi_0 \otimes n\pi_1) = (\pi_0^\circ m^\circ \otimes \pi_1^\circ n^\circ)e^\circ e(m\pi_0 \otimes n\pi_1) = 1 \cap \pi_0 fg\pi_1^\circ$ as in:

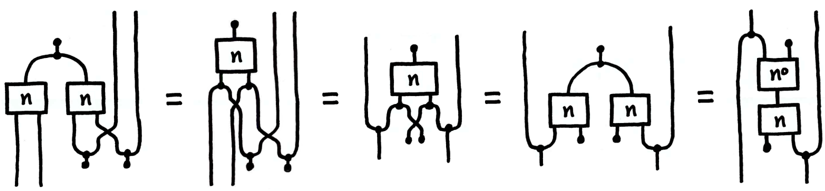


Then since q and p both split $1\cap\pi_0 fg\pi_1^\circ$ we know that they are the same subobject of $X \times Z$, which means that $\psi_{\mathbb{C}}(f)\psi_{\mathbb{C}}(g) = \psi_{\mathbb{C}}(fg)$. Thus, $\psi_{\mathbb{C}}$ is a functor.

It remains to show that $\psi_{\mathbb{C}}$ is full and faithful. To see that $\psi_{\mathbb{C}}$ is full, suppose $n \looparrowright f : X \to Y$ in $\mathsf{Rel}(\mathsf{Map}(\mathbb{C}))$. Consider $(n\pi_0)^\circ n\pi_1 : X \to Y$ in $\mathbb{C}$, and let $m \looparrowright \psi_{\mathbb{C}}((n\pi_0)^\circ n\pi_1)$ with m as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \times Y \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \times Y \arrow[u, "m^\circ"'] \arrow[ur, "1\cap\pi_0(n\pi_0)^\circ n\pi_1\pi_1^\circ"'] & \end{tikzcd}$$

Now $nn^\circ = 1_N$ because n is a monic map, and we have $n^\circ n = 1\cap\pi_0(n\pi_0)^\circ n\pi_1\pi_1^\circ = m^\circ m$ as in:



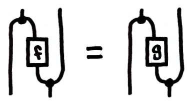
It follows that m and n are the same subobject of $X \times Y$, and so $\psi_{\mathbb{C}}((n\pi_0)^\circ n\pi_1) = f$ and $\psi_{\mathbb{C}}$ is full.

To see that $\psi_{\mathbb{C}}$ is faithful, suppose that we have $f, g : X \to Y$ in $\mathbb{C}$ with

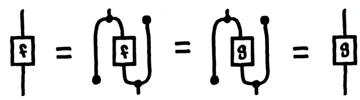
$\psi_{\mathbb{C}}(f) = \psi_{\mathbb{C}}(g)$, where $m \looparrowright \psi_{\mathbb{C}}(f)$ and $n \looparrowright \psi_{\mathbb{C}}(g)$ with m and n as in:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \times Y \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \times Y \arrow[u, "m^\circ"'] \arrow[ur, "1 \cap \pi_0 f \pi_1^\circ"'] & \end{tikzcd} \qquad \begin{tikzcd} & N \arrow[r, "n"] & X \times Y \\ N \arrow[ur, equal] \arrow[r, "n"'] & X \times Y \arrow[u, "n^\circ"'] \arrow[ur, "1 \cap \pi_0 n \pi_1^\circ"'] & \end{tikzcd}$$

Now since $\psi_{\mathbb{C}}(f) = \psi_{\mathbb{C}}(g)$ we know that n and m are the same subobject of $X \times Y$, which is to say that there is an isomorphism $\alpha : M \to N$ with $\alpha n = m$. But then $1 \cap \pi_0 f \pi_1^\circ = m^\circ m = (\alpha n)^\circ \alpha n = n^\circ \alpha^\circ \alpha n = n^\circ n = 1 \cap \pi_0 g \pi_1^\circ$. As an equation of string diagrams, this is:



But then we have $f = g$ as in:



and so $\psi_{\mathbb{C}}$ is faithful. Thus, $\psi_{\mathbb{C}}$ is an isomorphism. $\square$

With the natural isomorphism itself as in:

Lemma 4.5.11. *There is an invertible strict 2-natural transformation* $\psi : 1_{\mathsf{CW}_{\mathsf{cor}}} \to \mathsf{Rel} \circ \mathsf{Map}$ *with components* $\psi_{\mathbb{X}} : \mathbb{X} \to \mathsf{Rel}(\mathsf{Map}(\mathbb{X}))$.

Proof. We show that this defines a strict 2-natural transformation. To that end, suppose $F : \mathbb{X} \to \mathbb{Y}$ is a 1-cell of $\mathsf{CW}_{\mathsf{cor}}$. Then we have:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\psi_{\mathbb{X}}"] \arrow[d, "F"'] & \mathsf{Rel}(\mathsf{Map}(\mathbb{X})) \arrow[d, "\mathsf{Rel}(\mathsf{Map}(F))"] \\ \mathbb{Y} \arrow[r, "\psi_{\mathbb{Y}}"'] & \mathsf{Rel}(\mathsf{Map}(\mathbb{Y})) \end{tikzcd}$$

as follows: on objects both composites map X in $\mathbb{X}$ to FX in $\mathsf{Rel}(\mathsf{Map}(\mathbb{Y}))$. On arrows $f : X \to Y$ of $\mathbb{X}$, let $1 \cap \pi_0 f \pi_1^\circ$ split as follows:

$$\begin{tikzcd} & M \arrow[r, "m"] & X \times Y \\ M \arrow[ur, equal] \arrow[r, "m"'] & X \times Y \arrow[u, "m^\circ"'] \arrow[ur, "1 \cap \pi_0 f \pi_1^\circ"'] & \end{tikzcd}$$

Then $\mathsf{Rel}(\mathsf{Map}(F))(\psi_{\mathbb{X}}(f)) = \mathsf{Rel}(\mathsf{Map}(F))(m \looparrowright \psi_{\mathbb{X}}(f)) = F(m) \looparrowright F(\psi_{\mathbb{X}}(f))$.

Now, let $n \looparrowright \psi_{\mathbb{Y}}(F(f)) : FX \to FY$. That is, let n split $1 \cap \pi_0 F(f)\pi_1^\circ$ as in:

$$\begin{tikzcd}
& N \arrow[r, "n"] & FX \times FY \\
N \arrow[ur, equal] \arrow[r, "n"'] & FX \times FY \arrow[u, "n^\circ"'] \arrow[ur, "1\cap\pi_0F(f)\pi_1^\circ"'] &
\end{tikzcd}$$

Then we have $F(m)F(m)^\circ = 1$ and $F(m)^\circ F(m) = F(m^\circ m) = F(1 \cap \pi_0 \cap \pi_1^\circ) = 1 \cap \pi_0 F(f)\pi_1^\circ$, so m and n are equal as subobjects of $FX \times FY$, which gives $\mathsf{Rel}(\mathsf{Map}(F))(\psi_{\mathbb{X}}(f)) = \psi_{\mathbb{Y}}(F(f))$ as required. For any 1-cells $F, G : \mathbb{C} \to \mathbb{D}$ and 2-cell $\beta : F \to G$ it is straightforward to verify that $1_{\psi_{\mathbb{X}}} \star \mathsf{Rel}(\mathsf{Map}(\beta)) = \beta \star 1_{\psi_{\mathbb{Y}}}$. The claim follows. □

The promised strict 2-equivalence between $\mathsf{CW_{cor}}$ and Reg is thus established. We record:

Theorem 4.5.12. *There is a strict 2-equivalence:*

$$\mathsf{CW_{cor}} \underset{\mathsf{Rel}}{\overset{\mathsf{Map}}{\underset{\simeq}{\rightleftarrows}}} \mathsf{Reg}$$

□

4.6 Varieties and Morita Equivalence

In this section we construct a variety theorem for relational algebraic theories. We first recall the notion of *exact category*, which is a regular category satisfying a condition. Next, we recapitulate the main results surrounding *definable categories* and their connection to regular and exact categories. Specifically there is an analogue of Gabriel-Ulmer duality in which exact categories are seen to correspond to definable categories. Our variety theorem then follows immediately from our development up to this point: the categories that arise as models and model morphisms of relational algebraic theories are precisely the definable categories. This gives an analogue of Theorem 2.4.28 for relational algebraic theories. We end with a discussion of Morita equivalence for relational algebraic theories, in which we find that two relational algebraic theories present the same category of models and model morphisms if and only if they have equivalent idempotent splitting completions. This gives an analogue of Theorem 2.4.30 for relational algebraic theories.

We begin by recalling the closely related notions of effectivity and exactness:

Definition 4.6.1 ([32])**.** A CW category $\mathbb{X}$ is *effective* in case all partial equivalence relations in $\mathbb{X}$ split.

Definition 4.6.2 ([13])**.** A regular category $\mathbb{C}$ is *exact* in case $\mathsf{Rel}(\mathbb{C})$ is effective. Let EX be the full 2-subcategory of Reg on the exact 0-cells.

It is straightforward to verify that Theorem 4.5.12 restricts to the effective case:

Proposition 4.6.3. *If $\mathbb{X}$ is an effective relational algebraic theory, then $\mathsf{Map}(\mathbb{X})$ is exact. Conversely, if $\mathbb{C}$ is an exact category, then $\mathsf{Rel}(\mathbb{C})$ is effective. This extends to a strict 2-equivalence:*

$$\begin{tikzcd} \mathsf{CW}_{\mathsf{per}} \arrow[r, shift left, "\mathsf{Map}"] \arrow[r, phantom, "\simeq"] & \mathsf{EX} \arrow[l, shift left, "\mathsf{Rel}"] \end{tikzcd}$$

We may therefore give the exact completion of a regular category as follows:

Proposition 4.6.4 ([13, 40])**.** *If $\mathbb{C}$ is regular, define the* exact completion *of $\mathbb{C}$ by*

$$\mathbb{C}_{\mathsf{ex/reg}} = \mathsf{Map}(\mathsf{Split}_{\mathsf{eq}}(\mathsf{Rel}(\mathbb{X})))$$

Then $\mathbb{C}_{\mathsf{ex/reg}}$ is exact. This extends to a biadjunction:

$$\begin{tikzcd} \mathsf{Reg} \arrow[r, shift left, "\mathsf{ex/reg}"] \arrow[r, phantom, "\perp"] & \mathsf{EX} \arrow[l, shift left, hook'] \end{tikzcd}$$

where the right adjoint is the evident forgetful 2-functor.

We summarize the relationship of regularity and exactness to CW categories:

Corollary 4.6.5. *The following diagram of left biadjoints commutes:*

$$\begin{tikzcd} \mathsf{CW}_{\mathsf{cor}} \arrow[r, "\mathsf{Map}", "\sim"'] \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"'] & \mathsf{Reg} \arrow[d, "\mathsf{ex/reg}"] \\ \mathsf{CW}_{\mathsf{per}} \arrow[r, "\sim", "\mathsf{Map}"'] & \mathsf{EX} \end{tikzcd}$$

where the arrows marked with $\sim$ are part of a biequivalence.

The final idea involved in our variety theorem is that of a definable category [38]. Definable categories come from categorical universal algebra. If we take regular categories as our notion of theory, regular functors into Set as our notion of model, and natural transformations as our model morphisms, then definable categories are the corresponding varieties. We follow the exposition of [40], and in particular we formulate definable categories via finite injectivity classes:

Definition 4.6.6 (Finite Injectivity Class)**.** Let $h : A \to B$ be an arrow of $\mathbb{X}$. Then an object C of $\mathbb{X}$ is said to be *h-injective* in case the function of hom-sets $\mathbb{X}(h, C) : \mathbb{X}(B, C) \to \mathbb{X}(A, C)$ defined by $X(h, C)(f) = hf$ is injective. If M is a finite set of arrows in $\mathbb{X}$, write $\mathsf{inj}(M)$ for the full subcategory on the objects C of $\mathbb{X}$

that are h-injective for each $h \in M$. We say that each $\mathsf{inj}(M)$ is a *finite injectivity class* in $\mathbb{X}$.

Definable categories are defined relative to an ambient locally finitely presentable category. It is an open problem to give a free-standing characterization [38].

Definition 4.6.7. A category is said to be *definable* if it arises as a finite injectivity class in some locally finitely presentable category. Every definable category has products and directed colimits [38]. If $\mathbb{X}$ and $\mathbb{Y}$ are definable categories, a functor $F : \mathbb{X} \to \mathbb{Y}$ is called an *interpretation* in case it preserves products and directed colimits. Let DEF be the 2-category with definable categories as 0-cells, interpretations as 1-cells, and natural transformations as 2-cells.

For example, any locally finitely presentable category is a definable category, and in particular this means that Set is definable. From any definable category we can obtain an exact category by considering its interpretations into Set.

Proposition 4.6.8 ([40])**.** *If $\mathbb{X}$ is a definable category then the functor category $\mathsf{DEF}(\mathbb{X}, \mathsf{Set})$ is an exact category. This extends to a 2-functor $\mathsf{DEF}(-, \mathsf{Set}) : \mathsf{DEF}^{\mathsf{op}} \to \mathsf{EX}$.*

Similarly, for any regular category the associated category of regular functors into Set is definable.

Proposition 4.6.9 ([40])**.** *If $\mathbb{C}$ is a regular category then the functor category*[1] *$\mathsf{Reg}(\mathbb{C}, \mathsf{Set})$ is definable. This extends to a 2-functor $\mathsf{Reg}(-, \mathsf{Set}) : \mathsf{Reg} \to \mathsf{DEF}^{\mathsf{op}}$.*

If the category in question is exact, then considering interpretations of the resulting definable category into Set yields the original exact category. This lifts to the 2-categorical setting.

Proposition 4.6.10 ([40])**.** *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{Reg} \arrow[r, shift left=1.5ex, "\mathsf{Reg}(-,\mathsf{Set})"] \arrow[r, phantom, "\perp"] & \mathsf{DEF}^{\mathsf{op}} \arrow[l, shift left=1.5ex, "\mathsf{DEF}(-,\mathsf{Set})"] \end{tikzcd}$$

Which specializes to a biequivalence:

$$\begin{tikzcd} \mathsf{EX} \arrow[r, shift left=1.5ex, "\mathsf{Reg}(-,\mathsf{Set})"] \arrow[r, phantom, "\simeq"] & \mathsf{DEF}^{\mathsf{op}} \arrow[l, shift left=1.5ex, "\mathsf{DEF}(-,\mathsf{Set})"] \end{tikzcd}$$

This gives another way to describe the exact completion of a regular category:

[1]Strictly speaking it does not make sense to write $\mathsf{Reg}(\mathbb{C}, \mathsf{Set})$, since the 0-cells of Reg are *small* and Set is not small. Nonetheless, there is a category of regular functors $\mathbb{C} \to \mathsf{Set}$, which we denote $\mathsf{Reg}(\mathbb{C}, \mathsf{Set})$.

Proposition 4.6.11 ([40]). *If $\mathbb{C}$ is regular then $\mathbb{C}_{\mathsf{ex/reg}} \simeq \mathsf{DEF}(\mathsf{Reg}(\mathbb{C}, \mathsf{Set}), \mathsf{Set})$.*

Thus, we may summarize the relationship between definable, regular, and exact categories as follows:

Corollary 4.6.12 ([40, Section 9,10]). *The following diagram of left biadjoints commutes.*

$$\begin{tikzcd} \mathsf{Reg} \arrow[d, "\mathsf{ex/reg}"'] \arrow[dr, "\mathsf{Reg}(-,\mathsf{Set})"] & \\ \mathsf{EX} \arrow[r, "\sim", "\mathsf{Reg}(-,\mathsf{Set})"'] & \mathsf{DEF}^{\mathsf{op}} \end{tikzcd}$$

where the arrow marked with $\sim$ is part of a biequivalence.

The ingredients of our variety theorem for relational algebraic theories are now assembled. Together, Corollary 4.1.33, Corollary 4.6.5, and Corollary 4.6.12 give:

Corollary 4.6.13. *The following diagram of left biadjoints commutes:*

$$\begin{tikzcd} \mathsf{CW} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[dr, "\mathsf{Split}_{\mathsf{per}}"'] & \mathsf{CW}_{\mathsf{cor}} \arrow[r, "\mathsf{Map}", "\sim"'] \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"] & \mathsf{Reg} \arrow[d, "\mathsf{ex/reg}"'] \arrow[dr, "\mathsf{Reg}(-,\mathsf{Set})"] & \\ & \mathsf{CW}_{\mathsf{per}} \arrow[r, "\sim", "\mathsf{Map}"'] & \mathsf{EX} \arrow[r, "\sim", "\mathsf{Reg}(-,\mathsf{Set})"'] & \mathsf{DEF}^{\mathsf{op}} \end{tikzcd}$$

where the arrows marked with $\sim$ are part of a biequivalence.

Now our variety theorem is an immediate consequence of Corollary 4.6.13:

Theorem 4.6.14. *There is a biadjunction:*

$$\begin{tikzcd} \mathsf{CW} \arrow[r, shift left, "\mathsf{Mod}"] \arrow[r, phantom, "\perp"] & \mathsf{DEF}^{\mathsf{op}} \arrow[l, shift left, "\mathsf{Th}"] \end{tikzcd}$$

It may not be immediately clear what this tells us about the category of models and model morphisms of a relational algebraic theory, so let us briefly discuss. Consider an arbitrary relational algebraic theory $\mathbb{X}$. Our universe of models Rel has split idempotents, so models of $\mathbb{X}$ and models of $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$ are the same thing since the image of any coreflexive in $\mathbb{X}$ already splits in Rel. Then the category of models of $\mathbb{X}$ and model morphisms thereof is $\mathsf{CW}_{\mathsf{cor}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}), \mathsf{Rel})$. When we transport this across the 2-equivalence $\mathsf{Map} : \mathsf{CW}_{\mathsf{cor}} \xrightarrow{\sim} \mathsf{Reg}$ it becomes $\mathsf{Reg}(\mathsf{Map}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})), \mathsf{Set})$, a definable category. Thus, categories of models and model morphisms of regular algebraic theories are definable categories.

Now, Set is exact, so Rel is effective, which means that much like the models of $\mathbb{X}$ and $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$, the models of $\mathbb{X}$ and $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$ are the same. We have shown that $\mathsf{CW}_{\mathsf{per}} \simeq \mathsf{EX} \simeq \mathsf{DEF}^{\mathsf{op}}$, and so the question of when two relational algebraic theories generate the same category of models and model morphisms can be answered as follows (using Corollary 4.1.31):

Theorem 4.6.15. *Two relational algebraic theories* $\mathbb{X}$ *and* $\mathbb{Y}$ *present equivalent definable categories if and only if* $\mathsf{Split}(\mathbb{X})$ *and* $\mathsf{Split}(\mathbb{Y})$ *are equivalent.*

This is encouraging, given that Morita equivalence for algebraic theories and partial algebraic theories is characterised in the same way.

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Abstract

Partial and Relational Algebraic Theories

This thesis introduces notions of partial algebraic theory and relational algebraic theory, in which operations are interpreted as partial functions and as relations, respectively. The development focuses on the monoidal category structure of partial functions and relations. In particular, partial and relational algebraic theories are both intuitively presentable by means of string diagrams for monoidal categories, which play the role of terms. The varieties — those categories that arise as the category of models of a given theory — are rigorously characterised for both partial algebraic theories and relational algebraic theories. Specifically, the varieties associated with partial algebraic theories are precisely the locally finitely presentable categories, while the varieties associated with relational algebraic theories are precisely the definable categories.

Kokkuvõte

Osalised ja relatsioonilised algebralised teooriad

Käesolev doktoritöö toob sisse osalise ning relatsioonilise algebralise teooria mõisted, kus tehete interpretatsiooniks on vastavalt osalised funktsioonid ning relatsioonid. Töö põhiosa keskendub osaliste funktsioonide ning relatsioonide kategooriate monoidilisele struktuurile. Osalisi ning relatsioonilisi algebralisi teooriaid saab kumbagi näitlikult esitada monoidiliste kategooriate nöördiagrammide abiga, mis täidavad siin termide rolli. Muutkondadele — kategooriatele, mis on mingi teooria mudelite kategooriaks — on leitud karakterisatsioonid nii osaliste kui ka relatsiooniliste algebraliste teooriate juhtumil. Selgub, et osalistele algebralistele teooriatele vastavad muutkonnad on täpselt lokaalselt lõplikult esitatavad kategooriad, samas kui relatsioonilistele algebralistele teooriatele vastavad muutkonnad on täpselt defineeritavad kategooriad.

Appendix A

Publications

A.1 A Foundation for Ledger Structures

C. Nester. A Foundation for Ledger Structures. In *International Conference on Blockchain Economics, Security and Protocols*, volume 82 of *Open Access Series in Informatics (OASIcs)*, pages 7:1-7:13, 2021.

A Foundation for Ledger Structures

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Abstract

This paper introduces an approach to constructing ledger structures for cryptocurrency systems with basic category theory. Compositional theories of resource convertibility allow us to express the material history of virtual goods, and ownership is modelled by a free construction. Our notion of ownership admits an intuitive graphical representation through string diagrams for monoidal functors.

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Keywords and phrases String Diagrams, Category Theory, Blockchains

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1 Introduction

Modern cryptocurrency systems consist of two largely orthogonal parts: A consensus protocol, and the ledger structure it is used to maintain. While consensus protocols have received a lot of attention (see e.g. [10, 7]), the design space of the accompanying ledger structures is barely explored. The recent interest in smart contracts has led to the development of sophisticated ledger structures with complex behaviour (see e.g. [1, 13]). These efforts have been largely *ad hoc*, and the resulting ledger structures are difficult to reason about. This difficulty also manifests in the larger system, which has contributed to several unfortunate incidents involving blockchain technology [2].

A strong mathematical foundation for ledger structures would enable more rigorous development of sophisticated blockchain systems. Further, the ability to reason about the ledger at a high level of abstraction would facilitate analysis of system behaviour. This is important: users of the system must understand it in order to use it with confidence. The formalism we propose has an intuitive graphical representation, which would make this kind of rigorous operational understanding possible on a far wider scale that it would otherwise be.

Blockchain systems are largely concerned with recording the material history of virtual objects, with a particular focus on changes in ownership. The resource theoretic interpretation of string diagrams for symmetric monoidal categories gives a precise mathematical meaning to this sort of material history. Building on this, we consider string diagrams augmented with extra information concerning the ownership of resources. We give these diagrams a precise mathematical meaning in terms of strong monoidal functors, drawing heavily on the work of [9], where our augmented diagrams originated. We show that an augmented resource theory has the same categorical structure as the original, in the sense that the two corresponding categories are equivalent. Finally, we give a simple example of a ledger structure using our machinery.

2nd International Conference on Blockchain Economics, Security and Protocols (Tokenomics 2020).
Editors: Emmanuelle Anceaume, Christophe Bisière, Matthieu Bouvard, Quentin Bramas, and Catherine Casamatta; Article No. 7; pp. 7:1–7:13
OpenAccess Series in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

2 Monoidal Categories as Resource Theories

We assume familiarity with some basic category theory, in particular with symmetric monoidal categories. A good reference is [8]. Throughout, we will write composition in diagrammatic order. That is, the composite of $f : X \to Y$ and $g : Y \to Z$ is written $fg : X \to Z$. We may also write $g \circ f : X \to Z$, but we will *never* write $gf : X \to Z$. We will make heavy use of string diagrams for monoidal categories (see e.g. [11]), which we read from top to bottom (for composition) and left to right (for the monoidal tensor). Our string diagrams for ownership are in fact the string diagrams for monoidal functors of [9].

2.1 Resource Theories

We begin by observing (after [4]) that a symmetric strict monoidal category can be interpreted as a theory of resource convertibility: Each object corresponds to collection of resources with $A \otimes B$ denoting the collection composed of both A and B and the unit I denoting the empty collection. Morphisms $f : A \to B$ are then understood as a way to convert the resources of A to those of B.

For example, consider the free symmetric strict monoidal category on the set

$$\{\mathtt{bread}, \mathtt{dough}, \mathtt{water}, \mathtt{flour}, \mathtt{oven}\}$$

of atomic objects, subject to the following additional axioms:

$$\mathtt{mix} : \mathtt{water} \otimes \mathtt{flour} \to \mathtt{dough} \qquad \mathtt{knead} : \mathtt{dough} \to \mathtt{dough}$$

$$\mathtt{bake} : \mathtt{dough} \otimes \mathtt{oven} \to \mathtt{bread} \otimes \mathtt{oven}$$

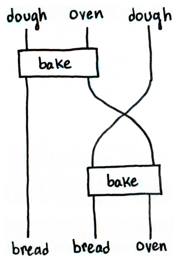
This category can be understood as a theory of resource convertibility for baking bread. The morphism `mix` represents the process of combining water and flour to form a bread dough, `knead` the process of kneading the dough, and `bake` the process of baking the dough in an oven to yield bread (and an oven). While this model has many failings as a theory of bread, it suffices to illustrate the idea. The axioms of a symmetric strict monoidal category provide a natural scaffolding for this theory to live in. For example, consider the morphism

$$(\mathtt{bake} \otimes 1_{\mathtt{dough}})(1_{\mathtt{bread}} \otimes \sigma_{\mathtt{oven},\mathtt{dough}}\mathtt{bake})$$

where $\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ is the braiding. This morphism has type

$$\mathtt{dough} \otimes \mathtt{oven} \otimes \mathtt{dough} \to \mathtt{bread} \otimes \mathtt{bread} \otimes \mathtt{oven}$$

and describes the transformation of two pieces of dough into two loaves of bread by baking them one after the other in an oven. We obtain a string diagram for this morphism by drawing our objects as wires, and our morphisms as boxes with inputs and outputs. Composition is represented by connecting output wires to input wires, and we represent the tensor product of two morphisms by placing them beside one another. Finally, the braiding is represented by crossing the involved wires. For the morphism in question, we obtain:



We will think of our ledger systems in terms of such string diagrams: The state of the system is a string diagram describing the *material history* of the resources involved, the available resources correspond to the output wires, and changes are effected by appending resource conversions to the bottom of the diagram. From now on we understand a *resource theory* to be a symmetric strict monoidal category with an implicit resource-theoretic interpretation.

2.2 How to Read Equality

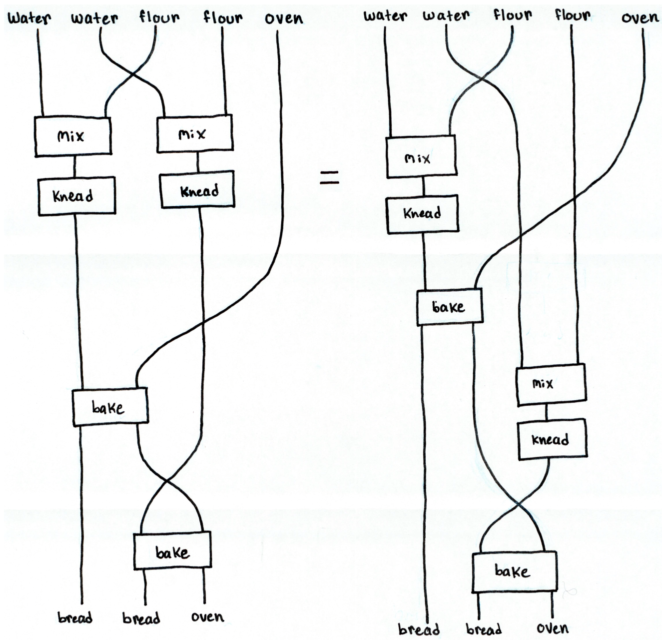
Suppose we have a resource theory $\mathbb{X}$, and two resource transformations $f, g : A \to B$. Each of f and g expresses a different way to transform an instance of resource A into an instance of resource B, but these may not have the same effect. For example, consider $\mathtt{knead} : \mathtt{dough} \to \mathtt{dough}$ and $1_{\mathtt{dough}} : \mathtt{dough} \to \mathtt{dough}$ from our resource theory of bread. Clearly these should not have the same effect on the input dough. This is reflected in our resource theory in the sense that they are not made equal by its axioms. For contrast, we can imagine a (somewhat) reasonable model of baking bread in which there is no difference between kneading the dough once and kneading it many times. We could capture this in our resource theory of baking bread by imposing the equation

$$\mathtt{knead} = \mathtt{knead} \circ \mathtt{knead}$$

In this new resource theory, our equation tells us that kneading dough once has the same effect as kneading it twice, or three times, and so on, since the corresponding morphisms of the resource theory are made equal by its axioms. Of course, the material history described by $\mathtt{knead} \circ \mathtt{knead}$ is not identical to that described by $\mathtt{knead}$. In the former case, the kneading process has been carried out twice in sequence, while in the latter case it has only been carried out once. That these morphisms are equal merely means that the effect of each sequence of events on the dough involved is the same.

We adopt the following general principle in our design and understanding of resource theories: *Two transformations should be equal precisely when they have the same effect on the resources involved.*

We further illustrate this by observing that, by the axioms of a symmetric monoidal category (specifically, by naturality of braiding), the following two transformations in the resource theory of baking (expressed as string diagrams) are equal. The transformation on the left describes baking two loaves of bread by first mixing and kneading two batches of dough before baking them in sequence, while the transformation on the right describes baking two loaves of bread by mixing, kneading, and baking the first batch of dough, and *then* mixing, kneading, and baking the second batch. Thus, according to our resource theory the two procedures will yield the same result – not an entirely unreasonable conclusion!



3 String Diagrams for Ownership

Ledgers used by blockchain systems are largely concerned with *ownership*. For example, in the Bitcoin system, each coin is associated with a computable function called the *validator*, which is used to control access to it. Anyone who wishes to use the coin must supply input data, called a *redeemer*, and the system only allows them to use the coin in question in case running the validator on the redeemer terminates in a fixed amount of time. If the validator is defined only on the data that results from `Alice` digitally signing a nonce generated by the system, then that coin can only be used by `Alice`, who then effectively owns it.

Different use cases call for different authentication schemes. For example, a proposed application of blockchain technology is to improve supply chain accountability by requiring participants to log any transfers and transformations of material on a public ledger (see e.g. [5, 12]). Here ownership implies responsibility, and so for Alice to log the transfer of, say, a ton of steel to Bob, *both* Alice and Bob must ratify the transfer via digital signature.

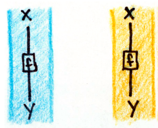
What different use cases have in common is that the resources of the ledger system are associated with ownership data. We leave the interpretation of this ownership data, including the specific details of the authentication scheme unspecified, instead giving a structural account of resource ownership. We develop our account of resource ownership intuitively, and somewhat informally, by introducing addtional features to string diagrams. This is made fully formal in the next section.

3.1 Ownership and Collection Management

Begin by assuming a theory of resources $\mathbb{X}$, and a collection $\mathcal{C}$ of potential resource owners, each of which we associate with a colour for use in our diagrams. Suppose for the remainder that `Alice`, `Bob`, and `Carol` range over $\mathcal{C}$, and are associated with colours as follows:

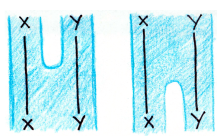
Our goal will be to construct a new theory of resources in which resources and transformations are associated with (owned and carried out by) elements of $\mathcal{C}$. The objects of our new resource theory will be collections of owned objects of $\mathbb{X}$. That is, for each object X of $\mathbb{X}$ and each $\texttt{Alice} \in \mathcal{C}$ we have an object $X^{\texttt{Alice}}$, which we interpret as an instance of resource X owned by `Alice`, along with the empty collection I and composite collections $X^{\texttt{Alice}} \otimes Y^{\texttt{Bob}}$, in which `Alice`'s instance of X exists alongside an instance of Y owned by `Bob`.

Similarly, for each transformation $f : X \to Y$ in $\mathbb{X}$, we ask for transformations $f^{\texttt{Alice}} : X^{\texttt{Alice}} \to Y^{\texttt{Alice}}$ and $f^{\texttt{Bob}} : X^{\texttt{Bob}} \to Y^{\texttt{Bob}}$ for all $\texttt{Alice}, \texttt{Bob} \in \mathcal{C}$, whose presence we interpret as the ability of each owner to effect all possible transformations of resources they own. We draw these annotated transformations as, respectively:



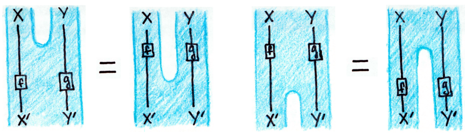
Since we are building a theory of resources we must end up with a symmetric monoidal category, so we also assume the presence of the associated morphisms, such as $f^{\texttt{Alice}} \otimes g^{\texttt{Bob}}$ and $\sigma_{X^{\texttt{Alice}}, Y^{\texttt{Bob}}}$.

Next, we account for the formal difference between $X^{\texttt{Alice}} \quad Y^{\texttt{Alice}}$ and $(X \quad Y)^{\texttt{Alice}}$. In both situations `Alice` owns an X and a Y, but in the former they are formally grouped together, while in the latter they are formally separated. We understand this formal grouping of `Alice`'s assets by analogy with physical currency. The situation in which `Alice`'s assets are separated is like `Alice` having two coins worth one euro, while the situation in which they are grouped together is like `Alice` having one coin worth two euros. In both cases, `Alice` posesses two euros, but the difference is important: `Alice` cannot give `Bob` half of the two euro coin, but can easily give `Bob` one of the two one euro coins. This distinction is also present in cryptocurrency systems, where there is an operational difference between having funds spread across many addresses and having them collected at one address. Reflecting both the reality of such systems and the principle that one ought to be able to freely reconfigure the formal grouping of things that they own, we ask that for each X, Y objects of $\mathbb{X}$ and each $\texttt{Alice} \in \mathcal{C}$ our new resource theory has morphisms $\phi_{X,Y} : X^{\texttt{Alice}} \otimes Y^{\texttt{Alice}} \to (X \otimes Y)^{\texttt{Alice}}$ and $\ _{X,Y} : (X \otimes Y)^{\texttt{Alice}} \to X^{\texttt{Alice}} \otimes Y^{\texttt{Alice}}$. We draw these morphisms, respectively, as follows:



These changes of formal grouping should not interact with the resource transformations of our original theory $\mathbb{X}$, since it ought not matter whether `Alice` combines (splits) her resources before or after transforming them. That is, we we require:

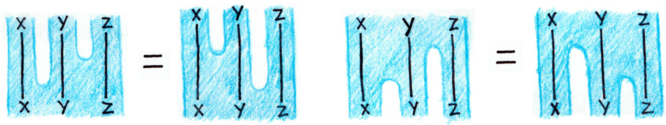
[G.1] $\phi^{\texttt{Alice}}_{X,Y}(f \otimes g)^{\texttt{Alice}} = (f^{\texttt{Alice}} \otimes g^{\texttt{Alice}})\phi^{\texttt{Alice}}_{X',Y'}$
[G.2] $(f \otimes g)^{\texttt{Alice}}\psi^{\texttt{Alice}}_{X',Y'} = \psi^{\texttt{Alice}}_{X,Y}(f^{\texttt{Alice}} \otimes g^{\texttt{Alice}})$



As it stands, there are many non-equal ways for `Alice` to reconfigure the formal grouping of their assets. Since these should all have the same effect, we need them all to be equal as morphisms in our resource theory. It suffices to ask that the $\phi^{\texttt{Alice}}$ and $\psi^{\texttt{Alice}}$ maps give, respectively, associative and coassociative operations, and that they are mutually inverse. That is (associativity and coassociativity):

[G.3] $(\phi^{\texttt{Alice}}_{X,Y} \otimes 1^{\texttt{Alice}}_{Z})\phi^{\texttt{Alice}}_{X\otimes Y,Z} = (1^{\texttt{Alice}}_{X} \otimes \phi^{\texttt{Alice}}_{Y,Z})\phi^{\texttt{Alice}}_{X,Y\otimes Z}$

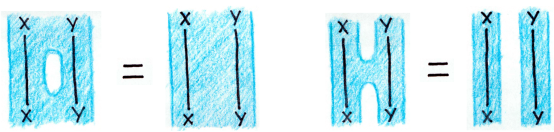
[G.4] $\psi^{\texttt{Alice}}_{X\otimes Y,Z}(\psi^{\texttt{Alice}}_{X,Y} \otimes 1^{\texttt{Alice}}_{Z}) = \psi^{\texttt{Alice}}_{X,Y\otimes Z}(1^{\texttt{Alice}}_{X} \otimes \psi^{\texttt{Alice}}_{Y,Z})$



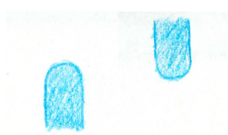
and (mutually inverse):

[G.5] $\psi^{\texttt{Alice}}_{X,Y}\phi^{\texttt{Alice}}_{X,Y} = 1^{\texttt{Alice}}_{X\otimes Y}$

[G.6] $\phi^{\texttt{Alice}}_{X,Y}\psi^{\texttt{Alice}}_{X,Y} = 1^{\texttt{Alice}}_{X} \otimes 1^{\texttt{Alice}}_{Y}$



To complete our treatment of these formal resource groupings, we must deal with the empty case $I^{\texttt{Alice}}$. We insist that `Alice` may freely create and destroy such empty collections via morphisms $\phi^{\texttt{Alice}}_{I} : I \to I^{\texttt{Alice}}$ and $\psi^{\texttt{Alice}}_{I} : I^{\texttt{Alice}} \to I$:



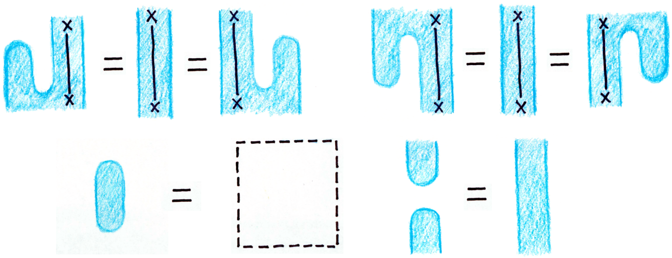
subject to the following axioms, which state that adding or removing nothing from a group or resources has the same effect as doing nothing, and that ϕ_I and ψ_I are mutually inverse, which together ensure that even with ϕ_I and ψ_I in the mix, any two formal regroupings with the same domain and codomain are equal.

[G.7] $(\phi^{\texttt{Alice}}_{I} \otimes 1^{\texttt{Alice}}_{X})\phi^{\texttt{Alice}}_{I,X} = 1^{\texttt{Alice}}_{X} = (1^{\texttt{Alice}}_{X} \otimes \phi^{\texttt{Alice}}_{I})\phi^{\texttt{Alice}}_{X,I}$

[G.8] $\psi^{\texttt{Alice}}_{I,X}(\psi^{\texttt{Alice}}_{I} \otimes 1^{\texttt{Alice}}_{X}) = 1^{\texttt{Alice}}_{X} = \psi^{\texttt{Alice}}_{X,I}(1^{\texttt{Alice}}_{X} \otimes \psi^{\texttt{Alice}}_{I})$

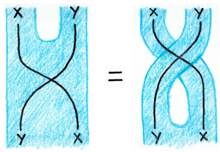
[G.9] $\phi^{\texttt{Alice}}_{I}\psi^{\texttt{Alice}}_{I} = 1_I$

[G.10] $\psi^{\texttt{Alice}}_{I}\phi^{\texttt{Alice}}_{I} = 1^{\texttt{Alice}}_{I}$



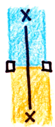
Finally, we ask that ϕ and ψ behave coherently with respect to the symmetry maps. It suffices to require that

[G.11] $\phi^{\texttt{Alice}}_{X,Y}\sigma^{\texttt{Alice}}_{X,Y} = \sigma_{X^{\texttt{Alice}},Y^{\texttt{Alice}}}\phi^{\texttt{Alice}}_{Y,X}$



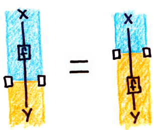
3.2 Change of Ownership

Of course, ownership is not static over time. We require the ability the *change* the owner of a given collection of resources. To this end we add morphisms $\gamma^{\texttt{Alice},\texttt{Bob}}_X : X^{\texttt{Alice}} \to X^{\texttt{Bob}}$ to our new resource theory for each object X of $\mathbb{X}$, each $\texttt{Alice}, \texttt{Bob} \in \mathcal{C}$. We depict these new morphisms in our string diagrams as follows:



As with regrouping, change of ownership should not interact with resource transformations, in the sense that:

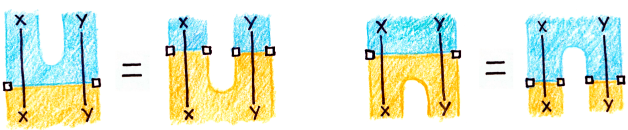
[O.1] $f^{\texttt{Alice}}\gamma^{\texttt{Alice},\texttt{Bob}}_Y = \gamma^{\texttt{Alice},\texttt{Bob}}_X f^{\texttt{Bob}}$



Further, change of ownership must behave coherently with respect to the regrouping morphisms in the sense that:

[O.2] $\phi^{\texttt{Alice}}_{X,Y}\gamma^{\texttt{Alice},\texttt{Bob}}_{X\otimes Y} = (\gamma^{\texttt{Alice},\texttt{Bob}}_X \otimes \gamma^{\texttt{Alice},\texttt{Bob}}_Y)\phi^{\texttt{Bob}}_{X,Y}$

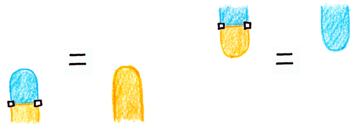
[O.3] $\gamma^{\texttt{Alice},\texttt{Bob}}_{X\otimes Y}\psi^{\texttt{Bob}}_{X,Y} = \psi^{\texttt{Alice}}_{X,Y}(\gamma^{\texttt{Alice},\texttt{Bob}}_X \otimes \gamma^{\texttt{Alice},\texttt{Bob}}_Y)$



For completeness, we axiomatize the interaction of change of ownership with empty collections by requiring that:

[O.4] $\phi_I^{\texttt{Alice}}\gamma_I^{\texttt{Alice,Bob}} = \phi_I^{\texttt{Bob}}$

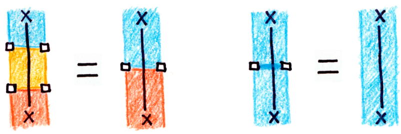
[O.5] $\gamma_I^{\texttt{Alice,Bob}}\psi_I^{\texttt{Bob}} = \psi_I^{\texttt{Alice}}$



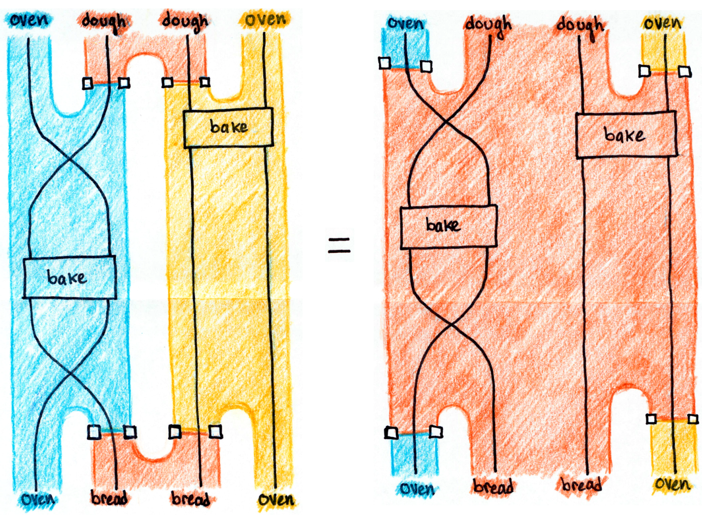
Finally, we insist that if `Alice` gives something to `Bob`, and `Bob` then gives it to `Carol`, this has the same effect as `Alice` giving the thing directly to `Carol`. Similarly, if `Alice` gives something to `Alice`, we insist that this has no effect.

[O.6] $\gamma_X^{\texttt{Alice,Bob}}\gamma_X^{\texttt{Bob,Carol}} = \gamma_X^{\texttt{Alice,Carol}}$

[O.7] $\gamma_X^{\texttt{Alice,Alice}} = 1_X^{\texttt{Alice}}$



We end up with a rather expressive diagrammatic language. For example, if we begin with the resource theory of bread, then our new resource theory is powerful enough to show:



which captures the fact that the sequence of events on the left in which `Carol` gives `Alice` and `Bob` each a portion of dough to bake in their ovens, after which they give the resulting bread to `Carol` *has the same effect* as the sequence of events on the right in which `Alice` and `Bob` give their ovens to `Carol`, who bakes the portions of dough herself before returning the ovens to their original owners. Notice that our diagrammatic representation of this is *much* easier to understand than the corresponding terms in linear syntax!

4 Categorical Semantics

In this section we show how our augmented string diagrams can be given precise mathematical meaning. Specifically, from a resource theory and a set whose elements we think of as entities capable of owning resources, we construct a new resource theory in which all resources are owned by some entity. We finish by showing how to model a simple cyrptocurrency ledger with our machinery.

4.1 Interpreting String Diagrams with Ownership

If $\mathbb{X}$ is a theory of resources and $\mathcal{C}$ is our set, we treat $\mathcal{C}$ as the corresponding discrete category, writing $A : A \to A$ for the identity maps, and form the product category $\mathbb{X} \times \mathcal{C}$. Write objects and maps of this product category as $X^A = (X, A)$ and $f^A = (f, A)$ respectively. Now, define $\mathcal{C}(\mathbb{X})$ to be the free strict symmetric monoidal category on $\mathbb{X} \times \mathcal{C}$ subject to the following additional axioms:

$$\frac{A \in \mathcal{C} \qquad X, Y \text{ objects of } \mathbb{X}}{\phi^A_{X,Y} : X^A \otimes Y^A \to (X \otimes Y)^A \text{ in } \mathcal{C}(\mathbb{X})} \qquad \frac{A \in \mathcal{C}}{\phi^A_I : I \to I^A \text{ in } \mathcal{C}(\mathbb{X})}$$

$$\frac{A \in \mathcal{C} \qquad X, Y \text{ objects of } \mathbb{X}}{{}^A_{X,Y} : (X \otimes Y)^A \to X^A \otimes Y^A \text{ in } \mathcal{C}(\mathbb{X})} \qquad \frac{A \in \mathcal{C}}{{}^A_I : I^A \to I \text{ in } \mathcal{C}(\mathbb{X})}$$

$$\frac{A, B \in \mathcal{C} \qquad X \text{ an object of } \mathbb{X}}{\gamma^{A,B}_X : X^A \to X^B \text{ in } \mathcal{C}(\mathbb{X})}$$

and subject to equations **[G.1–11]** and **[O.1–7]** for `Alice`, `Bob`, `Carol` $\in \mathcal{C}$, X, Y, Z objects of $\mathbb{X}$, and f, g morphisms of $\mathbb{X}$.

Clearly, $\mathcal{C}(\mathbb{X})$ is the new resource theory our coloured string diagrams live in. We think of objects X^A and morphisms f^A as being owned and carried out, respectively, by $A \in \mathcal{C}$. The free monoidal structue gives us the ability to compose such transformations sequentially and in parallel, and the additional axioms ensure our ownership interpretation of $\mathcal{C}(\mathbb{X})$ is reasonable.

We can characterize the category-theoretic effect of axioms **[G.1–11]** and **[O.1–5]** as follows:

▶ **Proposition 1.** *For any symmetric monoidal category $\mathbb{X}$ and any set $\mathcal{C}$, there is a strong symmetric monoidal functor*

$$A : \mathbb{X} \to \mathcal{C}(\mathbb{X})$$

for each $A \in \mathcal{C}$. Further, there is a monoidal and comonoidal natural transformation

$$\gamma^{A,B} : A \to B$$

between the functors corresponding to any two $A, B \in \mathcal{C}$.

Proof. Define $A : \mathbb{X} \to \mathcal{C}(\mathbb{X})$ by $A(X) = (X, A)$ on objects, and $A(f) = (f, A)$ on maps. For identity maps, $A(1_X) = (1_X, A) = 1_{(X,A)} = 1_{A(X)}$ since $(1_X, A)$ is the identity on (X, A) in $\mathbb{X} \times \mathcal{C}$. For composition, $A(fg) = (fg, A) = (f, A)(g, A) = A(f)A(g)$.Thus A defines a functor. A is strong symmetric monoidal via the ϕ^A and ψ^A maps together with **[G.1]** through **[G.11]**. Consider $A, B : \mathbb{X} \to \mathcal{C}(\mathbb{X})$ corresponding to $A, B \in \mathcal{C}$. Define $\gamma^{A,B} : A \to B$ to have components $\gamma^{A,B}_X$. Then $\gamma^{A,B}$ is a monoidal and comonoidal via **[O.1]** through **[O.5]**. ◀

Notice that we did not use [**O.6**–**7**] above. These axioms are motivated by our desire to model resource ownership, but they have an important, if subtle, effect on the theory: they allow us to show that $\mathbb{X}$ and $\mathcal{C}(\mathbb{X})$ are equivalent as categories. This means that any suitably categorical structure is present in $\mathbb{X}$ if and only if it is present in $\mathcal{C}(\mathbb{X})$ as well. For example, products in $\mathbb{X}$ manifest as products in $\mathcal{C}(\mathbb{X})$, morphisms that are monic in $\mathbb{X}$ remain monic in $\mathcal{C}(\mathbb{X})$, and so on. We may be confident that our addition of ownership information has not broken any of the structure of $\mathbb{X}$, or added anything superfluous!

▶ **Proposition 2.** *There is an adjoint equivalence between $\mathbb{X}$ and $\mathcal{C}(\mathbb{X})$ for each functor corresponding to some $A \in \mathcal{C}$.*

Proof. We show that each $A : \mathbb{X} \to \mathcal{C}(\mathbb{X})$ is fully faithful, and essentially surjective, beginning with the latter. To that end, suppose that P is an object of $\mathcal{C}(\mathbb{X})$. We proceed by structural induction: If P is I, then ϕ_0 witnesses $I \simeq A(I)$. If P is an atom (X, A), then $(X, A) = A(X)$. If P is $Q \quad R$ for some Q, R, then by induction we have that $Q \simeq A(X_1)$ and $R \simeq A(X_2)$ for some objects X_1, X_2 of $\mathbb{X}$. We may now form

$$Q \otimes R \simeq A(X_1) \otimes A(X_2) \xrightarrow{\phi^A_{X_1,X_2}} A(X_1 \otimes X_2)$$

which witnesses $P \simeq A(X_1 \otimes X_2)$. Thus, A is essentially surjective. To see that A is fully faithful, let $U : \mathcal{C}(\mathbb{X}) \to \mathbb{X}$ be the obvious forgetful functor. The required bijection $\mathbb{X}(X, Y) \simeq \mathcal{C}(\mathbb{X})(A(X), A(Y))$ is given by A in one direction and U in the other. It sufffices to show that any morphism $h : A(X) \to A(Y)$ with $U(h) = f$ is such that $h = A(f)$. Notice that since each $\gamma^{A,B}$ is a monoidal and comonoidal natural transformation, there is a term equal to h in which all γ morphisms occur before all other morphisms (in the sense that f occurs before g in fg). Since $h : A(X) \to A(Y)$ we know that in this equal term the composite of the γ must have type $A(X) \to A(X)$, and must therefore be the identity by repeated application of [**O.6**] and [**O.7**]. This gives a term h' containing no γ maps with $h' = h$. Similarly, since the various ϕ and ψ morphisms are natural transformations, we may construct a term h'' by collecting all instances of ϕ and ψ terms at the beginning of h'. Once collected there, the composite of all the ϕ and ψ must have type $A(X) \to A(X)$, and is therefore equal to the identity. At this point we know that $h'' : A(X) \to A(Y)$ is such that $h'' = A(f_1) \cdots A(f_n)$ for some $f_1, \ldots, f_n$ in $\mathbb{X}$. By assumption $f = U(h) = U(h'') = f_1 \cdots f_n$, and therefore $h'' = A(f)$. ◀

4.2 A Simple Example

In this section we attempt to demonstrate the relevance of the above techniques to the cryptocurrency world by building a resource theory that models a simple ledger structure along the lines of Bitcoin [10]. Let $\mathbb{1}$ be the trivial category, with one object, 1, and one morphism, the identity 1_1. Define $\mathbb{N}$ to be the free symmetric strict monoidal category on $\mathbb{1}$, write 0 for the monoidal unit of $\mathbb{N}$, and n for the n-fold tensor product of 1 with itself for all natural numbers $n \geq 1$. Notice that $n + m$ is $n \otimes m$. We will think of the objects n of $\mathbb{N}$ where $n \geq 1$ as *coins*. Of course, $0 = I$ represents the situation in which no coin in present.

Define $\mathbb{N}_\nu$ to be the result of formally adding a morphism $\nu : 0 \to 1$ to $\mathbb{N}$, write $\nu_0 = 1_0 : 0 \to 0$, and $\nu_n : 0 \to n$ for the n-fold tensor product of ν with itself for $n \geq 1$. These morphisms confer the ability to create new coins, so we imagine their use would be restricted in practice. We will not ask for the ability to destroy coins, although there would be no theoretical obstacle to doing so.

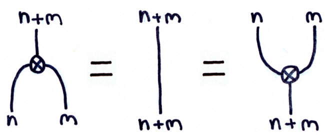
Now, let $\mathcal{C}$ be a collection of colours, which we can think of as standing in for cryptographic key pairs, or simply entities capable of owning coins. Consider $\mathcal{C}(\mathbb{N}_\nu)$. Objects are lists $n_1^{c_1} \otimes \cdots \otimes n_k^{c_k}$, which we interpret as lists of coins, where $n_i^{c_i}$ is a coin of value n_i belonging to

$c_i \in \mathcal{C}$. The morphisms are either ν_n^c for some $c \in \mathcal{C}$, the structural morphisms of a monoidal category, or the ϕ, ψ, and γ morphisms added by our construction. For $n, m \in \mathbb{N}$ and $\texttt{Alice}, \texttt{Bob} \in \mathcal{C}$, the maps $\phi_{n,m}^{\texttt{Alice}} : n^{\texttt{Alice}} \otimes m^{\texttt{Alice}} \to (n+m)^{\texttt{Alice}}$ and $\psi_{n,m}^{\texttt{Alice}} : (n+m)^{\texttt{Alice}} \to n^{\texttt{Alice}} \otimes m^{\texttt{Alice}}$ allow users to combine and split their coins in a value-preserving manner, and the $\gamma_n^{\texttt{Alice},\texttt{Bob}}$ maps allow them to exchange coins.

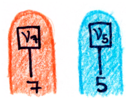
Now, a ledger is a (syntactic) morphism $a : I \to A$ of $\mathcal{C}(\mathbb{N}_\nu)$. A transaction to be included in a consists of a transformation $f : X \to Y$ of $\mathcal{C}(\mathbb{N}_\nu)$ along with information about which outputs of a are to be the inputs of the transformation, which we package as $t = \pi_t(1 \otimes f \otimes 1) : A \to B$. The result of including transaction t in ledger a is then the composite ledger $t \circ a : I \to B$. Put another way, a ledger is given by a list of transformations in $\mathcal{C}(\mathbb{N}_\nu)$:

$$I \xrightarrow{t_1} A_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} A_k$$

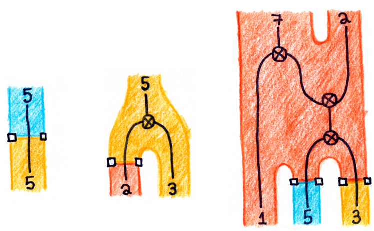
For the purpose of illustration, we differentiate between $m + n$ and $m \quad n$ in our string diagrams for $\mathbb{N}_\nu$. We do so by means of the string diagrams for (not necessarily strict) monoidal categories (see e.g. [3]), as in:



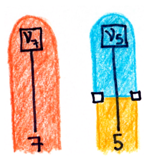
Now, suppose we have a ledger $a : I \to \nu_7^{\texttt{Carol}} \otimes \nu_5^{\texttt{Alice}}$:



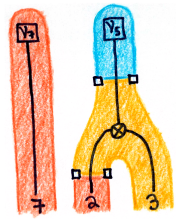
and resource transformations f_1, f_2, f_3 defined, respectively, by:



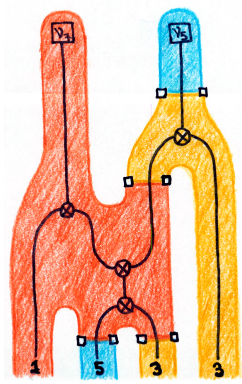
Now, form transaction $t_1 = (1_7^{\texttt{Carol}} \otimes f_1)$ and append it to a to obtain $t_1 \circ a$



Next, form transaction $t_2 = (1_7^{\texttt{Carol}} \otimes f_2)$ and append it to obtain $t_2 \circ t_1 \circ a$



Finally, form transaction $t_3 = (f_3 \otimes 1_3^{\mathsf{Bob}})$ and append it to obtain $t_3 \circ t_2 \circ t_1 \circ a$



In this manner, we capture the evolution of the ledger over time. Of course, we can also reason about whether two sequences of transactions result in the same ledger state by comparing the corresponding morphisms for equality, although in the case of $\mathcal{C}(\mathbb{N}_\nu)$ there isn't much point, since all morphisms $A \to B$ are necessarily equal.

5 Conclusions and Future Work

We have seen how the resource theoretic interpretation of monoidal categories, and in particular their string diagrams, captures the sort of material history that concerns ledger structures for blockchain systems. Additionally, we have shown how to freely add a notion of ownership to such a resource theory, and that the resulting category is equivalent to the original one. We have also shown that these resource theories with ownership admit an intuitive graphical calculus, which is more or less that of monoidal functors and natural transformations. Finally, we have used our machinery to construct a simple ledger structure and show how it might be used in practice.

While we do not claim to have solved the problem of providing a rigorous foundation for the development of ledger structures in its entirety, we feel that our approach shows promise. There are a few diffrent directions for future research. One is the development of categorical models for more sophisticated ledger structures, with the eventual goal being to give a rigorous formal account of smart contracts. Another is to explore the connections of the current work with formal treatments of accounting, such as [6].

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A.2 Functorial Semantics for Partial Theories

I. Di Liberti, F. Loregian, C. Nester, and P. Sobociński. Functorial Semantics for Partial Theories. In *Proc. ACM Program. Lang.*, 5(POPL), 2021.

Functorial Semantics for Partial Theories

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We provide a Lawvere-style definition for partial theories, extending the classical notion of equational theory by allowing partially defined operations. As in the classical case, our definition is syntactic: we use an appropriate class of string diagrams as terms. This allows for equational reasoning about the class of models defined by a partial theory. We demonstrate the expressivity of such equational theories by considering a number of examples, including partial combinatory algebras and cartesian closed categories. Moreover, despite the increase in expressivity of the syntax we retain a well-behaved notion of semantics: we show that our categories of models are precisely locally finitely presentable categories, and that free models exist.

CCS Concepts: • **Theory of computation** → **Categorical semantics**.

Additional Key Words and Phrases: Lawvere theory, categories of partial maps, syntax, semantics, variety theorem.

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1 INTRODUCTION

Mathematicians interested in, say, the theory of monoids or the theory of groups work in an axiomatic setting, asserting the presence of a collection of n-ary *operations* on an ambient set A — i.e. (total) functions $A^n \to A$ for some $n : \mathbb{N}$ — that satisfy a number of axioms. This data can be packaged up into an *equational theory*: a pair (Σ, E) where Σ is the *signature*, consisting of *operation symbols*, each with a specified arity, and E is a collection of *equations* — i.e. pairs of *terms* built up from the signature Σ and auxiliary variables — that provide the axioms. An ambient equational theory is thus the bread and butter of an algebraist, that together with the principles of equational reasoning provides the basic calculus of mathematical investigation into the structure of interest.

Birkhoff [Birkhoff 1935] discovered that a substantial amount of mathematics can be done at the level of generality of an equational theory. Given an equational theory (Σ, E), a *model* is a set together with an interpretation of the function symbols Σ that satisfies the equations E. A monoid is nothing but a model of the equational theory of monoids, a group is a model of the equational theory of groups, and so on. The semantics of an equational theory, i.e. its class of models, is called a *variety*. Birkhoff showed that certain results (e.g. the so-called isomorphism theorems, existence of free models) can be derived uniformly for generic varieties, independent of the equational theory

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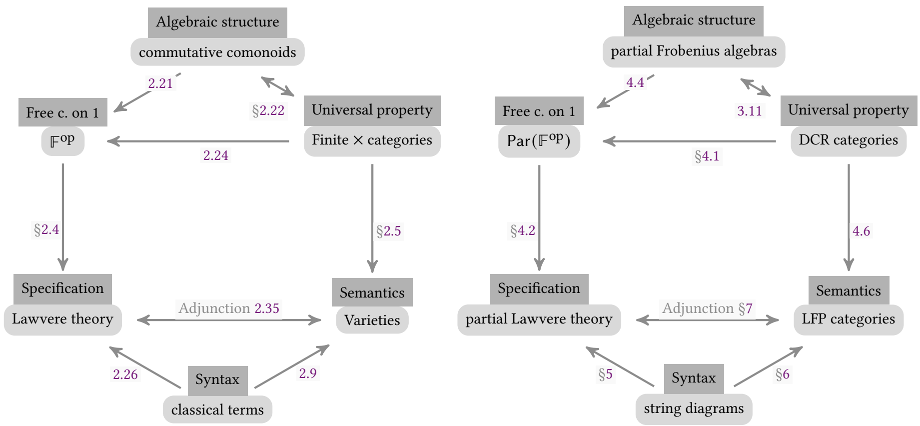


Fig. 1. Elements of classical functorial semantics on the left, and our contribution on the right.

at hand. Most spectacularly, a class of sets-with-structure can be determined to be a variety through purely structural means; this is often referred to as Birkhoff's Variety Theorem or the HSP Theorem.

The resulting field is known as *universal algebra*. Its mathematical objects of study are equational theories and varieties. Given its goal of uncovering methodological and technical similarities of a large swathe of contemporary algebra, universal algebra is in the intersection of mathematics and mathematical logic. It has influenced computer science, especially programming language theory, as a formal and generic treatment of syntax, terms, equational reasoning, etc.

Lawvere [Lawvere 1963], and the subsequent development of *categorical* universal algebra, addressed some of the perceived shortcomings of the classical account. It is well known that a single variety can have many different axiomatic presentations, and in this sense the choice of a particular presentation may seem ad hoc. The requirement that models be sets-with-structure is also restrictive, since algebraic structures appear in other mathematical contexts as well. A Lawvere theory is a category $\mathcal{L}$ that serves as a *presentation-independent* way of capturing the specification of a variety. A central conceptual role is played by *cartesian categories*, i.e. categories with finite products. The free cartesian category on one object often appears in the very definition of a Lawvere theory – the "one object" here capturing single-sortedness. Finite products track arities and ensure that operations are total functions. Functorial semantics gives us the correct generalisation of varieties: a model is cartesian functor $\mathcal{L} \to \mathrm{Set}$. This point of view is flexible (e.g. Set can be replaced with another cartesian category) and leads to a rich theory [Adámek et al. 2003; Hyland and Power 2007; Lawvere 1963], where the study of varieties and their specifications can take place at a high level of generality.

The beautiful abstract picture painted by Lawvere can be used to give a post-hoc explanation of the elements of *classical* equational theories. Every equational theory yields a Lawvere theory. Free equational theories, i.e. those where $E = \varnothing$, are Lawvere theories whose arrows can be concretely described as (tuples of) terms. Indeed, it is well-known that terms are closely connected to the universal property of products. The abstract mathematics, therefore, *explains* the structure of terms and *justifies* the use of ordinary equational reasoning. The elements of Lawvere's approach to universal algebra are illustrated in the left side of Fig. 1.

In this paper we are concerned with *partial* algebraic structures, i.e. those where the operations are not, in general, defined on their whole domain. Partiality is important in mathematics: the very notion of *category* itself is a partial algebraic structure, since only compatible pairs of arrows can be composed. Even more so, partiality is an essential property of computation, and partial functions play a role in many different parts of computer science, starting with initial forays into recursion theory at the birth of the subject, and being ever present in more recent developments, for example arising as an essential ingredient in the study of fixpoints [Bloom and Ésik 1993]. From the start it was clear that additional care is necessary for partial operations, the terms built up from them, and the associated principles of (partial) equational reasoning. An example is the principle of *Kleene equality*: using $s = t$ to assert that whenever one side is defined, so is the other, and they are equal, or the use of notation $-|_X$ to restrict the domain of definition of a function. In general, reasoning about partially defined terms can be quite subtle.

Our contribution is summarised on the right hand side of Fig. 1 and follows Lawvere's approach closely. A key question that we address is what replaces the central notion of cartesian category. This turns out to be the notion of *discrete cartesian restriction* (DCR) category, which arose from research on the structure of partiality [Cockett et al. 2012]. Just as the free cartesian category on one object plays a central role in the definition of Lawvere theory, the free DCR category on one object plays a central role in the definition of *partial* Lawvere theory that we propose. In our development, the category of sets and partial functions Par replaces Set as the universe of models. Much of the richness of the classical picture is unchanged: e.g. we obtain free models just as in the classical setting. Moreover, we prove a variety theorem that characterises partial varieties as locally finitely presentable (LFP) categories.

Props [Lack 2004; Mac Lane 1965] and their associated *string diagrams*, play a crucial technical role. Props are a convenient categorical structure that capture generic *monoidal* theories. Monoidal theories differ from equational theories in that, roughly speaking, in that we are able to consider more general monoidal structures other than the cartesian one. String diagrams are the syntax of props, and they are a bona fide syntax not far removed from traditional terms. For example, they can be recursively defined and enjoy similar properties as free objects, e.g. the principle of structural induction. The connective tissue between the classical story and string diagrams is Fox's Theorem [Fox 1976], which states that the structure of cartesian categories can be captured by the presence of local algebraic structure: a coherent and natural commutative comonoid structure on each object. This implies several things: (*i*) that classical terms can be seen as particular kinds of string diagrams, (*ii*) that classical equational reasoning can be seen as diagrammatic reasoning on these string diagrams and (*iii*) that the prop induced from the monoidal theory of commutative comonoids — well-known to coincide with $\mathbb{F}^{\mathrm{op}}$, the opposite of the prop of finite sets and functions — is the free cartesian category on one object. The correspondence goes further: as shown in [Bonchi et al. 2018], Lawvere theories are particular kinds of monoidal theories.

We are able to identify the nature of the free DCR category on one object by proving a result similar to Fox's Theorem, but for DCR categories instead of cartesian categories. Instead of commutative comonoids, we identify the algebraic structure of interest as *partial Frobenius algebras*. The free DCR category on one object is the prop induced from this monoidal theory, and it can be characterised as $\mathrm{Par}(\mathbb{F}^{\mathrm{op}})$: the prop of partial functions in $\mathbb{F}^{\mathrm{op}}$. This informs our definition of partial Lawvere theory. Crucially, just as the mathematics of ordinary Lawvere theories serves as a post hoc justification for equational theories, we identify the precise class of string diagrams that serve as *partial terms*, which lets us define a *partial equational theory* in a familiar way as pair of signature and equations. We give several examples, from partial commutative monoids, to several examples important in computer science, notably the theory of partial combinatory algebras [Bethke 1988], the theory of pairing functions, and the theory of cartesian closed categories.

To summarise, the original contributions of this paper are:

- A "Fox theorem" for DCR categories, which uses the notion of *partial Frobenius algebra* and leads to the characterisation of the free DCR category on one object as $\mathsf{Par}(\mathbb{F}^{\mathrm{op}})$;
- The definitions of partial Lawvere theory and partial equational theory, which use string diagrammatic syntax informed by the aforementioned Fox theorem;
- The coupling of these notions into a comprehensive framework for partial algebraic theories, analogous to the work of Lawvere on classical algebraic theories, as illustrated in Fig. 1;
- The existence of free models, and—more generally–a variety theorem, building on known results about DCR categories, and the Gabriel-Ulmer duality.

Related work. There are a number of formalisms in the literature that aim at providing a rigorous way of specifying partial algebraic structure. Freyd's *essentially algebraic theories* [Freyd 1972] were introduced informally, but were subsequently formalised and generalised in various ways [Adamek and Rosickỳ 1994; Adámek et al. 2011; Palmgren and Vickers 2007]. A different, but equally expressive approach is via *finite limit sketches* [Adamek and Rosickỳ 1994]. Nevertheless, none of these approaches can claim to have the foundational status of classical equational theories - e.g. they do not, per se, provide a canonical notion of syntax to replace classical terms, nor a calculus for (partial) equational reasoning about the categories of models they define. Tout court, none of them can claim to be *equational*. Interestingly, the semantic landscape (i.e. the corresponding notion of *partial variety*) is better understood than the syntax. The class of models of essentially algebraic theories and finite limit sketches are closely related to Gabriel-Ulmer duality [Centazzo 2004], which asserts a contravariant (bi)equivalence between the category of categories with finite limits and the category of locally finitely presentable (LFP) categories.

Partial Frobenius algebras, which arise in our characterisation of DCR categories, are special/separable Frobenius algebras without units. The version *with* units was originally studied in [Carboni and Walters 1987], is deeply connected to the relational algebra [Freyd and Scedrov 1990], characterises 2-dimensional TQFTs [Kock 2003], and has been used extensively in categorical approaches to the study of quantum information and quantum computing, such as the ZX calculus [Coecke and Duncan 2008]. In a similar way to the use of partial Frobenius algebras in this paper, they are used in the recently proposed *Frobenius theories* [Bonchi et al. 2017], which are algebraic theories that take their models in the category of relations Rel, and are guided by the structure of cartesian bicategories of relations [Carboni and Walters 1987].

Restriction categories were introduced in [Cockett and Lack 2002] as a general framework for reasoning about categories of partial maps. Cartesian restriction (CR) categories are those with a certain sort of formal finite product structure – restriction products – introduced in [Cockett and Lack 2007]. Notably, the p-categories of [Robinson and Rosolini 1988] arise as restriction categories with restriction products. Discrete cartesian restriction (DCR) categories – named for a similarity to categories of discrete topological spaces – arise in [Cockett et al. 2012] as the restriction categories with finite *latent* limits – again a sort of formal limit. DCR categories are closely connected to the *discrete inverse categories* considered in [Giles 2014] which are presentable in terms of *semi-Frobenius* algebras, being those special/separable commutative Frobenius algebras with neither a unit *nor a counit.*

Structure of the paper. In S2 we lay the foundations by recalling the basic concepts of universal algebra, props and string diagrams, Fox's theorem, and functorial semantics. In S3, after recalling the basics of restriction category theory, we prove Theorem 3.6, which is to DCR categories what Fox's theorem is to cartesian categories. In S4 we propose our original definitions: partial Lawvere theories and their varieties. Next, S5 is devoted to the associated notion of partial equational theory, and several examples, continued in S6 with multi-sorted examples. Our variety theorem is in S7 where we also treat other semantic aspects, e.g. the existence of free models.

2 BACKGROUND MATERIAL

2.1 Overview of Classical Universal Algebra

Universal algebra is the study of *equational theories* and of their semantics, *varieties*. In this section we recall the basic concepts and definitions.

Definition 2.1. A *signature* is a pair (Σ, α) where Σ is a set and α a function $\Sigma \to \mathbb{N}$ that assigns to every element $t : \Sigma$ a natural number $\alpha(n) : \mathbb{N}$ called the *arity* of the *function symbol* t.

Notation 2.2. The arity "slices" the set Σ of function symbols. The slice $\Sigma_n \subseteq \Sigma$ contains operations of arity n, and $t : \Sigma_n$ is a synonym for "t is a n-ary operation". We will sometimes write t_n for a generic element of Σ_n. We shall refer to the signature as just Σ if the arity function is understood from the context. For example the signature Σ_{M} of *monoids* is $\{m,\ e\}$, with $\alpha(m) = 2$ and $\alpha(e) = 0$.

Definition 2.3. An Σ-*algebra* is a pair $(A, [\![-]\!]_A)$ where A is a set and $[\![-]\!]_A$ is a function sending function symbols $t : \Sigma_n$ to functions $[\![t]\!]_A \colon A^n \to A$. The function $[\![t]\!]_A$ is called the n-ary operation on A *associated* to the function symbol $t : \Sigma_n$. We refer to A as the *carrier* of the Σ-algebra. A Σ-algebra homomorphism from $(A, [\![-]\!]_A)$ to $(B, [\![-]\!]_B)$ is a function $f \colon A \to B$ that respects the Σ structure: i.e. for every $n \in \mathbb{N}$ and $t_n : \Sigma_n$, the diagram on the right commutes:

$$\begin{tikzcd} A^n \arrow[r, "f^n"] \arrow[d, "{[\![t_n]\!]_A}"'] & B^n \arrow[d, "{[\![t_n]\!]_B}"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Remark 2.4. Σ-algebras and their homomorphisms define a category $\mathcal{V}_\Sigma$.

Of course, an algebraic structure isn't just about operations, but also about *properties* enjoyed by those operations. To express this we first need the notion of *term*. Fixing a signature Σ, we recall the usual recursive construction of the set of terms T_Σ^V, for some set of variables V:

$$T_\Sigma^V \ ::= \ V \ \mid \ t_0 \ \mid \ t_1(T_\Sigma^V) \ \mid \ t_2(T_\Sigma^V, T_\Sigma^V) \ \mid \ \ldots \ \mid \ t_n(T_\Sigma^V, \ldots, T_\Sigma^V) \ \mid \ \ldots$$

In the above, each t_i ranges over the function symbols in Σ_i. For any V, T_Σ^V carries a canonical Σ-algebra structure: $[\![t]\!](t_1, t_2, \ldots, t_{n_t}) \stackrel{\text{def}}{=} t(t_1, t_2, \ldots, t_{n_t})$. We call this the term Σ-algebra over V.

Observation 2.5. The term Σ-algebra T_Σ^V enjoys a universal property: given a Σ-algebra $(A, [\![-]\!]_A)$ and function $v \colon V \to A$, there is a unique extension to a homomorphism of algebras $\bar{v} \colon T_\Sigma^V \to A$. This is just the induction principle associated to the recursive definition of terms.

Definition 2.6 (Σ-equation). Fixing V, a Σ-*equation* is a pair $(s, t) \in T_\Sigma^V \times T_\Sigma^V$; we usually write '$s = t$'. A Σ-equation $s = t$ *holds* in Σ-algebra $(A, [\![-]\!]_A)$ if for all $v \colon V \to A$ we have $\bar{v}(s) = \bar{v}(t)$ in A.

Given the signature of monoids, we can express properties such as associativity: $m(x, m(y, z)) = m(m(x, y), z)$; or commutativity: $m(x, y) = m(y, x)$; etc. The idea is that a set of Σ-equations *constrains* the choice of algebras $(A, [\![-]\!]_A)$ to those where every equation holds.

Definition 2.7 (Equational Theory and Variety). A pair (Σ, E) where Σ is a signature and E a set of Σ-equations is called an *equational theory*. A *model* of (Σ, E) is a Σ-algebra where every $e : E$ holds. The class of models for an equational theory is called a *variety*.

Example 2.8. The equational theory of commutative monoids is

$$(\ \{m,\ e\},\ \{\ m(m(x, y), z) = m(x, m(y, z)),\ m(x, y) = m(y, x),\ m(e, x) = x\ \}\ ).$$

The corresponding variety is the class of commutative monoids.

Some of the most famous results of universal algebra characterise varieties. For example:

Theorem 2.9 (Birkhoff [Birkhoff 1935]). A class of Σ-algebras is a variety if and only if it is closed under homomorphic images, subalgebras and products.

2.2 Props and Monoidal Theories

Our development is informed by the differences between the algebraic structure of total functions and partial functions. Given the focus on algebra, the notion of prop is useful as a categorical gadget on which to hang an algebraic structure. Moreover, the associated notion of string diagram will lead us to a syntax with which to express partial equational theories by appropriately generalising classical terms. Here we recall the basic definitions of props [Lack 2004], string diagrams and some of the algebraic structures that are prominent in subsequent sections.

Definition 2.10 (Prop [Mac Lane 1965, Ch. 5]). A *prop* is a symmetric strict monoidal category with set of objects the natural numbers $\mathbb{N}$, where the monoidal product on objects is addition: $m \otimes n := m + n$. A *homomorphism of props* is an identity-on-objects symmetric strict monoidal functor.

Example 2.11. An important example is the prop $\mathbb{F}$ of finite ordinal numbers. In the following, $[m] \stackrel{\text{def}}{=} \{1, 2, \ldots, m\}$. The $\mathbb{F}$-arrows $m \to n$ are all functions $[m] \to [n]$: composition is function composition, and the monoidal product is "disjoint union"; i.e. for $f_1 : m_1 \to n_1$ and $f_2 : m_2 \to n_2$,

$$(f_1 \otimes f_2)(i) : m_1 + m_2 \to n_1 + n_2 \stackrel{\text{def}}{=} \begin{cases} f_1(i) & \text{if } i \leq m_1 \\ f_2(i - m_1) + n_1 & \text{otherwise.} \end{cases}$$

Free props generated from some signature of operations are of particular importance.

Definition 2.12 (Monoidal signature). A monoidal signature Γ is a collection of *generators* $\gamma : \Gamma$, each with an arity $ar(\gamma) : \mathbb{N}$ and coarity $coar(\gamma) : \mathbb{N}$.

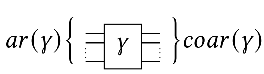
Concrete terms can be given a BNF description, as follows:

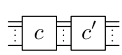
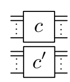
c ::= $\gamma \in \Gamma$ |  | |  | $c \otimes c$ | $c \,;\, c$ (1)

Arities and coarities are not handled in the BNF but with an associated sorting discipline, shown below. We only consider terms that have a sort, which is unique if it exists.

$\dfrac{}{\gamma : (ar(\gamma), coar(\gamma))}$  $: (0, 0)$ $: (1, 1)$  $: (2, 2)$ $\dfrac{c : (n, z) \quad d : (z, m)}{c \,;\, d : (n, m)}$ $\dfrac{c : (n, m) \quad d : (r, z)}{c \otimes d : (n{+}r, m{+}z)}$

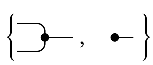
The idea is that the sort $c : (m, n)$ counts the number of "dangling wires" of each term. Every sortable term generated from (1) has a diagrammatic representation. The convention for $\gamma : \Gamma$ is to draw it as a box with $ar(\gamma)$ "dangling wires" on the left and $coar(\gamma)$ on the right:

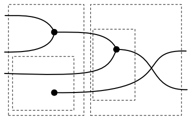


The conventions for the (1) operations are: $c \,;\, c'$ is drawn  and $c \otimes c'$ is drawn  .

The sorting discipline ensures that the convention for $;$ makes sense.

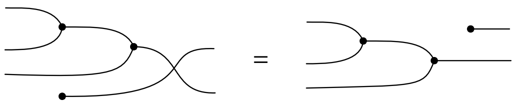
Example 2.13. Consider the following signature, where the (co)arities are apparent from the

$\Gamma \stackrel{\text{def}}{=}$  (CMG) glyphs. The term $(\; \otimes (\; \otimes \;)) \mathbin{;} ((\; \otimes \;) \mathbin{;} \;)$ has sort (3, 2) and diagram:



where the "dotted line" boxes serve the role of parentheses.

Terms of (1) are quotiented by the laws of symmetric strict monoidal categories. We do not go into the details here, but these are closely connected with the diagrammatic conventions. Indeed, they allow us to discard the "dotted line" boxes and focus *only* on the connectivity between the generators. For example the following two diagrams are in the same equivalence class of terms:



We refer to equivalence classes $[c] : m \to n$ as *string diagrams*.

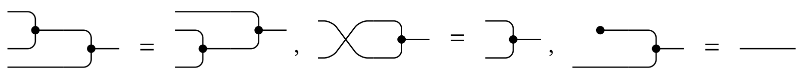
Definition 2.14. The free prop $\mathbb{X}_\Gamma$ on Γ has as arrows $m \to n$ string diagrams $[c] : (m, n)$.

String diagrams can be used to specify additional equations that specify algebraic structure.

Definition 2.15 (Monoidal theory [Lack 2004]). For a monoidal signature Γ, a Γ-equation is a pair $([c], [d])$ of equally-sorted string diagrams; we usually write '$[c] = [d]$'. A *monoidal theory* is a pair (Γ, F) where F is a set of Γ-equations.

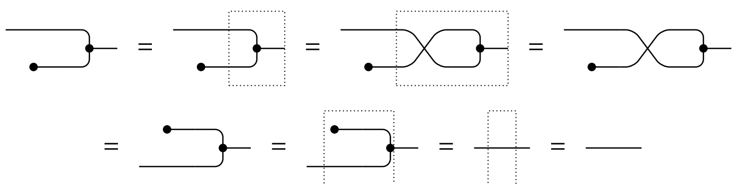
Given a monoidal theory (Γ, F), the induced prop $\mathbb{X}_{(\Gamma,F)}$ can be obtained by taking a coequaliser in Cat. It can alternatively be given an explicit description as follows: as arrows $[m] \to [n]$ it has arrows of $\mathbb{X}_\Gamma$ quotiented by the smallest congruence containing F.

Example 2.16. Consider the signature (CMG) and the following set of equations:

$E \stackrel{\text{def}}{=} \Big\{$  $\Big\}$. (CM)

The resulting prop $\mathbb{CM}$ is the prop of *commutative monoids*. The equations, from left to right, express associativity, commutativity and unitality.

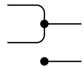
Remark 2.17. String diagrams in $\mathbb{X}_{(\Gamma,F)}$ are amenable to equational reasoning, often referred to as *diagrammatic reasoning* in this context: if $([c], [d]) \in F$ then substituting c for d inside any context is sound. For example in $\mathbb{CM}$ the set of equations contains only one of the unit laws. The other may be derived:



We typically omit the "dotted line" boxes in such chains of reasoning.

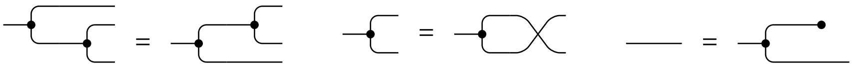
Interestingly, $\mathbb{CM}$ can be seen as the algebraic characterisation of $\mathbb{F}$.

Observation 2.18 ([Lack 2004]). As props, $\mathbb{F} \cong \mathbb{CM}$.

Remark 2.19. In fact, arrows of $\mathbb{CM}$ can be intuitively understood as "pictures of functions". For example, the function $f\colon 2 \to 2$ where $f(1) = f(2) = 1$ is drawn 

Example 2.20. The theory of commutative *comonoids* plays an important role for us. The data is:

(CCMG)



(CCM)

Let $\mathbb{CC}$ be the prop induced from the monoidal theory ((CCMG),(CCM)).

Given that (CCMG) and (CCM) are mirrored (CMG) and (CM), Observation 2.18 gives:

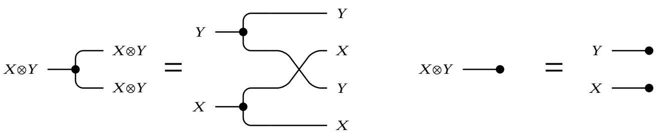
Observation 2.21. As props, $\mathbb{F}^{\mathrm{op}} \cong \mathbb{CC}$.

While we have specialised our discussion of string diagrams as the syntax of props, it is well-known that they can be used as a sound calculus in any symmetric (strict) monoidal category. Roughly speaking, objects are represented by wires, and morphisms by boxes.

2.3 Fox's Theorem

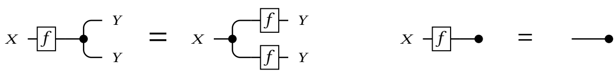
Equational and monoidal theories are linked by Fox's theorem ([Fox 1976]), recalled here – this will be explained in S2.6. *Cartesian* categories are categories with finite products, and *cartesian functors* preserve them. Fox showed that cartesian categories are exactly those that have a certain algebraic structure.

A commutative comonoid on an object X of a symmetric monoidal category $\mathbb{X}$ is a triple $(X, \delta_X, \varepsilon_X)$ s.t. $\delta_X : X \to X \otimes X$ and $\varepsilon_X : X \to I$, depicted as and respectively, and these satisfy (CCM). If all objects are so equipped, then the structures are *coherent* if for all objects X, Y:



(coherent)

Further, we say that the δ and ε are *natural* if for any arrow $f : X \to Y$ of $\mathbb{X}$, we have:



(natural)

Theorem 2.22 ([Fox 1976]). A cartesian category is the same thing as a symmetric monoidal category where every object is equipped with a (coherent) and (natural) commutative comonoid structure.

In light of Observation 2.21, we know that a commutative comonoid structure on X is equivalently a cartesian functor $\mathcal{X} : \mathbb{F}^{\mathrm{op}} \to \mathbb{X}$ where $\mathcal{X}[1] = X$. The action of $\mathcal{X}$ on objects is determined by its action on 1, and the generators give arrows $\mathcal{X}(\ ) = \delta_X : X \to X \otimes X$ and $\mathcal{X}(x\ ) = \varepsilon_X : X \to I$ of $\mathbb{X}$ which satisfy (CCM). Thus we may specialize Theorem 2.22, to props as follows:

Corollary 2.23. A prop $\mathbb{X}$ is cartesian (with categorical product the monoidal product) if and only if there is a homomorphism of props $\mathbb{F}^{\mathrm{op}} \to \mathbb{X}$ and the picked out comonoid structure is (natural).

It is easy to show that a coherent and natural commutative comonoid structure, if it exists, is unique. An easy consequence of Theorem 2.22 is that a cartesian functor is precisely a symmetric monoidal functor that preserves the comonoid structure. This, combined with Corollary 2.23, gives:

Proposition 2.24. The prop $\mathbb{F}^{\mathrm{op}}$ is the free cartesian category on a single object.

2.4 Lawvere Theories

We recall Lawvere's approach [Lawvere 1963] of *functorial semantics* of algebraic theories in the rest of the section. Lawvere's approach is centered on the theory of cartesian categories.

Definition 2.25 (Lawvere theory). A *Lawvere theory* is a cartesian prop. A morphism of Lawvere theories is a cartesian prop homomorphism. Lawvere theories and homomorphisms define the category Law.

Finite products do two jobs: they keep track of arities of operations, and—less obviously—they ensure the *totality* and *single-valuedness* of the interpretation of function symbols in any model.

Free categories with products play a leading role. Recall from Proposition 2.24 that $\mathbb{F}^{\mathrm{op}}$ is the free category with products on one object. Spelled out, a Lawvere theory is a cartesian category $\mathcal{L}$ and an identity-on-objects cartesian functor $\mathbb{F}^{\mathrm{op}} \to \mathcal{L}$. A morphism of Lawvere theories is a functor $h : \mathcal{L} \to \mathcal{M}$ s.t. the following triangle commutes:

$$\begin{tikzcd} & \mathbb{F}^{\mathrm{op}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{L} \arrow[rr, "h"'] & & \mathcal{M}. \end{tikzcd}$$

Remark 2.26. Every equational theory gives rise to a Lawvere theory. For the case of no equations $(\Sigma, \varnothing)$, this Lawvere theory $\mathcal{L}_\Sigma$ is the free category with products on Σ. It also has a simple, concrete description that uses Σ-terms. An arrow $m \to n$ is an n-tuple

$$(t_1,\ t_2,\ \ldots,\ t_n) \quad \text{where each} \quad t_i : T_\Sigma^{[m]} \tag{2}$$

i.e. where each term in the tuple may use formal variables from the set $\{1, \ldots, m\}$. Composition of $(s_1, \ldots, s_k) \colon m \to k$ with $(t_1, \ldots, t_n) \colon k \to n$ is via substitution:

$$(t_1[{}^{s_1}/_1, \ldots {}^{s_k}/_k], \ldots, t_n[{}^{s_1}/_1, \ldots, {}^{s_k}/_k]) \colon m \to n.$$

Given a set of equations E, ordering the variables in each $s = t : E$ induces pairs of arrows $s, t : m \to 1$ (where m is the number of variables appearing in s and t). Then $\mathcal{L}_{(\Sigma,E)}$ is obtained by "equating" s and t –this can be computed via a coequaliser, or directly as in (2) where terms are taken modulo the smallest congruence containing the required equations. We omit the details.

The Lawvere theory induced from the empty equational theory $(\varnothing, \varnothing)$ is $\mathbb{F}^{\mathrm{op}}$.

2.5 Semantics For Algebraic Theories

Here we recall some of the basic elements of functorial semantics.

Definition 2.27 (Model of a Lawvere theory). A *model* for a Lawvere theory $\mathcal{L}$ is a cartesian functor $L \colon \mathcal{L} \to \mathrm{Set}$. A model homomorphism $L \to L'$ is a natural transformation $\alpha \colon L \Rightarrow L'$. This defines the category of models $\mathrm{Mod}_{\mathcal{L}}$ of a Lawvere theory $\mathcal{L}$.

Remark 2.28. There are forgetful functors $U : \mathrm{Mod}_{\mathcal{L}} \to \mathrm{Set}$, given by evaluating on the terminal object $F \mapsto F(1)$. Intuitively, U forgets the algebraic structure, returning the underlying carrier set.

Definition 2.27 is compatible with its classical counterpart: the data required to give a model of (Σ, E), in the sense of Definition 2.7, is precisely that required to give a functor $\mathcal{L}_{(\Sigma,E)} \to \mathrm{Set}$. The functorial approach lends itself to generalisations: e.g. replacing Set with another cartesian category. Moreover, it allows for further structural analysis.

Observation 2.29. $\mathrm{Mod}_{\mathcal{L}}$ is closed under limits computed in the category of all functors $[\mathcal{L}, \mathrm{Set}]$, because limits commute with limits. For a similar reason it is closed under sifted colimits. Thus the inclusion $\mathrm{Mod}_{\mathcal{L}} \hookrightarrow [\mathcal{L}, \mathrm{Set}]$ creates limits and sifted colimits[1].

Remark 2.30 (Multi-sorted, unsorted). The codomain of $U\colon \mathrm{Mod}_{\mathcal{L}} \to \mathrm{Set}$ betrays that our presentation is *single sorted*. Indeed when $\mathcal{L}$ is S-sorted we obtain a functor $U\colon \mathrm{Mod}_{\mathcal{L}} \to [S, \mathrm{Set}]$. Historically, syntactic aspects are single sorted, while categorical variety theorems are most crisp in the *unsorted* case. It is thus worthwhile to focus on these concepts in more detail.

An S-sorted Lawvere theory is a cartesian S-coloured prop. Spelled out, it is an identity-on-objects cartesian functor $\mathbb{F}(S) \to \mathcal{L}$, where $\mathbb{F}(S)$ is the free cartesian category on S.

An *unsorted* Lawvere theory is simply a (small) category with products.

In the remainder of this section we focus on the unsorted version, because the treatment is notationally and technically simplified. Nevertheless, much of the following is sort-agnostic.

Observation 2.31. A theory morphism $h\colon \mathcal{L} \to \mathcal{L}'$ induces a (contravariant) functor

$$\mathrm{Mod}_h : \mathrm{Mod}_{\mathcal{L}'} \to \mathrm{Mod}_{\mathcal{L}}$$

taking $F\colon \mathcal{L}' \to \mathrm{Set}$ to $F \cdot h\colon \mathcal{L} \to \mathrm{Set}$. The functor Mod_h always admits a left adjoint: Mod_h preserves limits and sifted colimits because they are computed in the underlying functor category. The special adjoint functor theorem can now be used to obtain a left adjoint $L_h : \mathrm{Mod}_{\mathcal{L}} \to \mathrm{Mod}_{\mathcal{L}'}$.

Example 2.32. For intuition, we consider a concrete example. Consider the inclusion i of the theory of monoids in commutative monoids. Then Mod_i is the functor that "forgets" commutativity. Its left adjoint takes a monoid and "forces" commutativity by quotienting the underlying carrier set.

Observation 2.33. Lawvere theories have free models. Let $p : \mathbb{F}^{\mathrm{op}} \to \mathcal{L}$ be a Lawvere theory. Observation 2.31 gives an adjunction

$$F : \mathrm{Mod}_{\mathbb{F}^{\mathrm{op}}} \leftrightarrows \mathrm{Mod}_{\mathcal{L}} : \mathrm{Mod}_p.$$

Then Mod_p coincides with the forgetful functor of Remark 2.28. The left adjoint F gives free objects.

Because of Observation 2.31, it is natural to take adjunctions as *the* notion of variety morphism. Below, by *unsorted*-variety we mean a category equivalent to $\mathrm{Mod}_{\mathcal{L}}$ for $\mathcal{L}$ with finite products.

Definition 2.34. Let $\mathcal{V}, \mathcal{W}$ be two (unsorted) varieties; a *morphism of varieties* is a functor $R : \mathcal{V} \to \mathcal{W}$ satisfying the following:

MV1) R admits a left adjoint $L : \mathcal{W} \to \mathcal{V}$;
MV2) R commutes with sifted colimits.

Given that adjunctions compose, this data yields a category Var.

Let Prod be the 2-category whose objects are small cartesian categories, morphisms are cartesian functors and 2-cells are natural transformations. Then Observation 2.31 boils down to defining a 2-functor $\mathrm{Mod}\colon \mathrm{Prod}^{\mathrm{op}} \to \mathrm{Var}$. The following captures the relationship between Law and Var.

[1]Sifted $\mathcal{J}$-indexed colimits satisfy the following property: given a functor $E : I \times \mathcal{J} \to \mathrm{Set}$, s.t. the category I is discrete (namely, it is just a set), the following isomorphism holds:

$$\mathrm{colim}_{\mathcal{J}} \lim_I E(I, J) \cong \lim_I \mathrm{colim}_{\mathcal{J}} E(I, J).$$

In other words, sifted colimits are those that commute with finite products in Set.

Theorem 2.35 ([Adámek et al. 2003, Theorem 4.1]). There exists a 2-adjunction whose unit is an equivalence:

$$\mathsf{Th} : \mathsf{Var} \leftrightarrows \mathsf{Prod}^{\mathrm{op}} : \mathsf{Mod}$$

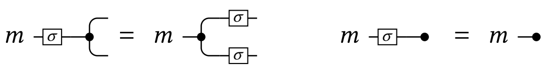
Remark 2.36. One obtains the S-sorted version of Theorem 2.35 by slicing on both sides over the free category with products on S. This is given in more detail for *partial* Lawvere theories in S7.2.

2.6 Equational Theories as Monoidal Theories

Given that Lawvere theories are cartesian props, Theorem 2.22 suggests how to consider them as monoidal theories. We recall the recipe from [Bonchi et al. 2018]: the idea is to characterise Σ-terms as certain string diagrams, and then—through this lens—turn any equational theory into a monoidal theory.

Recipe 2.37. Fix a signature Σ. A Σ-term $t : T_\Sigma^{[n]}$ is the same thing as a string diagram $n \to 1$ in the prop induced by the monoidal theory with

- generators $\Gamma \stackrel{\text{def}}{=} \Sigma + (\text{CCMG})$
- (CCM) together with equations that ensure naturality with respect to the comonoid structure. The latter can be easily added as two additional equations for each $\sigma : \Sigma$:

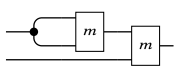


(SNσ)

The Lawvere theory induced by equational theory (Σ, E) can now be seen as the prop induced by the monoidal theory (Γ, F) where F is the set of equations obtained by translating the equations in E to string diagrams, together with (CCM), and (SNσ) for each $\sigma : \Sigma$.

It is important to build an intuition behind this translation. An obvious difference between terms and string diagrams is that the latter do not have named variables. The translation ensures that *wires* play the role of variables, and the comonoid structure plays the role of "variable management". We illustrate this with an example below.

Example 2.38. The prop corresponding to the Lawvere theory induced by the equational theory of commutative monoids (Example 2.8) is the same as the prop of commutative bialgebra. For example, the term $m(m(x, x), y)$ in the theory of commutative monoids can be depicted as



In the term we have considered, the variable x appears twice. In the corresponding diagram, the wire corresponding to x starts with a comultiplication that witnesses the "copying of x".

3 ALGEBRA OF PARTIAL MAPS

We have seen that finite products are central in classical universal algebra. It is therefore natural to begin our development of its partial analogue by identifying the corresponding universal property in the partial setting. We will see that this amounts to replacing the class of cartesian categories with the class of *discrete cartesian restriction categories* (DCR categories) [Cockett et al. 2012]. Next, we characterise DCR categories in terms of algebraic structure, analogous to Theorem 2.22 for cartesian categories.

3.1 Partial Functions

The starting point of our journey is the (2-)category Par of sets and partial functions. Just as Set was the semantic universe for ordinary equational theories, Par is the semantic universe for partial equational theories. We first recall an elementary, set theoretic presentation:

Definition 3.1. Par has sets as objects and *partial functions* $f\colon X \rightharpoonup Y$ as arrows, where a partial function f is a pair $(\mathrm{dom}f,\ \mathrm{def}f)$ where $\mathrm{dom}f \subseteq X$ is the *domain of definition* of f and $\mathrm{def}f\colon \mathrm{dom}f \to Y$ is a (total) function. Given a partial function $f\colon X \rightharpoonup Y$, and some $X' \subseteq X$ we write $f|_{X'}$ for the partial function $(\mathrm{dom}f \cap X',\ f')$ where $f'\colon \mathrm{dom}f \cap X' \to Y$ is $\mathrm{def}f$ restricted to the (potentially smaller) domain of definition $\mathrm{dom}f \cap X'$. Similarly, given $Y' \subseteq Y$, write $f^{-1}(Y') = \{x \in \mathrm{dom}f \mid \mathrm{def}f(x) \in Y'\}$. Given $f\colon X \rightharpoonup Y$ and $g\colon Y \rightharpoonup Z$, their composite is defined by $f \mathbin{⨟} g = (f^{-1}(\mathrm{dom}g),\ (\mathrm{def}f|_{f^{-1}(\mathrm{dom}g)} \mathbin{⨟} \mathrm{def}g)$. The identity on X is (X, id_X).

There is a natural partial order between partial functions $X \rightharpoonup Y$:

$$f \leq g \quad \stackrel{\text{def}}{=} \quad \mathrm{dom}f \subseteq \mathrm{dom}g \ \wedge\ g|_{\mathrm{dom}f} = f.$$

It is straightforward to verify that this data makes Par a category, and with $\leq$, a 2-category.

Categorifying partiality has long history (see e.g., [Cockett and Lack 2002; Robinson and Rosolini 1988]). We recall a classical approach:

Definition 3.2. Suppose that $\mathbb{C}$ has finite limits. Its 2-category of *partial maps*, $\mathrm{Par}(\mathbb{C})$ has:

1) **objects** are objects of $\mathbb{C}$.
2) **arrows** are equivalence classes $[m, f] : X \to Y$ of spans $X \xleftarrow{m} A \xrightarrow{f} Y$ where m is monic. We equate $(m, f) \sim (m', f')$ iff there is an isomorphism α s.t. the following diagram commutes:

$$\begin{tikzcd} & A \arrow[r, "\alpha"] \arrow[dl, "m"'] \arrow[drr, "f"] & A' \arrow[dll, "m'"] \arrow[dr, "f'"] & \\ X & & & Y. \end{tikzcd}$$

3) **2-cells**: $[m, f] \leq [m', f']$ when there exists *any* α that makes the diagram commute.
4) **composition** is defined by pullback. Explicitly, the composite of $(m, f) : A \to B$ and $(m', g) : B \to C$ is the outer span of the diagram on the left

$$\begin{tikzcd} & & X \wedge X' \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & \\ & X \arrow[dl, "m"'] \arrow[dr, "f"] & & X' \arrow[dl, "m'"'] \arrow[dr, "g"] & \\ A & & B & & C \end{tikzcd}$$

where the square with top $X \wedge X'$ is a pullback in $\mathbb{C}$. Note that it doesn't matter which pullback, since any two choices will give isomorphic spans, and therefore equal morphisms.
5) **Identities** are diagonal spans $(1_A, 1_A) : A \to A$.

Given a morphism $(m, f) : A \to B$ in $\mathrm{Par}(\mathbb{C})$, we think of the monic $m : X \to A$ as a subobject, specifying which part of A the morphism is defined on, and then $f : X \to B$ tells us what it does.

The following is a straightforward sanity check:

Observation 3.3. There is an isomorphism of (2-)categories $\mathrm{Par} \cong \mathrm{Par}(\mathrm{Set})$.

Just as a model of a total operation of arity n is a function $A^n \to A$ (an arrow in Set), a model of a partial operation ought to be a partial function $A^n \rightharpoonup A$ (an arrow in Par). For this reason, it is important to understand the mathematical status of the cartesian product in Par. Interestingly, Par has categorical products, but these *do not* correspond to the cartesian product of sets.[2]

3.2 Cartesian Restriction Categories

It is by focusing on the universal property of the cartesian product in Par that we are able to identify a generalisation of Lawvere's approach to partial operations. This is the goal of this section.

Restriction categories were devised to study partial phenomena in an axiomatic setting. Here we give a whirlwind tour, more details can be found in [Cockett et al. 2012; Cockett and Lack 2002, 2007]. In a restriction category, every arrow $f : A \to B$ has an associated idempotent $\overline{f} : A \to A$, thought of as the identity function restricted to the domain of definition of f. We call them *domain idempotents*. Arrows where $\overline{f} = 1_A$ are called *total*, and form a subcategory. Further, we have:

Remark 3.4. Any restriction category is poset-enriched, with the ordering defined by

$$f \leq g \Leftrightarrow \overline{f} \mathbin{;} g = f$$

Functors F that preserve domain idempotents ($F\overline{f} = \overline{Ff}$) are called *restriction functors*. Restriction categories and restriction functors form a category. This extends to a 2-category in which the 2-cells are *lax transformations*. A lax transformation $\alpha : F \to G$ of restriction functors $F, G : \mathbb{X} \to \mathbb{Y}$ consists of a family of total maps $\alpha_A : FA \to GA$ in $\mathbb{Y}$ indexed by the objects A of $\mathbb{X}$ s.t. for every $f : A \to B$ of $\mathbb{X}$ the usual naturality square *commutes up to inequality*:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] & GB \end{tikzcd}$$

where $\leq$ is the ordering introduced above. Call this 2-category $\mathrm{RCat}^{\leq}$. Just as categories with finite products enjoy a universal property in the 2-category Cat, those with finite restriction products have a universal property in $\mathrm{RCat}^{\leq}$. In general, formal limits in $\mathrm{RCat}^{\leq}$ are called *restriction limits* [Cockett and Lack 2007]. A *cartesian restriction* (CR) category is a restriction category with finite restriction products.

Observation 3.5 ([Cockett and Lack 2002]). Par is a CR category, with the cartesian product as restriction product.

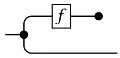
CR categories have appeared in the literature under a variety of different names, including *p-category with a one-element object* [Robinson and Rosolini 1988] and *partially cartesian category* [Curien and Obtułowicz 1989]. For our development, it is crucial that the data of CR categories can be equivalently captured as follows:

Theorem 3.6 ([Cockett and Lack 2007]). A CR category is the same thing as a symmetric monoidal category where every object is equipped with a commutative comonoid structure that is (coherent) and the comultiplication is natural. That is, for any $f : A \to B$ we have $f \mathbin{;} \delta_B = \delta_A \mathbin{;} (f \otimes f)$.

From this perspective a CR category is very close to a cartesian category viewed as a monoidal category through Theorem 2.22. The difference is that we do not ask for the counit of the comonoid to be natural. This has profound consequences: for instance, the same symmetric monoidal category may have more than one such chosen comonoid structure, thus definining different CR categories.

[2]The categorical product of A and B in Par is $(A + \{\star\}) \times (B + \{\star\}) - \{(\star, \star)\}$. This can be seen via the equivalence $1/\mathrm{Set} \simeq \mathrm{Par}$. Limits in the coslice category $1/\mathrm{Set}$ are calculated pointwise, and the functor $1/\mathrm{Set} \to \mathrm{Par}$ removes the point

Given the algebraic data, the domain idempotent $\overline{f} : A \to A$ of an arrow $f : A \to B$ in a CR category is recovered as:

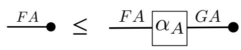
$\overline{f}$ = 

and so in particular the subcategory of $\mathbb{X}$ for which the counit is natural is precisely the subcategory of total maps. Notice that this means the subcategory of total maps of a CR category is cartesian.

Definition 3.7. A *CR functor* between two CR categories $F : \mathbb{X} \to \mathbb{Y}$ is a functor that preserves the algebraic structure. That is, $F(A \otimes B) = FA \otimes FB$, $F1 = 1$, $F\delta_A = \delta_{FA}$ and $F\varepsilon_A = \varepsilon_{FA}$.

Remark 3.8. A lax transformation of CR functors may be equivalently specified as a family of maps $\alpha_A : FA \to GA$ indexed by the objects A of $\mathbb{X}$ s.t. for every $f : A \to B$ we have $Ff \fatsemi \alpha_B \leq \alpha_A \fatsemi Gf$. We do not need to ask that each α_A is total, since if F and G preserve the cartesian restriction structure, then they are *automatically* total. In particular the diagram on the left gives the inequality on the right, which gives that α_A is total:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "F\varepsilon_A"'] \arrow[dr, phantom, "\leq"] & GA \arrow[d, "G\varepsilon_A"] \\ FI \arrow[r, "\alpha_I"'] & GI \end{tikzcd}$$

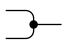


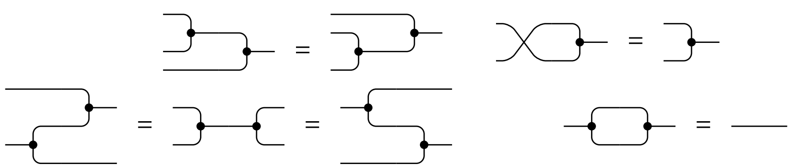
3.3 Discrete Cartesian Restriction Categories

Restriction products do not quite capture all the properties of Par needed for partial universal algebra. In particular, we require CR categories with the following extra structure:

Definition 3.9. A CR category is said to be *discrete* (DCR category [Cockett et al. 2012]) if for each object A there is an arrow $\mu_A : A \otimes A \to A$ that is *partial inverse* to δ_A. That is, $\delta_A \fatsemi \mu_A = \overline{\delta_A} = 1_A$ and $\mu_A \fatsemi \delta_A = \overline{\mu_A}$.

We give a novel presentation of DCR categories, inspired by the work of [Giles 2014]. Central to our presentation is the notion of a commutative special Frobenius algebra in which the monoid does not have a unit, which we call a *partial Frobenius algebra*. More precisely:

Definition 3.10. A *partial Frobenius algebra* $(A, \delta_A, \mu_A, \varepsilon_A)$ in a symmetric monoidal category consists of a commutative comonoid $(A, \delta_A, \varepsilon_A)$ and a commutative semigroup (A, μ_A) s.t. (A, δ_A, μ_A) is a semi-Frobenius algebra. Diagramatically, this is the comonoid structure we have already seen together μ_A, which we depict as  in our string diagrams, subject to the following equations:



(MCA)

(SFROB)

Note that there is some redundancy in the equational presentation above, as discussed in [Carboni 1991]. We now extend the characterisation of CR categories given in Theorem 3.6 to DCR categories:

Theorem 3.11. A DCR category is the same thing as a symmetric monoidal category where every object A is equipped with a coherent partial Frobenius algebra structure $(A, \delta_A, \varepsilon_A, \mu_A)$ s.t. the comultiplication is natural. That is, for any $f : A \to B$ we have $f \fatsemi \delta_B = \delta_A \fatsemi (f \otimes f)$.

DCR categories are intimately connected to categories with finite limits [Cockett et al. 2012]. In particular:

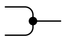
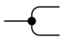
Proposition 3.12. If $\mathbb{C}$ is a category with finite limits, $\mathsf{Par}(\mathbb{C})$ is a DCR category.

Definition 3.13 (the 2-category $\mathsf{DCRC}^{\leq}$*).* It follows that for any CR functor $F : \mathbb{X} \to \mathbb{Y}$ between DCR categories, we have $F\mu_A = \mu_{FA}$. CR functors therefore give the notion of morphism between DCR categories. We consider the strict 2-category of DCR categories, restriction functors, and lax transformations, which we call $\mathsf{DCRC}^{\leq}$.

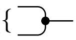
4 PARTIAL LAWVERE THEORIES

In this section we develop a Lawvere-style approach to *partial algebraic theories*, where operations may be partial. Ordinary Lawvere theories are determined by the free cartesian category on a single object $\mathbb{F}^{\mathrm{op}}$; we are thus interested in the analogue of $\mathbb{F}^{\mathrm{op}}$ in the world of DCR categories.

4.1 The Free DCR Category on One Object

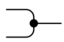
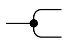
Given Theorem 3.11, we have an explicit description for the DCR category on one object: it is the prop $\mathbb{PF}$ induced from the monoidal theory of partial Frobenius algebras. That is, it has generators {  ,  , } and equations (MCA), (CCM) and (SFROB).

It turns out that one can gain a precise intuition on what $\mathbb{PF}$ looks like by mimicking the way in which the props $\mathbb{F}$ and its opposite $\mathbb{F}^{\mathrm{op}}$ describe familiar algebraic structures. In fact, the prop $\mathbb{CM}$ of commutative monoids is isomorphic to the prop $\mathbb{F}$ (see Observation 2.18), and similarly, the prop $\mathbb{CC}$ of commutative comonoids is isomorphic to $\mathbb{F}^{\mathrm{op}}$ (Observation 2.21).

The prop $\mathbb{PF}$ of partial Frobenius algebras that we want to describe here can be given a similar "combinatorial" characterisation relying on the insights of Lack [Lack 2004]. First, we note that the prop $\mathbb{CAM}$ induced by the monoidal theory of commutative semigroups ({  } and equations (MCA)) is isomorphic to sub-prop $\mathbb{F}_s \subset \mathbb{F}$ of finite sets and *surjective functions.*

Observation 4.1. As props, $\mathbb{CAM} \cong \mathbb{F}_s$.

This is intuitive: as observed in Remark 2.19, string diagrams of $\mathbb{CM}$ allow one to "draw" all functions $[m] \to [n]$. Doing without the unit means that we can express exactly the surjective ones.

Next, we know from [Lack 2004] that the prop $\mathbb{FROB}$ induced by the monoidal theory of special Frobenius algebras with generators {  , ,  , } and equations (CM), (CCM) and (SFROB) is isomorphic to the prop of *cospans* of finite sets $\mathsf{Cospan}(\mathbb{F})$. An arrow $m \to n$ here is (an isomorphism class of) a cospan of functions $[m] \xrightarrow{f} [k] \xleftarrow{g} [n]$, and composition is by pushout.

Proposition 4.2 ([Lack 2004]). As props, $\mathbb{FROB} \cong \mathsf{Cospan}(\mathbb{F})$.

Given that surjective functions are closed under composition and pushouts in $\mathbb{F}$, we can consider the subprop $\mathsf{Cospan}_s(\mathbb{F})$ of $\mathsf{Cospan}(\mathbb{F})$ with arrows those cospans where the left leg is surjective. Now, combining Observation 4.1 and Proposition 4.2 yields:

Proposition 4.3. As props, $\mathbb{PF} \cong \mathsf{Cospan}_s(\mathbb{F})$.

This gives us a combinatorial characterisation of $\mathbb{PF}$. But there is a more familiar and satisfactory way of describing $\mathsf{Cospan}_s(\mathbb{F})$. Given that cospans in $\mathbf{C}$ are spans in $\mathbf{C}^{\mathrm{op}}$, and epimorphisms in $\mathbf{C}$ are monomorphisms in $\mathbf{C}^{\mathrm{op}}$, we see that $\mathsf{Cospan}_s(\mathbb{F}) = \mathsf{Par}(\mathbb{F}^{\mathrm{op}})$, since a cospan in $\mathbb{F}$ with left leg surjective is the same thing as a span in $\mathbb{F}^{\mathrm{op}}$ with left leg a monomorphism. Therefore, we obtain:

Proposition 4.4. As props, $\mathbb{PF} \cong \mathsf{Par}(\mathbb{F}^{\mathrm{op}})$.

4.2 Partial Lawvere Theories

We have seen that $\mathbb{F}^{\mathrm{op}}$ is central to the definition of Lawvere theories, being the free cartesian category on one object. The prop $\mathsf{Par}(\mathbb{F}^{\mathrm{op}})$, being the free DCR on one object, plays the corresponding role in the definition of partial Lawvere theories.

Definition 4.5. A partial Lawvere theory $\mathcal{L}$ is a DCR prop.

Spelled out, a partial Lawvere theory is a DCR category $\mathcal{L}$ for which there is an identity-on-objects CR functor $\mathsf{Par}(\mathbb{F}^{\mathrm{op}}) \to \mathcal{L}$. A morphism of partial Lawvere theories is a functor $h : \mathcal{L} \to \mathcal{M}$ s.t. the following triangle commutes:

$$\begin{tikzcd} & \mathsf{Par}(\mathbb{F}^{\mathrm{op}}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{L} \arrow[rr, "h"'] & & \mathcal{M}. \end{tikzcd}$$

This defines the category pLaw of partial Lawvere theories.

Mimicking also the definition of model of a Lawvere theory, we obtain at once the notion of model of a *partial* Lawvere theory:

Definition 4.6 (Model of a partial Lawvere theory). A *model* for a partial Lawvere theory $\mathcal{L}$ is a CR functor $L\colon \mathcal{L} \to \mathsf{Par}$. A homomorphism $L \to L'$ is a *lax* natural transformation $\alpha\colon L \Rightarrow L'$.

Definition 4.7. The category of models and homomorphisms of a partial Lawvere theory $\mathcal{L}$ is denoted $\mathsf{pMod}_{\mathcal{L}}$. As explained in Remark 3.8, the homomorphisms are *total* functions.

5 PARTIAL EQUATIONAL THEORIES

In order to consider interesting examples of partial Lawvere theories, we need to introduce the notion of *partial equational theory*. For partial structures, these are the syntactic analogue of equational theories, and yield partial Lawvere theories in a similar way to how equational theories yield Lawvere theories.

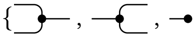
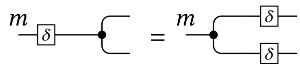
Monoidal signatures (Definition 2.12) Γ have unrestricted arities and coarities. Instead, a signature Σ of an equational theory (Definition 2.1) has function symbols of arbitrary arities, but—considered as a monoidal signature—all coarities are 1. Partial signatures are an intermediate concept: as for equational theories, coarities > 1 are redundant, but we need to admit symbols of coarity 0.

Definition 5.1. A partial signature $\Delta \stackrel{\mathrm{def}}{=} \Delta_0 + \Delta_1$, where Δ_0 is the set of operations of coarity 0, and Δ_1 is the set of operations of coarity 1. Each $\delta : \Delta_i$ comes with an arity $ar(\delta) : \mathbb{N}$.

Differently from ordinary equational theories, we cannot use classical terms, which—as discussed in Remark 2.26—are tied to an underlying cartesian structure. Instead, we adapt Recipe 2.37 to DCR categories, obtaining *partial* terms as particular string diagrams. Before we launch into formal definitions, and illustrate them with a variety of examples, let us establish some intuitions for how to read the string diagrams.

- string diagrams represent partial terms, obtained through composing partial operations,
- equalities and inequalities between them are understood in the sense of Kleene,
- the comonoid structure { , } plays a similar role to that described in S2.6.

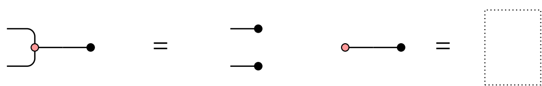
Recipe 5.2. Given a partial signature Δ, the free DCR prop $\mathcal{L}_\Delta$ on Δ is the prop induced from the monoidal theory with

- generators $(\Delta + \{$  $\})$
- equations (MCA), (CCM) and (SFROB), together with  for each $\delta : \Delta_1$, where $ar(\delta) = m$.

Definition 5.3 (Partial Equational Theory). A *partial equation* is a pair (s, t) where $s, t : \mathcal{L}_\Delta(m, n)$ for some $m, n : \mathbb{N}$; we usually write '$s = t$'. A *partial equational theory* is a pair (Δ, G) where Δ is a partial signature and G is a set of partial equations.

We first return to a familiar example.

Example 5.4 ((Partial) Commutative Monoids). We start with the monoidal theory of commutative monoids (Example 2.16), where the multiplication and unit generators are re-coloured to red to avoid a clash. In models, the multiplication operation may be partially defined *and* the unit may be undefined. To define the partial theory of *total* commutative monoids, we'd need to add equations:

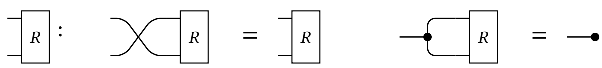


(3)

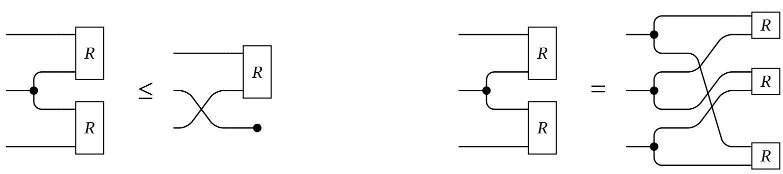
Example 5.5 (Equational Theories). Any equational theory (Σ, E) is an example. One follows Recipe 5.2, adding equations analogous to (3) to specify that every generator in Σ is total. The category of models of this partial theory then agrees with that of the Lawvere theory $\mathcal{L}_{(\Sigma,E)}$.

The following elementary examples illustrate the novel features of partial Lawvere theories, highlighting the way in which they differ from classical (i.e., total) Lawvere theories.

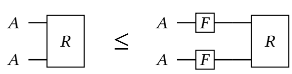
Example 5.6 (Equivalence Relations). Consider the partial Lawvere theory consisting of a single binary operation R with coarity 0, together with equations expressing symmetry and reflexivity:



Note that *inequations* of terms, as in Remark 3.4, do not add expressivity. As such, we may use them freely when specifying partial Lawvere theories. Transitivity is intuitively captured by the inequation on the left, which, unfolding the definition of $\leq$, is precisely the equation on the right:



A model $\mathcal{A}$ of this theory consists of a set A together with an equivalence relation $=_A \subseteq A \times A$ corresponding to the domain of definition of $\mathcal{A}(R)$. A morphism $F : \mathcal{A} \to \mathcal{B}$ is a function $F : A \to B$ with $a =_A b \Rightarrow Fa =_B Fb$, which arises from the requirement that F is a lax transformation:

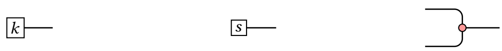


Thus, the variety corresponding to this theory is the category of *Bishop sets* (*setoids*) [Palmgren 2009].

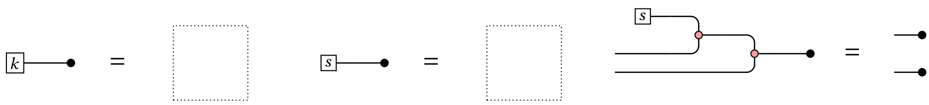
Example 5.7 (Partial Combinatory Algebras). A *partial combinatory algebra* (PCA) is a set A with a binary partial operation $\_ \bullet \_ : A \times A \longrightarrow A$, and elements $\mathrm{s}, \mathrm{k} \in A$ s.t. for any $x, y, z \in A$:

(i) $(\mathrm{k} \bullet x) \bullet y = x$
(ii) $((\mathrm{s} \bullet x) \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z)$
(iii) $(\mathrm{s} \bullet x) \bullet y$ is defined

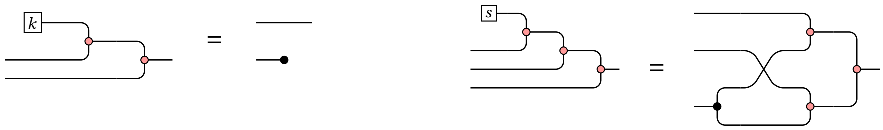
where "=" is Kleene equality. The partial Lawvere theory of PCAs has three generators:



and equations that ensure the totality of k, s, i.e. they define elements of the carrier, and (iii):



as well as equations for (i) and (ii):



The variety here is the category of PCAs and homomorphisms preserving the applicative structure.

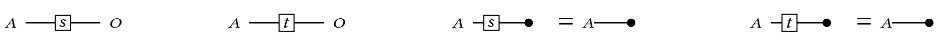
Example 5.8 (Pairing Functions). Consider the partial Lawvere theory with two operations which we think of as *pairing* and *unpairing* respectively, subject to the equation on the right:

Models are sets equipped with a *pairing function*, and model morphisms map pairs to pairs. For example, $\mathbb{N}$ and Cantor's pairing function, or Λ – the set of untyped λ-terms – with the usual pairing and projection functions. Note that our equation makes pairing a section, and so it is total.

6 MULTI-SORTED EQUATIONAL THEORIES

In this section we present a progression of multi-sorted partial Lawvere theories for categories with different kinds of structure. While our development of partial Lawvere theories has thus far focused on the single-sorted case, the move to multi-sorted theories contains no surprises, so we omit the details. The short version is that props are replaced with *coloured* props, and the sorting discipline changes accordingly. The examples that follow are developed incrementally: Each step adds more categorical structure to the models by adding the appropriate operations and equations to the theory, culminating in the partial Lawvere theory of cartesian closed categories.

Example 6.1 (Directed Graphs). We begin with the partial Lawvere theory of *directed graphs*, which has a sort O of vertices and a sort A of edges, together with source and target operations:

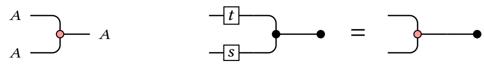


The associated variety is the category of directed graphs, as model morphisms F must satisfy:

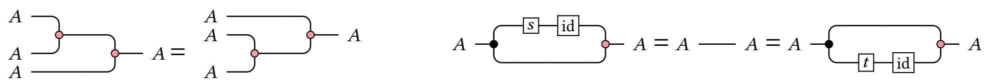
Example 6.2 (Reflexive Graphs). Extending Example 6.1, we ask that each vertex has a self-loop:

then morphisms of models are required to preserve the self-loop, so the associated variety is the category of *reflexive graphs*. Notice that along with Example 6.1, this could also be presented as a (total) 2-sorted Lawvere theory, since all the operations are total.

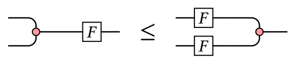
Example 6.3 (Categories). To capture *categories* we extend Example 6.2 with a composition operator, which is defined when the target of the first arrow matches the source of the second:



and equations insisting composition is associative and unital, with identities given by the self-loops:

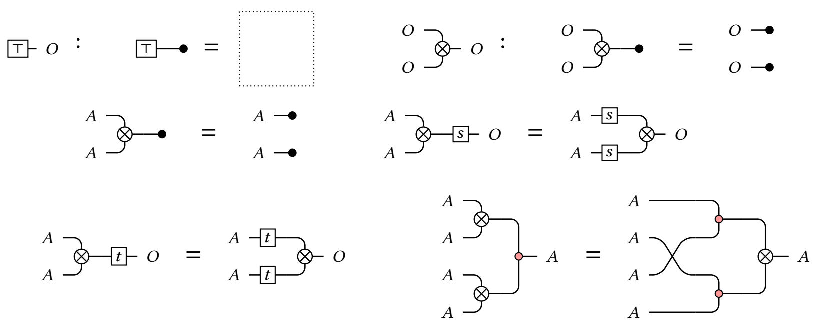


Model morphisms are precisely functors. It is worth noting that this involves an inequality:

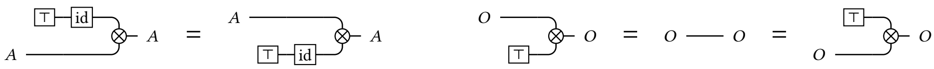


This states that if f and g are composable then so are Ff and Fg, and in particular $F(f \mathbin{;} g) = Ff \mathbin{;} Fg$. If this were an equality, it would insist also that if Ff and Fg are composable, then so are f and g, which is not always the case. Of course, the associated variety is the category of small categories.

Example 6.4 (Strict Monoidal Categories). Next, we extend Example 6.3 by asking for a functorial binary operation $\otimes$ on O and A together with a unit constant $\top$ of O:

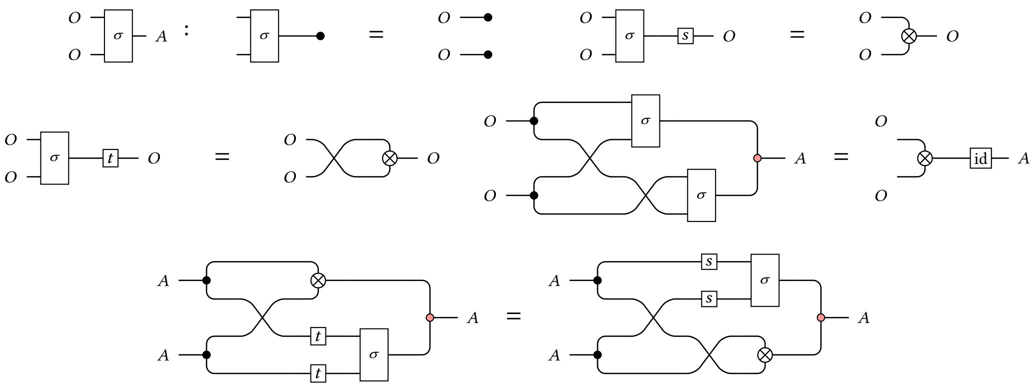


Additionally, we require equations to the effect that $\otimes$ is associative and unital:



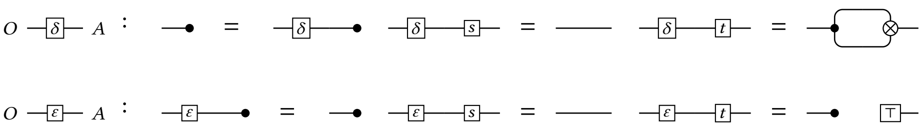
Now the associated variety is the category of strict monoidal categories and strict monoidal functors.

Example 6.5 (Symmetric Strict Monoidal Categories). To capture *symmetric monoidal categories*, we extend Example 6.4 with a binary operation $\sigma : O \otimes O \to A$ for the braiding maps, subject to:

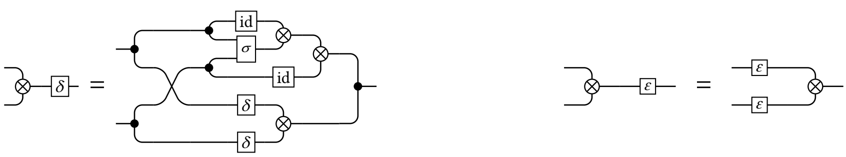


This gives the variety of strict monoidal categories and symmetric strict monoidal functors.

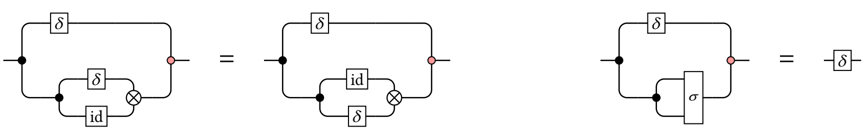
Example 6.6 (Cartesian Restriction Categories). In light of Theorem 3.6, we can capture *CR categories* by extending Example 6.5 with operations $\delta : O \to A$ and $\varepsilon : O \to A$ corresponding to the comultiplication and counit of the comonoid structure:

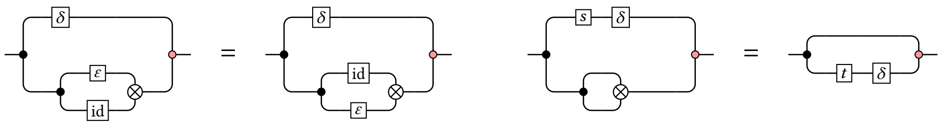


along with equations insisting that δ and ε are coherent with respect to the monoidal structure:



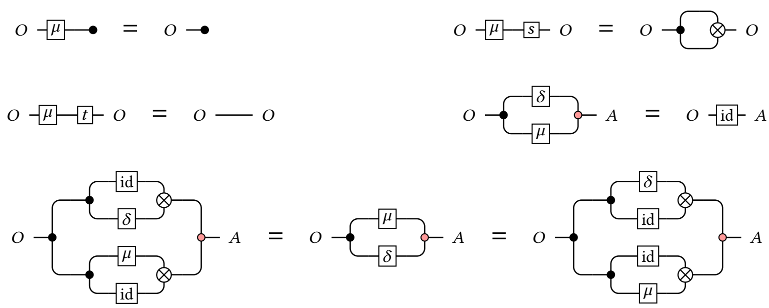
And finally equations for the commutative comonoid axioms, and naturality of δ:





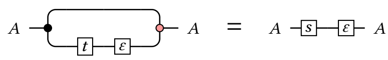
The associated variety is the category of CR categories and CR functors.

Example 6.7 (Discrete Cartesian Restriction Categories). Theorem 3.11 makes it easy to capture *DCR categories* by extending Example 6.6 with $\mu : O \to A$ satisfying the Frobenius and special equations: there is a O $\boxed{\mu}$ A such that



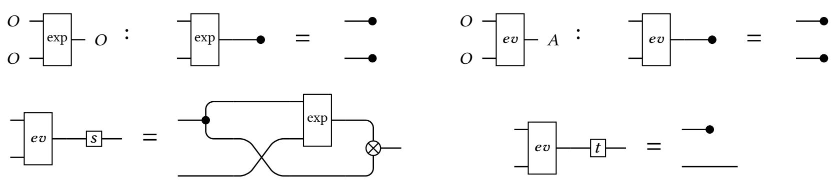
The variety is the category of strict DRC categories and strict CR functors (since they preserve μ).

Example 6.8 (Cartesian Categories). To capture *cartesian categories* instead, we can extend Example 6.6 with one equation, ensuring that ε is natural:

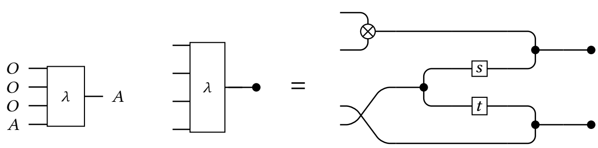


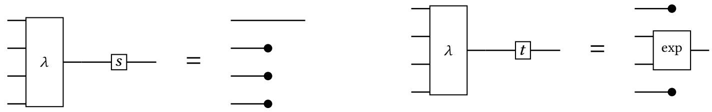
Then by Theorem 2.22, this gives the variety of strict cartesian categories and strict cartesian functors.

Example 6.9 (Cartesian Closed Categories). Finally, to capture *cartesian closed categories* we extend Example 6.8 with an operator $\exp : O \otimes O \to O$, the idea being that $\exp(A, B)$ is the internal hom $[A, B]$, along with an operator $\mathrm{ev} : O \otimes O \to O$ that gives the corresponding evaluation map:

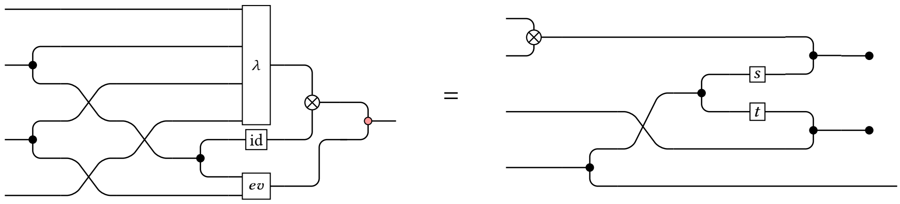


along with an operation λ and equations stating, intuitively, that $\lambda(X, A, B, f)$ is defined precisely in case $f : X \times A \to B$, and yields a map $\lambda(X, A, B, f) : X \to [A, B]$ as in:

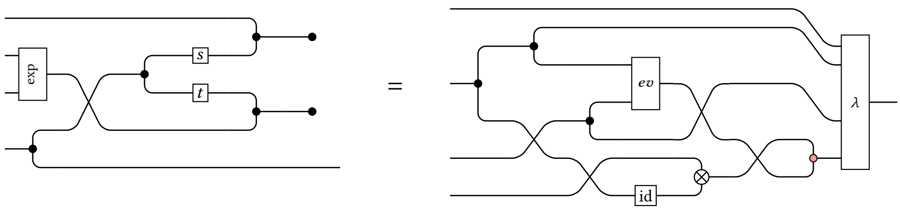




also equations insisting that if $f : X \times A \longrightarrow B$ then $(\lambda(X, A, B, f) \times 1) \mathbin{⨟} \mathrm{ev} = f$ holds:



and that if $g : X \longrightarrow [A, B]$ then $\lambda(X, A, B, (g \times 1) \mathbin{⨟} \mathrm{ev}) = g$ holds:



Now the associated variety is the category of strict cartesian closed categories and strict cartesian closed functors: these preserve hom-objects and, when $\lambda(X, A, B, f)$ is defined, satisfy $F\lambda(X, A, B, f) = \lambda(FX, FA, FB, Ff)$. This presentation of cartesian closed categories is essentially due to Freyd: a version of it is given immediately after the first appearance of the notion of essentially algebraic theory in [Freyd 1972], albeit somewhat informally, and using very different syntax.

7 THE VARIETY THEOREM FOR PARTIAL THEORIES

Here we classify the categories of models of partial Lawvere theories. These turn out to be exactly the locally finitely presentable (LFP) categories [Adamek and Rosický 1994, 1A]. LFP categories have an important position in categorical algebra, due to deep connections with model theory [Adamek and Rosický 1994, Ch. 5] and [Makkai and Paré 1989], homotopy theory [Dugger 2001], and universal algebra [Adamek and Rosický 1994, Ch. 3].

7.1 The Unsorted Case

Definition 7.1. In a category C, an object C is *finitely presentable* if the hom-functor $C(C, \_)$ preserves directed colimits (see [Adamek and Rosický 1994, 1.1] for the definition).

This notion might appear obscure to the reader unfamiliar with categorical logic; [Adamek and Rosický 1994, 1.2] contains lots of examples to help the reader build their intuition: for instance, an object of the category of sets is finitely presentable if and only if it is finite, and a (commutative) monoid is finitely presentable if and only if it admits a presentation $\langle G \mid R \rangle$ where both G (set of generators) and R (set of relations) are finite sets: this happens for many other algebraic structures, and thus motivates the definition.

Definition 7.2 (Locally finitely presentable category). [Adamek and Rosický 1994, Def. 1.9] A locally finitely presentable (LFP) category $\mathcal{K}$ is a cocomplete category s.t. there is a small full subcategory $\mathcal{A} \subset \mathcal{K}$ of finitely presentable objects, and such that every object of $\mathcal{K}$ is a directed colimit of objects of $\mathcal{A}$.

As in the classical case (Remark 2.30), the most crisp statement of the variety theorem is for the unsorted case. Just as an unsorted Lawvere theory is exactly a (small) cartesian category, we define an *unsorted partial Lawvere theory* to be a (small) DCR category, and the corresponding notion of morphism to be a CR functor. Then:

$$\boxed{\text{Categories of models of partial theories are exactly LFP categories.}}$$

Indeed, we have a similar contravariant adjunction to that of Theorem 2.35, if LFP is a 2-category having 1-cells $R : \mathcal{K} \to \mathcal{K}'$ the right adjoint functors R preserving directed colimits, and 2-cells all natural transformations $\alpha : R \Rightarrow R'$. A motivation for this apparently peculiar choice of 1-cells can be found in our Observation 2.31; it is exactly as our Definition 2.34, provided one replaces "sifted" colimit with "directed".

Theorem 7.3. There is a 2-adjunction

$$\mathsf{Th} : \mathsf{LFP} \leftrightarrows (\mathsf{DCRC}^{\leq})^{\mathrm{op}} : \mathsf{Mod}, \tag{4}$$

where $\mathsf{DCRC}^{\leq}$ is the 2-category of DCR categories of our Definition 3.13 and LFP is the 2-category of LFP categories. Moreover, the unit of this adjunction is an equivalence, namely there is a natural equivalence of categories between $\mathcal{K} \in \mathsf{LFP}$ and $\mathsf{Mod}(\mathsf{Th}(\mathcal{K}))$, i.e. each LFP category is equivalent to the category of models of its induced theory.

The proof of Theorem 7.3 is split into two parts, as illustrated below:

$$\begin{tikzcd}
(\mathsf{DCRC}^{\leq})^{\mathrm{op}} \arrow[rr, shift left, leftarrow] \arrow[rr, shift right] \arrow[rr, phantom, "\boxed{1}"] \arrow[dr, dashed, shift left, leftarrow] \arrow[dr, dashed, shift right] & & (\mathsf{Lex})^{\mathrm{op}} \arrow[dl, shift left] \arrow[dl, shift right, leftarrow] \arrow[dl, phantom, "\boxed{2}"] \\
& \mathsf{LFP} &
\end{tikzcd} \tag{5}$$

1) we show that Lex –the 2-category of categories $\mathcal{A}$ with finite limits, functors $\mathcal{A} \to \mathcal{A}'$ preserving finite limits, and natural transformations– is reflective in the 2-category $\mathsf{DCRC}^{\leq}$. This is the original, technical core of Theorem 7.3.
2) we connect $\mathsf{Lex}^{\mathrm{op}}$ and LFP with a contravariant biequivalence of 2-categories. This is a classical result called *Gabriel-Ulmer duality.*

Composing the two, we obtain Theorem 7.3.

We will start from the first of the two tasks, providing an adjunction of 2-categories as follows.

$$K_t : \mathsf{DCRC}^{\leq} \leftrightarrows \mathsf{Lex} : \mathsf{Par}$$

We first describe the left adjoint K_t, then the right adjoint Par, and conclude by showing that they define an adjunction.

Splitting Domain Idempotents. The functor K_t arises via a modified Karoubi envelope, also called *Cauchy completion* in [Borceux and Dejean 1986]. Recall that an idempotent $a : A \to A$ in a category *splits* if there is a commutative diagram

$$\begin{tikzcd}
& X \arrow[r, "s"] & A \\
X \arrow[ur, "1_X"] \arrow[r, "s"'] & A \arrow[u, "r"] \arrow[ur, "a"'] &
\end{tikzcd}$$

Restriction categories in which all of the domain idempotents split are called *split* restriction categories. An example is $\mathsf{Par}(\mathbb{C})$: for any arrow $(m, f) : A \to B$, $\overline{(m, f)} = (m, m) : A \to A$ splits with $s = (1, m)$ and $r = (m, 1)$. Notice that this means the domain of definition of (m, f) is a subobject of A. This is a good way to think of split domain idempotents in general: for $\overline{f}$ to be split in a restriction category is for the domain of definition of $f : A \to B$ to be a subobject of A.

For any restriction category $\mathbb{X}$ we can construct a split restriction category $K(\mathbb{X})$ that contains $\mathbb{X}$ as a subcategory. Its subcategory of *total* maps $K_t(\mathbb{X})$ is of particular interest.

Definition 7.4. Let $\mathbb{X}$ be a DCRC. Then $K_t(\mathbb{X})$ is the category where

1) objects are pairs (A, a) with A an object of $\mathbb{X}$ and $a : A \to A$ a domain idempotent in $\mathbb{X}$;
2) arrows $f : (A, a) \to (B, b)$ are arrows $f : A \to B$ of $\mathbb{X}$ such that $\overline{f} = a$ and $f \fatsemi b = f$;
3) composition is given by composition in $\mathbb{X}$;
4) The identity on (A, a) is given by a.

It is routine to verify that this forms a category. Crucially, if $\mathbb{X}$ is a DCRC, then the subcategory $K_t(\mathbb{X})$ of *total* maps of $K(\mathbb{X})$ has finite limits [Cockett et al. 2012]:

Lemma 7.5. For any DCRC $\mathbb{X}$, $K_t(\mathbb{X})$ has finite limits.

We now show that this extends to a 2-functor $K_t : \mathsf{DCRC}^{\leq} \to \mathsf{Lex}$. If $\mathbb{X}$ and $\mathbb{Y}$ are DCRCs and $F : \mathbb{X} \to \mathbb{Y}$ is a CR functor, then there is a functor $K_t(F) : K_t(\mathbb{X}) \to K_t(\mathbb{Y})$ defined by $K_t(F)(A, a) = (FA, Fa)$ on objects and $K_t(F)(f) = f$ on arrows. It follows from our characterization of CR functors in terms of the partial Frobenius algebra structure that $K_t(F)$ preserves finite limits, giving the action of our 2-functor K_t on 1-cells. The action of K_t on 2-cells is given as follows:

Lemma 7.6. If $F, G : \mathbb{X} \to \mathbb{Y}$ are CR functors between DCR categories and $\alpha : F \to G$ is a lax transformation, define $K_t(\alpha) : K_t(F) \to K_t(G)$ by letting $K_t(\alpha)$ at (A, a) in $K_t(\mathbb{X})$ be:

$$K_t(\alpha)_{(A,a)} = Fa \fatsemi \alpha_A : (FA, Fa) \to (GA, Ga)$$

Then $K_t(\alpha)$ is a natural transformation.

At this point we need only show that K_t preserves composition and identities for 1-cells and 2-cells, which in both cases is straightforward.

Lemma 7.7. $K_t : \mathsf{DCRC}^{\leq} \to \mathsf{Lex}$ is a 2-functor.

Partial Functions Revisited. Here we show that the Par construction (S3.1) also extends to a 2-functor $\mathsf{Par} : \mathsf{Lex} \to \mathsf{DCRC}^{\leq}$. If $\mathbb{C}$ and $\mathbb{D}$ are categories with finite limits and $F : \mathbb{C} \to \mathbb{D}$ is a finite-limit preserving functor, then we obtain a CR functor $\mathsf{Par}(F) : \mathsf{Par}(\mathbb{C}) \to \mathsf{Par}(\mathbb{D})$, defined on objects by $\mathsf{Par}(F)(A) = F(A)$, and on arrows by

$$\begin{tikzcd} & X \arrow[dl, "m"'] \arrow[dr, "f"] & & & & FX \arrow[dl, "Fm"'] \arrow[dr, "Ff"] & \\ A & & B & \overset{\mathsf{Par}(F)}{\longmapsto} & FA & & FB \end{tikzcd}$$

Since F preserves finite limits, we have that $\mathsf{Par}(F)(\delta_A) = (F1_A, F\Delta_A) = (1_{FA}, \Delta_{FA}) = \delta_{FA} = \delta_{\mathsf{Par}(F)(A)}$ and $\mathsf{Par}(F)(\varepsilon_A) = (F1_A, F!_A) = (1_{FA}, !_{FA}) = \varepsilon_{\mathsf{Par}(F)(A)}$, so $\mathsf{Par}(F)$ preserves the CR structure. This defines the action of Par on 1-cells. We present the action of Par on 2-cells as a lemma:

Lemma 7.8. If $F, G : \mathbb{C} \to \mathbb{D}$ are finite limit preserving functors between categories with finite limits and $\alpha : F \to G$ is a natural transformation, define $\mathsf{Par}(\alpha) : \mathsf{Par}(F) \to \mathsf{Par}(G)$ by defining the component of $\mathsf{Par}(\alpha)$ at A in $\mathbb{C}$ to be:

$$\begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] \arrow[d, phantom, "\mathsf{Par}(\alpha)_A"] & \\ FA & {} & GA \end{tikzcd}$$

Then $\mathsf{Par}(\alpha) : \mathsf{Par}(F) \to \mathsf{Par}(G)$ is a lax transformation.

It remains only show that Par preserves composition and identities at the level of 1-cells and 2-cells, which is immediate in both cases. We therefore have:

Lemma 7.9. $\mathsf{Par} : \mathsf{Lex} \to \mathsf{DCRC}^{\leq}$ is a 2-functor.

Adjointness. The following result is original, and builds on [Cockett and Lack 2002, Corollary 3.5]; however, there the 2-cells of the categories involved are different.

Theorem 7.10. There is a 2-adjunction $K_t : \mathsf{DCRC}^{\leq} \leftrightarrows \mathsf{Lex} : \mathsf{Par}$.

It is worth describing the unit and counit of our adjunction. The unit $\eta : 1 \to \mathsf{Par} \cdot K_t$ is given by the canonical inclusion $\eta_{\mathbb{X}} : \mathbb{X} \to \mathsf{Par}(K_t(\mathbb{X}))$ defined by

$$A \xrightarrow{f} B \qquad \overset{\eta_{\mathbb{X}}}{\longmapsto} \qquad \begin{tikzcd} & (A, \overline{f}) \arrow[dl, "\overline{f}"'] \arrow[dr, "f"] & \\ (A, 1_A) & & (B, 1_B) \end{tikzcd}$$

The counit $\varepsilon : K_t \cdot \mathsf{Par} \to 1$ is defined in terms of the equivalence of categories $K(\mathbb{X}) \simeq \mathbb{X}$ between any split restriction category $\mathbb{X}$ and the result of formally splitting its domain idempotents. In particular, since $\mathsf{Par}(\mathbb{C})$ is always split, we obtain an equivalence of categories $K(\mathsf{Par}(\mathbb{C})) \simeq \mathsf{Par}(\mathbb{C})$. Restricting this to the subcategories of total maps gives defines our counit $\varepsilon_{\mathbb{C}} : K_t(\mathsf{Par}(\mathbb{C})) \simeq \mathbb{C}$. In particular, the fact that the counit is a natural equivalence gives:

Lemma 7.11. Lex is a reflective (2-)subcategory of $\mathsf{DCRC}^{\leq}$.

Gabriel-Ulmer duality. To complete the triangle (5), we recall a theorem first shown by P. Gabriel and F. Ulmer [Gabriel and Ulmer 1971], establishing a contravariant equivalence between the 2-category LFP of locally finitely presentable categories and the 2-category Lex of categories with finite limits.

The duality asserts that a locally finitely presentable category $\mathcal{K}$ can be reconstructed from its subcategory $\mathcal{K}_\omega$ of finitely presentable objects. A good reference for the proof is [Centazzo and Vitale 2002, Th. 3.1].

Theorem 7.12 (Gabriel-Ulmer duality). There is a biequivalence of 2-categories

$$\mathsf{Lex}^{\mathrm{op}} \leftrightarrows \mathsf{LFP}$$

between Lex, the 2-category of small categories with finite limits, where 1-cells are functors preserving finite limits and 2-cells are the natural transformations, and LFP, the 2-category of locally finitely presentable categories, where 1-cells are right adjoints preserving directed colimits.

7.2 Sorted Gabriel-Ulmer Duality

A similar version of the above theorem holds if, instead of considering theories of all possible sorts, we fix once and for all a single cardinality for the sorts $\mathfrak{S}$. Such "relative" version of Gabriel-Ulmer duality is useful to recover the classical Lawvere-style approach of single- and many-sorted theories.

Definition 7.13. We call $L\mathfrak{S}$ the free category with finite limits over the discrete set $\mathfrak{S}$. When $\mathfrak{S}$ is the singleton we will use the shortened notation $L1$.

Definition 7.14. A $\mathfrak{S}$-sorted category with finite limits $(\mathcal{A}, p)$ is an object in $(\mathsf{Lex})^{\mathrm{op}}/L\mathfrak{S}$ whose specifying functor $p : L\mathfrak{S} \to \mathcal{A}$ is bijective on objects. $(\mathfrak{S}\text{-}\mathsf{Lex})^{\mathrm{op}}$ is the full 2-subcategory of $\mathfrak{S}$-sorted categories with finite limits.

Definition 7.15. A $\mathfrak{S}$-sorted locally finitely presentable category $(\mathcal{K}, U)$ is an object in $\mathsf{LFP}/[\mathfrak{S}, \mathsf{Set}]$ whose specifying functor $U : \mathcal{K} \to [\mathfrak{S}, \mathsf{Set}]$ is conservative. $(\mathfrak{S}\text{-}\mathsf{LFP})^{\mathrm{op}}$ is the full 2-subcategory of $\mathfrak{S}$-sorted locally finitely presentable categories.

Theorem 7.16 (Sorted Gabriel-Ulmer duality). There is a biequivalence of 2-categories

$$\mathsf{Mod}_{\mathfrak{S}} : (\mathfrak{S}\text{-}\mathsf{Lex})^{\mathrm{op}} \leftrightarrows \mathfrak{S}\text{-}\mathsf{LFP} : \mathsf{Th}_{\mathfrak{S}}.$$

We can use the sorted version of Gabriel-Ulmer duality to infer the sorted version of the syntax-semantics duality for multi-sorted partial Lawvere theories.

Theorem 7.17. There is an 2-adjunction, whose unit is an equivalence,

$$\mathfrak{S}\text{-}\mathsf{LFP} \leftrightarrows (\mathfrak{S}\text{-}\mathsf{pLaw})^{\mathrm{op}},$$

where $\mathfrak{S}$-pLaw is the 2-category of "$\mathfrak{S}$-sorted partial Lawvere theories", understood as the analogue of Remark 2.30 for partial theories (see Definition 4.5), and $\mathfrak{S}$-LFP is the 2-category of $\mathfrak{S}$-sorted locally finitely presentable categories.

Sketch of proof. The proof is divided into intermediate steps: each tag on the following two diagrams indicates the section where the proof of the adjunction, or equivalence, is given.

$$\begin{tikzcd}
(\mathfrak{S}\text{-}\mathsf{pLaw})^{\mathrm{op}} \arrow[rr, shift right, "\star" description] \arrow[dr, dashed, shift right] & & (\mathfrak{S}\text{-}\mathsf{Lex})^{\mathrm{op}} \arrow[ll, shift right] \arrow[dl, shift right, "7.16" description] \\
& \mathfrak{S}\text{-}\mathsf{LFP} \arrow[ul, dashed, shift right] \arrow[ur, shift right] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathsf{DCRC}^{\leq})^{\mathrm{op}} \arrow[rr, shift right, "7.10" description] \arrow[dr, dashed, shift right] & & (\mathsf{Lex})^{\mathrm{op}} \arrow[ll, shift right] \arrow[dl, shift right, "7.12" description] \\
& \mathsf{LFP} \arrow[ul, dashed, shift right] \arrow[ur, shift right] &
\end{tikzcd}$$

The claim in ($\star$) is the only one that needs to be proven. Yet it is also the most trivial one. We will deduce it directly from Theorem 7.10. Indeed if Lex is reflective in $\mathsf{DCRC}^{\leq}$, $(\mathsf{Lex})^{\mathrm{op}}/L\mathfrak{S}$ is coreflective in $(\mathsf{DCRC}^{\leq})^{\mathrm{op}}/\mathsf{Par}(L\mathfrak{S})$, now observe that $\mathsf{Par}(L\mathfrak{S})$ is precisely the free discrete cartesian restriction category over $\mathfrak{S}$. The desired result follows passing to functors bijective on objects in the slice. □

Observation 7.18. In analogy with 2.33, we can show that sorted partial Lawvere theories have free models. For the single-sorted case, let $p : \mathsf{Par}(\mathbb{F}^{\mathrm{op}}) \to \mathcal{L}$ be a partial Lawvere theory. Indeed we can look at it as a morphism in $\mathsf{DCRC}^{\leq}$, then the previous theorem produces an adjunction $F \dashv \mathsf{Mod}_p$

$$F : \mathsf{Mod}_{\mathsf{Par}(\mathbb{F}^{\mathrm{op}})} \leftrightarrows \mathsf{Mod}_{\mathcal{L}} : \mathsf{Mod}_p.$$

The functor Mod_p coincides with the *forgetful functor*. Its left adjoint F provides free objects.

8 CONCLUSIONS AND FUTURE WORK

We introduced partial Lawvere theories and their associated notion of partial equational theory. Our definitions are guided by the appropriate universal property, replacing cartesian categories with discrete cartesian restriction categories. Knowing the right universal property determines our choice of syntax, isolating the correct class of string diagrams that replace classical terms. This enables the standard methodology of presenting a theory by means of a signature and equations, while avoiding ad-hoc notations and eliminating the subtleties of reasoning about partial structures.

The extension is conservative: every equational theory yields a partial equational theory such that the categories of models coincide, even though our models are in Par rather than in Set. The recently proposed *Frobenius theories* [Bonchi et al. 2017] take their models in the category of relations Rel, and are guided by the structure of cartesian bicategories of relations [Carboni and Walters 1987]. Every partial equational theory yields a Frobenius theory and again, the categories of models coincide. We feel that our notion is a sweet-spot. First, we have shown that our notion of partial theories is expressive, capturing a number of important examples. Second, we retain much of the richness of the semantic picture, via a canonical variety theorem and existence of free models.

There is much future work. The fact that the syntax introduced here is inherently partial makes it well-suited to applications in computing. In particular there is an evident notion of *computable model* for partial Lawvere theories, namely those models valued in the category of sets and partial recursive functions. The corresponding *computable varieties* seem to be interesting for programming language semantics, and therefore worthy of study. A further step would be the lifting of this situation to a more synthetic category of computable functions, such as a *Turing category* [Cockett and Hofstra 2008] or *monoidal computer* [Pavlovic 2013].

An important part of categorical universal algebra is played by monads, a point of view that we have not considered here. Indeed, Lawvere theories can be seen as *finitary monads* [Linton 1966], with the category of algebras giving the associated variety. This connection has been a fruitful one, relating areas of research that are, on the surface, very different, see e.g. [Cheng 2020; Loday and Vallette 2012; Markl et al. 2002]. A natural question is whether there is an analogous approach for partial algebraic theories. We conjecture that there is, with certain formal monads [Street 1972] in the 2-category $\mathsf{DCRC}^{\leq}$ playing the role of finitary monads. We expect that other constructions of categorical universal algebra (e.g. [Freyd 1966; Power 2006]) will have corresponding partial accounts.

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A.3 The Structure of Concurrent Process Histories

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The Structure of Concurrent Process Histories

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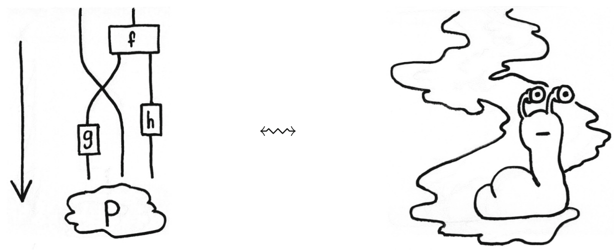
Abstract. We identify the algebraic structure of the material histories generated by concurrent processes. Specifically, we extend existing categorical theories of resource convertibility to capture concurrent interaction. Our formalism admits an intuitive graphical presentation via string diagrams for proarrow equipments.

1 Introduction

Concurrent systems are abundant in computing, and indeed in the world at large. Despite the large amount of attention paid to the modelling of concurrency in recent decades (e.g., [1,10,16–18]), a canonical mathematical account has yet to emerge, and the basic structure of concurrent systems remains elusive.

In this paper we present a basic structure that captures what we will call the *material* aspect of concurrent systems: As a process unfolds in time it leaves behind a material history of effects on the world, like the way a slug moving through space leaves a trail of slime. This slime is captured in a natural way by *resource theories* in the sense of [4], in which morphisms of symmetric monoidal categories – conveniently expressed as string diagrams – are understood as transformations of resources.



From the resource theoretic perspective, objects of a symmetric monoidal category are understood as collections of resources, with the unit object denoting the empty collection and the tensor product of two collections consisting of their combined contents. Morphisms are understood as ways to transform one collection of resources into another, which may be combined sequentially via

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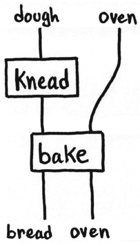
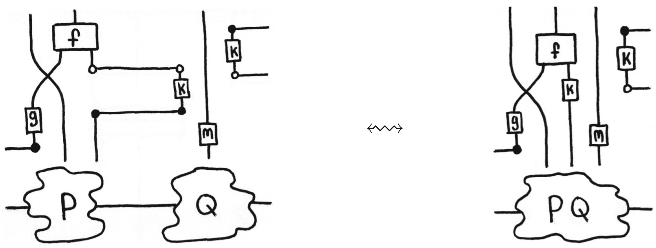
composition, and in parallel via the tensor product. For example, the process of baking bread might generate the following material history:



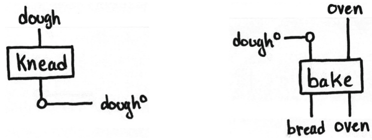
meaning that the baking process involved kneading dough and baking it in an oven to obtain bread (and also the oven).

This approach to expressing the material history of a process has many advantages: It is general, in that it assumes minimal structure; canonical, in that monoidal categories are well-studied as mathematical objects; and relatively friendly, as it admits an intuitive graphical calculus (string diagrams). However, it is unable to capture the interaction between components of a concurrent process. For example, consider our hypothetical baking process and suppose that the kneading and baking of the dough are handled by separate subsystems, with control of the dough being handed to the baking subsystem once the kneading is complete. Such interaction of parts is a fundamental aspect of concurrency, but is not expressible in this framework – we can only describe the effects of the system as a whole.

We remedy this by extending a given resource theory to allow the decomposition of material histories into concurrent components. Specifically, we augment the string diagrams for symmetric monoidal categories with *corners*, through which resources may flow between different components of a transformation.



Returning to our baking example, we might express the material history of the kneading and baking subsystems *separately* with the following diagrams, which may be composed horizontally to obtain the material history of the baking process as a whole.



These augmented diagrams denote cells of a single object double category constructed from the original resource theory. The corners make this double category into a proarrow equipment, which turns out to be all the additional structure we need in order to express concurrent interaction. From only this structure, we obtain a theory of exchanges – a sort of minimal system of behavioural types – that conforms to our intuition about how such things ought to work remarkably well.

Our approach to these concurrent material histories retains the aforementioned advantages of the resource-theoretic perspective: We lose no generality, since our construction applies to any resource theory; It is canonical, with proarrow equipments being a fundamental structure in formal category theory – although not usually seen in such concrete circumstances; Finally, it remains relatively friendly, since the string diagrams for monoidal categories extend in a natural way to string diagrams for proarrow equipments [11].

1.1 Contributions and Related Work

Related Work. Monoidal categories are ubiquitous – if often implicit – in theoretical computer science. An example from the theory of concurrency is [15], in which monoidal categories serve a purpose similar to their purpose here. String diagrams for monoidal categories seem to have been invented independently a number of times, but until recently were uncommon in printed material due to technical limitations. The usual reference is [12]. We credit the resource-theoretic interpretation of monoidal categories and their string diagrams to [4]. Double categories first appear in [6]. Free double categories are considered in [5] and again in [7]. The idea of a proarrow equipment first appears in [22], albeit in a rather different form. Proarrow equipments have subsequently appeared under many names in formal category theory (see e.g., [9,20]). String diagrams for double categories and proarrow equipments are treated precisely in [11]. We have been inspired by work on message passing and behavioural types, in particular [2], from which we have adopted our notation for exchanges.

Contributions. Our main contribution is the resource-theoretic interpretation of certain proarrow equipments, which we call *cornerings*, and the observation that they capture exactly the structure of concurrent process histories. Our mathematical contributions are minor, most significantly the identification of crossing cells in the free cornering of a resource theory and the corresponding Lemma 2, which we believe to be novel. We do not claim the other lemmas of the paper as significant mathematical contributions. Instead, they serve to flesh out the structure of the free cornering.

1.2 Organization and Prerequisites

Prerequisites. This paper is largely self-contained, but we assume some familiarity with category theory, in particular with monoidal categories and their string diagrams. Some good references are [8,14,19].

Organization. In Sect. 2 we review the resource-theoretic interpretation of symmetric monoidal categories. We continue by reviewing the theory of double categories in Sect. 3, specialized to the single object case. In Sect. 4 we introduce cornerings of a resource theory, in particular the free such cornering, and exhibit the existence of crossing cells in the free cornering. In Sect. 5 we show how the free cornering of a resource theory inherits its resource-theoretic interpretation while enabling the concurrent decomposition of resource transformations. In Sect. 6 we conclude and consider directions for future work.

2 Monoidal Categories as Resource Theories

Symmetric strict monoidal categories can be understood as theories of resource transformation. Objects are interpreted as collections of resources, with $A \otimes B$ the collection consisting of both A and B, and I the empty collection. Arrows $f : A \to B$ are understood as ways to transform the resources of A into those of B. We call symmetric strict monoidal categories *resource theories* when we have this sort of interpretation in mind.

For example, let $\mathfrak{B}$ be the free symmetric strict monoidal category with generating objects

$$\{\mathtt{bread}, \mathtt{dough}, \mathtt{water}, \mathtt{flour}, \mathtt{oven}\}$$

and with generating arrows

$$\mathtt{mix} : \mathtt{water} \otimes \mathtt{flour} \to \mathtt{dough} \qquad \mathtt{knead} : \mathtt{dough} \to \mathtt{dough}$$

$$\mathtt{bake} : \mathtt{dough} \otimes \mathtt{oven} \to \mathtt{bread} \otimes \mathtt{oven}$$

subject to no equations. $\mathfrak{B}$ can be understood as a resource theory of baking bread. The arrow `mix` represents the process of combining water and flour to form a bread dough, `knead` represents kneading dough, and `bake` represents baking dough in an oven to obtain bread (and an oven).

The structure of symmetric strict monoidal categories provides natural algebraic scaffolding for composite transformations. For example, consider the following arrow of $\mathfrak{B}$:

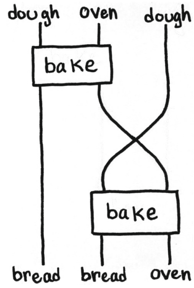
$$(\mathtt{bake} \otimes 1_{\mathtt{dough}}); (1_{\mathtt{bread}} \otimes \sigma_{\mathtt{oven},\mathtt{dough}}; \mathtt{bake})$$

of type

$$\mathtt{dough} \otimes \mathtt{oven} \otimes \mathtt{dough} \to \mathtt{bread} \otimes \mathtt{bread} \otimes \mathtt{oven}$$

where $\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ is the braiding. This arrow describes the transformation of two units of dough into loaves of bread by baking them one after the other in an oven.

It is often more intuitive to write composite arrows like this as string diagrams: Objects are depicted as wires, and arrows as boxes with inputs and outputs. Composition is represented by connecting output wires to input wires, and we represent the tensor product of two morphisms by placing them beside one another. Finally, the braiding is represented by crossing the wires involved. For the morphism discussed above, the corresponding string diagram is:

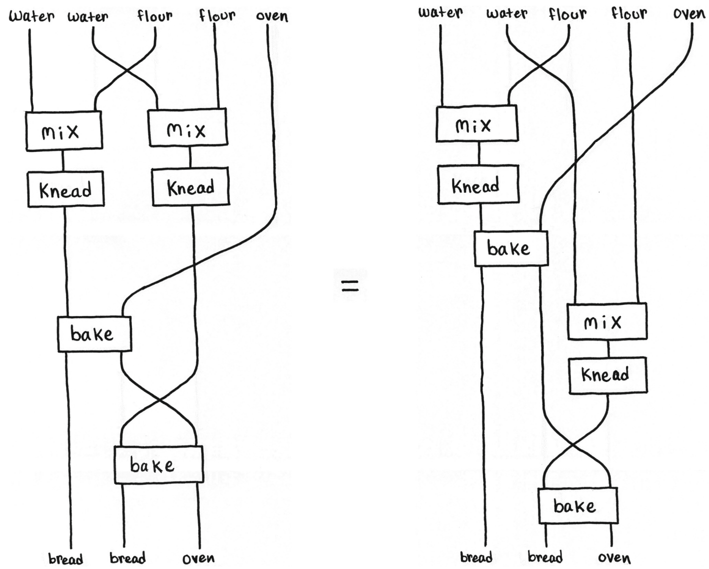


Notice how the topology of the diagram captures the logical flow of resources.

Given a pair of parallel arrows $f, g : A \to B$ in some resource theory, both f and g are ways to obtain B from A, but they may not have the same effect on the resources involved. We explain by example: Consider the parallel arrows $1_{\mathtt{dough}}, \mathtt{knead} : \mathtt{dough} \to \mathtt{dough}$ of $\mathfrak{B}$. Clearly these should not be understood to have the same effect on the dough in question, and this is reflected in $\mathfrak{B}$ by the fact that they are not made equal by its axioms. Similarly, `knead` and $\mathtt{knead} \circ \mathtt{knead}$ are not equal in $\mathfrak{B}$, which we understand to mean that kneading dough twice does not have the same effect as kneading it once, and that in turn any `bread` produced from twice-kneaded dough will be different from once-kneaded bread in our model.

Consider a hypothetical resource theory constructed from $\mathfrak{B}$ by imposing the equation $\mathtt{knead} \circ \mathtt{knead} = \mathtt{knead}$. In this new setting we understand kneading dough once to have the same effect as kneading it twice, three times, and so on, because the corresponding arrows are all equal. Of course, the sequence of events described by `knead` is not the one described by $\mathtt{knead} \circ \mathtt{knead}$: In the former the dough has been kneaded only once, while in the latter it has been kneaded twice. The equality of the two arrows indicates that these two different processes would have the same effect on the dough involved. We adopt as a general principle in our design and understanding of resource theories that transformations should be equal if and only if they have the same effect on the resources involved.

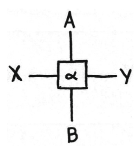
For the sake of further illustration, observe that by naturality of the braiding maps the following two resource transformations are equal in $\mathfrak{B}$:



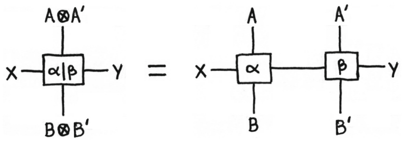
Each transformation gives a method of baking two loaves of bread. On the left, two batches of dough are mixed and kneaded before being baked one after the other. On the right, first one batch of dough is mixed, kneaded and baked and only then is the second batch mixed, kneaded, and baked. Their equality tells us that, according to $\mathfrak{B}$, the two procedures will have the same effect, resulting in the same bread when applied to the same ingredients with the same oven.

3 Single Object Double Categories

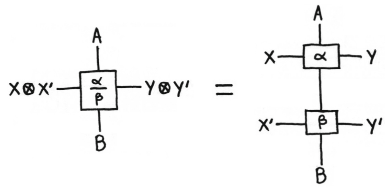
In this section we set up the rest of our development by presenting the theory of *single object double categories*, being those double categories $\mathbb{D}$ with exactly one object. In this case $\mathbb{D}$ consists of a *horizontal edge monoid* $\mathbb{D}_H = (\mathbb{D}_H, \otimes, I)$, a *vertical edge monoid* $\mathbb{D}_V = (\mathbb{D}_V, \otimes, I)$, and a collection of *cells*



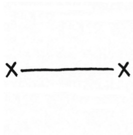
where $A, B \in \mathbb{D}_H$ and $X, Y \in \mathbb{D}_V$. Given cells α, β where the right boundary of α matches the left boundary of β we may form a cell $\alpha|\beta$ – their *horizontal composite* – and similarly if the bottom boundary of α matches the top boundary of β we may form $\frac{\alpha}{\beta}$ – their *vertical composite* – with the boundaries of the composite cell formed from those of the component cells using $\otimes$. We depict horizontal and vertical composition, respectively, as in:



and

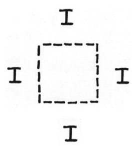


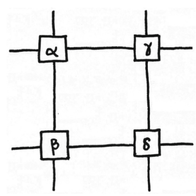
Horizontal and vertical composition of cells are required to be associative and unital. We omit wires of sort I in our depictions of cells, allowing us to draw horizontal and vertical identity cells, respectively, as in:



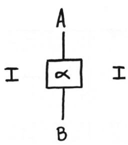
and

Finally, the horizontal and vertical identity cells of type I must coincide – we write this cell as $\square_I$ and depict it as empty space, see below on the left – and vertical and horizontal composition must satisfy the interchange law. That is, $\frac{\alpha}{\beta}|\frac{\gamma}{\delta} = \frac{\alpha|\gamma}{\beta|\delta}$, allowing us to unambiguously interpret the diagram below on the right:

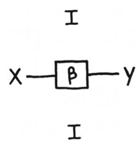




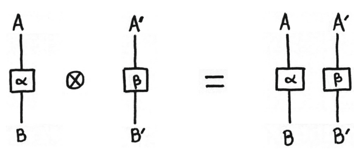
Every single object double category $\mathbb{D}$ defines strict monoidal categories $\mathbf{V}\mathbb{D}$ and $\mathbf{H}\mathbb{D}$, consisting of the cells for which the $\mathbb{D}_H$ and $\mathbb{D}_V$ valued boundaries respectively are all I, as in:



and



That is, the collection of objects of $\mathbf{V}\mathbb{D}$ is $\mathbb{D}_H$, composition in $\mathbf{V}\mathbb{D}$ is vertical composition of cells, and the tensor product in $\mathbf{V}\mathbb{D}$ is given by horizontal composition:



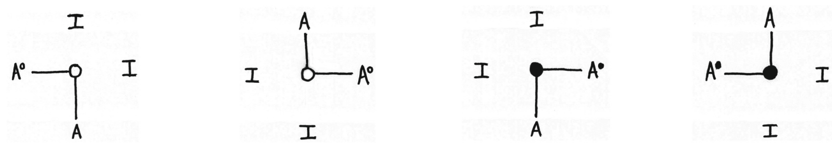
In this way, $\mathbf{V}\mathbb{D}$ forms a strict monoidal category, which we call the category of *vertical cells* of $\mathbb{D}$. Similarly, $\mathbf{H}\mathbb{D}$ is also a strict monoidal category (with collection of objects $\mathbb{D}_V$) which we call the *horizontal cells* of $\mathbb{D}$.

4 Cornerings and Crossings

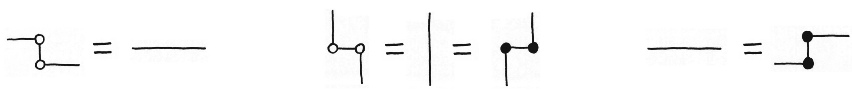
Next, we define cornerings, our primary technical device. In particular we discuss the free cornering of a resource theory, which we show contains special crossing cells with nice formal properties. Tersely, a cornering of a resource theory $\mathbb{A}$ is a single object proarrow equipment with $\mathbb{A}$ as its vertical cells. Explicitly:

Definition 1. *Let $\mathbb{A}$ be a symmetric strict monoidal category. Then a* cornering *of $\mathbb{A}$ is a single object double category $\mathbb{D}$ such that:*

(i) The vertical cells of $\mathbb{D}$ are $\mathbb{A}$. That is, there is an isomorphism of categories $\mathbf{V}\mathbb{D} \cong \mathbb{A}$.
(ii) For each A in $\mathbb{A}_0 \cong \mathbb{D}_H$, there are distinguished elements A° and $A^\bullet$ of $\mathbb{D}_V$ along with distinguished cells of $\mathbb{D}$



called $\circ$-corners *and* $\bullet$-corners *respectively, which must satisfy the* yanking equations*:*

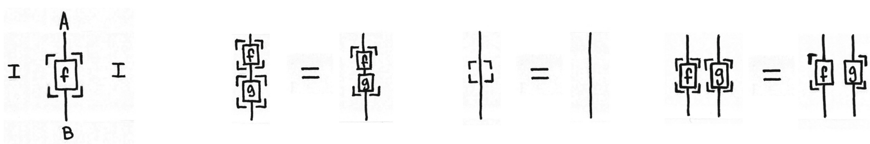


Intuitively, A° denotes an instance of A moving from left to right, and $A^\bullet$ denotes an instance of A moving from right to left (see Sect. 5).

Of particular interest is the free cornering of a resource theory:

Definition 2. *Let $\mathbb{A}$ be a resource theory. Then the* free cornering of $\mathbb{A}$, *written $\ulcorner\mathbb{A}\lrcorner$, is the free single object double category defined as follows:*

- *The horizontal edge monoid $\ulcorner\mathbb{A}\lrcorner_H = (\mathbb{A}_0, \otimes, I)$ is given by the objects of $\mathbb{A}$.*
- *The vertical edge monoid $\ulcorner\mathbb{A}\lrcorner_V = (\mathbb{A}_0 \times \{\circ, \bullet\})^*$ is the free monoid on the set $\mathbb{A}_0 \times \{\circ, \bullet\}$ of polarized objects of $\mathbb{A}$ – whose elements we write A° and $A^\bullet$.*
- *The generating cells consist of corners for each object A of $\mathbb{A}$ as above, subject to the yanking equations, along with a vertical cell $\ulcorner f \lrcorner$ for each morphism $f : A \to B$ of $\mathbb{A}$ subject to equations as in:*



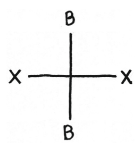
For a precise development of free double categories see [7]. In brief: cells are formed from the generating cells by horizontal and vertical composition, subject to the axioms of a double category in addition to any generating equations. The free cornering is free both in the sense that it is freely generated, and in the sense that for any cornering $\mathbb{D}$ of $\mathbb{A}$ there is exactly one double functor $\ulcorner \mathbb{A} \lrcorner \to \mathbb{D}$ that sends corner cells to corner cells and restricts to the identity on $\mathbb{A} \cong \mathbf{V}\mathbb{D}$. That is, diagrams in $\ulcorner \mathbb{A} \lrcorner$ have a canonical interpretation in any cornering of $\mathbb{A}$.

Proposition 1. *$\ulcorner \mathbb{A} \lrcorner$ is a cornering of $\mathbb{A}$.*

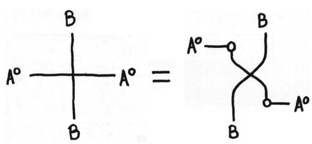
Proof. Intuitively $\mathbf{V}\ulcorner \mathbb{A} \lrcorner \cong \mathbb{A}$ because in a composite vertical cell every wire bent by a corner must eventually be un-bent by the matching corner, which by yanking is the identity. The only other generators are the cells $\ulcorner f \lrcorner$, and so any vertical cell in $\ulcorner A \lrcorner$ can be written as $\ulcorner g \lrcorner$ for some morphism g of $\mathbb{A}$. A more rigorous treatment of corner cells can be found in [11], to the same effect.

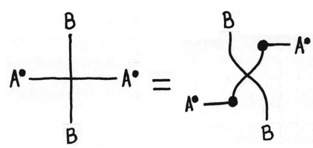
□

Before we properly explain our interest in $\ulcorner \mathbb{A} \lrcorner$ we develop a convenient bit of structure: *crossing cells*. For each B of $\ulcorner \mathbb{A} \lrcorner_H$ and each X of $\ulcorner \mathbb{A} \lrcorner_V$ we define a cell

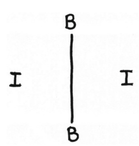


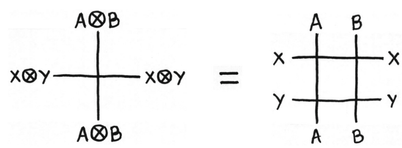
of $\ulcorner \mathbb{A} \lrcorner$ inductively as follows: In the case where X is A° or $A^\bullet$, respectively, define the crossing cell as in the diagrams below on the left and right, respectively:





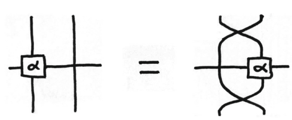
in the case where X is I, define the crossing cell as in the diagram below on the left, and in the composite case define the crossing cell as in the diagram below on the right:



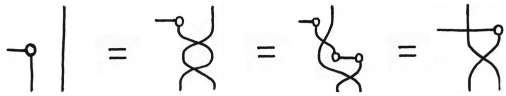


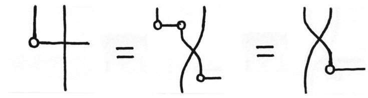
We prove a technical lemma:

Lemma 1. *For any cell α of $\ulcorner \mathbb{A} \lrcorner$ we have*

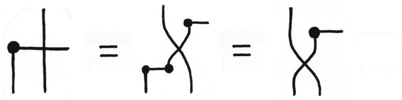


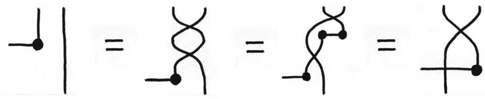
Proof. By structural induction on cells of $\ulcorner \mathbb{A} \urcorner$. For the $\circ$-corners we have:



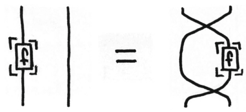


and for the $\bullet$-corners, similarly:

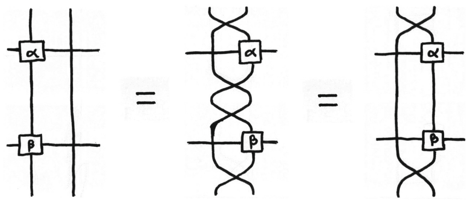




the final base cases are the $\ulcorner f \urcorner$ maps:



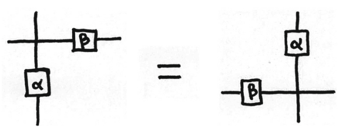
There are two inductive cases. For vertical composition, we have:



Horizontal composition is similarly straightforward, and the claim follows by induction. $\square$

From this we obtain a "non-interaction" property of our crossing cells, similar to the naturality of braiding in symmetric monoidal categories:

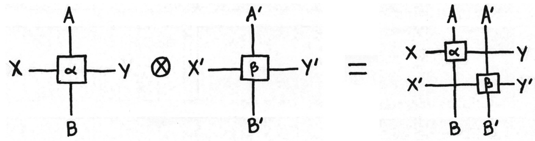
Corollary 1. *For cells α of $\mathbf{V}\ulcorner \mathbb{A} \urcorner$ and β of $\mathbf{H}\ulcorner \mathbb{A} \urcorner$, the following equation holds in $\ulcorner \mathbb{A} \urcorner$:*



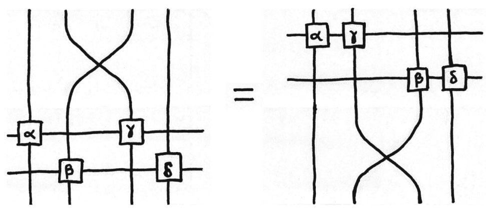
These crossing cells greatly aid in the legibility of diagrams corresponding to cells in $\ulcorner \mathbb{A} \urcorner$, but also tell us something about the categorical structure of $\ulcorner \mathbb{A} \urcorner$, namely that it is a monoidal double category in the sense of [21]:

Lemma 2. *If $\mathbb{A}$ is a symmetric strict monoidal category then $\ulcorner \mathbb{A} \urcorner$ is a monoidal double category. That is, $\ulcorner \mathbb{A} \urcorner$ is a pseudo-monoid object in the strict 2-category* $\mathsf{VDblCat}$ *of double categories, lax double functors, and vertical transformations.*

Proof. We give the action of the tensor product on cells:



This defines a pseudofunctor, with the component of the required vertical transformation given by exchanging the two middle wires as in:

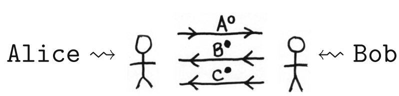


Notice that $\otimes$ is strictly associative and unital, in spite of being only pseudofunctorial.

□

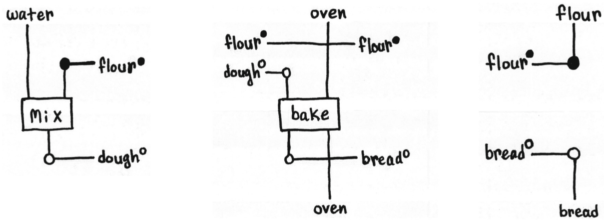
5 Concurrency Through Cornering

We next proceed to extend the resource-theoretic interpretation of some symmetric strict monoidal category $\mathbb{A}$ to its free cornering $\ulcorner \mathbb{A} \lrcorner$. Interpret elements of $\ulcorner \mathbb{A} \lrcorner_V$ as $\mathbb{A}$-*valued exchanges*. Each exchange $X_1 \otimes \cdots \otimes X_n$ involves a left participant and a right participant giving each other resources in sequence, with A° indicating that the left participant should give the right participant an instance of A, and $A^\bullet$ indicating the opposite. For example say the left participant is `Alice` and the right participant is `Bob`. Then we can picture the exchange $A^\circ \otimes B^\bullet \otimes C^\bullet$ as:

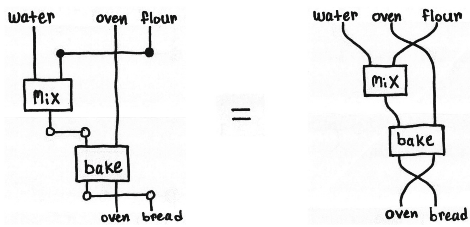


Think of these exchanges as happening *in order*. For example the exchange pictured above demands that first `Alice` gives `Bob` an instance of A, then `Bob` gives `Alice` an instance of B, and then finally `Bob` gives `Alice` an instance of C.

We interpret cells of $\ulcorner \mathbb{A} \lrcorner$ as *concurrent transformations*. Each cell describes a way to transform the collection of resources given by the top boundary into that given by the bottom boundary, via participating in $\mathbb{A}$-valued exchanges along the left and right boundaries. For example, consider the following cells of $\ulcorner \mathfrak{B} \lrcorner$:



From left to right, these describe: A procedure for transforming `water` into nothing by `mix`ing it with `flour` obtained by exchange along the right boundary, then sending the resulting `dough` away along the right boundary; A procedure for transforming an `oven` into an `oven`, receiving `flour` along the right boundary and sending it out the left boundary, then receiving `dough` along the left boundary, which is `bake`d in the `oven`, with the resulting `bread` sent out along the right boundary; Finally, a procedure for turning `flour` into `bread` by giving it away and then receiving `bread` along the left boundary. When we compose these concurrent transformations horizontally in the evident way, they give a transformation of resources in the usual sense, i.e., a morphism of $\mathbb{A} \cong \mathbf{V}\ulcorner\mathbb{A}\urcorner$:



We understand equality of cells in $\ulcorner\mathbb{A}\urcorner$ much as we understand equality of morphisms in a resource theory: two cells should be equal in case the transformations they describe would have the same effect on the resources involved. In this way, cells of $\ulcorner\mathbb{A}\urcorner$ allow us to break a transformation into many concurrent parts. Note that with the crossing cells, it is possible to exchange resources "across" cells.

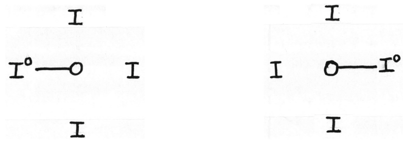
Consider the category $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ of horizontal cells. If the vertical cells $\mathbf{V}\ulcorner\mathbb{A}\urcorner$ are concerned entirely with the transformation of resources, then our interpretation tells us that the horizontal cells are concerned entirely with exchange. Just as isomorphic objects in $\mathbf{V}\ulcorner\mathbb{A}\urcorner \cong \mathbb{A}$ can be thought of as equivalent collections of resources – being freely transformable into each other – we understand isomorphic objects in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ as *equivalent exchanges.* For example, There are many ways for `Alice` to give `Bob` an A and a B: Simultaneously, as $A \otimes B$; one after the other, as A and then B; or in the other order, as B and then A. While these are different sequences of events, they achieve the same thing, and are thus equivalent. Similarly, for `Alice` to give `Bob` an instance of I is equivalent to nobody doing anything. Formally, we have:

Lemma 3. *In $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ we have for any A, B of $\mathbb{A}$:*

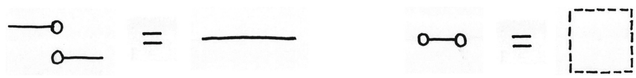
(i) $I^\circ \cong I \cong I^\bullet$.

(ii) $A^\circ \otimes B^\circ \cong B^\circ \otimes A^\circ$ *and* $A^\bullet \otimes B^\bullet \cong B^\bullet \otimes A^\bullet$.
(iii) $(A \otimes B)^\circ \cong A^\circ \otimes B^\circ$ *and* $(A \otimes B)^\bullet \cong A^\bullet \otimes B^\bullet$

Proof. (i) For $I \cong I^\circ$, consider the $\circ$-corners corresponding to I:

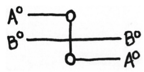


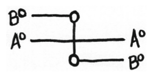
we know that these satisfy the yanking equations:



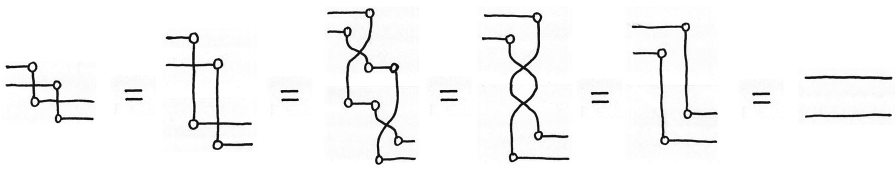
which exhibits an isomorphism $I \cong I^\circ$. Similarly, $I \cong I^\bullet$. Thus, we see formally that exchanging nothing is the same as doing nothing.

(ii) The $\circ$-corner case is the interesting one: Define the components of our isomorphism to be:



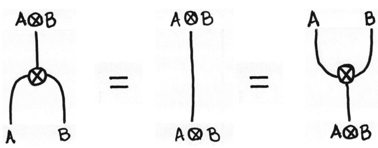
and


then for both of the required composites we have:

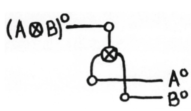


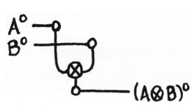
and so $A^\circ \otimes B^\circ \cong B^\circ \otimes A^\circ$. Similarly $A^\bullet \otimes B^\bullet \cong B^\bullet \otimes A^\bullet$. This captures formally the fact that if `Alice` is going to give `Bob` an A and a B, it doesn't really matter which order she does it in.

(iii) Here it is convenient to switch between depicting a single wire of sort $A \otimes B$ and two wires of sort A and B respectively in our string diagrams. To this end, we allow ourselves to depict the identity on $A \otimes B$ in multiple ways, using the notation of [3]:



Then the components of our isomorphism $(A \otimes B)^\circ \cong A^\circ \otimes B^\circ$ are:

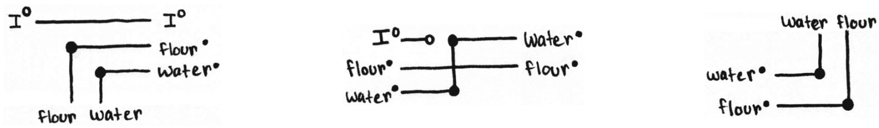


and


and, much as in (ii), it is easy to see that the two possible composites are both identity maps. Similarly, $(A \otimes B)^\bullet \cong (A^\bullet \otimes B^\bullet)$. This captures formally the fact that giving away a collection is the same thing as giving away its components.

□

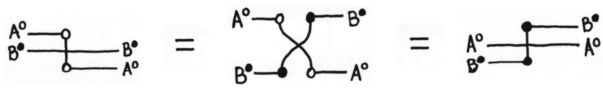
For example, we should be able to compose the cells on the left and right below horizontally, since their right and left boundaries, respectively, indicate equivalent exchanges:



Our lemma tells us that there will always be a canonical isomorphism, as above in the middle, making composition possible.

It is worth noting that we *do not* have $A^\circ \otimes B^\bullet \cong B^\bullet \otimes A^\circ$:

Observation 1. *There is a morphism* $d^\circ_\bullet : A^\circ \otimes B^\bullet \to B^\bullet \otimes A^\circ$ *in one direction, defined by*



but there is need not be a morphism in the other direction, and this is not in general invertible. In particular, $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ *is monoidal, but need not be symmetric.*

This observation reflects formally the intuition that if I receive some resources before I am required to send any, then I can send some of the resources that I receive. However, if I must send the resources first, this is not the case. In this way, $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ contains a sort of causal structure.

6 Conclusions and Future Work

We have shown how to decompose the material history of a process into concurrent components by working in the free cornering of an appropriate resource theory. We have explored the structure of the free cornering in light of this interpretation and found that it is consistent with our intuition about how this sort of thing ought to work. We do not claim to have solved all problems in the modelling of concurrency, but we feel that our formalism captures the material aspect of concurrent systems very well.

We find it quite surprising that the structure required to model concurrent resource transformations is precisely the structure of a proarrow equipment. This structure is already known to be important in formal category theory, and we are appropriately intrigued by its apparent relevance to models of concurrency

– a far more concrete setting than the usual context in which one encounters proarrow equipments!

There are of course many directions for future work. For one, our work is inspired by the message passing logic of [2], which has its categorical semantics in *linear actegories*. Any cornering defines an category – although not quite a *linear* actegory – and we speculate that cornerings are equivalent to some class of actegories, which would connect our work to the literature on behavioural types. Another direction for future work is to connect our material histories to a theory of concurrent processes – the slugs to our slime – with the goal of a formalism accounting for both. The category of spans of reflexive graphs, interpreted as open transition systems, seems especially promising here [13]. More generally, we would like to know how the perspective presented here can be integrated into other approaches to modelling concurrent systems.

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A.4 Situated Transition Systems

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Situated Transition Systems

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We construct a monoidal category of open transition systems that generate material history as transitions unfold, which we call situated transition systems. The material history generated by a composite system is composed of the material history generated by each component. The construction is parameterized by a symmetric strict monoidal category, understood as a resource theory, from which material histories are drawn. We pay special attention to the case in which this category is compact closed. In particular, if we begin with a compact closed category of integers then the resulting situated transition systems can be understood as systems of double-entry bookkeeping accounts.

1 Introduction

Graphs have been used to model the states and state changes (transitions) of systems for hundreds of years [7]. Today, graphs can be found everywhere in the scientific literature, and entire fields of study are concerned with specific kinds of graph models. In common practice, to model something as a graph is to treat is as a *closed system* — that is, the surrounding context is ignored by the model. The closed nature of these models is a failure of compositionality: it prevents us from explaining large systems as the combination of smaller components. This sort of compositionality is all but required if our modelling techniques are to apply to the complex systems we encounter in the world.

A promising compositional approach is the algebra of transition systems with boundary given by the category $\mathsf{Span}(\mathsf{RGraph})$ of spans of reflexive graphs [14]. In this formalism, each transition manifests as an event at the boundaries of a system, and composing systems along a common boundary constrains their behaviour to be consistent with the events observed there. This allows us to consider graph models of *open systems*, and to use these as components in the construction of a larger whole. For example, the authors of [9] have constructed a simplified model of the heart system in the $\mathsf{Span}(\mathsf{RGraph})$ setting.

In an unpublished and — it seems — largely unknown paper [15], the category $\mathsf{Span}(\mathsf{RGraph})$ is modified to give a category of systems of partita-doppia (double-entry bookkeeping) accounts. These systems have an account balance, which may change as the result of vaule entering or leaving the system during a transition. The resulting category $\mathsf{Accounts}$ allows us to model a system of partita-doppia accounts in context, as one part of a notional system of all accounts. This is more exciting than may be immediately apparent. From [15]:

> ”The aim of accounting is the measurement of a distributed concurrent system, and it is our contention that it is one of the earliest and most successful mathematical theories of concurrency.”

The present work arose from a desire to generalize the category $\mathsf{Accounts}$. In a sense, models in $\mathsf{Span}(\mathsf{RGraph})$ (indeed, graph models more generally) are detached from any sort of material reality. The states and state transitions are specified, but the material effect of a given sequence of transitions is left informal, specified as vague intuition. In the category of $\mathsf{Accounts}$, transitions come equipped with a

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material effect on the partita-doppia ledger associated with that system. The abstract, conceptual world of graphs is thus *situated* in the world of accouting.

Our point of departure is to replace the theory of partita-doppia ledgers with an arbitrary *resource theory* (symmetric strict monoidal category) in the sense of [3]. Augmenting our resource theories with *corners* [19] allows us to assign material history to a transition in a compositional way: material history generated by a composite transition system is the composite of material history generated by its components. We call the resulting notion a *situated transition system*, and we show that for any resource theory $\mathbb{A}$ the $\mathbb{A}$-situated transition systems form a monoidal category.

We show that our formalism specializes to capture its inspiration: if we begin with a compact closed category $\mathbb{Z}$ of integers, the category of $\mathbb{Z}$-situated transition systems is a category of systems of partita doppia accounts in the sense of [15]. Further, we show that for any compact closed category $\mathbb{A}$, the catgory of $\mathbb{A}$-situated transition systems is also compact closed. This generalizes the main theorem of [15], which is that $\mathsf{Accounts}$ is a compact closed category.

1.1 Contributions and Related Work

Related Work. We credit the resource-theoretic interpretation of monoidal categories and their string diagrams to [3]. String diagrams for monoidal categories are dealt with rigorously in [12]. The use of "corners" in single-object double categories to allow the concurrent decomposition of resource transformations is due to [19]. Double categories first appear in [5]. Free double categories are considered in [4] and again in [8]. The corner structure we use is in fact the structure of a proarrow equipment. The idea of a proarrow equipment first appears in [22], albeit in a rather different form. Proarrow equipments have subsequently appeared under many names in formal category theory [20, 10]. String diagrams for double categories and proarrow equipments are treated precisely in [17]. The original work on the category of spans of reflexive graphs as a setting for modelling concurrent systems is [14]. Our work is directly inspired by earlier efforts to eqiup such models with accounting information [15]. An excellent mathematical exposition of double-entry bookkeeping is [6]. Compact closed categories were introduced in [13], along with the compact closed category $\mathbb{Z}$ of integers. More on compact closed categories, and specifically on compact closed categories of integers, can be found in [1].

Contributions. The main contribution of this paper is the notion of situated transition system, accompanied by the construction of the monoidal category $\mathsf{S}(\mathbb{A})$ of situated transition systems over an arbitrary monoidal category $\mathbb{A}$ (Propositions 1, 2). Other contributions are our investigation into the effect of compact closed structure in $\mathbb{A}$ on $\mathsf{S}(\mathbb{A})$ (Lemmas 1, 2, 3), and the observation that $\mathsf{S}(\mathbb{Z})$ captures the systems of partita-doppia accounts of [15] (Corollary 1). To our knowledge the compact closed perspective on double-entry bookkeeping is also novel, and so may be viewed as a modest contribution.

2 Preliminaries

2.1 Monoidal Categories as Resource Theories

Symmetric strict monoidal categories can be understood as theories of resource transformation [3]. Objects are interpreted as collections of resources, with $A \otimes B$ the collection consisting of both A and B, and I the empty collection. Arrows $f : A \to B$ are understood as ways to transform the resources of A into those of B, or equivalently as parts of a larger *material history* involving those resources. We call symmetric strict monoidal categories *resource theories* when we have this sort of interpretation in mind.

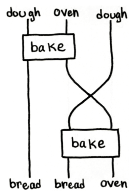
For example, let $\mathfrak{B}$ be the free symmetric strict monoidal category generated by:

$$\{\mathsf{bread}, \mathsf{dough}, \mathsf{flour}, \mathsf{oven}\}$$

$$\mathsf{knead} : \mathsf{flour} \to \mathsf{dough} \qquad \mathsf{bake} : \mathsf{dough} \otimes \mathsf{oven} \to \mathsf{bread} \otimes \mathsf{oven} \qquad \mathsf{eat} : \mathsf{bread} \to I$$

subject to no equations. $\mathfrak{B}$ can be understood as a resource theory of bread. The arrow knead represents the process of making dough from flour, bake represents baking dough in an oven to obtain bread (and an oven), and eat represents the consumption of bread.

The structure of symmetric strict monoidal categories provides natural algebraic scaffolding for composite transformations, with the associated string diagrams acting as a convenient syntax for expressing material histories. For example in the following string diagram over $\mathfrak{B}$ we see two units of dough made into loaves of bread by baking one after the other in an oven.

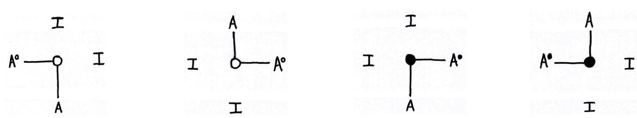


Notice how the topology of the diagram captures the logical flow of resources.

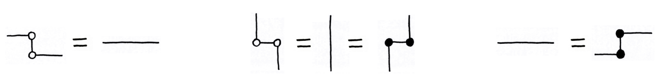
Given a parallel pair $f, g : A \to B$ of material histories in some resource theory $\mathbb{A}$, we understand equality of f and g to mean that both have the same effect on the resources involved. For example, suppose we add a generating morphism $\mathsf{sift} : \mathsf{flour} \to \mathsf{flour}$ to our resource theory $\mathfrak{B}$, subject to the equation $\mathsf{sift} \circ \mathsf{sift} = \mathsf{sift}$. Call the resulting resource theory $\mathfrak{B}_{\mathsf{sift}}$. In this new theory the material histories sift and $\mathsf{sift} \circ \mathsf{sift}$ express different sequences of events, with the flour being sifted once in the former, but twice in the latter. They are made equal by our new equation, which means that in $\mathfrak{B}_{\mathsf{sift}}$, sifting flour twice has the same effect as sifting it once. Contrast this to 1_{flour} and $\mathsf{sift} : \mathsf{flour} \to \mathsf{flour}$. Identity morphisms have no effect on the resources involved, so intuitively these two material histories should not denote equal morphisms of $\mathfrak{B}_{\mathsf{sift}}$, and indeed they do not. We adopt this understanding of equality as a general principle in our design and understanding of resource theories.

2.2 Cornering and Concurrent Transformations

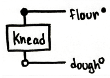
The resource theoretic interpretation of symmetric strict monoidal categories can be extended to allow the decomposition of material histories into their concurrent components [19]. Specifically, we augment the string diagrams for a given resource theory $\mathbb{A}$ with *corners* for each object A of $\mathbb{A}$:

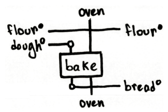


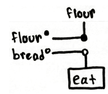
Corners allow us to express resources flowing into and out of a system. A° denotes an instance of A flowing from left to right, and $A^\bullet$ denotes an instance of A flowing from right to left. Our corners must satisfy the *yanking identities*, which ensure that this movement has no effect on the resources themselves:



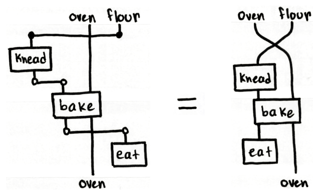
For example, adding corners to our resource theory $\mathfrak{B}$ allows the following decomposition of the baking process. The transformation below on the left begins with no resources, then flour enters along the right boundary and is kneaded into dough, which leaves along the right boundary. The transformation below in the middle begins with an oven, then flour passes through from right to left, dough is received along the left boundary and is baked, and the resulting bread leaves along the right boundary, with the oven staying put. Finally, the transformation below on the right begins with flour, which leaves the system along the left boundary, after which bread enters from the left, and is eaten.



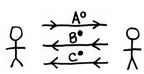




These transformations may be composed horizontally to obtain a single transformation of resources:

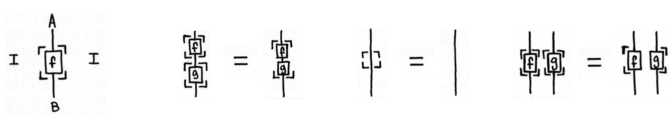


Formally, these augmented string diagrams denote cells of a single-object double category $\ulcorner\mathbb{A}\urcorner$ which we call the *free cornering of* $\mathbb{A}$. This double category has one object, so in particular the horizontal and vertical edge categories are necessarily monoids (single-object categories). The horizontal edge monoid $(\mathbb{A}_0, \otimes, I)$ is given by the monoidal structure on the objects of $\mathbb{A}$. The vertical edge monoid $\mathbb{A}^{\circ\bullet} = (\mathbb{A}_0 \times \{\circ, \bullet\})^*$ is the free monoid of polarized objects of $\mathbb{A}$, written as in A° and $A^\bullet$. Elements of $\mathbb{A}^{\circ\bullet}$ are sequences of polarized objects of $\mathbb{A}$, which we understand as $\mathbb{A}$-*valued exchanges*. The monoid operation is given by concatenation (denoted by $\otimes$) and the empty sequence (denoted by I) is the unit of the monoid. Each exchange $X_1 \otimes \cdots \otimes X_n \in \mathbb{A}^{\circ\bullet}$ involves a left participant and a right participant giving each other resources in sequence, with A° indicating that the left participant should give the right participant an instance of A, and $A^\bullet$ indicating that the right participant should give the left participant an instance of A. For example if `Alice` is the left participant and `Bob` is the right participant, then we can picture the exchange $A^\circ \otimes B^\bullet \otimes C^\bullet \in \mathbb{A}^{\circ\bullet}$ as

`Alice` $\rightsquigarrow$  $\leftsquigarrow$ `Bob`

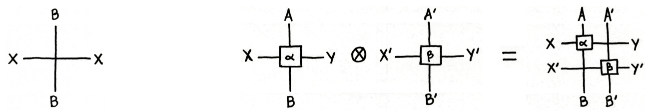
These exchanges happen in order. The exchange pictured above demands that first `Alice` gives `Bob` an instance of A, then `Bob` gives `Alice` an instance of B, and then finally `Bob` gives `Alice` an instance of C.

The generating cells of $\ulcorner\mathbb{A}\urcorner$ are the corners discussed above, subject to the yanking equations, together with cells $\ulcorner f \urcorner$ for each arrow $f : A \to B$ of $\mathbb{A}$, subject to the following equations:



Now $\ulcorner\mathbb{A}\urcorner$ is the free double category generated by this data, with arbitrary cells of $\ulcorner\mathbb{A}\urcorner$ being obtained by vertical and horizontal composition of the generators, subject to the equations of a double category (see [8, 4] for more on free double categories).

The double category $\ulcorner\mathbb{A}\urcorner$ is more thoroughly investigated in [19]. For our purposes we need only mention that $\ulcorner\mathbb{A}\urcorner$ always contains *crossing cells*, pictured below on the left for an arbitrary $B \in \mathbb{A}_0$ and $X \in \mathbb{A}^{\circ\bullet}$. These crossing cells make $\ulcorner\mathbb{A}\urcorner$ into a monoidal double category in the sense of [21], with the tensor product of cells given given below on the right.



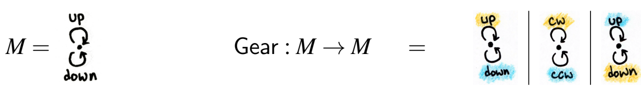
This is all the resource-theoretic machinery we will need to give a compositional account of the material histories generated by our transition systems. We turn now to the transition systems themselves.

2.3 The Algebra of Transition Systems with Boundary

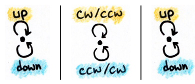
For our purposes a *transition system R* consists of a collection of *states*, R_0, and a collection of *transitions* $t : A \to B \in R_1$ where $A, B \in R_0$. We ask further that for each $A \in R_0$ there is a *trivial transition* $\varepsilon_A : A \to A \in R_1$. In other words, a transition system is precisely a *reflexive graph* (states are vertices, transitions are edges). A *morphism* $F : R \to S$ of transition systems is a morphism of reflexive graphs: It consists of a mapping of vertices $F_0 : R_0 \to S_0$ together with a mapping of edges $F_1 : R_1 \to S_1$ and must preserve the source and target of edges in the sense that if $t : A \to B$ then $F_1(t) : F_0(A) \to F_0(B)$. Further, it must preserve the trivial edges in the sense that $F_1(\varepsilon_A) = \varepsilon_{F_0(A)}$. Reflexive graphs and reflexive graph morphisms form a cartesian category RGraph, which will play a supporting role in our development.

The algebra of transition systems with boundary is captured by the category $\mathsf{Span}(\mathsf{RGraph})$ of spans in RGraph [14]. If U and V are reflexive graphs, then a morphism $R : U \to V$ of $\mathsf{Span}(\mathsf{RGraph})$ consists of another reflexive graph R (the *apex*) with morphisms $\delta_0 : R \to U$ and $\delta_1 : R \to V$ of RGraph (the *legs*). We understand this as a transition system R with *boundaries* U and V. Every transition $t : A \to B$ of R corresponds to an *event* at each boundary — $\delta_0(t)$ at U and $\delta_1(t)$ at V. Span composition is given by pullback: If $R : U \to V$ and $S : V \to W$ in $\mathsf{Span}(\mathsf{RGraph})$, a transition of $S \circ R : U \to W$ consists of a pair of transitions $(t, t') \in R_1 \times S_1$ which correspond to the same event $\delta_1(t) = \delta_0(t')$ at the shared boundary V. In the composite each of the components constrains the behaviour of the other. We consider spans modulo the equivalence relation generated by span isomorphism.

For example, let M be the reflexive graph with a single vertex and two nontrivial edges up and down, pictured below on the left. The diagram below on the right indicates a morphism $\mathsf{Gear} : M \to M$ of $\mathsf{Span}(\mathsf{RGraph})$. The apex has a single vertex and two nontrivial edges cw and ccw, and the legs of the span are indicated by the colouring. The idea is that our gear can rotate clockwise (cw), in which case the teeth along the left and right boundary move up and down respectively, or may rotate counterclockwise (ccw), with the boundary teeth moving in the opposite directions. We omit the trivial edges from our diagrams but nonetheless consider them to be present, so our gear system can also do nothing via ε.



Now the composite system $\mathsf{Gear} \circ \mathsf{Gear}$ represents two interlocking gears. The teeth interlock at the shared boundary, where they must move in unison. Our notion of composition captures this formally: the apex of our composite span has a single vertex and two nontrivial edges, one in which the gear on the left rotates clockwise and the gear on the right rotates counterclockwise, and one representing the opposite situation. The case where both gears rotate in the same direction is not present as it would be inconsistent along the shared boundary. In fact $\mathsf{Gear} \circ \mathsf{Gear} = 1_M$, reflecting a similar property of physical gears.

$\mathsf{Gear} \circ \mathsf{Gear} : M \to M \quad =$ 

$\mathsf{Span(RGraph)}$ is a symmetric monoidal category. The tensor product is defined on objects by $U \otimes V = U \times V$, and the unit 1 is the graph with a single vertex and no nontrivial edges. On arrows $R : U \to V$ and $S : U' \to V'$ the tensor product $R \otimes S : U \otimes U' \to V \otimes V'$ has apex $R \times S$ with left and right leg given by the product of the left and right legs of R and S, respectively. A transition in the tensor product of two systems is simply a transition from each component. Intuitively, the components function independently of each other. Further, notice that the component systems may function asynchronously via the ε transitions: If $t \in R_1$ and $t' \in S_1$ then $(t,t'), (t,\varepsilon), (\varepsilon,t')$, and $(\varepsilon,\varepsilon)$ are all transitions of $R \otimes S$.

There is also a lot of other structure in $\mathsf{Span(RGraph)}$. Relevant to our purposes here is the fact that $\mathsf{Span(RGraph)}$ is compact closed. The dual of X is given by X itself, and the unit and counit are defined in terms of the finite product structure on RGraph: $\eta_X : 1 \to X \otimes X$ is given by the span with apex X, left leg $!_X : X \to 1$, and right leg $\Delta_X : X \to X \times X$, with $\varepsilon_X : X \otimes X \to 1$ constructed similarly.

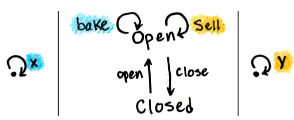
We conclude our discussion of $\mathsf{Span(RGraph)}$ with a bread-themed example. Define objects U, V of $\mathsf{Span(RGraph)}$ as follows — again omitting the trivial edges from our diagrams:

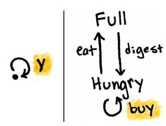
$U =$ 

$V =$ 

We understand the event $x \in U_1$ to indicate that the system on the right is obtaining ingredients for baking from the system on the left, and the $y \in V_1$ indicates that the system on the left is selling bread to the system on the right.

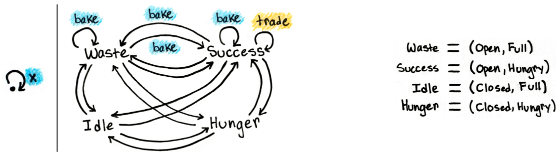
Let Baker be the morphism of $\mathsf{Span(RGraph)}$ pictured below on the left. The apex has two vertices, one in which the system is open for business, and another in which it is closed. There are edges allowing the system to transition from being open to being closed, and vice versa. When it is open, the system may bake and sell bread. The legs of the span are indicated by the colouring: The bake transition corresponds to the event x at the left boundary, and the transition sell corresponds to the event y at the right boundary. An absence of colour indicates the trivial event ε, so for example the transition open corresponds to the trivial event at both boundaries, and bake corresponds to the trivial event at the right boundary.

$\mathsf{Baker} : U \to V \;=$ 

$\mathsf{Eater} : V \to 1 \;=$ 

Let Eater be the morphism of $\mathsf{Span(RGraph)}$ pictured above on the right. The apex has two vertices, one in which the system is hungry, and another in which it is full. If hungry, the system may eat to become full, and if full may digest to become hungry. Finally, when it is hungry the system may buy food. The legs are again indicated by the colouring, with the right leg omitted entirely since in this case there is nothing to indicate. The transition buy corresponds to event y at the left boundary, and that is all.

Now, composing our two systems along their shared boundary V yields:

$\mathsf{Eater} \circ \mathsf{Baker} : U \to 1 \;=\;$ 

The unlabelled transitions arise from combinations of open, close, eat, and digest — those transitions corresponding to the trivial event at the boundaries. The bake transitions are those in which the Baker system bakes, and the trade transition corresponds to the Baker subsystem selling bread and the Eater subsystem buying it — activities which must be synchronised in the composite system. The legs of the span are indicated by the colouring, and we see that every bake transition involves the event x along the left boundary. The transition trade is coloured yellow to draw attention to the fact that it is the coincidence of the two yellow transitions in the component systems, and it has trivial boundary events.

3 Situated Transition Systems

Given a resource theory $\mathbb{A}$, in this section we show how transition systems with boundary can be equipped to generate $\mathbb{A}$-valued material histories as transitions occur. The double category $\ulcorner\mathbb{A}\lrcorner$ of concurrent transformations plays an essential role, allowing us to combine the histories generated by component spans into the history generated by their composite through horizontal composition in $\ulcorner\mathbb{A}\lrcorner$.

We begin by situating the boundaries of our transition systems. In $\mathsf{Span}(\mathsf{RGraph})$ the possible events (edges) along a boundary (reflexive graph) serve to synchronise and constrain the behaviour of the larger system. From the material point of view, the relevant part of a boundary event is whether or not any resources leave or enter the system, and if so which ones. This information is captured by the monoid $\mathbb{A}^{\circ\bullet}$ of $\mathbb{A}$-valued exchanges, which is equivalently a reflexive graph with a single vertex where the unit I of the monoid is the trivial edge.

Definition 1. Let $\mathbb{A}$ be a resource theory. Then an $\mathbb{A}$-*situated boundary* (U, ϕ_U) consists of a reflexive graph U together with a reflexive graph homomorphism $\phi_U : U \to \mathbb{A}^{\circ\bullet}$. Call ϕ_U the *situation of U in $\mathbb{A}$*.

We understand $\phi_U(x)$ to describe the resources that cross the boundary as part of the event x, and thus constitute its material effect. We will depict $\mathbb{A}$-situated boundaries as graphs with edge labels drawn from $\mathbb{A}^{\circ\bullet}$, defining the situation of the boundary in $\mathbb{A}$. Since $\mathbb{A}^{\circ\bullet}$ has only one vertex, we do not need to label the vertices. Edges with no label are understood as having label I, and we continue to omit the trivial edges from our depictions. For $X \in \mathbb{A}^{\circ\bullet}$ we adopt the convention of writing X for the $\mathbb{A}$-situated boundary with a single vertex and a single nontrivial edge, which is mapped to X by the situation. For example the $\mathfrak{B}$-situated boundary flour° is depicted below on the left. The boundary with two vertices and two nontrivial edges — one from each vertex to the other — which are both mapped to I by the situation is depicted below on the right.

Now to situate entire transition systems we associate each transition with a cell of $\ulcorner\mathbb{A}\lrcorner$ describing the corresponding material effect. The left and right boundaries of this cell must match the labels in $\mathbb{A}^{\circ\bullet}$ of the left and right boundary events, respectively, so that any material exchanges entailed by those events are present in the material history of the transition. In order to make this precise we view $\ulcorner\mathbb{A}\lrcorner$ as a span of reflexive graphs. Specifically, define $\langle \mathbb{A} \rangle$ to be the reflexive graph with vertex set $\mathbb{A}_0$ in which an edge $\alpha : A \to B$ is a cell α of $\ulcorner \mathbb{A} \urcorner$ with top boundary A and bottom boundary B. Then there is a span

$$\mathbb{A}^{\circ\bullet} \xleftarrow{\delta_0} \langle \mathbb{A} \rangle \xrightarrow{\delta_1} \mathbb{A}^{\circ\bullet}$$

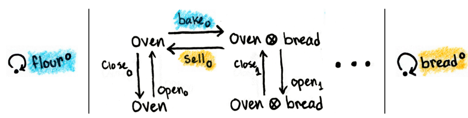
where $\delta_0(\alpha)$ and $\delta_1(\alpha)$ are the left and right boundary of α, respectively. The trivial edges of $\langle \mathbb{A} \rangle$ are given by the vertical identity cells. Situated transition systems are now defined as follows.

Definition 2. Let $\mathbb{A}$ be a resource theory, and let (U, ϕ_U) and (V, ϕ_V) be $\mathbb{A}$-situated boundaries. Then an $\mathbb{A}$-*situated transition system* $(R, \phi_R) : (U, \phi_U) \to (V, \phi_V)$ consists of a morphism $U \leftarrow R \to V$ of $\mathsf{Span}(\mathsf{RGraph})$ together with a reflexive graph homomorphism $\phi_R : R \to \langle \mathbb{A} \rangle$ that we call the *situation of* R *in* $\mathbb{A}$. We require ϕ_R to be *coherent* with respect to ϕ_U and ϕ_V in the sense that the following diagram of reflexive graph homomorphisms commutes:

$$\begin{tikzcd}
U \arrow[d, "\phi_U"'] & R \arrow[l] \arrow[r] \arrow[d, "\phi_R"] & V \arrow[d, "\phi_V"] \\
\mathbb{A}^{\circ\bullet} & \langle \mathbb{A} \rangle \arrow[l, "\delta_0"] \arrow[r, "\delta_1"'] & \mathbb{A}^{\circ\bullet}
\end{tikzcd}$$

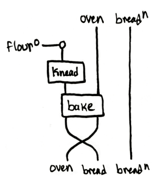
We understand ϕ_R as assigning a collection of resources to each state of R, and assigning to each transition of R a concurrent transformation of resources whose left and right boundary coincide with the material effect of the left and right boundary events. We depict situated transition systems by giving the underlying span of reflexive graphs as before, with the legs indicated by the colouring. We indicate the action of ϕ_R by labelling the vertices (resp. edges) of the apex with the object of $\mathbb{A}$ (resp. cell of $\ulcorner \mathbb{A} \urcorner$) that ϕ_R maps them to. For example we can refine our earlier bread-themed example to be $\mathfrak{B}$-situated, with the new Baker system given by:

$$\mathsf{Baker} : \mathsf{flour}^\circ \to \mathsf{bread}^\circ \quad =$$

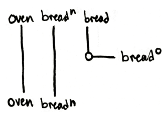


where the edge labels are the following cells of $\ulcorner \mathfrak{B} \urcorner$:

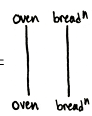
$\mathsf{bake}_n =$



$\mathsf{sell}_n =$

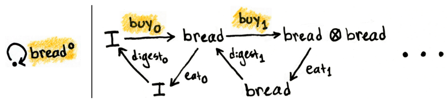


$\mathsf{close}_n = \mathsf{open}_n =$

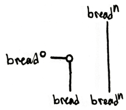


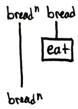
The left boundary is given by the graph with a single vertex and one nontrivial edge, which is mapped to flour° by the situation, indicating that flour enters the system as part of that event. The right boundary is similar, with the single nontrivial edge mapped to bread° by the situation, indicating that bread leaves the system. The apex has two vertices for each $n \in \mathbb{N}$ which indicate whether the system is open for business or not, and that it currently has n units of bread in stock. The two states in which the sytem has n units of bread are mapped to $\mathsf{oven} \otimes \mathsf{bread}^n$ by the situation. The edges are similarly indexed: the system may open and close while retaining its stores of bread via open_n and close_n. When open the system may bake bread via bake_n, in which case we see that flour enters the system from the left, and may also sell any bread it has via sell_n, in which case bread leaves from the right.

We continue by defining a $\mathfrak{B}$-situated Eater as follows:

$\mathsf{Eater} : \mathsf{bread}^\circ \to I \quad =$ 

where the edge labels are the following cells of $\ulcorner \mathfrak{B} \urcorner$:

$\mathsf{buy}_n =$ 

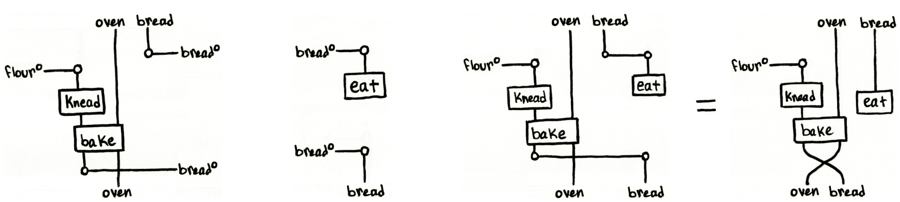
$\mathsf{eat}_n =$ 

$\mathsf{digest}_n =$ 

There are two states for each $n \in \mathbb{N}$ in which the system is hungry, and one in which it is full. In the nth iteration of each of these states, the system posesses n units of bread. If in a hungry state and posessing at least one bread, the eat_n transitions allow it to eat and enter a full state. From a full state the digest_n transitions allow the system to become hungry, leaving the amount of bread unchanged, and finally if the system is hungry then the buy_n transitions allow it to acquire more bread along the left boundary, with the legs of the span indicating that when this happens bread must enter the system along the left boundary.

To compose $\mathbb{A}$-situated transition systems $(R, \phi_R) : (U, \phi_U) \to (V, \phi_V)$ and $(S, \phi_S) : (V, \phi_V) \to (W, \phi_W)$ we compose the underlying spans by pullback as in $\mathsf{Span}(\mathsf{RGraph})$, and define the composite situation $\phi_{S \circ R} : S \circ R \to \langle \mathbb{A} \rangle$ by horizontal composition: $\phi_{S \circ R}(t, t') = \phi_R(t) \mid \phi_S(t')$. This is well-defined because the situations are coherent. In particular this means that $\delta_1 \circ \phi_R = \delta_0 \circ \phi_S$, which says precisely that the right boundary of $\phi_R(t)$ is the left boundary of $\phi_S(t')$ for edges (t, t') of $S \circ R$. Composition of situated transition systems is associative because composition in $\mathsf{Span}(\mathsf{RGraph})$ and horizontal composition in $\ulcorner \mathbb{A} \urcorner$ are both associative. Notice also that paths in a situated transition system have vertically composable material effects, with the composite giving the effect of the entire sequence of transitions.

Continuing our example, we may compose our $\mathfrak{B}$-situated Eater and Baker transition systems to obtain $\mathsf{Eater} \circ \mathsf{Baker} : \mathsf{flour}^\circ \to I$. This transition system has four vertices for each pair n, m of natural numbers, being those states in which the Baker has n bread and the Eater has m bread. The transitions of this new system are mostly pairs of transitions of the components, the exception being that when the Baker sells the Eater must buy due to the fact that these transitions are assigned to the same event along the shared boundary bread°. Now, suppose that in our composite system the Baker begins with one bread and that the Eater begins with none. Suppose further that events unfold as follows: First, the Baker sells its bread to the Eater, which promptly eats it. Then, the Baker bakes more bread, and finally sells the new bread to the Eater. This sequence of transitions corresponds to the following material history: below on the left we see the history generated by the Baker, below in the middle the history generated by the Eater, and below on the right we see the composite history generated by the system as a whole.



Situated transition systems are now easily seen to form a category. We record:

Proposition 1. Let $\mathbb{A}$ be a resource theory. Then there is a category $\mathsf{S}(\mathbb{A})$ of situated transition systems, defined as follows:

objects are $\mathbb{A}$-situated boundaries.

arrows are $\mathbb{A}$-situated transition systems, modulo coherent isomorphism of the underlying spans. That is, for two $\mathbb{A}$-situated transition systems $(R,\phi_R),(S,\phi_S) : (U,\phi_U) \to (V,\phi_V)$, say that $(R,\phi_R) \sim (S,\phi_S)$ in case there exists a reflexive graph isomorphism $\alpha : R \xrightarrow{\sim} S$ such that

(i) $\alpha : R \xrightarrow{\sim} S$ is an isomorphism of spans, in the sense that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[dl] \arrow[d, "\alpha"] \arrow[dr] & \\ U & S \arrow[l] \arrow[r] & V \end{tikzcd}$$

(ii) $\alpha : R \xrightarrow{\sim} S$ preserves material histories, in the sense that there is a natural isomorphism $\iota : \phi_R \xrightarrow{\sim} \phi_S \circ \alpha$ (see Remark 1).

Now an arrow of $\mathsf{S}(\mathbb{A})$ is a $\sim$-equivalence class of situated transition systems.

the **identity** arrow on (U,ϕ_U) is given by the identity span $U \xleftarrow{1_U} U \xrightarrow{1_U} U$ and the situation map $\phi_{1_U} : U \to \langle\mathbb{A}\rangle$ sends $t : A \to B$ in U to the horizontal identity cell for $\phi_U(t)$.

composition is as discussed above.

□

Remark 1. In the definition of $\mathsf{S}(\mathbb{A})$, an equilvalence $(R,\phi_R) \sim (S,\phi_S)$ requires a natural isomorphism $\iota : \phi_R \to \phi_S \circ \alpha$, where ϕ_R and $\phi_S \circ \alpha$ are reflexive graph homomorphisms of type $R \to \langle\mathbb{A}\rangle$. Natural transformations are defined between *functors*, so the reader would be justified in thinking that we have made a fatal mistake! All is in fact well, as we explain presently.

There is a well-known adjunction $F : \mathsf{RGraph} \dashv \mathsf{Cat} : U$ with $F(G)$ being the category of paths in a reflexive graph G, and $U(\mathbb{C})$ being the underlying graph of a category $\mathbb{C}$. Given two reflexive graph homomorphisms $f,g : G \to U(\mathbb{C})$ define a *natural transformation* $\iota : f \to g$ to consist of a morphism $\iota_A : f(A) \to g(A)$ of $\mathbb{C}$ for each vertex A of G such that for every edge $t : A \to B$ of G, $\iota_B \circ f(t) = g(t) \circ \iota_A$ in $\mathbb{C}$. Thus, the definition of natural transformation applies unchanged to reflexive graph homomorphisms whose codomain happens to be a category. Further, applying F to this situation yields a natural transformation in the usual sense. Now $\langle\mathbb{A}\rangle$ is clearly the underlying graph of a category, so in particular it makes sense to ask for a natural isomorphism $\iota : \phi_R \to \phi_S \circ \alpha$. Every isomorphism in $\langle\mathbb{A}\rangle$ has trivial left and right boundary. We therefore require an isomorphism $\iota_A : \phi_R(A) \xrightarrow{\sim} \phi_S(\alpha(A))$ in $\mathbb{A} \cong \mathbf{V}\ulcorner\mathbb{A}\urcorner$ for each vertex A of R such that $\phi_R(t)\iota_B = \iota_A \phi_S(\alpha(t))$ in $\ulcorner\mathbb{A}\urcorner$ for each edge $t : A \to B$ of R.

Intuitively, isomorphic objects of $\mathbb{A}$ denote the same collection of resources, only orgainzed differently. Understood this way, our notion of equivalence $(R,\phi_R) \sim (S,\phi_S)$ identifies situated transition systems that differ only in the internal organization of their resources. More concretely, asking for strict equality $\phi_R = \phi_S \circ \alpha$ does not result in a monoidal category. We would like $\mathsf{S}(\mathbb{A})$ to be monoidal, and our notion of equality is just flexible enough to make this the case.

Proposition 2. If $\mathbb{A}$ is a resource theory then $\mathsf{S}(\mathbb{A})$ is a monoidal category.

□

4 Compact Closure and Accounting

In this section we consider the case in which our resource theory $\mathbb{A}$ is compact closed. From the perspective of accountancy, string diagrams over a resource theory are like *ledgers*, recording the material history of the resources they concern [18]. In the partita-doppia (double-entry) method of accounting every change to a ledger must consist of a matching credit (positive change) and debit (negative change), so that the ledger remains *balanced*. This serves as a kind of integrity check: given a ledger we may attempt to *balance* it by matching credits with debits and cancelling them out, and the ledger is well-formed in case all entries may be cancelled in this way.

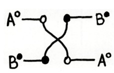
While the credits and debits of partita-doppia accounting are usually positive and negative integers, the technique applies in the context of any compact closed resource theory. The units $\eta_A : I \to A \otimes A^*$ create matching credits and debits, and the cancellative process of balancing is performed via the counits $\varepsilon_A : A^* \otimes A \to I$. The traditional setting [6] is captured by the compact closed category $\mathbb{Z}$ whose objects are the group of differences construction of the integers and in which there is a morphism between two objects if and only if the corresponding integers are equal [13, 1].

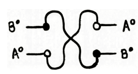
The cells of $\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ with I as their top and bottom boundary are called *horizontal cells*. The horizontal cells of $\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ form a monoidal category $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$, with composition given by horizontal composition in $\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ and the tensor product given by vertical composition in $\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$. Think of $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ as a category of *exchanges* — a point of view is developed in [19]. Isomorphic objects of $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ correspond to equivalent exchanges ([19], Lemma 3). If $\mathbb{A}$ is compact closed we encounter a formal version of the fact that if `Alice` gives `Bob` negative five dollars, this is equivalent to `Bob` giving `Alice` positive five dollars. More generally, that to get rid of a debit is in many ways the same thing as receiving a credit, and vice-versa.

Lemma 1. If $\mathbb{A}$ is compact closed then $A^\circ \cong (A^*)^\bullet$ and $A^\bullet \cong (A^*)^\circ$ in $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$.

□

There is a kind of causal structure present in $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$. The corners allow us to bend wires down, but not up, a formal reflection of the fact that I cannot give something away unless I have it. In particular this means that $\mathsf{S}(\mathbb{A})$ need not be symmetric monoidal: For any A, B there is always a morphism of type $A^\circ \otimes B^\bullet \to B^\bullet \otimes A^\circ$, pictured below on the left, but this is not always an isomorphism.





If our resource theory $\mathbb{A}$ is compact closed, then $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ is symmetric monoidal, with the inverse to the problematic morphism given above on the right. This is a formal reflection of the way that debits allow us to violate causality in everyday life: by incurring a debit I may give something away before I have it. For similar reasons, $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ need not be rigid, but if $\mathbb{A}$ is compact closed then it is.

Lemma 2. If $\mathbb{A}$ is compact closed then so is $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$.

□

In fact, if $\mathbb{A}$ is compact closed, then $\mathsf{S}(\mathbb{A})$ is as well. While we might expect $\mathsf{S}(\mathbb{A})$ to be compact closed for every $\mathbb{A}$ — inheriting the compact closed structure of $\mathsf{Span}(\mathsf{RGraph})$ — the geometry of $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ prevents this. Both $\mathsf{Span}(\mathsf{RGraph})$ and $\mathbf{H}\ulcorner\!\llcorner\mathbb{A}\lrcorner\!\urcorner$ occur as subcategories of $\mathsf{S}(\mathbb{A})$, and it seems that structure must be present in both of them in order to manifest in $\mathsf{S}(\mathbb{A})$. It is interesting that for compact closed resource theories the more flexible compact closed geometry is also present in the category of situated transition systems. Perhaps the use of partita-doppia style debits and credits allows more flexible "wiring" of real-world accounting systems than would otherwise be the case.

Lemma 3. If $\mathbb{A}$ is compact closed, so is $\mathsf{S}(\mathbb{A})$.

$\square$

Now, the category $\mathsf{S}(\mathbb{Z})$ of $\mathbb{Z}$-situated transition systems describes systems of partita-doppia accounts in the sense of [15]. The situation maps each state to an integer-valued account balance, and similarly each transition corresponds to a cell of $\ulcorner\!\!\llcorner\mathbb{Z}\lrcorner\!\!\urcorner$ with top and bottom boundary the balance of the source and target states, respectively. This ensures that any change in the account balance is reflected by value entering or leaving the system along the boundaries, and vice-versa. Since $\mathbb{Z}$ is compact closed, we obtain an analogue of the main theorem of [15] as a special case of Lemma 3, as promised:

Corollary 1. $\mathsf{S}(\mathbb{Z})$ is compact closed.

$\square$

5 Conclusions and Future Work

We have introduced the idea of situating a transition system with boundary in a resource theory and constructed a monoidal category $\mathsf{S}(\mathbb{A})$ of such systems over an arbitrary resource theory $\mathbb{A}$. Further, we have shown that when $\mathbb{A}$ is compact closed, $\mathsf{S}(\mathbb{A})$ is also compact closed, generalizing existing work concering systems of partita-doppia accounts [15]. We feel that this in a promising new direction in the study of concurrent systems, and have many ideas for future work.

If $\mathbb{A}$ is a model of a functional programming language, then an object of $\mathsf{S}(\mathbb{A})$ can be understood as a very general sort of behavioural type. There is an extensive literature on behavioural types, and we speculate that situated transition systems would be a good way to place this work in the wider context of entire systems. If $\mathbb{A}$ is a model of a ledger system in the sense of [18], then the material history generated by an $\mathbb{A}$-situated transition system can be seen as a sequence of ledger transactions. It seems that this is relevant to the study of smart contracts, since the ability to transact on the blockchain as they execute is one of their defining features. More ambitiously, we wish to construct compositional models of the systems one encounters in molecular biology, and we imagine that situated transition systems over a resource theory of biomolecules would be a good setting for this.

It is currently rather painful to specify a situated transition system, and it would be worthwhile to investigate various kinds of syntax that can be given semantics in $\mathsf{S}(\mathbb{A})$. A promising approach is interpret arrows of $\mathbf{H}\ulcorner\!\!\llcorner\mathbb{A}\lrcorner\!\!\urcorner$ as a sort of resource transducer using ideas developed in [2] — we hope to elaborate on this in a future paper. Finally, “spancospans” of reflexive graphs allow us to talk about transition systems with boundary in which the shape of the boundary may change over time [16]. It should be possible to formulate situated transition systems with this capabilty, presumably by working with the intercategory of spancospans [11].

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A.5 A Variety Theorem for Relational Universal Algebra

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A Variety Theorem for Relational Universal Algebra

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Abstract. We consider an analogue of universal algebra in which generating symbols are interpreted as relations. We prove a variety theorem for these relational algebraic theories, in which we find that their categories of models are precisely the definable categories. The syntax of our relational algebraic theories is string-diagrammatic, and can be seen as an extension of the usual term syntax for algebraic theories.

1 Introduction

Universal algebra is the study of what is common to algebraic structures, such as groups and rings, by algebraic means. The central idea of universal algebra is that of a *theory*, which is a syntactic description of some class of structures in terms of generating symbols and equations involving them. A *model* of a theory is then a set equipped with a function for each generating symbol in a way that satisfies the equations. There is a further notion of *model morphism*, and together the models and model morphisms of a given theory form a category. These categories of models are called *varieties*. Much of classical algebra can be understood as the study of specific varieties. For example, group theory is the study of the variety of groups, which arises from the theory of groups in the manner outlined above.

A given variety will in general arise as the models of more than one theory. A natural question to ask, then, is when two theories present the same variety. To obtain a satisfying answer to this question it is helpful to adopt a more abstract perspective. Theories become categories with finite products, models become functors, and model morphisms become natural transformations. Our reward for this shift in perspective is the following answer to our question: two theories present equivalent varieties in case they have equivalent idempotent splitting completions. Thus, from a certain point of view universal algebra is the study of categories with finite products.

This point of view has developed into *categorical* universal algebra. For any sort of categorical structure we can treat categories with that structure as theories, functors that preserve it as models, and natural transformations thereof as model morphisms. The aim is then to figure out what sort of categories arise as

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models and model morphisms of this kind – that is, to determine the appropriate notion of variety. For example, if we take categories with finite limits to be our theories, then varieties correspond to locally finitely presentable categories [2].

The familiar syntax of classical algebra – consisting of *terms* built out of variables by application of the generating symbols – is inextricably bound to finite product structure. In leaving finite products behind for more richly-structured settings, categorical universal algebra also leaves behind much of the syntactic elegance of its classical counterpart. While methods of specifying various sorts of theory (categories with structure) exist, these are often cumbersome, lacking the intuitive flavour of classical universal algebra.

The present paper concerns an analogue of classical universal algebra in which the generating symbols are understood as *relations* instead of functions. The role of classical terms is instead played by string diagrams, and categories with finite products become cartesian bicategories of relations in the sense of [10] – an idea that first appears in [7]. This allows us to present relational algebraic theories in terms of generators and equations, in the style of classical universal algebra. In fact, this approach to syntax for relational theories extends the classical syntax for algebraic theories, which admits a similar diagrammatic presentation.

Our development is best understood in the context of recent work on partial algebraic theories [11], in which the string-diagrammatic syntax for algebraic theories is modified to capture partial functions. This modification of the basic syntax coincides with an increase in the expressive power of the framework, corresponding roughly to the equalizer completion of a category with finite products [8]. The move to relational algebraic theories involves a further modification of the string-diagrammatic syntax, corresponding roughly to the regular completion of a category with finite limits [9]. Put another way, in [11] the (string-diagrammatic) syntax for algebraic theories is extended to express a certain kind of equality, and the resulting terms denote partial functions. In this paper, we further extend the string-diagrammatic syntax to express existential quantification, and the resulting terms denote relations.

Contributions. The central contribution of this paper is a variety theorem characterizing the categories that arise as the models and model morphisms of some relational algebraic theory (Theorem 48). Specifically, we will see that these are precisely the *definable* categories of [19]. As a consequence we obtain that two relational algebraic theories present the same definable category if and only if splitting the partial equivalence relations in each yields equivalent categories (Theorem 49). We illustrate the use of our framework with a number of examples, including the theory of regular semigroups [16] and the theory of effectoids [24]. Lemma 10 is also novel, and we consider it to be a minor contribution

Related Work. The study of universal algebra began with the work of Birkhoff [6]. A few decades later, Lawvere introduced the categorical perspective in his doctoral thesis [22]. A modern account of universal algebra from the categorical perspective is [3]. A highlight of this account is the variety theorem for algebraic theories [1], which our variety theorem for relational algebraic theories is explicitly modelled on. An important result in categorical algebra is Gabriel-Ulmer duality [15], which tells us that if we consider categories with finite limits as our notion of algebraic theory, then the corresponding notion of variety is that of a locally finitely presentable category [2]. Our development relies on the related notion of a definable category [19,20], which recently arose in the development of an analogue of Gabriel-Ulmer duality for regular categories.

We use cartesian bicategories of relations [10] as our notion of relational algebraic theory. Our development relies on several results from the theory of allegories [14], in which cartesian bicategories of relations coincide with the notion of a unitary pre-tabular allegory. We also make use of the theory of regular and exact completions [9]. Of course, all of this relies on the theory of regular and exact categories [5]. The idea of using string diagrams as terms in more general notions of algebraic theories is relatively recent, and relies on the work of Fox [13]. The present paper can be considered a generalisation of recent work on partial theories [11] to include relations. The idea to treat cartesian bicategories of relations as theories with models in the category of sets and relations originally appeared in [7], although no variety theorem is provided therein.

Organization and Prerequisites. In Sect. 2 we introduce categories of abstract relations. In Sect. 3 we give the definition of a relational algebraic theory, and provide a number of examples. Section 4 contains the proof of the variety theorem. We assume familiarity with category theory, including regular categories [5], string diagrams for monoidal categories [17] and their connection to algebraic theories [3], and some 2-category theory [18]. We will behave as though all monoidal categories are *strict* monoidal categories, justifying this behaviour in the usual way by appealing to the coherence theorem for monoidal categories [21].

2 The Algebra of Relations

In the context of algebraic theories, finite product structure serves as an algebra of functions. In this section, we consider an analogous algebra of relations. There are two perspectives from which to consider this algebra of relations: As internal relations in a regular category, or through cartesian bicategories of relations. The two perspectives are very closely related, and we require both: it is through regular categories that our development connects to the wider literature on categorical algebra, but our syntax for relational theories will be the string-diagrammatic syntax for cartesian bicategories of relations.

To begin, we recall the category Rel of sets and relations, which will serve as the universe of models for relational theories in the same way that the category Set of sets and functions is the universe of models for classical algebraic theories.

Definition 1. *The category* Rel *has sets as objects, with arrows* $f : X \to Y$ *given by binary relations* $f \subseteq X \times Y$*. The composite of arrows* $f : X \to Y$*,* $g : Y \to Z$ *is defined by* $fg = \{(x, z) \mid \exists y \in Y.(x, y) \in f \wedge (y, z) \in g\}$*, and the identity relation on* X *is* $\{(x, x) \mid x \in X\}$*.*

2.1 Categories of Internal Relations

In any regular category we can construct an abstract analogue of Definition 1. Instead of sub*sets* $R \subseteq A \times B$, we represent relations as sub*objects* $R \rightarrowtail A \times B$. This approach to categorifying the theory of relations has a relatively long history [14], and integrates well with standard categorical logic due to the ubiquity of regular categories there.

Definition 2. *Let* $\mathbb{C}$ *be a regular category. The associated category of* internal relations, $\mathsf{Rel}(\mathbb{C})$, *is defined as follows:*

objects *are objects of* $\mathbb{C}$
arrows $r : A \to B$ *are jointly monic spans* $r = \langle f, g\rangle : R \rightarrowtail A \times B$ *modulo equivalence as subobjects of* $A \times B$*. That is,* $r : R \rightarrowtail A \times B$ *and* $r' : R' \rightarrowtail A \times B$ *are equivalent (and thus define the same arrow of* $\mathsf{Rel}(\mathbb{C})$*) in case there exists an isomorphism* $\alpha : R \to R'$ *such that* $\alpha r' = r$*.*
composition *of two arrows* $r : A \to B$ *and* $s : B \to C$ *given respectively by* $\langle f, g\rangle : R \rightarrowtail A \times B$ *and* $\langle h, k\rangle : S \rightarrowtail B \times C$ *is defined by first constructing the pullback of* h *along* g*, pictured below on the left. This defines an arrow* $\langle h'f, g'k\rangle : R \times_B S \to A \times C$*. The composite* $rs : A \to C$ *is defined to be the monic part of the image factorization of this arrow, pictured below on the right.*

$$
\begin{tikzcd}
R \times_B S \arrow[r, "g'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "h"] \\
R \arrow[r, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
R \times_B S \arrow[rr, "\langle h'f, g'k\rangle"] \arrow[dr, two heads] & & A \times C \\
& RS \arrow[ur, tail, "rs"'] &
\end{tikzcd}
$$

identities $1_A : A \to A$ *are are given by diagonal maps* $\Delta_A : A \rightarrowtail A \times A$*.*

Example 3. Set is a regular category, and the category of internal relations in $\mathsf{Rel}(\mathsf{Set})$ is precisely the usual category of sets and relations Rel.

2.2 Cartesian Bicategories of Relations

It is difficult to work with relations internal to a regular category directly. Routine calculations often involve complex interaction between pullbacks and image factorizations, and this quickly becomes intractable. A much more tractable setting for working with relations is provided by cartesian bicategories of relations, which admit a convenient graphical syntax.

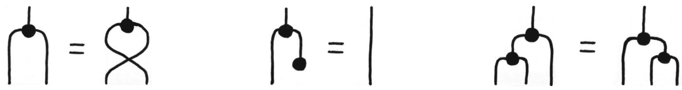
Cartesian bicategories of relations are defined in terms of commutative special frobenius algebras, which provide the basic syntactic scaffolding of our approach:

Definition 4. *Let* $\mathbb{X}$ *be a symmetric strict monoidal category. A* commutative special frobenius algebra *in* $\mathbb{X}$ *is a 5-tuple* $(X, \delta_X, \mu_X, \varepsilon_X, \eta_X)$*, as in*

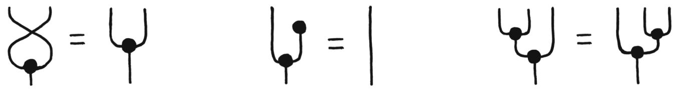
$\delta_X \rightsquigarrow$  $\mu_X \rightsquigarrow$  $\varepsilon_X \rightsquigarrow$  $\eta_X \rightsquigarrow$ 

such that

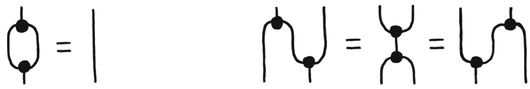
(i) $(X, \delta_X, \varepsilon_X)$ is a commutative comonoid:



(ii) (X, μ_X, η_X) is a commutative monoid:



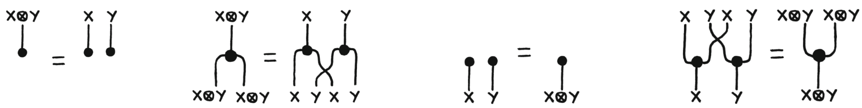
(iii) μ_X and δ_X satisfy the special and frobenius equations:



An intermediate notion is that of a hypergraph category, in which objects are coherently equipped with commutative special frobenius algebra structure:

Definition 5. *A symmetric strict monoidal category $\mathbb{X}$ is called a* hypergraph category *[12] in case:*

(i) Each object X of $\mathbb{X}$ is equipped with a commutative special frobenius algebra.
(ii) The frobenius algebra structure is coherent, i. e., for all X, Y we have:



Now a cartesian bicategory of relations is a hypergraph category enjoying certain additional structure:

Definition 6. *A* cartesian bicategory of relations *[10] is a poset-enriched hypergraph category $\mathbb{X}$ such that:*

(i) The comonoid structure is lax natural. *That is, for all arrows f of $\mathbb{X}$:*

(ii) Each of the frobenius algebras satisfy:

Example 7. The category Rel is a cartesian bicategory of relations with

$$\delta_X = \{(x, (x, x)) \mid x \in X\} \qquad \mu_X = \{((x, x), x) \mid x \in X\}$$

$$\varepsilon_X = \{(x, *) \mid x \in X\} \qquad \eta_X = \{(*, x) \mid x \in X\}$$

where $*$ is the unique element of the singleton set $I = \{*\}$.

Example 8. If $\mathbb{C}$ is a regular category then $\mathsf{Rel}(\mathbb{C})$ is a cartesian bicategory of relations with $X \otimes Y = X \times Y$, $I = 1$, and

$$\delta_X = \langle 1_X, \Delta_X \rangle : X \rightarrowtail X \times (X \times X) \qquad \mu_X = \langle \Delta_X, 1_X \rangle : X \rightarrowtail (X \times X) \times X$$

$$\varepsilon_X = \langle 1_X, !_X \rangle : X \rightarrowtail X \times 1 \qquad \eta_X = \langle !_X, 1_X \rangle : X \rightarrowtail 1 \times X$$

Where Δ_X is the diagonal morphism and $!_X$ is the unique morphism into the terminal object 1 of $\mathbb{C}$.

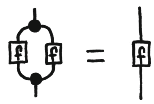
Cartesian bicategories of relations admit meets of hom-sets:

Lemma 9 ([7]). *Every cartesian bicategory of relations has meets of parallel arrows, with $f \cap g$ for $f, g : X \to Y$ defined by*

Further, the meet determines the poset-enrichment in that $f \leq g \Leftrightarrow f \cap g = f$.

We point out this allows for a much simpler presentation, as in:

Lemma 10. *A hypergraph category $\mathbb{X}$ is a cartesian bicategory of relations if and only if for each arrow f:*



We will require a 2-category of cartesian bicategories of relations in our development. Our notion of 1-cell is a structure-preserving functor as in:

Definition 11. *A* morphism *of cartesian bicategories of relations $F : \mathbb{X} \to \mathbb{Y}$ is a strict monoidal functor that preserves the frobenius algebra structure:*

$$F(\delta_X) = \delta_{FX} \qquad F(\mu_X) = \mu_{FX} \qquad F(\varepsilon_X) = \varepsilon_{FX} \qquad F(\eta_X) = \eta_{FX}$$

and the correct sort of 2-cell turns out to be a *lax* natural transformation:

Definition 12. *Let $\mathbb{X}, \mathbb{Y}$ be cartesian bicategories of relations, and let $F, G : \mathbb{X} \to \mathbb{Y}$ be morphisms thereof. Then a* lax transformation *$\alpha : F \to G$ consists of an $\mathbb{X}_0$-indexed family of arrows $\alpha_X : F(X) \to G(X)$ such that for each arrow $f : X \to Y$ of $\mathbb{X}$ we have $F(f)\alpha_Y \leq \alpha_X G(f)$ in $\mathbb{Y}$.*

Definition 13. *Let* RAT *be the 2-category with cartesian bicategories of relations as 0-cells, their morphisms as 1-cells, and lax transformations as 2-cells.*

An important class of arrows in a cartesian bicateory of relations are the *maps*, which should be thought of as those relations that happen to be functions.

Definition 14 (Maps). *An arrow $f : X \to Y$ in a cartesian bicategory of relations is called:*

(i) simple *in case the equation below on the left holds.*
(ii) total *in case the equation below on the right holds.*

(iii) A map *in case it is both simple and total.*

The maps of a cartesian bicategory of relations always form a subcategory $\mathsf{Map}(\mathbb{X})$. For example, $\mathsf{Map}(\mathsf{Rel}) \cong \mathsf{Set}$. More generally:

Theorem 15 ([14]). *For $\mathbb{C}$ a regular category, there is an equivalence of categories $\mathbb{C} \simeq \mathsf{Map}(\mathsf{Rel}(\mathbb{C}))$.*

Remarkably, the components of lax transformations are always maps:

Lemma 16 ([7]). *If $\mathbb{X}, \mathbb{Y}$ are cartesian bicategories of relations, $F, G : \mathbb{X} \to \mathbb{Y}$ are morphisms thereof and $\alpha : F \to G$ is a lax transformation, then each component $\alpha_X : FX \to GX$ of α is necessarily a map.*

3 Relational Algebraic Theories

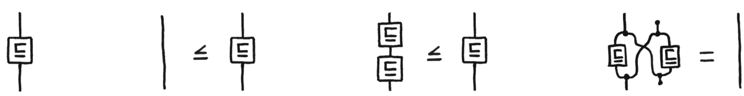
In this section we define relational algebraic theories along with the models and model morphisms, and consider a number of examples.

Definition 17. [7] *A* relational algebraic theory *is a cartesian bicategory of relations. A* model *of a relational algebraic theory $\mathbb{X}$ is a morphism of cartesian bicategories of relations $F : \mathbb{X} \to \mathsf{Rel}$. A* model morphism *$\alpha : F \to G$ is a lax transformation.*

It is convenient to present relational algebraic theories somewhat informally in terms of string-diagrammatic generators and (in)equations between them, with the structure of a cartesian bicategory of relations implicitly present. A more formal account would proceed in terms of monoidal equational theories, from which the cartesian bicategory of relations giving the associated relational algebraic theory may be freely constructed [7].

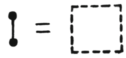
Example 18 (Sets). The relational algebraic theory with no generators and no equations has sets as models and functions as model morphisms (see Lemma 16), and so the associated category of models is Set.

Example 19 (Posets). Consider the relational theory with a single generator (below left) which is required to be reflexive, transitive, and antisymmetric:



The associated category of models is the category of posets and monotone maps.

Example 20 (Nonempty Sets). Consider the relational theory with no generating symbols and a single equation:



Models of the associated relational algebraic theory are sets X such that the generating equation is satisfied in Rel:

$$\eta_X \varepsilon_X = \{(*, *)\} = \square_I$$

where η_X and ε_X are defined as in Definition 1. If we calculate the relational composite, we find that:

$$\eta_X \varepsilon_X = \{(*, *) \mid \exists x \in X.(*, x) \in \eta_X \wedge (x, *) \in \varepsilon_X\} = \{(*, *) \mid \exists x \in X\}$$

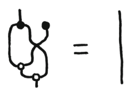
and so models are nonempty sets. The theory of nonempty sets contains no generating morphisms, and so model morphisms are simply functions. Contrast this to the category of *pointed* sets, in which morphisms must preserve the point.

Example 21 (Regular Semigroups). A *semigroup* is a set equipped with an associative binary operation, denoted by juxtaposition. A semigroup S is *regular* [16] in case

$$\forall a \in S. \exists x \in S. axa = a$$

The relational theory of semigroups has a single generating symbol (below left) which is required to be simple, total, and associative:

To capture the regular semigroups we include the following equation:



The associated category of models is the category of regular semigroups and semigroup homomorphisms.

Example 22 (Effectoids). An *effectoid* [24] is a set A equipped with a unary relation $\not\varepsilon \mapsto \_ \subseteq A$, a binary relation $\_ \preceq \_ \subseteq A \times A$, and a ternary relation $\_ ; \_ \mapsto \_ \subseteq A \times A \times A$ satisfying:

(Identity) For all $a, a' \in A$,

$$\exists x \in A.(\not\varepsilon \mapsto x) \wedge (x \,; a \mapsto a') \Leftrightarrow a \preceq a' \Leftrightarrow \exists y \in A.(\not\varepsilon \mapsto y) \wedge (a \,; y \mapsto a')$$

(Associativity) For all $a, b, c, d \in A$,

$$\exists x.(a \,; b \mapsto x) \wedge (x \,; c \mapsto d) \Leftrightarrow \exists y.(b \,; c \mapsto y) \wedge (a \,; y \mapsto d)$$

(Reflexive Congruence 1) For all $a \in A$, $a \preceq a$.
(Reflexive Congruence 2) For all $a, a' \in A$, $(\not\in \mapsto a) \wedge (a \preceq a') \Rightarrow (\not\in \mapsto a')$
(Reflexive Congruence 3) For all $a, b, c \in A$, $\exists x.(a \,; b \mapsto x) \wedge (x \preceq c) \Rightarrow (a \,; b \preceq c)$

To obtain a relational theory of effectoids, we ask for three generating symbols corresponding respectively to the unary, binary, and ternary relation:

Then the identity and associativity axioms become:

And the reflexive congruence axioms become:

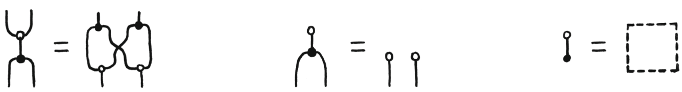
The models of this relational theory are precisely the effectoids.

Example 23 (Generalized Separation Algebras). A *generalized separation algebra* [4] is a partial monoid satisfying the left and right cancellativity axioms, which further satisfies the conjugation axiom:

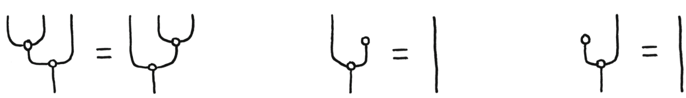
$$\forall x, y.(\exists z.x \circ z = y) \Leftrightarrow (\exists w.w \circ x = y)$$

To capture generalized separation algebras as a relational algebraic theory, we require two generating symbols in the generating monoidal equational theory, corresponding to the monoid operation and the unit:

Both are required to be simple, and the unit is required to be total:

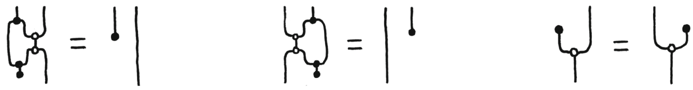


The associativity and unitality axioms become:



Now, define upside-down versions of the generators as in:

Then left cancellativity, right cancellativity, and conjugation are, respectively:



The corresponding category of models is the category of generalized separation algebras and partial monoid homomorphisms.

Example 24 (Algebraic Theories). Let $\mathbb{X}$ be an algebraic theory, and let $(\mathbb{X}_{\mathsf{eq}})_{\mathsf{reg/lex}}$ be the regular completion of $\mathbb{X}$ [8,9]. $\mathsf{Rel}((\mathbb{X}_{\mathsf{eq}})_{\mathsf{reg/lex}})$ is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of $\mathbb{X}$ (as an algebraic theory). Conversely, if $\mathbb{X}$ is a relational algebraic theory, then the maps of $\mathbb{X}$ form a subcategory $\mathsf{Map}(\mathbb{X})$. $\mathsf{Map}(\mathbb{X})$ has finite products, and so defines an algebraic theory in the usual sense. Further, the notions of model and model morphism for relational algebraic theories restrict to the usual notions for algebraic theories on the category of maps.

Example 25 (Essentially Algebraic Theories). An *essentially algebraic theory* [23] is (among many equivalent presentations) a category $\mathbb{X}$ with finite limits. Models are the finite-limit preserving functors $\mathbb{X} \to \mathsf{Set}$, and model morphisms are natural transformations. For $\mathbb{X}$ an essentially algebraic theory let $\mathbb{X}_{\mathsf{reg/lex}}$ be the regular completion of $\mathbb{X}$ [9]. Then $\mathsf{Rel}(\mathbb{X}_{\mathsf{reg/lex}})$ is a relational algebraic theory. Further, its models and model morphisms (as a relational algebraic theory) coincide with the models and model morphisms of $\mathbb{X}$ (as an essentially algebraic theory). Conversely, if $\mathbb{X}$ is a relational algebraic theory then the simple maps of $\mathbb{X}$ are a partial algebraic theory in the sense of [11] – which turn out to be equivalent to essentially algebraic theories. The notions of model and model morphism for relational theories restrict to the corresponding notions for partial theories.

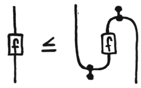
4 The Variety Theorem

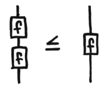
In this section we prove the variety theorem for relational algebraic theories. We do this in phases: first we introduce some necessary terminology concerning classes of idempotents, and recall some details of the idempotent splitting completion. Next, we make the relationship between bicategories of relations and regular categories precise. We then show how the situation extends to include exact categories, this being necessary because exactness is the difference between regular categories and definable categories. Finally, we introduce definable categories, which end up being the varieties of our relational theories. This is structured so that the variety theorem follows immediately. We end by showing precisely when two relational theories present the same definable category.

4.1 Flavours of Idempotent Splitting

We begin by introducing some important kinds of arrow in a relational theory:

Definition 26. *An arrow $f : A \to A$ of a relational algebraic theory is called* reflexive *in case* $1 \leq f$, coreflexive *in case* $f \leq 1$, *a* partial equivalence relation *in case it is symmetric and transitive as in:*





and is called an equivalence relation *if it is reflexive, symmetric, and transitive.*

Notice in particular that every partial equivalence relation is idempotent, that every coreflexive arrow is a partial equivalence relation, and that every equivalence relation is a partial equivalence relation. We also recall the idempotent splitting completion relative to a class of idempotents in a category:

Definition 27. *Let $\mathbb{X}$ be a category, and let $\mathcal{E}$ be a collection of idempotents in $\mathbb{X}$. Define a category $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ in which objects are pairs (X, a) where X is a object of $\mathbb{X}$ and $a : X \to X$ is in $\mathcal{E}$, and arrows $f : (X, a) \to (Y, b)$ are arrows $f : X \to Y$ of $\mathbb{X}$ such that $afb = f$. Composition is composition in $\mathbb{X}$, and identities are given by $a = 1_{(X,a)} : (X, a) \to (X, a)$.*

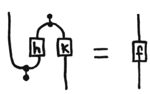
Every member of $\mathcal{E}$ splits in $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$. It turns out that splitting partial equivalence relations works well with cartesian bicategories of relations:

Proposition 28 ([14]). *If $\mathbb{X}$ is a relational algebraic theory and $\mathcal{E}$ is a class of partial equivalence relations in $\mathbb{X}$, then $\mathsf{Split}_{\mathcal{E}}(\mathbb{X})$ is a relational algebraic theory.*

4.2 Tabulation and Regular Categories

We begin our exposition of the correspondence between regular categories and relational algebraic theories by recalling the notion of tabulation [10]. Intuitively, a tabulation of an arrow represents it as a subobject in the category of maps.

Definition 29. *A* tabulation *of an arrow $f : X \to Y$ in a relational algebraic theory $\mathbb{X}$ consists of a pair of maps (h, k) such that the equation below on the left holds in $\mathbb{X}$, and the map below on the right is monic in $\mathsf{Map}(\mathbb{X})$:*



$\mathbb{X}$ *is* tabular *in case every arrow of $\mathbb{X}$ admits a tabulation. Further, define $\mathsf{RAT}_{\mathsf{tab}}$ to be the full 2-subcategory of RAT (Definition 13) on the tabular 0-cells.*

The category of maps of a tabular relational algebraic theory is regular, and conversely the category of internal relations in a regular category is tabular:

Proposition 30. *Let* REG *be the 2-category of regular categories, regular functors, and natural transformation. Then:*

(i) If $\mathbb{X}$ *is a tabular relational algebraic theory then* $\mathsf{Map}(\mathbb{X})$ *is regular. This extends to a 2-functor* $\mathsf{Map} : \mathsf{RAT}_{\mathsf{tab}} \to \mathsf{REG}$*.*
(ii) If $\mathbb{C}$ *is a regular category, then* $\mathsf{Rel}(\mathbb{C})$ *is tabular. This extends to a 2-functor* $\mathsf{Rel} : \mathsf{REG} \to \mathsf{RAT}_{\mathsf{tab}}$*.*

Tabular relational theories and regular categories are thus interchangeable:[1]

Theorem 31. *There is an equivalence of 2-categories* $\mathsf{Map} : \mathsf{RAT}_{\mathsf{tab}} \simeq \mathsf{REG} : \mathsf{Rel}$*.*

Finally, any relational theory can be made tabular by splitting the coreflexives:

Proposition 32. *Let* $\mathbb{X}$ *be a relational algebraic theory, and let* cor *be the collection of coreflexives in* $\mathbb{X}$*. Then* $\mathbb{X}$ *is tabular if and only if every member of* cor *splits. In particular,* $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$ *is always tabular. This extends to a 2-adjunction* $\mathsf{Split}_{\mathsf{cor}} : \mathsf{RAT} \dashv \mathsf{RAT}_{\mathsf{tab}} : U$ *where* U *is the evident forgetful functor.*

4.3 Effectivity and Exact Categories

We begin by recalling the closely related notions of effectivity and exactness:

Definition 33 ([14]). *A relational algebraic theory* $\mathbb{X}$ *is* effective *in case all partial equivalence relations in* $\mathbb{X}$ *split. Let* $\mathsf{RAT}_{\mathsf{eff}}$ *be the full 2-subcategory of* RAT *on the effective 0-cells.*

Definition 34 ([9]). *A regular category* $\mathbb{C}$ *is* exact *in case* $\mathsf{Rel}(\mathbb{C})$ *is effective. Let* EX *be the full 2-subcategory of* REG *on the exact 0-cells.*

It is straightforward to verify that Theorem 31 restricts to the effective case:

Proposition 35. *If* $\mathbb{X}$ *is an effective relational algebraic theory, then* $\mathsf{Map}(\mathbb{X})$ *is exact. Conversely, if* $\mathbb{C}$ *is an exact category, then* $\mathsf{Rel}(\mathbb{C})$ *is effective. This extends to an equivalence of 2-categories* $\mathsf{Map} : \mathsf{RAT}_{\mathsf{eff}} \simeq \mathsf{EX} : \mathsf{Rel}$*.*

Splitting equivalence relations makes tabular relational theories effective:

Proposition 36. *Let* $\mathbb{X}$ *be a tabular relational algebraic theory, and let* eq *be the collection of equivalence relations in* $\mathbb{X}$*. Then* $\mathsf{Split}_{\mathsf{eq}}(\mathbb{X})$ *is effective. This extends to a 2-adjunction* $\mathsf{Split}_{\mathsf{eq}}\mathsf{RAT}_{\mathsf{tab}} \dashv \mathsf{RAT}_{\mathsf{eff}} : U$ *where* U *is the evident forgetful functor.*

We may therefore give the exact completion of a regular category as follows:

[1] We note that we restrict our attention to the 0- and 1-cells then this is proven in [10]. Our contribution is to extend this to include 2-cells.

Proposition 37 ([9,20]). *If $\mathbb{C}$ is regular, define the* exact completion *of $\mathbb{C}$ by*

$$\mathbb{C}_{\mathsf{ex/reg}} = \mathsf{Map}(\mathsf{Split}_{\mathsf{eq}}(\mathsf{Rel}(\mathbb{X})))$$

Then $\mathbb{C}_{\mathsf{ex/reg}}$ is exact. This extends to a 2-adjunction $\mathsf{ex/reg} : \mathsf{REG} \dashv \mathsf{EX} : U$ where U is the evident forgetful functor.

We summarize the relationship of regularity and exactness to relational theories:

Corollary 38. *The following diagram of left 2-adjoint commutes:*

$$\begin{tikzcd}
\mathsf{RAT}_{\mathsf{tab}} \arrow[r, "\mathsf{Map}", "\sim"'] \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"'] & \mathsf{REG} \arrow[d, "\mathsf{ex/reg}"] \\
\mathsf{RAT}_{\mathsf{eff}} \arrow[r, "\sim", "\mathsf{Map}"'] & \mathsf{EX}
\end{tikzcd}$$

where the arrows marked with $\sim$ are part of a 2-equivalence.

Similarly, splitting partial equivalence relations allows us to summarize the role of the idempotent splitting completion:

Proposition 39. *Write per to denote the collection of partial equivalence relations in a relational algebraic theory. There is a 2-adjunction $\mathsf{Split}_{\mathsf{per}} : \mathsf{RAT} \dashv \mathsf{RAT}_{\mathsf{eff}} : U$ where U is the evident forgetful functor. Further, for any relational algebraic theory $\mathbb{X}$, we have $\mathsf{Split}_{\mathsf{per}}(\mathbb{X}) \simeq \mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$, and so the following diagram of left 2-adjoints commutes:*

$$\begin{tikzcd}
\mathsf{RAT} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[dr, "\mathsf{Split}_{\mathsf{per}}"'] & \mathsf{RAT}_{\mathsf{tab}} \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"] \\
& \mathsf{RAT}_{\mathsf{eff}}
\end{tikzcd}$$

Proof. The proof that $\mathsf{Split}_{\mathsf{per}}$ defines a 2-functor which is left adjoint to the forgetful 2-functor is straightforward, and similar to Proposition 32. A proof that $\mathsf{Split}_{\mathsf{per}}(\mathbb{X}) \simeq \mathsf{Split}_{\mathsf{eq}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}))$ can be found in [14, 2.169], it follows immediately that our diagram of left 2-adjoints commutes.

4.4 Definable Categories

The final idea involved in our variety theorem is that of a definable category [19]. Definable categories come from categorical universal algebra. If we take regular categories as our notion of theory, regular functors into Set as our notion of model, and natural transformations as our model morphisms, then definable categories are the corresponding varieties. We follow the exposition of [20], and in particular we formulate definable categories via finite injectivity classes:

Definition 40 (Finite Injectivity Class). *Let $h : A \to B$ be an arrow of $\mathbb{X}$. Then an object C of $\mathbb{X}$ is said to be* h-injective *in case the function of hom-sets $\mathbb{X}(h, C) : \mathbb{X}(B, C) \to \mathbb{X}(A, C)$ defined by $X(h, C)(f) = hf$ is injective. If M is a finite set of arrows in $\mathbb{X}$, write $\mathsf{inj}(M)$ for the full subcategory on the objects C of $\mathbb{X}$ that are h-injective for each $h \in M$. We say that each $\mathsf{inj}(M)$ is a* finite injectivity class *in $\mathbb{X}$.*

Definable categories are defined relative to an ambient locally finitely presentable category. It is an open problem to give a free-standing characterization [19].

Definition 41. *A category is said to be* definable *if it arises as a finite injectivity class in some locally finitely presentable category. If $\mathbb{X}$ and $\mathbb{Y}$ are definable categories, a functor $F : \mathbb{X} \to \mathbb{Y}$ is called an* interpretation *in case it preserves products and directed colimits. Let DEF be the 2-category with definable categories as 0-cells, interpretations as 1-cells, and natural transformations as 2-cells.*

From any definable category we can obtain an exact category by considering its interpretations into Set.

Proposition 42 ([20]). *If $\mathbb{X}$ is a definable category then the functor category $\mathsf{DEF}(\mathbb{X}, \mathsf{Set})$ is an exact category. This extends to a 2-functor $\mathsf{DEF}(\_, \mathsf{Set}) : \mathsf{DEF}^{\mathsf{op}} \to \mathsf{EX}$.*

Similarly, for any regular category the associated category of regular functors into Set is definable.

Proposition 43 ([20]). *If $\mathbb{C}$ is a regular category then the functor category $\mathsf{REG}(\mathbb{C}, \mathsf{Set})$ is definable. This extends to a 2-functor $\mathsf{REG}(\_, \mathsf{Set}) : \mathsf{REG} \to \mathsf{DEF}^{\mathsf{op}}$.*

If the category in question is exact, then considering interpretations of the resulting definable category into Set yields the original exact category. This lifts to the 2-categorical setting.

Proposition 44 ([20]). *There is an adjunction of 2-categories $\mathsf{REG}(-, \mathsf{Set}) : \mathsf{REG} \dashv \mathsf{DEF}^{\mathsf{op}} : \mathsf{DEF}(-, \mathsf{Set})$ which specializes to an equivalence of 2-categories $\mathsf{REG}(-, \mathsf{Set}) : \mathsf{EX} \simeq \mathsf{DEF}^{\mathsf{op}} : \mathsf{DEF}(-, \mathsf{Set})$.*

This gives another way to describe the exact completion of a regular category:

Proposition 45 ([20]). *If $\mathbb{C}$ is regular then $\mathbb{C}_{\mathsf{ex/reg}} \simeq \mathsf{DEF}(\mathsf{REG}(\mathbb{C}, \mathsf{Set}), \mathsf{Set})$.*

Thus, we may summarize the relationship between definable, regular, and exact categories as follows:

Corollary 46 ([20, Sect. 9,10]). *The following diagram of left 2-adjoints commutes.*

$$\begin{tikzcd} \mathsf{REG} \arrow[d, "\mathsf{ex/reg}"'] \arrow[dr, "\mathsf{REG}(-{,}\mathsf{Set})"] & \\ \mathsf{EX} \arrow[r, "\sim", "\mathsf{REG}(-{,}\mathsf{Set})"'] & \mathsf{DEF}^{\mathsf{op}} \end{tikzcd}$$

where the arrow marked with $\sim$ is part of a 2-equivalence.

The ingredients of our variety theorem for relational algebraic theories are now assembled. Together, Proposition 39, Corollary 38, and Corollary 46 give:

Corollary 47. *There following diagram of left 2-adjoints commutes:*

$$\begin{tikzcd}
\mathsf{RAT} \arrow[r, "\mathsf{Split}_{\mathsf{cor}}"] \arrow[dr, "\mathsf{Split}_{\mathsf{per}}"'] & \mathsf{RAT}_{\mathsf{tab}} \arrow[r, "\mathsf{Map}", "\sim"'] \arrow[d, "\mathsf{Split}_{\mathsf{eq}}"] & \mathsf{REG} \arrow[d, "\mathsf{ex/reg}"'] \arrow[dr, "\mathsf{REG}(-,\mathsf{Set})"] & \\
 & \mathsf{RAT}_{\mathsf{eff}} \arrow[r, "\sim", "\mathsf{Map}"'] & \mathsf{EX} \arrow[r, "\sim", "\mathsf{REG}(-,\mathsf{Set})"'] & \mathsf{DEF}^{\mathsf{op}}
\end{tikzcd}$$

where the arrows marked with $\sim$ *are part of a 2-equivalence.*

Now our variety theorem is an immediate consequence of Corollary 47:

Theorem 48. *There is an adjunction of 2-categories* $\mathsf{Mod} : \mathsf{RAT} \dashv \mathsf{DEF}^{\mathsf{op}} : \mathsf{Th}$

It may not be immediately clear what this tells us about the category of models and model morphisms of a relational algebraic theory, so let us briefly discuss. Consider an arbitrary relational algebraic theory $\mathbb{X}$. Our universe of models Rel is tabular, so models of $\mathbb{X}$ and models of $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$ are the same thing since the image of any coreflexive in $\mathbb{X}$ already splits in Rel. Then the category of models of $\mathbb{X}$ and model morphisms thereof is $\mathsf{RAT}_{\mathsf{tab}}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X}), \mathsf{Rel})$. When we transport this across the 2-equivalence $\mathsf{Map} : \mathsf{RAT}_{\mathsf{tab}} \xrightarrow{\sim} \mathsf{REG}$ it becomes $\mathsf{REG}(\mathsf{Map}(\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})), \mathsf{Set})$, a definable category. Thus, categories of models and model morphisms of regular algebraic theories are definable categories.

Now, Set is exact, so Rel is effective, which means that much like the models of $\mathbb{X}$ and $\mathsf{Split}_{\mathsf{cor}}(\mathbb{X})$, the models of $\mathbb{X}$ and $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$ are the same. We have shown that $\mathsf{RAT}_{\mathsf{eff}} \simeq \mathsf{EX} \simeq \mathsf{DEF}^{\mathsf{op}}$, and so the question of when two relational algebraic theories generate the same category of models and model morphisms can be answered as follows:

Theorem 49. *Two relational algebraic theories* $\mathbb{X}$ *and* $\mathbb{Y}$ *present equivalent definable categories if and only if* $\mathsf{Split}_{\mathsf{per}}(\mathbb{X})$ *and* $\mathsf{Split}_{\mathsf{per}}(\mathbb{Y})$ *are equivalent.*

Compare this to the case of algebraic theories, in which two theories present the same variety in case splitting *all* idempotents yields equivalent categories [1].

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A.6 Cornering Optics

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Cornering Optics

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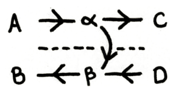
We show that the category of optics in a monoidal category arises naturally from the free cornering of that category. Further, we show that the free cornering of a monoidal category is a natural setting in which to work with comb diagrams over that category. The free cornering admits an intuitive graphical calculus, which in light of our work may be used to reason about optics and comb diagrams.

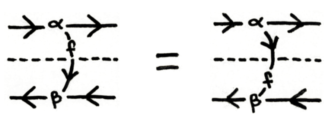
Introduction

Optics in a monoidal category are a notion of bidirectional transformation, and have been something of a hot topic in recent years. In particular *lenses*, which are optics in a cartesian monoidal category, play an important role in the theory of open games [8], compositional machine learning [5], dialectica categories [16], functional programming [17, 3], the theory of polynomial functors [21], and of course in the study of bidirectional transformations [15, 6].

We recall the elementary presentation of the category $\mathsf{Optic}_{\mathbb{A}}$ of optics in a monoidal category $\mathbb{A}$. Objects (A,B) are pairs of objects of $\mathbb{A}$. Arrows $\langle \alpha \mid \beta \rangle_M : (A,B) \to (C,D)$ consist of arrows $\alpha : A \to M \otimes C$ and $\beta : M \otimes D \to B$ of $\mathbb{A}$. It is helpful to visualize this as follows:



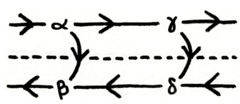
Arrows are subject to equations of the form $\langle \alpha(f \otimes 1_C) \mid \beta \rangle_N = \langle \alpha \mid (f \otimes 1_D)\beta \rangle_M$ for $f : M \to N$ in $\mathbb{A}$. This is often visualized as a sort of sliding between components, as in:



Equivalently, the hom-sets of $\mathsf{Optic}_{\mathbb{A}}$ can be given as a coend of hom-functors of $\mathbb{A}$:

$$\mathsf{Optic}_{\mathbb{A}}((A,B),(C,D)) \cong \int^{M} \mathbb{A}(A, M \otimes C) \times \mathbb{A}(M \otimes D, B)$$

Composition is given by $\langle \alpha \mid \beta \rangle_M \langle \gamma \mid \delta \rangle_N = \langle \alpha(1_M \otimes \gamma) \mid (1_M \otimes \delta)\beta \rangle_{M \otimes N}$. Visually:



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Identity arrows are given by $1_{(A,B)} = \langle 1_A \mid 1_B \rangle_I$.

Originally studied as an approach to concurrency by Nester [14], the *free cornering* of a monoidal category is the double category obtained by freely adding companion and conjoint structure to it. The usual string diagrams for monoidal categories extend to an intuitive graphical calculus for the free cornering. The free cornering is the main piece of mathematical machinery in our development, and we give a detailed introduction to it in Section 1.

Our main contribution is a characterisation of optics in a monoidal category in terms of its free cornering. More exactly, in Theorem 1 we show that the category of optics is a full subcategory of the horizontal cells of the free cornering. In addition to shedding some light on the nature of optics, this allows us to reason about them using the graphical calculus of the free cornering. We demonstrate this by using the graphical calculus to prove Lemmas 3, 4, 5, and 6, which are a series of results originally due to Riley [18] concerning the lens laws. This occupies Section 3.

Optics in a monoidal category can be seen as a special case of *comb diagrams* in that category. Comb diagrams arose in the theory of quantum circuits [2], and have since appeared in algebraic investigations of causal structure [11, 10]. We suspect comb diagrams to be widely applicable, but there is not yet a commonly accepted algebra of comb diagrams. In Section 4 we give a notion of (single-sided) comb diagram in terms of the free cornering that coincides with the notion of comb diagram present in the work of Román [19]. We demonstrate that the free cornering is a natural setting in which to work with comb diagrams, and consider this a further contribution of the present work.

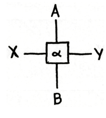
Our results are consequences of Lemma 2, which characterises cells of the free cornering with a certain boundary shape in terms of coends. In particular, we make use of the soundness result for the graphical calculus of the free cornering due to Myers [13]. The relevant definitions and the lemma itself are presented in Section 2. The reader need not be familiar with coends to follow our development. While coends connect the free cornering to the wider literature through Lemma 2, our work offers an alternate perspective that is conceptually simpler.

In summary, we give a novel characterisation of optics and comb diagrams in a monoidal category in terms of the free cornering of that category. The graphical calculus of the free cornering allows one to work with these structures more easily. In addition to telling us something about the nature of optics and comb diagrams, our results suggest that the free cornering is worthy of further study in its own right.

1 Double Categories and the Free Cornering

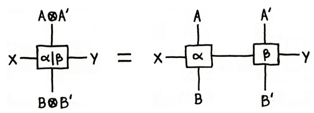
In this section we set up the rest of our development by presenting the theory of single object double categories and the free cornering of a monoidal category. In this paper we consider only *strict* monoidal categories, and in our development the term "monoidal category" should be read as "strict monoidal category". That said, we imagine that our results will hold in some form for arbitrary monoidal categories via the coherence theorem for monoidal categories [12].

A *single object double category* is a double category $\mathbb{D}$ with exactly one object. In this case $\mathbb{D}$ consists of a *horizontal edge monoid* $\mathbb{D}_H = (\mathbb{D}_H, \otimes, I)$, a *vertical edge monoid* $\mathbb{D}_V = (\mathbb{D}_V, \otimes, I)$, and a collection of *cells*

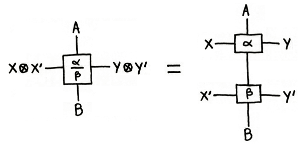


where $A, B \in \mathbb{D}_H$ and $X, Y \in \mathbb{D}_V$. We write $\mathbb{D}(X\,{}^{A}_{B}\,Y)$ for the *cell-set* of all such cells in $\mathbb{D}$. Given cells α, β

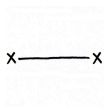
where the right boundary of α matches the left boundary of β we may form a cell $\alpha|\beta$ – their *horizontal composite* – and similarly if the bottom boundary of α matches the top boundary of β we may form $\frac{\alpha}{\beta}$ – their *vertical composite* – with the boundaries of the composite cell formed from those of the component cells using $\otimes$. We depict horizontal and vertical composition, respectively, as in:



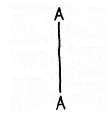
and



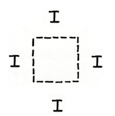
Horizontal and vertical composition of cells are required to be associative and unital. We omit wires of sort I in our depictions of cells, allowing us to draw horizontal and vertical identity cells, respectively, as in:

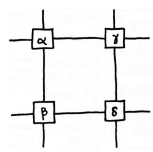


and

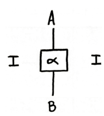


Finally, the horizontal and vertical identity cells of type I must coincide – we write this cell as $\square_I$ and depict it as empty space, see below on the left – and vertical and horizontal composition must satisfy the interchange law. That is, $\frac{\alpha}{\beta}|\frac{\gamma}{\delta} = \frac{\alpha|\gamma}{\beta|\delta}$, allowing us to unambiguously interpret the diagram below on the right:

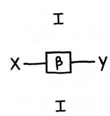




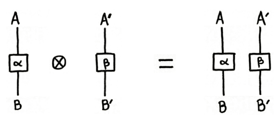
Every single object double category $\mathbb{D}$ defines strict monoidal categories $\mathbf{V}\mathbb{D}$ and $\mathbf{H}\mathbb{D}$, consisting of the cells for which the $\mathbb{D}_H$ and $\mathbb{D}_V$ valued boundaries respectively are all I, as in:



and



That is, the collection of objects of $\mathbf{V}\mathbb{D}$ is $\mathbb{D}_H$, composition in $\mathbf{V}\mathbb{D}$ is vertical composition of cells, and the tensor product in $\mathbf{V}\mathbb{D}$ is given by horizontal composition:

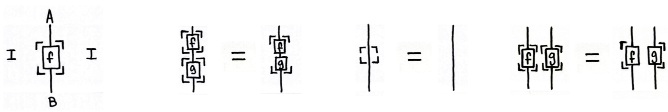


In this way, $\mathbf{V}\mathbb{D}$ forms a strict monoidal category, which we call the category of *vertical cells* of $\mathbb{D}$. Similarly, $\mathbf{H}\mathbb{D}$ is also a strict monoidal category (with collection of objects $\mathbb{D}_V$) which we call the *horizontal cells* of $\mathbb{D}$.

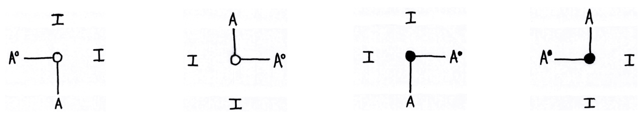
Next, we introduce the free cornering of a monoidal category.

Definition 1 ([14])**.** Let $\mathbb{A}$ be a monoidal category. We define the *free cornering* of $\mathbb{A}$, written $\ulcorner\mathbb{A}\lrcorner$, to be the free single object double category on the following data:

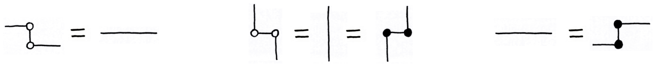
- The horizontal edge monoid $\ulcorner\mathbb{A}\lrcorner_H = (\mathbb{A}_0, \otimes, I)$ is given by the objects of $\mathbb{A}$.
- The vertical edge monoid $\ulcorner\mathbb{A}\lrcorner_V = (\mathbb{A}_0 \times \{\circ, \bullet\})^*$ is the free monoid on the set $\mathbb{A}_0 \times \{\circ, \bullet\}$ of polarized objects of $\mathbb{A}$ – whose elements we write A° and $A^\bullet$.
- The generating cells consist of vertical cells $\ulcorner f \lrcorner$ for each morphism $f : A \to B$ of $\mathbb{A}$ subject to equations as in:

  along with the following *corner cells* for each object A of $\mathbb{A}$:

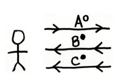
  

  which are subject to the *yanking equations*:

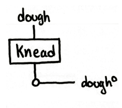
For a precise development of free double categories see [4]. Briefly, cells are formed from the generating cells by horizontal and vertical composition, subject to the axioms of a double category in addition to any generating equations. The corner structure has been heavily studied under various names including *proarrow equipment*, *connection structure*, and *companion and conjoint structure*. A good resource is the appendix of [20].

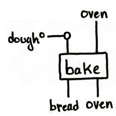
We understand elements of $\ulcorner\mathbb{A}\lrcorner_V$ as $\mathbb{A}$-*valued exchanges*. Each exchange $X_1 \otimes \cdots \otimes X_n$ involves a left participant and a right participant giving each other resources in sequence, with A° indicating that the left participant should give the right participant an instance of A, and $A^\bullet$ indicating the opposite. For example say the left participant is `Alice` and the right participant is `Bob`. Then we can picture the exchange $A^\circ \otimes B^\bullet \otimes C^\bullet$ as:

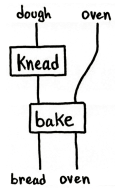
`Alice` ⇝  ⇜ `Bob`

Think of these exchanges as happening *in order*. For example the exchange pictured above demands that first `Alice` gives `Bob` an instance of A, then `Bob` gives `Alice` an instance of B, and then finally `Bob` gives `Alice` an instance of C.

Cells of $\ulcorner\mathbb{A}\lrcorner$ can be understood as *interacting* morphisms of $\mathbb{A}$. Each cell is a method of obtaining the bottom boundary from the top boundary by participating in $\mathbb{A}$-valued exchanges along the left and right boundaries in addition to using the arrows of $\mathbb{A}$. For example, if the morphisms of $\mathbb{A}$ describe processes involved in baking bread, we might have the following cells of $\ulcorner\mathbb{A}\urcorner$:







The cell on the left describes a procedure for transforming `dough` into nothing by `kneading` it and sending the result away along the right boundary, and the cell in the middle describes a procedure for transforming an `oven` into `bread` and an `oven` by receiving `dough` along the left boundary and then using the `oven` to `bake` it. Composing these cells horizontally results in the cell on the right via the yanking equations. In this way the free cornering models concurrent interaction, with the corner cells capturing the flow of information across different components.

The vertical cells of the free cornering involve no exchanges, and as such are the cells of the original monoidal category:

Lemma 1 ([14])**.** There is an isomorphism of categories $\mathbf{V}\ulcorner\mathbb{A}\urcorner \cong \mathbb{A}$.

In comparison, the horizontal cells of the free cornering are not well understood. In the sequel we will see that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ contains $\mathsf{Optic}_{\mathbb{A}}$ as a full subcategory.

2 Alternation and Coends

In this section we prove a technical lemma characterizing certain cell-sets of $\ulcorner\mathbb{A}\urcorner$ as coends.

Definition 2. An element of $\ulcorner\mathbb{A}\urcorner_V$ is said to be $\bullet\circ$-*alternating* in case it is of the form $A_1^\bullet \otimes B_1^\circ \otimes \cdots \otimes A_n^\bullet \otimes B_n^\circ$ for some $n \in \mathbb{N}$ such that $n > 0$. The *alternation length* of a $\bullet\circ$-alternating element is defined to be the evident $n \in \mathbb{N}$. For example:

- $B^\bullet \otimes A^\circ$ is $\bullet\circ$-alternating with alternation length 1.
- $A^\bullet \otimes B^\circ \otimes C^\bullet \otimes A^\circ$ is $\bullet\circ$-alternating with alternation length 2.
- $(A \otimes B)^\bullet \otimes I^\circ$ is $\bullet\circ$-alternating with alternation length 1.
- None of the following are $\bullet\circ$-alternating:

$$I \qquad A^\bullet \otimes B^\circ \otimes C^\circ \qquad A^\bullet \otimes B^\bullet \qquad A^\bullet \qquad (A\otimes B)^\circ \otimes B^\bullet \qquad A^\bullet \otimes B^\circ \otimes C^\bullet$$

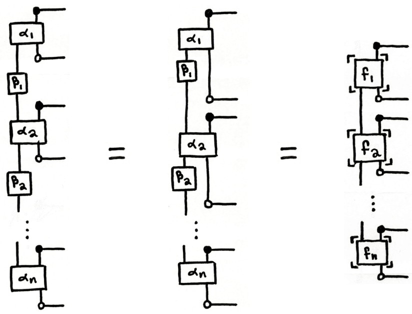
Definition 3. A cell-set of the form $\ulcorner\mathbb{A}\urcorner({}_I{}^I_I X)$ is said to be *right-$\bullet\circ$-alternating* in case X is $\bullet\circ$-alternating. The *alternation depth* of a right-$\bullet\circ$-alternating cell-set is the alternation length of its right boundary.

Lemma 2. If $\ulcorner\mathbb{A}\urcorner({}_I{}^I_I X)$ is right-$\bullet\circ$-alternating with alternation depth n and $X = A_1^\bullet \otimes B_1^\circ \otimes \cdots \otimes A_n^\bullet \otimes B_n^\circ$ then

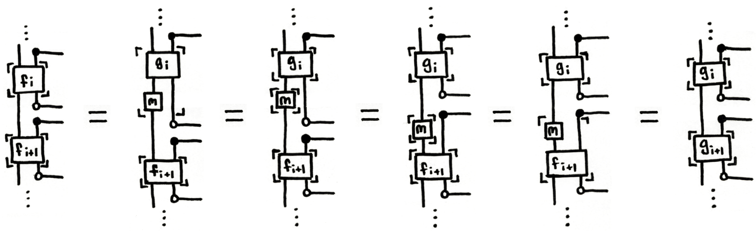
$$\ulcorner\mathbb{A}\urcorner \left(I \, {}^{I}_{I} \, X \right) \cong \int^{M_1,\ldots,M_{n-1}} \prod_{i=1}^{n} \mathbb{A}(M_{i-1} \otimes A_i, M_i \otimes B_i)$$

where $M_0 = M_n = I$.

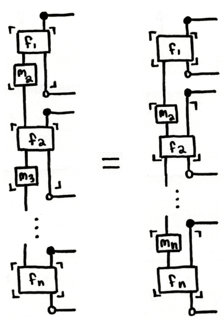
Proof. By inspecting the generating cells of $\ulcorner\mathbb{A}\urcorner$ and making use of Lemma 1 we find that any cell of $\ulcorner\mathbb{A}\urcorner(\iota^I_I X)$ is necessarily of the form:



Thus cells of $\ulcorner\mathbb{A}\urcorner(\iota^I_I X)$ may be written as n-tuples $\langle f_1 \mid \cdots \mid f_n\rangle$. As a consequence of Myers' soundness result for the graphical calculus [13], we know that two cells $\langle f_1 \mid \cdots \mid f_n\rangle$ and $\langle g_1 \mid \cdots \mid g_n\rangle$ of $\ulcorner\mathbb{A}\urcorner(\iota^I_I X)$ are equal iff they are deformable into each other modulo the equations of $\mathbb{A}$. Consider that all *local* deformations $\langle \cdots \mid f_i \mid f_{i+1} \mid \cdots\rangle = \langle \cdots \mid g_i \mid g_{i+1} \mid \cdots\rangle$ are of the form:



where $f_i = g_i(m \otimes 1)$ and $g_{i+1} = (m \otimes 1)f_{i+1}$. Now, the only way $\langle f_1 \mid \cdots \mid f_n\rangle$ and $\langle g_1 \mid \cdots \mid g_n\rangle$ can be equal is by (repeated) parallel local deformation of the associated diagrams, as in:



Thus, $\ulcorner\mathbb{A}\urcorner(\iota^I_I X)$ is the set of (appropriately typed) n-tuples $\langle f_1 \mid \cdots \mid f_n\rangle$ of morphisms of $\mathbb{A}$, quotiented by equations of the form:

$$\langle f_1(m_2 \otimes 1) \mid f_2(m_3 \otimes 1) \mid \cdots \mid f_n\rangle = \langle f_1 \mid (m_2 \otimes 1)f_2 \mid \cdots \mid (m_n \otimes 1)f_n\rangle$$

which is precisely to say that the claim holds. $\square$

Remark 1. There is an obvious dual notion of *left-○●-alternating* cell-set for which a version of Lemma 2 holds.

3 Optics and the Free Cornering

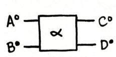
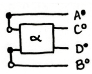
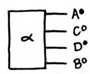
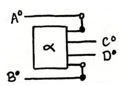
In this section we use Lemma 2 to show that $\mathsf{Optic}_{\mathbb{A}}$ is a full subcategory of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ for any monoidal category $\mathbb{A}$. We then briefly discuss lenses, and illustrate the power of the graphical calculus for $\ulcorner\mathbb{A}\urcorner$ by reproving a correspondence between lenses satisfying the the lens laws and lenses that are comonoid homomorphisms with respect to a certain comonoid structure. These results about lenses are originally due to Riley [18], and were also used to demonstrate Boisseau's approach to string diagrams for optics [1]. We end with Observation 1, which discusses the relation of teleological categories [9] to the free cornering.

Theorem 1. Let $\mathbb{A}$ be a monoidal category. Then $\mathsf{Optic}_{\mathbb{A}}$ is the full subcategory of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ on objects of the form $A^\circ \otimes B^\bullet$ for $A, B \in \mathbb{A}_0$.

Proof. We begin by noticing that

$$\mathbf{H}\ulcorner\mathbb{A}\urcorner(A^\circ \otimes B^\bullet, C^\circ \otimes D^\bullet) \cong \ulcorner\mathbb{A}\urcorner\left(I \begin{smallmatrix} I \\ \\ I \end{smallmatrix} A^\bullet\otimes C^\circ\otimes D^\bullet\otimes B^\circ\right)$$

via:

 $\mapsto$  and  $\mapsto$ 

This cell-set is right-●○-alternating of depth 2, and so we have:

$$\ulcorner\mathbb{A}\urcorner\left(I \begin{smallmatrix} I \\ \\ I \end{smallmatrix} A^\bullet\otimes C^\circ\otimes D^\bullet\otimes B^\circ\right) \cong \int^{M\in\mathbb{A}} \mathbb{A}(A, M\otimes C) \times \mathbb{A}(M \otimes D, B)$$

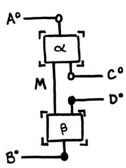
Now we already know that

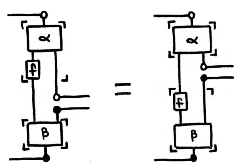
$$\int^{M\in\mathbb{A}} \mathbb{A}(A, M\otimes C) \times \mathbb{A}(M \otimes D, B) \cong \mathsf{Optic}_{\mathbb{A}}((A,B),(C,D))$$

and so we have a correspondence between arrows of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ and arrows of $\mathsf{Optic}_{\mathbb{A}}$:

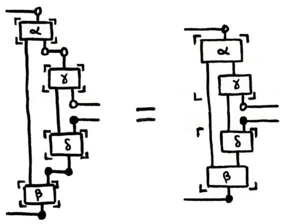
$$\mathbf{H}\ulcorner\mathbb{A}\urcorner(A^\circ \otimes B^\bullet, C^\circ \otimes D^\bullet) \cong \mathsf{Optic}_{\mathbb{A}}((A,B),(C,D))$$

In particular, we know that arrows in $\mathbf{H}\ulcorner\mathbb{A}\urcorner(A^\circ \otimes B^\bullet, C^\circ \otimes D^\bullet)$ are equivalently optics $\langle \alpha \mid \beta \rangle_M$ as below left, and that the equations between optics – below right – capture all equations in $\mathbf{H}\ulcorner\mathbb{A}\urcorner(A^\circ \otimes B^\bullet, C^\circ \otimes D^\bullet)$:

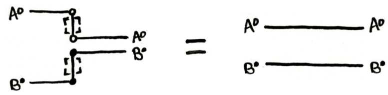




Next, given arrows $\langle \alpha \mid \beta \rangle_M : (A,B) \to (C,D)$ and $\langle \gamma \mid \delta \rangle_N : (C,D) \to (E,F)$ of $\mathsf{Optic}_{\mathbb{A}}$, we find that composing the corresponding arrows of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ yields the arrow corresponding to $\langle \alpha(1_M \otimes \gamma) \mid (1_M \otimes \delta)\beta \rangle_{M \otimes N} = \langle \alpha \mid \beta \rangle_M \langle \gamma \mid \delta \rangle_N$ as in:



Further, the identity on $A^\circ \otimes B^\bullet$ in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ corresponds to the $1_{(A,B)} = \langle 1_A \mid 1_B \rangle_I$ in $\mathsf{Optic}_{\mathbb{A}}$ as in:

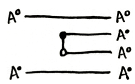


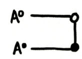
The result is thus proven. □

Remark 2. Following Remark 1, a similar argument gives that if A is symmetric monoidal then $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{op}$ also contains $\mathsf{Optic}_{\mathbb{A}}$ as the full subcategory on those objects of the form $A^\bullet \otimes B^\circ$.

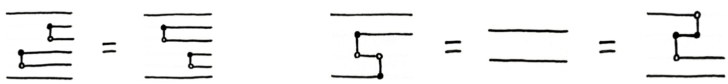
Remark 3. If $\mathbb{A}$ is a *symmetric* monoidal category then $\mathsf{Optic}_{\mathbb{A}}$ is itself monoidal [18]. We remark that while $\mathsf{Optic}_{\mathbb{A}}$ remains a subcategory of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ in this case, it is not a *monoidal* subcategory. That is, the tensor product of optics is *not* given by the tensor product in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$.

As an illustration of our approach, we consider the characterisation of the lens laws given in [18]. Say that an optic is *homogeneous* in case it is contained in the full subcategory of $\mathsf{Optic}_{\mathbb{A}}$ on objects (A,A) for some $A \in \mathbb{A}_0$. Notice that every object of this subcategory is a comonoid in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$, with the comultiplication and counit given as in:





where the comonoid axioms hold as in:

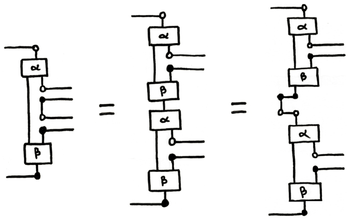


Definition 4 ([18]). A homogeneous optic $h : (A,A) \to (B,B)$ of $\mathsf{Optic}_{\mathbb{A}}$ is called *lawful* in case the following equations hold in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$:

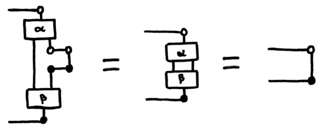
That is, in case h is a comonoid homomorphism with respect to the comonoid structure given above.

Lemma 3 ([18]). If $h = \langle \alpha \mid \beta \rangle_M : (A,A) \to (B,B)$ in $\mathsf{Optic}_{\mathbb{A}}$ with α and β mutually inverse, then h is lawful.

Proof.



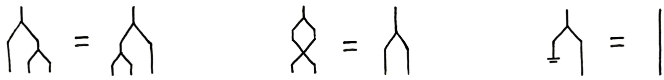
and



□

Recalling the algebraic characterisation of cartesian monoidal categories [7], we denote the commutative comonoid structure in a cartesian monoidal category as follows:

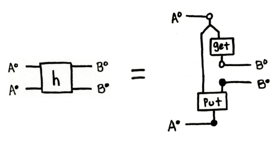
This structure must satisfy the commutative comonoid axioms:



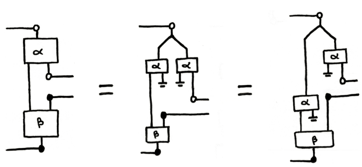
Must further be coherent with respect to the monoidal structure:

And every morphism f of the category in question must be a comonoid homomorphism:

Lemma 4 ([18]). Let $\mathbb{A}$ be a cartesian monoidal category, and let $h = \langle \alpha \mid \beta \rangle_M : (A,A) \to (B,B)$ be a homogeneous optic in $\mathbb{A}$. Then there exist arrows $\mathrm{get} : A \to B$ and $\mathrm{put} : A \otimes B \to A$ of $\mathbb{A}$ such that:



Proof. We have:

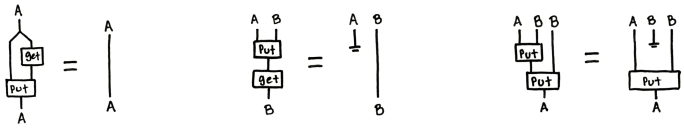


and so the claim follows via:

□

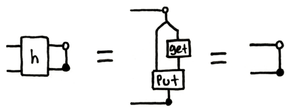
Homogeneous optics in cartesian monoidal categories are called *lenses*. We write [put | get] : $(A,A) \to (B,B)$ for the lens specified by appropriate put and get arrows in the above manner.

Definition 5 ([6])**.** A lens [put | get] : $(A,A) \to (B,B)$ is is said satisfy the *lens laws* in case:

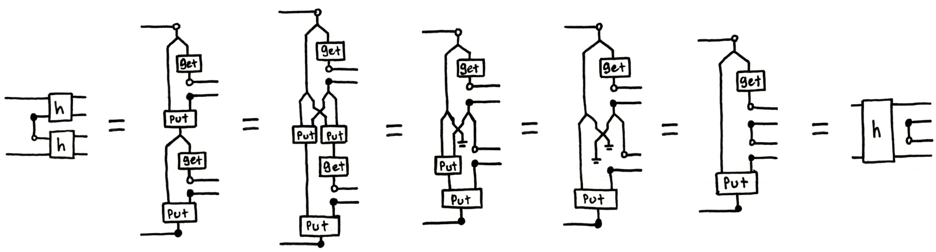


Lemma 5 ([18])**.** If a lens $h =$ [put | get] : $(A,A) \to (B,B)$ satisfies the lens laws then it is lawful.

Proof. For the counit we have:



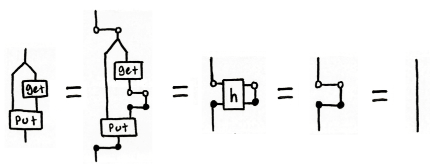
And for the comultiplication:



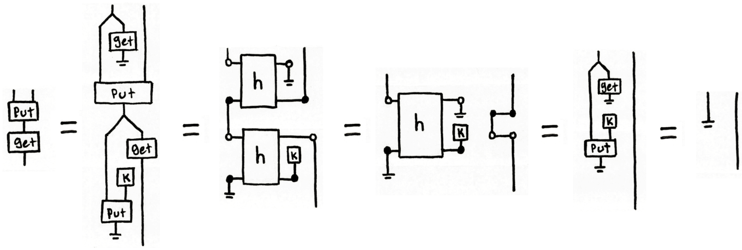
□

Lemma 6 ([18])**.** If a lens $h =$ [put | get] : $(A,A) \to (B,B)$ is lawful and B is inhabited in the sense that there is an arrow $k : 1 \to B$ in $\mathbb{A}$, then it satisfies the lens laws.

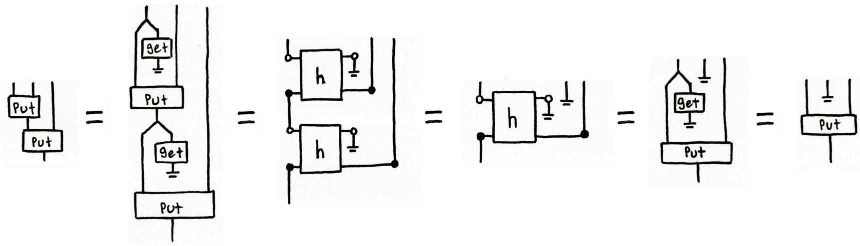
Proof. The first lens law holds as in:



The second lens law holds as in:



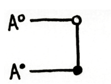
and the third lens law holds as in:



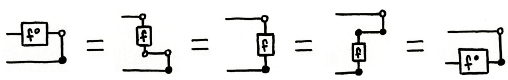
□

Observation 1 (Teleological Categories)**.** $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ contains structure reminiscent of *teleological categories* [9], which were introduced to allow well-founded diagrammatic reasoning about lenses. Analogous to the *dualizable* morphisms of a teleological category are those of the form f°, defined as below left, with duals $f^\bullet$, defined as below right:

Standing in for the *counits* of a teleological category we have the following cell for each $A \in \mathbb{A}$:



We then obtain an analogue of the condition that the counits be *extranatural* as in:



Notice that all arrows $A^\circ \to B^\circ$ of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ are of the form f° for some $f : A \to B$ in $\mathbb{A}$ and that dually all arrows $B^\bullet \to A^\bullet$ are of the form $f^\bullet$, further characterising our analogue of the dualizable morphisms.

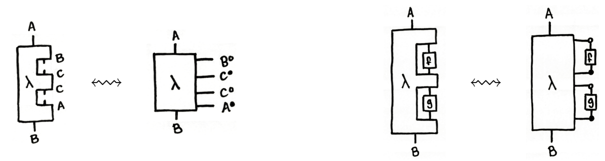
In light of this, we suggest that teleological categories are a shadow of the fact that A° is formally left adjoint to $A^\bullet$ in $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$. We also point out that teleological categories do not contain enough of the relevant structure to prove Lemmas 5 and 6, which require the unit of the formal adjunction between A° and $A^\bullet$ as well as the counit.

4 Comb Diagrams

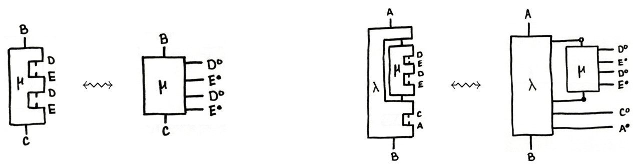
In this section we discuss comb diagrams in the free cornering. The basic idea is that we would like to have *higher-order* diagrams for our monoidal categories, pictured below on the left. Supplying the appropriate first-order string diagrams to a higher-order diagram results in a first-order diagram, pictured below on the right:

These higher-order diagrams have been called *(right) comb diagrams* due to their appearance.

In the free cornering of a monoidal category $\mathbb{A}$, elements of right-•○-alternating cell-sets are a good notion of right comb diagram, with the alternation depth corresponding to the number of gaps between the teeth:



Lemma 2 tells us that this notion of comb diagram coincides with the notion of comb diagram developed by Román in the more general framework of open diagrams [19]. The free cornering admits common comb diagram operations beyond inserting morphisms into the gaps. First, we may insert a comb diagram into one of the gaps to form another comb diagram:



Next, following Remarks 1 and 2 there is an dual notion of *left comb diagrams* in the free cornering corresponding to the left-○•-alternating cell-sets. In certain cases it makes sense to compose a right comb diagram with a left comb diagram by interleaving their teeth. The free cornering supports this as well:

Thus, the free cornering is a natural setting in which to work with comb diagrams in a monoidal category.

References

[1] G. Boisseau (2020): *String Diagrams for Optics*. In Zena M. Ariola, editor: *5th International Conference on Formal Structures for Computation and Deduction, FSCD 2020, June 29-July 6, 2020, Paris, France (Virtual Conference)*, *LIPIcs* 167, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, pp. 17:1–17:18, doi:10.4230/LIPIcs.FSCD.2020.17.

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A.7 Concurrent Process Histories and Resource Transducers

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CONCURRENT PROCESS HISTORIES AND RESOURCE TRANSDUCERS

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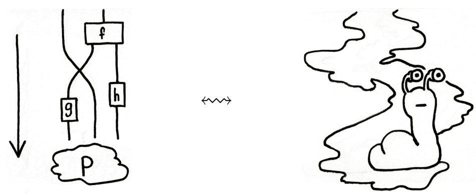
Abstract. We identify the algebraic structure of the material histories generated by concurrent processes. Specifically, we extend existing categorical theories of resource convertibility to capture concurrent interaction. Our formalism admits an intuitive graphical presentation via string diagrams for proarrow equipments. We also consider certain induced categories of resource transducers, which are of independent interest due to their unusual structure.

1. Introduction

Concurrent systems are abundant in computing, and indeed in the world at large. Despite the large amount of attention paid to the modelling of concurrency in recent decades (e.g., [Hoa78, Mil80, Pet66, Mil99, Abr14]), a canonical mathematical account has yet to emerge, and the basic structure of concurrent systems remains elusive.

In this paper we present a basic structure that captures what we will call the *material* aspect of concurrent systems: As a process unfolds in time it leaves behind a material history of effects on the world, like the way a slug moving through space leaves a trail of slime. This slime is captured in a natural way by *resource theories* in the sense of [CFS16], in which morphisms of symmetric monoidal categories — conveniently expressed as string diagrams — are understood as transformations of resources.



From the resource theoretic perspective, objects of a symmetric monoidal category are understood as collections of resources, with the unit object denoting the empty collection and the tensor product of two collections consisting of their combined contents. Morphisms are understood as ways to transform one collection of resources into another, which may be

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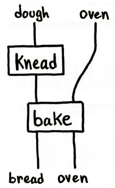
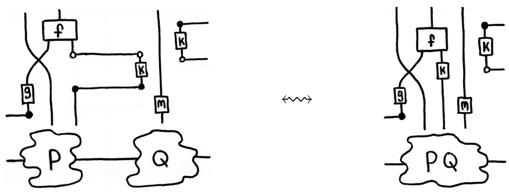
combined sequentially via composition, and in parallel via the tensor product. For example, the process of baking bread might generate the following material history:



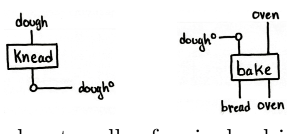
meaning that the baking process involved kneading dough and baking it in an oven to obtain bread (and also the oven).

This approach to expressing the material history of a process has many advantages: It is general, in that it assumes minimal structure; canonical, in that monoidal categories are well-studied as mathematical objects; and relatively friendly, as it admits an intuitive graphical calculus (string diagrams). However, it is unable to capture the interaction between components of a concurrent process. For example, consider our hypothetical baking process and suppose that the kneading and baking of the dough are handled by separate subsystems, with control of the dough being handed to the baking subsystem once the kneading is complete. Such interaction of parts is a fundamental aspect of concurrency, but is not expressible in this framework — we can only describe the effects of the system as a whole.

We remedy this by extending a given resource theory to allow the decomposition of material histories into concurrent components. Specifically, we augment the string diagrams for symmetric monoidal categories with *corners*, through which resources may flow between different components of a transformation.



Returning to our baking example, we might express the material history of the kneading and baking subsystems *separately* with the following diagrams, which may be composed horizontally to obtain the material history of the baking process as a whole.



These augmented diagrams denote cells of a single-object double category constructed from the original resource theory. The corners make this double category into a proarrow equipment, which turns out to be all the additional structure we need in order to express concurrent interaction. From only this structure, we obtain a theory of exchanges — a sort of minimal system of behavioural types — that conforms to our intuition about how such things ought to work remarkably well.

Our approach to these concurrent material histories retains the aforementioned advantages of the resource-theoretic perspective: We lose no generality, since our construction applies to any resource theory; It is canonical, with proarrow equipments being a fundamental structure in formal category theory — although not usually seen in such concrete circumstances; Finally, it remains relatively friendly, since the string diagrams for monoidal categories extend in a natural way to string diagrams for proarrow equipments [Mye16].

Every single-object double category defines two monoidal categories: one composed of cells with trivial left and right boundary, and one composed of cells with trivial top and bottom boundary. For the double category obtained by adding corners to a resource theory the induced monoidal categories are, respectively, the resource theory itself and a category of *resource transducers* — being an alternative interpretation of concurrent transformations that neither begin nor end with any resources. This category of resource transducers is rich in structure, exhibiting unusual features that make it an interesting object of study in its own right. We establish some elementary properties of this category and axiomatize it directly — that is, we give a monoidal signature and a collection of equations that characterize the category of resource transducers.

This paper is an extended version of [Nes21b], including additional examples and an exploration of the aforementioned categories of resource transducers.

1.1. **Contributions and Related Work.** *Related Work.* Monoidal categories are ubiquitous — if often implicit — in theoretical computer science. An example from the theory of concurrency is [MM90], in which monoidal categories serve a purpose similar to their purpose here. String diagrams for monoidal categories seem to have been invented independently a number of times, but until recently were uncommon in printed material due to technical limitations. The usual reference is [JS91]. We credit the resource-theoretic interpretation of monoidal categories and their string diagrams to [CFS16]. Double categories first appear in [Ehr63]. Free double categories are considered in [DP02] and again in [FPP08]. The idea of a proarrow equipment first appears in [Woo82], albeit in a rather different form. Proarrow equipments have subsequently appeared under many names in formal category theory (see e.g., [Shu08, GP04]). String diagrams for double categories and proarrow equipments are treated precisely in [Mye16]. We have been inspired by work on message passing and behavioural types, in particular [CP09], from which we have adopted our notation for exchanges.

Contributions. The main contribution of this paper is the resource-theoretic interpretation of the free cornering and the observation that it captures the structure of concurrent process histories. Other contributions concern the categorical structure of the free cornering of a resource theory: we show that it has *crossing cells* and is consequently a monoidal double category in Lemma 4.5 and Lemma 4.7, argue that the vertical cells are the original monoidal category in Proposition 4.4, show that the induced monoidal category of horizontal cells can be understood as a category of *resource transducers*, and establish Lemma 6.2, Lemma 6.3, Observation 6.4, Lemma 6.5, Lemma 6.6, and Proposition 6.8 — all of which concern the structure of this category of horizontal cells. Finally, we give an axiomatization of the category of horizontal cells in terms of equations over a monoidal signature in Section 7. The original contributions of this paper over [Nes21b] are Lemma 6.2, Lemma 6.5, Lemma 6.6, Proposition 6.8, and the axiom scheme of Section 7.

1.2. **Organization and Prerequisites.** *Prerequisites.* This paper is largely self-contained, but we assume some familiarity with category theory, in particular with monoidal categories and their string diagrams. Some good references are [Mac71, Sel10, FS19].
Organization. In Section 2 we review the resource-theoretic interpretation of symmetric monoidal categories. We continue by reviewing the theory of double categories in Section 3, specialized to the single object case. In Section 4 we recall the notion of proarrow equipment, introduce the free cornering of a resource theory, and exhibit the existence of crossing cells in the free cornering. In Section 5 we show how the free cornering of a resource theory inherits its resource-theoretic interpretation while enabling the concurrent decomposition of resource transformations. In Section 6 we consider the category of resource transducers and investigate its structure, and in Section 7 we give an axiom scheme for it. In Section 8 we conclude and consider directions for future work.

2. Monoidal Categories as Resource Theories

Symmetric strict[1] monoidal categories can be understood as theories of resource transformation. Objects are interpreted as collections of resources, with $A \otimes B$ the collection consisting of both A and B, and I the empty collection. Arrows $f : A \to B$ are understood as ways to transform the resources of A into those of B. We call symmetric strict monoidal categories *resource theories* when we have this sort of interpretation in mind.

For example, let $\mathfrak{B}$ be the free symmetric strict monoidal category with generating objects

$$\{\texttt{bread}, \texttt{dough}, \texttt{water}, \texttt{flour}, \texttt{oven}\}$$

and with generating arrows

$$\texttt{mix} : \texttt{water} \otimes \texttt{flour} \to \texttt{dough} \qquad\qquad \texttt{knead} : \texttt{dough} \to \texttt{dough}$$

$$\texttt{bake} : \texttt{dough} \otimes \texttt{oven} \to \texttt{bread} \otimes \texttt{oven}$$

subject to no equations. $\mathfrak{B}$ can be understood as a resource theory of baking bread. The arrow `mix` represents the process of combining water and flour to form a bread dough, `knead` represents kneading dough, and `bake` represents baking dough in an oven to obtain bread (and an oven).

The structure of symmetric strict monoidal categories provides natural algebraic scaffolding for composite transformations. For example, consider the following arrow of $\mathfrak{B}$:

$$(\texttt{bake} \otimes 1_{\texttt{dough}}); (1_{\texttt{bread}} \otimes \sigma_{\texttt{oven},\texttt{dough}}; \texttt{bake})$$

of type

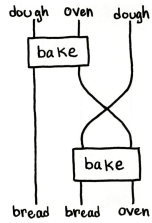
$$\texttt{dough} \otimes \texttt{oven} \otimes \texttt{dough} \to \texttt{bread} \otimes \texttt{bread} \otimes \texttt{oven}$$

where $\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ is the braiding. This arrow describes the transformation of two units of dough into loaves of bread by baking them one after the other in an oven.

It is often more intuitive to write composite arrows like this as string diagrams: Objects are depicted as wires, and arrows as boxes with inputs and outputs. Composition is represented by connecting output wires to input wires, and we represent the tensor product of two morphisms by placing them beside one another. Finally, the braiding is represented

[1]We work with strict monoidal categories for the sake of convenience and readability. We expect the present development to apply equally well to the general case, and if pressed would appeal to the coherence theorem for monoidal categories [Mac71].

by crossing the wires involved. For the morphism discussed above, the corresponding string diagram is:

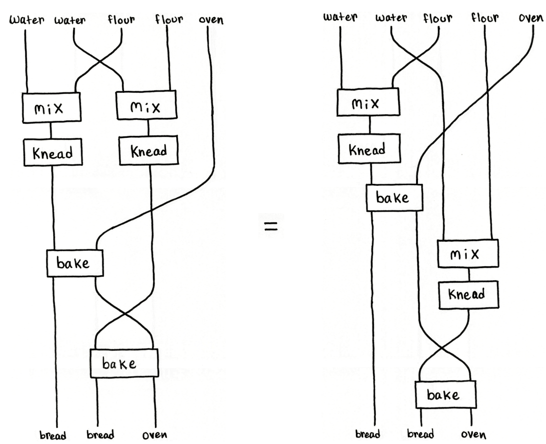


Notice how the topology of the diagram captures the logical flow of resources.

Given a pair of parallel arrows $f, g : A \to B$ in some resource theory, both f and g are ways to obtain B from A, but they may not have the same effect on the resources involved. We explain by example: Consider the parallel arrows $1_{\texttt{dough}}, \texttt{knead} : \texttt{dough} \to \texttt{dough}$ of $\mathfrak{B}$. Clearly these should not be understood to have the same effect on the dough in question, and this is reflected in $\mathfrak{B}$ by the fact that they are not made equal by its axioms. Similarly, `knead` and $\texttt{knead} \circ \texttt{knead}$ are not equal in $\mathfrak{B}$, which we understand to mean that kneading dough twice does not have the same effect as kneading it once, and that in turn any `bread` produced from twice-kneaded dough will be different from once-kneaded bread in our model.

Consider a hypothetical resource theory constructed from $\mathfrak{B}$ by imposing the equation $\texttt{knead} \circ \texttt{knead} = \texttt{knead}$. In this new setting we understand kneading dough once to have the same effect as kneading it twice, three times, and so on, because the corresponding arrows are all equal. Of course, the sequence of events described by `knead` is not the one described by $\texttt{knead} \circ \texttt{knead}$: In the former the dough has been kneaded only once, while in the latter it has been kneaded twice. The equality of the two string diagrams indicates that these two different processes would have the same effect on the dough involved. We adopt as a general principle in our design and understanding of resource theories that transformations should be equal as morphisms if and only if they have the same effect on the resources involved.

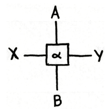
For the sake of further illustration, observe that by naturality of the braiding maps the following two resource transformations are equal in $\mathfrak{B}$:



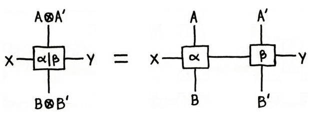
Each transformation gives a method of baking two loaves of bread. On the left, two batches of dough are mixed and kneaded before being baked one after the other. On the right, first one batch of dough is mixed, kneaded and baked and only then is the second batch mixed, kneaded, and baked. Their equality tells us that, according to $\mathfrak{B}$, the two procedures will have the same effect, resulting in the same bread when applied to the same ingredients with the same oven.

3. Single-Object Double Categories

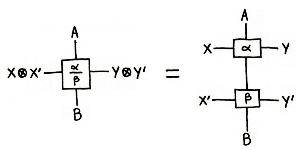
In this section we set up the rest of our development by presenting the theory of *single-object double categories*, being those double categories $\mathbb{D}$ with exactly one object. In this case $\mathbb{D}$ consists of a *horizontal edge monoid* $\mathbb{D}_H = (\mathbb{D}_H, \otimes, I)$, a *vertical edge monoid* $\mathbb{D}_V = (\mathbb{D}_V, \otimes, I)$, and a collection of *cells*



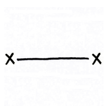
where $A, B \in \mathbb{D}_H$ and $X, Y \in \mathbb{D}_V$. Given cells α, β where the right boundary of α matches the left boundary of β we may form a cell $\alpha|\beta$ — their *horizontal composite* — and similarly if the bottom boundary of α matches the top boundary of β we may form $\frac{\alpha}{\beta}$ — their *vertical composite* — with the boundaries of the composite cell formed from those of the component cells using $\otimes$. We depict horizontal and vertical composition, respectively, as in:



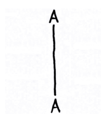
and



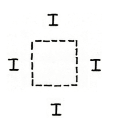
Horizontal and vertical composition of cells are required to be associative and unital. We omit wires of sort I in our depictions of cells, allowing us to draw horizontal and vertical identity cells, respectively, as in:

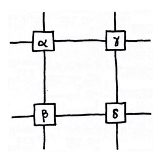


and

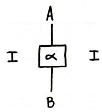


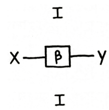
Finally, the horizontal and vertical identity cells of type I must coincide — we write this cell as $\square_I$ and depict it as empty space, see below on the left — and vertical and horizontal composition must satisfy the interchange law. That is, $\frac{\alpha}{\beta}|\frac{\gamma}{\delta} = \frac{\alpha|\gamma}{\beta|\delta}$, allowing us to unambiguously interpret the diagram below on the right:



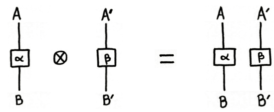


Every single-object double category $\mathbb{D}$ defines strict monoidal categories $\mathbf{V}\mathbb{D}$ and $\mathbf{H}\mathbb{D}$, consisting of the cells for which the $\mathbb{D}_H$ and $\mathbb{D}_V$ valued boundaries respectively are all I, as in:



and


That is, the collection of objects of $\mathbf{V}\mathbb{D}$ is $\mathbb{D}_H$, composition in $\mathbf{V}\mathbb{D}$ is vertical composition of cells, and the tensor product in $\mathbf{V}\mathbb{D}$ is given by horizontal composition:

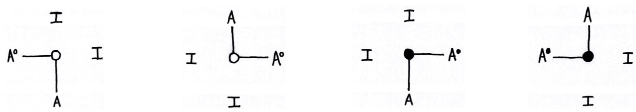


In this way, $\mathbf{V}\mathbb{D}$ forms a strict monoidal category, which we call the category of *vertical cells* of $\mathbb{D}$. Similarly, $\mathbf{H}\mathbb{D}$ is also a strict monoidal category (with collection of objects $\mathbb{D}_V$) which we call the *horizontal cells* of $\mathbb{D}$.

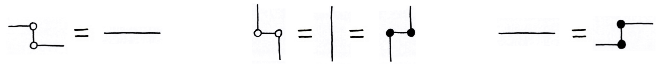
4. Cornerings and Crossings

In this section we introduce the free cornering of a resource theory, our primary technical device, and show that the free cornering contains special crossing cells with nice formal properties. We begin by recalling the notion of proarrow equipment, specialised to the case of single-object double categories:

Definition 4.1. Let $\mathbb{D}$ be a single-object double category. $\mathbb{D}$ is called a *proarrow equipment* in case for each $A \in \mathbb{D}_H$ there are distinguished elements A° and $A^\bullet$ of $\mathbb{D}_V$ along with distinguished cells of $\mathbb{D}$:



called $\circ$-*corners* and $\bullet$-*corners* respectively, which satisfy the *yanking equations*:



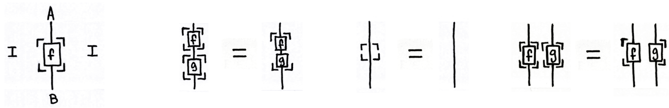
Tersely, the free cornering of a resource theory is the proarrow equipment obtained by freely adding corner cells. Explicitly, we define:

Definition 4.2. Let $\mathbb{A}$ be a resource theory. Then the monoid $\mathbb{A}^{\circ\bullet}$ of $\mathbb{A}$-*valued exchanges* is defined by $\mathbb{A}^{\circ\bullet} = (\mathbb{A}_0 \times \{\circ, \bullet\})^*$. That is, $\mathbb{A}^{\circ\bullet}$ is the free monoid on the set $\mathbb{A}_0 \times \{\circ, \bullet\}$ of polarized objects of $\mathbb{A}$, whose elements we write A° and $A^\bullet$. Intuitively, elements of $\mathbb{A}^{\circ\bullet}$ describe a sequence of resources moving between participants in the exchange, where A° denotes an instance of A moving from left to right, and $A^\bullet$ denotes an instance of A moving from right to left (see Section 5).

Now the free cornering is given as follows:

Definition 4.3. Let $\mathbb{A}$ be a resource theory. Then the *free cornering of* $\mathbb{A}$, written $\ulcorner\mathbb{A}\urcorner$, is the free single-object double category determined by the following data:

- The horizontal edge monoid $\ulcorner \mathbb{A} \urcorner_H = (\mathbb{A}_0, \otimes, I)$ is given by the objects of $\mathbb{A}$.
- The vertical edge monoid $\ulcorner \mathbb{A} \urcorner_V = \mathbb{A}^{\circ\bullet}$ is the monoid of $\mathbb{A}$-valued exchanges.
- The generating cells consist of corners for each object A of $\mathbb{A}$ as in Definition 4.1, subject to the yanking equations, along with a vertical cell $\ulcorner f \urcorner$ for each morphism $f : A \to B$ of $\mathbb{A}$ subject to equations as in:



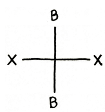
For a precise development of free double categories see [FPP08]. In brief: cells are formed from the generating cells by horizontal and vertical composition, subject to the axioms of a double category in addition to any generating equations. We call this the "free" cornering both because it is freely generated, and because we imagine there is an adjunction relating proarrow equipments and arbitrary double categories under which $\ulcorner \mathbb{A} \urcorner$ is "free" in a more principled sense. We leave the construction of such an adjunction for future work.

An important property of the free cornering is that the vertical cells are the original resource theory:

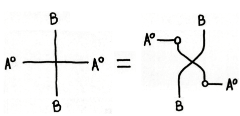
Proposition 4.4. *There is an isomorphism of categories* $\mathbf{V}\ulcorner \mathbb{A} \urcorner \cong \mathbb{A}$.

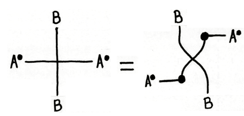
Proof. Intuitively $\mathbf{V}\ulcorner \mathbb{A} \urcorner \cong \mathbb{A}$ because in a composite vertical cell every wire bent by a corner must eventually be un-bent by the matching corner, which by yanking is the identity. The only other generators are the cells $\ulcorner f \urcorner$, and so any vertical cell in $\ulcorner A \urcorner$ can be written as $\ulcorner g \urcorner$ for some morphism g of $\mathbb{A}$. A more rigorous treatment of corner cells can be found in [Mye16], to the same effect. □

Before we properly explain our interest in $\ulcorner \mathbb{A} \urcorner$ we develop a convenient bit of structure: *crossing cells.* For each B of $\ulcorner \mathbb{A} \urcorner_H$ and each X of $\ulcorner \mathbb{A} \urcorner_V$ we define a cell

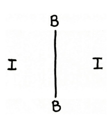


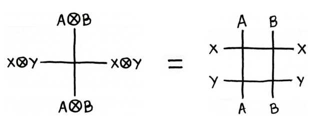
of $\ulcorner \mathbb{A} \urcorner$ inductively as follows: In the case where X is A° or $A^\bullet$, respectively, define the crossing cell as in the diagrams below on the left and right, respectively:





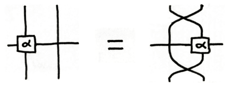
in the case where X is I, define the crossing cell as in the diagram below on the left, and in the composite case define the crossing cell as in the diagram below on the right:



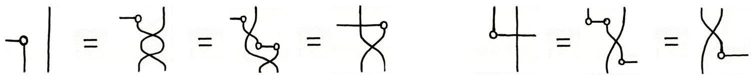


We prove a technical lemma:

Lemma 4.5. *For any cell α of $\ulcorner\mathbb{A}\urcorner$ we have*

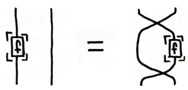


Proof. By structural induction on cells of $\ulcorner\mathbb{A}\urcorner$. For the $\circ$-corners we have:

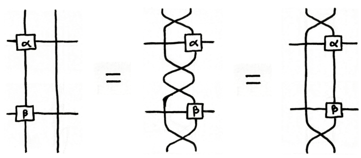


and for the $\bullet$-corners, similarly:

the final base cases are the $\ulcorner f \urcorner$ maps:



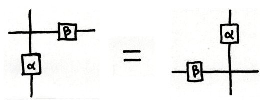
There are two inductive cases. For vertical composition, we have:



Horizontal composition is similarly straightforward, and the claim follows by induction. □

From this we obtain a "non-interaction" property of our crossing cells, similar to the naturality of braiding in symmetric monoidal categories:

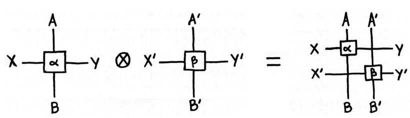
Corollary 4.6. *For cells α of $\mathbf{V}\ulcorner\mathbb{A}\urcorner$ and β of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$, the following equation holds in $\ulcorner\mathbb{A}\urcorner$:*



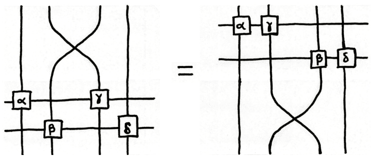
These crossing cells greatly aid in the legibility of diagrams corresponding to cells in $\ulcorner\mathbb{A}\urcorner$, but also tell us something about the categorical structure of $\ulcorner\mathbb{A}\urcorner$, namely that it is a monoidal double category in the sense of [Shu10]:

Lemma 4.7. *If $\mathbb{A}$ is a symmetric strict monoidal category then $\ulcorner\mathbb{A}\urcorner$ is a monoidal double category. That is, $\ulcorner\mathbb{A}\urcorner$ is a pseudo-monoid object in the strict 2-category* $\mathsf{VDblCat}$ *of double categories, lax double functors, and vertical transformations.*

Proof. We give the action of the tensor product on cells:



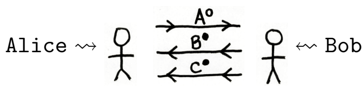
This defines a pseudofunctor, with the component of the required vertical transformation given by exchanging the two middle wires as in:



Notice that $\otimes$ is strictly associative and unital, in spite of being only pseudo-functorial. $\square$

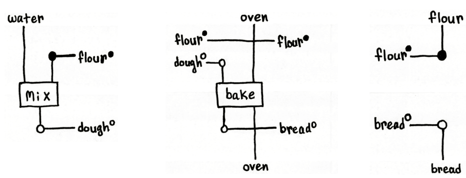
5. Concurrency Through Cornering

We proceed to extend the resource-theoretic interpretation of some symmetric strict monoidal category $\mathbb{A}$ to its free cornering $\ulcorner\mathbb{A}\lrcorner$. We interpret elements of $\ulcorner\mathbb{A}\lrcorner_V = \mathbb{A}^{\circ\bullet}$ as $\mathbb{A}$-*valued exchanges*. Each exchange $X_1 \otimes \cdots \otimes X_n$ involves a left participant and a right participant giving each other resources in sequence, with A° indicating that the left participant should give the right participant an instance of A, and $A^\bullet$ indicating the opposite. For example say the left participant is `Alice` and the right participant is `Bob`. Then we can picture the exchange $A^\circ \otimes B^\bullet \otimes C^\bullet$ as:

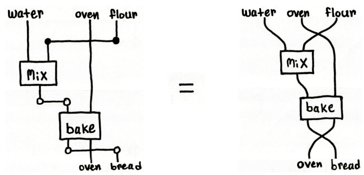


Think of these exchanges as happening *in order*. For example the exchange pictured above demands that first `Alice` gives `Bob` an instance of A, then `Bob` gives `Alice` an instance of B, and then finally `Bob` gives `Alice` an instance of C.

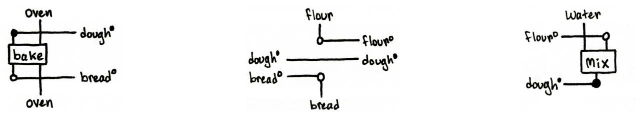
We interpret cells of $\ulcorner\mathbb{A}\lrcorner$ as *concurrent transformations*. Each cell describes a way to transform the collection of resources given by the top boundary into that given by the bottom boundary, via participating in $\mathbb{A}$-valued exchanges along the left and right boundaries. For example, consider the following cells of $\ulcorner\mathfrak{B}\lrcorner$:



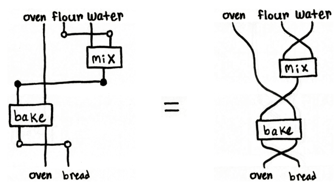
From left to right, these describe: A procedure for transforming `water` into nothing by `mixing` it with `flour` obtained by exchange along the right boundary, then sending the resulting `dough` away along the right boundary; A procedure for transforming an `oven` into an `oven`, receiving `flour` along the right boundary and sending it out the left boundary, then receiving `dough` along the left boundary, which is `bake`d in the `oven`, with the resulting `bread` sent out along the right boundary; Finally, a procedure for turning `flour` into `bread` by giving it away and then receiving `bread` along the left boundary. When we compose these concurrent transformations horizontally in the evident way, they give a transformation of resources in the usual sense, i.e., a morphism of $\mathbb{A} \cong \mathbf{V}\ulcorner\mathbb{A}\lrcorner$:



We understand equality of cells in $\ulcorner\mathbb{A}\lrcorner$ much as we understand equality of morphisms in a resource theory: two cells should be equal in case the transformations they describe would have the same effect on the resources involved. In this way, cells of $\ulcorner\mathbb{A}\lrcorner$ allow us to break a transformation into many concurrent parts. Note that with the crossing cells, it is possible for cells that are not immediately adjacent to exchange resource across the cells in between them. In the above example, `flour` is sent from the rightmost cell to the leftmost cell across the middle cell. This makes the double-categorical structure less constraining that it may seem at first. For example we might rearrange our previous example into the following horizontally composable cells of $\ulcorner\mathfrak{B}\lrcorner$:



When composed, we obtain a similar morphism of $\mathbb{A}$:



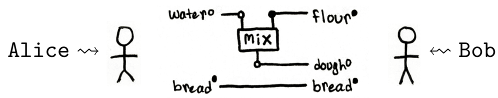
It is worth mentioning that the difference between `oven` $\otimes$ `flour` $\otimes$ `water` and `water` $\otimes$ `oven` $\otimes$ `flour` is negligible since any permutation of a collection of resources is naturally isomorphic to the original collection as an object of $\mathbb{A}$.

6. Horizontal Cells as Resource Transducers

If $\mathbb{A}$ is a resource theory, then the category $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ of horizontal cells of the free cornering can be understood as a category of ($\mathbb{A}$-valued) *resource transducers*.[2] Specifically, recall our interpretation of $\mathbb{A}^{\circ\bullet} = (\mathbf{H}\ulcorner\mathbb{A}\lrcorner)_0$ as $\mathbb{A}$-valued exchanges, in which two parties `Alice` and `Bob` must supply or retreive the resources involved in the exchange in the order specified, with who gives whom what determined by the polarity of the resources (see Section 5). Let $h : X \to Y$ be an arrow of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$. We can understand h as a machine operated by a left and right participant, again called `Alice` and `Bob` respectively. To operate the machine, `Alice` must play the left hand role of the domain exchange X and `Bob` must play the right hand

[2]The word "transducer" is derived from the latin words *trans* — meaning "across" and *ducere* — meaning "lead". We feel this is a good fit for the horizontal cells of the free cornering, which can be understood as a method of leading resources across the cell in question.

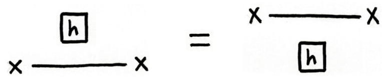
role of the codomain exchange Y. The morphism h describes the internals of the machine. For example, consider the following morphism of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$:



To operate the transducer, `Alice` must supply water and then receive bread, while `Bob` must supply flour, receive dough, and then supply bread. The effect of the machine is to mix the flour and water initially supplied into the dough `Bob` receives, and then to send the bread `Bob` supplies to `Alice`.

The transducer interpretation (along with our previous interpretation of the whole of $\ulcorner\mathbb{A}\urcorner$) makes $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ into a category of independent interest, and in this section we will study it. Compounding our interest is the fact that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ is rather unusual. It is of course a monoidal category (see Section 3) but fails to have any of the properties common to monoidal categories. Selinger's survey paper [Sel10] lists many such properties, for example:

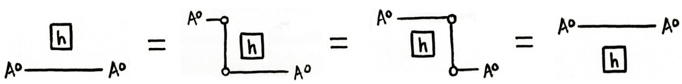
Definition 6.1 [Sel10]**.** A monoidal category is *spatial* in case for all objects X and arrows $h : I \to I$ we have:



It is easy to see that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ has the property of being spatial:

Lemma 6.2. $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ *is spatial.*

Proof. We use the fact that every symmetric monoidal category is spatial. The proof is by induction on the type X of the wire. If X is A° we have:



and so the spatial axiom holds. Similarly the spatial axiom holds if X is $A^\bullet$. If X is I the spatial axiom holds trivially, and the inductive case is immediate. □

We note that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ has *no other property* found in the aforementioned survey paper.

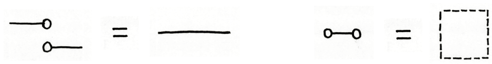
Much of the structure that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ does have consists of isomorphisms formed of corner cells. While isomorphic objects in $\mathbf{V}\ulcorner\mathbb{A}\urcorner \cong \mathbb{A}$ can be thought of as equivalent collections of resources — being freely transformable into each other — we understand isomorphic objects in $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ as *equivalent exchanges*. For example, there are many ways for `Alice` to give `Bob` an A and a B: Simultaneously, as $A \otimes B$; one after the other, as A and then B; or in the other order, as B and then A. While these are different sequences of events, they achieve the same thing, and are thus equivalent. Similarly, for `Alice` to give `Bob` an instance of I is equivalent to nobody doing anything. Formally, we have:

Lemma 6.3. *In* $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ *we have for any* A, B *of* $\mathbb{A}$*:*

(1) $I^\circ \cong I \cong I^\bullet$.
(2) $A^\circ \otimes B^\circ \cong B^\circ \otimes A^\circ$ *and* $A^\bullet \otimes B^\bullet \cong B^\bullet \otimes A^\bullet$.
(3) $(A \otimes B)^\circ \cong A^\circ \otimes B^\circ$ *and* $(A \otimes B)^\bullet \cong A^\bullet \otimes B^\bullet$

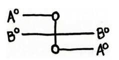
Proof. (1) For $I \cong I^\circ$, consider the $\circ$-corners corresponding to I:

we know that these satisfy the yanking equations:

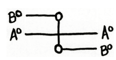


which exhibits an isomorphism $I \cong I^\circ$. Similarly, $I \cong I^\bullet$. Thus, we see formally that exchanging nothing is the same as doing nothing.

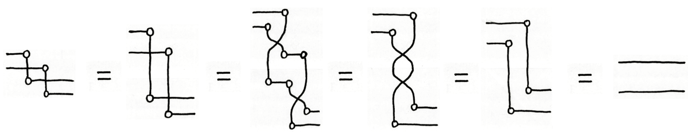
(2) The $\circ$-corner case is the interesting one: Define the components of our isomorphism to be:



and

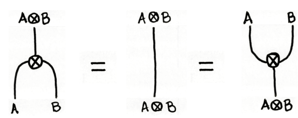


then for both of the required composites we have:

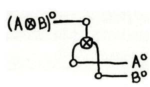


and so $A^\circ \otimes B^\circ \cong B^\circ \otimes A^\circ$. Similarly $A^\bullet \otimes B^\bullet \cong B^\bullet \otimes A^\bullet$. This captures formally the fact that if `Alice` is going to give `Bob` an A and a B, it doesn't really matter which order she does it in.

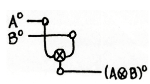
(3) Here it is convenient to switch between depicting a single wire of sort $A \otimes B$ and two wires of sort A and B respectively in our string diagrams. To this end, we allow ourselves to depict the identity on $A \otimes B$ in multiple ways, using the notation of [CS17]:



Then the components of our isomorphism $(A \otimes B)^\circ \cong A^\circ \otimes B^\circ$ are:

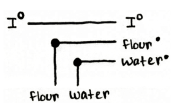


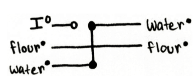
and

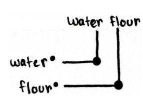


and, much as in (ii), it is easy to see that the two possible composites are both identity maps. Similarly, $(A \otimes B)^\bullet \cong (A^\bullet \otimes B^\bullet)$. This captures formally the fact that giving away a collection is the same thing as giving away its components. $\square$

For example, we should be able to compose the cells on the left and right below horizontally, since their right and left boundaries, respectively, indicate equivalent exchanges:



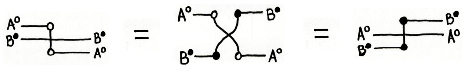




Our lemma tells us that in cases like this there will be a mediating isomorphism, as above in the middle, making composition possible.

It is worth noting that we *do not* have $A^\circ \otimes B^\bullet \cong B^\bullet \otimes A^\circ$:

Observation 6.4. *There is a morphism $d^\circ_\bullet : A^\circ \otimes B^\bullet \to B^\bullet \otimes A^\circ$ in one direction, defined by*

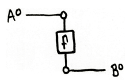


but there need not be a morphism in the other direction, and this is not in general invertible. In particular, $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ is monoidal, but need not be symmetric.

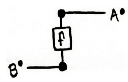
This observation reflects formally the intuition that if I receive some resources before I am required to send any, then I can send some of the resources that I receive. However, if I must send the resources first, this is not the case. In this way, $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ contains a sort of causal structure.

Next, we find that $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ contains the original resource theory $\mathbb{A}$ as a subcategory in two different ways, one for each polarity:

Lemma 6.5. *There are strong monoidal functors $(-)^\circ : \mathbb{A} \to \mathbf{H}\ulcorner\mathbb{A}\urcorner$ and $(-)^\bullet : \mathbb{A}^{\mathsf{op}} \to \mathbf{H}\ulcorner\mathbb{A}\urcorner$ defined respectively on $f : A \to B$ of $\mathbb{A}$ by:*

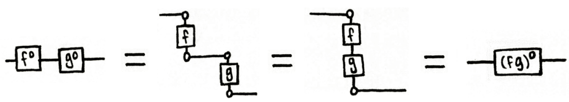


and

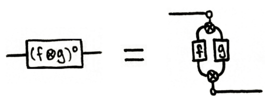


Further, each of these functors is full and faithful.

Proof. $(-)^\circ$ is functorial as in:



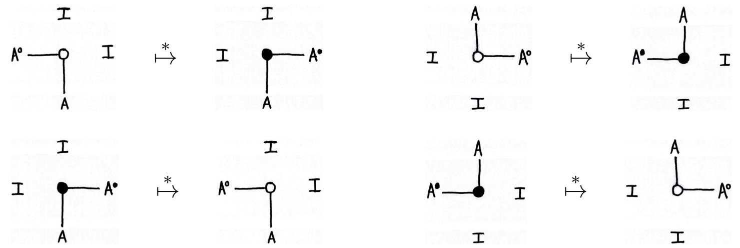
It interacts with the tensor product in $\mathbb{A}$ as in:



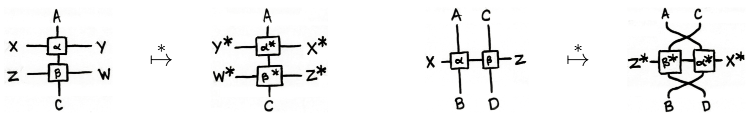
and is therefore strong monoidal as a consequence of Lemma 6.3. Further $(-)^\circ$ is faithful because $\ulcorner\mathbb{A}\urcorner$ is freely generated. It is full because of the coherence theorem of [Mye16], which implies that for any horizontal cell (morphism of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$) $h : A^\circ \to B^\circ$ we may yank all of the wires straight to obtain an equal morphism $f^\circ = h$ for some $f : A \to B$ of $\mathbb{A}$. Similarly, $(-)^\bullet$ is functorial, strong monoidal, full, and faithful. □

There is also a contravariant involution $(-)^* : \mathbf{H}\ulcorner\mathbb{A}\urcorner^{\mathsf{op}} \to \mathbf{H}\ulcorner\mathbb{A}\urcorner$. As an intermediate step we define an operation on the cells of $\ulcorner\mathbb{A}\urcorner$ as follows: For $A \in \mathbb{A}_0 = \mathbf{V}\ulcorner\mathbb{A}\urcorner$ let $A^* = A$. For $X \in \mathbb{A}^{\circ\bullet} = \mathbf{H}\ulcorner\mathbb{A}\urcorner$ define X^* inductively: $I^* = I$, $(A^\circ)^* = A^\bullet$, $(A^\bullet)^* = A^\circ$, and $(X \otimes Y)^* = X^* \otimes Y^*$. On cells of $\ulcorner\mathbb{A}\urcorner$ we also define $(-)^*$ inductively: The base cases are

$\ulcorner f \urcorner^* = \ulcorner f \urcorner$ along with:



and the inductive cases are:



Informally, α^* is the mirror image of α. It is easy to see that we have $\alpha^{**} = \alpha$ for any cell α of $\ulcorner \mathbb{A} \urcorner$. Thus, restricting $(-)^*$ to $\mathbf{H}\ulcorner \mathbb{A} \urcorner$ gives:

Lemma 6.6. *There is a contravariant involution* $(-)^* : \mathbf{H}\ulcorner \mathbb{A} \urcorner^{\mathsf{op}} \to \mathbf{H}\ulcorner \mathbb{A} \urcorner$ *with the property that* $(f \otimes g)^* = f^* \otimes g^*$.[3]

We discuss one final bit of structure in $\mathbf{H}\ulcorner \mathbb{A} \urcorner$, concerning the following arrows:

These are reminiscent of the string diagrams for rigid monoidal categories, these arrows make A° into the left dual of $A^\bullet$ (and so make $A^\bullet$ into the right dual of A°). However, $\mathbf{H}\ulcorner \mathbb{A} \urcorner$ is neither left nor right rigid: for example $A^\circ \otimes B^\bullet$ has neither a left nor right dual. It is natural to ask whether the arrows introduced above carry significant categorical structure. We give one answer, and in doing so connect the present work to Cockett and Pastro's logic of message passing [CP09]. In particular, the categorical semantics of this logic of message passing is given by *linear actegories*. If $\mathbb{A}$ is a symmetric monoidal category, a linear $\mathbb{A}$-actegory is given by a linearly distributive category $\mathbb{X}$ (see e.g., [CS17]) together with two functors:

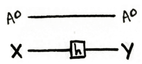
$$\circ : \mathbb{A} \times \mathbb{X} \to \mathbb{X} \qquad\qquad \bullet : \mathbb{A}^{\mathsf{op}} \times \mathbb{X} \to \mathbb{X}$$

such that $\circ$ is the paramaterised left adjoint of $\bullet$ — that is, for all $A \in \mathbb{A}_0$ we have $A \circ - \dashv A \bullet -$ — along with nine natural families of arrows subject to a large number of coherence conditions.

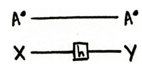
The category $\mathbf{H}\ulcorner \mathbb{A} \urcorner$ exhibits similar, if much simpler, structure. In particular the strong monoidal functors $(-)^\circ$ and $(-)^\bullet$ of Lemma 6.5 allow us to define $\circ : \mathbb{A} \times \mathbf{H}\ulcorner \mathbb{A} \urcorner \to \mathbf{H}\ulcorner \mathbb{A} \urcorner$ and $\bullet : \mathbb{A}^{\mathsf{op}} \times \mathbf{H}\ulcorner \mathbb{A} \urcorner \to \mathbf{H}\ulcorner \mathbb{A} \urcorner$ by $f \circ h = f^\circ \otimes h$ and $f \bullet h = f^\bullet \otimes h$. Echoing the definition of a linear actegory, we have:

[3]It is tempting to call this a *contravariant monoidal involution*, but in the covariant case a *monoidal involution* $(-)^\iota$ has the property that $(f \otimes g)^\iota = g^\iota \otimes f^\iota$, twisting the tensor product [Egg11]. We refrain from coining any new technical terms lest a "contravariant monoidal involution" turn out to be better suited to describing contravariant involutions that twist the tensor product instead of those that do not.

Lemma 6.7. *$\circ$ is the paramaterised left adjoint of $\bullet$. That is, for all $A \in \mathbb{A}$ the functors $A \circ - : \mathbf{H}{\ulcorner\mathbb{A}\lrcorner} \to \mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ and $A \bullet - : \mathbf{H}{\ulcorner\mathbb{A}\lrcorner} \to \mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ defined on $h : X \to Y$ by, respectively:*

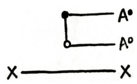


and

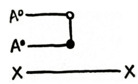


are such that $A \circ - \dashv A \bullet -$.

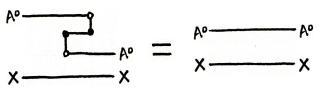
Proof. Fix an object $A \in \mathbb{A}$. We require natural families of morphisms $\eta_{A,X} : X \to A \bullet (A \circ X)$ and $\varepsilon_{A,X} : A \circ (A \bullet X) \to X$ in $\mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ that satisfy the triangle identities. Define $\eta_{A,X}$ and $\varepsilon_{A,X}$, respectively, by



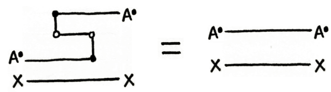
and



Now the triangle identities hold by repeated yanking, as in:



and



We therefore conclude that $A \circ - \dashv A \bullet -$, as required. □

Now, every monoidal category is a linearly distributive category (with both monoidal operations given by $\otimes$), and it turns out that $\mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ forms a (somewhat degenerate) linear actegory. Of the nine natural families of arrows required by the definition, four are accounted for by the isomorphisms of Lemma 6.3, a further four become identities in our setting, and the final one is given by the $d^{\circ}_{\bullet}$ morphisms from Observation 6.4. The coherence conditions all hold trivially. We record:

Proposition 6.8. *Let $\mathbb{A}$ be a resource theory. Then $\mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ is a linear actegory.*

This is intriguing insofar as it exhibits a formal connection between the free cornering of a resource theory and existing work on behavioural types. For example, the message-passing interpretation of classical linear logic presented by Wadler in [Wad14] corresponds to the message-passing interpretation of linear actegories in the special case of a *-autonomous category acting on itself (Example 4.2(4) of [CP09]). There may be an even stronger connection to the behavioural type interpretation of intuitionistic linear logic due to Caires and Pfenning [CP10], although here the connection to the logic of message passing is weaker (Example 4.2(1) of [CP09]). We leave the full investigation of these connections for future work.

7. Axioms for Resource Transducers

We have seen that the category of horizontal cells of the free cornering of a resource theory is an interesting object of study in its own right: it is a planar monoidal category that arises naturally and is different from those typically considered. In this section we give a direct presentation of $\mathbf{H}{\ulcorner\mathbb{A}\lrcorner}$ both to deepen our understanding of its structure and to facilitate its use as an example (or counterexample) in the future. While there are many axioms, they are mostly intuitive, and are conveniently organized into pairs by the contravariant involution $(-)^*$ of Lemma 6.6.

Let $\mathbb{A}$ be a resource theory. Define $\mathsf{T}(\mathbb{A})$ to be the free spatial strict monoidal category with the generating objects as in:

$$\frac{A \in \mathbb{A}_0}{A^\circ \quad \mathsf{obj}} \qquad\qquad \frac{A \in \mathbb{A}_0}{A^\bullet \quad \mathsf{obj}}$$

and the generating morphisms given by:

$$\frac{f : A \to B \in \mathbb{A}_1}{f^\circ : A^\circ \to B^\circ}\ \circ \qquad\qquad \frac{f : A \to B \in \mathbb{A}_1}{f^\bullet : B^\bullet \to A^\bullet}\ \bullet$$

$$\frac{A, B \in \mathbb{A}_0}{\triangleleft_{A,B} : (A \otimes B)^\circ \to A^\circ \otimes B^\circ}\ \triangleleft \qquad\qquad \frac{A, B \in \mathbb{A}_0}{\blacktriangleright_{A,B} : A^\bullet \otimes B^\bullet \to (A \otimes B)^\bullet}\ \blacktriangleright$$

$$\frac{A, B \in \mathbb{A}_0}{\triangleright_{A,B} : A^\circ \otimes B^\circ \to (A \otimes B)^\circ}\ \triangleright \qquad\qquad \frac{A, B \in \mathbb{A}_0}{\blacktriangleleft_{A,B} : (A \otimes B)^\bullet \to A^\bullet \otimes B^\bullet}\ \blacktriangleleft$$

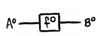
$$\frac{}{\circ\!\!- : I \to I^\circ}\ \circ\!\!- \qquad \frac{}{-\!\!\circ : I^\circ \to I}\ -\!\!\circ \qquad \frac{}{-\!\!\bullet : I^\bullet \to I}\ -\!\!\bullet \qquad \frac{}{\bullet\!\!- : I \to I^\bullet}\ \bullet\!\!-$$

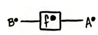
$$\frac{A, B \in \mathbb{A}_0}{\sigma^\circ_{A,B} : A^\circ \otimes B^\circ \to B^\circ \otimes A^\circ}\ \sigma\circ \qquad\qquad \frac{A, B \in \mathbb{A}_0}{\sigma^\bullet_{A,B} : A^\bullet \otimes B^\bullet \to B^\bullet \otimes A^\bullet}\ \sigma\bullet$$

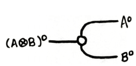
$$\frac{A \in \mathbb{A}_0}{\eta_A : I \to A^\bullet \otimes A^\circ}\ \eta \qquad\qquad \frac{A \in \mathbb{A}_0}{\varepsilon_A : A^\circ \otimes A^\bullet \to I}\ \varepsilon$$

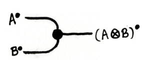
The rules $\circ$ and $\bullet$ correspond to the image of the functors from Lemma 6.5. All of $\triangleleft, \triangleright, \blacktriangleleft, \blacktriangleright, -\!\!\circ, \circ\!\!-, -\!\!\bullet, \bullet\!\!-, \sigma\circ, \sigma\bullet$ correspond to the isomorphisms of Lemma 6.3, and the η and ε rules correspond to the morphisms considered at the end of Section 6 that lead to Proposition 6.8.

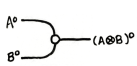
Before presenting the equations for $\mathsf{T}(\mathbb{A})$ we give the following string-diagrammatic conventions for our generators:

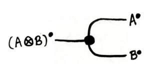
f° $\leftrightsquigarrow$ 

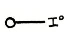
$f^\bullet$ $\leftrightsquigarrow$ 

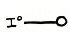
$\triangleleft_{A,B}$ $\leftrightsquigarrow$ 

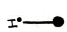
$\blacktriangleright_{A,B}$ $\leftrightsquigarrow$ 

$\triangleright_{A,B}$ $\leftrightsquigarrow$ 

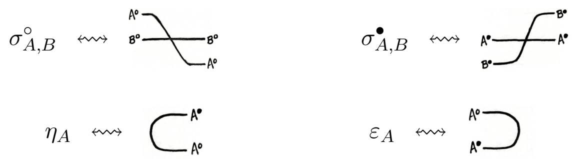
$\blacktriangleleft_{A,B}$ $\leftrightsquigarrow$ 

$\circ\!\!-$ $\leftrightsquigarrow$ 

$-\!\!\circ$ $\leftrightsquigarrow$ 

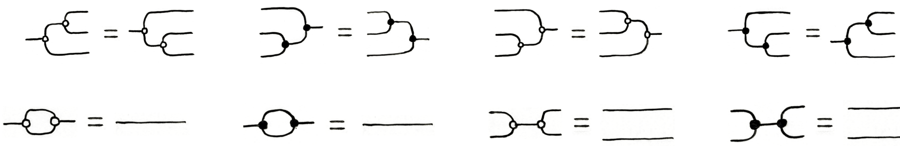
$-\!\!\bullet$ $\leftrightsquigarrow$ 

$\bullet\!\!-$ $\leftrightsquigarrow$ 



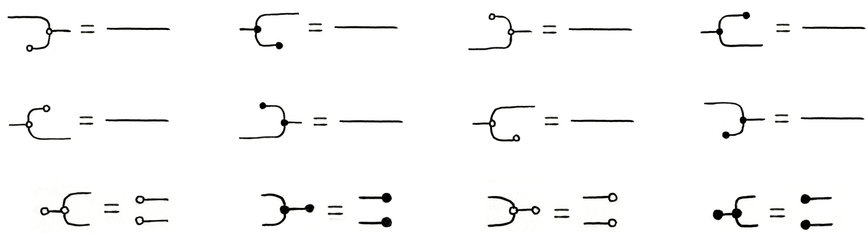
While it is initially difficult to keep track of the polarity of each wire, particularly in the diagrams for the $\sigma\circ$, $\sigma\bullet$, η, and ε morphisms, this is alleviated by the fact that resources may flow down but not up. Keeping this in mind allows us to omit any sort of directional information from the wires of our diagrams, which we feel makes them more readable.

Now, we impose the following equations in addition to those of a spatial strict monoidal category, and those inherited from $\mathbb{A}$. First, concerning the interaction of the $\triangleleft, \triangleright$ and $\blacktriangleleft, \blacktriangleright$ morphisms we require:

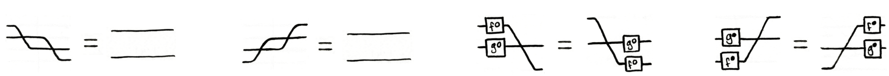


We note that this is a polarized version of the axioms for "dividers" and "gatherers" found in the SZX calculus [CHP19]. We continue with axioms concerning the interaction of the $\multimap$, $\circ\!\!-$ and $-\!\!\bullet$, $\bullet\!\!-$ morphisms:

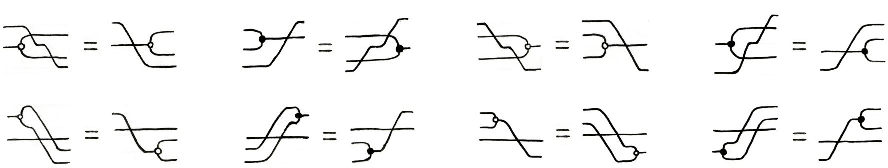
For the remaining interactions of $\triangleleft, \triangleright$, $\multimap$, $\circ\!\!-$ and $\blacktriangleleft, \blacktriangleright$, $-\!\!\bullet$, $\bullet\!\!-$ we require:



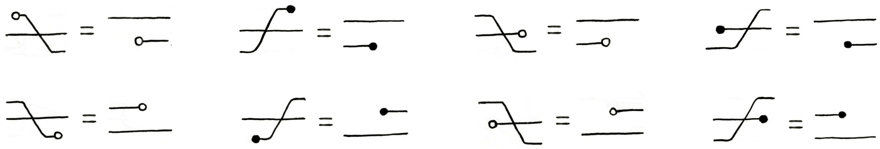
Next, the interaction between $\sigma\circ, \circ$ and $\sigma\bullet, \bullet$ is captured by:



For the interaction between $\sigma\circ, \triangleleft, \triangleright$ and $\sigma\bullet, \blacktriangleleft, \blacktriangleright$ we require:



and the interaction between $\sigma\circ, \multimap, \circ\!\!-$ and $\sigma\bullet, -\!\bullet, \bullet\!-$ is captured by:



For the interaction between $\eta, \varepsilon, \circ, \bullet$ we require:

The interaction between $\sigma\circ, \sigma\bullet, \eta, \varepsilon$ is captured by:

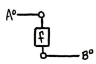
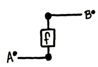
And for the interaction between $\eta, \varepsilon, \triangleleft, \triangleright, \blacktriangleleft, \blacktriangleright$ we ask that:

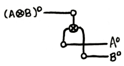
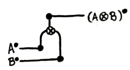
Finally, we require the following axioms concerning f° and $f^\bullet$:

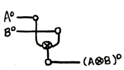
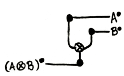
$$f^\circ g^\circ = (fg)^\circ \qquad \triangleleft\,(f^\circ \otimes g^\circ)\,\triangleright = (f \otimes g)^\circ \qquad (1_A)^\circ = 1_{A^\circ} \qquad \triangleright\,(\sigma_{A,B})^\circ \triangleleft = \sigma^\circ_{A,B}$$

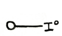
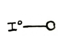
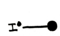
$$g^\bullet f^\bullet = (fg)^\bullet \qquad \blacktriangleleft\,(f^\bullet \otimes g^\bullet)\,\blacktriangleright = (f \otimes g)^\bullet \qquad (1_A)^\bullet = 1_{A^\bullet} \qquad \blacktriangleright\,(\sigma_{A,B})^\bullet \blacktriangleleft = \sigma^\bullet_{A,B}$$

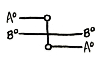
This concludes the presentation of $\mathsf{T}(\mathbb{A})$. We proceed to define a strict monoidal functor $M : \mathsf{T}(\mathbb{A}) \to \mathbf{H}\ulcorner\mathbb{A}\lrcorner$ on objects by $M(X) = X$ (since $\mathsf{T}(\mathbb{A})$ and $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ have the same objects) and on the generators by:

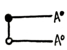
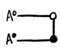
$M(f^\circ) =$  $\qquad M(f^\bullet) =$ 

$M(\triangleleft_{A,B}) =$  $\qquad M(\blacktriangleright_{A,B}) =$ 

$M(\triangleright_{A,B}) =$  $\qquad M(\blacktriangleleft_{A,B}) =$ 

$M(\circ\!\!-) =$  $\qquad M(\multimap) =$  $\qquad M(-\!\bullet) =$  $\qquad M(\bullet\!-) =$ 

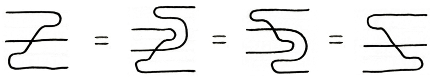
$M(\sigma^\circ_{A,B}) =$  $\qquad M(\sigma^\bullet_{A,B}) =$ 

$M(\eta_A) =$  $\qquad M(\varepsilon_A) =$ 

It is straightforward to verify that M is a strict monoidal functor. Additionally, we have:

Proposition 7.1. $M : \mathsf{T}(\mathbb{A}) \to \mathbf{H}\ulcorner\mathbb{A}\lrcorner$ *is full, faithful, and identity-on-objects.*

Proof. That M is full follows from the cohrerence theorem for string diagrams for proarrow equipments [Mye16]. Intuitively, every arrow of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ is either in the image of $(-)^\circ$ or $(-)^\bullet$, or is built out of corner cells and crossing cells. Every horizontal cell of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ that can be built out of only corner cells and does not decompose into multiple such cells is the image of one of the generators of $\mathsf{T}(\mathbb{A})$, and so we know that M is full. Perhaps surprising is that the horizontal cell $d^\circ_\bullet : A^\circ \otimes B^\bullet \to B^\bullet \otimes A^\circ$ of Observation 6.4 decomposes in this way, being the image under M of the following morphism in $\mathsf{T}(\mathbb{A})$:



To show that M is faithful is to show that the equations of $\mathsf{T}(\mathbb{A})$ capture all equations between horizontal cells of $\ulcorner\mathbb{A}\lrcorner$ when taken together with the equations of a spatial strict monoidal category. Recall that all of the equations of $\ulcorner\mathbb{A}\lrcorner$ are generated by the yanking equations, along with any equations of $\mathbb{A}$. The yanking equations are local, in that each instance of one of the yanking equations involves exactly two cells of $\ulcorner\mathbb{A}\lrcorner$, so we need only consider local interactions of cells of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ in our analysis. It is relatively straightforward to verify that the defining equations of $\mathsf{T}(\mathbb{A})$ are precisely the equations that arise in this way, and so M is faithful.[4] Finally, M is clearly identity-on-objects. □

It follows that our axiomatization of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$ is correct. We record:

Corollary 7.2. *There is an isomorphism of categories* $\mathbf{H}\ulcorner\mathbb{A}\lrcorner \cong \mathsf{T}(\mathbb{A})$.

8. Conclusions and Future Work

We have shown how to decompose the material history of a process into concurrent components by working in the free cornering of an appropriate resource theory. We have explored the structure of the free cornering in light of this interpretation and found that it is consistent with our intuition about how this sort of thing ought to work. We do not claim to have solved all problems in the modelling of concurrency, but we feel that our formalism captures the material aspect of concurrent systems very well.

We find it quite surprising that the structure required to model concurrent resource transformations is precisely the structure of a proarrow equipment. This structure is already known to be important in formal category theory, and we are appropriately intrigued by its apparent relevance to models of concurrency — a far more concrete setting than the usual context in which one encounters proarrow equipments!

Further, we have considered categories of resource transducers that are induced by our construction. We have identified some structure they do and do not exhibit, and have provided a more direct axiomatization of them. We are not aware of any categories with similar structure, which we feel makes these categories of resource transducers worthy of further study, and of potential value as a counterexample.

There are of course many directions for future work. For one, it would be nice to connect the development here to the wider literature on concurrent processes. An obstacle to this is that the free cornering does not allow us to express branching or recursion, both of which feature heavily in more general theories of process communication. If we assume that

[4]Given the large number of equations involved, it is of course possible that we have missed some. That said, we are reasonably confident that the equations we have given are complete in this sense.

our monoidal category $\mathbb{A}$ has binary coproducts then we may represent a limited sort of branching computation in which $(A+B)^\circ$ and $(A+B)^\bullet$ represent choices to be made by the left and right participant respectively, but this is less flexible than the protocol-level choice that one finds in e.g. session types or the nondeterminism of process calculi. We speculate that this is best approached through the "situated transition systems" introduced in [Nes21a], in which the concurrent resource transformations developed in [Nes21b] (which this paper extends) are used to augment the category of spans of reflexive graphs — interpreted as open transition systems [KSW97] — to generate material history over some resource theory as transitions unfold in time. Alternatively, one might impose additional structure on the free cornering to allow nondeterministic choice and repetition.

Another direction for future work is to pursue the connection with the message passing logic of Cockett and Pastro [CP09] (established in Proposition 6.8) and the wider programme of behavioural types influenced by linear logic including [Wad14] and [CP10]. Finally, the presence of proarrow equipments here is rather mysterious, and we wonder if some deeper reason for it might exist.

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A.8 Protocol Choice and Iteration for the Free Cornering

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Protocol Choice and Iteration for the Free Cornering

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Abstract

We extend the free cornering of a symmetric monoidal category, a double categorical model of concurrent interaction, to support branching communication protocols and iterated communication protocols. We validate our constructions by showing that they inherit significant categorical structure from the free cornering, including that they form monoidal double categories. We also establish some elementary properties of the novel structure they contain. Further, we give a model of the free cornering in terms of strong functors and strong natural transformations, inspired by the literature on computational effects.

Keywords: Category Theory, Concurrency, Double Categories, Computational Effects

1. Introduction

While there are many theories of concurrent computation, none may yet claim to be the *canonical* such theory. In the words of Abramsky [2]:

> It is too easy to cook up yet another variant of process calculus or algebra; there are too few constraints ... The mathematician André Weil apparently compared finding the right definitions in algebraic number theory — which was like carving adamantine rock — to making definitions in the theory of uniform spaces ... which was like sculpting with snow. In concurrency theory we are very much at the snow-sculpture end of the spectrum. We lack the kind of external reality ... which is hard and obdurate, and resistant to our definitions.

This motivates the search for categorical models of concurrency, with category theory playing the role of a suitably stubborn external reality against which to test our definitions. In this we adopt the perspective suggested by Maddy [21] on the relationship of category theory to set theory. Loosely, set theory serves among other things as a *generous arena* in which the full array of

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mathematical construction techniques are permissible, but with no way of discerning the mathematically promising structures from the rest (sculpting with snow). The role of category theory is to provide *essential guidance*, with the idea being that the mathematically promising structures are precisely those which fit smoothly into the category-theoretic landscape (carving rock). Whether or not category theory can provide this sort of essential guidance in the theory of concurrent computation has not yet been conclusively established, but the idea is a compelling one, and we work under the assumption that it can and will.

This paper concerns the free cornering of a symmetric monoidal category, a categorical model of concurrent interaction proposed by Nester [25, 27]. The model builds on the resource-theoretic interpretation of symmetric monoidal categories as a kind of process theory (see e.g., [10]) by defining the *free cornering* of a given symmetric monoidal category to be a certain double category in which the processes represented by the base are augmented with *corner cells*. The cells of the free cornering admit interpretation as *interacting processes*, with the corner cells embodying a notion of message passing. The corner cells are precisely what is required to make the free cornering into a *proarrrow equipment*, which is a kind structured double category that plays a fundamental role in formal category theory. We remark that for the structure of a proarrow equipment to coincide with a notion of message passing would seem to be essential guidance of the highest quality. That said, the free cornering is currently far from being a canonical model of concurrent computation. Much work remains to be done, and many connections remain unexplored.

More specifically, this paper addresses the connection of the free cornering to notions of *session type*. Roughly, session types are to communication protocols what data types are to structured data. Much like the way that data types constrain the possible data values a process operates on, session types constrain the behaviour of a process so that it conforms to the corresponding communication protocol. In the free cornering of a symmetric monoidal category, process interaction is governed by the *monoid of exchanges*, whose elements may be understood as a basic sort of session type. The communication protocols expressible in this way are those in which messages of a predetermined type are sent and received in a predetermined sequence. Viewed this way, the monoid of exchanges is missing a number of features that systems of session types are expected to have. In particular, *branching protocols* — in which one of the participants chooses which of two possible continuations of the current protocol will happen and the other participant must react — and *iterated protocols*, in which all or part of the protocol is carried out some number of times based on choices made by the participants.

In an effort to rectify the situation, we construct the *free cornering with choice* and *free cornering with iteration* of a distributive monoidal category. The free cornering with choice supports branching communication protocols in addition to those of the free cornering, and the free cornering with iteration supports iterated communication protocols in addition to those of the free cornering with choice. To ask that the base category is distributive monoidal is to ask that it supports a kind of sequential branching, analogous to the "if then else" statements present in many programming languages. In the free cornering with choice, this sequential branching structure interacts nontrivially with branching communication protocols, and is what allows a process to decide which branch of a protocol to select based on its inputs.

We prove some elementary results concerning the monoidal category of *horizontal cells* of the free cornering with choice and the free cornering with iteration. The objects of the category of horizontal cells correspond to communication protocols. We show that the category of horizontal cells of the free cornering with choice has binary products and coproducts given by branching communication protocols. Further, we characterize iterated communication protocols in the category of horizontal cells of the free cornering with iteration by showing that they arise as initial algebras (or dually, final coalgebras) of a suitable endofunctor, and that process iteration forms a monad (or dually, a comonad).

An important feature of the free cornering is the existence of well-behaved *crossing cells*, which among other things carry the structure of a *monoidal double category* in the sense of Shulman [33]. We extend the construction of crossing cells in the free cornering to construct crossing cells in the free cornering with choice and the free cornering with iteration, and show that these crossing cells remain well-behaved. It follows that the free cornering with choice and free cornering with iteration also form monoidal double categories. We view the presence of these well-behaved crossing cells as a kind of sanity check.

As an additional sanity check, we construct a model of the free cornering, and extend it to give a model of the free cornering with choice and the free cornering with iteration. Specifically, from a cartesian closed category we construct a double category of *stateful transformations*, in which cells are given by strong natural transformations between strong endofunctors on the base category (see e.g. [22]). This double category is a model of the free cornering in the sense that there is a structure-preserving double functor from the free cornering of the base category into the category of stateful transformations. We show that under additional assumptions on the base category, the double category of stateful transformations gives a model of the free cornering with choice and the free cornering with iteration in this sense. The existence of such a model is reassuring in that it tells us the axioms of the free cornering with choice and free cornering with iteration do not collapse. Strong functors play an important role in the categorical semantics of effectful computations (see e.g,[23, 29]), and so the double category of stateful transformations may be of independent interest.

Contributions. The central contributions of this paper are the construction of the free cornering with choice (Definitions 9 and 10) and the free cornering with iteration (Definitions 12 and 13). In this we include the results validating the two constructions, specifically the contents of Sections 3.3,3.4,4.2, and 4.3. A further contribution is the construction of the double category of stateful transformations together with the fact that it models the free cornering (Section 2.4), free cornering with choice (Section 3.5), and free cornering with iteration (Section 4.4) in the presence of suitable assumptions on the base category. Finally, Lemma 1 is a minor contribution to the theory of the free cornering (without choice or iteration): while in previous work on the free cornering it has been part of the definition of the crossing cells, in recapitulating the material on crossing cells we realised that it was in fact a consequence of the slightly weaker definition used here.

Related Work. Double categories first appear in [12]. Free double categories are considered in [11] and again in [13]. The idea of a proarrow equipment first appears in [38], albeit in a rather different form. Proarrow equipments have subsequently appeared under many names in formal category theory (see e.g., [32, 15]). The string diagrams for double categories and proarrow equipments that we will use without comment are given a detailed treatment in [24]. The free cornering was introduced in [25], and has been developed in [27, 26, 5]. Session types were introduced by Honda [17] and the idea has since been developed by a number of authors. While the purpose of this paper is to develop more sophisticated session types in the free cornering, we are primarily influenced not by the literature on session types after Honda but by the *logic of message passing* of Cockett and Pastro [8], in which process communication is modelled categorically by *linear actegories* (the semantics of a kind of augmented linear logic). A particular point of difference between these two lines of research is that in the logic of message passing, like in the free cornering, the protocols that our types describe are *two-sided*, requiring a left and right participant. In the literature on session types after Honda the protocol types are *one-sided*, and two participants must each conform to a session type *dual* to that of the other if they wish to interact. While this may seem like a large difference, it is purely formal. The connection between session types and linear logic is explored from slightly different angles in [36] and [7], and all of this seems to have been heavily influenced by the early work of Bellin and Scott [4]. In our use of distributive monoidal categories to model branching programs and datatypes we follow Walters [37]. The definitions of protocols in the model of stateful transformations are based on the theory of strong functors and monads for describing computational effects by Moggi [23]. Stateful transformations themselves are inspired by *stateful runners* [34], and *interaction laws* as described in [18].

Organisation. In Section 2 we give an introduction to single-object double categories (Section 2.1); recapitulate the construction of the free cornering of a symmetric monoidal category (Section 2.2); recall the construction of crossing cells in the free cornering along with certain properties of crossing cells (Section 2.3); and introduce the double category of stateful transformations and its relationship to the free cornering (Section 2.4). Section 3 concerns the free cornering with choice. We introduce distributive monoidal categories and discuss the way in which they model branching sequential processes (Section 3.1); introduce the free cornering with choice of a distributive monoidal category together with its interpretation (Section 3.2); establish a few elementary properties of the resulting single-object double category (Section 3.3); show that the construction of crossing cells in the free cornering extends to the free cornering with choice, and that the attendant properties of crossing cells hold in the larger setting (Section 3.4); and show that when the base cartesian closed category is distributive the category of stateful transformations gives a model of the free cornering with choice (Section 3.5). Section 4 concerns the free cornering with iteration, and its organisation is similar to that of Section 3. We introduce the free cornering with iteration of a distributive monoidal category together with its interpretation (Section 4.1); establish a few elementary properties of the resulting single-object double category, in particular that our notion of iteration is (co)monadic (Section 4.2); show that the construction of crossing cells in the free cornering with choice extends to the free cornering with iteration, and that the attendant properties of crossing cells conitnue to hold in the larger setting (Section 4.3); and show that when we consider only the part of the double category of stateful transformations given by certain *container functors* on the base category Set this gives a model of the free cornering with iteration (Section 4.4). We conclude and discuss a number of directions for future work in Section 5.

Prerequisites. We assume some familiarity with elementary category theory (see e.g., [20]), cartesian closed categories (see e.g., [19]), and in particular with symmetric monoidal categories and their string diagrams (see e.g., [31]). While knowledge of the theory of double categories would certainly be helpful, it is not strictly required, and we provide a brief technical introduction in Section 2.1 that covers everything we will need for our development. The sections concerning the double category of stateful transformations make heavy use of the notion of tensorial strength and the associated notion of strong natural transformation. We give the necessary definitions in Section 2.4, but prior familiarity would, of course, be helpful (see e.g., [9]).

2. The Free Cornering

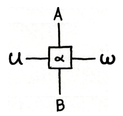
The aim of this section is to introduce the free cornering of a symmetric monoidal category. We begin by recapitulating some basic double category theory in the single-object case, which occupies Section 2.1. This done, in Section 2.2 we recapitulate the free cornering construction and its interactive interpretation. In Section 2.3 we recall the *crossing cells* of the free cornering. We recall certain important properties of the crossing cells, the continuing validity of which will serve as a kind of litmus test for the soundness of our notions of choice and iteration in the sequel. Finally, in Section 2.4 we construct a double category of *stateful transformations* over a cartesian closed category, and moreover show that it is a model of the free cornering of that category, possessing corner cells and crossing cells in an interesting fashion. This will serve as a running example throughout the paper.

Before we begin, we must briefly discuss strictness and notation. We write composition of arrows in a category in *diagrammatic order*. That is, the composite of $f : A \to B$ and $g : B \to C$ is written $fg : A \to C$. While we may write $g \circ f : A \to C$, we will *never* write $gf : A \to C$. Moreover, in this paper we consider only *strict* monoidal categories, and in our development the term "monoidal category" should be read as "strict monoidal category".

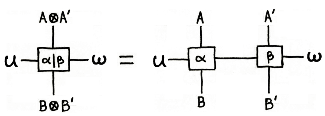
That said, we imagine that our results will hold in some form for arbitrary monoidal categories via the coherence theorem for monoidal categories [20]. Similarly, our double categories are what some authors call *strict double categories.* The braiding maps in a symmetric monoidal category will be written $\sigma_{A,B} : A \otimes B \to B \otimes A$. Further notational conventions will be introduced as needed.

2.1. Single-Object Double Categories

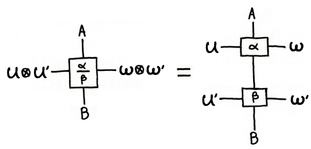
In this section we set up the rest of our development by recalling the theory of single-object double categories, being those double categories $\mathbb{D}$ with exactly one object. In this case $\mathbb{D}$ consists of a *horizontal edge monoid* $\mathbb{D}_H = (\mathbb{D}_H, \otimes, I)$, a *vertical edge monoid* $\mathbb{D}_V = (\mathbb{D}_V, \otimes, I)$, and a collection of *cells*



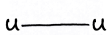
where $A, B \in \mathbb{D}_H$ and $U, W \in \mathbb{D}_V$. We write $\mathbb{D}(U\,{}^{A}_{B}\,W)$ for the *cell-set* of all such cells in $\mathbb{D}$, and write $\alpha : \mathbb{D}(U\,{}^{A}_{B}\,W)$ to indicate the membership of α in a cell-set. When $\mathbb{D}$ is clear from context, we write $(U\,{}^{A}_{B}\,W)$ instead of $\mathbb{D}(U\,{}^{A}_{B}\,W)$. Given cells $\alpha : (U\,{}^{A}_{B}\,V)$ and $\beta : (V\,{}^{A'}_{B'}\,W)$ for which the right boundary of α matches the left boundary of β we may form a cell $\alpha|\beta : (U\,{}^{A\otimes A'}_{B\otimes B'}\,W)$ – their *horizontal composite* – and similarly if the bottom boundary of $\alpha : (U\,{}^{A}_{C}\,W)$ matches the top boundary of $\beta : (U'\,{}^{C}_{B}\,W')$ we may form $\frac{\alpha}{\beta} : (U{\otimes}U'\,{}^{A}_{B}\,W{\otimes}W')$ – their *vertical composite* – with the boundaries of the composite cell formed from those of the component cells using the binary operation associated with the appropriate monoid (both written $\otimes$). We depict horizontal and vertical composition, respectively, as in:



and

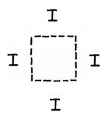


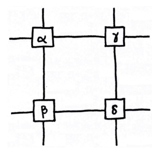
Horizontal and vertical composition of cells are required to be associative and unital. We write $id_U : (U\,{}^{I}_{I}\,U)$ and $1_A : (I\,{}^{A}_{A}\,I)$ for units of horizontal and vertical composition, respectively. We omit wires of sort I in our depictions of cells, allowing us to depict horizontal and vertical identity cells, respectively, as in:



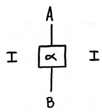
and

Finally, the horizontal and vertical identity cells of type I must coincide – we call this cell $\square_I = 1_I = id_I : (I\,{}^{I}_{I}\,I)$ and depict it as empty space, see below on the left – and vertical and horizontal composition must satisfy the interchange law. That is, $\frac{\alpha}{\beta}|\frac{\gamma}{\delta} = \frac{\alpha|\gamma}{\beta|\delta}$, allowing us to unambiguously interpret the diagram below on the right:

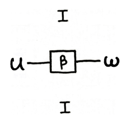




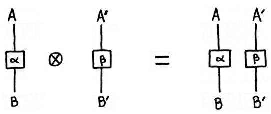
Every single-object double category $\mathbb{D}$ defines monoidal categories $\mathbf{V}\mathbb{D}$ and $\mathbf{H}\mathbb{D}$, consisting of the cells for which the $\mathbb{D}_V$ and $\mathbb{D}_H$ valued boundaries respectively are all I, as in:



and



That is, the collection of objects of $\mathbf{V}\mathbb{D}$ is $\mathbb{D}_H$, composition in $\mathbf{V}\mathbb{D}$ is vertical composition of cells, and the tensor product in $\mathbf{V}\mathbb{D}$ is given by horizontal composition:



In this way, $\mathbf{V}\mathbb{D}$ forms a monoidal category, which we call the category of *vertical cells* of $\mathbb{D}$. Similarly, $\mathbf{H}\mathbb{D}$ is also a monoidal category (with collection of objects $\mathbb{D}_V$) which we call the *horizontal cells* of $\mathbb{D}$.

2.2. The Free Cornering

In this section we introduce the free cornering of a symmetric monoidal category. It is useful to frame this construction in terms of the resource-theoretic understanding of symmetric monoidal categories [10]. That is, objects are understood of as collections of resources. The tensor product $A \otimes B$ of two objects is the collection consisting of A and B, and the unit I is the empty collection. Morphisms are understood as *transformations*, with $f : A \to B$ being understood as a way to transform the resources of A to the resources of B. We adopt this perspective here and use the associated vocabulary to elucidate our development.

We begin with the monoid of exchanges over a symmetric monoidal category:

Definition 1. Let $\mathbb{A}$ be a symmetric monoidal category. Define the monoid $\mathbb{A}^{\circ\bullet}$ of $\mathbb{A}$-*valued exchanges* to be the free monoid on the set of polarized objects of $\mathbb{A}$, as in $\mathbb{A}^{\circ\bullet} = \mathbb{A}_0 \times \{\circ, \bullet\}^*$. Explicitly, $\mathbb{A}^{\circ\bullet}$ has elements given by:

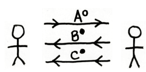
$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}^{\circ\bullet}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}^{\circ\bullet}} \qquad \frac{}{I \in \mathbb{A}^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}^{\circ\bullet} \qquad W \in \mathbb{A}^{\circ\bullet}}{U \otimes W \in \mathbb{A}^{\circ\bullet}}$$

subject to the following equations:

$$I \otimes U = U \qquad U \otimes I = U \qquad (U \otimes W) \otimes V = U \otimes (W \otimes V)$$

We may omit brackets as in $A^\circ \otimes B^\circ \otimes C^\bullet$, as associativity of $\otimes$ ensures that this denotes an element of $\mathbb{A}^{\circ\bullet}$ unambiguously.

The $\mathbb{A}$-valued exchanges are interpreted as follows: each $X_1 \otimes \cdots \otimes X_n \in \mathbb{A}^{\circ\bullet}$ involves a left participant and a right participant giving each other resources in sequence, with A° indicating that the left participant should give the right participant an instance of A, and $A^\bullet$ indicating the opposite. For example say the left participant is `Alice` and the right participant is `Bob`. Then we can picture the exchange $A^\circ \otimes B^\bullet \otimes C^\bullet$ as:

`Alice` $\rightsquigarrow$  $\leftsquigarrow$ `Bob`

These exchanges happen *in order*. For example the exchange pictured above demands that first `Alice` gives `Bob` an instance of A, then `Bob` gives `Alice` an instance of B, and then finally `Bob` gives `Alice` an instance of C.

The monoid of $\mathbb{A}$-valued exchanges plays an important role in the free cornering of $\mathbb{A}$, which we introduce presently:

Definition 2 ([25])**.** Let $\mathbb{A}$ be a monoidal category. We define the *free cornering* of $\mathbb{A}$, written $\ulcorner\mathbb{A}\lrcorner$, to be the free single-object double category with horizontal edge monoid $(\mathbb{A}_0, \otimes, I)$, vertical edge monoid $\mathbb{A}^{\circ\bullet}$, and generating cells and equations consisting of:

- For each $f : A \to B$ of $\mathbb{A}$ a cell $\ulcorner f \lrcorner : \ulcorner\mathbb{A}\lrcorner(I\,{}^{A}_{B}\,I)$ subject to equations:

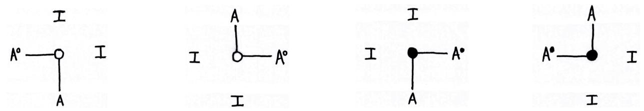
  $$\ulcorner fg \lrcorner = \frac{\ulcorner f \lrcorner}{\ulcorner g \lrcorner} \qquad \ulcorner 1_A \lrcorner = 1_A \qquad \ulcorner f \otimes g \lrcorner = \ulcorner f \lrcorner \mid \ulcorner g \lrcorner$$

  One way to understand these is to notice that they allow us to interpret string diagrams denoting morphisms of $\mathbb{A}$ as cells of $\ulcorner\mathbb{A}\lrcorner$ unambiguously:

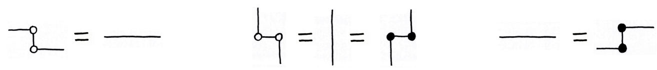
  

  We write $\ulcorner f \lrcorner = f$ when it is clear in context that f denotes a cell of $\ulcorner\mathbb{A}\lrcorner$.

- For each object A of $\mathbb{A}$, *corner cells* $\mathsf{get}^A_{\mathsf{L}} : \ulcorner\mathbb{A}\lrcorner(A^\circ \begin{smallmatrix} I \\ A \end{smallmatrix} I)$, $\mathsf{put}^A_{\mathsf{R}} : \ulcorner\mathbb{A}\lrcorner(I \begin{smallmatrix} A \\ I \end{smallmatrix} A^\circ)$, $\mathsf{get}^A_{\mathsf{R}} : \ulcorner\mathbb{A}\lrcorner(I \begin{smallmatrix} I \\ A \end{smallmatrix} A^\bullet)$, and $\mathsf{put}^A_{\mathsf{L}} : \ulcorner\mathbb{A}\lrcorner(A^\bullet \begin{smallmatrix} A \\ I \end{smallmatrix} I)$, which we depict as follows:

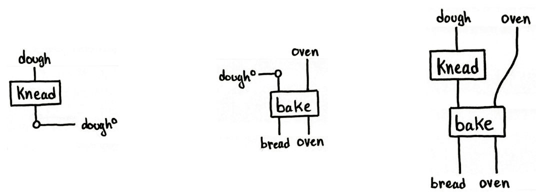
  The corner cells are subject to the *yanking equations*:

  Intuitively, the corner cells send and receive resources along the left and right boundaries. The yanking equations allow us to carry out exchanges between horizontally composed cells, and tell us that being exchanged has no effect on the resources involved.

For a precise development of free double categories see [13]. Briefly, cells are formed from the generating cells by horizontal and vertical composition, subject to the axioms of a double category in addition to any generating equations. The corner structure has been heavily studied under various names including *proarrow equipment*, *framed bicategory*, *connection structure*, and *companion and conjoint structure*. A good resource is the appendix of [32].

Cells of $\ulcorner\mathbb{A}\lrcorner$ can be understood as *interacting* morphisms of $\mathbb{A}$. Each cell is a method of obtaining the resources of bottom boundary from those of the top boundary by participating in $\mathbb{A}$-valued exchanges along the left and right boundaries in addition to using the resource transformations supplied by $\mathbb{A}$. For example, if the morphisms of $\mathbb{A}$ describe the procedures involved in baking bread, we might have the following cells of $\ulcorner\mathbb{A}\lrcorner$:

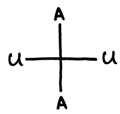


The cell on the left describes a procedure for transforming `dough` into nothing by `kneading` it and sending the result away along the right boundary, and the cell in the middle describes a procedure for transforming an `oven` into `bread` and an `oven` by receiving `dough` along the left boundary and then using the `oven` to `bake` it. Composing these cells horizontally results in the cell on the right via the yanking equations. In this way the free cornering models process interaction, with the corner cells capturing the flow of information across components.

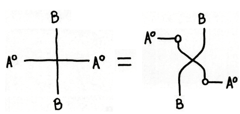
2.3. Crossing Cells

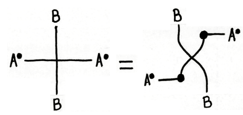
In this section we recall *crossing cells*, an interesting bit of structure that exists in the free cornering of any symmetric monoidal category. We recall a few results concerning the crossing cells, which are quite well-behaved. Our purpose in doing so is mainly to extend these results later on when we add choice and iteration to the free cornering. The crossing cells will remain well-behaved, which is a sign that our notions of choice and iteration are formally coherent.

Definition 3 ([25])**.** Let $\mathbb{A}$ be a symmetric monoidal category. For each $A \in {\ulcorner\mathbb{A}\lrcorner}_H$ and each $U \in {\ulcorner\mathbb{A}\lrcorner}_V$ we define a crossing cell $\chi_{U,A}$, pictured as in:

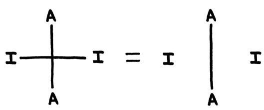


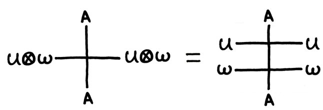
inductively as follows: define $\chi_{A^\circ,B}$ and $\chi_{A^\bullet,B}$ as in the diagrams below on the left and right, respectively:





further, define $\chi_{I,A} = 1_A$ and $\chi_{U\otimes W,A} = \frac{\chi_{U,A}}{\chi_{W,A}}$, as in:





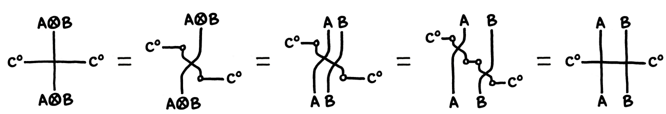
We note that this definition differs slightly from that given in [25, 27]. In previous work on the free cornering, the definition of crossing cells included the assumption that they were coherent with respect to horizontal composition. We show that in fact, this can be derived:

Lemma 1. *Let $\mathbb{A}$ be a symmetric monoidal category. For $U \in A^{\circ\bullet}$ and $A, B \in \mathbb{A}_0$ the following equations hold in ${\ulcorner\mathbb{A}\lrcorner}$:*

(i) $\chi_{U,A\otimes B} = \chi_{U,A} \mid \chi_{U,B}$

(ii) $\chi_{U,I} = id_U$

Proof. (i) By structural induction on U. In case $U = C^\circ$ we have:



as required. The case for $U = C^\bullet$ is similar. If $U = I$ then we have:

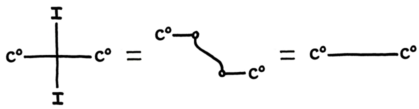
$$\chi_{I,A\otimes B} = \ulcorner 1_{A\otimes B} \urcorner = \ulcorner 1_A \urcorner \mid \ulcorner 1_B \urcorner = \chi_{I,A} \mid \chi_{I,B}$$

For the inductive case $U \otimes W$ we have

$$\chi_{U\otimes W,A\otimes B} = \frac{\chi_{U,A\otimes B}}{\chi_{W,A\otimes B}} = \frac{\chi_{U,A} \mid \chi_{U,B}}{\chi_{W,A} \mid \chi_{W,B}} = \frac{\chi_{U,A}}{\chi_{W,A}} \mid \frac{\chi_{U,B}}{\chi_{W,B}} = \chi_{U\otimes W,A} \mid \chi_{U\otimes W,B}$$

The claim follows.

(ii) By induction on the structure of U. If $U = A^\circ$ then we have:



as required. The case for $U = A^\bullet$ is similar. If $U = I$ then we have:

$$\chi_{U,I} = \chi_{I,I} = 1_I = id_I$$

For $U \otimes W$, we have:

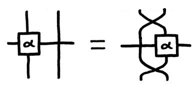
$$\chi_{U\otimes W,I} = \frac{\chi_{U,I}}{\chi_{W,I}} = \frac{id_U}{id_W} = id_{U\otimes W}$$

The claim follows.

□

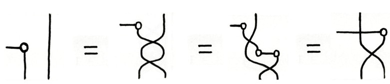
Additionally, the crossing cells carry interesting categorical structure. The core technical lemma underpinning this structure is as follows:

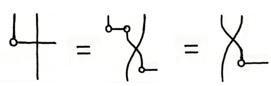
Lemma 2 ([25])**.** *For any cell α of $\ulcorner \mathbb{A} \urcorner$ we have*



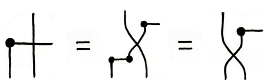
We recapitulate the proof of this, as we will refer to it later on, when we extend the above lemma to the setting with choice and iteration.

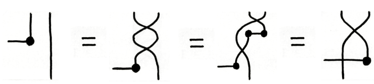
Proof. By structural induction on cells of $\ulcorner \mathbb{A} \urcorner$. For the $\circ$-corners we have:



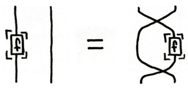


and for the $\bullet$-corners, similarly:

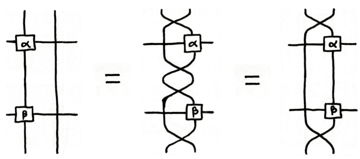




the final base cases are the $\ulcorner f \urcorner$ maps:



There are two inductive cases. For vertical composition, we have:

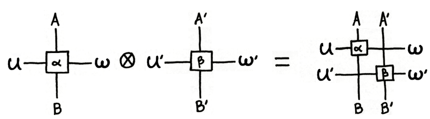


Horizontal composition is similarly straightforward, and the claim follows by induction. ☐

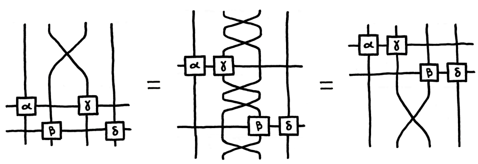
A particularly interesting consequence of Lemma 2 is that for any symmetric monoidal category $\mathbb{A}$, $\ulcorner \mathbb{A} \urcorner$ is a monoidal double category in the sense of Shulman [33]. That is:

Lemma 3 ([25])**.** *If $\mathbb{A}$ is a symmetric monoidal category then $\ulcorner \mathbb{A} \urcorner$ is a monoidal double category. That is, $\ulcorner \mathbb{A} \urcorner$ is a pseudo-monoid object in the strict 2-category* $\mathsf{VDblCat}$ *of double categories, lax double functors, and vertical transformations.*

Proof. We give the action of the tensor product on cells:



This defines a pseudofunctor, with the component of the required vertical transformation given by exchanging the two middle wires as in:



Notice that $\otimes$ is strictly associative and unital, in spite of being only pseudofunctorial. ☐

This concludes our treatment of crossing cells in the free cornering. We proceed to give a model of the free cornering that we have recovered from the mathematical wilderness.

2.4. A Model: Stateful Transformations

In this section we construct a single-object double category of *stateful transformations*, named for their resemblance to the *stateful runners* studied by Uustalu in the context of monadic computational effects [34]. Our interest in the double category of stateful transformations is that it gives a model of the free cornering, exemplifying the corner cells and crossing structure in a more familiar setting. Stateful transformations will serve as a running example throughout our development. First, we require the notion of strong functor. Recall:

Definition 4. Let $(\mathbb{C}, \otimes, I)$ be a monoidal category, and let $F : \mathbb{C} \to \mathbb{C}$ be a functor. Then a *tensorial strength for* F consists of a natural transformation:

$$\tau^F_{X,Y} : FX \otimes Y \to F(X \otimes Y)$$

satisfying $\tau^F_{X,I} = 1_{FX}$ and also:

$$\begin{tikzcd}
FX \otimes Y \otimes Z \arrow[d, "\tau^F_{X,Y} \otimes 1_Z"'] \arrow[dr, "\tau^F_{X,Y\otimes Z}"] & \\
F(X \otimes Y) \otimes Z \arrow[r, "\tau^F_{X\otimes Y, Z}"'] & F(X \otimes Y \otimes Z)
\end{tikzcd}$$

A *strong functor* $(F, \tau^F) : \mathbb{C} \to \mathbb{C}$ consists of a functor $F : \mathbb{C} \to \mathbb{C}$ together with a tensorial strength τ^F for F.

Strong functors $\mathbb{C} \to \mathbb{C}$ form a monoid $(\mathbb{C}^{\mathbb{C}}, \circ, I)$. Given two strong functors $(F, \tau^F), (G, \tau^G) : \mathbb{C} \to \mathbb{C}$ we define $(F, \tau^F) \circ (G, \tau^G) = (G \circ F, \tau^{G \circ F})$ where $\tau^{G\circ F}_{X,Y} = \tau^G_{FX,Y} G(\tau^F_{X,Y})$. The unit I is given by the identity functor $1_{\mathbb{C}}$ with strength $\tau^{1_{\mathbb{C}}}_{X,Y} = 1_{X \otimes Y}$. We write F instead of (F, τ^F) when confusion is unlikely. The accompanying notion of natural transformation is:

Definition 5. Let $\mathbb{C}$ be a monoidal category, and let $(F, \tau^F), (G, \tau^G) : \mathbb{C} \to \mathbb{C}$ be strong functors. A *strong natural transformation* $\alpha : (F, \tau^F) \to (G, \tau^G)$ is a natural transformation $\alpha : F \to G$ satisfying:

$$\begin{tikzcd}
FX \otimes Y \arrow[r, "\alpha_X \otimes 1_Y"] \arrow[d, "\tau^F_{X,Y}"'] & GX \otimes Y \arrow[d, "\tau^G_{X,Y}"] \\
F(X \otimes Y) \arrow[r, "\alpha_{X \otimes Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

We will be concerned with strong functors over a cartesian closed category $(\mathbb{C}, \otimes, I)$. We write X^A for the exponential, $\mathsf{ev}^B_A : B^A \otimes A \to A$ for the evaluation maps, and $\lambda[f] : B \to C^A$ for the name of $f : B \otimes A \to C$. Given an object A of $\mathbb{C}$, we define endofunctors $A^\circ = (- \otimes A)$ and $A^\bullet = (-)^A$ of $\mathbb{C}$. The tensorial strengths for A° and $A^\bullet$ have components as in:

$$\tau^{A^\circ}_{X,Y} = 1_X \otimes \sigma_{A,Y} : X \otimes A \otimes Y \to X \otimes Y \otimes A$$

$$\tau^{A^\bullet}_{X,Y} = \lambda[(1_{X^A} \otimes \sigma_{Y,A})(\mathsf{ev}^A_X \otimes 1_Y)] : X^A \otimes Y \to (X \otimes Y)^A$$

Note that A° is left adjoint to $A^\bullet$.

We may now assemble the double category of stateful transformations:

Definition 6. Let $(\mathbb{C}, \otimes, I)$ be a cartesian closed category. The single-object double category $\mathsf{S}(\mathbb{C})$ of *stateful transformations in* $\mathbb{C}$ has horizontal edge monoid $(\mathbb{C}_0, \otimes, I)$ given by the cartesian product structure of $\mathbb{C}$, and has vertical edge monoid $(\mathbb{C}^{\mathbb{C}}, \circ, I)$ the monoid of strong endofunctors on $\mathbb{C}$. The cells $\alpha : \mathsf{S}(\mathbb{C})(U \genfrac{}{}{0pt}{}{A}{B} W)$ are strong natural transformations:

$$\alpha : (A^\circ \circ U, \tau^{A^\circ \circ U}) \to (W \circ B^\circ, \tau^{W \circ B^\circ})$$

so in particular the components are of the form:

$$\alpha_X : UX \otimes A \to W(X \otimes B)$$

For horizontal composition, if we have $\alpha : (F \genfrac{}{}{0pt}{}{A}{B} G)$ and $\beta : (G \genfrac{}{}{0pt}{}{A'}{B'} H)$ then their horizontal composite $(\alpha \mid \beta) : (F \genfrac{}{}{0pt}{}{A \otimes A'}{B \otimes B'} H)$ is given by:

$$(\alpha|\beta)_X = (\alpha_X \otimes 1_{A'})(\beta_{X \otimes B}) : FX \otimes A \otimes A' \to H(X \otimes B \otimes B')$$

and the horizontal identity cells $id_F : (F \genfrac{}{}{0pt}{}{I}{I} F)$ are given by:

$$(id_F)_X = 1_{FX} : FX \otimes 1 = FX \to FX = F(X \otimes 1)$$

For vertical composition, if we have $\alpha : (F \genfrac{}{}{0pt}{}{A}{B} G)$ and $\beta : (H \genfrac{}{}{0pt}{}{B}{C} K)$ then their vertical composite $\frac{\alpha}{\beta} : (F \circ H \genfrac{}{}{0pt}{}{A}{C} G \circ K)$ is given by

$$\left(\frac{\alpha}{\beta}\right)_X = \alpha_{HX} G(\beta_X) : F(H(X)) \otimes A \to G(K(X \otimes C))$$

and the vertical identity cells $1_A : (I \genfrac{}{}{0pt}{}{A}{A} I)$ are given by:

$$(1_A)_X = 1_{X \otimes A} : 1_{\mathbb{C}}(X) \otimes A = X \otimes A \to X \otimes A = 1_{\mathbb{C}}(X \otimes A)$$

We show that this is indeed a double category:

Lemma 4. $\mathsf{S}(\mathbb{C})$ *is well-defined.*

Proof. Horizontal and vertical composition are associative and unital because both composition of natural transformations and the cartesian product structure are associative and unital. The rest of the requirements on a double category are easily seen to hold, with the most involved being interchange, which we show holds explicitly. Given $\alpha : (F \genfrac{}{}{0pt}{}{A}{B} G)$, $\beta : (G \genfrac{}{}{0pt}{}{A'}{B'} H)$, $\gamma : (F' \genfrac{}{}{0pt}{}{B}{C} G')$ and $\delta : (G' \genfrac{}{}{0pt}{}{B'}{C'} H')$, then the interchange law $(\frac{\alpha}{\gamma}|\frac{\beta}{\delta}) = (\frac{\alpha|\beta}{\gamma|\delta})$ holds as follows:

$$
\begin{tikzcd}
 & & F(F'(X)) \times A \times A' \arrow[ddl, bend right, "(\frac{\alpha}{\gamma})_X \times A'"'] \arrow[d, "\alpha_{F'(X)} \times A'"] \arrow[ddr, bend left, "(\alpha|\beta)_{F'X}"] \arrow[dddd, bend right=90, looseness=1.5, "(\frac{\alpha}{\gamma}|\frac{\beta}{\delta})_X"'] \arrow[dddd, bend left=90, looseness=1.5, "(\frac{\alpha|\beta}{\gamma|\delta})_X"] & & \\
 & & G(F'(X) \times B) \times A' \arrow[dl, "G(\gamma_X) \times A'"] \arrow[dr, "\beta_{F'X \times B}"] & & \\
 & G(G'(X \times C)) \times A' \arrow[dr, "\beta_{G'(X \times C)}"] \arrow[ddr, bend right, "(\frac{\beta}{\delta})_{X \times C}"'] & & H(F'(X) \times B \times B') \arrow[dl, "H(\gamma_X \times B')"'] \arrow[ddl, bend left, "H(\gamma|\delta)"] & \\
 & & H(G'(X \times C) \times B') \arrow[d, "H(\delta_{X \times C})"] & & \\
 & & H(H'(X \times C \times C')) & &
\end{tikzcd}
$$

where the middle diamond commutes by naturality of β and the rest of the diagram is obtained by unfolding definitions. $\square$

A first observation concerning $\mathsf{S}(\mathbb{C})$ is that for each $f : A \to B$ of $\mathbb{C}$ there is a cell $[f] : \mathsf{S}(\mathbb{C})(I\,{}^{A}_{B}\,I)$ given by $[f]_X : (1_X \otimes f) : X \otimes A \to X \otimes B$, and that this defines an embedding $[-] : \mathbb{C} \to \mathsf{S}(\mathbb{C})$. Moreover, the category of strong endofunctors of $\mathbb{C}$ and strong natural transformations embeds into $\mathbf{H}\,\mathsf{S}(\mathbb{C})$, since a strong natural transformation $\alpha : (F, \tau^F) \to (G, \tau^G)$ is equivalently a cell $\alpha : \mathsf{S}(\mathbb{C})(F\,{}^{I}_{I}\,G)$.

We define $\circ$-corner cells $\mathsf{put}^A_{\mathsf{R}} : \mathsf{S}(\mathbb{C})(I\,{}^{A}_{I}\,A^\circ)$ and $\mathsf{get}^A_{\mathsf{L}} : \mathsf{S}(\mathbb{C})(A^\circ\,{}^{I}_{A}\,I)$ to have identity maps as components, as in:

$$(\mathsf{put}^A_{\mathsf{R}})_X = 1_{X \otimes A} : 1_{\mathbb{C}}(X) \otimes A = X \otimes A \to X \otimes A = A^\circ(X \otimes I)$$

$$(\mathsf{get}^A_{\mathsf{L}})_X = 1_{X \otimes A} : A^\circ(X) \otimes I = X \otimes A \to X \otimes A = 1_{\mathbb{C}}(X \otimes A)$$

That the yanking equations hold is immediate. Next, we define $\bullet$-corner cells $\mathsf{put}^A_{\mathsf{L}} : \mathsf{S}(\mathbb{C})(A^\bullet\,{}^{A}_{I}\,I)$ and $\mathsf{get}^A_{\mathsf{R}} : \mathsf{S}(\mathbb{C})(I\,{}^{I}_{A}\,A^\bullet)$ using the closed structure, as in:

$$(\mathsf{put}^A_{\mathsf{L}})_X = \mathsf{ev}^A_X : A^\bullet(X) \otimes A = X^A \otimes A \to X = 1_{\mathbb{C}}(X \otimes I)$$

$$(\mathsf{get}^A_{\mathsf{R}})_X = \lambda[1_{X \otimes A}] : 1_{\mathbb{C}}(X) \otimes I = X \to (X \otimes A)^A = A^\bullet(X \otimes A)$$

In other words, $(\mathsf{put}^A_{\mathsf{L}})_X : A^\circ(A^\bullet X) \to X$ is and $(\mathsf{get}^A_{\mathsf{R}})_X : X \to A^\bullet(A^\circ X)$ are the unit and counit of the adjunction $A^\circ \dashv A^\bullet$ given by the cartesian closed structure. The yanking equations are then the triangle equations of this adjunction. Explicitly:

$$(\mathsf{get}^A_{\mathsf{R}} \mid \mathsf{put}^A_{\mathsf{L}})_X = (\lambda[1_{X\otimes A}] \otimes 1_A)\mathsf{ev}^A_{X\otimes A} = 1_{X\times A} = (id_{A^\bullet})_X$$

$$\left(\frac{\mathsf{get}^A_{\mathsf{R}}}{\mathsf{put}^A_{\mathsf{L}}}\right)_X = \lambda[1_{X^A\otimes A}](\mathsf{ev}^A_X)^A = \lambda[1_{X^A\otimes A}\mathsf{ev}^A_X] = 1_{X^A} = (1_A)_X$$

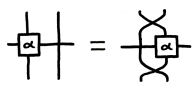
That the corner cells constitute strong natural transformations is straightforward, if slightly tedious, to verify.

The fact that $\mathsf{S}(\mathbb{C})$ is constructed from *strong* functors is closely connected to the nature of crossing cells there. Given a strong functor $(F, \tau^F) : \mathbb{C} \to \mathbb{C}$ and an object A of $\mathbb{C}$ the tensorial strength of F defines a crossing cell $\chi_{F,A} : \mathsf{S}(\mathbb{C})(F\,{}^{A}_{A}\,F)$ with components $(\chi_{F,A})_X = \tau^F_{X,A} : FX \otimes A \to F(X \otimes A)$. These crossing cells are coherent with respect to both horizontal and vertical composition in $\mathsf{S}(\mathbb{C})$ in the sense that:

$$\chi_{F,I} = id_F \qquad \chi_{F,A\otimes B} = \chi_{F,A} \mid \chi_{F,B} \qquad \chi_{I,A} = 1_A \qquad \chi_{F\circ G,A} = \frac{\chi_{F,A}}{\chi_{G,A}}$$

The crossing cells $\chi_{B^\circ,A}$ and $\chi_{B^\bullet,A}$ obtained in this manner are equal to those defined in terms of the corner cells and braiding, as in Definition 3.

Moreover, the equation from Lemma 2 concerning crossing cells:



holds for all cells α of $\mathsf{S}(\mathbb{C})$ precisely because cells α are required to be *strong* natural transformations.

We have seen that the double category of stateful transformations has corner and crossing cells, much as the free cornering does. We have previously mentioned that stateful transformations give a *model* of the free cornering. What we mean by this is that there is a structure-preserving double functor from the free cornering of a cartesian closed category into the associated category of stateful transformations, which moreover preserves the relevant structure. Explicitly, define a double functor $D : {}^{\ulcorner}_{\llcorner}\mathbb{C}{}^{\urcorner}_{\lrcorner} \to \mathsf{S}(\mathbb{C})$ as follows: On the horizontal edge monoid D acts as the identity. On the vertical edge monoid D sends A° and $A^\bullet$ in $\mathbb{A}^{\circ\bullet}$ to the strong functors A° and $A^\bullet$, and is defined inductively as in $D(I) = I$ and $D(U \otimes W) = DU \circ DW$ to capture all elements of $\mathbb{A}^{\circ\bullet}$. On cells, $D({}^{\ulcorner}_{\llcorner}f{}^{\urcorner}_{\lrcorner}) = [f]$, and the corner cells of ${}^{\ulcorner}_{\llcorner}\mathbb{C}{}^{\urcorner}_{\lrcorner}$ are sent to those of $\mathsf{S}(\mathbb{C})$. Clearly D preserves the corner and crossing cells, and preserves the embedding of $\mathbb{C}$ in the sense that $D({}^{\ulcorner}_{\llcorner}f{}^{\urcorner}_{\lrcorner}) = [f]$. The presence of well-behaved crossing cells in $\mathsf{S}(\mathbb{C})$ makes it a monoidal double category (as in Lemma 3), and then because D preserves the crossing cells it also preserves the monoidal double category structure.

Remark 1. Strong functors, and in particular strong monads, are often used to model computational effects (see e.g., [23, 29]). For example $A^\bullet$ is sometimes called the *reader monad*: arrows $f : B \to C$ in the Kleisli category are given by arrows $f : B \to C^A = A^\bullet(C)$, which are understood as arrows $B \to C$ that *read input* of type A. This input is understood to be provided by the environment of the program. Moreover, A° is the *writer comonad*, $A^\bullet \circ A^\circ$ is the *state monad*, and $A^\circ \circ A^\bullet$ is the *store comonad*.

One way to think of cells α of $\mathsf{S}(\mathbb{C})$ from the perspective of computational effects is as follows: the right boundary of α represents its *environment*, that is, the context in which α executes. For example if the right boundary of α is $A^\bullet$ then α will read a value (supplied by the environment). The left boundary of α represents the *interior* of α, in the sense that α acts as the environment of its interior. For example if the left boundary of α is $A^\bullet$ then the interior of α will read a value, which α must supply. Here effects are understood to be triggered from the left, propagating outwards until resolved.

This concludes our discussion of $\mathsf{S}(\mathbb{C})$ for the time being. We proceed to discuss the addition of choice to the free cornering.

3. Adding Choice to the Free Cornering

In this section we extend the free cornering of a symmetric monoidal category with a notion of protocol choice. In addition to symmetric monoidal structure we will require the base category to have distributive binary coproducts, which we review in Section 3.1. We also discuss the way in which this sort of category can be seen as an algebra of sequential branching programs. In Section 3.2 we construct the *free cornering with choice* over a suitable base and discuss its interpretation. In Section 3.3 we establish a number of elementary properties of the free cornering with choice. In Section 3.4 we define crossing cells in the free cornering with choice, and show that they are well-behaved. This is mostly an extension of Section 2.3 to the new setting, with the exception of Lemma 13. Finally, in Section 3.5 we show that with additional assumptions on the base category the double category of stateful transformations from Section 2.4 gives a model of the free cornering with choice.

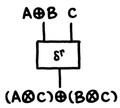
3.1. Distributive Monoidal Categories and Branching Programs

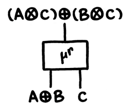
We begin by recapitulating the notion of a category with binary coproducts, largely in order to establish our notation for them:

Definition 7. A category $\mathbb{A}$ is said to *have binary coproducts* in case for each pair A, B of objects of $\mathbb{A}$ there is is an object $A \oplus B$ of $\mathbb{A}$ together with morphisms $\sigma_0^{A,B} : A \to A \oplus B$ and $\sigma_1^{A,B} : B \to A \oplus B$ such that for any pair of morphims $f : A \to C$ and $g : B \to C$ there exists a unique morphism $[f, g] : A \oplus B \to C$ with the property that $\sigma_0^{A,B}[f, g] = f$ and $\sigma_1^{A,B}[f, g] = g$. We call $A \oplus B$ the *coproduct of A and B*, and call $[f, g]$ the *copairing of f and g*. We write $\sigma_0^{A,B} = \sigma_0$ and $\sigma_1^{A,B} = \sigma_1$ when it is unlikely to result in confusion. Note that a category with binary coproducts need not have an initial object.

Next, we recall the notion of a distributive monoidal category, being a monoidal category with distributive binary coproducts:

Definition 8. A *distributive monoidal category* $(\mathbb{A}, \otimes, \oplus, I)$ is a symmetric monoidal category $(\mathbb{A}, \otimes, I)$ with binary coproducts $A \oplus B$ such that $\otimes$ distributes over $\oplus$. That is, for all objects A, B, C of $\mathbb{A}$ the arrow $\mu^r = [(\sigma_0 \otimes 1_C), (\sigma_1 \otimes 1_C)] : (A \otimes C) \oplus (B \otimes C) \to (A \oplus B) \otimes C$ has an inverse $\delta^r : (A \oplus B) \otimes C \to (A \otimes C) \oplus (B \otimes C)$. Diagrammatically:





Note that in any distributive monoidal category there is necessarily an inverse $\delta^l : C \otimes (A \oplus B) \to (C \otimes A) \oplus (C \otimes B)$ to the arrow $\mu^l = [(1_C \otimes \sigma_0), (1_C \otimes \sigma_1)] : (C \otimes A) \oplus (C \otimes B) \to C \otimes (A \oplus B)$.

The resource-theoretic understanding of symmetric monoidal categories extends to distributive monoidal categories, with $A \oplus B$ understood as the collection consisting of the contents of A *or* the contents of B. Notice tha tthis interpretation is only really coherent in the presence of distributivity.

Another way to understand distributive monoidal categories is that they model *branching programs*. In any distributive monoidal category we may define an object $\mathsf{Bool} = I \oplus I$ of booleans with elements $\top = \sigma_0 : I \to \mathsf{Bool}$ and $\bot = \sigma_1 : I \to \mathsf{Bool}$ given by the coproduct injections. Then for any $f, g : A \to B$ the morphism $\delta^r[f, g] : \mathsf{Bool} \otimes A \to B$ models the conditional statement:

$$\texttt{if } b \texttt{ then } f(x) \texttt{ else } g(x)$$

In particular we have both of (see Lemma 5):

$$(\top \otimes 1_A)\delta^r[f, g] = \sigma_0^{A,A}[f, g] = f \qquad (\bot \otimes 1_A)\delta^r[f, g] = \sigma_1^{A,A}[f, g] = g$$

which we should think of as program equivalences:

$$\texttt{if } \top \texttt{ then } f(x) \texttt{ else } g(x) = f(x) \qquad \texttt{if } \bot \texttt{ then } f(x) \texttt{ else } g(x) = g(x)$$

Before moving on we record a useful fact about coproduct injections in distributive monoidal categories for later use:

Lemma 5. *In distributive monoidal category:*

$$(\sigma_0^{A,B} \otimes 1_C)\delta^r = \sigma_0^{A \otimes C, B \otimes C} \qquad \textit{and} \qquad (\sigma_1^{A,B} \otimes 1_C)\delta^r = \sigma_1^{A \otimes C, B \otimes C}$$

Proof. We have:

$$\sigma_0^{A \otimes C, B \otimes C}\mu^r = \sigma_0^{A \otimes C, B \otimes C}[(\sigma_0^{A,B} \otimes 1_C), (\sigma_1^{A,B} \otimes 1_C)] = \sigma_0^{A,B} \otimes 1_C$$

It follows immediately that:

$$\sigma_0^{A \otimes C, B \otimes C} = \sigma_0^{A \otimes C, B \otimes C}\mu^r\delta^r = (\sigma_0^{A,B} \otimes 1_C)\delta^r$$

Similarly, we have $(\sigma_1^{A,B} \otimes 1_C)\delta^r = \sigma_1^{A \otimes C, B \otimes C}$. $\square$

Distributive monoidal categories are also a good place to model datatypes. For example, if $\mathbb{A}$ is distributive monoidal and A is an object of $\mathbb{A}$, then we may model *stacks of type* A as an object S_A of $\mathbb{A}$ equipped with an isomorphism $S_A \cong I \oplus (A \otimes S_A)$ with components $\mathsf{pop} : S_A \to I \oplus (A \otimes S_A)$ and $[\mathsf{nil}, \mathsf{push}] : I \oplus (A \otimes S_A) \to S_A$. Then for example the object S_I of stacks of type I is a model of the natural numbers with $\mathsf{nil} = \mathsf{zero}$ and $\mathsf{push} = \mathsf{succ}$. See [37] for a more in-depth discussion.

3.2. The Free Cornering With Choice

In this section we extend the free cornering of a monoidal category with a notion of protocol choice. We begin by extending the monoid of exchanges (Definition 1) with binary operations $- + -$ and $- \times -$ representing branching protocols:

Definition 9. Let $\mathbb{A}$ be a symmetric monoidal category. The monoid $A^{\circ\bullet}_{\oplus}$ of $\mathbb{A}$-*valued exchanges with choice* has elements generated by:

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}^{\circ\bullet}_{\oplus}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}^{\circ\bullet}_{\oplus}} \qquad \frac{}{I \in \mathbb{A}^{\circ\bullet}_{\oplus}} \qquad \frac{U \in \mathbb{A}^{\circ\bullet}_{\oplus} \qquad W \in \mathbb{A}^{\circ\bullet}_{\oplus}}{U \otimes W \in \mathbb{A}^{\circ\bullet}_{\oplus}}$$

$$\frac{U \in \mathbb{A}^{\circ\bullet}_{\oplus} \qquad W \in \mathbb{A}^{\circ\bullet}_{\oplus}}{U \times W \in \mathbb{A}^{\circ\bullet}_{\oplus}} \qquad \frac{U \in \mathbb{A}^{\circ\bullet}_{\oplus} \qquad W \in \mathbb{A}^{\circ\bullet}_{\oplus}}{U + W \in \mathbb{A}^{\circ\bullet}_{\oplus}}$$

subject to the following equations:

$$I \otimes U = U \qquad U \otimes I = U \qquad (U \otimes W) \otimes V = U \otimes (W \otimes V)$$

We extend the interpretation of $\mathbb{A}^{\circ\bullet}$ from Section 2.2 to an interpretation of $\mathbb{A}^{\circ\bullet}_{\oplus}$, interpreting $- + -$ and $- \times -$ as *choices*. Specifically, For any $U, W \in A^{\circ\bullet}_{\oplus}$ we interpret $U + W \in A^{\circ\bullet}_{\oplus}$ as an exchange which begins with the *left* participant choosing whether the rest of the exchange will be of the form U or of the form W, after which the exchange proceeds according to this choice. Dually, we interpret $U \times W \in A^{\circ\bullet}_{\oplus}$ in the same way, except that the *right* participant chooses instead of the left participant.

For example, suppose $A, B \in \mathbb{A}_0$. For each of the following exchanges, call the left participant `Alice` and the right participant `Bob`, as before. Now, consider:

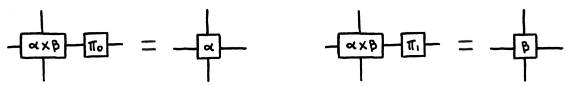
- To carry out $A^\circ \times A^\bullet$, first `Bob` chooses which of A° and $A^\bullet$ will happen. If `Bob` chooses A° then `Alice` sends him an instance of A and the exchange ends. If `Bob` chooses A° then he sends `Alice` an instance of A and the exchange ends.
- To carry our $(A^\circ \times A^\bullet) \otimes B^\bullet$, first `Alice` and `Bob` carry out $A^\circ \times A^\bullet$ as above, and then `Bob` gives `Alice` an instance of B.
- To carry out $A^\bullet + I$, first `Alice` chooses which of $A^\bullet$ and I will happen. If `Alice` chooses $A^\bullet$ the `Bob` sends her an instance of A and the exchange ends. If `Alice` chooses I then the exchange ends immediately.

- To carry out $A^\circ + (A^\circ \times B^\circ) \in A^{\circ\bullet}_{\oplus}$, first `Alice` chooses which of A° and $(A^\circ \times B^\circ)$ will happen. If `Alice` chooses $\mathbb{A}^\circ$, then she sends `Bob` an instance of $\mathbb{A}$ and the exchange ends. If `Alice` chooses $A^\circ \times B^\circ$, then next `Bob` chooses which of $\mathbb{A}^\circ$ and B° will happen. If `Bob` chooses A° then `Alice` sends him an instance of $\mathbb{A}$ and the exchange ends. If `Bob` chooses B° then instead `Alice` sends an instance of B and the exchange ends.

We proceed to extend the rest of the free cornering construction with choice:

Definition 10. Let $\mathbb{A}$ be a distributive monoidal category. We define the *free cornering with choice* of $\mathbb{A}$, written $\ulcorner\mathbb{A}\urcorner^{\oplus}$, to be the free single-object double category with horizontal edge monoid $(\mathbb{A}_0, \otimes, I)$, vertical edge monoid $A^{\circ\bullet}_{\oplus}$, and generating cells and equations consisting of:

- The generating cells and equations of $\ulcorner\mathbb{A}\urcorner$ (Definition 2).
- For each $U, W \in \mathbb{A}^{\circ\bullet}_{\oplus}$, horizontal projection cells $\pi_0 : ({}_{U\times W}\,{}^{I}_{I}\,{}_{U})$ and $\pi_1 : ({}_{U\times W}\,{}^{I}_{I}\,{}_{W})$. Further, for each pair of cells $\alpha \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{V}\,{}^{A}_{B}\,{}_{U})$ and $\beta \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{V}\,{}^{A}_{B}\,{}_{W})$ a unique cell $\alpha \times \beta \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{V}\,{}^{A}_{B}\,{}_{U\times W})$ satisfying:



- Dually, for each $U, W \in \mathbb{A}^{\circ\bullet}_{\oplus}$, horizontal injection cells $\amalg_0 : ({}_{U}\,{}^{I}_{I}\,{}_{U+W})$ and $\amalg_1 : ({}_{W}\,{}^{I}_{I}\,{}_{U+W})$. Further, for each pair of cells $\alpha \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{U}\,{}^{A}_{B}\,{}_{V})$ and $\beta \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{W}\,{}^{A}_{B}\,{}_{V})$ a unique cell $\alpha + \beta \in \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{U+W}\,{}^{A}_{B}\,{}_{V})$ satisfying:

- Finally, for each pair of cells $\alpha : \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{U}\,{}^{A}_{W}\,{}_{C})$ and $\beta : \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{U}\,{}^{B}_{W}\,{}_{C})$ a unique cell $[\alpha, \beta] : \ulcorner\mathbb{A}\urcorner^{\oplus}({}_{U}\,{}^{A\oplus B}_{W}\,{}_{C})$ satisfying:

  where $\ulcorner\sigma_0\urcorner$ and $\ulcorner\sigma_1\urcorner$ are given by the coproduct injections in $\mathbb{A}$.

We extend our interpretation of cells of $\ulcorner\mathbb{A}\urcorner$ as interacting processes to cells of $\ulcorner\mathbb{A}\urcorner^{\oplus}$. Recall that $U \times W$ is the exchange in which the right participant chooses whether the exchange will be of the form U or W. The projection cells $\pi_0 : ({}_{U\times W}\,{}^{I}_{I}\,{}_{U})$ and $\pi_1 : ({}_{U\times W}\,{}^{I}_{I}\,{}_{W})$ allow a cell, acting as the right participant in the exchange $U \times W$, to *make* such choices. The corresponding cells $\alpha \times \beta : ({}_{V}\,{}^{A}_{B}\,{}_{U\times W})$ allow a cell, acting as the left participant in the exchange $U \times W$, to *react* to such choices by specifying a response to each of the two possible choices. Similarly, $U + W$ is the exchange in which the left participant chooses whether the exchange will be of the form U or W. The injections allow a cell, acting as the left participant, to make such choices, and the corresponding cells $\alpha + \beta : (U{+}W \overset{A}{\underset{B}{}} V)$ to react to such choices by specifying a response to each possibility.

Example 1. Suppose $\mathbb{A}$ contains a morphism $\texttt{bake} : \texttt{dough} \otimes \texttt{oven} \to \texttt{bread} \otimes \texttt{oven}$ (as in Section 2.2), and define:

$$\texttt{react} = \left(\mathsf{get}^{\texttt{bread}}_{\mathsf{L}} \mid 1_C\right) + \left(\frac{\mathsf{get}^{\texttt{dough}}_{\mathsf{L}} \mid 1_C}{\texttt{bake}}\right) : \left(\texttt{bread}^\circ{+}\texttt{dough}^\circ \overset{\texttt{oven}}{\underset{\texttt{bread}\otimes\texttt{oven}}{}} I\right)$$

Then $\texttt{react}$ describes a procedure for obtaining $\texttt{bread}$, assuming one posesses an $\texttt{oven}$, by participating in an exchange along the left boundary in which the counterparty chooses whether to supply $\texttt{bread}$ or $\texttt{dough}$. If they choose to supply $\texttt{bread}$ (via $\amalg_0$), then the bread has been obtained. If they choose to supply $\texttt{dough}$ (via $\amalg_1$), then we instead $\texttt{bake}$ the $\texttt{dough}$ to obtain $\texttt{bread}$. So for example:

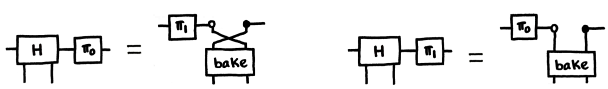
$$(\mathsf{put}^{\texttt{bread}}_{\mathsf{R}} \mid \amalg_0) \mid \texttt{react} = 1_{\texttt{bread}\otimes\texttt{oven}} \qquad\qquad (\mathsf{put}^{\texttt{dough}}_{\mathsf{R}} \mid \amalg_1) \mid \texttt{react} = \texttt{bake}$$

or, diagrammatically:

Example 2. Consider $H : \left(\texttt{dough}^\circ{\times}\texttt{oven}^\circ \overset{I}{\underset{\texttt{bread}\otimes\texttt{oven}}{}} \texttt{dough}^\bullet{\times}\texttt{oven}^\bullet\right)$ defined by

$$H = \frac{\left(\pi_1 \mid \frac{\mathsf{get}^{\texttt{oven}}_{\mathsf{L}} \mid \mathsf{get}^{\texttt{dough}}_{\mathsf{R}}}{\sigma_{\texttt{oven},\texttt{dough}}}\right) \times \left(\pi_0 \mid \mathsf{get}^{\texttt{dough}}_{\mathsf{L}} \mid \mathsf{get}^{\texttt{oven}}_{\mathsf{R}}\right)}{\texttt{bake}}$$

Then we have:



That is, if $\texttt{dough}$ is supplied along the right boundary, then H chooses to obtain an $\texttt{oven}$ along the left boundary, and bakes bread. Otherwise an $\texttt{oven}$ is supplied along the right boundary, in which case H chooses to obtain $\texttt{dough}$ along the left boundary, and bakes bread anyway.

In this way, cells $\alpha + \beta$ and $\alpha \times \beta$ are understood as procedures that branch according to choices made externally as part of the exchange along their left an right boundary, respectively. We compare this to cells $[\alpha, \beta] : (U \overset{A\oplus B}{\underset{C}{}} W)$, which we understand as procedures that branch according to their input, much as in Section 3.1. An important feature of $\ulcorner \mathbb{A} \lrcorner^{\oplus}$ is that when such a procedure branches according to its input, this may be reflected in choices made along the left and right boundary. Explicitly, let $\alpha : \left(U \,{}^{A}_{C}\, W\right)$ and $\beta : \left(U' \,{}^{B}_{C}\, W'\right)$, and consider $[(\pi_0 \mid \alpha \mid \amalg_0), (\pi_1 \mid \beta \mid \amalg_1)] : \left(U{+}U' \,{}^{A\oplus B}_{C}\, W{+}W'\right)$. Then we have:

$$\frac{\sigma_0}{[(\pi_0 \mid \alpha \mid \amalg_0), (\pi_1 \mid \beta \mid \amalg_1)]} = \pi_0 \mid \alpha \mid \amalg_0$$

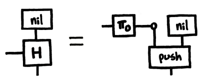
$$\frac{\sigma_1}{[(\pi_0 \mid \alpha \mid \amalg_0), (\pi_1 \mid \beta \mid \amalg_1)]} = \pi_1 \mid \beta \mid \amalg_1$$

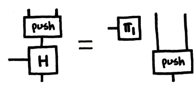
and in this way the choice a procedure makes as part of some exchange along its left and/or right boundary may be determined by its inputs.

Example 3. Suppose our base category has both an object `bread` as well as an object $S_{\texttt{bread}} \cong I \oplus (\texttt{bread} \otimes S_{\texttt{bread}})$ of stacks of bread as in Section 3.1). Then consider the cell $H : \left(\texttt{bread}^{\circ} \times I \,{}^{S_{\texttt{bread}}}_{S_{\texttt{bread}}}\, I\right)$ defined as in:

$$H = \frac{\texttt{pop}}{\dfrac{\left[(\pi_0 \mid \mathsf{get}^{\texttt{bread}}_{\mathsf{L}} \mid \texttt{nil}), (\pi_1 \mid 1_{\texttt{bread}\otimes S_{\texttt{bread}}})\right]}{\texttt{push}}}$$

Then we have:





That is, if the input stack of `bread` is empty then H chooses to obtain `bread` along the left boundary. If the input stack of `bread` is nonempty then H chooses to do nothing along the left boundary (presumably since it already has bread and does not need any more).

3.3. Elementary Properties

In this section we establish a number of elementary properties of the free cornering with choice. First, we observe that where our formation rule for cells $[\alpha, \beta]$ in $\ulcorner \mathbb{A} \lrcorner^{\oplus}$ overlaps with the formation rule for copairing maps in $\mathbb{A}$, the two coincide:

Lemma 6. *For any $f : A \to C$ and $g : B \to C$ in $\mathbb{A}$, $\left[\ulcorner f \lrcorner, \ulcorner g \lrcorner\right] = \ulcorner [f, g] \lrcorner$ in $\ulcorner \mathbb{A} \lrcorner^{\oplus}$.*

Proof. We have:

$$\frac{\sigma_0}{\ulcorner [f,g] \lrcorner} = \ulcorner f \lrcorner = \frac{\sigma_0}{[\ulcorner f \lrcorner, \ulcorner g \lrcorner]} \qquad\qquad \frac{\sigma_1}{\ulcorner [f,g] \lrcorner} = \ulcorner g \lrcorner = \frac{\sigma_1}{[\ulcorner f \lrcorner, \ulcorner g \lrcorner]}$$

and the claim follows. □

Next, we find that cells $[\alpha, \beta]$ enjoy certain absorption properties in $\ulcorner \mathbb{A} \lrcorner^{\oplus}$:

Lemma 7. *In* $\ulcorner\mathbb{A}\urcorner^{\oplus}$*:*

(i) $\gamma \mid [\alpha, \beta] = \frac{\delta^l}{[(\gamma|\alpha),(\gamma|\beta)]}$

(ii) $[\alpha, \beta] \mid \gamma = \frac{\delta^r}{[(\alpha|\gamma),(\beta|\gamma)]}$

(iii) $\frac{[\alpha,\beta]}{\gamma} = [\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}]$

Proof. (i) We have:

$$\frac{\sigma_0}{\frac{\mu^l}{\gamma|[\alpha,\beta]}} = \gamma \mid \frac{\sigma_0}{[\alpha, \beta]} = \gamma \mid \alpha = \frac{\sigma_0}{[(\gamma \mid \alpha), (\gamma \mid \beta)]}$$

$$\frac{\sigma_1}{\frac{\mu^l}{\gamma|[\alpha,\beta]}} = \gamma \mid \frac{\sigma_1}{[\alpha, \beta]} = \gamma \mid \beta = \frac{\sigma_1}{[(\gamma \mid \alpha), (\gamma \mid \beta)]}$$

and so we have $\frac{\mu^l}{\gamma|[\alpha,\beta]} = [(\gamma \mid \alpha), (\gamma \mid \beta)]$. Precomposing vertically with δ^l on both sides proves the claim.

(ii) Similar to (i).

(iii) We have:

$$\frac{\sigma_0}{\frac{[\alpha,\beta]}{\gamma}} = \frac{\alpha}{\gamma} = \frac{\sigma_0}{[\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}]} \qquad \frac{\sigma_1}{\frac{[\alpha,\beta]}{\gamma}} = \frac{\beta}{\gamma} = \frac{\sigma_1}{[\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}]}$$

and the claim follows.

□

While (iii) is analogous to the naturality of the codiagonal map in a category with finite coproducts, (i) and (ii) do not seem to admit similar analogies.

When restricted to the category of horizontal cells, the axioms concerning cells $\alpha \times \beta$ and $\alpha + \beta$ are precisely the axioms for binary products and binary coproducts. That is, we have:

Lemma 8. *We have:*

(i) $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{\oplus}$ *has binary products* $U \xleftarrow{\pi_0} U \times W \xrightarrow{\pi_1} W$.

(ii) $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{\oplus}$ *has binary coproducts* $U \xrightarrow{\amalg_0} U + W \xleftarrow{\amalg_1} W$.

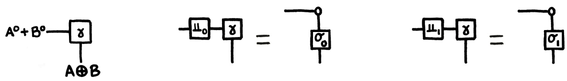
The category of horizontal cells $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ of the free cornering can be understood as a category of exchanges, a perspective developed in [25, 27]. In particular, isomorphic objects of $\mathbf{H}\ulcorner\mathbb{A}\urcorner$ correspond to exchanges of $\mathbb{A}^{\circ\bullet}$ that are *morally equivalent* (Lemma 3 of [25]). We show that $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{\oplus}$ contains two novel pairs of such morally equivalent exchanges:

Lemma 9. *In* $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{\oplus}$*:*

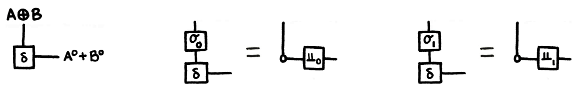
(i) $(A \oplus B)^\circ \cong A^\circ + B^\circ$ *and* $(A \oplus B)^\bullet \cong A^\bullet \times B^\bullet$

(ii) $(A \times B) \times C \cong A \times (B \times C)$ *and* $(A + B) + C \cong A + (B + C)$

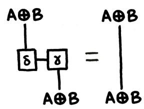
Proof. (i) Let $\gamma = \frac{\mathsf{get}_{\mathsf{L}}^{A}}{\sigma_0} + \frac{\mathsf{get}_{\mathsf{L}}^{B}}{\sigma_1} : \left(A^\circ{+}B^\circ \;{}^{\;\;I}_{A\oplus B}\, I\right)$. That is, γ is the unique cell such that:



Next, define $\delta = \left[\mathsf{put}_{\mathsf{R}}^{A} \mid \amalg_0, \mathsf{put}_{\mathsf{R}}^{B} \mid \amalg_1\right] : \left(I \;{}^{A\oplus B}_{\;\;I}\; A^\circ{\otimes}B^\circ\right)$. That is, δ is the unique cell such that:

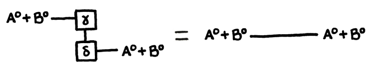


Then we have

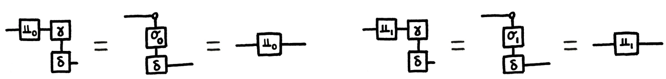


by the universal property of $\oplus$, since we have both of:

Similarly, we have:



by the universal property of $+$, as in:



Then the following arrows of $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{\oplus}$ are mutually inverse:

and the claim follows. The proof that $(A \oplus B)^\bullet = A^\bullet \times B^\bullet$ is similar.

(ii) Follows immediately from Lemma 8.

□

This makes intuitive sense: from the resource-theoretic perspective, an instance of $A \oplus B$ is either an instance of A or an instance of B. Then it certainly ought to be the case that `Alice` giving `Bob` an instance of $A \oplus B$ is the same as `Alice` choosing whether to give `Bob` an instance of A or to give `Bob` an instance of B. The above lemma tells us that this is indeed the case.

3.4. Crossing Cells

We extend Definition 3 to obtain crossing cells in the free cornering with choice:

Definition 11. Let $\mathbb{A}$ be a distributive monoidal category. For each $A \in \ulcorner\mathbb{A}\urcorner^{\oplus}_{H}$ and each $U \in \ulcorner\mathbb{A}\urcorner^{\oplus}_{V}$ we define a crossing cell $\chi_{U,A} : (U \, {}^{A}_{A} \, U)$ by induction on the structure of U. The cases for A°, $A^\bullet$, I, and $U \otimes W$ are as in Definition 3. For $U + W$ we define $\chi_{U+W,A} = (\chi_{U,A} \mid \perp\!\!\!\perp_0) + (\chi_{W,A} \mid \perp\!\!\!\perp_1)$. That is, $\chi_{U+W,A}$ is the unique cell satisfying:

Similarly, for $U \times W$ we define $\chi_{U \times W,A} = (\pi_0 \mid \chi_{U,A}) \times (\pi_1 \mid \chi_{W,A})$, so $\chi_{U \times W,A}$ is the unique cell satisfying:

We show that the crossing cells remain well-behaved. First, the crossing cells remain coherent with respect to horizontal composition in the free cornering with choice:

Lemma 10. *For $U \in A^{\circ\bullet}_{\oplus}$ and $A, B \in \mathbb{A}_0$ we have*

(i) $\chi_{U,A\otimes B} = \chi_{U,A} \mid \chi_{U,B}$

(ii) $\chi_{U,I} = id_U$

Proof. We provide the inductive cases necessary to extend the proof of Lemma 1 to account for the new structure.

(i) In the inductive case for $U + W$ we have:

$$\perp\!\!\!\perp_0 \mid \chi_{U+W,A\otimes B} = \chi_{U,A\otimes B} \mid \perp\!\!\!\perp_0 = \chi_{U,A} \mid \chi_{U,B} \mid \perp\!\!\!\perp_0 = \perp\!\!\!\perp_0 \mid \chi_{U+W,A} \mid \chi_{U+W,B}$$

Similarly, we have $\perp\!\!\!\perp_1 \mid \chi_{U+W,A\otimes B} = \perp\!\!\!\perp_1 \mid \chi_{U+W,A} \mid \chi_{U+W,B}$. It follows by the universal property of $- + -$ that $\chi_{U+W,A\otimes B} = \chi_{U+W,A} \mid \chi_{U+W,B}$. The inductive case for $U \times W$ is similar.

(ii) In the inductive case for $U + W$ we have:

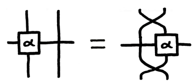
$$\perp\!\!\!\perp_0 \mid \chi_{U+W,I} = \chi_{U,I} \mid \perp\!\!\!\perp_0 = id_U \mid \perp\!\!\!\perp_0 = \perp\!\!\!\perp_0 \mid id_{U+W}$$

Similarly, we have $\perp\!\!\!\perp_1 \mid \chi_{U+W,I} = \perp\!\!\!\perp_1 \mid id_{U+W}$. It follows that $\chi_{U+W,I} = id_{U+W}$, as required. The case for $U \times W$ is similar.

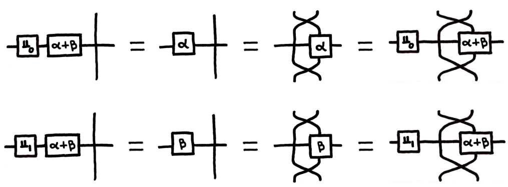
□

Further, we find that the core technical lemma concerning crossing cells still holds:

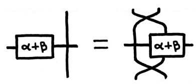
Lemma 11. *For any cell α of $\ulcorner \mathbb{A} \urcorner^{\oplus}$ we have*



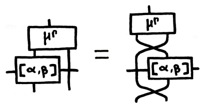
Proof. By structural induction on cells of $\ulcorner \mathbb{A} \urcorner^{\oplus}$. The base cases and the (inductive) cases for cells $\alpha \mid \beta$ and $\frac{\alpha}{\beta}$ are as in the proof of Lemma 2. The remaining inductive cases are as follows: For cells $\alpha + \beta$, we have:



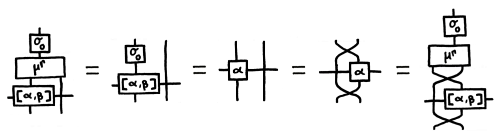
and then by the universal property of $+$ we have



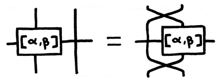
as required. The case for cells $\alpha \times \beta$ is similar. In the inductive case for cells $[\alpha, \beta]$ we have:



by the universal property of $\oplus$ as in:



with the case for σ_1 being similar. Precomposing vertically with δ_r yields:



$\square$

Consequently, $\ulcorner\mathbb{A}\urcorner^{\oplus}$ is a monoidal double category with the tensor product of cells and proof as in Lemma 3. We record:

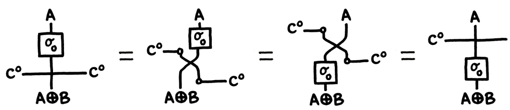
Lemma 12. *If $\mathbb{A}$ is a distributive monoidal category then $\ulcorner\mathbb{A}\urcorner^{\oplus}$ is a monoidal double category.*

Further, we find that crossing cells in the free cornering with choice are coherent with respect to $\oplus$ in the following sense:

Lemma 13. *In $\ulcorner\mathbb{A}\urcorner^{\oplus}$, $\chi_{U,A\oplus B} = \left[\frac{\chi_{U,A}}{\sigma_0}, \frac{\chi_{U,B}}{\sigma_1}\right]$. That is, $\chi_{U,A\oplus B}$ is the unique cell such that:*

$$\frac{\sigma_0}{\chi_{U,A\oplus B}} = \frac{\chi_{U,A}}{\sigma_0} \qquad\qquad \frac{\sigma_1}{\chi_{U,A\oplus B}} = \frac{\chi_{U,B}}{\sigma_1}$$

Proof. By structural induction on U. In case $U = C^\circ$ we have $\frac{\sigma_0}{\chi_{C^\circ,A\oplus B}} = \frac{\chi_{C^\circ,A}}{\sigma_0}$ as in:



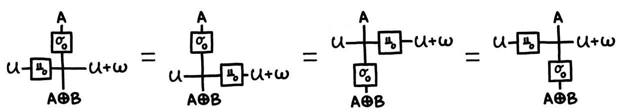
Similarly $\frac{\sigma_1}{\chi_{C^\circ,A\oplus B}} = \frac{\chi_{C^\circ,B}}{\sigma_1}$, and an analogous argument can be made for $U = C^\bullet$. If $U = I$ then we have:

$$\frac{\sigma_0}{\chi_{I,A\oplus B}} = \frac{\sigma_0}{1_{A\oplus B}} = \frac{1_A}{\sigma_0} = \frac{\chi_{I,A}}{\sigma_0}$$

Similarly $\frac{\sigma_1}{\chi_{I,A\oplus B}} = \frac{\chi_{I,B}}{\sigma_1}$. For $U \otimes W$ we have:

$$\frac{\sigma_0}{\chi_{U\otimes W,A\oplus B}} = \frac{\sigma_0}{\frac{\chi_{U,A\oplus B}}{\chi_{W,A\oplus B}}} = \frac{\frac{\chi_{U,A}}{\chi_{W,A}}}{\sigma_0} = \frac{\chi_{U\otimes W,A}}{\sigma_0}$$

Similarly, we have $\frac{\sigma_1}{\chi_{U\otimes W,A\oplus B}} = \frac{\chi_{U\otimes W,B}}{\sigma_1}$. For $U + W$ we have $\text{ɥ}_0 \mid \frac{\sigma_0}{\chi_{U+W,A\oplus B}} = \text{ɥ}_0 \mid \frac{\chi_{U+W,A}}{\sigma_0}$ as in:



An analogous argument gives $\text{ɥ}_1 \mid \frac{\sigma_0}{\chi_{U+W,A\oplus B}} = \text{ɥ}_1 \mid \frac{\chi_{U+W,A}}{\sigma_0}$, and so $\frac{\sigma_0}{\chi_{U+W,A\oplus B}} = \frac{\chi_{U+W,A}}{\sigma_0}$ by the universal property of $+$. Similarly, we have $\frac{\sigma_1}{\chi_{U+W,A\oplus B}} = \frac{\chi_{U+W,B}}{\sigma_1}$. The case for $U \times W$ is similar to the case for $U + W$. $\square$

3.5. A Model: Stateful Choice

In this section we return to the double category of stateful transformations defined in Section 2.4 over a cartesian closed category $\mathbb{C}$. We show that if $\mathbb{C}$ has binary coproducts $\oplus$ which are distributive with respect to the cartesian product $\otimes$ then we can define the branching protocols of the free cornering with chocie in the double category $\mathsf{S}(\mathbb{C})$.

Let $p_0^{X,Y} : X \otimes Y \to X$ and $p_1^{X,Y} : X \otimes Y \to Y$ be the projection maps, and note that they are natural in X and Y. For $f : A \to B$ and $g : A \to C$, we write $\langle f, g\rangle : A \to B \otimes C$ to be the unique morphism such that $\langle f, g\rangle p_0 = f$ and $\langle f, g\rangle p_1 = g$. We note that the injections $\sigma_0^{X,Y} : X \to X \oplus Y$, $\sigma_1^{X,Y} : Y \to X \oplus Y$ given by the coproduct structure are also natural in X and Y. Finally, both $\otimes$ and $\oplus$ give bifunctors $\mathbb{C}$ sending $f : A \to C$ and $g : B \to D$ to $(f \otimes g) = \langle p_0\, f, p_1\, g\rangle : A \otimes B \to C \otimes D$ and $(f \oplus g) = [f\, \sigma_0, g\, \sigma_1]$.

Given two strong endofunctors (F, τ^F) and (G, τ^F), we define their product $(F \times G, \tau^{F \times G})$ as follows: the endofunctor $F \times G$ is given by $(F \times G)(X) = FX \otimes GX$ and $(F \times G)(f) = F(f) \otimes G(f)$; and the strength $\tau^{F\times G}$ is given by:

$$(F\times G)(X)\otimes A \xrightarrow{\langle (p_0 \otimes A), (p_1 \otimes A)\rangle} FX\otimes A\otimes GX\otimes A \xrightarrow{\tau^F_{X,A} \otimes \tau^G_{X,A}} (F\times G)(X\otimes A)$$

Similarly, the endofunctor $F + G$ is given by $(F + G)(X) = FX \oplus GX$ and $(F + G)(f) = F(f) \oplus G(f)$. Its strength τ^{F+G} is given by:

$$(F + G)(X) \otimes A \xrightarrow{\delta^r_{FX,GX,A}} (FX \otimes A) \oplus (GX \otimes A) \xrightarrow{\tau^F_{X,A} \oplus \tau^G_{X,A}} (F + G)(X \otimes A)$$

These two constructions have the same universal properties as the corresponding definitions of choice protocols in Definition 10. Specifically, we have 2-cells $\pi_0^{F,G} : (F{\times}G\,{}^{1}_{1}\,F)$ and $\pi_1^{F,G} : (F{\times}G\,{}^{1}_{1}\,F)$ with components $p_0^{FX,GX}$ and $p_1^{FX,GX}$ respectively. Given $\alpha : (F\,{}^{A}_{B}\,G)$ and $\beta : (F\,{}^{A}_{B}\,H)$, we define $(\alpha \times \beta) : (F\,{}^{A}_{B}\,G{\times}H)$ by:

$$(\alpha \times \beta)_X = \langle \alpha_X, \beta_X\rangle : FX \otimes A \to (G \times H)(X \otimes B)$$

Dually, we have 2-cells $\rotatebox{180}{$\pi$}_0^{F,G} : (F\,{}^{1}_{1}\,F{+}G)$ and $\rotatebox{180}{$\pi$}_1^{F,G} : (G\,{}^{1}_{1}\,F{+}G)$ with components $\sigma_0^{FX,GX}$ and $\sigma_1^{FX,GX}$ respectively. Given $\alpha : (F\,{}^{A}_{B}\,H)$ and $\beta : (G\,{}^{A}_{B}\,H)$, we define $(\alpha \times \beta) : (F{+}G\,{}^{A}_{B}\,H)$ by using the distributivity natural transformation:

$$(\alpha + \beta)_X = \delta^r[\alpha_X, \beta_X] : (F + G)(X) \otimes A \to H(X \otimes B)$$

Finally, we have vertical injection cells $[\sigma_0^{A,B}] : (I\,{}^{A}_{A\oplus B}\,I)$ and $[\sigma_1^{A,B}] : (I\,{}^{B}_{A\oplus B}\,I)$. Given $\alpha : (F\,{}^{A}_{C}\,G)$ and $\beta : (F\,{}^{B}_{C}\,G)$, we define the corresponding copairing cell $[\alpha, \beta] : (F\,{}^{A\oplus B}_{C}\,H)$ as in:

$$[\alpha, \beta]_X = \delta^l[\alpha_X, \beta_X] : FX \otimes (A \oplus B) \to G(X \otimes C)$$

These constructions have universal properties inherited from the distributive cartesian structure of $\mathbb{C}$, and it is straightforward to show that the projections and injections are strong. We show that these constructions are well-defined in the sense that they give 2-cells of $\mathsf{S}(\mathbb{C})$:

Lemma 14. *If α and β are 2-cells of $\mathsf{S}(\mathbb{C})$, then $\alpha \times \beta$, $\alpha + \beta$ and $[\alpha, \beta]$ are 2-cells of $\mathsf{S}(\mathbb{C})$.*

Proof. We show that the results of the operations are strong by showing that they are a composition of strong natural transformations. First, note that the pairing and copairing operations $\langle -, - \rangle$ and $[-, -]$ when applied to strong natural transformations α, β create strong natural transformations, as in:

$$\begin{tikzcd}
FX \otimes A \arrow[rr, "\langle \alpha_X, \beta_X \rangle \otimes 1_A"] \arrow[dr, "\langle \alpha_X \otimes 1_A, \beta_X \otimes 1_A \rangle"] \arrow[dd, "\tau^F_{X,A}"'] & & (G \times H)(X) \otimes A \arrow[d, "\langle p_0 \otimes 1_A, p_1 \otimes 1_A \rangle"] \\
& & GX \otimes A \otimes HX \otimes A \arrow[d, "\tau^G_{X,A} \otimes \tau^H_{X,A}"] \\
F(X \otimes A) \arrow[rr, "\langle \alpha_{X \otimes A}, \beta_{X \otimes A} \rangle"'] & & (G \otimes H)(X \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
(F + G)(X) \otimes A \arrow[rr, "{[\alpha_X, \beta_X] \otimes 1_A}"] \arrow[d, "\delta^r"'] & & HX \otimes A \arrow[dd, "\tau^H_{X,A}"] \\
(FX \otimes A) \oplus (GX \otimes A) \arrow[urr, "{[\alpha_X \otimes 1_A, \beta_X \otimes 1_A]}"] \arrow[d, "{[\tau^F_{X,A}, \tau^G_{X,A}]}"'] & & \\
(F + G)(X \otimes A) \arrow[rr, "{[\alpha_{X \otimes A}, \beta_{X \otimes A}]}"'] & & H(X \otimes A)
\end{tikzcd}$$

Secondly, given strong endofunctors F and G, and objects A and B, both of the natural transformations $\delta^r : (FX \oplus GX) \otimes A \to (FX \otimes A) \oplus (GX \otimes A)$ and $\delta^l : FX \otimes (A \oplus B) \to (FX \otimes A) \oplus (FX \otimes B)$ are strong since their inverses μ^r and μ^l can be constructed using σ_0, σ_1 and $[-, -]$. We conclude that $\alpha \times \beta$, $\alpha + \beta$ and $[\alpha, \beta]$ are strong since they are compositions of strong natural transformations. $\square$

Now the structure-preserving double functor $D : \ulcorner\mathbb{C}\lrcorner \to \mathsf{S}(\mathbb{C})$ of Section 2.4 extends to a double functor $D : \ulcorner\mathbb{C}\lrcorner^{\oplus} \to \mathsf{S}(\mathbb{C})$ with $D(A \oplus B) = DA \oplus DB$, $D(U + W) = DU + DW$, $D(U \times W) = DU \times DW$, $D([\alpha, \beta]) = [D(\alpha), D(\beta)]$, $D(\alpha + \beta) = D(\alpha) + D(\beta)$, and $D(\alpha \times \beta) = D(\alpha) \times D(\beta)$. Clearly this double functor preserves the branching protocol structure in addition to the corner cells and monoidal double category structure, and so gives a model of $\ulcorner\mathbb{C}\lrcorner^{\oplus}$.

Remark 2. We consider the interpretation of branching protocols given by $- + -$ and $- \times -$ from the perspective of computational effects, extending Remark 1. As a computational effect, $F + G$ may be triggered by any program which would trigger F or G. Dually, to resolve an effect $F + G$ its environment must be able to resolve both F and G independently. Similarly, to trigger $F \times G$ a program must be ready for the environment to resolve either of F or G, and dually to resolve $F \times G$ the environment must be able to resolve either F or G independently. For $F + G$ the choice comes from the interior, while for $F \times G$ the choice comes from the environment.

This concludes our discussion of the free cornering with choice. We proceed to add a notion of protocol iteration the the free

4. Adding Iteration to the Free Cornering

In this section we extend the free cornering with choice to include a notion of protocol iteration. In Section 4.1 we construct the *free cornering with iteration* over a distributice monoidal category and discuss its interpretation. In

Section 4.2 we establish a number of elementary properties of the free cornering with iteration, and in Section 4.3 we define crossing cells in the free cornering with iteration, and show that they are well-behaved (extending Section 3.4 to the new setting). Finally, in Section 4.4 we discuss iteration in terms of the double category of stateful transformations.

4.1. The Free Cornering with Iteration

In this section we extend the free cornering with choice to include a notion of protocol iteration. We begin by extending the monoid of exchanges with choice (Definition 9) with unary operations $(-)^+$ and $(-)^\times$ representing iterated protocols:

Definition 12. Let $\mathbb{A}$ be a symmetric monoidal category. The monoid $A_*^{\circ\bullet}$ of $\mathbb{A}$-*valued exchanges with choice and iteration* has elements generated by:

$$\frac{A \in \mathbb{A}_0}{A^\circ \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{A \in \mathbb{A}_0}{A^\bullet \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{}{I \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U \otimes W \in \mathbb{A}_*^{\circ\bullet}}$$

$$\frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U \times W \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet} \qquad W \in \mathbb{A}_*^{\circ\bullet}}{U + W \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet}}{U^\times \in \mathbb{A}_*^{\circ\bullet}} \qquad \frac{U \in \mathbb{A}_*^{\circ\bullet}}{U^+ \in \mathbb{A}_*^{\circ\bullet}}$$

subject to the following equations:

$$I \otimes U = U \qquad U \otimes I = U \qquad (U \otimes W) \otimes V = U \otimes (W \otimes V)$$

$$U^\times = I \times (U \otimes U^\times) \qquad U^+ = I + (U \otimes U^+)$$

Notice in particular that $\mathbb{A}_\oplus^{\circ\bullet}$ embeds into $\mathbb{A}_*^{\circ\bullet}$.

We extend our interpretation of elements of $\mathbb{A}_\oplus^{\circ\bullet}$ to elements of $\mathbb{A}_*^{\circ\bullet}$. To do so we must interpret $U^\times$ and U^+. We begin with $U^\times$. Our interpretation of $U^\times$ is informed by the equation $U^\times = I \times (U \otimes U^\times)$. Recall that $V \times W$ is the protocol in which the right participant chooses whether to continue as V or W. It follows that $U^\times$ is the protocol in which the right participant chooses whether to continue as I or $U \otimes U^\times$. Thus, $U^\times$ is the protocol in which the right participant chooses whether to do nothing, or to do U and then $U^\times$ again. Put another way, $U^\times$ is the iterated version of U, in which the right participant decides when to stop iterating. Our interpretation of U^+ is dual. Everything is as above, except that the roles of the right and left participant are swapped.

For example, suppose $A, B \in \mathbb{A}_0$. For each of the following exchanges, call the left participant `Alice` and the right participant `Bob`, as before. Now, consider:

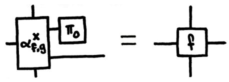
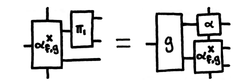
- To carry out $(A^\circ)^\times = I \times (A^\circ \otimes (A^\circ)^\times) \in \mathbb{A}_*^{\circ\bullet}$, first `Bob` chooses which of I and $A^\circ \otimes (A^\circ)^\times$ will happen. If `Bob` chooses I then the exchange ends. If `Bob` chooses $A^\circ \otimes (A^\circ)^\times$ then `Alice` sends `Bob` and instance of A and the two of them carry out $(A^\circ)^\times$ again from the beginning. In other words, `Bob` can request any number of instances of A from `Alice`.

- To carry out $(A^\circ \times B^\bullet)^+ \in \mathbb{A}_*^{\circ\bullet}$, first `Alice` chooses which of I and $(A^\circ \times B^\bullet) \otimes (A^\circ \times B^\bullet)^+$ will happen. If `Alice` chooses I then the exchange ends. If `Alice` chooses $(A^\circ \times B^\bullet) \otimes (A^\circ \times B^\bullet)^+$ then `Bob` chooses which of A° and $B^\bullet$ will happen, with `Alice` sending `Bob` an instance of A if `Bob` chooses A° and `Bob` sending `Alice` an instance of B if he chooses $B^\bullet$. Then, the two of them carry out $(A^\circ \times B^\bullet)^+$ again from the beginning.

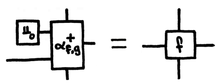
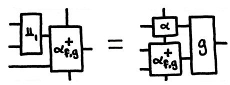
The key idea in our notion of iteration is the equation $U^\times = I \times (U \otimes U^\times)$, which allows us to use $\pi_0 : U^\times \to I$ and $\pi_1 : U^\times \to U \otimes U^\times$ to express properties of $U^\times$. Dually, we will be able to use $\amalg_0 : I \to U^+$ and $\amalg_1 : U \otimes U^+ \to U^+$ to express properties of U^+. We proceed to extend the free cornering with the notion of iteration suggested by this:

Definition 13. Let $\mathbb{A}$ be a distributive monoidal category. We define the *free cornering with iteration of* $\mathbb{A}$, written $\ulcorner\mathbb{A}\lrcorner^*$, to be the free single-object double category with horizontal edge monoid $(\mathbb{A}_0, \otimes, I)$, vertical edge monoid $\mathbb{A}_*^{\circ\bullet}$, and generating cells and equations consisting of:

- The generating cells and equations of $\ulcorner\mathbb{A}\lrcorner^\oplus$ (Definition 10).
- For each trio of cells $\alpha : \ulcorner\mathbb{A}\lrcorner^*(V\,{}^{A}_{A}\,U)$, $f : \ulcorner\mathbb{A}\lrcorner^*(W\,{}^{A}_{B}\,K)$, and $g : \ulcorner\mathbb{A}\lrcorner^*(W\,{}^{I}_{I}\,V{\otimes}W)$ a unique cell $\alpha^\times_{f,g} : \ulcorner\mathbb{A}\lrcorner^*(W\,{}^{A}_{B}\,U^\times{\otimes}K)$ satisfying:

- Dually, for each trio of cells $\alpha : \ulcorner\mathbb{A}\lrcorner^*(U\,{}^{A}_{A}\,V)$, $f : \ulcorner\mathbb{A}\lrcorner^*(K\,{}^{A}_{B}\,W)$, and $g : \ulcorner\mathbb{A}\lrcorner^*(V{\otimes}W\,{}^{I}_{I}\,W)$ a unique cell $\alpha^+_{f,g} : \ulcorner\mathbb{A}\lrcorner^*(U^+{\otimes}K\,{}^{A}_{B}\,W)$ satisfying:

We extend our interpretation of cells of $\ulcorner\mathbb{A}\lrcorner^\oplus$ as interacting processes to cells of $\ulcorner\mathbb{A}\lrcorner^*$. Recall that $U^\times = I \times (U \otimes U^\times)$ is the exchange in which the right participant chooses whether the exchange is over (via π_0), or is to continue as $U \otimes U^\times$ (via π_1). The cells $\alpha^\times_{f,g}$ enable our procedures to be the left participant of such exchanges, reacting to the choices of the right participant as specified by the equations. Of course, $\alpha^+_{f,g}$ is the dual version for exchanges U^+, with the roles of the left and right participants swapped.

Remark 3. In developing an intuition about this it is helpful to consider the following, simpler forms of these cell formation rules: Let $\alpha : (U\,{}^{A}_{A}\,W)$. We define:

$$\alpha^\times = \alpha^\times_{\pi_0,\pi_1} : (U^\times\,{}^{A}_{A}\,W^\times) \qquad\qquad \alpha^+ = \alpha^+_{\amalg_0,\amalg_1} : (U^+\,{}^{A}_{A}\,W^+)$$

Then $\alpha^\times$ and is the unique cell such that:

$$\alpha^\times \mid \pi_0 = \pi_0 \mid 1_A \qquad\qquad \alpha^\times \mid \pi_1 = \pi_1 \mid \frac{\alpha}{\alpha^\times}$$

and similarly α^+ is the unique cell such that:

$$\amalg_0 \mid \alpha^+ = 1_A \mid \amalg_0 \qquad\qquad \amalg_1 \mid \alpha^+ = \frac{\alpha}{\alpha^+} \mid \amalg_1$$

Interpreted as an interacting process, $\alpha^\times$ reacts to the choice (made along the right boundary) to stop iterating by doing nothing to its inputs and propagating this choice along its left boundary. Similarly, $\alpha^\times$ reacts the the choice to continue iterating, performing U once before performing $U^\times$ again, by acting as α once and then continuing as $\alpha^\times$, propagating this choice along its left boundary. The interpretation of α^+ is similar. We note that both $(-)^+$ and $(-)^\times$ give functors $\mathbf{H}\ulcorner\mathbb{A}\urcorner^* \to \mathbf{H}\ulcorner\mathbb{A}\urcorner^*$.

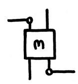
Remark 4. Before moving on we note that cells $\alpha^\times_{f,g}$, and $\alpha^+_{f,g}$ admit a coinductive reasoning principle. For $\alpha^\times_{f,g}$, we have that if γ satisfies:

$$\gamma \mid \frac{\pi_0}{id} = f \qquad\qquad \gamma \mid \frac{\pi_1}{id} = g \mid \frac{\alpha}{\gamma}$$

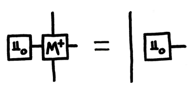
then $\gamma = \alpha^\times_{f,g}$, which we say holds by *coinduction*. The coinductive reasoning principles for cells $\alpha^+_{f,g}$ is similar.

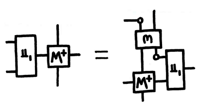
Example 4 (Mealy Machines)**.** Say that a *mealy machine in a monoidal category* $\mathbb{A}$ consists of a morphism $m : A \otimes S \to S \otimes B$. Then the classical notion of mealy machine is recoverable by considering mealy machine in the category of finite sets, with S the set of states, A the input alphabet, and B the output alphabet. Mealy machines are usually understood to operate on a sequence of inputs drawn from A, producing a sequence of outputs drawn from B. The state of the machine is fed forward to future iterations.

If $m : A \otimes S \to S \otimes B$ in $\mathbb{A}$ then let $M : ({A^\circ}\,{}^{S}_{S}\,{B^\circ})$ be the cell:



Then the cell $M^+ : \left((A^\circ)^+\,{}^{S}_{S}\,(B^\circ)^+\right)$ exhibits the behaviour of the process that the Mealy machine m is intended to define, as in:



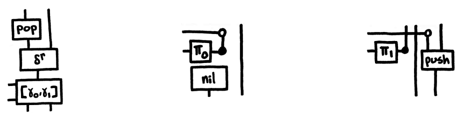


that is, if the there is no more input then the machine produces no more output, and if there is further input then the machine produces output accoring to m and updates its internal state.

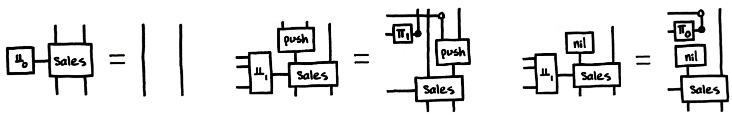
Example 5 (Memory Cell)**.** Consider the cell $H = \left(\frac{\mathsf{put}^A_{\mathsf{R}}}{\mathsf{get}^A_{\mathsf{R}}}\right)^\times : \left(I\,{}^{A}_{A}\,(A^\circ \otimes A^\bullet)^\times\right)$. This cell behaves as follows:

Think of H as a simple sort of memory cell that stores a value of type A. When called upon by the environment along its right boundary, the cells outputs its contents, waits for its new contents to be supplied, and waits for further instructions (above right). The cell can also be told to stop (above left).

Example 6. Suppose our base category $\mathbb{A}$ has objects `bread` and `$`, as well as objects representing stacks of each: $S_{\mathtt{bread}} \cong I \oplus (\mathtt{bread} \otimes S_{\mathtt{bread}})$ and $S_{\$} \cong I \oplus (\$ \otimes S_{\$})$ as in Section 3.1. We construct a process that sells bread, $\mathtt{sales} : \ulcorner\mathbb{A}\lrcorner^* \left((\$^\circ \otimes (\$^\bullet \times \mathtt{bread}))^+ \, {}^{S_{\mathtt{bread}} \otimes S_{\$}}_{S_{\mathtt{bread}} \otimes S_{\$}} \, I\right)$ as follows: Let $\mathtt{sale} : \ulcorner\mathbb{A}\lrcorner^* \left(\$^\circ \otimes (\$^\bullet \times \mathtt{bread}) \, {}^{S_{\mathtt{bread}} \otimes S_{\$}}_{S_{\mathtt{bread}} \otimes S_{\$}} \, I\right)$ be the cell below on the left, with γ_0 and γ_1 the cells below on the right.



Then define $\mathtt{sales} = \mathtt{sale}^+_{1_{S_{\mathtt{bread}} \otimes S_{\$}}, \Box_I}$. We have:



We imagine the left boundary of `sales` as aa sort of queue of customers, waiting to purchase bread. If there are no more customers ($\amalg_0$) then `sales` simply retains its stacks of `bread` and `$`. If there is at least one more customer (π_1) then `sales` receives `$` from the first customer in line. If no bread is available (the stack of bread is `nil`) then the money is returned. Otherwise the customer receives the firs tpiece of bread on the stack. This process is repeated until no customers remain.

4.2. Elementary Properties

We proceed to establish some elementary properties of the free cornering with iteration. First we note that the properties of $\ulcorner\mathbb{A}\lrcorner^\oplus$ established in Section 3.3 all hold in the free cornering with iteration, specifically Lemmas 6,7,8, and 9 all hold in $\ulcorner\mathbb{A}\lrcorner^*$.

Moving on to elementary properties of $\ulcorner\mathbb{A}\lrcorner^*$ specifically, we show that $U^\times$ and U^+ arise as (co)algebras of a functor on the category of horizontal cells:

Lemma 15. *Consider the category $\mathbf{H}\ulcorner\mathbb{A}\lrcorner^*$ of horizontal cells of the free cornering with iteration. For all objects U of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner^*$, we have:*

(i) $(U^{\times}, id_{U^{\times}} : U^{\times} \to U^{\times} = I \times (U \otimes U^{\times}))$ *is the final coalgebra of the functor*

$$I \times (U \otimes -) : \mathbf{H}\ulcorner\mathbb{A}\urcorner^{*} \to \mathbf{H}\ulcorner\mathbb{A}\urcorner^{*}$$

(ii) $(U^{+}, id_{U^{+}} : U^{+} \to U^{+} = I + (U \otimes U^{+}))$ *is the initial algebra of the functor*

$$I + (U \otimes -) : \mathbf{H}\ulcorner\mathbb{A}\urcorner^{*} \to \mathbf{H}\ulcorner\mathbb{A}\urcorner^{*}$$

Proof. (i) Suppose $(W, h : W \to I \times (U \otimes W))$ is a coalgebra for $(I \times (U \otimes -)$. We must show that there is a unique coalgebra morphism $(W, h) \to (U^{\times}, id_{U^{\times}})$ in $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{*}$. Define $\alpha = (id_U)^{\times}_{(h|\pi_0),(h|\pi_1)} : (W\,{}^{I}_{I}\,U^{\times})$. We must show that α gives a morphism of coalgebras. That is, we must show that in $\mathbf{H}\ulcorner\mathbb{A}\urcorner^{*}$ we have:

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "\alpha"'] & I \times (U \otimes W) \arrow[d, "(\pi_0 \times \pi_1 | \frac{id_U}{\alpha}) = (I \times (U \otimes -))(\alpha)"] \\ U^{\times} \arrow[r, "id_{U^{\times}}"'] & U^{\times} = I \times (U \otimes U^{\times}) \end{tikzcd}$$

This is because we have:

$$h \mid (\pi_0 \times \pi_1 \mid \frac{id_U}{\alpha}) \mid \pi_0 = h \mid \pi_0$$

and

$$h \mid (\pi_0 \times \pi_1 \mid \frac{id_U}{\alpha}) \mid \pi_1 = h \mid \pi_1 \mid \frac{id_U}{\alpha}$$

and so by coinduction we have that $h \mid (\pi_0 \times \pi_1 \frac{id_U}{\alpha}) = (1_U)^{\times}_{(h|\pi_0),(h|\pi_1)} = \alpha$, and so $\alpha : (W, h) \to (U^{\times}, id_{U^{\times}})$ is a morphism of coalgebras. It remains to show that α is the unique such coalgebra morphism. To that end, suppose that $\beta : (W, h) \to (U^{\times}, id_{U^{\times}})$ is a coalgebra morphism. That is, suppose we have:

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "\beta"'] & I \times (U \otimes W) \arrow[d, "(\pi_0 \times \pi_1 | \frac{id_U}{\beta}) = (I \times (U \otimes -))(\beta)"] \\ U^{\times} \arrow[r, "id_{U^{\times}}"] & U^{\times} = I \times (U \otimes U^{\times}) \end{tikzcd}$$

Then we have:

$$h \mid (\pi_0 \times \pi_1 \mid \frac{id_U}{\beta}) \mid \pi_0 = h \mid \pi_0$$

and

$$h \mid (\pi_0 \times \pi_1 \mid \frac{id_U}{\beta}) \mid \pi_1 = h \mid \pi_1 \mid \frac{id_U}{\beta}$$

and so by coinduction we have that $\beta = (1_U)^{\times}_{(h|\pi_0),(h|\pi_1)} = \alpha$, as required.

(ii) Similar to the proof of (i).

∎

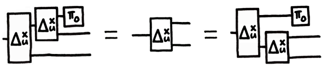
Further, we exhibit (co)monoid structures on our iterated protocol types and show that they enjoy a kind of naturality:

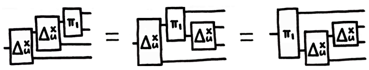
Lemma 16. *For all objects U of $\mathbf{H}\ulcorner\mathbb{A}\urcorner^*$:*

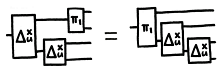
(i) $(U^\times, \Delta_U^\times, \pi_0)$ *is a comonoid in* $\mathbf{H}\ulcorner\mathbb{A}\urcorner^*$ *where* $\Delta_U^\times = (id_U)^\times_{id_{U^\times}, \pi_1}$. *Dually,* $(U^+, \nabla_U^+, \amalg_0)$ *is a monoid in* $\mathbf{H}\ulcorner\mathbb{A}\urcorner^*$ *where* $\nabla_U^+ = (id_U)^+_{id_{U^+}, \amalg_1}$.

(ii) For any $h : (U\,{}^{I}_{I}\,W)$, $\Delta_U^\times \mid \frac{h^\times}{h^\times} = h^\times \mid \Delta_W^\times$. *Dually,* $\nabla_U^+ \mid h^+ = \frac{h^+}{h^+} \mid \nabla_W^+$.

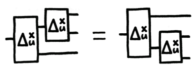
Proof. (i) We must show that $(U^\times, \Delta^\times, \pi_0)$ is coassociative and counital. For coassociativity, we have:



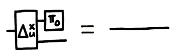




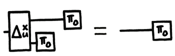
and then by coinduction we have:

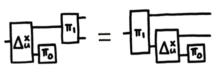


as required. The first counitality axiom holds immediately:

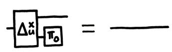


For the second counitality axiom, we have:





and so, since $id_{U^\times} \mid \pi_1 = \pi_1 \mid \frac{id_{U^\times}}{id_{U^\times}}$, we have by coinduction that the second unitality axiom holds, as in:



It follows that $(U^\times, \Delta_U^\times, \pi_0)$ is a comonoid. The proof that $(U^+, \nabla_U^+, \amalg_0)$ is a monoid is similar.

(ii) Suppose $h : (U \,{}^{I}_{I}\, W)$. Then we have:

$$h^\times \mid \Delta^\times_W \mid \frac{\pi_0}{id_{U^\times}} = h^\times = \Delta^\times_U \mid \frac{\pi_0}{h^\times} = \Delta^\times_U \mid \frac{h^\times}{h^\times} \mid \frac{\pi_0}{id_{U^\times}}$$

$$h^\times \mid \Delta^\times_W \mid \frac{\pi_1}{id_{W^\times}} = h^\times \mid \pi_1 \mid \frac{id_W}{\Delta^\times_W} = \pi_1 \mid \frac{h}{h^\times} \mid \frac{id_W}{\Delta^\times_W} = \pi_1 \mid \frac{h}{h^\times \mid \Delta^\times_W}$$

$$\Delta^\times_U \mid \frac{h^\times}{h^\times} \mid \frac{\pi_1}{id_{U^\times}} = \Delta^\times_U \mid \frac{\pi_1 \mid \frac{h}{h^\times}}{h^\times} = \pi_1 \mid \frac{h}{\Delta^\times_U \mid \frac{h^\times}{h^\times}}$$

and so by coinduction $\Delta^\times_U \mid \frac{h^\times}{h^\times} = h^\times \mid \Delta^\times_W$, as promised. The proof that $\nabla^+_U \mid h^+ = \frac{h^+}{h^+} \mid \nabla^+_W$ is similar.

□

We end our discussion of the elementary properties of $\ulcorner\mathbb{A}\lrcorner^*$ by showing that the functors $(-)^\times$ and $(-)^+$ of Remark 3 are (co)monadic:

Lemma 17. *Consider the category* $\mathbf{H}\ulcorner\mathbb{A}\lrcorner^*$. *We have:*

(i) The functor

$$(-)^\times : \mathbf{H}\ulcorner\mathbb{A}\lrcorner^* \to \mathbf{H}\ulcorner\mathbb{A}\lrcorner^*$$

is a comonad with counit $\varepsilon^\times : (-)^\times \to 1_{\mathbf{H}\ulcorner\mathbb{A}\lrcorner^*}$ *given by components* $\varepsilon^\times_U = \pi_1 \mid \frac{id_U}{\pi_0} : (U^\times \,{}^{I}_{I}\, U)$ *and comultiplication* $\delta^\times : (-)^\times \to (-)^{\times\times}$ *given by components* $\delta^\times_U = (id_{U^\times})^\times_{\pi_0, \Delta^\times_U} : (U^\times \,{}^{I}_{I}\, U^{\times\times})$.

(ii) The functor

$$(-)^+ : \mathbf{H}\ulcorner\mathbb{A}\lrcorner^* \to \mathbf{H}\ulcorner\mathbb{A}\lrcorner^*$$

is a monad with unit $\eta^+ : 1_{\mathbf{H}\ulcorner\mathbb{A}\lrcorner^*} \to (-)^+$ *given by components* $\eta^+_U = \frac{id_U}{\amalg_0} \mid \amalg_1 : (U \,{}^{I}_{I}\, U^+)$ *and comultiplication* $\mu^+ : (-)^{++} \to (-)^+$ *given by components* $\mu^+_U = (id_{U^+})^+_{\amalg_0, \nabla^+_U} : (U^{++} \,{}^{I}_{I}\, U^+)$.

Proof. (i) It is straightforward to verify that $(-)^\times$ is a functor. In order to prove that it is a comonad we first show that $\varepsilon^\times$ and $\delta^\times$ are natural. Explicitly, we require:

$$\begin{tikzcd} U^\times \arrow[r, "\varepsilon^\times_U"] \arrow[d, "h^\times"'] & U \arrow[d, "h"] \\ W^\times \arrow[r, "\varepsilon^\times_W"'] & W \end{tikzcd} \qquad \begin{tikzcd} U^\times \arrow[r, "\delta^\times_U"] \arrow[d, "h^\times"'] & U^{\times\times} \arrow[d, "h^{\times\times}"] \\ W^\times \arrow[r, "\delta^\times_W"'] & W^{\times\times} \end{tikzcd}$$

for any $h : U \to W$ of $\mathbf{H}\ulcorner\mathbb{A}\lrcorner$. For $\varepsilon^\times$ we have:

$$h^\times \mid \varepsilon^\times_W = h^\times \mid \pi_1 \mid \frac{id_W}{\pi_0} = \pi_1 \mid \frac{h}{h^\times} \mid \frac{id_U}{\pi_0} = \pi_1 \mid \frac{h}{\pi_0} = \varepsilon^\times_U \mid h$$

as required. For $\delta^\times$ we have:

$$h^\times \mid \delta^\times_W \mid \pi_0 = h^\times \mid \pi_0 = \pi_0 = \delta^\times_U \mid \pi_0 = \delta^\times_U \mid h^{\times\times} \mid \pi_0$$

$$h^\times \mid \delta^\times_W \mid \pi_1 = h^\times \mid \Delta^\times_W \mid \frac{id_{W^\times}}{\delta^\times_W} = \Delta^\times_U \mid \frac{h^\times}{h^\times} \mid \frac{id_{W^\times}}{\delta^\times_W} = \Delta^\times_U \mid \frac{h^\times}{h^\times \mid \delta^\times_W}$$

$$\delta^\times_U \mid h^{\times\times} \mid \pi_1 = \delta^\times_U \mid \pi_1 \mid \frac{h^\times}{h^{\times\times}} = \Delta^\times_U \mid \frac{id_{U^\times}}{\delta^\times_U} \mid \frac{h^\times}{h^{\times\times}} = \Delta^\times_U \mid \frac{h^\times}{\delta^\times_U \mid h^{\times\times}}$$

and then by coinduction we have $h^\times \mid \delta^\times_W = \delta^\times_U \mid h^{\times\times}$ as required.

It remains to show that the comonad axioms are satisfied. That is, we require:

$$\begin{tikzcd}
U^\times \arrow[r, "\delta^\times_U"] \arrow[d, "\delta^\times_U"'] & U^{\times\times} \arrow[d, "\delta^\times_{U^\times}"] \\
U^{\times\times} \arrow[r, "(\delta^\times_U)^\times"'] & U^{\times\times\times}
\end{tikzcd}
\qquad
\begin{tikzcd}
U^\times & U^{\times\times} \arrow[l, "\varepsilon^\times_{U^\times}"'] \arrow[r, "(\varepsilon^\times_U)^\times"] & U^\times \\
& U^\times \arrow[ul, "id_{U^\times}"] \arrow[u, "\delta^\times_U"] \arrow[ur, "id_{U^\times}"'] &
\end{tikzcd}$$

Notice that by coinduction $\delta^\times_U \mid \Delta^\times_{U^\times} = \Delta^\times_U \mid \frac{\delta^\times_U}{\delta^\times_U}$ although, somewhat misleadingly, not as a consequence of Lemma 16. For coassociativity of $\delta^\times$, we have:

$$\delta^\times_U \mid \delta^\times_{U^\times} \mid \pi_0 = \delta^\times_U \mid \pi_0 = \delta^\times_U \mid (\delta^\times_U)^\times \mid \pi_0$$

$$\delta^\times_U \mid \delta^\times_{U^\times} \mid \pi_1 = \delta^\times_U \mid \Delta^\times_{U^\times} \mid \frac{id_{U^{\times\times}}}{\delta^\times_{U^\times}} = \Delta^\times_U \mid \frac{\delta^\times_U}{\delta^\times_U} \mid \frac{id_{U^{\times\times}}}{\delta^\times_{U^\times}} = \Delta^\times_U \mid \frac{\delta^\times_U}{\delta^\times_U \mid \delta^\times_{U^\times}}$$

$$\delta^\times_U \mid (\delta^\times_U)^\times \mid \pi_1 = \delta^\times_U \mid \pi_1 \mid \frac{\delta^\times_U}{(\delta^\times_U)^\times} = \Delta^\times_U \mid \frac{id_{U^\times}}{\delta^\times_U} \mid \frac{\delta^\times_U}{(\delta^\times_U)^\times} = \Delta^\times_U \mid \frac{\delta^\times_U}{\delta^\times_U \mid (\delta^\times_U)^\times}$$

and so by coinduction we have $\delta^\times_U \mid \delta^\times_{U^\times} = \delta^\times_U \mid (\delta^\times_U)^\times$ as required. For the first counit law, we have:

$$\delta^\times_U \mid \varepsilon^\times_{U^\times} \mid \pi_0 = \delta^\times_U \mid \pi_1 \mid \frac{\pi_0}{\pi_0} = \Delta^\times_U \mid \frac{\pi_0}{\delta^\times_U \mid \pi_0} = \pi_0 = id_{U^\times} \mid \pi_0$$

$$\delta^\times_U \mid \varepsilon^\times_{U^\times} \mid \pi_1 = \delta^\times_U \mid \pi_1 \mid \frac{\pi_1}{\pi_0} = \Delta^\times_U \mid \frac{\pi_1}{\delta^\times_U \mid \pi_0} = \pi_1 \mid \frac{id_U}{\Delta^\times_U \mid \frac{id_{u^\times}}{\pi_0}} = \pi_1 =$$

$$id_{U^\times} \mid \pi_1$$

and then since $id_{U^\times} \mid \pi_1 = \pi_1 \mid \frac{id_U}{id_{U^\times}}$ we have $\delta^\times \mid \varepsilon^\times_{U^\times} = id_{U^\times}$ by coindution. For the second counit law, we have:

$$\delta^\times_U \mid (\varepsilon^\times_U)^\times \mid \pi_0 = \delta^\times_U \mid \pi_0 = \pi_0 = id_{U^\times} \mid \pi_0$$

$$\delta^\times_U \mid (\varepsilon^\times_U)^\times \mid \pi_1 = \delta^\times_U \mid \pi_1 \mid \frac{\varepsilon^\times_U}{(\varepsilon^\times_U)^\times} = \Delta^\times_U \mid \frac{id_{U^\times}}{\delta^\times_U} \mid \frac{\varepsilon^\times_U}{(\varepsilon^\times_U)^\times}$$

$$= \Delta^\times_U \mid \frac{\pi_1 \mid \frac{id_U}{\pi_0}}{\delta^\times_U \mid (\varepsilon^\times_U)^\times} = \pi_1 \mid \frac{id_U}{\Delta^\times_U \mid \frac{\pi_0}{\delta^\times_U \mid (\varepsilon^\times_U)^\times}} = \pi_1 \mid \frac{id_U}{\delta^\times_U \mid (\varepsilon^\times_U)^\times}$$

and then since $id_{U^\times} \mid \pi_1 = \pi_1 \mid \frac{id_U}{id_{U^\times}}$ we have that $\delta^\times_U(\varepsilon^\times_U)^\times = id_{U^\times}$ by coinduction. Thus, $((-)^\times, \delta^\times, \varepsilon^\times)$ is a comonad on $\mathbf{H}\ulcorner\mathbb{A}\urcorner^*$.

(ii) Similar to the proof of (i).

$\square$

4.3. Crossing Cells

We extend Definition 11 to obtain crossing cells in the free cornering with iteration:

Definition 14. Let $\mathbb{A}$ be a distributive monoidal category. For each $A \in \ulcorner\mathbb{A}\urcorner^*_H$ and each $U \in \ulcorner\mathbb{A}\urcorner^*_V$ We define crossing a crossing cell $\chi_{U,A} : (U\,{}^{A}_{A}\,U)$ by induction on the structure of U. The cases for $A^\circ, A^\bullet, I, U \otimes W, U \times W$, and $U + W$ are as in Definition 11. For $U^\times$ we define $\chi_{U^\times,A} = (\chi_{U,A})^\times$ and for U^+ we define $\chi_{U^+,A} = (\chi_{U,A})^+$. That is, $\chi_{U^\times,A}$ is the unique cell such that:

Similarly, $\chi_{U^+,A}$ is the unique cell such that:

The crossing cells remain coherent with respect to horizontal composition:

Lemma 18. *For $U \in A^{\circ\bullet}_*$ and $A, B \in \mathbb{A}_0$ we have*

(i) $\chi_{U,A\otimes B} = \chi_{U,A} \mid \chi_{U,B}$

(ii) $\chi_{U,I} = 1_U$

Proof. We extend the proof of Lemma 10 with the necessary inductive cases:

(i) For $U^{\times}$ we have:

$$\chi_{U^{\times},A\otimes B} \mid \pi_0 = \pi_0 \mid 1_{A\otimes B} = \pi_0 \mid 1_A \mid 1_B = \chi_{U^{\times},A} \mid \chi_{U^{\times},B} \mid \pi_0$$

$$\chi_{U^{\times},A\otimes B} \mid \pi_1 = \pi_1 \mid \frac{\chi_{U,A\otimes B}}{\chi_{U^{\times},A\otimes B}} = \pi_1 \mid \frac{\chi_{U,A} \mid \chi_{U,B}}{\chi_{U^{\times},A\otimes B}}$$

$$\chi_{U^{\times},A} \mid \chi_{U^{\times},B} \mid \pi_1 = \pi_1 \mid \frac{\chi_{U,A}}{\chi_{U^{\times},A}} \mid \frac{\chi_{U,B}}{\chi_{U^{\times},B}} = \pi_1 \mid \frac{\chi_{U,A} \mid \chi_{U,B}}{\chi_{U^{\times},A} \mid \chi_{U^{\times},B}}$$

and so $\chi_{U^{\times},A\otimes B} = \chi_{U^{\times},A} \mid \chi_{U^{\times},B}$ by coinduction. A similar argument gives $\chi_{U^{+},A\otimes B} = \chi_{U^{+},A} \mid \chi_{U^{+},B}$.

(ii) For $U^{\times}$ we have:

$$\chi_{U^{\times},I} \mid \pi_0 = \pi_0 \mid 1_I = \pi_0 = id_{U^{\times}} \mid \pi_0$$

$$\chi_{U^{\times},I} \mid \pi_1 = \pi_1 \mid \frac{\chi_{U,I}}{\chi_{U^{\times},I}} = \pi_1 \mid \frac{id_U}{\chi_{U^{\times},I}}$$

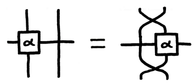
$$id_{U^{\times}} \mid \pi_1 = \pi_1 = \pi_1 \mid id_{U\otimes U^{\times}} = \pi_1 \mid \frac{id_U}{id_{U^{\times}}}$$

It follows that $\chi_{U^{\times},I} = id_{U^{\times}}$. The case for U^{+} is similar.

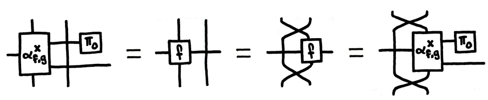
□

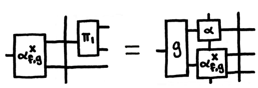
Next, we show that the technical lemma concerning crossing cells holds in the free cornering with iteration:

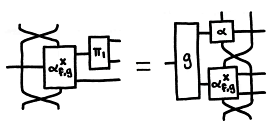
Lemma 19. *For any cell α of $\ulcorner\mathbb{A}\urcorner^{*}$ we have*



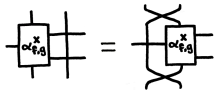
Proof. We extend the proof of Lemma 11 with the necessary inductive cases. For cells $\alpha^{\times}_{f,g}$ we have:







which gives, using the proof technique of Remark 4:



as required. The case for cells $\alpha^{+}_{f,g}$ is similar. $\square$

Consequently, $\ulcorner\mathbb{A}\urcorner^*$ is a monoidal double category with the tensor product of cells and proof as in Lemma 3. We record:

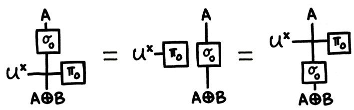
Lemma 20. *If $\mathbb{A}$ is a distributive monoidal category then $\ulcorner\mathbb{A}\urcorner^*$ is a monoidal double category.*

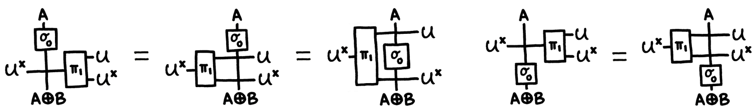
Further, the crossing cells remain coherent with respect to $\oplus$ in $\ulcorner\mathbb{A}\urcorner^*$:

Lemma 21. *In $\ulcorner\mathbb{A}\urcorner^*$, $\chi_{U,A\oplus B} = \left[\frac{\chi_{U,A}}{\sigma_0}, \frac{\chi_{U,B}}{\sigma_1}\right]$. That is, $\chi_{U,A\oplus B}$ is the unique cell such that:*

$$\frac{\sigma_0}{\chi_{U,A\oplus B}} = \frac{\chi_{U,A}}{\sigma_0} \qquad\qquad \frac{\sigma_1}{\chi_{U,A\oplus B}} = \frac{\chi_{U,B}}{\sigma_1}$$

Proof. By structural induction on U. We supply the necessary inductive cases to extend the proof of Lemma 13 to a proof of the present claim. For $U^\times$ we have:





and so by coinduction we have $\frac{\sigma_0}{\chi_{U^\times,A\oplus B}} = \frac{\chi_{U^\times,A}}{\sigma_0}$. Similarly, $\frac{\sigma_1}{\chi_{U^\times,A\oplus B}} = \frac{\chi_{U^\times,B}}{\sigma_1}$. The case for U^+ is analogous. $\square$

4.4. A Model: Iteration in Stateful Transformations

We return to the double category $\mathsf{S}(\mathbb{C})$ of stateful transformations over a cartesian closed category with distributive binary coproducts. In order to accommodate the iteration protocols in $\mathsf{S}(\mathbb{C})$ we require additional structure on the base category as follows:

- If the final coalgebra over the functor $Y \mapsto (X \otimes FY)$ exists for a strong F, we define $F^\times$ to be the functor sending X to the carrier of that final coalgebra. So there must be a final coalgebra $c^{\times}_{F,X} : F^\times X \to X \otimes F(F^\times X)$.

- If the initial algebra over the functor $Y \mapsto X \oplus FY$ exists for any strong F, we define F^+ to be the functor sending X to the carrier of that algebra. So there must be an initial algebra $a^+_{F,X} : X \oplus F(F^+X) \to F^+X$.

For example, *container functors* are strong and closed under these constructions and the choice constructions. Moreover, A°, $A^\bullet$ can be given by container functors too. So a possible model to work in would be the double category using the category of sets as horizontal edges, container functors as vertical edges, and the relevant container morphisms as 2-cells, which we call the double category of *stateful container transformations*. See [1, 3] for more details on container functors and their closure properties.

In any case, if the endofunctor $F^\times$ as given above exists, it is strong. This can be shown by constructing a coalgebra for $(X \otimes Y \otimes F(-))$ with carrier $F^\times X \otimes Y$, thereby constructing a coalgebra morphism: $\tau^{F^\times}_{X,Y} : F^\times X \otimes Y \to F^\times(X \otimes Y)$. The coalgebra is defined as follows:

$$F^\times X \otimes Y \xrightarrow{c^\times \otimes Y} X \otimes F(F^\times X) \otimes Y \xrightarrow{X \otimes \langle p_1, \tau^F \rangle} (X \otimes Y \otimes F(F^\times X \otimes Y))$$

The endofunctor F^+ is proven to be strong by defining an algebra for $(X \oplus F(-))$ with carrier $(F^+(X \otimes Y))^Y$, thereby constructing an algebra morphism $m : F^+X \to (F^+(X \otimes Y))^Y$ which induces a natural transformation for strength $\tau^{F^+}_{X,Y} = (m \otimes Y)\,\mathsf{ev}^Y : F^+X \otimes Y \to F^+(X \otimes Y)$. The algebra is defined as

$$[\lambda[l_{X,Y}], \lambda[r_{X,Y}]] : X \oplus F((F^+(X \otimes Y))^Y) \to (F^+(X \otimes Y))^Y \text{ where}$$

$$l_{X,Y} : X \otimes Y \xrightarrow{\sigma_0} (X \otimes Y) \oplus F(F^+(X \otimes Y)) \xrightarrow{a^+_{F,X \otimes Y}} F^+(X \otimes Y)$$

$$r_{X,Y} : F((F^+(X \otimes Y))^Y) \otimes Y \xrightarrow{\tau^F\ F(\mathsf{ev}^Y)} F(F^+(X \otimes Y)) \xrightarrow{\sigma_1\ a^+_{F,X \otimes Y}} F^+(X \otimes Y)$$

Given $\alpha : (G\,{}^{A}_{A}\,H)$, $g : (F\,{}^{1}_{1}\,G{\circ}F)$ and $f : (F\,{}^{A}_{B}\,K)$, we define $\alpha^\times_{f,g} : (F\,{}^{A}_{B}\,H^\times{\circ}K)$ as the unique coalgebra morphism $\alpha^\times_{f,g,X} : FX \otimes A \to H^\times K(X \otimes B)$ from the coalgebra $\langle f, (g \otimes A)\alpha \rangle : FX \otimes A \to K(X \otimes B) \otimes H(FX \otimes A)$ to $c^\times_{H,K(X \otimes B)}$.

The dual case is slightly more involved. Suppose given $\alpha : (F\,{}^{A}_{A}\,G)$, $g : (G{\circ}H\,{}^{1}_{1}\,H)$ and $f : (K\,{}^{A}_{B}\,H)$. We construct the map:

$$h := \quad F((H(X \otimes B))^A) \otimes A \xrightarrow{\alpha\, G(\mathsf{ev}^A_{X \otimes B})} GH(X \otimes B) \xrightarrow{g_{X \otimes B}} H(X \otimes B)$$

Then $\lambda[f] : KX \to (H(X \otimes B))^A$ and $\lambda[h] : F((H(X \otimes B))^A) \to (H(X \otimes B))^A$ gives us an algebra: $[\lambda[f], \lambda[h]] : KX \oplus F((H(X \otimes B))^A) \to (H(X \otimes B))^A$. This gives us an algebra morphism from $a^+_{F,KX}$ to $[\lambda[f], \lambda[h]]$, which we can evaluate to $\alpha^+_{f,g} = (a^+_{F,KX} \otimes A)\mathsf{ev}^A : F^+(KX) \otimes A \to H(X \otimes B)$.

To show that the constructed natural transformations are strong, we show that the components that they are constructed from are strong. Firstly, for a constant A, the natural transformation ev^A_X is strong by definition of strength of the $(-)^A$ functor. Similarly, $c^\times$ and a^+ are strong natural transformations by definition of strength on $F^\times$ and F^+. It remains to be verified that $\lambda[-]$ preserves strength of its argument.

Lemma 22. *Given a strong natural transformation $\alpha_X : FX \otimes A \to GX$, we prove that $\lambda[\alpha_X] : FX \to (GX)^A$ is strong.*

Proof. We show that the proper diagram commutes:

$$\begin{tikzcd}
FX \otimes B \arrow[r, "\tau^F_{X,B}"] \arrow[d, "\lambda[\alpha_X]\otimes B"'] & (GX)^A \otimes B \arrow[d, "\tau^{A^\bullet G}_{X,B}"] \\
F(X \otimes B) \arrow[r, "\lambda[\alpha_{X\otimes B}]"'] & G(X \otimes B)^A
\end{tikzcd}$$

We use that if $(f \otimes A)\mathsf{ev}^A = (g \otimes A)\mathsf{ev}^A$, then $f = \lambda[(f \otimes A)\mathsf{ev}^A] = \lambda[(g \otimes A)\mathsf{ev}^A] = g$ to show the above diagram commutes.

$$\begin{tikzcd}
FX\otimes B\otimes A \arrow[r, "\lambda[\alpha_X]\otimes B\otimes A"] \arrow[d, "FX\otimes\gamma_{B,A}"] & (GX)^A\otimes B\otimes A \arrow[r, "\tau^{A^\bullet}\otimes A"] \arrow[d, "(GX)^A\otimes\gamma_{B,A}"] & (GX\otimes B)^A\otimes A \arrow[r, "(\tau^G_{X,B})^A\otimes A"] \arrow[d, "\mathsf{ev}^A_{(GX\otimes B)^A}"] & (G(X\otimes B))^A\otimes A \arrow[dd, "\mathsf{ev}^A_{G(X\otimes B)}"] \\
FX\otimes A\otimes B \arrow[r, "\lambda[\alpha_X]\otimes B\otimes A"] \arrow[d, "FX\otimes\gamma_{A,B}"] & (GX)^A\otimes A\otimes B \arrow[r, "\mathsf{ev}^A_{GX}\otimes B"] & GX\otimes B \arrow[dr, "\tau^G_{X,B}"] & \\
FX\otimes B\otimes A \arrow[r, "\tau^F_{X,B}\otimes A"] & F(X\otimes B)\otimes A \arrow[rr, "\alpha_{X\otimes B}"] & & G(X\otimes B)
\end{tikzcd}$$

□

For certain categories $\mathbb{C}$ there is model of $\ulcorner\mathbb{C}\urcorner^*$ given by part of the double category of stateful transformations. Specifically, this model exists when $\mathbb{C}$ is a distributive cartesian closed category, and there is a collection of strong endofunctors on $\mathbb{C}$ closed under composition, $+$, $\times$, final coalgebra and initial algebra constructions, and this collection contains the identity functor and the endofunctors A° and $A^\bullet$ for each object A of $\mathbb{C}$. The model is then given by restricting $\mathsf{S}(\mathbb{C})$ to the part of the vertical edge monoid given by endofunctors in the collection discussed above. For example if $\mathbb{C}$ is the category Set of sets, then taking the collection to be the container endofunctors on Set gives a model of $\ulcorner\mathsf{Set}\urcorner^*$.

Remark 5. We consider the interpretation of iteration protocols given by $(-)^+$ and $(-)^\times$ from the perspective of computational effects, extending Remarks 1 and 2. As a computational effect, F^+ enables a program to trigger the effect F any number of times it chooses. Dually, to resolve an effect F^+ its environment must be able to resolve F any number of times. Similarly, to trigger $F^\times$ a program must provide a method for triggering F any number of times, as required by the environment. Dually, to resolve $F^\times$ the environment must commit to resolving F a number of times of its choosing, and then resolve those effects.

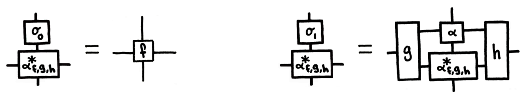
5. Concluding Remarks

We have shown how to extend the free cornering of a symmetric monoidal category to support both branching communication protocols and iterated communication protocols, bringing it closer to existing systems of session types. Specifically, we have constructed the free cornering with choice (Definition 10) and the free cornering with iteration (Definition 13) of a distributive monoidal category, shown that they inherit significant categorical structure from the free cornering, and provided some evidence that they fit well into the categorical landscape. Further, we have constructed the double category of stateful transformations (Definition 6) — a model of the structure found in the free cornering, free cornering with choice, and free cornering with iteration.

While our work constitutes a significant step, the path is long, and we envision our work here as a small part of a much larger research project surrounding the free cornering. In this final section we elucidate this project by outlining a number of directions for future work:

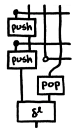
Active Iteration. There is a mismatch between our constructions of the free cornering with choice and the free cornering with iteration. In the free cornering with choice, we have a pair of dual operations $- + -$ and $- \times -$ on cells corresponding to *reactive* protocol choice, and a third operation $[-, -]$ which allows *active* protocol choice. In the free cornering with iteration we again have a pair of dual operations $(-)^{+}$ and $(-)^{\times}$ corresponding to *reactive* protocol iteration, but we are missing their active counterpart. That is, in the free cornering with iteration we cannot model processes that choose whether or not to continue iterating a given protocol as a function of their input. Put another way, we ought to be able to control the iteration of a communication process with a "while loop", but this would require a notion of "while loop" in the vertical direction.

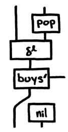
We briefly speculate about the form that the axioms for active iteration ought to take. For each quartet of cells $\alpha : \ulcorner\mathbb{A}\urcorner^{*}(U \, {}^{A}_{B\oplus A} \, W)$, $f : \ulcorner\mathbb{A}\urcorner^{*}(S \, {}^{B}_{C} \, K)$, $g : \ulcorner\mathbb{A}\urcorner^{*}(S \, {}^{I}_{I} \, U\otimes S)$, and $h : \ulcorner\mathbb{A}\urcorner^{*}(W\otimes K \, {}^{I}_{I} \, K)$ we seem to require a cell $\alpha^{*}_{f,g,h} : \ulcorner\mathbb{A}\urcorner^{*}(S \, {}^{B\oplus A}_{C} \, K)$ satisfying:



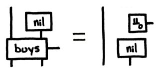
Significantly, asking for $\alpha^{*}_{f,g,h}$ to be the unique such cell is too strong, and collapses the hom-sets of the resulting category of vertical cells. While this sort of active iteration is convenient for constructing examples, it is unclear what sort of properties we ought to ask for in order to obtain e.g., well-behaved crossing cells. We speculate that a double-categorical analogue of the notion of *uniform trace operator* (see e.g, [16]) will suffice, but how such an analogue should look has not yet been fully worked out.

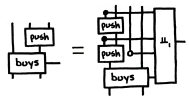
In the presence of cells $\alpha^{*}_{f,g,h}$ we may extend Example 6 as follows: let $\texttt{buy} : (I \, {}^{S_{\texttt{bread}}\otimes\$\otimes S_{\$}}_{S_{\texttt{bread}}\oplus(S_{\texttt{bread}}\otimes\$\otimes S_{\$})} \, \texttt{bread}^{\bullet}\otimes\texttt{bread}^{\bullet}\otimes\$^{\circ})$ be the cell below on the left, then define $\mathtt{buys'} = \mathtt{buy}^*_{1_{S_{\mathtt{bread}}},\square_I,\mu_1} : \left({}_I{}^{S_{\mathtt{bread}}\oplus(S_{\mathtt{bread}}\otimes\mathtt{\$}\otimes S_{\$})}_{S_{\mathtt{bread}}}(\mathtt{bread}^\bullet\otimes\mathtt{bread}^\bullet\otimes\mathtt{\$}^\circ)^+\right)$, and let $\mathtt{buys} : \left({}_I{}^{S_{\mathtt{bread}}\otimes S_{\$}}_{S_{\mathtt{bread}}\otimes S_{\$}}(\mathtt{bread}^\bullet\otimes\mathtt{bread}^\bullet\otimes\mathtt{\$}^\circ)^+\right)$ be the cell below on the right:





Now the behaviour of `buys` depends on how much money it has. Specifically, we have:





so `buys` buys bread until it is out of money. Now, we may also consider $\frac{\mathtt{sales}}{\mathtt{buys}}$, the process which first sells bread until instructed to switch modes, and then buys bread until it is out of money.

Abstract Definitions. In the free cornering, the corner cells carry the structure of a proarrow equipment. A natural question is what structure is carried by the cells of the free cornering with choice and free cornering with iteration. While the structure of the free cornering with choice is clearly some sort of product or coproduct on a single-object double category, it is not clear what sort of limit this is. We remark that it does not seem to be directly related to the double-categorical limits studied in [14]. Similarly, it is unclear what double-categorical structure the cells of the free cornering with iteration carry. We suspect this to be a fruitful direction for future work.

Term Logic for Cells. Any comparison of the free cornering (with or without choice and iteration) to existing models of concurrent computation is made somewhat awkward by the lack of a term calculus and accompanying term rewriting system for the free cornering. Most existing process calculi and process algebras are first and foremost term calculi, and do not tend to have an accompanying categorical semantics. The free cornering exists only as categorical semantics. Thus, in order to better situate our work in the literature on concurrent computation we would seem to require a term calculus for the free cornering.

While the terms of a rewriting system often form a category, we are not aware of any rewriting systems in which the terms form a double category. In particular, while systems of *tile logic* (see e.g., [6]) form double categories, there the cells of the double category in question correspond to the *rewrites*, while for us the cells must correspond to the *terms*. There is an evident notion of Cat-*enriched double category*, in which the cell-sets of the double category in question are in fact categories — in which morphisms correspond to rewrites — and the composition operations are given by functors. Cat-enriched categories are known to model rewriting systems in which the terms form a category [30], and so we expect that the rewriting systems appropriate to our setting will form Cat-enriched double categories. This requirement should guide future developments in in the direction of a term logic for the free cornering.

Coherence and Vertical Cells. An important property of the free cornering is that the vertical cells are the base category:

Proposition 1 ([25])**.** *Let A be a symmetric monoidal category. Then there is an isomorphism of categories $\mathbf{V}{\ulcorner\mathbb{A}\urcorner} \cong \mathbb{A}$.*

We think of this as a kind of coherence. We conjecture that the free cornering with choice and free cornering with iteration are also coherent in this way:

Conjecture 1. *Let $\mathbb{A}$ be a distributive monoidal category. Then:*

(i) There is an isomorphism of categories $\mathbf{V}{\ulcorner\mathbb{A}\urcorner}^{\oplus} \cong \mathbb{A}$.

(ii) There is an isomorphism of categories $\mathbf{V}{\ulcorner\mathbb{A}\urcorner}^{*} \cong \mathbb{A}$.

While we believe it to be true, we currently lack the machinery necessary to prove our conjecture. The most promising approach looks to be through the sort of term calculus and rewriting system for the free cornering discussed above, further motivating its development.

Additional Effects of Stateful Transformations. We have given a model of the free cornering in terms of strong functors and strong natural transformations. Besides the protocols in the image of the double functor from the free cornering of $\mathbb{C}$ into $\mathsf{S}(\mathbb{C})$, other effects modelled by strong monads occur as protocols in $\mathsf{S}(\mathbb{C})$ as well, such as monads for nondeterminism and probability. One direction for future work would be to axiomatise selected effects directly in the free cornering. Furthermore, the representation of computational effects in the form of a double category could help us describe both operational and denotational semantics of effectful programs. On the operational side, double cells function as a kind of effect handler [28], translating effects received from its interior into effects invoked in its environment. On the denotational side, models related to stateful nondeterministic runners [35] and other program-environment interaction laws [18] seem particularly suitable for interpretation in this model.

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Curriculum Vitae

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Scientific work

Papers

1. *Protocol Choice and Iteration for the Free Corneirng* with Niels Voorneveld. Submitted to JLAMP.

2. *Concurrent Process Histories and Resource Transducers.* LMCS Vol. 19, Issue 1, Part 7. 2023.

3. *Cornering Optics* with Mario Román and Guillaume Boisseau. To Appear in EPTCS.

4. *A Variety Theorem for Relational Universal Algebra.* LNCS Vol. 13027, pages 362–377. 2021.

5. *Situated Transition Systems.* EPTCS Vol. 372, pages 103–116. 2021.

6. *The Structure of Concurrent Process Histories.* LNCS Vol. 12717, pages 209–224.

7. *Functorial Semantics for Partial Theories* with Ivan Di Liberti, Fosco Loregian and Pawel Sobocinski. POPL 2021. Proc. ACM Prog. Lang., Vol. 5, No POPL, Article 57. 2021.

8. *A Foundation for Ledger Structures.* OASIcs Vol. 82, pages 7:1 – 7:13. 2020.

9. *System F in Agda for fun and profit* with James Chapman, Roman Kireev and Philip Wadler. LNCS Vol. 11825, pages 255–297. 2019.

10. *Unravelling Recursion: Compiling an IR with recursion to System F* with Roman Kireev, Michael Peyton Jones, Philip Wadler, Vasilis Gkoumas and Kenneth MacKenzie. LNCS Vol. 11825, pages 414–443. 2019.

Conference presentations

1. At the 2023 Bob Walters Memorial Meeting in Tallinn I spoke about my work on situated transition systems.

2. At the 2022 Nordic Workshop on Programming Theory (NWPT) I spoke about my work on double categories and interacting processes.

3. At the 2022 Applied Category Theory (ACT) conference, I spoke about my proceedings paper “Cornering Optics”.

4. At the 2022 Foundational Methods in Computer Science (FMCS) workshop I spoke about my work on optics in the free cornering of a monoidal category.

5. At the 2021 Relational and Algebraic Methods in Computer Science (RAMICS) conference, I spoke about my proceedings paper "A Variety Theorem for Universal Algebra".

6. At the 2021 International Category Theory (CT) conference, I spoke about my work on partial and relational algebraic theories.

7. At the 2021 Categories and Companions Symposium (CACS), I spoke about my work on partial and relational algebraic theories.

8. At the 2021 Applied Category Theory (ACT) conference, I spoke about my proceedings paper "Situated Transition Systems".

9. At the 2021 International Conference on Coordination Models and Languages (COORDINATION), I spoke about my proceedings paper "The Structure of Concurrent Process Histories".

10. At the 2021 Foundational Methods in Computer Science (FMCS) workshop I spoke about my work on situated transition systems and accounting.

11. At the 2021 Symposium on Principles of Programming Languages (POPL), I spoke about my proceedings paper "Functorial Semantics for Partial Theories".

12. At the 2020 International Conference on Blockchain Economics, Security, and Protocols (TOKENOMICS), I spoke about my proceedings paper "A Foundation for Ledger Structures".

13. At the 2020 Estonian Winter School in Computer Science (EWSCS), I spoke about my work on string diagrams for categories of partial maps.

14. In 2019 at the Fifth Symposium on Compositional Structures (SYCO 5), I spoke about my work on string diagrams for categories of partial maps.

15. In 2019 at the Scottish Programming Languages Seminar (SPLS), I spoke about my work on string diagrams for categories of partial maps.

16. In 2019 at the Third Symposium on Compositional Structures (SYCO 3), I spoke about my work on the categorical sematics of financial ledgers.

17. At the 2017 Category Theory Octoberfest, I spoke about my work in Turing categories and Realizability.

18. At the 2016 International Category Theory Conference (CT), I spoke about my work on Turing categories and realizability.
19. At the 2016 Foundational Methods in Computer Science (FMCS) workshop I spoke about my work on Turing categories and realizability.
20. At the 2014 Foundational Methods in Computer Science (FMCS) workshop I spoke about Gödel incompleteness in Turing categories.

Elulookirjeldus

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Hariduskäik

2019 - ...	Tallinna Tehnikaülikool, infotehnoloogia teaduskond, doktoriõpe
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2017, *Turing Categories and Realizability*, MSc, juhendaja Prof. Robin Cockett, Calgary Ülikool, Kanada.

Teadustegevus

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