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Inverse Problems for Parabolic Integro-Differential Equations with Instant and Integral Conditions

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Declaration: Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for any doctoral or equivalent academic degree.

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Contents

LIST OF PUBLICATIONS

I J. Janno, K. Kasemets, A positivity principle for parabolic integro-differential equations and inverse problems with final overdetermination. *Inverse Problems and Imaging*, **3**, 2009, 1, 17–41.

II K. Kasemets, J. Janno, Inverse problems for a parabolic integro-differential equation in a convolutional weak form. *Abstract and Applied Analysis*, Article ID 297104, 2013, 16pp.

III K. Kasemets, J. Janno, A weak inverse problem for a parabolic integro-differential equation containing two kernels. *Electronic Journal of Differential Equations*, **2014**, 2014, 176, 1–19.

Applicant's contribution in these publications

I Performing technical computations of the proofs, participation in writing the manuscript.

II Performing the proofs, participation in writing the manuscript.

III Contributing in ideas of proofs and formulation of statements, performing the proofs, preparing the manuscript.

In addition to I - III, the following publication related to the topic of the thesis has appeared:

IIa K. Kasemets, J. Janno, Recovery of a Source Term in a Parabolic Integro-Differential Equation from Final Data. *Mathematical Modelling and Analysis*, **16**, 2011, 2, 199–219.

Publication II contains results that generalize results of IIa. Therefore, Publication IIa doesn't form a basis of the present thesis.

INTRODUCTION

There are several definitions of direct and inverse problems. In physics, the direct problem means the determination of states using model parameters and the inverse problem means the determination of parameters of models using given states. Often the model is described by differential or integro-differential equations. Then the direct problem means the solution of the equation subject to proper boundary and/or initial conditions. In this narrower mathematical sense, the inverse problem means the identification of parameters of the equation (e.g. coefficients, free terms, kernels) or boundary or initial conditions on the basis of information available on the solution of the direct problem. Different mathematical and physical aspects of inverse problems for partial differential equations can be found in monographs and articles [2, 8, 20, 22, 23, 26, 38, 39, 51, 59].

An important issue is the well-posedness of a posed problem. A problem is well-posed in the sense of Hadamard [17] if the solution exists, is unique and continuously depends on the data (the latter one is the so-called stability requirement). In case the problem is not well-posed, it is called ill-posed. In case of nonexistence the problem is over-determined and contradictive. The non-uniqueness or non-stability indicates lack of information. Provided the solution exist, usually it is possible to find functional spaces where it is unique and stable. However, for an inverse problem those spaces often contain derivatives of data, which are not directly measurable. This means ill-posedness from the practical viewpoint. In case the spaces where the problem is well-posed contain derivatives up to some finite order from the data then the problem is called moderately ill-posed. The highest degree of the derivative involved in such a space is called the degree of ill-posedness of the problem. In case such spaces involve all derivatives of the data, the problem is called severely ill-posed. To solve ill-posed problems, regularization techniques are used [9, 19, 38, 62].

Starting from ca 1970 models with memory to describe heat processes were introduced and developed [6, 16, 36, 37, 49, 50, 52] (see also the monographs [1, 57]). In those models the temperature satisfies parabolic integro-differential equations that contain integral terms with kernels related to the "memory" of the material. Incorporation of memory terms brings along an inertia to the heat process and such models are more relevant from the practical viewpoint. In parallel, models with memory were introduced for mechanical processes, too (viscoelastic materials), leading to hyperbolic integro-differential equations [10, 58].

The study of inverse problems for parabolic and hyperbolic integro-differential equations with memory terms started in the middle of 1980s. First series of papers [12, 14, 24, 30, 31, 32, 33, 34, 48, 64, 66] was devoted to the identification of time-dependent kernels using information about certain

traces of the solutions of the direct problems over the time. Such problems are moderately ill-posed and can be reduced to Volterra equations of the second kind. In case the parabolic equation is linear, the corresponding Volterra equation contains nonlinearities that are only of convolution type. In [24] a method of norms with exponential weights was proposed to prove global (in time) existence and stability of the solutions of such problems. This method was exploited in many subsequent papers.

Approximately in the middle of 1990s the study of inverse problems to determine space- and time-dependent memory kernels in parabolic integro-differential equations started. One class of treated problems consists in identification of kernels that depend only on some part of the space variables or have radial or else symmetries under information about traces of the solutions of the direct problems over the time, again [7, 11, 28]. Into this group of results we can put also papers dealing with determination of kernels representable in the form of finite sums of products of known space-dependent and unknown time-dependent functions [35, 53, 54, 55]. It turns out that those problems still admit the reduction to Volterra equations of the second kind and can be treated as before, in particular the method of weighted norms enables to prove global existence and stability.

In case the kernel depends on all space variables, the inverse problem may be posed on the basis of the Dirichlet-to-Neumann map. The uniqueness of the solution of such a problem was proved in [25]. The proof adjustes the celebrated method of Sylvester and Uhlmann [61] to the integro-differential case.

Another direction is the treatment of problems to determine other space-dependent parameters than the memory kernels. In case the unknown parameter of the equation depends only on space variables, it is natural to use an additional information of the same structure in the inverse problem, e.g. traces of solutions of direct problems at fixed time values (instant conditions) or integrals over fixed time domains (integral conditions). Those problem are not of Volterra type any more. For example, the problem to determine a space-dependent free term in a usual parabolic equation by means of final over-determination of the solution can be reduced to a Fredholm equation of the second kind [21]. There are two possibilities to handle such type of problems. One way consists in applying the fixed point argument under certain smallness restrictions (local results). This was exploited in papers [45, 47]. Another way is to avoid smallness restrictions and to apply the Fredholm alternative. Then the uniqueness implies the existence and stability. Actually, the latter one is the starting point of the investigations of the present thesis.

The first aim is to prove uniqueness of the solution of the inverse problem to determine a space-dependent component of a source term of a parabolic integro-differential equation in case the solution of the direct

problem is over-determined at the final moment of time (problem IP1) without assuming smallness restrictions (global result). This was previously proved for the usual parabolic equation in [21] (see also [4] for the semi-linear case). The proof uses a positivity principle for the direct problem (e.g. positivity of data implies the positivity of the solution) that is an immediate consequence of the well-know maximum principle. Thus, in the first step, we prove such a principle for parabolic integro-differential equations. Further, by means of the positivity principle the uniqueness of the mentioned inverse problem is shown. The assumptions contain certain positivity and monotonicity restrictions on the time-dependent component of the source term. Making use of the proved uniqueness and Fredholm-type results for an analogous problem for the usual parabolic results, we prove the existence and stability of the solution of the inverse problem. We mention that the existence and stability was previously proved in [46] under the assumption that the memory kernel is positive and the solution of inverse problem is unique. We do not need the positivity of the kernel in the Fredholm-type result.

The next aim is to prove the global uniqueness and local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient of the time derivative involved in a parabolic integro-differential equation from the final data concerning the solution of the direct problem (problems IP2 and IP3). In this connection the previously obtained results for the inverse source problem and the Banach fixed-point theorem can be applied.

In addition to mentioned positivity and monotonicity assumptions, these results require also sufficient smoothness of the data. The stability estimates contain derivatives of the data up to the second order. Therefore, these inverse problems are moderately ill-posed. This complicates the generalization of the results to non-smooth models (e.g. transmission problems). Inverse transmission problems for time-dependent kernels and given additional information along the time axis can be treated assuming additional regularity of the problem in a neighborhood of the trace where the additional information is given [29]. But this is not the case when this information is given in an instant form over the space domain where the direct problem is not regular.

In the second part of the thesis we consider inverse problems to determine parameters in parabolic integro-differential equations using instant and integral additional data in such a manner that the solution is understood in a non-exact sense, namely we deal with quasi-solutions of these problems that minimize certain cost functionals. In this connection we pose the direct problem in a non-regular (weak) form and treat inverse problems to determine several parameters simultaneously.

More precisely, we consider:

- an inverse problem to reconstruct several components of a free term depending either on space or time variables making use of instant conditions given at different time levels (IP4);
- an inverse problem to determine space-dependent components of the free term and an initial condition from integral conditions over time containing different weights (IP5);
- an inverse problem to determine two kernels and a lower order coefficient from a final condition and two conditions for traces of the solution of the direct problem over time (IP6).

First two problems are linear and the latter one is nonlinear. The existence of the quasi-solutions may be proved making use of the Weierstrass existence theorem [65]. The proof is easy for the problems IP4 and IP5, but more complicated for the problem IP6. In latter case we have to show the weak continuity of the solution of the direct problem with respect to the parameters to be recovered. We will do it in the one-dimensional case. In general, the uniqueness of the quasi-solutions may be proved using the strict convexity of the cost functional. Unfortunately, the latter one may not hold for the problems under consideration. However, corresponding regularized problems have unique quasi-solutions due to the strict convexity. The stability issue of quasi-solutions falls outside of the content of the present thesis.

In addition, we prove the Fréchet differentiability of the cost functionals and deduce formulas for the Fréchet derivatives in terms of solutions of certain adjoint problems. To this end we introduce an integrated convolutional form for the weak direct problem. This form does not contain the time derivative of a test function. Operating with such a form of direct problem, we develop a general method to derive adjoint problems and apply it in particular cases.

Finally, we discuss issues related to the gradient method to find the quasi-solutions. The components of the gradient are expressed in terms of the mentioned solutions of the adjoint problems. We will show monotone convergence of the gradient method. This result is a generalization of the former work [18] related to inverse problems for the usual parabolic equation.

The main novelties of the thesis are:

1. a positivity principle for parabolic integro-differential equations is proved;
2. the global existence, uniqueness and stability for an inverse problem to determine a space-dependent component of a free term of a parabolic integro-differential equation in case of given final data are proved;

3. the global uniqueness and local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient of a time derivative of a parabolic integro-differential equation in case of given final data are proved;
4. a general method to derive adjoint problems for Fréchet derivatives of cost functionals corresponding to inverse problems for parabolic integro-differential equations in a weak form is developed and applied particular inverse problems;
5. the existence of quasi-solutions to particular inverse problems with instant and integral additional conditions for parabolic integro-differential equations in a weak form is proved in special cases.

Summing up, the thesis contains a systematical theoretical study of inverse problems for parabolic integro-differential equations with instant and integral conditions, which has not been done before.

The results of the thesis have been presented in the following international meetings:

1. the conference Direct, Inverse and Control Problems for PDE's - DICOP, Cortona (Italy), 22 - 26.09.2008;
2. Chemnitz–RICAM Symposium on Inverse Problems, Linz (Austria), 14-15.07.2009;
3. 17th International Conference Mathematical Modelling and Analysis, Tallinn, 6-9.06.2012;
4. 18th International Conference Mathematical Modelling and Analysis and 4th International Conference Approximation Methods and Orthogonal Expansions, Tartu, 27-30.05.2013.

Let us give an *overview of the contents of the thesis.* Thesis contains three chapters.

In Chapter I physical background of the problem is discussed and basic parabolic integrodifferential equation is deduced. Moreover, notation used throughout the thesis is introduced.

Chapter 2 contains results obtained in the smooth case when all terms in the parabolic equation are regular functions. We start by proving basic well-posedness results for the direct problem and establish the positivity principle (§2.1, 2.2). Thereupon, in §2.3 we study the problem to determine the space-dependent component of a free term and in §2.4 we treat the inverse coefficient problems.

Chapter 3 is devoted to the non-smooth case when the parabolic equation contains singular distributions. In §3.1 we prove well-posedness results for the direct problem and introduce the weak convolutional form of the direct problem. Further, in §3.2 we formulate three particular inverse problems in the sense of quasi-solutions, propose the general method to deduce adjoint problems for Fréchet derivatives of cost functionals and apply this method to the posed inverse problems. §3.3 is devoted to the existence of quasi-solutions. In §3.4 briefly the discretization discussed.

1 PHYSICAL BACKGROUND AND NOTATION

1.1 Physical background and the integro-differential equation

Let us deduce the basic parabolic integro-differential equation that will appear in our inverse problems in next chapters.

In linear theory of heat conduction with memory in a medium that is generally inhomogeneous and anisotropic the following constitutive relations are assumed [3, 6, 13, 14, 16, 33, 50, 52, 54]:

$$q_i(x,t) = -\sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,t) + \int_{-\infty}^{t} m(t-\tau)\sum_{j=1}^{n} a_{ij}(x)u_{x_j}(x,\tau)d\tau, \quad i = 1,\dots,n, \tag{1.1}$$

$$e(x,t) = \beta(x)\Big[u(x,t) + \int_{-\infty}^{t} \mu(t-\tau)u(x,\tau)d\tau\Big], \tag{1.2}$$

where x is the space variable, t is the time, $q = (q_1,\dots,q_n)$ is the heat flux, e is the internal energy and u is the temperature. Moreover, a_{ij} is the conductivity matrix that in the isotropic case has the form $a_{ij}(x) = \alpha(x)I$ with some function α and the unity matrix I, and β is the heat capacity. The functions m and μ are the heat flux relaxation kernel and the internal energy relaxation kernel, respectively. They express the memory of the material. We assume that the memory is synchronous in all points of the medium, i.e. m and μ depend only on the time.

Further, we make use of the continuity equation

$$e_t(x,t) + \operatorname{div} q(x,t) = \chi(x,t), \tag{1.3}$$

where χ is the source term. Inserting (1.1) and (1.2) into (1.3) and assuming that $u = 0$ for $t < 0$ we come to the following parabolic integro-differential equation:

$$\beta[u + \mu * u]_t \;=\; Au - m * Au + \chi, \tag{1.4}$$

where $A = \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial}{\partial x_j}\Big)$.

Here and in the sequel the symbol $*$ stands for the time convolution, i.e.

$$v_1 * v_2(t) = \int_0^t v_1(t-\tau)v_2(\tau)d\tau.$$

In this thesis we generalize this model mathematically. Namely, we assume the operator A in Eq. (1.4) be of the form

$$A = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x)\frac{\partial}{\partial x_j} + a(x,t), \tag{1.5}$$

where a_{ij}, a_j and a are some coefficients. If necessary, we write $A(t)$ to indicate the dependence of A on t.

We will consider the solution u of the integro-differential equation (1.4) for the arguments

$$(x,t) \in Q = \Omega \times (0,T),$$

where $\Omega \in \mathbb{R}^n$ is an n-dimensional open domain and $T > 0$ is a fixed number.

We are not going to specify the regularity of Ω. We simply suppose that Ω is sufficiently smooth in order to guarantee our statements to hold. We denote by Γ the boundary of Ω, by $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ the outer normal of Γ at the point $x \in \Gamma$ and by S the boundary cylinder, i.e.

$$S = \Gamma \times (0,T).$$

Throughout the thesis we assume that the x-dependent coefficient matrix a_{ij} of the higher order part of the operator A is uniformly elliptic, i.e.

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geq \epsilon|\lambda|^2 \quad \text{in } \Omega \ \text{ for any } \lambda \in \mathbb{R}^n \quad \text{and some } \epsilon \in (0,\infty) \tag{1.6}$$

and x-dependent coefficient β is strictly positive:

$$\beta \geq \beta_0 \quad \text{in } \Omega \ \text{ with some } \beta_0 \in (0,\infty). \tag{1.7}$$

(In Sections 2.1 and 2.2 the relations (1.6) and (1.7) will be assumed in a more general form).

1.2 Functional spaces

In this section we define most important functional spaces to be used in the study of inverse problems and give some notation.

Firstly, we introduce the Lebesgue spaces of functions defined on a set $U \subset \mathbb{R}^l$, $l \in \mathbb{N}$. They are

$$L^p(U) = \{v \,:\, \|v\|_{L^p(U)} = \Big[\int_U |v(y)|^p dy\Big]^{1/p} < \infty\}, \quad 1 \leq p < \infty,$$
$$L^\infty(U) = \{v \,:\, \|v\|_{L^\infty(U)} = \operatorname*{ess\,sup}_{y\in U} |v(y)| < \infty\}.$$

The space $L^2(U)$ is a Hilbert space with the inner product $\langle v, w\rangle_{L^2(U)} = \int_U v(y)w(y)dy$. The symbol $C(U)$ stands for the space of functions, continuous on U. In case U is compact, $C(U)$ is a Banach space endowed with the usual maximum norm $\|v\|_{C(U)} = \max_{y\in U} |v(y)|$.

Next, let X be a Banach space. We generalize the Lebesgue spaces to abstract functions v defined on the interval $(0,T)$ and having values in X:

$$L^p(0,T;X) = \{v \,:\, \|v\|_{L^p(0,T;X)} = \Big[\int_0^T \|v(t)\|_X^p dt\Big]^{1/p} < \infty\},\ 1 \le p < \infty,$$
$$L^\infty(0,T;X) = \{v \,:\, \|v\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{t\in(0,T)} \|v(t)\|_X < \infty\}$$

and for $1 \le p \le \infty$ and $l \in \mathbb{N}$ define the abstract Sobolev spaces:

$$W_p^l(0,T;X) = \Big\{v \,:\, \|v\|_{W_p^l(0,T;X)} := \sum_{j=0}^{l} \|v^{(j)}\|_{L^p(0,T;X)} < \infty\Big\}.$$

In case $X = \mathbb{R}$, we write merely $W_p^l(0,T;\mathbb{R}) = W_p^l(0,T)$.

Moreover, by $C([0,T];X)$ we denote the Banach space of abstract functions, continuous on $[0,T]$.

The symbol $\mathscr{L}(X,Y)$ stands for the space of linear bounded operators from a Banach space X to another Banach space Y. In case $X = Y$ we write merely $\mathscr{L}(X)$.

In the first part of the thesis (devoted to smooth problems) we need some spaces of fractional order and anisotropic spaces of x- and (x,t)-dependent real-valued functions. To defined them, let us first introduce the following notation for difference quotients of such functions with powers:

$$\langle v\rangle_p(x_1,x_2) := \tfrac{v(x_1)-v(x_2)}{|x_1-x_2|^p}\,, \quad \langle v\rangle_p(x_1,x_2;t) := \tfrac{v(x_1,t)-v(x_2,t)}{|x_1-x_2|^p}\,,$$
$$\langle v\rangle_p(x;t_1,t_2) := \tfrac{v(x,t_1)-v(x,t_2)}{|t_1-t_2|^p}.$$

For any real numbers $p \in [1,\infty)$ and $l \in [0,\infty)$ we define the Sobolev-Slobodeckij spaces (cf. [43, 60])

$$W_p^l(\Omega) = \Big\{v \,:\, \|v\|_{W_p^l(\Omega)} := \sum_{|\alpha|\le[l]} \big[\textstyle\int_\Omega |D^\alpha v(x)|^p dx\big]^{\frac1p}$$
$$+\Theta_l \sum_{|\alpha|=[l]} \Big[\int\limits_{\Omega\times\Omega} |\langle D^\alpha v\rangle_{\frac np + l - [l]}(x_1,x_2)|^p dx_1 dx_2\Big]^{\frac1p} < \infty\Big\},$$

$$W_p^{l,\frac l2}(Q) = \Big\{v \,:\, \|v\|_{W_p^{l,\frac l2}(Q)} := \sum_{2j+|\alpha|\le[l]} \Big[\int\limits_{\Omega\times[0,T]} |D_t^j D_x^\alpha v(x,t)|^p dx dt\Big]^{\frac1p}$$
$$+\Theta_l \sum_{2j+|\alpha|=[l]} \Big[\int\limits_{\substack{\Omega\times\Omega\\ \times[0,T]}} \big|\langle D_t^j D_x^\alpha v\rangle_{\frac np + l - [l]}(x_1,x_2;t)\big|^p dx_1 dx_2 dt\Big]^{\frac1p}$$
$$+\Theta_{\frac l2} \sum_{0<l-2j-|\alpha|<2} \Big[\int\limits_{\substack{\Omega\times[0,T]\\ \times[0,T]}} \big|\langle D_t^j D_x^\alpha v\rangle_{\frac1p + \frac{l-2j-|\alpha|}{2}}(x;t_1,t_2)\big|^p dx dt_1 dt_2\Big]^{\frac1p} < \infty\Big\}.$$

Here $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ is the multi-index with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $[l]$ is the greatest integer $\le l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer

l and non-integer l, respectively. We mention that in the case of integer l, $W_p^l(\Omega)$ is the usual Sobolev space of functions defined on Ω.

Furthermore, for any non-integer $l > 0$ we define the Hölder spaces

$$C^l(\Omega) = \Big\{ v \,:\, D^\alpha v \in C(\Omega) \ \text{ for } \ |\alpha| \le [l],$$
$$\|v\|_l := \sum_{|\alpha|\le[l]} \Big[\sup_{x\in\Omega} |D^\alpha v(x)| + \sup_{x_1,x_2\in\Omega} \big|\langle D^\alpha v\rangle_{l-[l]}(x_1,x_2)\big| \Big] < \infty \Big\},$$

$$C^{l,\frac{l}{2}}(Q) = \Big\{ v \,:\, D_t^j D_x^\alpha v \in C(Q) \ \text{ for } \ 2j+|\alpha| \le [l],$$
$$\|v\|_{l,\frac{l}{2}} := \sum_{2j+|\alpha|\le[l]} \Big[\sup_{\substack{(x,t)\in\\ \Omega\times[0,T]}} \Big|D_t^j D_x^\alpha v(x,t)\Big| + \sup_{\substack{(x_1,x_2,t)\in\\ \Omega\times\Omega\times[0,T]}} \Big|\langle D_t^j D^\alpha v\rangle_{l-[l]}(x_1,x_2;t)\Big|$$
$$+ \sup_{\substack{(x,t_1,t_2)\in\\ \Omega\times[0,T]\times[0,T]}} \Big|\langle D_t^j D^\alpha v\rangle_{\frac{l-[l]}{2}}(x;t_1,t_2)\Big| \Big] < \infty \Big\}.$$

The definitions of $W_p^{l,\frac{l}{2}}$ and $C^{l,\frac{l}{2}}$ are in a standard manner extended from Q to the boundary cylinder S (for details see [43]). For integer $l \ge 0$ we define

$$C^{2l,l}(\overline{Q}) = \{ v \,:\, D_t^j D_x^\alpha v \in C(\overline{Q}) \ \text{ for } \ 2j+|\alpha| \le 2l \}.$$

Finally, we introduce specific notation related to comparison of real-valued functions. Let U be an open subset or a closer of an open subset in $\mathbb{R}^l$, $l \in \mathbb{N}$, and $f, g : U \to \mathbb{R}$. We write

$$\begin{aligned} &f \ge g \ \text{ in } U \quad \text{if } f(x) \ge g(x) \text{ a.e. } x \in U, \\ &f > g \ \text{ in } U \quad \text{if for any open set } U_1 \text{ such that } \overline{U}_1 \subseteq U \\ &\text{there exists } \varepsilon_{U_1} > 0 \text{ such that } f \ge g + \varepsilon_{U_1} \text{ in } U_1. \end{aligned} \tag{1.8}$$

2 SMOOTH PROBLEMS

In this chapter we will pose and study problems for (1.4) under the assumption that this equation holds in the classical sense. The results with some modifications are taken from Publication I.

The starting point is the following initial-boundary value problem

$$\beta[u+\mu*u]_t \;=\; Au - m*Au + \chi \quad \text{in } Q, \tag{2.1}$$

$$u = u_0 \;\text{ in } \Omega\times\{0\}\,,\;\; Bu = b \;\text{ in } S\,, \tag{2.2}$$

where u_0, b are given functions, B is the boundary operator defined either by

$$Bu = u \qquad (\text{we call it } \textit{case} \text{ I}) \tag{2.3}$$

or by

$$Bu = \omega\cdot\nabla u - m*\omega\cdot\nabla u \quad (\text{we call it } \textit{case} \text{ II}), \tag{2.4}$$

the operator ∇ stands for the gradient with respect to the vector of space variables $x\in\Omega$ and $\omega(x) = (\omega_1(x),\ldots,\omega_n(x))$ is an x-dependent vector satisfying the condition $\omega\cdot\nu > 0$. We assume that $\omega\in(C^1(\Gamma))^n$.

We define a number ϑ that depends on cases I and II as follows:

$$\vartheta = \begin{cases} 0 & \text{in case I} \\ 1 & \text{in case II.} \end{cases}$$

This enables unified formulation of statements for direct and inverse problems in both cases (see e.g. Theorem 2.1).

Note that in case II the condition $Bu = g$ is a generalized boundary condition of the second kind. Indeed, if $\omega_j = \sum_{i=1}^n a_{ij}\nu_i$ then $Bu = b$ takes the form $-q\cdot\nu = b$. This is the physical flux condition.

Let us formulate the following inverse problems that use over-determined final data at $t = T$ of the solution of (2.1), (2.2) (an inverse free term problem and two inverse coefficient problems).

IP1: Let the free term be of the following form:

$$\chi(x,t) \;=\; z(x)\phi(x,t) + \chi_0(x,t). \tag{2.5}$$

Given $\mu, m, \beta, a_{ij}, a_j, a, u_0, b, \phi, \chi_0$ and a function $u_T(x)$, $x\in\Omega$, find z and u so that the relations (2.1), (2.2), (2.5) and

$$u \;=\; u_T \;\text{ in } \Omega\times\{T\} \tag{2.6}$$

hold.

IP2: Let $a_t = 0$. Given $\mu, m, \beta, a_{ij}, a_j, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find a and u so that the relations (2.1), (2.2) and (2.6) hold.

IP3: Given $\mu, m, a_{ij}, a_j, a, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find β and u so that the relations (2.1), (2.2) and (2.6) hold.

It turns out that it is more convenient to treat the direct problem (2.1), (2.2) in case the convolution is removed from the operator A. Let us transform (2.1), (2.2) to such a form. Define the resolvent kernel $\widehat{m}$ of the kernel m as the solution of the following Volterra integral equation:

$$\widehat{m}(t) - \int_0^t m(t-\tau)\widehat{m}(\tau)d\tau = m(t), \quad t \in (0,T). \tag{2.7}$$

It is well-known that in case $m \in L^p(0,T)$ with $p > 1$ the solution $\widehat{m}$ of(2.7) exists, is unique and belongs to $L^p(0,T)$ (see e.g. [15]).

The equality (2.7) implies the following operator relation:

$$(I + \widehat{m}*)(I - m*) = I,$$

where I is the unity operator. Bringing the derivative with respect to t into the integral $\mu * u$ and applying the operator $I + \widehat{m}$ to the equation (2.1) and the boundary condition (2.2) in case II we transform the relations (2.1), (2.2) to the following form:

$$\beta(u_t + k * u_t) = Au + f \ \text{ in } Q\,, \quad u = u_0 \ \text{ in } \Omega \times \{0\}\,, \quad B_1 u = g \ \text{ in } S, \tag{2.8}$$

where

$$k = \mu + \mu * \widehat{m} + \widehat{m}, \tag{2.9}$$

$$f \ = \ \chi - \beta\mu u_0 + \widehat{m} * (\chi - \beta\mu u_0), \tag{2.10}$$

$$B_1 = B\,, \ \ g = b \ \ \text{in case I}\,, \tag{2.11}$$

$$B_1 u = \omega \cdot \nabla u\,, \ \ g = b + \widehat{m} * b \ \ \text{in case II}. \tag{2.12}$$

Summing up, we can formulate the following lemma.

Lemma 2.1 *In case all t-dependent data in* (2.1) *and* (2.2) *belong to the space $L^p(0,T)$ with respect to t for any x with some $p > 1$, the problem* (2.1), (2.2) *is equivalent to the problem* (2.8) *in a class of functions u such that $u, u_t, u_{x_i}, u_{x_i,x_j}$ belong to $L^p(0,T)$ with respect to t for any x.*

2.1 Well-posedness results for direct problem

In present and the next section we prove existence, uniqueness, stability and a positivity principle for the solution of the direct problem (2.8). These results are used in the study of inverse problems in the subsequent sections.

To the author's opinion, the positivity principle has a scientific value independently of the inverse problems. Therefore, we try to prove it as generally as we can. For that reason we allow the kernel k and the coefficients β, a_{ij} and a_j to depend both on the variables x and t in Sections 2.1 and 2.2. This means that A has the form

$$A = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x,t)\frac{\partial}{\partial x_j} + a(x,t), \tag{2.13}$$

and the basic assumptions (1.6), (1.7) read

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j \geq \epsilon|\lambda|^2 \quad \text{in } Q \text{ for any } \lambda \in \mathbb{R}^n \text{ and some } \epsilon \in (0,\infty), \tag{2.14}$$

$$\beta \geq \beta_0 \quad \text{in } Q \quad \text{with some } \beta_0 \in (0,\infty). \tag{2.15}$$

We start by formulating without a proof a technical lemma. It gives two integral inequalities. Proofs of these inequalities are contained in Publication I, i.e. [27], p. 21 - 23.

Lemma 2.1 *Define $Q_t = \Omega \times (0,t)$, $t > 0$. The following assertions are valid.*

(i) *Let $k \in L^1(0,T;L^\infty(\Omega))$ and $v \in L^p(Q)$ with some $p \in (1,\infty)$. Then $k * v \in L^p(Q)$ and*

$$\|k * v\|_{L^p(Q_t)} \leq \int_0^t \|k(\cdot,t-\tau)\|_{L^\infty(\Omega)}\|v\|_{L^p(Q_\tau)}d\tau, \quad t \in (0,T). \tag{2.16}$$

(ii) *Let $k \in L^{\frac{2}{2-l}}(0,T;C^l(\Omega))$ and $v \in C^{l,\frac{l}{2}}(Q)$ with some $l \in (0,1)$. Then $k * v \in C^{l,\frac{l}{2}}(Q)$ and*

$$\|k * v\|_{C^{l,\frac{l}{2}}(Q_t)} \leq C_0\left[\int_0^t \left\{\|k(\cdot,t-\tau)\|_l \|v\|_{C^{l,\frac{l}{2}}(Q_\tau)}\right\}^{\frac{2}{2-l}} d\tau\right]^{\frac{2-l}{2}}, \quad t \in (0,T) \tag{2.17}$$

with some constant C_0.

Now let us prove the existence, uniqueness and stability theorem for the direct problem.

Theorem 2.1 *Assume (2.14), (2.15). Then the following assertions are valid.*

(i) *Let* $\beta, a_{ij}, a_j, a \in C(\overline{Q})$, $k \in L^1(0,T;L^\infty(\Omega))$, $f \in L^p(Q)$, $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ *and* $g \in W_p^{2-\frac{1}{p}-\vartheta,1-\frac{1}{2p}-\frac{\vartheta}{2}}(S)$ *with some* $p \in (1,\infty)$. *Moreover, in case* I *let* $p \neq \frac{3}{2}$ *and the consistency condition* $u_0 = g$ *hold in* $\Gamma \times \{0\}$ *if* $p > \frac{3}{2}$ *and in case* II *let* $p \neq 3$ *and the consistency condition* $\omega \cdot \nabla u_0 = g$ *hold in* $\Gamma \times \{0\}$ *if* $p > 3$. *Then the problem* (2.8) *has a unique solution in the space* $W_p^{2,1}(Q)$. *This solution satisfies the estimate*

$$\|u\|_{W_p^{2,1}(Q)} \le C_1 \Big\{ \|f\|_{L^p(Q)} + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|g\|_{W_p^{2-\vartheta-\frac{1}{p},1-\frac{\vartheta}{2}-\frac{1}{2p}}(S)} \Big\}, \tag{2.18}$$

where C_1 *is a constant depending on* β, a_{ij}, a_j, a *and* k.

(ii) *Let* $\beta, a_{ij}, a_j, a \in C^{l,\frac{l}{2}}(Q)$ *and* $k \in L^{\frac{2}{2-l}}(0,T;C^l(\Omega))$ *with some* $l \in (0,1)$. *Moreover, let* $f \in C^{l,\frac{l}{2}}(Q)$, $u_0 \in C^{2+l}(\Omega)$, $g \in C^{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}}(S)$ *and in case* I *the consistency conditions* $u_0 = g$, $\beta g_t = Au_0 + f$ *hold in* $\Gamma \times \{0\}$ *and in case* II *the consistency condition* $\omega \cdot \nabla u_0 = g$ *hold in* $\Gamma \times \{0\}$. *Then the solution of* (2.8) *belongs to* $C^{2+l,1+\frac{l}{2}}(Q)$ *and satisfies the estimate*

$$\|u\|_{2+l,1+\frac{l}{2}} \le C_2 \Big\{ \|f\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} \Big\} \tag{2.19}$$

with some constant C_2 *depending on* β, a_{ij}, a_j, a *and* k.

Proof. The assertions (i) and (ii) in the usual parabolic case when $k = 0$ were proved in [60]. The proof of the present theorem is based on the Banach fixed point theorem considering the problem (2.8) as a perturbation of the problem in case $k = 0$. The contraction is achieved in norms with exponential weights.

Let us start with the assertion (i). By Theorem 5.4 in [60], under the assumptions of (i) problem (2.8) in case $k = 0$ has a unique solution $\widehat{u}$ in the space $W_p^{2,1}(Q)$. Thus, (2.8) for $u \in W_p^{2,1}(Q)$ in case $k \neq 0$ is equivalent to the following problem for the difference $v = u - \widehat{u} \in W_p^{2,1}(Q)$:

$$\beta v_t = Av - \beta(k * (v_t + \widehat{u}_t)) \ \text{ in } Q\,, \quad v = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 v = 0 \ \text{ in } S\,. \tag{2.20}$$

Let F stand for the operator that assigns to a function ψ the solution w of the problem

$$\beta w_t = Aw + \psi \ \text{ in } Q\,, \quad w = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 w = 0 \ \text{ in } S\,. \tag{2.21}$$

By Theorem 5.4 in [60], it holds $F \in \mathscr{L}(L^p(Q), W_p^{2,1}(Q))$. On the other hand, due to the properties of β and the assertion (i) of Lemma 2.1 we have

$\beta(k * v_t) \in L^p(Q)$ for any $v \in W_p^{2,1}(Q)$. Consequently, the problem (2.20) in $W_p^{2,1}(Q_t)$ is equivalent to the following fixed-point equation:

$$v = \mathcal{F}v + \mathcal{F}\widehat{u}, \quad \text{where } \mathcal{F}v = F(-\beta(k * v_t)) \tag{2.22}$$

and $\mathcal{F} \in \mathscr{L}(W_p^{2,1}(Q))$.

Let $t \in (0,T)$ and define a cutting operator P_t by the formula $P_t v = \begin{cases} v & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases}$. Observing that $Fv = FP_t v$ in Q_t and using (2.16) we deduce the estimate

$$\begin{aligned}
\|\mathcal{F}v\|_{W_p^{2,1}(Q_t)} &= \|F(\beta(k * v_t))\|_{W_p^{2,1}(Q_t)} = \|FP_t(\beta(k * v_t))\|_{W_p^{2,1}(Q_t)} \\
&\le \|FP_t(\beta(k * v_t))\|_{W_p^{2,1}(Q)} \le \|F\| \, \|P_t(\beta(k * v_t))\|_{L^p(Q)} \\
&= \|F\| \, \|\beta(k * v_t)\|_{L^p(Q_t)} \le C_3 \int_0^t \|k(\cdot, t-\tau)\|_{L^\infty(\Omega)} \|v\|_{W_p^{2,1}(Q_\tau)} d\tau
\end{aligned}$$

with $C_3 = \|F\| \, \|\beta\|_{C(\overline{Q})}$. Now we define the weighted norms $\|v\|_\sigma = \sup\limits_{0<t<T} e^{-\sigma t} \|v\|_{W_p^{2,1}(Q_t)}$, $\sigma > 0$, in the space $W_p^{2,1}(Q)$ and deduce the estimate

$$\begin{aligned}
\|\mathcal{F}v\|_\sigma &\le C_3 \sup_{0<t<T} e^{-\sigma t} \int_0^t \|k(\cdot, t-\tau)\|_{L^\infty(\Omega)} \|v\|_{W_p^{2,1}(Q_\tau)} d\tau \qquad (2.23) \\
&= C_3 \sup_{0<t<T} \int_0^t e^{-\sigma(t-\tau)} \|k(\cdot, t-\tau)\|_{L^\infty(\Omega)} e^{-\sigma\tau} \|v\|_{W_p^{2,1}(Q_\tau)} d\tau \le c_\sigma \|v\|_\sigma,
\end{aligned}$$

where $c_\sigma = C_3 \int_0^T e^{-\sigma\tau} \|k(\cdot,\tau)\|_{L^\infty(\Omega)} d\tau$. Since $k \in L^1(0,T; L^\infty(\Omega))$, by the dominated convergence theorem it holds $c_\sigma \to 0$ as $\sigma \to \infty$. Thus, there exists $\sigma_0 > 0$ such that $c_{\sigma_0} < \frac{1}{2}$. Consequently, (2.23) shows that $\mathcal{F}$ is a contraction. We conclude that the equation (2.22), which is equivalent to (2.8), has a unique solution in $W_p^{2,1}(Q)$.

Further, from (2.22) and (2.23) we deduce

$$\|v\|_{\sigma_0} \le \|\mathcal{F}v\|_{\sigma_0} + \|\mathcal{F}\widehat{u}\|_{\sigma_0} \le \frac{1}{2}(\|v\|_{\sigma_0} + \|\widehat{u}\|_{\sigma_0}).$$

This implies $\|v\|_{\sigma_0} \le \|\widehat{u}\|_{\sigma_0}$. Taking into account the equivalency relations of norms $e^{-\sigma T}\|\cdot\|_0 \le \|\cdot\|_\sigma \le \|\cdot\|_0 = \|\cdot\|_{W_p^{2,1}(Q)}$ we further have $\|v\|_{W_p^{2,1}(Q)} \le e^{\sigma_0 T} \|\widehat{u}\|_{W_p^{2,1}(Q)}$ and by $u = v + \widehat{u}$ we get $\|u\|_{W_p^{2,1}(Q)} \le (1 + e^{\sigma_0 T}) \|\widehat{u}\|_{W_p^{2,1}(Q)}$. Recall that $\widehat{u}$ is the solution of (2.8) in case $k = 0$. According to Theorem 5.4 in [60], $\|\widehat{u}\|_{W_p^{2,1}(Q)}$ is bounded by the right-hand side of (2.18). Consequently, we obtain the estimate (2.18).

Secondly, we prove (ii). Here, in general words, we repeat the part (i) of the proof, but in different spaces. Consider again the above-defined

function $\widehat{u}$. By Theorem 4.9 in [60] it holds $\widehat{u} \in C^{2+l,1+\frac{l}{2}}(Q)$. This implies that the problem (2.8) for $u \in C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the problem (2.20) for the difference $v = u - \widehat{u} \in C^{2+l,1+\frac{l}{2}}(Q)$.

Moreover, recall that F is the operator that assigns to an element ψ the solution w of the problem (2.21). In order to guarantee that $F\psi \in C^{2+l,1+\frac{l}{2}}(Q)$, the element ψ must satisfy the consistency condition $\psi = 0$ in $\Gamma \times \{0\}$ in case I. To this end, let us define the following Banach space (that is a subspace of $C^{l,\frac{l}{2}}(Q)$ in case I):

$$\begin{aligned} C_o^{l,\frac{l}{2}}(Q) = \{\psi \in C^{l,\frac{l}{2}}(Q) \,:\, \psi = 0 \text{ in } \Gamma \times \{0\} \text{ in case I}\} \\ \text{with the norm } \|\psi\|_{C_o^{l,\frac{l}{2}}(Q)} = \|\psi\|_{l,\frac{l}{2}}. \end{aligned} \tag{2.24}$$

Due to Theorem 4.9 in [60], again, we have $F \in \mathscr{L}(C_o^{l,\frac{l}{2}}(Q), C^{2+l,1+\frac{l}{2}}(Q))$. According to the assumed properties of β, k and the assertion (ii) of Lemma 2.1 we have $\beta(k * v_t) \in C^{l,\frac{l}{2}}(Q)$ for any $v \in C^{2+l,1+\frac{l}{2}}(Q)$. Moreover, $\beta(k * v_t) \in C_o^{l,\frac{l}{2}}(Q)$ for any $v \in C^{2+l,1+\frac{l}{2}}(Q)$, because the time convolution is zero for $t = 0$. Now we see that the problem (2.8) for $u \in C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the fixed-point equation (2.22) for $v = u - \widehat{u} \in C^{2+l,1+\frac{l}{2}}(Q)$, where $\mathcal{F} \in \mathscr{L}(C^{2+l,1+\frac{l}{2}}(Q))$.

Let $t \in (0,T)$ and define a continuation operator $\widetilde{P}_t$ by the formula $\widetilde{P}_t v(x,\tau) = \begin{cases} v(x,\tau) & \text{for } (x,\tau) \in Q_t \\ v(x,t) & \text{for } (x,\tau) \in Q \setminus Q_t \end{cases}$. Then, it hold $\|\widetilde{P}_t v\|_{l,\frac{l}{2}} = \|v\|_{C^{l,\frac{l}{2}}(Q_t)}$ and $Fv = F\widetilde{P}_t v$ in Q_t. Thus, using (2.17) we obtain

$$\begin{aligned} \|\mathcal{F}v\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} &= \|F(\beta(k * v_t))\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} \\ &= \|F\widetilde{P}_t(\beta(k * v_t))\|_{C^{2+l,1+\frac{l}{2}}(Q_t)} \le \|F\widetilde{P}_t(\beta(k * v_t))\|_{2+l,1+\frac{l}{2}} \\ &\le \|F\| \, \|\widetilde{P}_t(\beta(k * v_t))\|_{l,\frac{l}{2}} = \|F\| \, \|\beta(k * v_t)\|_{C^{l,\frac{l}{2}}(Q_t)} \\ &\le C_4 \left[\int_0^t \left\{ \|k(\cdot, t-\tau)\|_l \, \|v\|_{C^{2+l,1+\frac{l}{2}}(Q_\tau)} \right\}^{\frac{2}{2-l}} d\tau \right]^{\frac{2-l}{2}}, \end{aligned}$$

where $C_4 = C_0 \|F\| \|\beta\|_{l,\frac{l}{2}}$. Defining the weighted norms $\|w\|_\sigma$

$= \sup_{0<t<T} e^{-\sigma t}\|w\|_{C^{2+l,1+\frac{l}{2}}(Q_t)}$, $\sigma > 0$, in the space $C^{2+l,1+\frac{l}{2}}(Q)$, we deduce

$$\begin{aligned}\|\mathcal{F}v\|_\sigma &\le C_4 \sup_{0<t<T} e^{-\sigma t}\Big[\int_0^t \Big\{\|k(\cdot,t-\tau)\|_l \|v\|_{C^{2+l,1+\frac{l}{2}}(Q_\tau)}\Big\}^{\frac{2}{2-l}} d\tau\Big]^{\frac{2-l}{2}} \\ &= C_4 \sup_{0<t<T}\Big[\int_0^t \Big\{e^{-\sigma(t-\tau)}\|k(\cdot,t-\tau)\|_l \\ &\qquad \times e^{-\sigma\tau}\|v\|_{C^{2+l,1+\frac{l}{2}}(Q_\tau)}\Big\}^{\frac{2}{2-l}} d\tau\Big]^{\frac{2-l}{2}} \\ &\le C_4 \left[\int_0^T \left\{e^{-\sigma\tau}\|k(\cdot,t-\tau)\|_l\right\}^{\frac{2}{2-l}} d\tau\right]^{\frac{2-l}{2}} \|v\|_\sigma.\end{aligned}$$

Since $k \in L^{\frac{2}{2-l}}(0,T;C^l(\Omega))$, by the dominated convergence theorem, the coefficient $\left[\int_0^T \{e^{-\sigma\tau}\|k(\cdot,t-\tau)\|_l\}^{\frac{2}{2-l}} d\tau\right]^{\frac{2-l}{2}}$ is small for large σ. Owing to this, the proof can be finished as in case (i) making use of the fixed-point argument and an estimate for $\|\widehat{u}\|_{2+l,1+\frac{l}{2}}$ in Theorem 4.9 of [60]. ■

2.2 Positivity principle

In this section we prove a positivity principle for the solution of (2.8).

Theorem 2.2 *Assume* (2.14), (2.15), $k \in W_1^1(0,T;L^\infty(\Omega))$, $\beta, a_{ij}, a_j, a \in C(\overline{Q})$ *and*

$$k \ge 0, \quad k_t \le 0. \tag{2.25}$$

Let $u \in W_p^{2,1}(Q)$ *with some* $p \in (1,\infty)$ *solve the problem* (2.8) *and* $u_0 \ge 0$, $g \ge 0$, $f \ge 0$. *Then the following assertions are valid:*

(i) $u \ge 0$;

(ii) *if, in addition,* $\beta, a_{ij}, a_j, a \in C^{l,\frac{l}{2}}(Q)$ *with some* $l \in (0,1)$ *and there exists an open subset* Q_f *of* Q *such that* $f > 0$ *in* Q_f, *then* $u(\cdot,T) > 0$ *in* Ω *in case* I *and* $u(\cdot,T) > 0$ *in* $\overline{\Omega}$ *in case* II.

Proof. It consists of 4 steps.

1. step. We prove the assertion (i) under the additional assumptions

$$u \in C^{2,1}(\overline{Q}),\ k \in W_1^1(0,T;C(\overline{\Omega})),\ a \le 0. \tag{2.26}$$

Since $u = u_0 \ge 0$ in $\Omega \times \{0\}$, there exists

$$t_0 = \sup\{t :\ u(x,\tau) \ge 0 \ \text{ for } \ (x,\tau) \in \Omega \times [0,t],\ 0 \le t \le T\}.$$

In case the assertion $u \geq 0$ holds in Q, we have $t_0 = T$. Suppose on the contrary that $t_0 < T$. Then, we fix some $h \in (0, T - t_0]$ and define the set $V_{t_0,h} = \overline{\Omega} \times (t_0, t_0 + h]$. Note that the closure of $V_{t_0,h}$ is $\overline{V_{t_0,h}} = \overline{\Omega} \times [t_0, t_0 + h]$. By the definition of t_0 and $V_{t_0,h}$, there exists $(x_h^*, t_h^*) \in V_{t_0,h}$ such that $u(x_h^*, t_h^*) < 0$. Let us introduce the following function:

$$v(x,t) = u(x,t) + \mu_h(t - t_0 - h), \quad \text{where} \quad \mu_h = -\frac{u(x_h^*, t_h^*)}{2h} > 0. \quad (2.27)$$

Since $-\mu_h h \leq \mu_h(t - t_0 - h) \leq 0$ for $t \in [t_0, t_0 + h]$, we have

$$u(x,t) - \mu_h h \leq v(x,t) \leq u(x,t) \ \text{ for } \ (x,t) \in \overline{V_{t_0,h}}. \quad (2.28)$$

Observing (2.28), the definition of μ_h and the inequality $u(x_h^*, t_h^*) < 0$, we see that for all $(x,t) \in \overline{V_{t_0,h}}$ such that $u(x,t) \geq 0$, the relations

$$v(x,t) \geq u(x,t) - \mu_h h \geq -\mu_h h = \frac{u(x_h^*, t_h^*)}{2} \ > \ u(x_h^*, t_h^*) \geq v(x_h^*, t_h^*)$$

are valid. They imply that

$$\begin{array}{l} \text{function } v \text{ cannot attain its minimum over } \overline{V_{t_0,h}} \\ \text{in a point } (x,t) \text{ where } u(x,t) \geq 0. \end{array} \quad (2.29)$$

In particular, (2.29) implies that v cannot attain its minimum over $\overline{V_{t_0,h}}$ on the subset $\overline{V_{t_0,h}} \setminus V_{t_0,h} = \overline{\Omega} \times \{t_0\}$, because there $u \geq 0$ in view of the definition of t_0. Therefore,

$$\exists (x_h, t_h) \in V_{t_0,h} \ : \quad v(x_h, t_h) \leq v(x,t) \quad \text{for all} \quad (x,t) \in \overline{V_{t_0,h}}\,.$$

Moreover, $v(x_h, t_h) < 0$, because $v(x_h, t_h) \leq v(x_h^*, t_h^*) \leq u(x_h^*, t_h^*)$ and $u(x_h^*, t_h^*) < 0$.

Let us show that $x = x_h$ is the stationary minimum point of the x-dependent function $v(x, t_h)$, i.e.

$$\nabla v(x_h, t_h) = 0\,. \quad (2.30)$$

This relation may fail only in case the minimum occurs in the lateral boundary of $V_{t_0,h}$, i.e. when $x_h \in \Gamma$. In case I we have $u = g \geq 0$ for $x \in \Gamma$ and, by statement (2.29), x_h cannot belong to Γ. Thus, it remains to show (2.30) for the case II when $x_h \in \Gamma$. In this case $\omega \cdot \nabla v = \omega \cdot \nabla u = g$ in Γ. Note that then the inequality $g(x_h, t_h) > 0$ cannot hold, because otherwise v is strictly decreasing in the inner direction $-\omega(x_h)$ at (x_h, t_h) which implies that $v(x_h, t_h)$ cannot be the minimum of v. Consequently, due to the assumption $g \geq 0$, it holds $g(x_h, t_h) = 0$ and we have $\omega(x_h) \cdot \nabla v(x_h, t_h) = 0$. In addition, in case $n \geq 2$ we also have $\tau \cdot \nabla v(x_h, t_h) = 0$, where τ is an

arbitrary tangential direction of Γ at x_h, because $x = x_h$ is the minimum point of the x-dependent function $v(x, t_h)$ over the set Γ. Summing up, $\xi \cdot \nabla v(x_h, t_h) = 0$, where ξ is any direction. We obtain (2.30).

Now we are going to estimate the operator $Lu = \beta(u_t + k * u_t) - Au$ of the equation (2.8) termwise at the point $(x, t) = (x_h, t_h)$. By (2.27) we have $u_t(x_h, t_h) = v_t(x_h, t_h) - \mu_h$. Since t_h is the minimum point of the $t-$dependent function $v(x_h, t)$ in the half-interval $(t_0, t_0 + h]$, it holds $v_t(x_h, t_h) \leq 0$. Thus, we obtain

$$u_t(x_h, t_h) \leq -\mu_h. \tag{2.31}$$

Substituting u by $v - \mu_h(t - t_0 - h)$ in the right-hand side of (2.13) we have $Au = \sum\limits_{i,j=1}^{N} a_{ij} v_{x_i x_j} + \sum\limits_{j=1}^{n} a_j v_{x_j} + a[v - \mu_h(t - t_0 - h)]$. Since $x = x_h$ is the stationary minimum point of $v(x, t_h)$ and the principal part of A is elliptic (see (2.14)), the relations $\sum\limits_{j=1}^{n} a_j v_{x_j}$ and $\sum\limits_{i,j=1}^{n} a_{ij} v_{x_i x_j} \geq 0$ are valid at $(x, t) = (x_h, t_h)$. Thus, $-Au(x_h, t_h) \leq a(x_h, t_h)[\mu_h(t_h - t_0 - h) - v(x_h, t_h)]$. By the additional assumption $a \leq 0$ and and the inequality $v(x_h, t_h) < 0$ we further get $-Au(x_h, t_h) \leq a(x_h, t_h)\mu_h(t_h - t_0 - h)$. Since $|t_h - t_0 - h| \leq h$ we deduce the estimate

$$-Au(x_h, t_h) \leq C_5 \mu_h h\,, \tag{2.32}$$

where $C_5 = \|a\|_{C(\overline{Q})}$.

Finally, we estimate the term $(k * u_t)(x_h, t_h)$ in $Lu(x_h, t_h)$. Integrating by parts we have

$$\begin{aligned}\int_0^{t_h} k(x_h, t_h - \tau) u_\tau(x_h, \tau) d\tau &= k(x_h, 0) u(x_h, t_h) - k(x_h, t_h) u_0(x_h) \\ &+ \int_0^{t_0} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau.\end{aligned}$$

Here $-k(x_h, t_h) u_0(x_h) \leq 0$ and $\int_0^{t_0} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau \leq 0$, because $k \geq 0$, $k_t \leq 0$, $u_0 \geq 0$ and $u(x_h, \tau) \geq 0$ for $\tau \in [0, t_0]$ by the definition of t_0. Consequently, we can estimate as follows:

$$\int_0^{t_h} k(x_h, t_h - \tau) u_\tau(x_h, \tau) d\tau \leq k(x_h, 0) u(x_h, t_h) + \int_{t_0}^{t_h} k_t(x_h, t_h - \tau) u(x_h, \tau) d\tau.$$

Substituting in the right-hand side $u(x_h,t)$ by $v(x_h,t)-\mu_h(t-t_0-h)$, we get

$$\begin{aligned}\int_0^{t_h} k(x_h,t_h-\tau)u_\tau(x_h,\tau)d\tau \qquad\qquad (2.33)\\ \leq k(x_h,0)v(x_h,t_h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau -\\ -\mu_h\left[k(x_h,0)(t_h-t_0-h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right].\end{aligned}$$

In this relation we analyze separately the term $\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau$. To this end, introduce the following subsets of $[t_0,t_h]$:

$$U_h^+=\{\tau\in[t_0,t_h]\ :\ v(x_h,\tau)\geq 0\},\ \ U_h^-=\{\tau\in[t_0,t_h]\ :\ v(x_h,\tau)<0\}.$$

Taking account of $k_t\leq 0$ and the fact that $v(x_h,t_h)<0$ is the minimum of $v(x_h,t)$ on the interval $[t_0,t_h]$, we deduce

$$\begin{aligned}&\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\\ &=\int_{U_h^+} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau+\int_{U_h^-} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\\ &\leq\int_{U_h^-} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\leq\int_{U_h^-} k_t(x_h,t_h-\tau)d\tau\cdot v(x_h,t_h)\\ &\leq\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)d\tau\cdot v(x_h,t_h)=\left(k(x_h,t_h-t_0)-k(x_h,0)\right)v(x_h,t_h).\end{aligned}$$

Using this estimate in (2.33) and taking into account $k\geq 0$, $v(x_h,t_h)<0$ we obtain

$$\begin{aligned}\int_0^{t_h} k(x_h,t_h-\tau)u_\tau(x_h,\tau)d\tau\leq k(x_h,0)v(x_h,t_h)\\ +\left(k(x_h,t_h-t_0)-k(x_h,0)\right)v(x_h,t_h)\\ -\mu_h\left[k(x_h,0)(t_h-t_0-h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right]\\ \leq-\mu_h\left[k(x_h,0)(t_h-t_0-h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right].\end{aligned}\qquad (2.34)$$

Let us put all pieces together. Making use of (2.31), (2.32), (2.34), the assumption (2.15) and the relations $\mu_h>0$ and $|\tau-t_0-h|\leq h$ for

$\tau \in [t_0, t_h]$ we obtain

$$
\begin{aligned}
Lu(x_h,t_h) &\le \mu_h \Bigg\{ -\beta(x_h,t_h) - \beta(x_h,t_h)\Big[k(x_h,0)(t_h-t_0-h) \\
&\quad + \int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\Big] + C_5 h \Bigg\} \\
&\le \mu_h \left\{ -\beta_0 + h\left(\|\beta\|_{C(\overline{Q})} \left[k(x_h,0) + \|k_t\|_{L^1(0,T;C(\overline{\Omega}))} \right] + C_5 \right) \right\}.
\end{aligned}
$$

In case $h > 0$ is sufficiently small, due to the inequalities $\mu_h > 0$ and $\beta_0 > 0$ the relation $Lu(x_h, t_h) < 0$ holds. But this contradicts to the assumption $Lu = f \ge 0$. Consequently, the supposition $t_0 < T$ was not right. It holds $t_0 = T$, which by the definition of t_0 implies $u \ge 0$.

2. step. We prove the assertion (i) under the additional assumptions

$$
u \in C^{2,1}(\overline{Q}),\ k \in W_1^1(0,T;C(\overline{\Omega})),\ u_0 = 0. \tag{2.35}
$$

Let us define $\widetilde{u} = e^{-\sigma t}u$, where $\sigma = \frac{\|a\|_{C(\overline{Q})}}{\beta_0}$ and insert $u = e^{\sigma t}\widetilde{u}$ to the equation (2.8). Expressing the t-derivative as follows: $u_t = e^{\sigma t}\widetilde{u}_t + \sigma e^{\sigma t}\widetilde{u}$ and dividing the equation by $e^{\sigma t}$ we obtain

$$
\beta\left(\widetilde{u}_t + \sigma\widetilde{u} + e^{-\sigma t}k * [e^{\sigma t}\widetilde{u}_t + \sigma e^{\sigma t}\widetilde{u}]\right) = A\widetilde{u} + e^{-\sigma t}f. \tag{2.36}
$$

Let us transform the term with k in this equation:

$$
\begin{aligned}
&e^{-\sigma t}k * [e^{\sigma t}\widetilde{u}_t + \sigma e^{\sigma t}\widetilde{u}] \\
&= \int_0^t e^{-\sigma(t-\tau)}k(x,t-\tau)\widetilde{u}_\tau(x,\tau)d\tau + \sigma\int_0^t e^{-\sigma(t-\tau)}k(x,t-\tau)\widetilde{u}(x,\tau)d\tau \\
&= \int_0^t e^{-\sigma(t-\tau)}k(x,t-\tau)\widetilde{u}_\tau(x,\tau)d\tau + \sigma\int_0^t\int_0^{t-\tau} e^{-\sigma\eta}k(\eta)d\eta\,\widetilde{u}_\tau(x,\tau)d\tau \\
&= \widetilde{k} * \widetilde{u}_t,
\end{aligned}
$$

where $\widetilde{k}(x,t) = e^{-\sigma t}k(x,t) + \sigma\int_0^t e^{-\sigma\eta}k(x,\eta)d\eta$. Here we used the relation $\widetilde{u}(\cdot,0) = u_0 = 0$ during the integration by parts.

Thus, the equation (2.36) takes the form $\beta(\widetilde{u}_t + \widetilde{k} * \widetilde{u}_t) = \widetilde{A}\widetilde{u} + \widetilde{f}$, where $\widetilde{f} = e^{-\sigma t}f$, $\widetilde{A}\widetilde{u} = \sum\limits_{i,j=1}^n a_{ij}\widetilde{u}_{x_i x_j} + \sum\limits_{j=1}^n a_j\widetilde{u}_{x_j} + \widetilde{a}\widetilde{u}$ and $\widetilde{a} = a - \sigma\beta$.

Summing up, $\widetilde{u}$ solves the problem

$$
\beta(\widetilde{u}_t + \widetilde{k} * \widetilde{u}_t) = \widetilde{A}\widetilde{u} + \widetilde{f}\ \text{ in } Q\,,\ \ \widetilde{u} = 0\ \text{ in } \Omega\times\{0\}\,,\ \ B_1\widetilde{u} = \widetilde{g}\ \text{ in } S,
$$

where $\widetilde{g} = e^{-\sigma t}g$. Since $k \in W_1^1(0,T;C(\overline{\Omega}))$, it holds $\widetilde{k} \in W_1^1(0,T;C(\overline{\Omega}))$. The inequalities (2.25) yield $\widetilde{k} \ge 0$, $\widetilde{k}_t \le 0$. Moreover, $\widetilde{f} \ge 0$, $\widetilde{g} \ge 0$, and

due to the choice of σ and the condition (2.15), it holds $\widetilde{a} \le 0$. By means of the part 1 of the proof, we get $\widetilde{u} \ge 0$. This implies $u \ge 0$.

3. step. We prove the assertion (i) in the general case. It is enough to prove this assertion for $p \in (1, \frac{3}{2})$. Indeed, a solution u of (2.8) that belongs to the space $W_p^{2,1}(Q)$ for some $p \in [\frac{3}{2}, \infty)$ belongs to such a space for any $p \in (1, \frac{3}{2})$, too. Operating with solutions in $W_p^{2,1}(Q)$, $p \in (1, \frac{3}{2})$, we have not to deal with consistency conditions.

Let us formulate two problems:

$$\beta \bar{u}_t = (A - k(\cdot,0)\beta)\bar{u} \ \text{ in } Q\,, \ \ \bar{u} = u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 \bar{u} = g \ \text{ in } S, \quad (2.37)$$

$$\beta(\tilde{u}_t + k * \tilde{u}_t) = A\tilde{u} + f_{\bar{u}} \ \text{ in } Q\,, \ \ \tilde{u} = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 \tilde{u} = 0 \ \text{ in } S, \quad (2.38)$$

where $f_{\bar{u}} = f - \beta(k_t * \bar{u}) + \beta k u_0$.

Firstly, we establish the existence of solutions of these problems. Due to $u \in W_p^{2,1}(Q)$ and embedding theorems [43] we have $u_0 = u|_{\Omega\times\{0\}} \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g = B_1 u|_S \in W_p^{2-\frac{1}{p}-\vartheta, 1-\frac{1}{2p}-\frac{\vartheta}{2}}(S)$. Since $k \in W_1^1(0,T;L^\infty(\Omega))$ and $\beta \in C(\overline{Q})$, the coefficient of the lower order term $k(\cdot,0)\beta$ of the parabolic equation in (2.37) belongs to $L^\infty(Q)$. Using Theorem 5.4 in [60] we conclude that problem (2.37) has a unique solution $\bar{u} \in W_p^{2,1}(Q)$. Further, due to $\beta \in C(\overline{Q})$, $k \in W_1^1(0,T;L^\infty(\Omega)) \subset L^\infty(Q)$ and $u_0 \in W_p^{2-\frac{2}{p}}(\Omega) \subset L^p(\Omega)$ we have $\beta k u_0 \in L^p(Q)$. Moreover, in view of $k_t \in L^1(0,T;L^\infty(\Omega))$, $\bar{u} \in W_p^{2,1}(Q) \subset L^p(Q)$ and the Young's theorem for convolutions, we get $k_t * \bar{u} \in L^p(Q)$, which implies $\beta(k_t * \bar{u}) \in L^p(Q))$. Consequently, due to $f \in L^p(Q)$ we obtain $f_{\bar{u}} \in L^p(Q)$. By Theorem 2.1 (i), problem (2.38) has a unique solution $\tilde{u} \in W_p^{2,1}(Q)$.

Adding the problems (2.37) and (2.38) and integrating by parts the convolution term in $f_{\bar{u}}$, we see that the sum $\bar{u} + \tilde{u}$ solves (2.8). Hence, by the uniqueness, it holds $u = \bar{u} + \tilde{u}$. According to the well-known extremum principle for parabolic equations (e.g. [43] Ch. 3, Theorem 7.2), we have $\bar{u} \ge 0$. This together with the assumptions of theorem implies $f_{\bar{u}} \ge 0$. In order to complete the step 3 it remains to show that $\tilde{u} \ge 0$.

The idea to prove $\tilde{u} \ge 0$ consists in approximation of the problem (2.38) by a sequence of smooth problems and applying the result of step 2 to latter ones. Let us choose some functions $\beta^m, a_{ij}^m, a_j^m, a^m, \hat{f}^m, \hat{k}^m \in C^\infty(\overline{Q})$ such that

$$\begin{aligned} &\|\beta^m - \beta\|_{C(\overline{Q})},\ \|a_{ij}^m - a_{ij}\|_{C(\overline{Q})},\ \|a_j^m - a_j\|_{C(\overline{Q})},\ \|a^m - a\|_{C(\overline{Q})}, \\ &\|\hat{f}^m - f_{\bar{u}}\|_{L^p(Q)},\ \|\hat{k}^m - k\|_{W_1^1(0,T;L^\infty(\Omega))} \to 0 \quad \text{as} \quad m \to \infty \end{aligned} \quad (2.39)$$

and $\hat{f}^m = 0$ in $\Gamma \times \{0\}$. Due to (2.39), for sufficiently large $m \ge M_{\epsilon,\beta_0}$ the relations $\max\limits_{i,j} \|a_{ij}^m - a_{ij}\|_{C(\overline{Q})} \le \frac{\epsilon}{2n}$ and $\|\beta^m - \beta\|_{C(\overline{Q})} \le \frac{\beta_0}{2}$ are valid. Therefore, from (2.14) and (2.15) we get the following ellipticity and positivity

inequalities for the matrix a_{ij}^m and the coefficient β^m in case $m \geq M_{\epsilon,\beta_0}$:

$$\sum_{i,j=1}^{n} a_{ij}^m(x,t)\,\lambda_i\lambda_j \geq \tfrac{\epsilon}{2}|\lambda|^2 \quad \text{for any } x \in \Omega,\ t \in (0,T),\ \ \lambda \in \mathbb{R}^n\,,$$
$$\beta^m(x,t) \geq \tfrac{\beta_0}{2} \quad \text{for any } x \in \Omega,\ t \in (0,T).$$

Further, on the basis of $\hat{f}^m$, let us define the nonnegative approximation $f^m(x,t) = \frac{|\hat{f}^m(x,t)|+\hat{f}^m(x,t)}{2}$ for the function $f_{\bar{u}}$. Due to $f_{\bar{u}} \geq 0$ the inequality $|f^m - f_{\bar{u}}| \leq |\hat{f}^m - f_{\bar{u}}|$ holds. Therefore, in view of (2.39) we obtain

$$\|f^m - f_{\bar{u}}\|_{L^p(Q)} \leq \|\hat{f}^m - f_{\bar{u}}\|_{L^p(Q)} \to 0 \quad \text{as} \quad m \to \infty. \tag{2.40}$$

The operation of taking the absolute value preserves the Hölder-continuity of a function. Therefore, $f^m \in C^{l,\frac{l}{2}}(Q)$ for any $l \in (0,1)$. The relation $\hat{f}^m = 0$ in $\Gamma \times \{0\}$ yields $f^m = 0$ in $\Gamma \times \{0\}$.

Finally, on the basis of $\hat{k}^m$ we define the new approximation for k:

$$k^m(x,t) = \textstyle\int_T^t \varkappa^m(x,\tau)d\tau + q^m(x),$$

where $\varkappa^m(x,t) = \frac{\hat{k}_t^m(x,t)-|\hat{k}_t^m(x,t)|}{2}$ and $q^m(x) = \frac{|\hat{k}^m(x,T)|+\hat{k}^m(x,T)}{2}$. Then $k^m \in W_1^1(0,T;C^l(\Omega))$ for any $l \in (0,1)$ and $k^m \geq 0$, $k_t^m \leq 0$. Moreover, since $k_t \leq 0$ and $k \geq 0$, we obtain $|\varkappa^m - k_t| \leq |\hat{k}_t^m - k_t|$ and $|q^m - k(\cdot,T)| \leq |\hat{k}^m(\cdot,T) - k(\cdot,T)|$. Observing these inequalities, the relation $k(x,t) = \int_T^t k_\tau(x,\tau)d\tau + k(x,T)$ and (2.39) we deduce

$$\begin{aligned}
&\|k^m - k\|_{L^1(0,T;L^\infty(\Omega))} \\
&\leq \textstyle\int_0^T \operatorname{ess\,sup}_{x\in\Omega} \left[\int_t^T |\varkappa^m(x,\tau) - k_\tau(x,\tau)|d\tau + |q^m(x) - k(x,T)|\right] dt \\
&\leq T\left[\|\hat{k}_t^m - k_t\|_{L^1(0,T;L^\infty(\Omega))} + \|\hat{k}^m(\cdot,T) - k(\cdot,T)\|_{L^\infty(\Omega)}\right] \to 0 \\
&\quad \text{as} \quad m \to \infty.
\end{aligned} \tag{2.41}$$

Now let us formulate the following sequence of approximating problems for (2.38):

$$\begin{gathered}
\beta^m(u_t^m + k^m * u_t^m) = A^m u^m + f^m \ \text{ in } Q\,, \\
u^m = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 u^m = 0 \ \text{ in } S,
\end{gathered} \tag{2.42}$$

where $A^m v = \sum_{i,j=1}^{n} a_{ij}^m v_{x_i x_j} + \sum_{j=1}^{n} a_j^m v_{x_j} + a^m v$. Observing the proved regularity of the data of these problems and Theorem 2.1 (ii) we conclude that for each integer $m \geq M_{\epsilon,\beta_0}$, (2.42) has the unique solution $u^m \in C^{2+l,1+\frac{l}{2}}(Q)$.

Subtracting (2.38) from (2.42) we obtain the following problems for the differences $v^m = u^m - \tilde{u}$:

$$\beta(v_t^m + k * v_t^m) = Av^m + \phi^m \ \text{ in } Q\,, \ \ v^m = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 v^m = 0 \ \text{ in } S$$

where

$$\begin{aligned}\phi^m = (A^m - A)(v^m + \tilde{u}) - (\beta^m - \beta)[v_t^m + \tilde{u}_t + k * (v_t^m + \tilde{u}_t)]\\ -(\beta + \beta^m - \beta)[(k^m - k) * (v_t^m + \tilde{u}_t)] + f^m - f_{\bar{u}}.\end{aligned}$$

Using the estimate (2.18) and the Young's inequality for convolutions we deduce

$$\begin{aligned}\|v^m\|_{W_p^{2,1}(Q)} \le C_1\|\phi^m\|_{L^p(Q)} \le C_1\|f^m - f_{\bar{u}}\|_{L^p(Q)}\\ +\widetilde{C}(m)(\|v^m\|_{W_p^{2,1}(Q)} + \|\tilde{u}\|_{W_p^{2,1}(Q)}), \ \text{ where}\end{aligned}$$

$$\begin{aligned}\widetilde{C}(m) = C_6\Big[\|a_{ij}^m - a_{ij}\|_{(C(\overline{Q}))^{n\times n}} + \|a_j^m - a_j\|_{(C(\overline{Q}))^n} + \|a^m - a\|_{C(\overline{Q})}\\ +\|\beta^m - \beta\|_{C(\overline{Q})} + (\|\beta\|_{C(\overline{Q})} + \|\beta^m - \beta\|_{C(\overline{Q})})\|k^m - k\|_{L^1(0,T;L^\infty(\Omega))}\Big]\end{aligned}$$

and C_6 is a constant independent of m. By virtue of (2.39) and (2.41) we have $\widetilde{C}(m) \to 0$ as $m \to \infty$. Hence, there exists $\widetilde{M} \ge M_{\epsilon,\beta_0}$ such that $\widetilde{C}(m) \le \frac{1}{2}$ for $m \ge \widetilde{M}$. We get $\|v^m\|_{W_p^{2,1}(Q)} \le 2C_1\|f^m - f_{\bar{u}}\|_{L^p(Q)} + 2\widetilde{C}(m)\|\tilde{u}\|_{W_p^{2,1}(Q)}$ for $m \ge \widetilde{M}$. This in view of (2.40) implies

$$\|v^m\|_{W_p^{2,1}(Q)} = \|u^m - \tilde{u}\|_{W_p^{2,1}(Q)} \to 0 \quad \text{as} \quad m \to \infty. \tag{2.43}$$

Recall that $f^m \ge 0$, $k^m \ge 0$ and $k_t^m \le 0$. Thus, according to step 2 of the proof, the solution of the problem (2.42) satisfies $u^m \ge 0$. This with (2.43) implies $\tilde{u} \ge 0$. This yields $u \ge 0$. Step 3 is completed. The assertion (i) of the theorem is proved.

4. step. We prove (ii). According to (1.8), there exists an open ball $U \subset Q_f$ and $\varepsilon > 0$ such that $f \ge \varepsilon$ in U. We can choose some $f^\dagger \in C^\infty(\overline{Q})$ so that $f^\dagger = 0$ in $Q \setminus U$ and $0 < f^\dagger \le \frac{\varepsilon}{2}$ in U. (For instance, we can set $f^\dagger(y) = \frac{\varepsilon}{2} e^{-\frac{|y-y_0|^2}{\rho^2 - |y-y_0|^2}}$ in U, where $y = (x,t)$ and y_0 and ρ are the center and the radius of the ball U, respectively.) Then $f^\dagger \le f$ in Q.

Further, let us define $c_k = \operatorname{ess\,sup}_{x\in\Omega} k(x,0)$ and formulate the following problems:

$$\beta u_t^\dagger = (A - c_k\beta)u^\dagger + f^\dagger \text{ in } Q, \ \ u^\dagger = 0 \text{ in } \Omega \times \{0\}, \ \ B_1 u^\dagger = 0 \text{ in } S, \tag{2.44}$$

$$\beta(\hat{u}_t + k * \hat{u}_t) = A\hat{u} + \hat{f}_{u^\dagger} \text{ in } Q, \ \ \hat{u} = u_0 \text{ in } \Omega \times \{0\}, \ \ B_1\hat{u} = g \text{ in } S, \tag{2.45}$$

where $\hat{f}_{u^\dagger} = f - f^\dagger - \beta(k_t * u^\dagger) + \beta(c_k - k(\cdot,0))u^\dagger$. By Theorem 2.1 (ii), problem (2.44) has a unique solution $u^\dagger \in C^{2+l,1+\frac{l}{2}}(Q)$. Observing that

$f^\dagger \geq 0$ in Q and $f^\dagger > 0$ in U and using the well-known strong extremum principles for parabolic equations (see e.g. Theorem 6.1.1 (ii) in [20]), we obtain the relation

$$u^\dagger(\cdot, T) > 0 \ \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II)}. \tag{2.46}$$

As in step 3 we assume without restriction of generality that $p \in (1, \frac{3}{2})$. The components $f^\dagger, \beta, u^\dagger$ of the function $\hat{f}_{u^\dagger}$ are continuous in $\overline{Q}$. Moreover, $k_t \in L^1(0,T; L^\infty(\Omega))$, $k(\cdot, 0) \in L^\infty(\Omega)$ and $f \in L^p(Q)$. Thus, we have $\hat{f}_{u^\dagger} \in L^p(Q)$. Moreover, as in step 3 we can show (by embedding theorems) that $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g \in W_p^{2-\frac{1}{p}-\vartheta, 1-\frac{1}{2p}-\frac{\vartheta}{2}}(S)$. Therefore, due to Theorem 2.1 (i) the problem (2.45) has a unique solution $\hat{u} \in W_p^{2,1}(Q)$. Because of $f - f^\dagger \geq 0$, $\beta > 0$, $k_t \leq 0$, $u^\dagger \geq 0$, $c_k - k(\cdot, 0) \geq 0$ we have $\hat{f}_{u^\dagger} \geq 0$. Moreover, $u_0, g \geq 0$. Thus, the part (i) of the present theorem yields $\hat{u} \geq 0$.

Adding (2.44) and (2.45) and integrating by parts the term $k_t * u^\dagger$ we see that the function $u^\dagger + \hat{u}$ solves the problem (2.8). By uniqueness we get $u = u^\dagger + \hat{u}$. The relations (2.46) and $\hat{u} \geq 0$ imply the assertion (ii). Theorem is completely proved ■

Remark 2.1 For k, depending only on t, the assumptions (2.25) read

$$k \in W_1^1(0,T)\,, \quad k \geq 0\,, \quad k' \leq 0. \tag{2.47}$$

A natural question is: which sufficient conditions should satisfy the original relaxation kernels m and μ in order to guarantee (2.47)?

The simplest particular case occurs when $m = 0$. Then $\mu = k$ and the conditions for k and μ coincide. In general case, let us firstly assume that $m, \mu \in W_1^1(0,T)$ and $m, \mu \geq 0$. Then $m \in C[0,T]$ and the solution $\widehat{m}$ of (2.7) also belongs to $C[0,T]$ (see e.g. [15]). From (2.7) we deduce the following relations for $\widehat{m}$ and its derivative:

$$\widehat{m} = m * \widehat{m} + m\,, \quad \widehat{m}' = m' * \widehat{m} + m(0)\widehat{m} + m'. \tag{2.48}$$

Since $m' \in L^1(0,T)$ and $m \in C[0,T]$, the right-hand side of the second equality in (2.48) belongs to $L^1(0,T)$. Therefore, $\widehat{m} \in W_1^1(0,T)$. Iterating the first relation in (2.48) we get the representation of $\widehat{m}$ in the form of the Neumann series: $\widehat{m} = \sum_{i=0}^{\infty} m(*m)^i$ that converges in $C[0,T]$. Due to $m \geq 0$, this series is nonnegative, hence $\widehat{m} \geq 0$. From (2.9) in view of $\mu, \widehat{m} \in W_1^1(0,T)$, $\mu, \widehat{m} \geq 0$ we obtain the desired relations $k \in W_1^1(0,T)$, $k \geq 0$. It remains to deduce sufficient conditions for the relation $k' \leq 0$. To this end, let us perform the following computations. Observing (2.48) we substitute the last addend $\widehat{m}$ in (2.9) by $m * \widehat{m} + m$ to get $k = m + \mu + (m + \mu) * \widehat{m}$. Then we differentiate: $k' = m' + \mu' + (m' + \mu') * \widehat{m} + (m(0) + \mu(0))\widehat{m}$. Substituting the term $\widehat{m}$ in the last addend again by $m * \widehat{m} + m$ we finally

obtain $k' = l + l * \widehat{m}$, where $l = m' + \mu' + (m(0) + \mu(0))m$. Since $\widehat{m} \geq 0$, the sufficient condition for $k' \leq 0$ is $l \leq 0$. Summing up, *the sufficient conditions for* (2.47) *in terms of the original kernels* m *and* μ *are*

$$m, \mu \in W_1^1(0,T), \;\; m, \mu \geq 0\,, \quad m' + \mu' \leq -(m(0) + \mu(0))m. \tag{2.49}$$

For instance, the widely used exponential kernels $m(t) = \sum_{i=1}^N \alpha_i e^{-\gamma_i t}$, $\mu(t) = \sum_{i=1}^N \beta_i e^{-\gamma_i t}$ satisfy the conditions (2.49) provided $\alpha_i, \beta_i \geq 0$, $\gamma_i \geq \frac{\alpha_i}{\alpha_i + \beta_i} \sum_{j=1}^N (\alpha_j + \beta_j)$, $i = 1, \ldots, N$.

2.3 Results for IP1

From now on, let the coefficients β, a_{ij}, a_j depend only on x and k depend only on t.

Due to Lemma 2.1, IP1 is in the class of pairs (z, u) of functions, whose second component u together with its derivatives u_t, u_{x_i}, $u_{x_i x_j}$ belongs to $L^p(0,T)$, $p > 1$, for any x, equivalent to the following inverse problem:

$$\begin{aligned} &\beta(u_t + k * u_t) = Au + zr + f_0 \;\text{ in } Q, \\ &u = u_0 \;\text{ in } \Omega \times \{0\}, \;\; B_1 u = g \;\text{ in } S, \end{aligned} \tag{2.50}$$

$$u \;=\; u_T \;\text{ in } \Omega \times \{T\}, \tag{2.51}$$

where B_1, g are given by (2.11), (2.12) and

$$r \;=\; \phi + \widehat{m} * \phi\,, \;\; f_0 = \chi_0 - \beta\mu u_0 + \widehat{m} * (\chi_0 - \beta\mu u_0). \tag{2.52}$$

We continue to study the problem (2.50), (2.51).

2.3.1 Uniqueness

Firstly, we formulate a technical result.

Lemma 2.2 *Let* (1.6), (1.7) *hold. Assume* $\beta \in C^l(\Omega)$ *with some* $l \in (0,1)$, $a_{ij}, a_j \in C(\overline{\Omega})$, $a \in C(\overline{Q})$, $a_t \in L^p(Q)$, $k \in L^p(0,T)$ *with some* $p \in (1, \min\{\frac{3}{2}, \frac{2}{2-l}\})$ *and the problem* (2.8) *has a solution* $u \in W_p^{2,1}(Q)$ *such that* $B_1 u$ *is continuous in a neighborhood of* $\Gamma \times \{0\}$. *If* $f_t \in L^p(Q)$, $u_0 \in W_p^2(\Omega)$, $A(0)u_0 + f(\cdot, 0) \in W_p^{2-\frac{2}{p}}(\Omega)$ *and* $g_t \in W_p^{2-\frac{1}{p}-\vartheta, 1-\frac{1}{2p}-\frac{\vartheta}{2}}(S)$ *then* $u_t \in W_p^{2,1}(Q)$.

This lemma is proved in Publication I, i.e. [27], p. 30-31.

Now we formulate a uniqueness theorem for the inverse problem (2.50), (2.51).

Theorem 2.3 *Let* (1.6), (1.7), (2.47) *hold and* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ *with some* $l \in (0,1), p \in (1,\infty)$. *Moreover, let* $a_t \geq 0$ *in* Q, $r \in C^{l,\frac{l}{2}}(Q), r_t \in L^p(Q)$ *and*

$$r \geq 0\,, \quad r_t + k * r_t - \theta r \geq 0 \quad \text{in } Q, \tag{2.53}$$

where

$$\theta = \sup_{x\in\Omega} \tfrac{a(x,T)}{\beta(x)}. \tag{2.54}$$

Finally, assume that

$$\begin{aligned} &\text{for all } x \in \Omega \text{ there exists an open subset } Q_x \text{ of } Q \text{ such that} \\ &\exists t_x \in (0,T) : (x,t_x) \in Q_x \quad \text{and} \quad r_t + k * r_t - \theta r > 0 \text{ in } Q_x. \end{aligned} \tag{2.55}$$

If $(z,u) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *solves* (2.50), (2.51) *and* $f_0, u_0, g, u_T = 0$ *then* $z = 0$, $u = 0$.

Before proving this theorem we formulate and prove an additional technical lemma.

Lemma 2.3 *Under the assumptions of Theorem* 2.3 *the following assertion is valid:*

$$\begin{aligned} &\text{for all } x \in \Omega \text{ there exists an open subset } \widehat{Q}_x \text{ of } Q \text{ such that} \\ &(x,t_x) \in \widehat{Q}_x \quad \text{and} \quad r > 0 \text{ in } \widehat{Q}_x. \end{aligned} \tag{2.56}$$

Proof. Denote $q = r_t + k * r_t - \theta r$. Integrating by parts the term $k * r_t$ we can transform this equality to the following ordinary differential equation for r: $r_t + (k(0) - \theta)r = r(x,0)k - k' * r + q$. The solution is

$$r = r(x,0)e^{(\theta - k(0))t} + e^{(\theta-k(0))t} * [r(x,0)k - k' * r] + e^{(\theta-k(0))t} * q. \tag{2.57}$$

Let x be an arbitrary point in Ω. We can choose an open cylinder $\widehat{Q}_x = U \times (t_1, \tilde{t}_1) \subset Q_x$ such that $(x,t_x) \in \widehat{Q}_x$. Due to (2.47), the continuity of r and the inequality $r \geq 0$ the first two addends in the right-hand side of (2.57) are continuous nonnegative functions. Thus, $r(y,t) \geq e^{(\theta-k(0))t} * q(y,t)$ for any $(y,t) \in Q_x$. According to the assumptions (2.53), (2.55) and (1.8) it holds $q \geq 0$ in Q and there exists $\varepsilon > 0$ such that $q \geq \varepsilon$ in $\widehat{Q}_x$. Inside the equivalence class corresponding to q, we can choose such a q that satisfies $q(y,t) \geq 0$ for any $(y,t) \in Q$ and $q(y,t) \geq \varepsilon$ for any $(y,t) \in \widehat{Q}_x$. Now for any $(y,t) \in \widehat{Q}_x$ we estimate

$$\begin{aligned} r(y,t) &\geq e^{(\theta-k(0))t} * q(y,t) = \textstyle\int_0^t e^{(\theta-k(0))(t-\tau)} q(y,\tau)\,d\tau \\ &\geq \textstyle\int_{t_1}^t e^{(\theta-k(0))(t-\tau)} q(y,\tau)\,d\tau \geq \min\{1\,;\, e^{(\theta-k(0))(t-t_1)}\}\varepsilon \cdot (t - t_1). \end{aligned}$$

This yields (2.56). ■

Proof of Theorem 2.3. The proof develops further a method that was previously applied for usual parabolic inverse problems with final overdetermination in [21].

Suppose contrary that $z \neq 0$ and define $z^+ = \frac{|z|+z}{2}$, $z^- = \frac{|z|-z}{2}$. Note that $z^\pm \in C^l(\Omega)$. (The operation of taking absolute value preserves the Hölder-continuity of a function.) Firstly, let us show that

$$z^+ \neq 0 \ \text{ and } \ z^- \neq 0. \tag{2.58}$$

Let $z^- = 0$. Then $z = z^+ \geq 0$ and $zr \geq 0$. Moreover, by the supposition $z \neq 0$ and the continuity of z there exists an open ball U in Ω such that $z > 0$ in U. Let us choose some $x_1 \in U$. Then, by virtue of (2.56) it holds $zr > 0$ in the open set $[U \times (0,T)] \cap \widehat{Q}_{x_1}$ of Q. Observing the assumptions $f_0, u_0, g = 0$ and applying Theorem 2.2 to the solution u of the problem (2.50) we get $u(x,T) > 0$, $x \in \Omega$. But this contradicts to the assumption $u_T = 0$. Similarly, we reach the contradiction in case $z^+ = 0$ making use of Theorem 2.2 for $-u$.

Further, let us formulate the following problems for $u^\pm$:

$$\beta(u_t^\pm + k * u_t^\pm) = Au^\pm + z^\pm r \text{ in } Q, \ \ u^\pm = 0 \text{ in } \Omega \times \{0\}, \ \ B_1 u^\pm = 0 \text{ in } S. \tag{2.59}$$

By virtue of the assumptions, the free term of (2.59) has the smoothness property $z^\pm r \in C^{l,\frac{l}{2}}(Q)$. In order to apply Theorem 2.1 (ii) to the problem (2.59), it remains to show the consistency condition $z^\pm r = 0$ in $\Gamma \times \{0\}$ in case I. Due to $u \in C^{2+l,1+\frac{l}{2}}(Q)$, the equation in (2.50) can be extended to $\Gamma \times \{0\}$. In case I this implies the relation $g_t = Au_0 + zr + f_0$, and by $g, u_0, f_0 = 0$ the equality $zr = 0$ in $\Gamma \times \{0\}$. Since the null-set of $z^\pm$ is larger than the null-set of z, we get the desired relation $z^\pm r = 0$ in $\Gamma \times \{0\}$. Consequently, by Theorem 2.1 (ii) problems (2.59) have the unique solutions $u^\pm \in C^{2+l,1+\frac{l}{2}}(Q)$.

Next step is to prove the following inequalities:

$$u^\pm \geq 0\,, \ \ u^\pm(\cdot,T) > 0 \ \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II)}, \tag{2.60}$$

$$u_t^\pm + k * u_t^\pm - \theta u^\pm \geq 0, \ (u_t^\pm + k * u_t^\pm - \theta u^\pm)(\cdot,T) > 0 \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II)}. \tag{2.61}$$

By $z^\pm \geq 0$ and (2.53) we have $z^\pm r \geq 0$. Moreover, since $z^\pm$ are continuous and non-vanishing (see (2.58)), there exist open balls $U^\pm$ of Ω such that $z^\pm > 0$ in $U^\pm$. Let us choose some $x^\pm \in U^\pm$. By virtue of (2.56) it holds $z^\pm r > 0$ in the open subsets $[U^\pm \times (0,T)] \cap \widehat{Q}_{x^\pm}$ of Q. Using Theorem 2.2 for solutions of problems (2.59) we immediately obtain (2.60).

Let us prove (2.61). Assume without restriction of generality that $p \in (1, \min\{\frac{3}{2}; \frac{2}{2-l}\})$. Then the assumptions of Lemma 2.2 are satisfied

for the solutions $u^{\pm}$ of the problems (2.59). (In particular, the assumption $z^{\pm}r(\cdot,0) \in W_p^{2-\frac{2}{p}}(\Omega)$ follows from $z^{\pm}r(\cdot,0) \in C^l(\Omega)$ and the inequality $l > 2 - \frac{2}{p}$.) Thus, we obtain $u_t^{\pm} \in W_p^{2,1}(Q)$. We have immediately $u_{tt}^{\pm}, u_{tx_i}^{\pm}, u_{tx_ix_j}^{\pm} \in L^p(\Omega)$ and by Lemma 2.1 (i) we get $k * u_{tt}^{\pm}, k * u_{tx_i}^{\pm}, k * u_{tx_ix_j}^{\pm} \in L^p(\Omega)$, which implies $k * u_t^{\pm} \in W_p^{2,1}(Q)$. From (2.59) we deduce the following problems for the functions $v^{\pm} = u_t^{\pm} + k * u_t^{\pm} - \theta u^{\pm} \in W_p^{2,1}(Q)$:

$$\begin{aligned} &\beta(v_t^{\pm} + k * v_t^{\pm}) = Av^{\pm} + z^{\pm}[r_t + k * r_t - \theta r] + f_1^{\pm} \ \text{ in } Q\,, \\ &v^{\pm} = \frac{1}{\beta} z^{\pm} r \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 v^{\pm} = 0 \ \text{ in } S, \end{aligned} \tag{2.62}$$

where

$$f_1^{\pm}(x,t) = a_t(x,t)u^{\pm}(x,t) + \int_0^t k'(t-\tau)(a(x,\tau) - a(x,t))u^{\pm}(x,\tau)d\tau.$$

By virtue of the assumptions of theorem and $z^{\pm} \geq 0$ and $u^{\pm} \geq 0$ the free term and initial condition in (2.62) are non-negative. Therefore, Theorem 2.2 (i) implies $v^{\pm} \geq 0$, i.e. the left relation in (2.61). Moreover, according to (2.55) and the definition of $U^{\pm}$ we have $z^{\pm}[r_t + k * r_t - \theta r] > 0$ in the open subset $[U^{\pm} \times (0,T)] \cap Q_{x^{\pm}}$ of Q. Using Theorem 2.2 (ii) deduce the right relation in (2.61).

Since u^+ is continuous in $\overline{Q}$, there exists $x^* \in \overline{\Omega}$ such that

$$u^+(x,T) \leq u^+(x^*,T) \ \text{ for any } \ x \in \overline{\Omega}. \tag{2.63}$$

Observing the relation $u = u^+ - u^-$ and the assumption $u(\cdot,T) = u_T = 0$, we have $u^+(\cdot,T) = u^-(\cdot,T)$, and (2.63) implies

$$u^-(x,T) \leq u^-(x^*,T) \ \text{ for any } \ \ x \in \overline{\Omega}. \tag{2.64}$$

Let us show that the point x^* is the stationary maximum of $u^{\pm}(\cdot,T)$, i.e.

$$\nabla u^{\pm}(x^*,T) = 0. \tag{2.65}$$

The equality (2.65) may fail only when $x^* \in \Gamma$. In case I we have the boundary condition $u^+ = 0$ in Γ, hence in view of (2.60) the function $u^+(\cdot,T)$ cannot achieve its maximum on Γ, and we automatically get (2.65). It remains to show (2.65) for the case II when $x^* \in \Gamma$. In this case due the vanishing boundary condition we have $\omega(x^*) \cdot \nabla u^+(x^*,T) = 0$. (Recall that $\omega(x)$ is an outer direction at $x \in \Gamma$). Furthermore, since $u^*(\cdot,T)$ achieves its maximum over Γ in the point $x = x^*$, we have $\tau \cdot \nabla u^+(x^*,T) = 0$, where τ is any tangential direction at x^* (this applies when $n > 1$). Summing up, we get $\xi \cdot \nabla u^+(x^*,T) = 0$, where ξ is any direction. This yields (2.65).

Now we are ready to present the final part of the proof. By the definitions of z^+ and z^-, it holds either $z^+(x^*) = 0$ or $z^-(x^*) = 0$. In case $z^+(x^*) = 0$ we have $(z^+r)(x^*, T) = 0$ and from the equation (2.59) we obtain

$$\left[\beta(u_t^+ + k * u_t^+) - au^+\right](x^*, T) = A_0 u^+(x^*, T), \tag{2.66}$$

where $A_0 = A - a$. The left-hand side of (2.66) is strictly positive due to the inequalities (1.7), (2.60), (2.61) and the definition of θ. Indeed:

$$\begin{aligned}\left[\beta(u_t^+ + k * u_t^+) - au^+\right](x^*, T) &= \beta(x^*)\left[u_t^+ + k * u_t^+ - \frac{a}{\beta}u^+\right](x^*, T)\\ &\geq \beta_0\left[u_t^+ + k * u_t^+ - \theta u^+\right](x^*, T) > 0.\end{aligned}$$

Therefore, the right-hand side of (2.66) is also strictly positive, i.e.

$$A_0 u^+(x^*, T) > 0. \tag{2.67}$$

On the other hand, since $x = x^*$ is the stationary maximum point of $u^+(\cdot, T)$ and the principal part of A_0 is elliptic (see (1.6)), we obtain

$$A_0 u^+(x^*, T) = \sum_{i,j=1}^{n} a_{ij}(x^*) u^+_{x_i x_j}(x^*, T) + \sum_{j=1}^{n} a_j(x^*) u^+_{x_j}(x^*, T) \leq 0.$$

This contradicts (2.67). Analogously we come to a contradiction in case $z^-(x^*) = 0$. Hence, the assumption $z \neq 0$ was incorrect. We have $z = 0$. Finally, whereas $f_0, u_0, g = 0$ by assumption, problem (2.50) is homogeneous. Thus, by the uniqueness of the solution (see Theorem 2.1) it holds $u = 0$. Proof is complete. ■

2.3.2 Existence and stability

Let us impose the following additional assumptions on the function r:

$$\begin{aligned}&r \geq \delta \ \text{ in } \overline{\Omega} \times (T - \delta, T) \quad \text{with some } \delta \in (0, \tfrac{T}{2}) \text{ and}\\ &r = 0 \ \text{ in } \overline{\Omega} \times (0, \delta).\end{aligned} \tag{2.68}$$

Below we formulate and prove an existence and stability theorem for IP1. We note that a Fredholm-type result of this theorem (i.e. the assertion (i) of Theorem 3.8) was already obtained in [46], but under different assumptions. Namely, in [46] $\mu = 0$ was assumed and certain positivity conditions on the original kernel m were imposed. We do not need such assumptions in the assertion (i).

Theorem 2.4 *Let* (1.6), (1.7) *hold,* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$ *and* $a_t \in L^p(Q)$ *with some* $l \in (0,1), p \in (1,\infty)$. *Moreover, let* $a_t \geq 0$,

$r \in C^{l,\frac{l}{2}}(Q)$, $r_t \in L^p(Q)$ and (2.68) hold. In addition, let $f_0 \in C^{l,\frac{l}{2}}(Q)$, $u_0 \in C^{2+l}(\Omega)$, $g \in C^{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}}(S)$, $u_T \in C^{2+l}(\Omega)$ and the consistency conditions

$$\begin{aligned} &\text{(a)} \quad u_0 = g\,, \ \ \beta g_t = Au_0 + f_0 \quad \textit{in case I}, \\ &\qquad\ \ \omega \cdot \nabla u_0 = g \quad \textit{in case II} \quad \textit{in } \Gamma \times \{0\} \end{aligned} \tag{2.69}$$

$$\begin{aligned} &\text{(b)} \quad u_T = g \ \ \textit{in case I}, \\ &\qquad\ \ \omega \cdot \nabla u_T = g \quad \textit{in case II} \quad \textit{in } \Gamma \times \{T\} \end{aligned} \tag{2.70}$$

be satisfied. Then the following assertions are valid.

(i) (*Fredholm-type result*) *If* $k \in W^1_{\frac{2}{2-l}}(0,T)$,

$$\begin{aligned} &r \geq 0,\ r_t - \theta r \geq 0 \textit{ in } Q \textit{ with } \theta \textit{ given by } (2.54), \\ &\textit{for all } x \in \Omega \textit{ there exists an open subset } \tilde{Q}_x \textit{ of } Q \textit{ such that} \\ &\quad \exists \tilde{t}_x \in (0,T) : (x,\tilde{t}_x) \in \tilde{Q}_x \quad \textit{and} \quad r_t - \theta r > 0 \ \textit{ in } \tilde{Q}_x \\ &\textit{and the homogeneous inverse problem, i.e.} \end{aligned} \tag{2.71}$$

$$\begin{aligned} &\beta(v^0_t + k * v^0_t) = Av^0 + q^0 r \ \ \textit{in } Q, \\ &v^0 = 0 \ \ \textit{in } \Omega\times\{0\}, \ \ B_1 v^0 = 0 \ \ \textit{in } S\,, \ \ v^0 = 0 \ \ \textit{in } \Omega\times\{T\} \end{aligned} \tag{2.72}$$

has in $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ only the trivial solution $q^0 = 0$, $v^0 = 0$, then the inverse problem (2.50), (2.51) has a unique solution (z,u) in the space $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$. Moreover, the solution (z,u) satisfies the estimate

$$\begin{aligned} \|z\|_l + \|u\|_{2+l,1+\frac{l}{2}} \ &\leq \ \Lambda(\beta, a_{ij}, a_j, a, k, r) \\ &\times \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} + \|u_T\|_{2+l} \right\} \end{aligned} \tag{2.73}$$

with some constant Λ depending on the quantities shown in brackets.

(ii) (*Full existence, uniqueness and stability result*) *If*

$$k \in W^1_{\frac{2}{2-l}}(0,T)\,, \quad k \geq 0\,, \quad k' \leq 0 \tag{2.74}$$

and r satisfies (2.53), (2.55), (2.71) then the inverse problem (2.50), (2.51) has a unique solution (z,u) in the space $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$. The solution satisfies the estimate (2.73).

Proof. We note that the assertion (ii) follows from the assertion (i) and Theorem 2.3. Indeed, the assumptions of (ii) contain the assumptions of the uniqueness Theorem 2.3. This implies that the homogeneous problem (2.72) has in $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ only the trivial solution $q^0 = 0$, $v^0 = 0$.

As a result, all assumptions of (i) are satisfied, and the desired existence, uniqueness and stability result follows.

Thus, it remains to prove the assertion (i). We use a method that is based on a proved Fredholm-type result for the usual parabolic case (when $k = 0$). By means of this result we reduce the problem under consideration to an equation of the second kind, and apply the Fredholm's alternative.

So, let us start with the case $k = 0$. We mention that the general assumptions of the theorem preceding the statement (i) (except for the assumptions imposed on k) are sufficient for the following Fredholm-type result: the inverse problem in case $k = 0$, i.e. the problem

$$\begin{aligned} &\beta u^1_t = Au^1 + z^1 r + f_0 \text{ in } Q, \\ &u^1 = u_0 \text{ in } \Omega \times \{0\}, \;\; B_1 u^1 = g \text{ in } S, \;\; u^1 = u_T \text{ in } \Omega \times \{T\} \end{aligned} \tag{2.75}$$

has a solution (z^1, u^1) in the space $\mathcal{X}^l = C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and the estimate

$$\begin{aligned} \|z^1\|_l + \|u^1\|_{2+l,1+\frac{l}{2}} \le \; &\Lambda_1(\beta, a_{ij}, a_j, a, r) \\ &\times \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} + \|u_T\|_{2+l} \right\} \end{aligned} \tag{2.76}$$

holds with some constant Λ_1 depending on the quantities shown in brackets, provided the corresponding homogeneous problem (i.e. the problem with the data $f_0, u_0, g, u_T = 0$) has only the trivial solution $z^1 = 0, u^1 = 0$. This follows directly from Theorem 1.2 in [21]. But because of the additional assumption (2.71) of (i) Theorem 2.3 (in case $k = 0$) implies that such a homogeneous problem indeed has only the trivial solution. Therefore, we can conclude that the unique solution (z^1, u^1) of (2.75) exists in the space $\mathcal{X}^l$ and the estimate (2.76) is valid.

For further discussion we introduce an additional Banach space of pairs of functions, whose second components are zero at $t = 0$: $\mathcal{X}^l_0 = C^l(\Omega) \times C_0^{2+l,1+\frac{l}{2}}(Q)$, where

$$\begin{aligned} C_0^{2+l,1+\frac{l}{2}}(Q) = \{v \in C^{2+l,1+\frac{l}{2}}(Q) \,:\, v = 0 \text{ in } \Omega \times \{0\}\} \\ \text{with the norm} \|v\|_{C_0^{2+l,1+\frac{l}{2}}(Q)} = \|v\|_{2+l,1+\frac{l}{2}}, \end{aligned}$$

Let us denote $q = z - z^1$ and $v = u - u^1$. Then the inverse problem (2.50), (2.51) for $(z, u) \in \mathcal{X}^l$ is equivalent to the following inverse problem for the pair $X = (q, v) \in \mathcal{X}^l_0$:

$$\begin{aligned} &\beta v_t = Av + qr - \beta k * (u^1_t + v_t) \text{ in } Q, \\ &v = 0 \text{ in } \Omega \times \{0\}, \; B_1 v = 0 \text{ in } S, \; v = 0 \text{ in } \Omega \times \{T\}. \end{aligned} \tag{2.77}$$

Let $\mathcal{P}$ stand for the operator that assigns to a given right-hand side f_0 the solution of the problem (2.75) with $u_0, g, u_T = 0$. In view of the statements formulated before for (2.75), it holds $\mathcal{P} \in \mathcal{L}(C_o^{l,\frac{l}{2}}(Q), \mathcal{X}_0^l)$, where the space $C_o^{l,\frac{l}{2}}(Q)$ is defined in (2.24). (The domain of $\mathcal{P}$ is $C_o^{l,\frac{l}{2}}(Q)$, because the element $f_0 \in C^{l,\frac{l}{2}}(Q)$ must satisfy the consistency condition $f_0 = 0$ in $\Gamma \times \{0\}$ in case I (cf. (2.69) in case $u_0, g, u_T = 0$).) According to (2.76) $\|\mathcal{P}\| \le \Lambda_1(\beta, a_{ij}, a_j, a, r)$.

By virtue of Lemma 2.1 (ii) and $\beta \in C^l(\Omega)$ it holds $\beta k * w_t \in C^{l,\frac{l}{2}}(Q)$ for any $w \in C^{2+l,1+\frac{l}{2}}(Q)$ and $\beta k * w_t|_{t=0} = 0$. Thus, the operator $(q, w) \to \beta k * w_t$ is well-defined from $\mathcal{X}_0^l$ to $C_o^{l,\frac{l}{2}}(Q)$, and in turn the operator $\mathcal{T}$ defined by $\mathcal{T}(q, v) = \mathcal{P}(-\beta k * v_t)$ is well-defined from $\mathcal{X}_0^l$ to itself. Now we see that the problem (2.77) is in $\mathcal{X}_0^l$ equivalent to the operator equation

$$X \;=\; \mathcal{T}X + \Psi, \tag{2.78}$$

where $\Psi = \mathcal{P}(-\beta k * u_t^1)$.

But thanks to the assumption $k' \in L^{\frac{2}{2-l}}(0,T)$ we can even extend $\mathcal{T}$ to the space $\mathcal{L}(\mathcal{U}_0^{l,l'}, \mathcal{X}_0^l)$, where $\mathcal{U}_0^{l,l'} = C^{l'}(\Omega) \times C_0^{l,\frac{l}{2}}(Q)$,

$$C_0^{l,\frac{l}{2}}(Q) = \{v \in C^{l,\frac{l}{2}}(Q) \,:\, v = 0 \text{ in } \Omega \times \{0\}\}$$
$$\text{with the norm} \|v\|_{C_0^{l,\frac{l}{2}}(Q)} = \|v\|_{l,\frac{l}{2}},$$

and l' is an arbitrary number in the interval $(0, l)$. Indeed, taking into account the relation $\mathcal{T}(q, v) = \mathcal{P}(-\beta k * v_t) = \mathcal{P}(-\beta k' * v - \beta k(0)v)$, obtained by integration by parts, and using (2.17) we deduce

$$\begin{aligned}\|\mathcal{T}(q,v)\|_{\mathcal{X}_0^l} &\le \|\mathcal{P}\|\|\beta k' * v + \beta k(0)v\|_{C_o^{l,\frac{l}{2}}(Q)} = \|\mathcal{P}\|\|\beta k' * v + \beta k(0)v\|_{l,\frac{l}{2}} \\ &\le \Lambda_2\|v\|_{l,\frac{l}{2}} \le \Lambda_2(\|q\|_{C^{l'}(\Omega)} + \|v\|_{C_0^{l,\frac{l}{2}}(Q)}) = \Lambda_2\|(q,v)\|_{\mathcal{U}_0^{l,l'}}\end{aligned} \tag{2.79}$$

with $\Lambda_2 = \Lambda_1\|\beta\|_l\big\{C_0\|k'\|_{L^{\frac{2}{2-l}}(0,T)} + |k(0)|\big\}$. This proves $\mathcal{T} \in \mathcal{L}(\mathcal{U}_0^{l,l'}, \mathcal{X}_0^l)$.

Since $\mathcal{X}_0^l$ is compactly embedded in $\mathcal{U}_0^{l,l'}$, the operator $\mathcal{T}$ is compact in $\mathcal{U}_0^{l,l'}$. Moreover, 1 is not an eigenvalue of $\mathcal{T}$, because the equation $X^0 = \mathcal{T}X^0$ is in $\mathcal{X}_0^l$ equivalent to the problem (2.72), whose solution $X^0 = (q^0, v^0)$ is zero by the assumption. Consequently, by the Fredholm's alternative, the equation (2.78) has a unique solution in $\mathcal{X}_0^l$. This proves the existence assertion of (i).

It remains to prove (2.73). Since 1 belongs to the resolvent set of $\mathcal{T}$, it holds $(I - \mathcal{T})^{-1} \in \mathcal{L}(\mathcal{U}_0^{l,l'})$ and $\|(I - \mathcal{T})^{-1}\|_{\mathcal{L}(\mathcal{U}_0^{l,l'})} = \Lambda_3(\beta, a_{ij}, a_j, a, k, r)$

with some constant Λ_3 depending on the parameters shown in brackets. Thus, from (2.78) we immediately have

$$\|X\|_{\mathcal{U}_0^{l,l'}} = \|(I-\mathcal{T})^{-1}\Psi\|_{\mathcal{U}_0^{l,l'}} \le \Lambda_3\|\Psi\|_{\mathcal{U}_0^{l,l'}}. \tag{2.80}$$

Observing that $\|\mathcal{T}\|_{\mathcal{L}(\mathcal{U}_0^{l,l'},\mathcal{X}_0^l)} \le \Lambda_2$ (in view of (2.79)) and $\|\cdot\|_{\mathcal{U}_0^{l,l'}} \le C_7\|\cdot\|_{\mathcal{X}_0^l}$ with some constant C_7 (in view of the continuous embedding of $\mathcal{X}_0^l$ in $\mathcal{U}_0^{l,l'}$) and using (2.80) from (2.78) again we obtain

$$\begin{aligned}\|X\|_{\mathcal{X}_0^l} &\le \|\mathcal{T}\|_{\mathcal{L}(\mathcal{U}_0^{l,l'},\mathcal{X}_0^l)}\|X\|_{\mathcal{U}_0^{l,l'}} + \|\Psi\|_{\mathcal{X}_0^l} \le \Lambda_2\Lambda_3\|\Psi\|_{\mathcal{U}_0^{l,l'}} + \|\Psi\|_{\mathcal{X}_0^l}\\ &\le (C_7\Lambda_2\Lambda_3+1)\|\Psi\|_{\mathcal{X}_0^l}.\end{aligned} \tag{2.81}$$

Here $\|\Psi\|_{\mathcal{X}_0^l} = \|\mathcal{P}(-\beta k * u_t^1)\|_{\mathcal{X}_0^l} \le \|\mathcal{P}\|\|\beta k * u_t^1\|_{C_o^{l,\frac{l}{2}}(Q)} \le \Lambda_4\|u^1\|_{2+l,1+\frac{l}{2}}$ with $\Lambda_4 = \Lambda_1\|\beta\|_l C_0\|k\|_{L^{\frac{2}{2-l}}(0,T)}$ in view of (2.17). Consequently, from (2.81) we deduce

$$\|X\|_{\mathcal{X}_0^l} \le \Lambda_4(C_7\Lambda_2\Lambda_3+1)\|u^1\|_{2+l,1+\frac{l}{2}}. \tag{2.82}$$

Recall that $X=(q,v)$ with $q=z-z^1$ and $v=u-u^1$. Thus, by means of (2.82) we deduce

$$\begin{aligned}\|z\|_l + \|u\|_{2+l,1+\frac{l}{2}} &= \|(z,u)\|_{\mathcal{X}^l} \le \|(q,v)\|_{\mathcal{X}_0^l} + \|(z^1,u^1)\|_{\mathcal{X}^l}\\ &\le \big\{\Lambda_4(C_7\Lambda_2\Lambda_3+1)+1\big\}\|(z^1,u^1)\|_{\mathcal{X}^l}.\end{aligned}$$

Here the term $\|(z^1,u^1)\|_{\mathcal{X}^l}$ can be estimated by (2.76). We reach the desired estimate (2.73). ■

Remark 2.2 The relationship between the conditions (2.53)&(2.55) and (2.71) essentially depends on θ.

In case $\theta \ge 0$ provided $k \ge 0$, the conditions (2.71) imply (2.53)&(2.55). Indeed, then the relations $r \ge 0$ and $r_t - \theta r \ge 0$ in (2.71) immediately yield $r_t \ge 0$ and in turn $k * r_t \ge 0$. Thus, adding the nonnegative term $k * r_t$ to the left-hand side of the inequalities in (2.71) we immediately obtain (2.53)&(2.55). Consequently, the assumptions (2.53) and (2.55) in the formulation of the statement (ii) of Theorem 3.8 are redundant in case $\theta \ge 0$.

In case $\theta < 0$ the situation is more complicated. Then it is possible to find functions r that satisfy (2.71) but not (2.53)&(2.55). To construct such a counter-example, we can make use of following ideas. Firstly, we note that the kernel $k_\alpha(t) = \alpha e^{-\alpha t}$ with sufficiently large $\alpha > 0$ approaches

the Dirac delta-distribution. Indeed, let $t \in (0,T)$ and $w \in L^1(0,T)$ be a function satisfying $w' \in L^1(t-\varepsilon, t)$ with some $\varepsilon > 0$. Then

$$(k_\alpha * w)(t) - w(t) = \int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau + \int_{t-\varepsilon}^{t} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau - w(t)$$
$$= \int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau - \int_{t-\varepsilon}^{t} e^{-\alpha(t-\tau)} w'(\tau) d\tau - e^{-\alpha\varepsilon} w(t-\varepsilon).$$

Note that $\alpha e^{-\alpha(t-\tau)} w(\tau) \to 0$ as $\alpha \to \infty$ for a.e. $\tau \in (0, t-\varepsilon)$ and $|\alpha e^{-\alpha(t-\tau)} w(\tau)| \le \alpha e^{-\alpha\varepsilon} |w(\tau)| \le \frac{1}{e\varepsilon} |w(\tau)|$ for any $\alpha > 0$ and a.e. $\tau \in (0, t-\varepsilon)$, where $|w(\tau)|$ is integrable on $(0, t-\varepsilon)$. Due to the dominated convergence theorem we have $\int_0^{t-\varepsilon} \alpha e^{-\alpha(t-\tau)} w(\tau) d\tau \to 0$ as $\alpha \to \infty$. Similarly we get $\int_{t-\varepsilon}^{t} e^{-\alpha(t-\tau)} w'(\tau) d\tau \to 0$ as $\alpha \to \infty$. Moreover, $e^{-\alpha\varepsilon} w(t-\varepsilon) \to 0$ as $\alpha \to \infty$. Consequently, $|(k_\alpha * w)(t) - w(t)| \to 0$ as $\alpha \to \infty$. Now let us define the function $r = r(t)$ in the following manner:

$$r(t) = 0,\ t \in [0, \delta]\,, \quad r(t) = e^{-bt},\ t \in [T-\delta, T],$$
$$r(t) = \frac{e^{-b(T-\delta)}}{T-2\delta}(t-\delta),\ t \in (\delta, T-\delta),$$

where $b > 0$ is a constant. Then $r \ge 0$ in $(0,T)$ and $r_t \ge 0$ in $(0, T-\delta)$. This in view of $\theta < 0$ implies $r_t - \theta r \ge 0$ in $(0, T-\delta)$. Further, let us choose $b \in (-\frac{\theta}{2}, -\theta)$. Then $[r_t - \theta r](t) = (-b-\theta)e^{-bt} > 0$ for any $t \in (T-\delta, T)$. The deduced inequalities show that (2.71) is valid. On the other hand, with fixed $t \in (T-\delta, T)$ in view of the choice of b we have $[2r_t - \theta r](t) = (-2b-\theta)e^{-bt} < 0$. Therefore, in case of sufficiently large α it holds $[r_t + k_\alpha * r_t - \theta r](t) = [2r_t - \theta r](t) + (k_\alpha * r_t)(t) - r_t(t) < 0$. Consequently, (2.53) fails.

2.4 Results for inverse coefficient problems

In this section we deal with the nonlinear coefficient-type inverse problems IP2 and IP3 making use of previously proved results concerning the linear inverse free term problem (2.50), (2.51). For this purpose, we introduce the following notation.

Let $\hat{F}_{\beta,a,r}$ stand for the operator that assigns to the vector $d = (f_0, u_0, g, u_T)$ the solution of the inverse problem (2.50), (2.51). We have shown that in case β, a, r and also a_{ij}, a_j, k satisfy the assumptions of Theorem 3.8 (incl. the additional assumptions of the assertion (ii) of this theorem), the operator $\hat{F}_{\beta,a,r}$ is well-defined from the space

$$\mathscr{D} = \{d : d \in C^{l,\frac{l}{2}}(Q) \times C^{2+l}(\Omega) \times C^{2+l-\vartheta, 1+\frac{l}{2}-\frac{\vartheta}{2}}(S) \times C^{2+l}(\Omega)\,,$$
$$d \text{ satisfy the consistency conditions (2.69), (2.70)}\}$$

to the space $C^l_*(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and satisfies the estimate

$$\begin{aligned} \|\hat{F}_{\beta,a,r}(f_0,u_0,g,u_T)\|_{C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)} &\le \Lambda(\beta,a_{ij},a_j,a,k,r) \\ &\times \left\{\|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} + \|u_T\|_{2+l}\right\}. \end{aligned} \tag{2.83}$$

2.4.1 Results for IP2

Firstly, let us study IP2. Due to Lemma 2.1, IP2 is in the class of pairs (a,u) of functions, whose second component u together with its derivatives u_t, u_{x_i}, $u_{x_ix_j}$ belongs to $L^p(0,T)$, $p>1$, for any x, equivalent to the following inverse problem:

$$\begin{aligned} &\beta(u_t + k*u_t) = A_0u + au + f \ \text{ in } Q, \\ &u = u_0 \ \text{ in } \Omega\times\{0\},\ B_1u = g \ \text{ in } S, \end{aligned} \tag{2.84}$$

$$u = u_T \ \text{ in } \Omega\times\{T\}, \tag{2.85}$$

where f, B_1 and g are given by (2.10) - (2.12) and

$$A_0u = \sum_{i,j=1}^n a_{ij}u_{x_ix_j} + \sum_{j=1}^n a_ju_{x_j}.$$

We are going to prove existence, uniqueness and stability results for the inverse problem (2.84), (2.85) in spaces of pairs (a,u) whose first components a belong to the following sets that depend on l, β and a given number $\theta\in\mathbb{R}$:

$$\mathcal{A}^l_{\beta,\theta} = \{a\in C^l(\Omega) : \sup_{x\in\Omega}\tfrac{a(x)}{\beta(x)} \le \theta\}.$$

The next theorem comprises two results for (2.84), (2.85): (i) a global uniqueness; (ii) local conditional existence and stability. The meaning of the latter one is the following: assuming the existence of the solution of (2.84), (2.85) with certain data d, we prove the existence of solution to (2.84), (2.85) with data $\widetilde{d}$ that are sufficiently close to d and estimate the difference of these solutions in terms of $\widetilde{d}-d$.

Theorem 2.5 *Let* (1.6), (1.7) *hold,* $\beta, a_{ij}, a_j \in C^l(\Omega)$ *with some* $l\in(0,1)$ *and* $\theta\in\mathbb{R}$. *Then the following assertions are valid.*

(i) *If* k *satisfies* (2.47) *and the problem* (2.84), (2.85) *has the solutions* $(a_1,u_1)\in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)$, $(a_2,u_2)\in\mathcal{A}^l_{\beta,\theta}\times C^{2+l,1+\frac{l}{2}}(Q)$, *where* $u=u_1$ *satisfies the conditions*

$$\begin{aligned} &u\ge 0,\ u_t + k*u_t - \theta u \ge 0 \ \text{ in } Q, \\ &\text{for all } x\in\Omega \ \text{ there exists an open subset } Q_x \text{ of } Q \text{ such that} \\ &\exists t_x\in(0,T) : (x,t_x)\in Q_x \quad \text{and} \quad u_t + k*u_t - \theta u > 0 \ \text{ in } Q_x, \end{aligned} \tag{2.86}$$

then $a_1=a_2$ *and* $u_1=u_2$.

(ii) *If k satisfies* (2.74) *and* (2.84), (2.85) *has a solution* $(a,u)\in\mathcal{A}^l_{\beta,\theta}\times C^{2+l,1+\frac{l}{2}}(Q)$ *such that* u *fulfills the conditions* (2.86), *the conditions* (2.86) *with* k *replaced by* 0 *and the relations*

$$\begin{aligned} &u\geq\delta \ \text{ in } \overline{\Omega}\times(T-\delta,T),\\ &u=0 \ \text{ in } \overline{\Omega}\times(0,\delta) \text{ with some } \delta\in(0,\tfrac{T}{2}), \end{aligned} \tag{2.87}$$

then for any data vector $\widetilde{f},\widetilde{u}_0,\widetilde{g},\widetilde{u}_T$ *such that*

$$\begin{aligned} D:=\|\widetilde{f}-f\|_{l,\frac{l}{2}}+\|\widetilde{u}_0\|_{2+l}+\|\widetilde{g}-g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}}\\ +\|\widetilde{u}_T-u_T\|_{2+l}<\frac{1}{2\lambda^2} \quad \text{with} \quad \lambda=\Lambda(\beta,a_{ij},a_j,a,k,u), \end{aligned} \tag{2.88}$$

where Λ *is the coefficient of the estimate* (2.73) (*note that* $u_0=0$ *due to the second relation in* (2.88)),

$$\begin{aligned} &\text{(a)}\quad \widetilde{u}_0=\widetilde{g}\,,\ \ \beta\widetilde{g}_t=(A_0+a)\widetilde{u}_0+\widetilde{f} \ \text{ in case I},\\ &\qquad\ \omega\cdot\nabla\widetilde{u}_0=\widetilde{g} \ \text{ in case II} \ \text{ in } \Gamma\times\{0\} \end{aligned} \tag{2.89}$$

$$\begin{aligned} &\text{(b)}\quad \widetilde{u}_T=\widetilde{g} \ \text{ in case I},\\ &\qquad\ \omega\cdot\nabla\widetilde{u}_T=\widetilde{g} \ \text{ in case II} \ \text{ in } \Gamma\times\{T\} \end{aligned} \tag{2.90}$$

and $\widetilde{u}_0=0$ *in case* I *in* Γ, *the problem* (2.84), (2.85) *with* f_0,u_0,g,u_T *replaced by* $\widetilde{f}_0,\widetilde{u}_0,\widetilde{g},\widetilde{u}_T$ *has a unique solution* $(\widetilde{a},\widetilde{u})$ *in the ball*

$$\mathcal{U}=\Big\{(\widetilde{a},\widetilde{u}):\ \|\widetilde{a}-a\|_l+\|\widetilde{u}-u\|_{2+l,1+\frac{l}{2}}\leq\tfrac{1}{\lambda}\Big(1-\sqrt{1-2\lambda^2D}\Big)\Big\}. \tag{2.91}$$

Remark 2.3 Since $\frac{1}{\lambda}(1-\sqrt{1-2\lambda^2D})\sim\lambda D$ as $D\to 0^+$, the relation (2.91) implies that the solution operator of the problem (2.84), (2.85) is locally Lipschitz-continuous in the neighborhood of (a,u).

Proof of Theorem 2.5. Let us prove (i). Subtracting the problems (2.84), (2.85) for the pairs (a_1,u_1) and (a_2,u_2) we obtain the following problem for the pair of differences $z=a_1-a_2$, $u=u_1-u_2$:

$$\begin{aligned} &\beta(u_t+k*u_t)=(A_0+a_2)u+zu_1 \ \text{ in } Q,\\ &u=0 \ \text{ in } \Omega\times\{0\},\ \ B_1u=0 \ \text{ in } S,\\ &u=0 \ \text{ in } \Omega\times\{T\}. \end{aligned}$$

This problem satisfies the assumptions of Theorem 2.3. Indeed, we have $a_2\in C^l(\Omega)$ and $a_{2,t}=0$. Moreover, due to $a_2\in\mathcal{A}^l_{\beta,\theta}$ it holds $\theta_2:=\sup\limits_{x\in\Omega}\frac{a_2(x)}{\beta(x)}\leq\theta$. In view of this inequality, the assumption (2.86) for $u=$

u_1 holds with θ replaced by θ_2. This means that the assumptions (2.53) and (2.55) are satisfied for $r = u_1$. Applying Theorem 2.3 we obtain the assertion $z = a_1 - a_2 = 0$, $u = u_1 - u_2 = 0$.

Let us we prove (ii). The proof is based on Banach fixed-point principle.

Note that the problem for $(\widetilde{a}, \widetilde{u}) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the following problem for the differences $z = \widetilde{a} - a \in C^l(\Omega), w = \widetilde{u} - u \in C^{2+l,1+\frac{l}{2}}(Q)$:

$$\begin{aligned} &\beta(w_t + k * w_t) = (A_0 + a)w + zu + f_0[zw] \ \text{ in } Q\,, \\ &w = \widetilde{u}_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 w = \widetilde{g} - g \ \text{ in } S\,, \\ &w = \widetilde{u}_T - u_T \ \text{ in } \Omega \times \{T\} \end{aligned} \tag{2.92}$$

with $f_0[zw] = zw + \widetilde{f} - f$. Since $\widetilde{u}_0 = 0$ in case I in Γ, any solution $S = (z, w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ of (2.92) belongs to the following space:

$$\mathscr{S} = \{S = (z, w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q) \ : \ w = 0 \text{ in case I in } \Gamma \times \{0\}\}.$$

We will transform (2.92) to a fixed-point equation in $\mathscr{S}$.

Note that owing to the properties of β, a, u and also a_{ij}, a_j, k, the operator $\hat{F}_{\beta,a,u}$ is well-defined from $\mathscr{D}$ to $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$.

Let us show that the data vector of (2.92), i.e. $(f_0[zw], \widetilde{u}_0, \widetilde{g} - g, \widetilde{u}_T - u_T)$ belongs to $\mathscr{D}$ for any $S = (z, w) \in \mathscr{S}$. Assume $S \in \mathscr{S}$. From (2.88) and $z \in C^l(\Omega)$, $u \in C^{2+l,1+\frac{l}{2}}(Q)$ we immediately have $f_0[zw] \in C^{l,\frac{l}{2}}(Q)$, $\widetilde{u}_0, \widetilde{u}_T - u_T \in C^{2+l}(\Omega)$ and $\widetilde{g} - g \in C^{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}}(S)$. Moreover, from the smoothness of all terms of the equation (2.84) at the corners $\Gamma \times \{0\}$, $\Gamma \times \{T\}$ and the initial, boundary and final conditions in (2.84),(2.85) it follows that the consistency conditions (2.69), (2.70) (with $\beta g_t = f$ instead of $\beta g_t = Au_0 + f_0$ in case I in $\Gamma \times \{0\}$) are satisfied. Subtracting these conditions from (2.89), (2.90) and observing that $zw = 0$ in case I in $\Gamma \times \{0\}$ (see the definition of $\mathscr{S}$) we obtain the following consistency conditions:

$$\begin{aligned} &\widetilde{u}_0 = \widetilde{g} - g\,, \ \ \beta(\widetilde{g}_t - g_t) = (A_0 + a)\widetilde{u}_0 + f_0[zw] \ \text{ in case I,} \\ &\omega \cdot \nabla \widetilde{u}_0 = \widetilde{g} - g \ \text{ in case II } \ \text{ in } \Gamma \times \{0\}, \\ &\widetilde{u}_T - u_T = \widetilde{g} - g \ \text{ in case I,} \\ &\omega \cdot \nabla(\widetilde{u}_T - u_T) = \widetilde{g} - g \ \text{ in case II } \ \text{ in } \Gamma \times \{T\}. \end{aligned}$$

Consequently, the vector $(f_0[zw], \widetilde{u}_0, \widetilde{g} - g, \widetilde{u}_T - u_T)$ belongs to $\mathscr{D}$.

Now we see that the operator

$$\hat{\mathcal{F}}(S) = \hat{F}_{\beta,a,u}(zw + \widetilde{f} - f, \widetilde{u}_0, \widetilde{g} - g, \widetilde{u}_T - u_T) \tag{2.93}$$

is well-defined for any $S \in \mathscr{S}$. Moreover, it holds $\hat{\mathcal{F}}(\mathscr{S}) \subseteq \mathscr{S}$, because the second component of the element $\hat{\mathcal{F}}(S) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is zero at $\Gamma \times \{0\}$ in case I.

We can conclude that the problem (2.92) is in the space $\mathscr{S}$ equivalent to the fixed-point equation

$$S = \hat{\mathcal{F}}(S).$$

Define the norm $\|S\| = \|z\|_l + \|w\|_{2+l,1+\frac{l}{2}}$ in $\mathscr{S}$ and show that $\hat{\mathcal{F}}$ is a contraction in the ball

$$\mathcal{U}_0 = \Big\{ S \in \mathscr{S} \ : \ \|S\| \le \varrho := \tfrac{1}{\lambda}\left(1 - \sqrt{1 - 2\lambda^2 D}\right) \Big\}. \tag{2.94}$$

Using (2.93), (2.83) and the definitions of D and λ in (2.88) we deduce

$$\|\hat{\mathcal{F}}(S)\| \le \lambda \left\{ \|zw\|_{l,\frac{l}{2}} + D \right\} \le \lambda \left\{ \|z\|_l \|w\|_{2+l,1+\frac{l}{2}} + D \right\} \le \lambda \left\{ \tfrac{1}{2}\|S\|^2 + D \right\}.$$

In case $S \in \mathcal{U}_0$ it holds $\|\hat{\mathcal{F}}(S)\| \le \lambda \left\{ \frac{1}{2}\varrho^2 + D \right\}$. Note that ϱ defined in (2.94) solves the quadratic equation $\lambda \left\{ \frac{1}{2}\varrho^2 + D \right\} = \varrho$. Therefore, we have $\|\hat{\mathcal{F}}(S)\| \le \varrho$. Hence, $\hat{\mathcal{F}}(\mathcal{U}_0) \subseteq \mathcal{U}_0$.

Similarly, for $S^j = (z^j, w^j)$, $j = 1, 2$, in view of the relation

$$z^1 w^1 - z^2 w^2 = \tfrac{z^1 + z^2}{2}(w^1 - w^2) + (z^1 - z^2)\tfrac{w^1 + w^2}{2} \tag{2.95}$$

we obtain

$$\begin{aligned} \|\hat{\mathcal{F}}(S^1) - \hat{\mathcal{F}}(S^2)\| &= \|\hat{F}_{\beta,a,u}(z^1 w^1 - z^2 w^2, 0, 0, 0)\| \\ &\le \lambda \left\| \tfrac{S^1 + S^2}{2} \right\| \|S^1 - S^2\| \le \tfrac{\lambda}{2}(\|S^1\| + \|S^2\|)\|S^1 - S^2\|. \end{aligned}$$

In case $S^1, S^2 \in \mathcal{U}_0$ we have $\|\hat{\mathcal{F}}(S^1) - \hat{\mathcal{F}}(S^2)\| \le q\|S^1 - S^2\|$ with $q = \lambda\varrho = 1 - \sqrt{1 - 2\lambda^2 D} < 1$. Therefore, by the contraction principle, the equation $S = \hat{\mathcal{F}}(S)$ has a unique solution in the ball $\mathcal{U}_0$. This yields (ii). ■

Remark 2.4 It is possible to deduce sufficient conditions for the data of the direct problem (2.84) that imply the the conditions (2.86) and (2.87) for solution u. The second relation in (2.87) simply follows from uniqueness of the solution of the direct problem (2.84) restricted to $\Omega \times (0, \delta)$ if we assume $u_0 = 0$, $g = 0$ in $S \times (0, \delta)$, $f = 0$ in $\Omega \times (0, \delta)$. The first relation in (2.87) follows from Theorem 2.2 under the assumption that $u_0, g, f \ge 0$, there exists an open subset Q_f of $\Omega \times (0, T - \delta)$ such that $f > 0$ in Q_f and $g \ge \delta$ in $S \times (T - \delta, T)$ in case I. This theorem has to be applied for problems restricted to the domains $\Omega \times (0, T_1)$, where $T_1 \in [T - \delta, T]$ to get the desired result. Finally, the inequalities for $u_t + k * u_t - \theta u$ in (2.84) can be shown under certain assumptions on the data constructing a direct problem for $v = u_t + k * u_t - \theta u$ and applying Theorem 2.2 to this problem. Such a problem can be constructed similarly to the construction of problems (2.62) in the proof of Theorem 2.3.

2.4.2 Results for IP3

Finally, we study IP3. In view of 2.1, IP3 is in the class of pairs (β, u) of functions, whose second component u together with its derivatives u_t, u_{x_i}, $u_{x_i x_j}$ belongs to $L^p(0,T)$, $p>1$, for any x, equivalent to the problem

$$\begin{aligned} &\beta(u_t + k * u_t) = Au + f \ \text{ in } Q\,, \\ &u = u_0 \ \text{ in } \Omega\times\{0\}\,, \ \ B_1 u = g \ \text{ in } S\,, \end{aligned} \tag{2.96}$$

$$u = u_T \ \text{ in } \Omega\times\{T\}\,, \tag{2.97}$$

where f, B_1 and g are given by (2.10) - (2.12). We assume here $\mu = 0$.

Let us introduce the following set for the coefficients β that depends on $\beta_0 > 0$:

$$\mathcal{B}^l_{\beta_0} \,=\, \{\beta \in C^l(\Omega) \,:\, \inf_{x\in\Omega} \beta(x) \ge \beta_0\}$$

and define $\theta_{\beta_0} = \max\{0; \frac{1}{\beta_0}\sup_{x\in\Omega} a(x,T)\}$. Then we have $\sup_{x\in\Omega}\frac{a(x,T)}{\beta(x)} \le \theta_{\beta_0}$ for any $\beta \in \mathcal{B}^l_{\beta_0}$.

Theorem 2.6 *Let* (1.6) *hold,* $a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ *with some* $l \in (0,1)$, $p \in (1,\infty)$, $a_t \ge 0$ *and* $\beta_0 > 0$. *Then the following assertions are valid.*

(i) *If* k *satisfies* (2.47), *the problem* (2.96), (2.97) *has the solutions* $(\beta_1, u_1) \in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)$ *and* $(\beta_2, u_2) \in \mathcal{B}^l_{\beta_0}\times C^{2+l,1+\frac{l}{2}}(Q)$ *such that* $u = u_1$ *and* $\hat u := u_t + k * u_t$ *satisfy the conditions*

$$\begin{aligned} &u_{tt} \in L^p(\Omega)\,, \quad \hat u \ge 0\,, \ \ \hat u_t + k * \hat u_t - \theta_{\beta_0}\hat u \ge 0\,, \\ &\text{for all } x\in\Omega \text{ there exists an open subset } Q_x \text{ of } Q \text{ such that} \\ &\exists t_x \in (0,T) \,:\, (x,t_x) \in Q_x \ \text{ and } \ \hat u_t + k * \hat u_t - \theta_{\beta_0}\hat u > 0 \ \text{ in } Q_x, \end{aligned} \tag{2.98}$$

then $\beta_1 = \beta_2$ *and* $u_1 = u_2$.

(ii) *If* k *satisfies* (2.74), $Au_0 + f = 0$ *in case* I *in* $\Gamma\times\{0\}$ *and the problem* (2.96), (2.97) *has a solution* $(\beta, u) \in \mathcal{B}^l_{\beta_0}\times C^{2+l,1+\frac{l}{2}}(Q)$ *such that* u *and* $\hat u = u_t + k * u_t$ *fulfill the conditions* (2.98) *and the conditions* (2.98) *with* k *replaced by* 0,

$$\begin{aligned} &\hat u \ge \delta \ \text{ in } \overline{\Omega}\times(T-\delta, T) \ \text{ and} \\ &u_t = 0 \ \text{ in } \overline{\Omega}\times(0,\delta) \ \text{ with some } \delta \in (0, \tfrac{T}{2}), \end{aligned} \tag{2.99}$$

then for any data vector $\widetilde f, \widetilde u_0, \widetilde g, \widetilde u_T$ *such that*

$$\begin{aligned} D :=\ &\|\widetilde f - f\|_{l,\frac{l}{2}} + \|\widetilde u_0 - u_0\|_{2+l} + \|\widetilde g - g\|_{2+l-\vartheta,1+\frac{l}{2}-\frac{\vartheta}{2}} \\ &+ \|\widetilde u_T - u_T\|_{2+l} < \frac{1}{2\bar\lambda^2(1+\|k\|)} \ \text{ with } \ \bar\lambda = \Lambda(\beta, a_{ij}, a_j, a, k, \hat u), \end{aligned} \tag{2.100}$$

where Λ *is the coefficient of the estimate* (2.73), $\|k\| = \|k\|_{C[0,T]}$ *and*

$$\begin{aligned} &\text{(a)} \quad \widetilde{u}_0 = \widetilde{g}\,, \ \ \beta\widetilde{g}_t = A\widetilde{u}_0 + \widetilde{f} = 0 \quad \textit{in case} \text{ I}, \\ &\qquad \omega\cdot\nabla\widetilde{u}_0 = \widetilde{g} \quad \textit{in case} \text{ II} \quad \textit{in } \Gamma\times\{0\} \end{aligned} \tag{2.101}$$

$$\begin{aligned} &\text{(b)} \quad \widetilde{u}_T = \widetilde{g} \ \ \textit{in case} \text{ I}, \\ &\qquad \omega\cdot\nabla\widetilde{u}_T = \widetilde{g} \quad \textit{in case} \text{ II} \quad \textit{in } \Gamma\times\{T\}, \end{aligned} \tag{2.102}$$

then the problem (2.96), (2.97) *with* f_0, u_0, g, u_T *replaced by* $\widetilde{f}_0, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ *has a unique solution* $(\widetilde{\beta}, \widetilde{u})$ *in the ball*

$$\begin{aligned} \bar{\mathcal{U}} = \Big\{(\widetilde{\beta},\widetilde{u}) \,:\, &\|\widetilde{\beta}-\beta\|_l + \|\widetilde{u}-u\|_{2+l,1+\frac{l}{2}} \\ &\le \frac{1}{\bar{\lambda}(1+\|k\|)}\left(1-\sqrt{1-2\bar{\lambda}^2(1+\|k\|)D}\right)\Big\}. \end{aligned} \tag{2.103}$$

Remark 2.5 The relation (2.103) implies that the solution operator of the problem (2.96), (2.97) is locally Lipschitz-continuous in the neighborhood of (β, u).

Proof of Theorem 2.6. The proof is similar to the proof of Theorem 2.5. To prove (i), we subtract the problems for with the pairs (β_1, u_1) and (β_2, u_2). Then we obtain the problem (2.50), (2.51) for the difference $z = \beta_1 - \beta_2$, $u = u_1 - u_2$ that has the zero free term f_0, zero initial, boundary and final conditions and contains β_2 and $\hat{u} = u_{1,t} + k * u_{1,t}$ instead of β and r, respectively. Applying Theorem 2.3 to this problem, we immediately obtain $z = 0$, $u = 0$.

Let us prove (ii). Note that $(\beta, u) \in \bar{\mathscr{S}}$, where

$$\bar{\mathscr{S}} = \{S = (z,w) \in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q) \,:\, w_t = 0 \text{ in case I in } \Gamma\times\{0\}\}.$$

Indeed, $(\beta, u) \in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)$, by assumption. Setting $t = 0$ in the equation (2.96) we get $\beta u_t(\cdot,0) = A(0)u_0 + f(\cdot,0)$. This due to the assumption $Au_0 + f = 0$ in case I in $\Gamma\times\{0\}$ and $\beta > 0$ implies $u_t = 0$ in case I in $\Gamma\times\{0\}$. Furthermore, any solution $(\widetilde{\beta}, \widetilde{u}) \in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)$ also belongs to $\bar{\mathscr{S}}$. This can be shown analogously, making us of the assumption $A\widetilde{u}_0 + \widetilde{f} = 0$ in case I in $\Gamma\times\{0\}$ (see (2.101)).

Subtracting the problems for $(\widetilde{\beta}, \widetilde{u})$ and (β, u) we see that the problem for $(\widetilde{\beta}, \widetilde{u}) \in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)$ is equivalent to the following problem for the pair of differences $(z, w) = (\beta - \widetilde{\beta}, \widetilde{u} - u)$ in the space $\bar{\mathscr{S}}$:

$$\begin{aligned} &\beta(w_t + k * w_t) = Aw + z\hat{u} + \bar{f}_0[z,w] \ \text{ in } Q\,, \\ &w = \widetilde{u}_0 - u_0 \ \text{ in } \Omega\times\{0\}\,, \ \ B_1 w = \widetilde{g} - g \ \text{ in } S\,, \\ &w = \widetilde{u}_T - u_T \ \text{ in } \Omega\times\{T\}\,, \end{aligned} \tag{2.104}$$

where $\bar{f}_0[z,w] = z(w_t + k * w_t) + \widetilde{f} - f$ and $\hat{u} = u_t + k * u_t$, as defined above. Let us transform (2.104) to a fixed-point equation.

Note that the assumptions $r \in C^{l,\frac{l}{2}}(Q)$, $r_t \in L^p(\Omega)$, (2.53), (2.55), (2.68) and (2.71) are satisfied for the function $r = \hat{u}$. Indeed, the relation $u \in C^{2+l,1+\frac{l}{2}}(Q)$ and the assumption $u_{tt} \in L^p(Q)$ with Lemma 2.1 imply $\hat{u} \in C^{l,\frac{l}{2}}(Q)$, $\hat{u}_t \in L^p(\Omega)$. Further, (2.53), (2.55), (2.71) and the first relation in (2.68) for $r = \hat{u}$ automatically follow from the assumptions of the assertion (ii). The second relation in (2.68) follows from the second relation in (2.99). Observing also other assumptions of the present theorem, we see that the assumptions of Theorem 3.8 are satisfied for the set of parameters a_{ij}, a_j, k, β, a and $r = \hat{u}$. This means that the operator $\hat{F}_{\beta,a,\hat{u}}$ is well-defined from $\mathscr{D}$ to $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$.

Further, for any $S = (z,w) \in \bar{\mathscr{S}}$ the data vector $(f_0[z,w], \widetilde{u}_0 - u_0, \widetilde{g} - g, \widetilde{u}_T - u_T)$ of the problem (2.104) belongs to $\mathscr{D}$. This can be shown by means of arguments that are similar to arguments that we used in the proof of Theorem 2.5. In particular, the consistency condition $\beta(\widetilde{g} - g)_t = A(\widetilde{u}_0 - u_0) + \bar{f}_0[z,w]$ in case I in $\Gamma \times \{0\}$ follows from the relations $\beta g_t = Au_0 + f$, $\beta\widetilde{g}_t = A\widetilde{u}_0 + f_0$ and $w_t = 0$ in case I in $\Gamma \times \{0\}$.

Now we see that the operator

$$\bar{\mathcal{F}}(S) = \hat{F}_{\beta,a,\hat{u}}(\bar{f}_0[z,w], \widetilde{u}_0 - u_0, \widetilde{g} - g, \widetilde{u}_T - u_T) \tag{2.105}$$

is well-defined for any $S \in \bar{\mathscr{S}}$. Moreover, it holds $\bar{\mathcal{F}}(\bar{\mathscr{S}}) \subseteq \bar{\mathscr{S}}$, because time derivative of the second component w^1 of $(z^1, w^1) = \bar{\mathcal{F}}(z,w) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ is zero at $\Gamma \times \{0\}$ in case I for $(z,w) \in \bar{\mathscr{S}}$. The latter statement follows from the equality $\beta w^1_t = A(\widetilde{u}_0 - u_0) + z^1\hat{u} + \bar{f}_0[z,w]$ that is derived from the equation for w^1 at $\Gamma \times \{0\}$ and the relations $Au_0 + f = 0$, $A\widetilde{u}_0 + \widetilde{f} = 0$, $\hat{u} = 0$, $w_t = 0$ in case I in $\Gamma \times \{0\}$ and $\beta > 0$.

Summing up, the problem (2.104) is in $\bar{\mathscr{S}}$ equivalent to the following operator equation:

$$S = \bar{\mathcal{F}}(S).$$

We use the norm $\|S\| = \|(z,w)\| = \|z\|_l + \|w\|_{2+l,1+\frac{l}{2}}$ in $\bar{\mathscr{S}}$ and define the following ball in $\bar{\mathscr{S}}$:

$$\bar{\mathcal{U}}_0 = \left\{ S \in \bar{\mathscr{S}} \,:\, \|S\| \le \bar{\varrho} := \tfrac{1}{\bar{\lambda}(1+\|k\|)} \left(1 - \sqrt{1 - 2\bar{\lambda}^2(1+\|k\|)D}\right) \right\}.$$

By means of (2.105), (2.83) and (2.100) we obtain

$$\begin{aligned} \|\bar{\mathcal{F}}(S)\| &\le \bar{\lambda}\left\{ \|z(w_t + k * w_t)\|_{l,\frac{l}{2}} + D \right\} \le \bar{\lambda}\left\{ (1+\|k\|)\|z\|_l \|w_t\|_{l,\frac{l}{2}} + D \right\} \\ &\le \bar{\lambda}\left\{ \tfrac{1+\|k\|}{2}\|S\|^2 + D \right\}. \end{aligned}$$

In case $S \in \bar{\mathcal{U}}_0$ it holds $\|\bar{\mathcal{F}}(S)\| \le \bar{\lambda}\left\{\frac{1+\|k\|}{2}\bar{\varrho}^2 + D\right\}$. Since $\bar{\varrho}$ solves the quadratic equation $\bar{\lambda}\left\{\frac{1+\|k\|}{2}\bar{\varrho}^2 + D\right\} = \bar{\varrho}$, we have $\|\bar{\mathcal{F}}(S)\| \le \bar{\varrho}$. Consequently, $\bar{\mathcal{F}}(\bar{\mathcal{U}}_0) \subseteq \bar{\mathcal{U}}_0$.

Moreover, for $S^j = (z^j, w^j)$, $j = 1, 2$, in view of the relation (2.95) we deduce

$$\begin{aligned}
&\|\bar{\mathcal{F}}(S^1) - \bar{\mathcal{F}}(S^2)\| = \|\hat{F}_{\beta,a,\hat{u}}(z^1(w_t^1 + k * w_t^1) - z^2(w_t^2 + k * w_t^2), 0, 0, 0)\| \\
&\quad \le \bar{\lambda}\|z^1(w_t^1 + k * w_t^1) - z^2(w_t^2 + k * w_t^2\|_{l,\frac{l}{2}} \\
&\quad = \bar{\lambda}\Big\|\tfrac{z^1+z^2}{2}(w_t^1 - w_t^2 + k * (w_t^1 - w_t^2) + (z^1 - z^2)\big(\tfrac{w_t^1+w_t^2}{2} + k * \tfrac{w_t^1+w_t^2}{2}\big)\Big\|_{l,\frac{l}{2}} \\
&\quad \le \bar{\lambda}(1 + \|k\|)\Big\|\tfrac{S^1+S^2}{2}\Big\| \|S^1 - S^2\| \le \tfrac{\bar{\lambda}(1+\|k\|)}{2}(\|S^1\| + \|S^2\|)\|S^1 - S^2\|.
\end{aligned}$$

In case $S^j \in \bar{\mathcal{U}}_0$, $j = 1, 2$, it holds $\|\bar{\mathcal{F}}(S^1) - \bar{\mathcal{F}}(S^2)\| \le \bar{q}\|S^1 - S^2\|$ with $\bar{q} = \bar{\lambda}(1 + \|k\|)\bar{\varrho} = 1 - \sqrt{1 - 2\bar{\lambda}^2(1 + \|k\|)D} < 1$. By the contraction principle, the equation $S = \bar{\mathcal{F}}(S)$ has a unique solution in the ball $\bar{\mathcal{U}}_0$. This proves (ii). ■

Remark 2.6 It is possible to establish conditions that guarantee (2.86) and (2.87) for the solution u of the direct problem (2.96). To this end, initial-boundary value problems for involved functions u_t, $\hat{u}$ and $\hat{v} = \hat{u}_t + k * \hat{u}_t - \theta_{\beta_0}\hat{u}$ have to be constructed. The second condition in (2.99) follows if we assume the initial data and boundary data of the problem for u_t to be zero for $t \in (0, \delta)$ and other conditions in (2.86) and (2.87) follow from Theorem 2.2 under proper assumptions of of the data of the problems for $\hat{u}$ and $\hat{v}$.

3 NON-SMOOTH PROBLEMS

In this chapter we deal with inverse problems for Eq. (1.4) in case this equation holds in a weak sense and analyze quasi-solutions of these problems. Results are taken from Publications II and III.

Let (1.4) have the following form:

$$u_t + (\mu * u)_t = Au - m * Au + f + \nabla \cdot \phi + \varphi_t \quad \text{in } Q, \tag{3.1}$$

where f, φ are regular scalar functions and ϕ is a regular vector function. The functions ϕ and φ may not have classical derivatives with respect to the space variables and the time, respectively. This means that the free term $\chi = f + \nabla \cdot \phi + \varphi_t$ is generally a singular distribution. Since we are not planning to consider inverse problems to determine β, we assume $\beta = 1$, for the sake of simplicity. Moreover, we assume that A is of the divergence type and has symmetric principal part, i.e.

$$(Av)(x) = \sum_{i,j=1}^{n} \left(a_{ij}(x) v_{x_j}\right)_{x_i} + a(x)v(x), \quad a_{ij} = a_{ji}.$$

We are going to study problems with generally mixed boundary conditions. To this end, we split the boundary of Ω into two parts. Namely, let $\Gamma = \Gamma_1 \cup \Gamma_2$ and we assume that $\operatorname{meas} \Gamma_1 \cap \Gamma_2 = 0$ and for any $j \in \{1; 2\}$ either $\Gamma_j = \emptyset$ or $\operatorname{meas} \Gamma_j > 0$.

Let us return to Eq. (3.1) and complement it with the initial condition

$$u = u_0 \quad \text{in } \Omega \times \{0\} \tag{3.2}$$

and the boundary conditions

$$u = g \quad \text{in } \Gamma_1 \times (0, T), \tag{3.3}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = h + \nu \cdot \phi \quad \text{in } \Gamma_2 \times (0, T), \tag{3.4}$$

where the functions u_0, g, h are given and

$$\nu_A = \Big(\sum_{j=1}^{n} a_{ij} \nu_j \Big|_{i=1,\dots,n} \Big)$$

is the co-normal vector to Γ. In case $\Gamma_1 = \emptyset$ (or $\Gamma_2 = \emptyset$) the boundary condition (3.3) (or (3.4)) is omitted.

Summing up, (3.1) - (3.4) constitute a formal direct problem for the function u.

Let us pose formal inverse problems. They use instant and integral data of the solution of (3.1) - (3.4).

IP4: Let the component f of the free term be of the form

$$f(x,t) = f_0(x,t) + \sum_{j=1}^{N} \gamma_j(t)\omega_j(x) \tag{3.5}$$

and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, u_0, f_0, \phi, g, h, \gamma_j$, $j = 1, \ldots, N$, and functions $u_{T_i}(x)$, $x \in \Omega$, $i = 1, \ldots, N$ with $0 < T_1 < T_2 < \ldots < T_N \leq T$, find ω_j, $j = 1, \ldots, N$, such that the solution u of (3.1) - (3.4) satisfies the following instant additional conditions:

$$u = u_{T_i} \quad \text{in } \Omega \times \{T_i\}, \quad i = 1, 2, \ldots, N.$$

IP5: Let the component f of the free term be of the form (3.5) and $\mu = 0$, $\varphi = 0$. Given $m, a_{ij}, a, f_0, \phi, g, h, \gamma_j$, $j = 1, \ldots, N$, and functions $v_i(x)$, $x \in \Omega$, $i = 1, \ldots, N+1$, find ω_j, $j = 1, \ldots, N$, and u_0 such that the solution u of (3.1) - (3.4) satisfies the following integral additional conditions:

$$\int_0^T \kappa_i(x,t)u(x,t)dt = v_i(x)\,,\ x \in \Omega\,, \quad i = 1, 2, \ldots, N+1, \tag{3.6}$$

where κ_i, $i = 1, \ldots, N+1$ are given weights.

IP6: Let $\operatorname{meas}\Gamma_2 > 0$. Given $a_{ij}, u_0, f, \phi, \varphi, g, h$ and functions $u_T(x)$, $x \in \Omega$, $v_i(t)$, $t \in (0,T)$, $i = 1, 2$, find a, m and μ such that the solution of (3.1) - (3.4) satisfies the following final and integral additional conditions:

$$u = u_T \quad \text{in } \Omega \times \{T\}, \tag{3.7}$$

$$\int_{\Gamma_2} \kappa_i(x,t)u(x,t)d\Gamma \ = \ v_i(t)\,,\ t \in (0,T), \ \ i = 1, 2, \tag{3.8}$$

where κ_i, $i = 1, 2$, are given weights and $d\Gamma$ is the surface measure on Γ.

Remark 3.1 In case $n = 1$ and $\Omega = (c, d)$ the integral $\int_{\Gamma_2} z(x)d\Gamma$ is merely the sum $\sum_{l=1}^{L} z(x_l)$, where $x_l \in \Gamma_2 \subseteq \{c; d\}$ and L is the number of points in Γ_2 (i.e $L \in \{1; 2\}$). Then the conditions (3.8) read

$$\sum_{l=1}^{L} \kappa_i(x_l,t)u(x_l,t) \ = \ v_i(t)\,,\ t \in (0,T), \ \ i = 1, 2. \tag{3.9}$$

Remark 3.2 The conditions $\varphi = 0$ and $\mu = 0$ in IP4 and IP5 is assumed for the sake of simplicity. The inclusion of generally non-vanishing φ in IP6 is natural due to the method we will use. Namely, an adjoint problem corresponding to IP6 contains a singular time-derivative in a free term. We have to prove well-posedness results both for (3.1) - (3.4) and the adjoint problem. Therefore, it is natural to incorporate such a singular term in (3.1) already from the beginning.

3.1 Results concerning direct problem

3.1.1 Additional notation. Well-posedness of weak direct problem

In addition to cylinders Q, S and $Q_t = \Omega \times (0,t)$, $t > 0$, defined in §1.1 and Lemma 2.1, we introduce t-dependent cylinders

$$S_t = \Gamma \times (0,t), \quad \Gamma_{1,t} = \Gamma_1 \times (0,t),\ \Gamma_{2,t} = \Gamma_2 \times (0,t)$$

for $t > 0$.

In the treatment of the weak direct problem we make use of the following functional spaces:

$$\begin{aligned}
&\mathcal{U}(Q_t) \ = \ C([0,t]; L^2(\Omega)) \cap L^2(0,t; W_2^1(\Omega)),\\
&\mathcal{U}_0(Q_t) = \Big\{\eta \in \mathcal{U}(Q_t) \ : \ \eta|_{\Gamma_{1,t}} = 0 \ \text{ in case } \Gamma_1 \neq \emptyset\Big\},\\
&\mathcal{T}(Q_t) = \Big\{\eta \in L^2(0,t; W_2^1(\Omega)) \ : \ \eta_t \in L^2(0,t; L^2(\Omega))\Big\},\\
&\mathcal{T}_0(Q_t) = \Big\{\eta \in \mathcal{T}(Q_t) \ : \ \eta|_{\Gamma_{1,t}} = 0 \ \text{ in case } \Gamma_1 \neq \emptyset\Big\},
\end{aligned}$$

where $t \in (0,T]$. In case $t = T$ we write merely $\mathcal{U}(Q)$, $\mathcal{U}_0(Q)$, $\mathcal{T}(Q)$ and $\mathcal{T}_0(Q)$, because $Q_T = Q$.

We recall that the ellipticity condition (1.6) is assumed by default in this thesis. Let us collect other regularity assumptions on the data of the direct problem (3.1) - (3.4). They are

$$a_{ij} \in L^\infty(\Omega), \tag{3.10}$$

$$a \in L^{q_1}(\Omega), \ \text{ where } \ q_1 = 1 \ \text{ if } \ n = 1, \ \ q_1 > \frac{n}{2} \ \text{ if } \ n \geq 2, \tag{3.11}$$

$$\mu \in L^2(0,T), \tag{3.12}$$

$$m \in L^1(0,T)\,, \tag{3.13}$$

$$u_0 \in L^2(\Omega), \tag{3.14}$$

$$g \in \mathcal{T}(Q), \ \ h \in L^2(\Gamma_{2,T}), \tag{3.15}$$

$$\begin{aligned}
&f \in L^2(0,T; L^{q_2}(\Omega)), \ \text{ where}\\
&q_2 = 1 \ \text{ if } \ n = 1, \ \ q_2 \in (1,q_1) \ \text{ if } \ n = 2, \ \ q_2 = \tfrac{2n}{n+2} \ \text{ if } \ n \geq 3,
\end{aligned} \tag{3.16}$$

$$\phi = (\phi_1, \ldots, \phi_n) \in (L^2(Q))^n\,, \tag{3.17}$$

$$\varphi \in \mathcal{U}(Q) \ \text{ and in case } \ \Gamma_1 \neq \emptyset \ \ \exists g_\varphi \in \mathcal{T}(Q) \ : \ \ \varphi = g_\varphi \ \text{ in } \Gamma_{1,T}. \tag{3.18}$$

In case the additional conditions $a_{ij} \in W_2^1(\Omega)$, $\frac{\partial}{\partial x_i}\phi_i \in L^2(Q)$, $i = 1, \ldots, n$, $\varphi_t \in L^2(Q)$ hold and (3.1) - (3.4) has a classical solution $u \in L^2(Q)$ such that $u_t, u_{x_i}, u_{x_i x_j} \in L^2(Q)$, $i,j = 1, \ldots, n$, then multiplying (3.1) with

a test function $\eta \in \mathcal{T}_0(Q)$ and integrating by parts we come to the relation

$$\begin{aligned}
&\int_\Omega \Big[(u+\mu*u-\varphi)(x,T)\eta(x,T)-(u_0(x)-\varphi(x,0))\eta(x,0)\Big]dx \\
&\quad -\iint\limits_Q (u+\mu*u-\varphi)\eta_t\,dxdt \\
&\quad +\iint\limits_Q \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j}-m*u_{x_j})\eta_{x_i}-a(u-m*u)\eta\Big]dxdt \\
&\quad +\iint\limits_{\Gamma_{2,T}} h\eta\,d\Gamma dt-\iint\limits_Q (f\eta-\phi\cdot\nabla\eta)\,dxdt=0.
\end{aligned} \tag{3.19}$$

Note that this relation makes sense also in a more general case when a_{ij}, ϕ, φ satisfy (3.10), (3.17), (3.18) and $u \in \mathcal{U}(Q)$.

We call a *weak solution* of the problem (3.1) - (3.4) a function that belongs to $\mathcal{U}(Q)$, satisfies the relation (3.19) for any $\eta \in \mathcal{T}_0(Q)$ and, in case $\Gamma_1 \neq \emptyset$, fulfills the boundary condition (3.3).

The next aim is to show the existence, uniqueness and stability of the weak solution. But before we prove a technical lemma.

Lemma 3.1 *The following assertions are valid:*

(i) $\mathcal{U}(Q) \hookrightarrow L^2(0,T;L^{q_3}(\Omega))$ *where* $q_3=\infty$ *if* $n=1$, $q_3<\infty$ *if* $n=2$ *and* $q_3=\frac{2n}{n-2}$ *if* $n>2$; *in the sequel we assume* $q_3 \in \big(\frac{q_1q_2}{q_1-q_2},\infty\big)$ *in case* $n=2$, *where* q_1 *and* q_2 *are given in* (3.11) *and* (3.16), *respectively;*

(ii) *if* a *satisfies* (3.11) *then for any* $u \in L^2(0,T;L^{q_3}(\Omega))$ *it holds* $au \in L^2(0,T;L^{q_2}(\Omega))$ *and* $\|au\|_{L^2(0,T;L^{q_2}(\Omega))} \le \hat{C}_0\|a\|_{L^{q_1}(\Omega)}\|u\|_{L^2(0,T;L^{q_3}(\Omega))}$, *where* $\hat{C}_0$ *is a constant.*

Proof. Since $\mathcal{U}(Q) \hookrightarrow L^2(0,T;W_2^1(\Omega))$, the assertion (i) follows from the continuous embedding of $W_2^1(\Omega)$ in $L^{q_3}(\Omega)$. The assertion (ii) can be directly proved by means of the Hölder's inequality. ■

Now we formulate and prove the main theorem of the subsection.

Theorem 3.1 *Let* (3.10) - (3.18) *hold. Then the problem* (3.1) - (3.4) *has a unique weak solution. This solution satisfies the estimate*

$$\begin{aligned}
\|u\|_{\mathcal{U}(Q)} \;\le\; \hat{C}_1\Big[&\|u_0\|_{L^2(\Omega)}+\|f\|_{L^2(0,T;L^{q_2}(\Omega))}+\|\phi\|_{(L^2(Q))^n} \\
&+\|\varphi\|_{\mathcal{U}(Q)}+\theta_1\{\|g\|_{\mathcal{T}(Q)}+\|g_\varphi\|_{\mathcal{T}(Q)}\}+\theta_2\|h\|_{L^2(\Gamma_{2,T})}\Big],
\end{aligned} \tag{3.20}$$

where $\theta_j=0$ *in case* $\Gamma_j=\emptyset$, $\theta_j=1$ *in case* $\Gamma_j\neq\emptyset$ *and* $\hat{C}_1$ *is a constant depending on* a_{ij}, a, μ, m.

Note that according to Remark 3.1, $\|h\|_{L^2(\Gamma_{2,T})} = \Big[\int_0^T \sum_{l=1}^{L} h(x_l,t)^2 dt\Big]^{1/2}$ in case $n = 1$.

Proof. Firstly, we prove the assertion of the theorem in case $\mu = 0$ and $\varphi = 0$.

The assertion of the theorem in case $\mu = 0$, $\varphi = 0$ and $m = 0$ is well-known from the theory of parabolic equations (see e.g. [43]). Let $\mathcal{Z}$ be the operator that assigns to the data vector $d := (u_0, f, \phi, g, h)$ the weak solution of the problem (3.1) - (3.4) in case $\mu = 0$, $\varphi = 0$ and $m = 0$. Then it holds

$$\begin{aligned} \|\mathcal{Z}d\|_{\mathcal{U}(Q)} \;\le\; \hat{C}_2\Big[&\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^{q_2}(\Omega))} + \|\phi\|_{(L^2(Q))^n} \\ &+ \theta_1\|g\|_{\mathcal{T}(Q)} + \theta_2\|h\|_{L^2(\Gamma_{2,T})}\Big], \end{aligned} \tag{3.21}$$

where $\hat{C}_2$ is a constant depending on a_{ij}, a.

Further, let in general $m \neq 0$ and formulate the problem for the difference $v = u - \mathcal{Z}d$. Introducing the linear operator $\mathcal{A}$ by the formula

$$\mathcal{A}w = \mathcal{Z}\Big(0, -am * w, -\sum_{j=1}^{n} a_{ij} m * w_{x_j}, 0, 0\Big),$$

the problem (3.1) - (3.4) for the weak solution $u \in \mathcal{U}(Q)$ is equivalent to the following operator equation for the quantity $v \in \mathcal{U}(Q)$:

$$v = \mathcal{A}v + \mathcal{A}\mathcal{Z}d\,. \tag{3.22}$$

To analyze this equation, have to estimate the operator $\mathcal{A}$. To this end, we need the following auxiliary inequality:

$$\|m * y\|_{L^2(0,t;L^p(\Omega))} \le \int_0^t |m(t-\tau)|\, \|y\|_{L^2(0,\tau;L^p(\Omega))} d\tau,\ t \in (0,T), \tag{3.23}$$

for any $p \ge 1$ and $y \in L^2(0,T;L^p(\Omega))$. This was proved in Publication II, i.e. [42], p. 4.

Let t be an arbitrary number in the interval $(0,T)$. As in the proof of Theorem 2.1, we introduce the cutting operator $P_t w = \begin{cases} w & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases}$. Due to the causality we have $\mathcal{Z}(0, P_t f, P_t\phi, 0, 0)(x,t) = \mathcal{Z}(0, f, \phi, 0, 0)(x,t)$ for any $(x,t) \in \Omega_t$. Using these relations, the inequalities (3.21), (3.23) and

the boundedness of a_{ij}, we compute:

$$\begin{aligned}
\|\mathcal{A}w\|_{\mathcal{U}(Q_t)} &= \left\| \mathcal{Z}\left(0, -am*w, -\textstyle\sum_{j=1}^n a_{ij} m * w_{x_j}, 0, 0\right) \right\|_{\mathcal{U}(Q_t)} \\
&= \left\| \mathcal{Z}\left(0, -P_t[am*w], -P_t\left[\textstyle\sum_{j=1}^n a_{ij} m * w_{x_j}\right], 0, 0\right) \right\|_{\mathcal{U}(Q_t)} \\
&\le \left\| \mathcal{Z}\left(0, -P_t[am*w], -P_t\left[\textstyle\sum_{j=1}^n a_{ij} m * w_{x_j}\right], 0, 0\right) \right\|_{\mathcal{U}(Q)} \\
&\le \hat{C}_2 \left[\|P_t[am*w]\|_{L^2(0,T;L^{q_2}(\Omega))} + \textstyle\sum_{i=1}^n \left\|P_t[a_{ij} m * w_{x_j}]\right\|_{L^2(Q)} \right] \\
&= \hat{C}_2 \left[\|am*w\|_{L^2(0,t;L^{q_2}(\Omega))} + \textstyle\sum_{i=1}^n \left\|a_{ij} m * w_{x_j}\right\|_{L^2(Q_t)} \right] \\
&\le \hat{C}_3 \int_0^t |m(t-\tau)| \left[\|aw\|_{L^2(0,\tau;L^{q_2}(\Omega))} + \|\nabla w\|_{(L^2(Q_\tau))^n} \right] d\tau \qquad (3.24)
\end{aligned}$$

with some constant $\hat{C}_3$ depending on a_{ij}, a. Due to Lemma 3.1, obtain

$$\|aw\|_{L^2(0,\tau;L^{q_2}(\Omega))} \le \hat{C}_4 \|a\|_{L^{q_1}(\Omega)} \|w\|_{\mathcal{U}(Q_\tau)}$$

with some constant $\hat{C}_4$. Using this relation in (3.24), we arrive at the following basic estimate for $\mathcal{A}$:

$$\|\mathcal{A}w\|_{\mathcal{U}(Q_t)} \;\le\; \hat{C}_5 \int_0^t |m(t-\tau)| \|w\|_{\mathcal{U}(Q_\tau)} d\tau\,, \quad t \in (0,T)\,, \qquad (3.25)$$

where the constant $\hat{C}_5$ depends on a_{ij}, a.

Let us define the weighted norms in $\mathcal{U}(Q)$: $\|v\|_\sigma = \sup\limits_{0<t<T} e^{-\sigma t} \|v\|_{\mathcal{U}(Q_t)}$ where $\sigma \ge 0$. The estimate (3.25) implies the further estimate:

$$\begin{aligned}
&\|\mathcal{A}w\|_\sigma \le \hat{C}_5 \sup_{0<t<T} e^{-\sigma t} \textstyle\int_0^t |m(t-\tau)| \, \|w\|_{\mathcal{U}(Q_\tau)} d\tau \\
&= \hat{C}_5 \sup_{0<t<T} \textstyle\int_0^t e^{-\sigma(t-\tau)} |m(t-\tau)| \, e^{-\sigma\tau} \|w\|_{\mathcal{U}(Q_\tau)} d\tau \\
&\le \hat{C}_5 \textstyle\int_0^T e^{-\sigma s} |m(s)| ds \, \|w\|_\sigma .
\end{aligned}$$

Since $\int_0^T e^{-\sigma s}|m(s)|ds \to 0$ as $\sigma \to \infty$, there exists σ_0, depending on $\hat{C}_5$ and m, such that $\hat{C}_5 \int_0^T e^{-\sigma_0 s}|m(s)|ds \le \frac{1}{2}$. Thus, $\|\mathcal{A}w\|_{\sigma_0} \le \frac{1}{2}\|w\|_{\sigma_0}$. The operator $\mathcal{A}$ is a contraction in $\mathcal{U}(Q)$. This implies the existence and uniqueness assertions of the theorem in case $\mu = 0$, $\varphi = 0$.

To prove the estimate (3.20) in case $\mu = 0$, $\varphi = 0$, we deduce from (3.22) the inequality $\|v\|_{\sigma_0} \le \|\mathcal{A}v\|_{\sigma_0} + \|\mathcal{A}\mathcal{Z}d\|_{\sigma_0} \le \frac{1}{2}[\|v\|_{\sigma_0} + \|\mathcal{Z}d\|_{\sigma_0}]$. This implies $\|v\|_{\sigma_0} \le \|\mathcal{Z}d\|_{\sigma_0}$, hence $\|u\|_{\sigma_0} = \|v + \mathcal{Z}d\|_{\sigma_0} \le 2\|\mathcal{Z}d\|_{\sigma_0}$. Using the equivalence relations $e^{-\sigma_0 T}\|\cdot\|_0 \le \|\cdot\|_{\sigma_0} \le \|\cdot\|_0$, where $\|\cdot\|_0 = \|\cdot\|_{\mathcal{U}(Q)}$,

and (3.21), we reach

$$\begin{aligned}\|u\|_{\mathcal{U}(Q)} \;\le\; 2\hat{C}_2 e^{\sigma_0 T}\Big[&\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^{q_2}(\Omega))} + \|\phi\|_{(L^2(Q))^n} \\ &+ \theta_1\|g\|_{\mathcal{T}(Q)} + \theta_2\|h\|_{L^2(\Gamma_{2,T})}\Big].\end{aligned} \tag{3.26}$$

This is (3.20) in case $\mu = 0$, $\varphi = 0$.

Secondly, we prove the theorem in the general case when μ and φ may not vanish. Let us introduce the resolvent kernel $\widehat{\mu}$ of μ that is the solution of the Volterra equation of the second kind

$$\widehat{\mu} + \mu * \widehat{\mu} = \mu \quad \text{in } (0,T). \tag{3.27}$$

Since $\mu \in L^2(0,T)$, the equation (3.27) has a unique solution $\widehat{\mu} \in L^2(0,T)$ [15]. Note that the relations

$$(I - \widehat{\mu}*)(I + \mu*) = (I + \mu*)(I - \widehat{\mu}*) = I \tag{3.28}$$

are valid, where I is the unity operator. Moreover, we define the following one-to-one connection between functions $u \in \mathcal{U}(Q)$ and $\widehat{u} \in \mathcal{U}(Q)$:

$$\widehat{u} = u + \mu * u - \varphi \quad \Leftrightarrow \quad u = \widehat{u} + \varphi - \widehat{\mu} * (\widehat{u} + \varphi) \tag{3.29}$$

Now we note that the problem (3.1) - (3.4) for the weak solution $u \in \mathcal{U}(Q)$ is equivalent to the following problem for the weak solution $\widehat{u} \in \mathcal{U}(Q)$:

$$\begin{aligned}&\widehat{u}_t \;=\; A\widehat{u} - \widehat{m} * A\widehat{u} + \widehat{f} + \nabla \cdot \widehat{\phi} \quad \text{in } Q, \\ &\widehat{u} = \widehat{u}_0 \quad \text{in } \Omega \times \{0\}, \\ &\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T}, \\ &-\nu_A \cdot \nabla\widehat{u} + \widehat{m} * \nu_A \cdot \nabla\widehat{u} = h + \nu \cdot \widehat{\phi} \quad \text{in } \Gamma_{2,T},\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}&\widehat{m} = m + \widehat{\mu} - m * \widehat{\mu}\,, \quad \widehat{f} = f + a\varphi - \widehat{m} * a\varphi, \\ &\widehat{\phi}_i = \phi_i + \sum_{j=1}^{n} a_{ij}\varphi_{x_j} - \widehat{m} * \sum_{j=1}^{n} a_{ij}\varphi_{x_j}, \\ &\widehat{g} = g + \mu * g - g_\varphi\,, \quad \widehat{u}_0 = u_0 - \varphi(\cdot, 0).\end{aligned}$$

This can be directly verified, inserting u by the right formula of (3.29) to (3.19) and vice versa, inserting $\widehat{u}$ by the left formula of (3.29) to the weak form of the problem (3.30).

Taking the assumptions (3.10) - (3.18) and the relation $\widehat{\mu} \in L^2(0,T)$ into account and making use of Lemma 3.1 and Young's inequality of convolutions, we obtain $\widehat{m} \in L^1(0,T)$, $\widehat{f} \in L^2(0,T;L^{q_2}(\Omega))$, $\widehat{\phi} \in (L^2(Q))^n$,

$\widehat{u}_0 \in L^2(\Omega)$, $\widehat{g} \in \mathcal{U}(Q)$. Moreover, the estimates

$$\begin{aligned}
&\|\widehat{f}\|_{L^2(0,T;L^{q_2}(\Omega))} \le \|f\|_{L^2(0,T;L^{q_2}(\Omega))} + \hat{C}_6\|\varphi\|_{\mathcal{U}(Q)},\\
&\|\widehat{\phi}\|_{(L^2(Q))^n} \le \|\phi\|_{(L^2(Q))^n} + \hat{C}_6\|\varphi\|_{\mathcal{U}(Q)},\\
&\|\widehat{g}\|_{\mathcal{T}(Q)} \le \hat{C}_6\|g\|_{\mathcal{T}(Q)} + \|g_\varphi\|_{\mathcal{T}(Q)},\\
&\|\widehat{u}_0\|_{L^2(Q)} \le \|u_0\|_{L^2(Q)} + \hat{C}_6\|\varphi\|_{\mathcal{U}(Q)},
\end{aligned} \tag{3.31}$$

where the constant $\hat{C}_6$ depends on a_{ij}, a, μ, m. Thus, the first part of the proof applied to (3.30) implies that this problem has a unique weak solution $\widehat{w}$ in $\mathcal{U}(Q)$. By the mentioned equivalence of the problems (3.30) and (3.1) - (3.4) we conclude that the latter one has a unique solution u in $\mathcal{U}(Q)$. Finally, we write the estimate (3.26) for the solution of the problem (3.30) and use the relations (3.31) for the data in the right-hand side. This yields $\|\widehat{u}\|_{\mathcal{U}(Q)} \le$ RHS, where RHS is the right-hand side of the estimate (3.20). Since due to (3.29) $\|u\|_{\mathcal{U}(Q)} \le (1+\|\hat{\mu}\|_{L^1(0,T)})\big[\|\widehat{u}\|_{\mathcal{U}(Q)}+\|\varphi\|_{\mathcal{U}(Q)}\big]$, we reach (3.20). ■

We note the upper integration bound T in (3.19) can be released to be any number t from the interval $[0,T]$. Namely, (3.19) is equivalent to the following problem:

$$\begin{aligned}
&\int_\Omega \Big[(u+\mu*u-\varphi)(x,t)\eta(x,t) - (u_0(x)-\varphi(x,0))\eta(x,0)\Big]dx\\
&\quad - \iint\limits_{Q_t} (u+\mu*u-\varphi)\eta_t\,dxdt\\
&\quad + \iint\limits_{Q_t} \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m*u_{x_j})\eta_{x_i} - a(u-m*u)\eta\Big]dxdt\\
&\quad + \iint\limits_{\Gamma_{2,t}} h\eta\,d\Gamma dt - \iint\limits_{Q_t} (f\eta - \phi\cdot\nabla\eta)\,dxdt = 0, \quad t\in[0,T]
\end{aligned} \tag{3.32}$$

for any $\eta \in \mathcal{T}_0(Q)$. This assertion can be proved using the standard technique defining the test function as follows:

$$\eta^\epsilon(x,t) = \begin{cases} \eta(x,\tau) & \text{for } \tau\in[0,t],\\ \eta(x,\tau)\left(1-\frac{\tau-t}{\epsilon}\right) & \text{for } \tau\in(t,t+\epsilon),\\ 0 & \text{for } \tau\in[t+\epsilon,T], \end{cases}$$

and letting the parameter ϵ to approach 0.

3.1.2 Convolutional form of weak direct problem

The test function η in (3.19) has stronger smoothness requirements than the solution u: it must possess a regular time derivative. But in some

cases we must operate with a test function that has the same regularity as u, for instance in computations with adjoint problems in next sections. Therefore, we have to generalize the formulation of the weak problem in a proper manner. Since the original problem contains time convolutions, it is convenient to present such a generalization also in a convolutional form.

Theorem 3.2 *The function $u \in \mathcal{U}(Q)$ satisfies the relation* (3.19) *for any $\eta \in \mathcal{T}_0(Q)$ if and only if it satisfies the following relation*

$$\begin{aligned}
&\int_\Omega (u + \mu * u - \varphi) * \eta \, dx - \int_\Omega \int_0^t (u_0(x) - \varphi(x,0))\eta(x,\tau) d\tau dx \\
&\quad + \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \Big] dx \\
&\quad + \int_{\Gamma_2} 1 * h * \eta \, d\Gamma - \int_\Omega 1 * \Big(f * \eta - \sum_{i=1}^n \phi_i * \eta_{x_i} \Big) dx = 0, \quad t \in [0,T],
\end{aligned} \tag{3.33}$$

for any $\eta \in \mathcal{U}_0(Q)$.

Proof. It is sufficient to prove that $u \in \mathcal{U}(Q)$ satisfies (3.32) for any $\eta \in \mathcal{T}_0(Q)$ if and only if it satisfies (3.33) for any $\eta \in \mathcal{U}_0(Q)$. Suppose that $u \in \mathcal{U}(Q)$ satisfies (3.32) and choose an arbitrary $\eta \in \mathcal{T}_0(Q)$. Let t_1 be an arbitrary number on the interval $[0,T]$ and choose some function $\xi^{t_1} \in \mathcal{T}_0(Q)$ such that the relation

$$\xi^{t_1}(x,t) = \eta(x, t_1 - t) \quad \text{for} \quad t \in [0, t_1] \tag{3.34}$$

is valid. (For instance, it is possible to define ξ^{t_1} as a periodic function with respect to t, i.e. $\xi^{t_1}(x,t) = \eta(x, t_1 - t)$ for $t \in [0, t_1]$, $\xi^{t_1}(x,t) = \eta(x, t - t_1)$ for $t \in [t_1, 2t_1]$, $\xi^{t_1}(x,t) = \eta(x, 3t_1 - t)$ for $t \in [2t_1, 3t_1]$ and so on.) Using the relation (3.32) with η replaced by ξ^{t_1} and setting there $t = t_1$ we obtain the equality

$$K_1(t_1) + K_2(t_1) = 0, \tag{3.35}$$

where

$$\begin{aligned}
K_1(t) \;=\; &\int_\Omega \Big[(u + \mu * u - \varphi)(x,t)\eta(x,0) - (u_0(x) - \varphi(x,0))\eta(x,t) \Big] dx \\
&+ \int_\Omega \int_0^t (u + \mu * u - \varphi)(x,\tau)\eta_t(x, t - \tau) \, d\tau dx,
\end{aligned}$$

$$\begin{aligned}
K_2(t) = &\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \Big] dx \\
&+ \int_{\Gamma_2} h * \eta \, d\Gamma - \int_\Omega \Big(f * \eta - \sum_{i=1}^n \phi_i * \eta_{x_i} \Big) dx.
\end{aligned} \tag{3.36}$$

Note that the time derivative of η can be removed from K_1 by integration. Indeed, let $t_2 \in [0,T]$. Then with the notation $\widehat{u} = u + \mu * u - \varphi$ we get

$$\int_0^{t_2} K_1(t_1)dt_1 = \int_0^{t_2}\int_\Omega \widehat{u}(x,t_1)\eta(x,0)\,dxdt_1 - \int_0^{t_2}\int_\Omega \widehat{u}(x,0)\eta(x,t_1)\,dxdt_1 + \int_0^{t_2}\int_\Omega\int_0^{t_1} \widehat{u}(x,\tau)\eta_{t_1}(x,t_1-\tau)\,d\tau dxdt_1.$$

Changing the order of the integrals over τ and t_1 in the last term, we easily obtain

$$\int_0^{t_2} K_1(t_1)dt_1 = \int_\Omega\int_0^{t_2} \widehat{u}(x,\tau)\eta(x,t_2-\tau)\,d\tau dx - \int_0^{t_2}\int_\Omega \widehat{u}(x,0)\eta(x,t_1)\,dxdt_1. \tag{3.37}$$

Integrating now the whole equality (3.35) over t_1 from 0 to t_2, observing (3.36), (3.37) and finally re-denoting t_2 by t, we reach the desired relation (3.33). Thus, we have proved that (3.33) holds for any $\eta \in \mathcal{T}_0(Q)$. But all terms in the right-hand side of (3.33) are well-defined for $\eta \in \mathcal{U}_0(Q)$, too. Since $\mathcal{T}_0(Q)$ is densely embedded in $\mathcal{U}_0(Q)$, we conclude that (3.33) holds for any $\eta \in \mathcal{U}_0(Q)$.

It remains to show that (3.33) implies (3.32). Suppose that $u \in \mathcal{U}(Q)$ satisfies (3.33) and choose an arbitrary $\eta \in \mathcal{T}_0(Q)$ and $t_1 \in [0,T]$. Again, let ξ^{t_1} be a function from $\mathcal{T}_0(Q)$ such (3.34) is valid. Inserting ξ^{t_1} instead of η into (3.33), differentiating with respect to t and setting $t = t_1$ we come to the relation (3.32). ■

3.2 Quasi-solutions of inverse problems. Fréchet derivatives of cost functionals

3.2.1 Quasi-solutions

(1) Firstly, let us consider IP4. We look for the vector of unknowns $\omega = (\omega_1, \ldots, \omega_N)$ in the space $\mathcal{Z}_1 = (L^2(\Omega))^N$. Assume that $\mu = 0$, $\varphi = 0$, (3.10), (3.11), (3.13) - (3.15), (3.17) hold, f_0 satisfies (3.16) and $\gamma_j \in L^2(0,T)$, $j = 1, \ldots, N$. Then, by Theorem 3.1, the problem (3.1) - (3.4) with f of the form (3.5) has a unique weak solution $u \in \mathcal{U}(Q)$ for any $\omega \in \mathcal{Z}_1$. We denote this ω-dependent solution by $u(x,t;\omega)$. Since $\mathcal{U}(Q) \subset C([0,T];L^2(\Omega))$, the traces $u(\cdot,T_i;\omega)$ belong to $L^2(\Omega)$. Let $M \subseteq \mathcal{Z}_1$. Assume $u_{T_i} \in L^2(\Omega)$, $i = 1, \ldots, N$. The quasi-solution of IP4 in the set M is an element $\omega^* \in \arg\min\limits_{\omega \in M} J_1(\omega)$, where J_1 is the following cost functional:

$$J_1(\omega) = \sum_{i=1}^{N} \|u(x,T_i;\omega) - u_{T_i}(x)\|^2_{L^2(\Omega)}.$$

Similarly we define cost functionals and quasi-solutions for other inverse problems, too.

(2) In IP5 we search for vectors $z = (\omega, u_0) \in \mathcal{Z}_2 = (L^2(\Omega))^{N+1}$. Assume that $\mu = 0$, $\varphi = 0$, (3.10), (3.11), (3.13), (3.15), (3.17) hold, f_0 satisfies (3.16) and $\gamma_j \in L^2(0,T)$, $j = 1, \ldots, N$. Then the problem (3.1) - (3.4) with f of the form (3.5) has a unique weak solution $u = u(x,t;z) \in \mathcal{U}(Q)$ for any $z \in \mathcal{Z}_2$. Further, let $M \subseteq \mathcal{Z}_2$ and assume that $\kappa_i \in L^\infty(Q)$, $v_i \in L^2(\Omega)$, $i = 1, \ldots, N+1$. The quasi-solution of IP5 in the set M is $z^* \in \arg\min_{z \in M} J_2(z)$, where J_2 is the cost functional

$$J_2(z) = \sum_{i=1}^{N+1} \left\| \int_0^T \kappa_i(\cdot,t) u(\cdot,t;z) dt - v_i \right\|^2_{L^2(\Omega)}.$$

(3) In IP6 we look for the vector $z = (a, m, \mu) \in \mathcal{Z}_3 = L^2(\Omega) \times (L^2(0,T))^2$. Assume that $n \in \{1;2;3\}$. This guarantees that any $a \in L^2(\Omega)$ satisfies (3.11). Moreover, assume that (3.10), (3.14) - (3.18) hold, where $q_2 \in (1,2)$ in (3.16) in case $n = 2$. Under such assumptions the problem (3.1) - (3.4) has a unique weak solution $u = u(x,t;z) \in \mathcal{U}(Q)$ for any $z \in \mathcal{Z}_3$. The trace of this solution at $\Gamma_{2,T}$ belongs to $L^2(\Gamma_{2,T})$ (in case $n = 1$, $u(x_l, \cdot) \in L^2(0,T)$, $l = 1, \ldots, L$). Let $M \subseteq \mathcal{Z}_3$ and assume that $u_T \in L^2(\Omega)$, $\kappa_i \in L^\infty(\Gamma_{2,T})$, $v_i \in L^2(0,T)$, $i = 1, 2$. The quasi-solution of IP6 in the set M is $z^* \in \arg\min_{z \in M} J_3(z)$, where J_3 is the cost functional

$$J_3(z) = \|u(\cdot,T;z) - u_T\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \left\| \int_{\Gamma_2} \kappa_i(x,\cdot) u(x,\cdot;z) d\Gamma - v_i \right\|^2_{L^2(0,T)}.$$

3.2.2 General procedure to deduce adjoint problems

Suppose that the solution u of the direct problem (3.1) - (3.4) depends on a vector of parameters p that has to be determined in an inverse problem making use of certain measurements of u. Let a quasi-solution of the inverse problem be sought by a method involving the Fréchet derivative of a cost functional (i.e. some gradient-type method). Usually in practice, a solution of a proper adjoint problem (or solutions of adjoint problems) are used to represent the Fréchet derivative.

We introduce a general procedure to deduce such adjoint problems in case $\mu = 0$ and $\varphi = 0$. Assume that Δu is the difference of solutions of the direct problem corresponding to a difference of the vector of the parameters Δp. More precisely, we suppose that Δu is a solution of the

following problem:

$$\begin{aligned}
&\Delta u_t + (\mu * \Delta u)_t \;=\; A\Delta u - m * A\Delta u + f^\dagger + \nabla\cdot\phi^\dagger \quad \text{in } Q,\\
&\Delta u = \Delta u_0 \quad \text{in } \Omega\times\{0\},\\
&\Delta u = 0 \quad \text{in } \Gamma_{1,T},\\
&-\nu_A\cdot\nabla\Delta u + m * \nu_A\cdot\nabla\Delta u = h^\dagger + \nu\cdot\phi^\dagger \quad \text{in } \Gamma_{2,T},
\end{aligned} \tag{3.38}$$

with some data $f^\dagger$, $\phi^\dagger$, Δu_0, $h^\dagger$ depending on Δp. We restrict ourselves to the case when the Dirichlet boundary condition of the state u is independent on p. Therefore, $\Delta u|_{\Gamma_{1,T}} = 0$ in (3.38).

In practice, the adjoint parabolic problems are usually formulated as backward problems. In our context, it is better to pose adjoint problems in the forward form. The involved memory term with m is defined via a forward convolution and from the practical viewpoint, it is preferable to have the direct and adjoint problems in a similar form (e.g., to simplify parallelization of computations).

Namely, let an adjoint state ψ be a solution of the following problem:

$$\begin{aligned}
&\psi_t + (\mu * \psi)_t \;=\; A\psi - m * A\psi + f^\circ + \nabla\cdot\phi^\circ \quad \text{in } Q,\\
&\psi = u^\circ \quad \text{in } \Omega\times\{0\},\\
&\psi = 0 \quad \text{in } \Gamma_{1,T},\\
&-\nu_A\cdot\nabla\psi + m * \nu_A\cdot\nabla\psi = h^\circ + \nu\cdot\phi^\circ \quad \text{in } \Gamma_{2,T},
\end{aligned} \tag{3.39}$$

where f°, ϕ°, u° and h° are some data depending on Δu and the cost functional under consideration.

Assume that (3.10), (3.11), (3.13) hold and the quadruplets $f^\dagger$, $\phi^\dagger$, Δu_0, $h^\dagger$ and f°, ϕ°, u°, h° satisfy the conditions (3.14) - (3.17). Then, due to Theorem 1, the problems (3.38) and (3.39) have unique weak solutions in the space $\mathcal{U}(Q)$. Actually, it hold $\Delta u, \psi \in \mathcal{U}_0(Q)$ because of the homogeneous boundary conditions on $\Gamma_{1,T}$.

Let us consider the relation (3.33) for Δu and take the test function $\eta = \psi$. Then we obtain for any $t \in [0,T]$

$$\begin{aligned}
&\int_\Omega (\Delta u + \mu * \Delta u) * \psi \, dx - \int_\Omega \int_0^t \Delta u_0(x)\psi(x,\tau) d\tau dx\\
&+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(\Delta u_{x_j} - m * \Delta u_{x_j}) * \psi_{x_i} - a(\Delta u - m * \Delta u) * \psi \Big] dx\\
&+ \int_{\Gamma_2} 1 * h^\dagger * \psi \, d\Gamma - \int_\Omega 1 * \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i} \Big) dx = 0.
\end{aligned} \tag{3.40}$$

Secondly, let us consider (3.33) for ψ and take the test function $\eta = \Delta u$.

Then we have for any $t \in [0,T]$

$$\int_\Omega (\psi + \mu * \psi) * \Delta u \, dx - \int_\Omega \int_0^t u^\circ(x) \Delta u(x,\tau) d\tau dx$$
$$+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij} (\psi_{x_j} - m * \psi_{x_j}) * \Delta u_{x_i} - a(\psi - m * \psi) * \Delta u \Big] dx \quad (3.41)$$
$$+ \int_{\Gamma_2} 1 * h^\circ * \Delta u \, d\Gamma - \int_\Omega 1 * \Big(f^\circ * \Delta u - \sum_{i=1}^n \phi_i^\circ * \Delta u_{x_i} \Big) dx = 0.$$

Subtracting (3.40) from (3.41), using the commutativity of the convolution, the symmetricity relations $a_{ij} = a_{ji}$ and differentiating with respect to t, we arrive at the following basic equality:

$$\int_\Omega u^\circ(x) \Delta u(x,t) dx - \int_{\Gamma_2} h^\circ * \Delta u \, d\Gamma + \int_\Omega \Big(f^\circ * \Delta u - \sum_{i=1}^n \phi_i^\circ * \Delta u_{x_i} \Big) dx$$
$$= \int_\Omega \Delta u_0(x) \psi(x,t) dx - \int_{\Gamma_2} h^\dagger * \psi \, d\Gamma + \int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i} \Big) dx,$$
$$t \in [0,T]. \quad (3.42)$$

This relation can be used to deduce proper representations for Fréchet derivatives of cost functionals of various inverse problems. The procedure starts with a derivation of a usual expression for the Fréchet derivative, thereupon entries of the adjoint problem f°, ϕ°, u° and h° are chosen so that the left hand side of (3.42) equals this expression. We will demonstrate this scheme in next three subsections.

We mention that the last step of the derivation of the formula (3.42), i.e. the differentiation of the difference of (3.41) and (3.40), is problematic in case the free term of the integro-differential equation (3.40) contains a singular addend $\varphi_t^\dagger$, because in this case we have to differentiate a convolution $\varphi^\dagger * \psi$ that may have not a regular time derivative. However, the step of differentiation is necessary, because the antiderivative of (3.42) is useless for inverse problems with instant conditions.

3.2.3 Derivative of J_2

We start by analyzing J_2, because this is the simplest of the functionals J_1, J_2, J_3.

Theorem 3.3 *Let the assumptions listed in* §3.2.1 (2) *be satisfied. Then the functional J_2 is Fréchet differentiable in $\mathcal{Z}_2$ and $J_2'(z)\Delta z = \langle \varrho_2, \Delta z \rangle_{\mathcal{Z}_2}$,*

where the z-dependent vector $\varrho_2 = \varrho_2(x;z)$ consists of the components

$$\begin{aligned} \varrho_{2,j}(x;z) &= \int_0^T \gamma_j(t)\psi(x,T-t;z)dt,\ j=1,\dots,N, \\ \varrho_{2,N+1}(x;z) &= \psi(x,T;z), \end{aligned} \tag{3.43}$$

$\psi = \psi(x,t;z) \in \mathcal{U}(Q)$ is the unique z-dependent weak solution of the following (adjoint) problem:

$$\begin{aligned} &\psi_t \ = \ A\psi - m * A\psi \\ &\quad + 2\sum_{i=1}^{N+1} \kappa_i(x,T-t)\left[\int_0^T \kappa_i(x,\tau)u(x,\tau;z)d\tau - v_i(x)\right] \quad \text{in } Q, \\ &\psi = 0 \quad \text{in } \Omega\times\{0\}, \\ &\psi = 0 \quad \text{in } \Gamma_{1,T}, \\ &-\nu_A\cdot\nabla\psi + m * \nu_A\cdot\nabla\psi = 0 \quad \text{in } \Gamma_{2,T} \end{aligned} \tag{3.44}$$

and $\langle \varrho_2, z\rangle_{\mathcal{Z}_2} = \sum\limits_{j=1}^{N} \langle \varrho_{2,j}\,, \omega_j\rangle_{L^2(\Omega)} + \langle \varrho_{2,N+1}\,, u_0\rangle_{L^2(\Omega)}$ is the inner product of ϱ_2 and z in the space $\mathcal{Z}_2$.

Proof. Let us fix some $z = (\omega, u_0)$ and $\Delta z = (\Delta\omega, \Delta u_0)$ in the space $\mathcal{Z}_2$. One can immediately check that it holds

$$\begin{aligned} J_2(z+\Delta z) - J_2(z) \ &= \ 2\sum_{i=1}^{N+1}\int_\Omega\int_0^T \kappa_i(x,t) \\ &\times\left[\int_0^T \kappa_i(x,\tau)u(x,\tau;z)d\tau - v_i(x)\right]\Delta u(x,t;z)dtdx \\ &+ \sum_{i=1}^{N+1}\int_\Omega\left[\int_0^T \kappa_i(x,t)\Delta u(x,t;z)dt\right]^2 dx, \end{aligned} \tag{3.45}$$

where $\Delta u(x,t;z) = u(x,t;z+\Delta z) - u(x,t;z) \in \mathcal{U}_0(Q)$ is the weak solution of the following problem:

$$\begin{aligned} &\Delta u_t \ = \ A\Delta u - m * A\Delta u + \sum_{j=1}^{N}\gamma_j\Delta\omega_j \quad \text{in } Q, \\ &\Delta u = \Delta u_0 \quad \text{in } \Omega\times\{0\}, \\ &\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \\ &-\nu_A\cdot\nabla\Delta u + m * \nu_A\cdot\nabla\Delta u = 0 \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{3.46}$$

Using the Cauchy inequality, the assumptions $\kappa_i \in L^\infty(Q)$, $\gamma_j \in L^2(0,T)$ and the estimate (3.20) for the solution of the problem (3.46) we deduce

the relation

$$\begin{aligned}&\left|\sum_{i=1}^{N+1}\int_\Omega\left[\int_0^T\kappa_i(x,t)\Delta u(x,t;z)dt\right]^2dx\right|\le\hat C_7\|\Delta u\|^2_{\mathcal U(Q)}\\&\le\hat C_8\Big[\Big\|\sum_{j=1}^N\gamma_j\Delta\omega_j\Big\|^2_{L^2(0,T;L^{q_2}(\Omega))}+\|\Delta u_0\|^2_{L^2(\Omega)}\Big]\le\hat C_9\|\Delta z\|^2_{(L^2(\Omega))^{N+1}}\end{aligned}\tag{3.47}$$

with some constants $\hat C_7,\hat C_8,\hat C_9$. Therefore J_2 is Fréchet differentiable and the first term in the right-hand side of (3.45) represents the Fréchet derivative, i.e.

$$\begin{aligned}J_2'(z)\Delta z\;=\;&2\sum_{i=1}^{N+1}\int_\Omega\int_0^T\kappa_i(x,t)\\&\times\left[\int_0^T\kappa_i(x,\tau)u(x,\tau;z)d\tau-v_i(x)\right]\Delta u(x,t;z)dtdx.\end{aligned}\tag{3.48}$$

Comparing (3.46) with (3.38) we see that $f^\dagger=\sum_{j=1}^N\gamma_j\Delta\omega_j$, $\phi^\dagger=h^\dagger=0$. Consequently, the relation (3.42) has the form

$$\begin{aligned}&\int_\Omega u^\circ(x)\Delta u(x,t)dx-\int_{\Gamma_2}h^\circ*\Delta u\,d\Gamma+\int_\Omega\Big(f^\circ*\Delta u-\sum_{i=1}^n\phi_i^\circ*\Delta u_{x_i}\Big)dx\\&=\int_\Omega\Delta u_0(x)\psi(x,t)dx+\sum_{j=1}^N\int_\Omega\gamma_j\Delta\omega_j*\psi dx,\quad t\in[0,T].\end{aligned}\tag{3.49}$$

We note that the left hand side of (3.49) coincides with (3.48) if we define f° in the in the following manner:

$$f^\circ=2\sum_{i=1}^{N+1}\kappa_i(x,T-t)\left[\int_0^T\kappa_i(x,\tau)u(x,\tau;z)d\tau-v_i(x)\right],$$

let $u^\circ=h^\circ=\phi^\circ=0$ and set $t=T$ in (3.49). The problem (3.39) with such entries takes the form of (3.44). By Theorem 3.1, the latter one has a unique solution $\psi\in\mathcal U(Q)$. Finally, the right-hand side of (3.49) at $t=T$ equals $\langle\varrho_2,\Delta u\rangle_{\mathcal Z_2}$ where the components of ϱ_2 are given by (3.43). This yields the equality $J_2'(z)\Delta z=\langle\varrho_2,\Delta z\rangle_{\mathcal Z_2}$. ■

3.2.4 Derivative of J_1

Theorem 3.4 *Let the assumptions listed in* §3.2.1 (1) *be satisfied. Then the functional J_1 is Fréchet differentiable in $\mathcal Z_1$ and $J_1'(\omega)\Delta\omega=\langle\varrho_1,\Delta\omega\rangle_{\mathcal Z_1}$,*

where the ω-dependent vector $\varrho_1 = \varrho_1(x;\omega)$ consists of the components

$$\varrho_{1,j}(x;\omega) = \sum_{i=1}^{N} \int_0^{T_i} \gamma_j(t)\psi_i(x, T_i - t;\omega)dt,\ j = 1,\dots,N, \tag{3.50}$$

$\psi_i = \psi_i(x,t;\omega) \in \mathcal{U}(Q)$, $i = 1,\dots,N$, are the unique ω-dependent weak solutions of the following (adjoint) problems:

$$\begin{aligned}
&\psi_{i,t} = A\psi_i - m * A\psi_i \quad \text{in } Q_{T_i},\\
&\psi_i = 2[u(x,T_i;\omega) - u_{T_i}(x)] \quad \text{in } \Omega \times \{0\},\\
&\psi_i = 0 \quad \text{in } \Gamma_{1,T_i},\\
&-\nu_A \cdot \nabla\psi_i + m * \nu_A \cdot \nabla\psi_i = 0 \quad \text{in } \Gamma_{2,T_i}
\end{aligned} \tag{3.51}$$

and $\langle \varrho_1, \omega\rangle_{\mathcal{Z}_1} = \sum\limits_{j=1}^{N} \langle \varrho_{1,j}, \omega_j\rangle_{L^2(\Omega)}$ is the inner product of ϱ_1 and ω in the space $\mathcal{Z}_1$.

Proof. Let us fix some $\omega, \Delta\omega \in \mathcal{Z}_1$. It holds

$$\begin{aligned}
J_1(\omega + \Delta\omega) - J_1(\omega) \ =\ & 2\sum_{i=1}^{N} \int_\Omega [u(x,T_i;\omega) - u_{T_i}(x)]\Delta u(x,T_i;\omega)dx\\
&+ \sum_{i=1}^{N} \int_\Omega \Delta u(x,T_i;\omega)^2 dx,
\end{aligned} \tag{3.52}$$

where $\Delta u(x,t;\omega) = u(x,t;\omega+\Delta\omega) - u(x,t;\omega) \in \mathcal{U}_0(Q)$ is the weak solution of the following problem:

$$\begin{aligned}
&\Delta u_t = A\Delta u - m * A\Delta u + \sum_{j=1}^{N} \gamma_j \Delta\omega_j \quad \text{in } Q,\\
&\Delta u = 0 \quad \text{in } \Omega \times \{0\},\\
&\Delta u = 0 \quad \text{in } \Gamma_{1,T},\\
&-\nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u = 0 \quad \text{in } \Gamma_{2,T}.
\end{aligned} \tag{3.53}$$

Similarly to (3.47) we obtain the estimate

$$\sum_{i=1}^{N} \int_\Omega \Delta u(x,T_i;\omega)^2 dx \le \hat{C}_{10}\|\Delta u\|^2_{\mathcal{U}(Q)} \le \hat{C}_{11}\|\Delta\omega\|^2_{(L^2(\Omega))^N}$$

with some constants $\hat{C}_{10}, \hat{C}_{11}$. This implies that J_1 is Fréchet differentiable and the first term in the right-hand side of (3.52) represents the Fréchet

derivative, i.e.

$$J_1'(\omega)\Delta\omega = \sum_{i=1}^{N} \sigma_i \quad \text{with}$$
$$\sigma_i = 2\int_\Omega [u(x,T_i;\omega) - u_{T_i}(x)]\Delta u(x,T_i;\omega)dx. \tag{3.54}$$

We are going to deduce suitable representations for the addends σ_i. For this purpose, we make use the method presented in Subsection 3.2.2, again. Comparing (3.53) with (3.38) we see that $f^\dagger = \sum_{j=1}^{N} \gamma_j \Delta\omega_j$, $\phi^\dagger = \Delta u_0 = h^\dagger = 0$. Therefore, the relation (3.42) reads

$$\int_\Omega u^\circ(x)\Delta u(x,t)dx - \int_{\Gamma_2} h^\circ * \Delta u \, d\Gamma + \int_\Omega \Big(f^\circ * \Delta u - \sum_{i=1}^{n} \phi_i^\circ * \Delta u_{x_i} \Big) dx$$
$$= \sum_{j=1}^{N} \int_\Omega \gamma_j \Delta\omega_j * \psi dx, \quad t \in [0,T]. \tag{3.55}$$

Note that the left-hand side of (3.55) equals σ_i if we set $u_i^\circ = 2[u(x,T_i;\omega) - u_{T_i}(x)]$, $h^\circ = f^\circ = \phi^\circ = 0$ and $t = T_i$ in (3.55). In such a case the initial condition u_i° of the adjoint problem (3.39) depends on the index i, thus the solution ψ depends also on i. Let us denote this solution by ψ_i. Rewriting (3.39) for ψ_i we immediately get (3.51). Due to Theorem 3.1, the problem (3.51) has a unique solution in $\mathcal{U}(Q)$.

From (3.55) we immediately get

$$\sigma_i = \sum_{j=1}^{N} \int_\Omega \int_0^{T_i} \gamma_j(t)\psi_i(x, T_i - t;\omega)dt\Delta\omega_j(x)dx. \tag{3.56}$$

From (3.54) and (3.56) we obtain $J_1'(\omega)\Delta\omega = \langle \varrho_1, \Delta\omega \rangle_{\mathcal{Z}_1}$, where the components of ϱ_1 have the form (3.50). ■

We point out that the formulas of the components of ϱ_1 (3.50) contain the solutions of the problems (3.51) in cylinders $\Omega \times (0,T_i) = Q_{T_i}$ of increasing heights $T_1 < T_2 < \dots T_N$. It turns out that we can reduce the solution of such a family of solutions to a successive solution of a certain family of N problems posed on the layers $\Omega_{T_i - T_{i-1}}$, $i = 1, \dots, N$. The computational cost of the latter procedure is comparable with the cost of a solution of a single problem on $\Omega \times (0, T_N)$. Let us formulate a corresponding theorem.

Theorem 3.5 *Let the assumptions listed in* §3.2.1 (1) *be satisfied. The components of* ϱ_1 *can also be presented in the form*

$$\varrho_{1,j}(x;\omega) = \sum_{l=1}^{N} \int_{T_{l-1}}^{T_l} \gamma_j(t)\beta_l(x, T_l - t;\omega)dt, \; j = 1, \dots, N, \tag{3.57}$$

where $\beta_l \in \mathcal{U}_0(Q_{T_l-T_{l-1}})$ *are the unique* ω*-dependent weak solutions of the following sequence of recursive problems in the domains* $Q_{T_l-T_{l-1}}$*:*

$$\begin{aligned}
&\beta_{l,t} = A\beta_l - m * A\beta_l - af^l - \nabla\cdot\Phi^l \quad \text{in } Q_{T_l-T_{l-1}},\\
&\beta_l = u_0^l \quad \text{in } \Omega\times\{0\},\\
&\beta_l = 0 \quad \text{in } \Gamma_{1,T_l-T_{l-1}},\\
&-\nu_A\cdot\nabla\beta_l + m*\nu_A\cdot\nabla\beta_l = -\nu\cdot\Phi^l \quad \text{in } \Gamma_{2,T_l-T_{l-1}},
\end{aligned}$$

where $l = N, N-1, \ldots, 2, 1$*. Here*

$$u_0^l(x) = 2\left[u(x,T_l;\omega) - u_{T_l}(x)\right] + \Theta_l\beta_{l+1}(x, T_{l+1}-T_l;\omega)$$

and the function f^l *and the vector* Φ^l *are defined via* $\beta_N, \beta_{N-1}, \ldots, \beta_{l+1}$ *as follows:*

$$f^l = \Theta_l \sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} m(T_k - T_l + t + \tau)\beta_{k+1}(x, T_{k+1}-T_k-\tau;\omega)d\tau\,,$$

$\Phi^l = (\Phi_1^l, \ldots, \Phi_n^l)$*,* $\Phi_i^l = \sum_{j=1}^n a_{ij}\frac{\partial}{\partial x_j}f^l$ *and* $\Theta_N = 0$*,* $\Theta_l = 1$ *for* $l < N$*.*

Proof of this quite technical result is contained in Publication II, p. 9-11. The relation between ψ_i an β_l is $\beta_l(x,t;\omega) = \sum_{i=l}^N \psi_i(x, T_i - T_l + t;\omega)$ for $(x,t) \in Q_{T_l-T_{l-1}}$ (formula (71) in Publication II).

3.2.5 Derivative of J_3

We prove the Fréchet differentiability and deduce a proper representation formula for the cost functional J_3 of the nonlinear inverse problem IP6 in two steps.

Theorem 3.6 *Let the assumptions listed in* §3.2.1 (3) *be satisfied. Then the functional* J_3 *is Fréchet differentiable in* $\mathcal{Z}_3$ *and*

$$\begin{aligned}
J_3'(z)\Delta z \;=\; & 2\int_\Omega \left[u(x,T;z) - u_T(x)\right]\Delta u(x,T)dx\\
& + 2\sum_{i=1}^2 \int_0^T \left[\int_{\Gamma_2} \kappa_i(y,t)u(y,t;z)d\Gamma - v_i(t)\right]\int_{\Gamma_2}\kappa_i(x,t)\Delta u(x,t)d\Gamma dt,
\end{aligned} \tag{3.58}$$

for any where $z = (a,m,\mu), \Delta z = (\Delta a, \Delta m, \Delta\mu) \in \mathcal{Z}_3$*, where* $\Delta u \in \mathcal{U}(Q)$

is the z- and Δz-dependent weak solution of the following problem:

$$\begin{aligned}
&\Delta u_t + (\mu * \Delta u)_t \;=\; A\Delta u - m * A\Delta u + \Delta a[u - m * u] - \Delta m * au \\
&\qquad - \nabla \cdot \Big[\Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j}\Big] - (\Delta\mu * u)_t \quad \textit{in } Q, \\
&\Delta u = 0 \quad \textit{in } \Omega \times \{0\}, \\
&\Delta u = 0 \quad \textit{in } \Gamma_{1,T}, \\
&-\nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u = -\nu \cdot \Big[\Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j}\Big] \quad \textit{in } \Gamma_{2,T}.
\end{aligned} \tag{3.59}$$

Proof. Let us estimate the components of the free term of the integro-differential equation in (3.59). Observing the inclusion $u \in \mathcal{U}(Q)$, Lemma 3.1 and using the Young and Cauchy inequalities we obtain

$$\begin{aligned}
&\|\Delta a[u - m * u] - \Delta m * au\|_{L^2(0,T;L^{q_2}(\Omega))} \\
&\quad \le \hat{c}_1 \|u\|_{\mathcal{U}(Q)} \left[(1 + \|m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\right] \\
&\quad \le \hat{c}_2(z,u)\|\Delta z\|,
\end{aligned} \tag{3.60}$$

where $\hat{c}_1$ is a constant, $\hat{c}_2$ is a coefficient depending on z, u and $\|\cdot\|$ denotes the norm in $\mathcal{Z}_3$. Taking the boundedness of a_{ij} into account we similarly get

$$\|\Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j}\|_{(L^2(Q))^n} \le \hat{c}_3 \|u\|_{\mathcal{U}(Q)} \|\Delta m\|_{L^2(0,T)} \tag{3.61}$$

with a constant $\hat{c}_3$. Further, we estimate the term $\Delta\mu * u$. Since $u \in C([0,T]; L^2(\Omega))$ and $\Delta\mu \in L^2(0,T)$, it is easy to check that $\Delta\mu * u \in C([0,T]; L^2(\Omega))$ and $\|\Delta\mu * u\|_{C([0,T];L^2(\Omega))} \le T^{1/2}\|u\|_{C([0,T];L^2(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Similarly, $\|\Delta\mu * u\|_{L^2(0,T;W_2^1(\Omega))} \le T^{1/2} \times \|u\|_{L^2(0,T;W_2^1(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Putting these estimates together, we have

$$\|\Delta\mu * u\|_{\mathcal{U}(Q)} \le T^{1/2}\|u\|_{\mathcal{U}(Q)}\|\Delta\mu\|_{L^2(0,T)}. \tag{3.62}$$

Since $u = g$ in $\Gamma_{1,T}$, we find that

$$\Delta\mu * u \;=\; \Delta\mu * g \quad \text{in } \Gamma_{1,T}. \tag{3.63}$$

Using the assumption $g \in \mathcal{T}(Q)$ and the Young and Cauchy inequalities again, we deduce

$$\begin{aligned}
\|\Delta\mu * g\|_{\mathcal{T}(Q)} &= \|\Delta\mu * g\|_{L^2(0,T;W_2^1(\Omega))} + \|(\Delta\mu * g)_t\|_{L^2(0,T;L^2(\Omega))} \\
&= \|\Delta\mu * g\|_{L^2(0,T;W_2^1(\Omega))} + \|\Delta\mu * g_t\|_{L^2(0,T;L^2(\Omega))} \\
&+ \|\Delta\mu\, g(\cdot,0)\|_{L^2(0,T;L^2(\Omega))} \le \hat{c}_4\|\Delta\mu\|_{L^2(0,T)}
\end{aligned} \tag{3.64}$$

with a constant $\hat{c}_4$. The relations (3.60) - (3.64) show that the assumptions of Theorem 3.1 are satisfied for the problem (3.59). Consequently, (3.59) has a unique weak solution $\Delta u \in \mathcal{U}(Q)$. Moreover, applying the estimate (3.20) for the solution of (3.59) we obtain

$$\begin{aligned}\|\Delta u\|_{\mathcal{U}(Q)} \le \hat{C}_1 \Big[&\|\Delta a[u - m * u] + \Delta m * au\|_{L^2(0,T;L^{q_2}(\Omega))} \\ &+ \Big\|\Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j}\Big\|_{(L^2(Q))^n} + \|\Delta\mu * u\|_{\mathcal{U}(Q)} + \theta_1 \|\Delta\mu * g\|_{\mathcal{T}(Q)}\Big] \\ &\le \hat{c}_5(z,u)\|\Delta z\|\end{aligned} \tag{3.65}$$

with a coefficient $\hat{c}_5$ depending on z, u.

Next, let us denote $\widetilde{\Delta} u = u(x,t;z+\Delta z) - u(x,t;z)$ and define $\widehat{\Delta} u = \widetilde{\Delta} u - \Delta u$. Then we can represent the difference of J_3 as follows:

$$J_3(z+\Delta z) - J_3(z) \;=\; \mathrm{RHS} + \Theta, \tag{3.66}$$

where RHS is the right-hand side of the equality (3.58) and

$$\begin{aligned}\Theta \;=\; & 2\int_\Omega [u(x,T) - u_T(x)]\,\widehat{\Delta} u(x,T)dx \\ & + 2\sum_{i=1}^{2}\int_0^T \Big[\int_{\Gamma_2} \kappa_i(y,t)u(y,t)d\Gamma - v_i(t)\Big]\int_{\Gamma_2}\kappa_i(x,t)\widehat{\Delta} u(x,t)d\Gamma dt \\ & + \int_\Omega \Big\{(\Delta u + \widehat{\Delta} u)(x,T)\Big\}^2 dx + \sum_{i=1}^{2}\int_0^T \Big\{\int_{\Gamma_2}\kappa_i(x,t)(\Delta u + \widehat{\Delta} u)(x,t)d\Gamma\Big\}^2 dt.\end{aligned}$$

The function $\widehat{\Delta} u$ satisfies the following problem:

$$\begin{aligned}&\widehat{\Delta} u_t + (\mu * \widehat{\Delta} u)_t \;=\; A\widehat{\Delta} u - m * A\widehat{\Delta} u + \bar{f} + \widehat{f} \\ &\qquad + \nabla\cdot\bar{\phi} + \nabla\cdot\widehat{\phi} + \bar{\varphi}_t + \widehat{\varphi}_t \quad \text{in } Q, \\ &\widehat{\Delta} u = 0 \quad \text{in } \Omega\times\{0\}, \\ &\widehat{\Delta} u = 0 \quad \text{in } \Gamma_{1,T}, \\ &-\nu_A\cdot\nabla\widehat{\Delta} u + m * \nu_A\cdot\nabla\widehat{\Delta} u = \nu\cdot\bar{\phi} + \nu\cdot\widehat{\phi} \quad \text{in } \Gamma_{2,T},\end{aligned} \tag{3.67}$$

where

$$\begin{aligned}&\bar{f} = \Delta a\Delta u - (m+\Delta m) * \Delta a\Delta u - \Delta m * a\Delta u - \Delta m * \Delta a u, \\ &\widehat{f} = \Delta a\widehat{\Delta} u - (m+\Delta m) * \Delta a\widehat{\Delta} u - \Delta m * a\widehat{\Delta} u, \\ &\bar{\phi} = -\Delta m * \sum_{j=1}^{n} a_{ij}\Delta u_{x_j}, \quad \widehat{\phi} = -\Delta m * \sum_{j=1}^{n} a_{ij}\widehat{\Delta} u_{x_j}, \\ &\bar{\varphi} = -\Delta\mu * \Delta u, \quad \widehat{\varphi} = -\Delta\mu * \widehat{\Delta} u.\end{aligned}$$

Similarly to (3.60) - (3.62) we deduce the following estimates:

$$\begin{aligned}
&\|\bar f\|_{L^2(0,T;L^{q_2}(\Omega))} \le \hat c_6\Big\{(1+\|m\|_{L^2(0,T)}+\|\Delta m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)}\\
&\times\|\Delta u\|_{\mathcal{U}(Q)}+\|\Delta m\|_{L^2(0,T)}\|a\|_{L^2(\Omega)}\|\Delta u\|_{\mathcal{U}(Q)}+\|\Delta m\|_{L^2(0,T)}\|\Delta a\|_{L^2(\Omega)}\\
&\times\|u\|_{\mathcal{U}(Q)}\Big\}\le \hat c_7(z,u)\Big\{\left[\|\Delta z\|+\|\Delta z\|^2\right]\|\Delta u\|_{\mathcal{U}(Q)}+\|\Delta z\|^2\Big\},\\
&\|\widehat f\|_{L^2(0,T;L^{q_2}(\Omega))} \le \hat c_8(z)\left[\|\Delta z\|+\|\Delta z\|^2\right]\|\widehat\Delta u\|_{\mathcal{U}(Q)},\\
&\|\bar\phi\|_{(L^2(Q))^n} \le \hat c_9\|\Delta z\|\|\Delta u\|_{\mathcal{U}(Q)},\\
&\|\widehat\phi\|_{(L^2(Q))^n} \le \hat c_9\|\Delta z\|\|\widehat\Delta u\|_{\mathcal{U}(Q)},\\
&\|\bar\varphi\|_{\mathcal{U}(Q)} \le T^{1/2}\|\Delta z\|\|\Delta u\|_{\mathcal{U}(Q)},\\
&\|\widehat\varphi\|_{\mathcal{U}(Q)} \le T^{1/2}\|\Delta z\|\|\widehat\Delta u\|_{\mathcal{U}(Q)}
\end{aligned}$$

with some coefficients $\hat c_6,\dots,\hat c_9$. Moreover, since $\Delta u=\widehat\Delta u=0$ in $\Gamma_{1,T}$, we have $\bar\varphi=\widehat\varphi=0$ in $\Gamma_{1,T}$. Applying the estimate (3.20) to the solution of the problem (3.67) we get

$$\begin{aligned}
&\|\widehat\Delta u\|_{\mathcal{U}(Q)}\\
&\quad\le \hat c_{10}(z,u)\Big\{\left[\|\Delta z\|+\|\Delta z\|^2\right]\Big\{\|\Delta u\|_{\mathcal{U}(Q)}+\|\widehat\Delta u\|_{\mathcal{U}(Q)}\Big\}+\|\Delta z\|^2\Big\}
\end{aligned}$$

with a coefficient $\hat c_{10}$. Provided $\|\Delta z\|$ is sufficiently small, i.e. $\|\Delta z\|+\|\Delta z\|^2\le\frac{1}{2\hat c_{10}(z,u)}$, we have

$$\|\widehat\Delta u\|_{\mathcal{U}(Q)}\le 2\hat c_{10}(z,u)\Big\{\left[\|\Delta z\|+\|\Delta z\|^2\right]\|\Delta u\|_{\mathcal{U}(Q)}+\|\Delta z\|^2\Big\}.$$

Due to (3.65), this yields

$$\|\widehat\Delta u\|_{\mathcal{U}(Q)}\le\hat c_{11}(z,u)\left[\|\Delta z\|^2+\|\Delta z\|^3\right] \tag{3.68}$$

with a coefficient $\hat c_{11}$.

In view of (3.65), (3.68), the assumption $\kappa_j\in L^\infty(\Gamma_{2,T})$ and a trace theorem, the RHS and the quantity Θ satisfy the estimates

$$|\mathrm{RHS}|\le\hat c_{12}(z,u)\|\Delta z\|,\quad |\Theta|\le\hat c_{13}(z,u)\sum_{l=2}^{6}\|\Delta z\|^l,$$

where $\hat c_{12}$ and $\hat c_{13}$ are some coefficients. Moreover, RHS is linear with respect to Δz. This with (3.66) shows that J_3 is Fréchet differentiable in $\mathcal{Z}_3$ and $J_3'(z)\Delta z$ equals RHS. ■

Theorem 3.7 *Let the assumptions listed in* §3.2.1 (3) *be satisfied. Moreover, assume* $g=0$. *Then the Fréchet derivative of* J_3 *admits the form*

$J_3'(z)\Delta z = \langle \varrho_3, \Delta z\rangle_{\mathcal{Z}_3}$, *where* $\varrho_3 = \widetilde{\varrho}_3 + \overline{\varrho}_3$, *the* z*-dependent vectors* $\widetilde{\varrho}_3, \overline{\varrho}_3 \in \mathcal{Z}_3$ *have the components*

$$\widetilde{\varrho}_{3,1}(x;z) = \big[(u - m * u) * \psi\big](x,T), \quad \overline{\varrho}_{3,1} = 0, \tag{3.69}$$

$$\widetilde{\varrho}_{3,2}(t;z) = \int_\Omega \Big[\sum_{i,j=1}^{n} a_{ij}\psi_{x_i} * u_{x_j} - au * \psi\Big](x,T-t)dx, \quad \overline{\varrho}_{3,2} = 0, \tag{3.70}$$

$$\begin{aligned}\widetilde{\varrho}_{3,3}(t;z) = \int_\Omega \Big[&au * \psi * (\widehat{\mu} + m - m * \widehat{\mu}) - au * \psi \\ &+ \sum_{i,j=1}^{n} a_{ij}\psi_{x_i} * u_{x_j} - \sum_{i,j=1}^{n} a_{ij}\psi_{x_i} * u_{x_j} * (\widehat{\mu} + m - m * \widehat{\mu})\Big](x,T-t)dx,\end{aligned} \tag{3.71}$$

$$\begin{aligned}\overline{\varrho}_{3,3}(t;z) = &-2\int_\Omega \{u(x,T) - u_T(x)\}\big[u - \widehat{\mu} * u\big](x,T-t)dx \\ &- 2\sum_{i=1}^{2}\int_t^T \Big[\int_{\Gamma_2} \kappa_i(y,\tau)u(y,\tau)d\Gamma - v_i(\tau)\Big] \\ &\times \int_{\Gamma_2} \kappa_i(x,\tau)\big[u - \widehat{\mu} * u\big](x,\tau - t)d\Gamma d\tau,\end{aligned} \tag{3.72}$$

$\widehat{\mu}$ *is the solution of* (3.27), $\psi = \psi(x,t;z) \in \mathcal{U}(Q)$ *is the* z*-dependent weak solution of the following (adjoint) problem:*

$$\begin{aligned}&\Delta\psi_t + (\mu * \Delta\psi)_t = A\Delta\psi - m * A\Delta\psi \quad \text{in } Q,\\ &\Delta\psi = 2[u(x,T;z) - u_T(x)] \quad \text{in } \Omega \times \{0\},\\ &\Delta\psi = 0 \quad \text{in } \Gamma_{1,T},\\ &-\nu_A \cdot \nabla\Delta\psi + m * \nu_A \cdot \nabla\Delta\psi = h^\circ \quad \text{in } \Gamma_{2,T},\end{aligned} \tag{3.73}$$

where

$$\begin{aligned}&h^\circ(x,t)\\ &\quad = -2\sum_{i=1}^{2}\kappa_i(x,T-t)\left[\int_{\Gamma_2}\kappa_i(y,T-t)u(y,T-t)d\Gamma - v_i(T-t)\right]\end{aligned} \tag{3.74}$$

and $\langle \varrho_3, z\rangle_{\mathcal{Z}_3} = \langle \varrho_{3,1}, a\rangle_{L^2(\Omega)} + \langle \varrho_{3,2}, m\rangle_{L^2(0,T)} + \langle \varrho_{3,3}, \mu\rangle_{L^2(0,T)}$ *is the inner product of* ϱ_3 *and* $z = (a,m,\mu)$ *in* $\mathcal{Z}_3$.

Proof. We are going to make use of the method presented in Subsection 3.2.2, but firstly we have to eliminate the singular term $(\Delta\mu * u)_t$ from the integro-differential equation (3.59). For this purpose, let us define a new function Δw via is the weak solution of (3.59) Δu by means of the formula $\Delta w = \Delta u + \Delta\mu * u - \widehat{\mu} * \Delta\mu * u$ and deduce a problem for Δw. Since $u, \Delta u \in \mathcal{U}(Q)$, we have $\Delta w \in \mathcal{U}(Q)$. Moreover, using (3.28) it is easy to

check that $\Delta u+\mu*\Delta u+\Delta\mu*u=\Delta w+\mu*\Delta w$. Using this relation for the time derivatives in (3.59) and the equality $\Delta u=\Delta w-\Delta\mu*u+\widehat{\mu}*\Delta\mu*u$ for other terms containing Δu in (3.59) we see that Δw is the weak solution of the following problem:

$$\begin{aligned}
&\Delta w_t+(\mu*\Delta w)_t = A\Delta w-m*A\Delta w+f^\dagger+\nabla\cdot\phi^\dagger \quad \text{in } Q,\\
&\Delta w=0 \quad \text{in } \Omega\times\{0\},\\
&\Delta w=0 \quad \text{in } \Gamma_{1,T},\\
&-\nu_A\cdot\nabla\Delta u+m*\nu_A\cdot\nabla\Delta u=\nu\cdot\phi^\dagger \quad \text{in } \Gamma_{2,T},
\end{aligned} \tag{3.75}$$

where

$$\begin{aligned}
f^\dagger &= \Delta a[u-m*u]-a\Delta m*u-a\Delta\mu*u\\
&\quad +a\Delta\mu*u*[\widehat{\mu}+m-m*\widehat{\mu}],\\
\phi_i^\dagger &= -\Delta m*\sum_{j=1}^n a_{ij}u_{x_j}-\Delta\mu*\sum_{j=1}^n a_{ij}u_{x_j}\\
&\quad +\Delta\mu*\sum_{j=1}^n a_{ij}u_{x_j}*[\widehat{\mu}+m-m*\widehat{\mu}].
\end{aligned}$$

Note that the integro-differential equation (3.75) doesn't contain a singular time derivative in its free term. In addition, let us rewrite the expression of $J'(z)\Delta z$ (3.58) in terms of Δw. Using the formula $\Delta u=\Delta w-(u-\widehat{\mu}*u)*\Delta\mu$, again, we obtain

$$\begin{aligned}
&J_3'(z)\Delta z=\sigma_1+\sigma_2 \quad \text{with}\\
&\sigma_1 = 2\int_\Omega[u(x,T)-u_T(x)]\,\Delta w(x,T)dx\\
&+2\sum_{i=1}^2\int_0^T\left[\int_{\Gamma_2}\kappa_i(y,t)u(y,t)d\Gamma-v_i(t)\right]\int_{\Gamma_2}\kappa_i(x,t)\Delta w(x,t)d\Gamma dt,\\
&\sigma_2 = -2\int_\Omega[u(x,T)-u_T(x)]\,\big[(u-\widehat{\mu}*u)*\Delta\mu\big](x,T)dx\\
&-2\sum_{i=1}^2\int_0^T\left[\int_{\Gamma_2}\kappa_i(y,t)u(y,t)d\Gamma-v_i(t)\right]\\
&\quad\times\int_{\Gamma_2}\kappa_i(x,t)\big[(u-\widehat{\mu}*u)*\Delta\mu\big](x,t)d\Gamma dt.
\end{aligned}$$

The term σ_2 has the form of the inner product $\sigma_2=\langle\overline{\varrho}_3,\Delta z\rangle_{\mathcal{Z}_3}$, where $\overline{\varrho}_3=(0,0,\overline{\varrho}_{3,3})$ and $\overline{\varrho}_{3,3}$ is given by (3.72). It remains to represent σ_1 in a form of the inner product. Let us make use of the basic formula (3.42) with

Δu replaced by Δw. Comparing (3.75) with (3.38) we see that $\Delta u_0 = 0$ and $h^\dagger = 0$. Thus, (3.42) reads

$$\int_\Omega u^\circ(x)\Delta w(x,t)dx - \int_{\Gamma_2} h^\circ * \Delta w\, d\Gamma + \int_\Omega \Big(f^\circ * \Delta w - \sum_{i=1}^n \phi_i^\circ * \Delta w_{x_i}\Big)dx$$

$$= \int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big)dx, \quad t \in [0,T].$$

The left-hand side of this expression equals σ_1 if we choose $u^\circ = 2[u(x,T) - u_T(x)]$, define h° by (3.74), take $f^\circ = 0$, $\phi^\circ = 0$ and set $t = T$. In this case $\sigma_1 = \int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big)dx\big|_{t=T}$. Substituting here the quantities $f^\dagger$ and $\phi^\dagger$ and rearranging the terms we reach the relation $\sigma_1 = \langle \widetilde{\varrho}_3, \Delta z\rangle_{\mathcal{Z}_3}$, where the components of $\widetilde{\varrho}_3$ are given by (3.69) - (3.71). This proves the assertion $J_3'(z)\Delta z = \langle \varrho_3, \Delta z\rangle_{\mathcal{Z}_3}$, where $\varrho_3 = \widetilde{\varrho}_3 + \overline{\varrho}_3$.

Finally, with the mentioned choice of u°, h°, f° and ϕ° the adjoint problem (3.39) has the form (3.73). Since $u \in \mathcal{U}(Q)$ and $\kappa_i \in L^\infty(\Gamma_{2,T})$, by trace theorems we get $u^\circ \in L^2(\Omega)$ and $h^\circ \in L^2(\Gamma_{2,T})$. Therefore, due to Theorem 3.1, the problem (3.73) has a unique solution $\psi \in \mathcal{U}(Q)$. ■

3.3 Existence of quasi-solutions

Theorem 3.8 *Let the assumptions listed in* §3.2.1 (1) *be satisfied and* $M \subset \mathcal{Z}_1$ *be compact. Then* IP4 *has a quasi-solution in* M*. Similar assertions are valid for* IP5 *and* IP6*, too.*

Proof. Since J_1 is bounded from below, there exists $\mathrm{m} = \inf\limits_{\omega\in M} J_1(\omega) > -\infty$. Let $\omega_l \in M$ be a minimising sequence, i.e. $\lim J_1(\omega_l) = \mathrm{m}$. By the compactness, there exists a subsequence $\omega_{l_j} \in M$ such that $\lim \omega_{l_j} = \omega^* \in M$. Due to the continuity of J_1, following from the Fréchet differentiability, we have $\lim J_1(\omega_{l_j}) = J_1(\omega^*)$. Thus, $J_1(\omega^*) = \mathrm{m}$ and $\omega^* \in \arg\min\limits_{\omega\in M} J_1(\omega)$. The element ω^* is a quasi-solution. ■

Theorem 3.9 *Let the assumptions listed in* §3.2.1 (1) *be satisfied and* $M \subset \mathcal{Z}_1$ *be bounded, closed and convex. Then* IP4 *has a quasi-solution in* M*. The set of quasi-solutions is closed and convex. Similar assertion is valid for* IP5*, too.*

Proof. The existence assertion follows from Weierstrass existence theorem (see [65], Section 2.5, Thm 2D) once we have proved that J_1 is weakly sequentially lower semicontinuous in $\mathcal{F}$, i.e.

$$J_1(\omega) \le \liminf J_1(\omega_n) \quad \text{as } \omega_n \rightharpoonup \omega \text{ in } M. \tag{3.76}$$

On the other hand, (3.76) is a consequence of the continuity and convexity of J_1 in M [65]. As mentioned before, J_1 is continuous in $\mathcal{Z}_1$. Thus, it remains to show that J_1 is convex. In view of the linearity of $u(x,t;\omega)$ with respect to ω and the convexity of the quadratic function we obtain

$$J_1(\lambda\hat{\omega}+(1-\lambda)\tilde{\omega}) = \sum_{i=1}^{N}\int_0^T \Big\{u\big(x,T_i;\lambda\hat{\omega}+(1-\lambda)\tilde{\omega}\big)-u_{T_i}(x)\Big\}^2 dx$$

$$= \sum_{i=1}^{N}\int_0^T \Big\{\lambda u(x,T_i;\hat{\omega})+(1-\lambda)u(x,T_i;\tilde{\omega})-u_{T_i}(x)\Big\}^2 dx$$

$$= \sum_{i=1}^{N}\int_0^T \Big\{\lambda\big[u(x,T_i;\hat{\omega})-u_{T_i}(x)\big]+(1-\lambda)\big[u(x,T_i;\tilde{\omega})-u_{T_i}(x)\big]\Big\}^2 dx$$

$$\le \lambda\sum_{i=1}^{N}\int_0^T \big\{u(x,T_i,\hat{\omega})-u_{T_i}(x)\big\}^2 dx$$

$$+(1-\lambda)\sum_{i=1}^{N}\int_0^T \big\{u(x,T_i,\tilde{\omega})-u_{T_i}(x)\big\}^2 dx = \lambda J_1(\hat{\omega})+(1-\lambda)J_1(\tilde{\omega})$$

for any $\lambda\in[0,1]$ and $\hat{\omega},\tilde{\omega}\in\mathcal{Z}_1$. This shows the convexity of J_1. The closedness and convexity of the set of quasi-solutions also follows from the continuity and convexity of J_1. ■

Proof of a theorem analogous to Theorem 3.9 for IP6 is a more complicated task, because in this problem $u(x,t;z)$ is not linear with respect to z and J_3 may not be convex. We are able prove such a result in the particular case $n=1$.

Theorem 3.10 *Let the assumptions listed in* §3.2.1 (3) *be satisfied. Assume that* $n=1$, $\Omega=(c,d)$, $\varphi=0$, $g(\cdot,0)=0$ *and* M *be bounded, closed and convex. Then* IP6 *has a quasi-solution in* M.

Proof. Again, the assertion of the theorem follows from Weierstrass existence theorem [65] provided we are able to show that J_3 is weakly sequentially lower semi-continuous in M. We will prove that J_3 is even weakly sequentially continuous in M.

Let us choose some sequence $z_k=(a_k,m_k,\mu_k)\in M$ such that $z_k\rightharpoonup z=(a,m,\mu)\in M$. Our aim is to show that $J_3(z_k)\to J_3(z)$.

Firstly, we mention that the relation $z_k\rightharpoonup z$ immediately implies $a_k\rightharpoonup a$ in $L^2(c,d)$ and $m_k\rightharpoonup m$, $\mu_k\rightharpoonup\mu$ in $L^2(0,T)$. The subsequent part of the proof consists of several steps.

1. step. Let $\widehat{\mu}\in L^2(0,T)$ be the solution of (3.27) and $\widehat{\mu}_k\in L^2(0,T)$ be the solution of the equation

$$\widehat{\mu}_k+\mu_k*\widehat{\mu}_k=\mu_k \quad \text{in} \quad (0,T).$$

We are going to show that $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ in $L^2(0,T)$.

To prove this convergence relation, we start by verifying the boundedness of the sequence $\widehat{\mu}_k$ in $L^2(0,T)$. Multiplying the equation of $\widehat{\mu}_k$ by $e^{-\sigma t}$, $\sigma > 0$, observing that $e^{-\sigma t}(\mu_k * \mu) = (e^{-\sigma t}\mu_k) * (e^{-\sigma t}\mu)$ and estimating by means of the Young and Cauchy inequalities we get

$$\begin{aligned}
&\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} \le \|e^{-\sigma t}\mu_k * e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\
&\le \|e^{-\sigma t}\mu_k\|_{L^1(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\
&\le \|e^{-\sigma t}\|_{L^2(0,T)}\|\mu_k\|_{L^2(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)}.
\end{aligned}$$

Since $\|e^{-\sigma t}\|_{L^2(0,T)} \to 0$ as $\sigma \to \infty$ and the weakly converging sequence μ_k is bounded in $L^2(0,T)$, there exists $\sigma > 0$ such that $\|e^{-\sigma t}\|_{L^2(0,T)}\|\mu_k\|_{L^2(0,T)} \le \frac{1}{2}$. With such a σ we obtain

$$\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} \le 2\|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \;\Rightarrow\; \|\widehat{\mu}_k\|_{L^2(0,T)} \le 2e^{\sigma T}\sup\|\mu_k\|_{L^2(0,T)}.$$

This shows that the sequence $\widehat{\mu}_k$ is bounded in $L^2(0,T)$.

Further, the difference $\widehat{\mu}_k - \widehat{\mu}$ can be expressed as

$$\widehat{\mu}_k - \widehat{\mu} = \mu_k - \mu - v_k * (\mu_k - \mu),$$

where $v_k = \widehat{\mu} + \widehat{\mu}_k - \widehat{\mu} * \widehat{\mu}_k$ is a bounded sequence in $L^2(0,T)$. With an arbitrary $\zeta \in L^2(0,T)$ we have

$$\langle \widehat{\mu}_k - \widehat{\mu}, \zeta\rangle_{L^2(0,T)} = -\langle \mu_k - \mu, \zeta\rangle_{L^2(0,T)} - N_k, \tag{3.77}$$

$$N_k = \langle v_k * (\mu_k - \mu), \zeta\rangle_{L^2(0,T)} = \int_0^T v_k(\tau)\int_0^{T-\tau}(\mu_k - \mu)(s)\zeta(\tau + s)ds d\tau.$$

Since $\mu_k \rightharpoonup \mu$ and $\zeta(\tau + \cdot) \in L^2(0, T-\tau)$ for $\tau \in (0,T)$, it holds $\int_0^{T-\tau}(\mu_k - \mu)(s)\zeta(\tau + s)ds \to 0$ for any $\tau \in (0,T)$. Moreover, since μ_k is bounded in $L^2(0,T)$, the sequence of τ-dependent functions $|\int_0^{T-\tau}(\mu_k - \mu)(s)\zeta(\tau + s)ds|$ is bounded by a constant. In view of the Cauchy inequality and the dominated convergence theorem, we find

$$|N_k| \le \|v_k\|_{L^2(0,T)}\left\|\int_0^{T-\cdot}(\mu_k - \mu)(s)\zeta(\cdot + s)ds\right\|_{L^2(0,T)} \to 0.$$

Thus, from (3.77), due to $\mu_k \rightharpoonup \mu$, we obtain $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$.

2. step. We estimate $J_3(z_k) - J_3(z)$ in terms of the difference of $\widehat{u}_k$ and $\widehat{u}$, where

$$\widehat{u} = u + \mu * u\,, \quad \widehat{u}_k = u_k + \mu_k * u_k \tag{3.78}$$

and $u = u(x,t;z)$ and $u_k = u(x,t;z_k)$ are the weak solutions of (3.1) - (3.4) corresponding to the vectors z and z_k, respectively.

The relations $u, u_k \in \mathcal{U}(Q)$ and $\mu, \mu_k \in L^2(0,T)$ imply $\widehat{u}, \widehat{u}_k \in \mathcal{U}(Q)$. Applying the operators $I - \widehat{\mu}*$ and $I - \widehat{\mu}_k*$ to the left and right equality in (3.78), respectively, and taking the relations $(I - \widehat{\mu}*)(I + \mu*) = I$ and $(I - \widehat{\mu}_k*)(I + \mu_k*) = I$ into account, we deduce the formulas

$$u \,=\, \widehat{u} - \widehat{\mu} * \widehat{u}\,, \quad u_k \,=\, \widehat{u}_k - \widehat{\mu}_k * \widehat{u}_k.$$

Subtracting we have

$$u_k - u = \widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}.$$

Making use of the latter relation we express the difference of values of the functional J_3 as follows:

$$\begin{aligned}
J_3(z_k) - J_3(z) &= \int_c^d \big(u_k - u\big)^2(x,T)dx \\
&+ 2\int_c^d [u(x,T) - u_T(x)]\big(u_k - u\big)(x,T)dx \\
&+ \sum_{i=1}^2 \int_0^T \left[\sum_{l=1}^L \kappa_i(x_l,t)\big(u_k - u\big)(x_l,t)\right]^2 dt \\
&+ 2\sum_{i=1}^2 \int_0^T \left[\sum_{l=1}^L \kappa_i(x_l,t)u(x_l,t) - v_i(t)\right]\left[\sum_{l=1}^L \kappa_i(x_l,t)\big(u_k - u\big)(x_l,t)\right] dt \\
&= I_k^1 + I_k^2 + I_k^3 + I_k^4,
\end{aligned} \tag{3.79}$$

where

$$\begin{aligned}
I_k^1 &= \int_c^d \Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)^2(x,T)dx, \\
I_k^2 &= 2\int_c^d [u(x,T) - u_T(x)] \\
&\qquad \times \Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x,T)dx, \\
I_k^3 &= \sum_{i=1}^2 \int_0^T \Big[\sum_{l=1}^L \kappa_i(x_l,t) \\
&\qquad \times \Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x_l,t)\Big]^2 dt, \\
I_k^4 &= 2\sum_{i=1}^2 \int_0^T \left[\sum_{l=1}^L \kappa_i(x_l,t)u(x_l,t) - v_i(t)\right] \\
&\qquad \times \left[\sum_{l=1}^L \kappa_i(x_l,t)\Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x_l,t)\right] dt.
\end{aligned}$$

Let us estimate I_1. We split the ingredient of the integral up as $\Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) - (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)^2 = \Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\Big)^2 - 2\Big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\Big)(\widehat{\mu}_k - \widehat{\mu}) * \widehat{u} + \Big((\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)^2$. Using the Cauchy inequality, the inclusions $\widehat{\mu} \in L^2(0,T)$, $\widehat{u}_k, \widehat{u} \in \mathcal{U}(Q)$ and the boundedness of the sequence $\widehat{\mu}_k$ in $L^2(0,T)$ we obtain

$$
\begin{aligned}
|I_k^1| &\le \|\big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|^2_{L^2(c,d)} \\
&+2\|\big((\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\big)(\cdot,T)\|_{L^2(c,d)} \|\big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|_{L^2(c,d)} + R_k^1 \\
&\le \hat{C}_{12}\big(\|\widehat{u}_k - \widehat{u}\|^2_{\mathcal{U}(Q)} + \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)}\big) + R_k^1,
\end{aligned}
$$

where $\hat{C}_{12}$ is a constant and

$$
R_k^1 = \int_c^d \Big[\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x, T-\tau)d\tau\Big]^2 dx.
$$

Since $\widehat{u} \in \mathcal{U}(Q) \subset L^2(Q)$, by Tonelli's theorem it holds $\widehat{u}(x,\cdot) \in L^2(0,T)$ a.e. $x \in (c,d) \Rightarrow \widehat{u}(x, T-\cdot) \in L^2(0,T)$ a.e. $x \in (c,d)$. Thus, in view of $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ in $L^2(0,T)$ we have $\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau \to 0$ a.e. $x \in (c,d)$. Moreover, by the Cauchy inequality, the boundedness of $\widehat{\mu}_k$ in $L^2(0,T)$ and $\widehat{u} \in L^2(Q)$ we get $\Big[\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big]^2 \le \hat{C}_{13}\int_0^T [\widehat{u}(x,\tau)]^2 d\tau \in L^1(c,d)$ with a constant $\hat{C}_{13}$. Therefore, due to the dominated convergence theorem we obtain $R_k^1 \to 0$.

Similarly, for I_k^2 we get

$$
\begin{aligned}
|I_k^2| &\le 2\|u(\cdot,T) - u_T\|_{L^2(c,d)} \|\big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|_{L^2(c,d)} + R_k^2 \\
&\le \hat{C}_{14}\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)} + R_k^2,
\end{aligned}
$$

where $\hat{C}_{14}$ is a constant and

$$
R_k^2 = \int_c^d \big|u(x,T) - u_T(x)\big| \Big|\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big| dx.
$$

By the same reasons as above, it holds $R_k^2 \to 0$.

Next, let us estimate I_k^3. Performing the same splitting as in I_k^1 we deduce

$$
\begin{aligned}
|I_k^3| &\le L^2 \sum_{i=1}^2 \max_{1\le l\le L} \Big[\|\kappa_i(x_l,\cdot)\|^2_{L^\infty(0,T)} \|\big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(x_l,\cdot)\|^2_{L^2(0,T)}\Big] \\
&+2L^2 \sum_{i=1}^2 \max_{1\le l\le L} \Big[\|\kappa_i(x_l,\cdot)\|^2_{L^\infty(0,T)} \|\big((\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\big)(x_l,\cdot)\|_{L^2(0,T)} \\
&\quad \times \|\big(\widehat{u}_k - \widehat{u} - \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(x_l,\cdot)\|_{L^2(0,T)}\Big] + R_k^3 \\
&\le \hat{C}_{15}\big(\|\widehat{u}_k - \widehat{u}\|^2_{\mathcal{U}(Q)} + \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)}\big) + R_k^3,
\end{aligned}
$$

where $\hat{C}_{15}$ is a constant and

$$R_k^3 = L^2 \sum_{i=1}^{2} \max_{1\le l\le L} \Big\{ \|\kappa_i(x_l,\cdot)\|^2_{L^\infty(0,T)} \int_0^T \Big[\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau \Big]^2 dt \Big\}.$$

Since $\widehat{u}(x_l, t-\cdot) \in L^2(0,t)\ \forall t \in (0,T)$ we get $\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau \to 0$ $\forall t \in (0,T)$. Moreover, the sequence $|\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau|$ is bounded by a constant. Consequently, $R_k^3 \to 0$.

Finally, in an analogous manner we deduce the estimate for I_k^4:

$$|I_k^4| \le \hat{C}_{16}\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)} + R_k^4, \qquad \hat{C}_{16} \text{ - a constant,}$$

$$R_k^4 = 2L \sum_{i=1}^{2} \Big\| \sum_{l=1}^{L} \kappa_i(x_l,t)u(x_l,\cdot) - v_j \Big\|_{L^2(0,T)} \max_{1\le l\le L} \Big\{ \|\kappa_i(x_l,\cdot)\|_{L^\infty(0,T)} \times \Big[\int_0^T \Big[\int_0^t (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau \Big]^2 dt \Big]^{1/2} \Big\},$$

where $R_k^4 \to 0$.

In view of (3.79) and the deduced estimates of $I_k^1, \ldots, I_k^4$ it holds

$$|J_3(z_k) - J_3(z)| \le \hat{C}_{17}\big(\|\widehat{u}_k - \widehat{u}\|^2_{\mathcal{U}(Q)} + \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)}\big) + R_k,$$

where $\hat{C}_{17}$ is a constant and $R_k = R_k^1 + \ldots + R_k^4 \to 0$.

3. step. We prove that $\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(Q)} \to 0$. This would imply $|J_3(z_k) - J_3(z)| \to 0$ and complete the proof.

The functions $\widehat{u}$ and $\widehat{u}_k$ are the weak solutions of the following problems (cp. the derivation of (3.30) in the proof of Theorem 3.1):

$$\begin{aligned} &\widehat{u}_t = A\widehat{u} - \widehat{m} * A\widehat{u} + f + \phi_x \quad \text{in } Q,\\ &\widehat{u} = u_0 \quad \text{in } (c,d)\times\{0\},\\ &\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T},\\ &-\nu_A\cdot\nabla\widehat{u} + \widehat{m} * \nu_A\cdot\nabla\widehat{u} = h + \nu\cdot\phi \quad \text{in } \Gamma_{2,T}, \end{aligned} \tag{3.80}$$

$$\begin{aligned} &\widehat{u}_{k,t} = A_k\widehat{u}_k - \widehat{m}_k * A_k\widehat{u}_k + f + \phi_x \quad \text{in } Q,\\ &\widehat{u}_k = u_0 \quad \text{in } (c,d)\times\{0\},\\ &\widehat{u}_k = \widehat{g}_k \quad \text{in } \Gamma_{1,T},\\ &-\nu_A\cdot\nabla\widehat{u}_k + \widehat{m}_k * \nu_A\cdot\nabla\widehat{u}_k = h + \nu\cdot\phi \quad \text{in } \Gamma_{2,T}, \end{aligned} \tag{3.81}$$

where $A_k v = (a_{11}v_x)_x + a_k v$ and

$$\widehat{m} = m + \widehat{\mu} - m * \widehat{\mu}\,, \quad \widehat{m}_k = m_k + \widehat{\mu}_k - m_k * \widehat{\mu}_k,$$
$$\widehat{g} = g + \mu * g\,, \quad \widehat{g}_k = g + \mu_k * g.$$

Let us show that $\widehat{m}_k \rightharpoonup \widehat{m}$ in $L^2(0,T)$. With an arbitrary $\zeta \in L^2(0,T)$ we compute

$$\langle \widehat{m}_k - \widehat{m}, \zeta \rangle_{L^2(0,T)} = \langle m_k - m, \zeta \rangle_{L^2(0,T)} + \langle \widehat{\mu}_k - \widehat{\mu}, \zeta \rangle_{L^2(0,T)} - N_k^1,$$

where

$$\begin{aligned} N_k^1 = \langle m_k * \widehat{\mu}_k - m * \widehat{\mu}, \zeta \rangle_{L^2(0,T)} &= \int_0^T \widehat{\mu}_k(\tau) \int_0^{T-\tau} (m_k - m)(s) \\ &\quad \times \zeta(\tau + s) ds d\tau \\ &+ \int_0^T m(\tau) \int_0^{T-\tau} (\widehat{\mu}_k - \widehat{\mu})(s) \zeta(\tau + s) ds d\tau. \end{aligned}$$

We use the relations $m_k \rightharpoonup m$, $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ and treat the term N_k^1 similarly to the term N_k above to get $N_k^1 \to 0$. As a result we obtain $\langle \widehat{m}_k - \widehat{m}, \zeta \rangle \to 0$. This yields $\widehat{m}_k \rightharpoonup \widehat{m}$.

Subtracting the problem of $\widehat{u}$ from the problem of $\widehat{u}_k$ we see that $w_k := \widehat{u}_k - \widehat{u}$ is a weak solution of the following problem:

$$\begin{aligned} & w_{k,t} = A w_k - \widehat{m} * A w_k + \tilde{f}_k + \tilde{\phi}_{k,x} \quad \text{in } Q, \\ & \widehat{u} = 0 \quad \text{in } (c,d) \times \{0\}, \\ & \widehat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \\ & -\nu_A \cdot \nabla w_k + \widehat{m} * \nu_A \cdot \nabla w_k = \nu \cdot \tilde{\phi}_k \quad \text{in } \Gamma_{2,T}, \end{aligned} \tag{3.82}$$

where

$$\begin{aligned} & \tilde{f}_k = (a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k) - a(\widehat{m}_k - \widehat{m}) * \widehat{u}_k, \\ & \tilde{\phi}_k = -a_{11}(\widehat{m}_k - \widehat{m}) * \widehat{u}_{k,x}, \;\; \tilde{g}_k = (\mu_k - \mu) * g. \end{aligned}$$

In order to use the weak convergence $a_k \rightharpoonup a$ in forthcoming estimations we have to introduce the functions $\rho_k \in W_2^2(c,d)$ being the solutions of the following Neumann problems:

$$\rho_k'' - \rho_k = a_k - a \quad \text{in } (c,d), \qquad \rho_k'(c) = \rho_k'(d) = 0.$$

Then $\rho_k(x) = \int_c^d G(x,y)(a_k - a)(y) dy$, $x \in (c,d)$, where

$$G(x,y) = \frac{1}{2(e^{c-d} - e^{d-c})} \begin{cases} (e^{c-y} + e^{y-c})(e^{d-x} + e^{x-d}) & \text{for } y < x \\ (e^{c-x} + e^{x-c})(e^{d-y} + e^{y-d}) & \text{for } y > x \end{cases}$$

is a Green function that satisfies the properties $G, G_x \in L^\infty((c,d)^2)$. The weak convergence $a_k \rightharpoonup a$ in $L^2(c,d)$ implies [1]

$$\|\rho_k\|_{W_2^1(c,d)} \to 0. \tag{3.83}$$

[1] This is the point we essentially use the assumption $n = 1$, because in case $n \geq 2$ the partial derivatives $G_{x_i}(x, \cdot)$ are not elements of $L^2(\Omega)$.

Using the relation $a_k - a = \rho_k'' - \rho$ we rewrite the term $(a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)$ in $\tilde{f}_k$ as follows:

$$(a_k - a)(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k) = [\rho_k'(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)]_x - \rho_k'(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)_x \\ -\rho_k(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)$$

and shift the addend $[\rho_k'(\widehat{u}_k - \widehat{m}_k * \widehat{u}_k)]_x$ to the singular part $\tilde{\phi}_{k,x}$. As a result, the problem for w_k is transformed to the form

$$\begin{aligned} &w_{k,t} = Aw_k - \widehat{m} * Aw_k + \overline{f}_k + \overline{\phi}_{k,x} \quad \text{in } Q, \\ &\widehat{u} = 0 \quad \text{in } (c,d) \times \{0\}, \\ &\widehat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \\ &-\nu_A \cdot \nabla w_k + \widehat{m} * \nu_A \cdot \nabla w_k = \nu \cdot \overline{\phi}_k \quad \text{in } \Gamma_{2,T}, \end{aligned} \tag{3.84}$$

where

$$\overline{f}_k = -\rho_k'(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k)_x - \rho_k(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k) - a(\widehat{m}_k - \widehat{m}) * \widehat{u}_k, \\ \overline{\phi}_k = \rho_k'(\widehat{u}_k + \widehat{m}_k * \widehat{u}_k) - a_{11}(\widehat{m}_k - \widehat{m}) * \widehat{u}_{k,x}.$$

Let t be an arbitrary number in $(0,T)$. As in the proofs of Theorems 2.1 and 3.1 we make use of the cutting operator $P_t w = \begin{cases} w & \text{in } Q_t \\ 0 & \text{in } Q \setminus Q_t \end{cases}$. Let w_k^t stand for the weak solution of the problem (3.84) with $\overline{f}_k$ and $\overline{\phi}_k$ replaced by $P_t\overline{f}_k$ and $P_t\overline{\phi}_k$, respectively. Then, due to the causality $w_k^t = w_k$ in Q_t. Applying (3.20) to w_k^t we obtain

$$\begin{aligned} \|w_k\|_{\mathcal{U}(Q_t)} &= \|w_k^t\|_{\mathcal{U}(Q_t)} \le \|w_k^t\|_{\mathcal{U}(Q)} \\ &\le \hat{C}_1\Big[\|P_t\overline{f}_k\|_{L^2(0,T;L^1(c,d))} + \|P_t\overline{\phi}_k\|_{L^2(Q)} + \theta_1\|\tilde{g}_k\|_{\mathcal{T}(Q)}\Big] \\ &= \hat{C}_1\Big[\|\overline{f}_k\|_{L^2(0,t;L^1(c,d))} + \|\overline{\phi}_k\|_{L^2(Q_t)} + \theta_1\|\tilde{g}_k\|_{\mathcal{T}(Q)}\Big]. \end{aligned} \tag{3.85}$$

We estimate the right-hand side of (3.85) term-wise. Using the relation $a \in L^2(c,d)$, Cauchy inequality, the inequality (3.23), the assumption $g(x,0) = 0$, the embedding $W_2^1(c,d) \hookrightarrow C[c,d]$ and $\widehat{u}_k = w_k + \widehat{u}$ we deduce:

$$\|\overline{f}_k\|_{L^2(0,t;L^1(c,d))} \le \hat{C}_{18}\Big[\|(\widehat{m}_k - \widehat{m}) * \widehat{u}_k\|_{L^2(0,t;L^2(c,d))} + \|\rho_k\|_{W_2^1(c,d)} \\ \times\big(1 + \|\widehat{m}_k\|_{L^2(0,T)}\big)\|\widehat{u}_k\|_{\mathcal{U}(Q_t)}\Big] \le \hat{C}_{18}\Big[\int_0^t |(\widehat{m}_k - \widehat{m})(t-\tau)|\,\|w_k\|_{L^2(Q_\tau)}d\tau \\ +\|\rho_k\|_{W_2^1(c,d)}\big(1 + \|\widehat{m}_k\|_{L^2(0,T)}\big)\|w_k\|_{\mathcal{U}(Q_t)}\Big] + \overline{R}_k^1, \tag{3.86}$$

$$\|\overline{\phi}_k\|_{L^2(Q_t)} \le \hat{C}_{19}\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}_{k,x}\|_{L^2(Q_t)}+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)$$
$$\times\|\widehat{u}_k\|_{L^2(0,t;C[c,d])}\Big]\le \hat{C}_{19}\Big[\int_0^t |(\widehat{m}_k-\widehat{m})(t-\tau)|\,\|w_{k,x}\|_{L^2(Q_\tau)}d\tau$$
$$+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\|w_k\|_{L^2(0,t;C[c,d])}\Big]+\overline{R}_k^2, \tag{3.87}$$

$$\|\tilde{g}_k\|_{\mathcal{T}(Q)} \le \overline{R}_k^3, \tag{3.88}$$

where

$$\overline{R}_k^1 = \hat{C}_{18}\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}\|_{L^2(Q)}+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\|\widehat{u}\|_{\mathcal{U}(Q)}\Big],$$
$$\overline{R}_k^2 = \hat{C}_{19}\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}_x\|_{L^2(Q)}$$
$$+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\|\widehat{u}\|_{L^2(0,T;C[c,d])}\Big],$$
$$\overline{R}_k^3 = \|(\mu_k-\mu)*g\|_{L^2(Q)}+\|(\mu_k-\mu)*g_x\|_{L^2(Q)}+\|(\mu_k-\mu)*g_t\|_{L^2(Q)}$$

and $\hat{C}_{18}, \hat{C}_{19}$ are constants. The quantities $\overline{R}_k^j$, $j=1,2,3$ contain terms with the factor $\|\rho_k\|_{W_2^1(c,d)}$ and terms of the form $\|z_k*\hat{v}\|_{L^2(Q)}$, where z_k is one of the functions $\widehat{m}_k-\widehat{m}$, $\mu_k-\mu$ or $\widehat{\mu}_k-\widehat{\mu}$ and $\hat{v}\in L^2(Q)$ is one of the functions $\widehat{u}$, $\widehat{u}_x$, g, g_x or g_t. The former terms converge to zero because of the relation (3.83) and the boundedness of $\widehat{m}_k$ in $L^2(0,T)$ and the latter terms approach zero by virtue of the weak convergence $z_k \rightharpoonup 0$ in $L^2(0,T)$. More precisely, to prove that $\|z_k*\hat{v}\|_{L^2(Q)}\to 0$ we write

$$\|z_k*\hat{v}\|_{L^2(Q)} = \Big\{\int_0^T\int_c^d\Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2 dxdt\Big\}^{1/2}.$$

The component $\Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2$ is bounded by an integrable with respect to x in (c,d) function $\sup\limits_k \|z_k\|^2_{L^2(0,T)}\|\hat{v}(x,\cdot)\|^2_{L^2(0,T)}$ and tends to zero for all $t\in(0,T)$ and a.e. $x\in(c,d)$, because $z_k\rightharpoonup 0$ and $\hat{v}(x,t-\cdot)\in L^2(0,T)$ for all $t\in(0,T)$ and a.e. $x\in(c,d)$. (The latter relation follows from $\hat{v}\in L^2(Q)$ and Tonelli's theorem.) Thus, by the dominated convergence theorem, it holds $\|z_k*\hat{v}\|_{L^2(Q)}\to 0$. Summing up,

$$\overline{R}_k^j \to 0, \quad j=1,2,3. \tag{3.89}$$

As in proof of Theorem 3.1, we use the norms $\|w\|_\sigma = \sup\limits_{0<t<T} e^{-\sigma t}\|w\|_{\mathcal{U}(Q_t)}$ with the weights $\sigma\ge 0$ in the space $\mathcal{U}(Q)$. Then in view of (3.86) - (3.88)

from (3.85) we deduce

$$\|w_k\|_\sigma \le \hat{C}_{20}\Big[\sup_{0<t<T}\int_0^t e^{-\sigma(t-\tau)}|(\widehat{m}_k-\widehat{m})(t-\tau)|\,e^{-\sigma\tau}\|w_k\|_{\mathcal{U}(Q_\tau)}d\tau$$
$$+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\|w_k\|_\sigma+\sum_{j=1}^3\overline{R}_k^j\Big]$$
$$\le \hat{C}_{21}\Big[\Big\{\|e^{-\sigma t}\|_{L^2(0,T)}\|\widehat{m}_k-\widehat{m}\|_{L^2(0,T)}$$
$$+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\Big\}\|w_k\|_\sigma+\sum_{j=1}^3\overline{R}_k^j\Big], \tag{3.90}$$

where $\hat{C}_{20}$, $\hat{C}_{21}$ are constants. Since $\|e^{-\sigma t}\|_{L^2(0,T)} \to 0$ as $\sigma \to \infty$, $\|\rho_k\|_{W_2^1(c,d)} \to 0$ and the sequence $\|\widehat{m}_k\|_{L^2(0,T)}$ is bounded, there exist $\sigma > 0$ and $K_2 \in \mathbb{N}$ such that

$$\hat{C}_{21}\Big\{\|e^{-\sigma t}\|_{L^2(0,T)}\|\widehat{m}_k-\widehat{m}\|_{L^2(0,T)}+\|\rho_k\|_{W_2^1(c,d)}\big(1+\|\widehat{m}_k\|_{L^2(0,T)}\big)\Big\}\le\frac{1}{2}$$

for $k \ge K_2$. This with (3.90) implies

$$\|w_k\|_\sigma \le 2\hat{C}_{21}\sum_{j=1}^3\overline{R}_k^j \quad\text{and hence}\quad \|w_k\|_{\mathcal{U}(Q)} \le 2e^{\sigma T}\hat{C}_{21}\sum_{j=1}^3\overline{R}_k^j$$

for $k \ge K_2$. Taking (3.89) into account we obtain the desired convergence $\|\widehat{u}_k-\widehat{u}\|_{\mathcal{U}(Q)} = \|w_k\|_{\mathcal{U}(Q)} \to 0$. The theorem is proved. ■

3.4 Discretization and minimization

In the final section of the thesis let us discuss some aspects of the discretization and minimization of the cost functionals by means of the penalized gradient method.

Let us consider any of the problems IP4, IP5 or IP6 and search for its quasi-solution. This means that we have to minimize corresponding cost functional J_k over a given set $M \subseteq \mathcal{Z}_k$, where k is some number in the set $\{1;2;3\}$. In order to treat these problems in a common manner, we use the notation z also for the solution ω of IP4.

Clearly, there are many possibilities to discretize IP4 - IP6. Here we describe in detail an orthogonal discretization. Let us introduce a L-dimensional subspace $\mathcal{Z}_{k,L}$ of $\mathcal{Z}_k$ (here $L < \infty$). Let P_L stand for the orthogonal projection to $\mathcal{Z}_{k,L}$. Then the L-dimensional analogue of the set M is

$$M_L = P_L M \subseteq \mathcal{Z}_{k,L}.$$

Now we replace the problem of minimization of J_k over M by the following penalized discrete problem: find $z^\dagger$ such that

$$z^\dagger \in \arg\min_{z\in\mathcal{Z}_{k,L}} \Phi_{k,L}(z), \quad \text{where} \quad \Phi_{k,L} = \Pi_L + J_k \tag{3.91}$$

and Π_L is a penalty function corresponding to the set M_L. In order the problem to be relevant, Π_L has to be small inside M_L and large outside M_L. The mathematical conditions imposed on Π_L are

$$\begin{aligned} &\Pi_L \text{ is coercive, convex, Fréchet differentiable}\,, \\ &\text{The Fréchet derivative } \Pi'_L \text{ of } \Pi_L \text{ is} \\ &\qquad \text{uniformly Lipschitz continuous in } \mathcal{Z}^*_{k,L}. \end{aligned} \tag{3.92}$$

Theorem 3.11 *Let the assumptions listed in* §3.2.1 (k) *be satisfied. Moreover, let* (3.92) *hold. Then the problem* (3.91) *has a solution.*

Proof. The proof is similar to the proof of Theorem 3.8. By coercitivity and continuity, Π_L is bounded from below. Moreover, J_k is also bounded from below. Thus, there exists $\mathrm{m} = \inf_{z\in\mathcal{Z}_{k,L}} [\Pi_L(z) + J_k(z)] = \inf_{z\in\mathcal{Z}_{k,L}} \Phi_{k,L}(z) > -\infty$. Let $z_l \in \mathcal{Z}_{k,L}$ be a minimizing sequence, i.e. $\lim \Phi_{k,L}(z_l) = \mathrm{m}$. Due to the coercitivity of Π_l, the sequence z_l is bounded (in case z_l is not bounded, there is a subsequence z_{l_i} such that $\|z_{l_i}\| \to \infty$, hence by the coercitivity $\Phi_{k,L}(z_{l_i}) \to \infty$, but this is in contradiction with the relation $\lim \Phi_{k,L}(z_l) = \mathrm{m}$). In a finite-dimensional space every bounded sequence is compact. Consequently, there exists a subsequence z_{l_j} such that $\lim z_{l_j} = z^*$. Due to the continuity of $\Phi_{k,L}$, following from the Fréchet differentiability, we have $\lim \Phi_{k,L}(z_{l_j}) = \Phi_{k,L}(z^*)$. Thus, $\Phi_{k,L}(z^*) = \mathrm{m}$ and $z^* \in \arg\min_{z\in\mathcal{Z}_{k,L}} \Phi_{k,L}(z)$. The element z^* is a solution of (3.91). ■

Further, we formulate an algorithm of the gradient method for the minimization of $\Phi_{k,L}$. To this end, the representations of Fréchet derivatives of J_k obtained in Theorems 3.3 – 3.7 are useful. According to these theorems, $J'_k(z)\Delta z = \langle \varrho_k, \Delta z\rangle_{\mathcal{Z}_k}$, where the z-dependent element $\varrho_k = \varrho_k[z] \in \mathcal{Z}_k$ is given by (3.57), (3.43) and (3.69) - (3.72) in cases $k = 1, 2$ and 3, respectively. In order to construct an algorithm that remains inside the subspace $\mathcal{Z}_{k,L}$, we have to find an analogue of ϱ_k in $\mathcal{Z}_{k,L}$. This is $P_L\varrho_k$. Indeed, for any $\Delta z \in \mathcal{Z}_{k,L}$ we have

$$\langle \varrho_k, \Delta z\rangle_{\mathcal{Z}_k} = \langle P_L\varrho_k, \Delta z\rangle_{\mathcal{Z}_k} + \langle \varrho_k - P_L\varrho_k, \Delta z\rangle_{\mathcal{Z}_k} = \langle P_L\varrho_k, \Delta z\rangle_{\mathcal{Z}_k},$$

because $\varrho_k - P_L\varrho_k \perp \Delta z = 0$ since P_L is the orthogonal projection onto $\mathcal{Z}_{k,L}$. Thus, $P_L\varrho_k$ can be used as a representative of Fréchet derivative of J_k in $\mathcal{Z}_{k,L}$.

Let the representative of $\Pi'_L(z)$ in $\mathcal{Z}_{k,L}$ be $\pi_L[z]$, i.e. $\Pi'_L(z)\Delta z = \langle \pi_L[z], \Delta z\rangle_{\mathcal{Z}_k}$ for $\Delta z \in \mathcal{Z}_{k,L}$. Then the Fréchet derivative of $\Phi_{k,L}$ at z has the formula

$$\Phi'_{k,L}(z)\Delta z = \langle G_k[z], \Delta z\rangle_{\mathcal{Z}_k} \quad \text{for} \quad \Delta z \in \mathcal{Z}_{k,L},$$
$$\text{where } G_k[z] = \pi_L[z] + P_L\varrho_k[z].$$

The gradient method is as follows. We choose some initial guess $z_0 \in \mathcal{Z}_{k,L}$ and compute successive approximate solutions by means of the formula

$$z_{s+1} = z_s - c_s G[z_s], \tag{3.93}$$

where $s = 0, 1, 2, \ldots$ and $c_s > 0$.

Theorem 3.12 *Let $k \in \{1; 2\}$, i.e. we have either* IP4 *or* IP5. *Assume that the assumptions listed in* §3.2.1 (k) *are satisfied and* (3.92) *hold. Moreover, let c_s be chosen by the rule*

$$\inf_{c>0} \Phi_{k,L}\left(z_s - cG_k[z_s]\right)) \le \Phi_{k,L}\left(z_s - c_s G_k[z_s]\right)$$
$$\le \inf_{c>0} \Phi_{k,L}\left(z_s - cG_k[z_s]\right) + \delta_s,$$

where $\delta_s \ge 0$, $\sum_{s=0}^{\infty} \delta_s =: \delta < \infty$. Then it holds $\operatorname{dist}(z_s, S) \to 0$ *as $s \to \infty$, where S is the set of solutions of* (3.91)

Proof. Let us prove the theorem in case $k = 1$ (IP4). The proof in case $k = 2$ is similar.

The assertion follows from Theorem 5.1.2 of [63] once we have proved that G_1 is uniformly Lipschitz-continuous with respect to z, the functional $\Phi_{1,L}$ is convex and the set $M(z_0) = \{z \in \mathcal{Z}_{1,L} : \Phi_{1,L}(z) \le \Phi_{1,L}(z_0) + \delta\}$ is bounded. The convexity of $\Phi_{1,L}$ follows from the convexity of its addends Π_L and J_1 (the latter one was shown in the proof of Theorem 3.9). The boundedness of $M(z_0)$ is a direct consequence of the coercitivity of $\Phi_{1,L}$ following from the coercitivity of the addend Π_L.

It remains to show the uniform Lipschitz continuity of $G_1[z]$ with respect to z in $\mathcal{Z}_{1,L}$. The Lipschitz continuity of $\pi_L[z]$ in $\mathcal{Z}_{1,L}$ follows from the assumed Lipschitz continuity of $\Pi'_L[z]$ in $\mathcal{Z}^*_{1,L}$, because $\mathcal{Z}_{1,L}$ and $\mathcal{Z}^*_{1,L}$ are isomorphic and $\pi_L[z]$ is the analogue of $\Pi'_L[z]$ in $\mathcal{Z}_{1,L}$. Thus, we have still to prove the Lipschitz continuity of the term $P_L\varrho_1[z]$ with respect to z occurring in the formula of $G_1[z]$.

Using (3.50) we obtain for any $z, \tilde{z} \in \mathcal{Z}_{1,L}$

$$\begin{aligned}\|P_L\varrho_1[\tilde{z}] - P_L\varrho_1[z]\|_{\mathcal{Z}_1} &\le \|P_L\|\|\varrho_1[\tilde{z}] - \varrho_1[z]\|_{\mathcal{Z}_1} \\ &\le C_{22}\sum_{i=1}^{N}\|\psi_i(\cdot,\cdot;\tilde{z}) - \psi_i(\cdot,\cdot;z)\|_{L^2(Q)} \\ &\le C_{23}\sum_{i=1}^{N}\|\psi_i(\cdot,\cdot;\tilde{z}) - \psi_i(\cdot,\cdot;z)\|_{\mathcal{U}(Q)},\end{aligned}$$

where the constants C_{22} and C_{23} are independent of z and $\tilde{z}$. The functions $\psi_i(\cdot,\cdot;\tilde{z})$ and $\psi_i(\cdot,\cdot;z)$ solve the problems (3.51) with $\omega = \widetilde{z}$ and $\omega = z$, respectively. Applying the relation (3.20) of Theorem 3.1 to the problem for the difference $\psi_i(\cdot,\cdot;\tilde{z}) - \psi_i(\cdot,\cdot;z)$ we continue the estimate as follows:

$$\begin{aligned}\|P_L\varrho_1[\tilde{z}] - P_L\varrho_1[z]\|_{\mathcal{Z}_1} &\le 2C_{23}\hat{C}_1\sum_{i=1}^{N}\|u(\cdot,T_i;\tilde{z}) - u(\cdot,T_i;z)\|_{L^2(\Omega)} \\ &\le C_{24}\|u(\cdot,\cdot;\tilde{z}) - u(\cdot,\cdot;z)\|_{\mathcal{U}(Q)}\end{aligned}$$

with a constant C_{24} independent of z and $\tilde{z}$. The function $u(\cdot,\cdot;\tilde{z}) - u(\cdot,\cdot;z)$ is a weak solution of the problem (3.1) - (3.4) with $f(x,t) = \sum_{j=1}^{N}\gamma_j(t)[\tilde{z}_j(x) - z_j(x)]$, $\phi = 0$, $\varphi = 0$, $u_0 = 0$, $g = 0$, $h = 0$. Using again (3.20) we obtain

$$\begin{aligned}\|P_L\varrho_1[\tilde{z}] - P_L\varrho_1[z]\|_{\mathcal{Z}_1} &\le 2C_{24}\hat{C}_1\Big\|\sum_{j=1}^{N}\gamma_j[\tilde{z}_j - z_j]\Big\|_{L^2(0,T;L^{q_2}(\Omega))} \\ &\le C_{25}\|\tilde{z} - z\|_{\mathcal{Z}_1}\end{aligned}$$

with a constant C_{25} independent of z and $\tilde{z}$. This proves the uniform Lipschitz-continuity of $P_L\varrho_1$. □

The convergence of z_s in case $k = 3$ is an open issue. This case is more complex because IP6 is nonlinear and the Fréchet derivative of J_3 is not uniformly Lipschitz-continuous.

The quasi-solutions of IP4 - IP6 are not expected to be stable with respect to noise of the data, i.e. the problems under consideration may be ill-posed. Nevertheless, from the intuitive viewpoint a discretisation should regularize an ill-posed problem. Such a property of the discretization has been proved in many cases [40, 56]. Alternatively, the index s of the gradient method could be used as a regularization parameter (cf. [19]). Moreover, the addend Π_L can be defined to be the stabilizing term of the Tikhonov's method instead of the penalty function, i.e. $\Pi_L = \alpha\|z\|^2$, where $\alpha > 0$ is the regularization parameter. Such a Π_L satisfies (3.92).

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ABSTRACT

Inverse problems for parabolic integro-differential equations with instant and integral conditions. Kairi Kasemets. Doctoral thesis, 2016.

The aim of the thesis is to perform a systematical study of inverse problems for parabolic integro-differential equations containing time convolutions with memory kernels in case space-dependent factors of free terms or coefficients of the equations are unknown and observation conditions are given either in the form of instant measurements over the space or integrated with respect to time measurements over the space. In the second part of the thesis a problem to determine kernels is also considered.

First part of the thesis contains an analysis of problems that are smooth in the sense that all derivatives included in the integro-differential equations are regular functions. A positivity principle for parabolic integro-differential equations is established. In case final data for a solution of a direct problem are given, the global existence, uniqueness and stability for an inverse problem to determine a space-dependent component of a free term of the equation are proved. The proof of the uniqueness uses the positivity principle and the proof of existence and stability exploits the Fredholm alternative. Moreover, making use of results obtained for the inverse free term problem, global uniqueness, local existence and stability for inverse problems to determine a lower-order coefficient and a coefficient at the time derivative occurring in the equation from final data are proved. The main tool is the Banach fixed-point principle.

Second part of the thesis includes a treatment of problems in non-smooth case, i.e. when higher order derivatives involved in an integro-differential equation are singular distributions. A weak convolutional form of the direct problem is introduced. Such a form does not involve time derivatives of solutions or test functions. A general method to derive adjoint problems for Fréchet derivatives of cost functionals corresponding to inverse problems for this equation is proposed. The method makes use of the mentioned weak convolutional form. Further, this method is applied to deduce adjoint problems for 3 particular inverse problems: a problem to determine a finite number of space-dependent factors of a free term from instant measurements; a problem to determine a finite number of space-dependent factors of a free term and an initial condition from integral measurements; a problem to determine a lower-order coefficient and two kernels from combined instant and integral measurements. The existence of quasi-solutions for the mentioned 3 inverse problems is established. The existence result for the third inverse problem (that is nonlinear) in a non-compact set is proved in the one-dimensional case.

KOKKUVÕTE

Hetk- ja integraaltingimustega pöördülesanded paraboolsetele integrodiferentsiaalvõrranditele. Kairi Kasemets. Doktoriväitekiri, 2016.

Doktoritöö eesmärgiks on süstemaatiliselt analüüsida pöördülesandeid mälutuumadega konvolutsiooniliikmeid sisaldavatele paraboolsetele integrodiferentsiaalvõrranditele juhul, kui tundmatuteks on vabaliikmete ruumimuutujatest sõltuvad tegurid või kordajad ja vaatlused on antud kas hetktingimustena üle ruumi või integreeritud mõõtmistena aja suhtes üle ruumi. Töö teises osas on vaatluse all ka tuumade määramise ülesanne.

Töö esimeses osas analüüsitakse pöördülesandeid, mis on siledad selles mõttes, et võrrandis sisalduvad kõrgemat järku tuletised on regulaarsed. Tuletatakse positiivsusprintsiip paraboolsete integrodiferentsiaalvõrrandite jaoks. Juhul, kui ette on antud lõpptingimus, tõestatakse ruumimuutujatest sõltuva vabaliikme komponendi määramise pöördülesande lahendi globaalne olemasolu, ühesus ja stabiilsus. Ühesuse tõestamisel kasutatakse positiivsusprintsiipi ja olemasolu ning stabiilsuse tõestamisel rakendatakse Fredholmi alternatiivi. Kasutades vabaliikme pöördülesande jaoks saadud tulemusi tõestatakse madalamat järku liikmes sisalduva kordaja ja ajatuletise ees seisva kordaja määramise ülesannete lahendite globaalne ühesus ning lokaalne olemasolu ja stabiilsus, kui ette on antud lõpptingimus. Rakendatakse Banachi püsipunktiprintsiipi.

Töö teises osas käsitletakse pöördülesandeid mittesiledal juhul, kui võrrandis sisalduvad kõrgemat järku tuletised on singulaarsed distributsioonid. Tuuakse sisse otsese ülesande nõrk konvolutsioonitüüpi seade. Selline seade ei sisalda lahendi ega testfunktsiooni tuletisi ajamuutuja suhtes. Pakutakse välja üldine meetod pöördülesannetele vastavate sihifunktsionaalide Fréchet tuletiste leidmisel kasutatavate kaasülesannete tuletamiseks mainitud konvolutsioonitüüpi seadet kasutades. Meetodit rakendatake kolmele pöördülesandele: ülesanne lõpliku arvu ruumimuutujatest sõltuvate kordajate määramiseks hetktingimuste alusel; ülesanne lõpliku arvu ruumimuutujatest sõltuvate kordajate ja algtingimuse määramiseks integraaltingimuste alusel; ülesanne madalamat järku kordaja ja kahe tuuma määramiseks kombineeritud hetk- ja integraaltingimuste alusel. Tehakse kindlaks kõigi kolme ülesande kvaasilahendite olemasolu. Kolmanda ülesande (mis on mittelineaarne) kvaasilahendi olemasolu tõestatakse ühemõõtmelisel juhul.

Curriculum Vitae

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3. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential equations with instant and integral conditions. *17th International Conference on Mathematical Modelling and Analysis : June 6-9, 2012, Tallinn, Estonia, Abstracts.* p 68.
4. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential

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5. K. Kasemets, J. Janno, Inverse problems for a parabolic integro-differential equation in a convolutional weak form. *Abstract and Applied Analysis*, Article ID 297104, 2013, 16pp.
6. K. Kasemets, J. Janno, A weak inverse problem for a parabolic integro-differential equation containing two kernels. *Electronic Journal of Differential Equations*, **2014**, 2014, 176, 1–19.

6. Defended thesis

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2. K. Kasemets, J. Janno, Recovery of a Source Term in a Parabolic Integro-Differential Equation from Final Data. *Mathematical Modelling and Analysis*, **16**, 2011, 2, 199–219.
3. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential

equations with instant and integral conditions. *17th International Conference on Mathematical Modelling and Analysis : June 6-9, 2012, Tallinn, Estonia, Abstracts.* p 68.
4. K. Kasemets, J. Janno, Inverse problems for parabolic integro-differential equations with two kernels. *18th International Conference Mathematical Modelling and Analysis and 4th International Conference Approximation Methods and Orthogonal Expansions, Tartu, 27-30.05.2013.* p 57.
5. K. Kasemets, J. Janno, Inverse problems for a parabolic integro-differential equation in a convolutional weak form. *Abstract and Applied Analysis*, Article ID 297104, 2013, 16pp.
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7. Teadustöö põhisuunad

Diferentsiaal- ja integraalvõrrandid, pöördülesanded, mittekorrektsed ülesanded

Appendix

Publication I

J. Janno, K. Kasemets, A positivity principle for parabolic integro-differential equations and inverse problems with final overdetermination. *Inverse Problems and Imaging*, **3**, 2009, 1, 17–41.

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A POSITIVITY PRINCIPLE FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS AND INVERSE PROBLEMS WITH FINAL OVERDETERMINATION

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Abstract. A positivity principle for parabolic integro-differential equations is proved. By means of this principle, uniqueness, existence and stability for an inverse source problem and two inverse coefficient problems are established.

1. **Introduction.** Inverse problems for parabolic differential equations with final overdetermination have been studied by many authors [1, 2, 6, 7, 8, 12]. The present paper represents an attempt to generalize some of these results to the case of parabolic integro-differential equations. More precisely, we will be limited to inverse problems for parabolic equations in the diffusion processes with memory that involve time convolutions of a kernel m and an elliptic operator A of the concentration or temperature u (equation (1) below).

A linear inverse problem to determine a source term in a parabolic integro-differential equation by means of the final overdetermination was previously studied in [11]. More precisely, in this paper the authors prove the existence and uniqueness for the inverse problem under the assumption that the solution is unique. The proof is based on Fredholm alternative.

The uniqueness is a more difficult issue. Its study requires a suitable positivity principle for mentioned parabolic integro-differential equations. To the authors' knowledge, such a positivity result has not been known in the literature. Our idea to prove this principle is as follows. We transform the equation to a form that contains a time convolution of the resolvent k of m and the time derivative of u instead of Au. Then it is possible to integrate by parts in this convolution to remove completely the derivatives of u. Owing to this possibility, we can use the classical proof scheme of extremum principles. Handling the additional convolution term is not very difficult, because we can operate with the sign of u instead of Au (see part 1 of the proof of Theorem 2). We prove the positivity principle in the cases of first and second kind boundary conditions under the assumptions that the resolvent kernel

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satisfies the conditions $k \geq 0$ and $k_t \leq 0$. The problem with third kind boundary conditions cannot be handled by the technique presented in the paper.

It turns out that the presence of a non-vanishing kernel k with such properties weakens other assumptions of the positivity principle. For instance, for the source term χ, it is sufficient to assume $\chi + k * \chi \geq 0$. This assumption is weaker than the assumption $\chi \geq 0$ that occurs in the case of usual parabolic differential equation. Physically this is explained by an inertia of the medium. The same remark holds for the second kind boundary condition, too. Details are given in Remark 2 below.

The analysis of inverse problems in this paper uses very much the technique of paper [7]. In particular, the decomposition of z into z^+ and z^- in Theorem 3 comes from [7] and proof of Theorem 4 uses an analogous theorem from [7].

In this paper we prove existence, uniqueness and stability theorems for the linear inverse source problem and two nonlinear inverse coefficient problems for parabolic integro-differential equations. Existence and stability results for coefficient problems are local. Again, the assumptions are in case $k \neq 0$ weaker than in the case $k = 0$.

The method of the paper works only in case the memory kernel m is independent of spatial variables. Otherwise it falls into a divergence-nabla type operator and a time resolvent is not applicable.

2. **Formulation of problems. Notation.** Linear non-homogeneous diffusion processes with memory are subject to the constitutive relation [3, 4, 9, 13]

$$q_i(x,t) = -\sum_{j=1}^{n} a_{ij}(x) u_{x_j}(x,t) + \int_0^t m(t-\tau) \sum_{j=1}^{n} a_{ij}(x) u_{x_j}(x,\tau) d\tau, \quad i = 1, \ldots, n,$$

where q is the flux and u is the concentration (or temperature). The function m is the memory kernel. We will consider the case when m is independent of x.

Using this relation in the continuity equation $\beta u_t + \operatorname{div} q = \chi$ which is assumed to be valid in some open domain $(x,t) \in Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$, where $\beta : \Omega \to \mathbb{R}$ and $\chi : Q \to \mathbb{R}$ are given functions, we arrive at the parabolic integro-differential equation

$$\beta u_t = Au - m * Au + \chi \quad \text{in} \quad Q, \tag{1}$$

where $A = \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$ and $*$ stands for the time convolution, i.e.

$$v * w = \int_0^{\cdot} v(\cdot - \tau) w(\tau) d\tau.$$

In this paper we generalize a bit this physical model. Namely, we consider the equation (1) with the operator A of the form

$$A = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} + a(x,t) \tag{2}$$

where a_{ij}, a_j and a are given cofficients. We assume that the principal part of A is positive definite, i.e. the inequality

$$\sum_{i,j=1}^{N} a_{ij}\, \lambda_i \lambda_j \geq \epsilon |\lambda|^2 \quad \text{for any } x \in \Omega, \quad \lambda \in \mathbb{R}^n \quad \text{and some } \epsilon \in (0, \infty) \tag{3}$$

holds. Since the operator A depends on t, we will write $A = A(t)$, if necessary. Moreover, let the relation

$$\beta \geq \beta_0 > 0 \quad \text{for any } x \in \Omega \quad \text{and some } \beta_0 \in (0, \infty) \tag{4}$$

be valid.

We supplement the equation (1) with initial and boundary conditions

$$u = u_0 \ \text{ in } \Omega \times \{0\}\,, \quad Bu = b \ \text{ in } S = \Gamma \times (0, T) \tag{5}$$

where u_0, b are some given functions, Γ is the boundary of Ω and either

$$Bu = u \qquad (\textit{case I}) \tag{6}$$

or

$$Bu(x,t) = \omega(x) \cdot \nabla_x u(x,t) - \int_0^t m(t-\tau)\, \omega(x) \cdot \nabla_x u(x,\tau) d\tau \quad (\textit{case II}) \tag{7}$$

with $\omega(x) \cdot N(x) > 0$, $N(x)$ - outer normal of Γ at x. Note that case II is a generalization of the second kind boundary condition $-q \cdot N = b$. In the sequel we denote by ν the order of the boundary operator B, i.e $\nu = 0$ in case I and $\nu = 1$ in case II.

Throughout the paper we assume the boundary Γ and the function ω sufficiently smooth.

The relations (1) and (5) form the direct problem for u. In this paper we will study three following inverse problems:

IP1: Let the free term be of the following form:

$$\chi(x,t) \ = \ z(x)\phi(x,t) + \chi_0(x,t). \tag{8}$$

Given $m, \beta, a_{ij}, a_j, a, u_0, b, \phi, \chi_0$ and a function $u_T(x)$, $x \in \Omega$, find z and u so that the relations (1), (5), (8) and

$$u \ = \ u_T \ \text{ in } \Omega \times \{T\} \tag{9}$$

hold.

IP2: Let $a_t = 0$. Given $m, \beta, a_{ij}, a_j, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find a and u so that the relations (1), (5) and (9) hold.

IP3: Given $m, a_{ij}, a_j, a, u_0, b, \chi$ and a function $u_T(x)$, $x \in \Omega$, find β and u so that the relations (1), (5) and (9) hold.

Let us introduce the functional spaces we will use in the paper. Given the Banach space X, $p \in [1, \infty)$ and an integer $l \geq 0$, we define the following abstract Sobolev spaces:

$$W_p^l(0,T;X) = \left\{ v \in [0,T] \to X \ : \ \|v\|_{W_p^l(0,T;X)} := \sum_{j=0}^{l} \left[\int_0^T \|v^{(j)}(t)\|_X^p dx \right]^{\frac{1}{p}} < \infty \right\}.$$

In case $l = 0$ they coincide with the abstract Lebesgue spaces, i.e. $W_p^0(0,T;X) = L^p(0,T;X)$. If $X = \mathbb{R}$, we merely write $W_p^l(0,T;\mathbb{R}) = W_p^l(0,T)$. In addition, we need spaces of fractional order and anisotropic spaces. To this end, let us first introduce the following notation for difference quotients of x- and (x,t)-dependent functions with powers:

$$\langle v \rangle_p(x_1, x_2) := \tfrac{v(x_1) - v(x_2)}{|x_1 - x_2|^p}\,, \quad \langle v \rangle_p(x_1, x_2; t) := \tfrac{v(x_1,t) - v(x_2,t)}{|x_1 - x_2|^p}\,,$$
$$\langle v \rangle_p(x; t_1, t_2) := \tfrac{v(x,t_1) - v(x,t_2)}{|t_1 - t_2|^p}.$$

For any $p \in [1,\infty)$ and $l \geq 0$ we define the Sobolev-Slobodeckij spaces (cf. [10, 14])

$$W_p^l(\Omega) = \Bigg\{ v \,:\, \|v\|_{W_p^l(\Omega)} := \sum_{|\alpha|\leq[l]} \Big[\int_\Omega |D^\alpha v(x)|^p dx\Big]^{\frac{1}{p}} + \Theta_l \sum_{|\alpha|=[l]} \Big[\int_\Omega dx_1 \int_\Omega |\langle D^\alpha v\rangle_{\frac{n}{p}+l-[l]}(x_1,x_2)|^p dx_2\Big]^{\frac{1}{p}} < \infty \Bigg\},$$

$$W_p^{l,\frac{l}{2}}(Q) = \Bigg\{ v \,:\, \|v\|_{W_p^{l,\frac{l}{2}}(Q)} := \sum_{2j+|\alpha|\leq[l]} \Big[\int_0^T \int_\Omega |D_t^j D_x^\alpha v(x,t)|^p dx dt\Big]^{\frac{1}{p}} + \Theta_l \sum_{2j+|\alpha|=[l]} \Big[\int_0^T dt \int_\Omega dx_1 \int_\Omega \big|\langle D_t^j D_x^\alpha v\rangle_{\frac{n}{p}+l-[l]}(x_1,x_2;t)\big|^p dx_2\Big]^{\frac{1}{p}}$$
$$+\Theta_{\frac{l}{2}} \sum_{0<l-2j-|\alpha|<2} \Big[\int_\Omega dx \int_0^T dt_1 \int_0^T \big|\langle D_t^j D_x^\alpha v\rangle_{\frac{1}{p}+\frac{l-2j-|\alpha|}{2}}(x;t_1,t_2)\big|^p dt_2\Big]^{\frac{1}{p}} < \infty \Bigg\}.$$

Here $[l]$ is the greatest integer $\leq l$ and $\Theta_l = 0$ and $\Theta_l = 1$ in the cases of integer l and non-integer l, respectively. Further, for any non-integer $l > 0$ we define the Hölder spaces

$$C^l(\Omega) = \Bigg\{ v \,:\, D^\alpha v \in C(\Omega) \ \text{ for } \ |\alpha| \leq [l], \quad \|v\|_l := \sum_{|\alpha|\leq[l]} \Big[\sup_{x\in\Omega} |D^\alpha v(x)| + \sup_{x_1,x_2\in\Omega} \big|\langle D^\alpha v\rangle_{l-[l]}(x_1,x_2)\big|\Big] < \infty \Bigg\},$$

$$C^{l,\frac{l}{2}}(Q) = \Bigg\{ v \,:\, D_t^j D_x^\alpha v \in C(Q) \ \text{ for } \ 2j+|\alpha| \leq [l],$$
$$\|v\|_{l,\frac{l}{2}} := \sum_{2j+|\alpha|\leq[l]} \Bigg[\sup_{(x,t)\in Q} \Big|D_t^j D_x^\alpha v(x,t)\Big| + \sup_{(x_1,t),(x_2,t)\in Q} \Big|\langle D_t^j D^\alpha v\rangle_{l-[l]}(x_1,x_2;t)\Big| + \sup_{(x,t_1),(x,t_2)\in Q} \Big|\langle D_t^j D^\alpha v\rangle_{\frac{l-[l]}{2}}(x;t_1,t_2)\Big|\Bigg] < \infty \Bigg\}.$$

The definitions of $W_p^{l,\frac{l}{2}}$ and $C^{l,\frac{l}{2}}$ are in a standard manner extended from Q to the boundary cylinder S (for details see [10]). For integer $l \geq 0$ we define

$$C^{2l,l}(\overline{Q}) = \{v \,:\, D_t^j D_x^\alpha v \in C(\overline{Q}) \ \text{ for } \ 2j+|\alpha| \leq 2l\}.$$

Finally, we introduce some additional notation. Let U be a finite-dimensional manifold and $f, g \in L^1(U)$. We write

$$\begin{aligned} &f \geq g \ \text{ in } U \quad \text{if } f(x) \geq g(x) \text{ a.e. } x \in U, \\ &f > g \ \text{ in } U \quad \text{if } \forall U_1\colon \overline{U}_1 \subseteq U \ \ \exists \varepsilon_{U_1} \in \mathbb{R}, \varepsilon_{U_1} > 0\colon f \geq g + \varepsilon_{U_1} \text{ in } U_1. \end{aligned} \tag{10}$$

It is not difficult to prove that

$$f \geq g, f \neq g \ \text{ in } U \quad \Rightarrow \quad \exists U_2 \subseteq U \,:\, \operatorname{meas} U_2 \neq 0,\ f > g \text{ in } U_2. \tag{11}$$

3. **Results for direct problem. Positivity principle.** The technique of the paper is based on the representation of the memory kernel m in terms of its resolvent k. Namely, we define k to be a solution of the following Volterra integral equation:

$$k(t) - \int_0^t m(t-\tau)k(\tau)d\tau = m(t), \quad t \in (0,T) \tag{12}$$

with given m. It is well-known that in case $m \in L^p(0,T)$ with $p>1$ the solution k exists, is unique and belongs to $L^p(0,T)$ (see e.g. [5]).

It is possible to transform the direct problem (1), (5) to a form that contains the kernel k instead of m. Indeed, observing the formula $(I+k*)(I-m*)=I$, where I is the unity operator we get

Lemma 1. *In case all t-dependent functions and their derivatives involved in equations (1) and (5) belong to the space $L^p(0,T)$ with respect to t for any x with some $p>1$, these equations are equivalent to*

$$\beta(u_t + k*u_t) = Au + f \quad \text{in } Q\,, \quad u = u_0 \quad \text{in } \Omega\times\{0\}\,, \quad B_1 u = g \quad \text{in } S\,, \tag{13}$$

where

$$f \;=\; \chi + k*\chi, \tag{14}$$

$$B_1 = B\,, \quad g = b \quad \text{in case I}\,, \tag{15}$$

$$B_1 u = \omega\cdot\nabla_x u\,, \quad g = b + k*b \quad \text{in case II}. \tag{16}$$

Now we are going to prove certain basic results for the direct problem of the form (13). For the sake of generality, we allow *the kernel k and the coefficients β, a_{ij} and a_j to depend on both x and t* in Theorems 1 and 2 below.

Theorem 1. *Assume (3) and (4) in Q. Then the following assertions are valid.*

(i) *Let $\beta, a_{ij}, a_j, a \in C(\overline{Q})$, $k \in L^1(0,T;L^\infty(\Omega))$, $f\in L^p(Q)$, $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g \in W_p^{2-\frac{1}{p}-\nu,1-\frac{1}{2p}-\frac{\nu}{2}}(S)$ with some $p\in(1,\infty)$. Moreover, in case* I *let $p\neq\frac{3}{2}$ and the consistency condition $u_0 = g$ hold in $\Gamma\times\{0\}$ if $p>\frac{3}{2}$ and in case* II *let $p\neq 3$ and the consistency condition $\omega\cdot\nabla_x u_0 = g$ hold in $\Gamma\times\{0\}$ if $p>3$. Then the problem (13) has a unique solution in the space $W_p^{2,1}(Q)$. This solution satisfies the estimate*

$$\|u\|_{W_p^{2,1}(Q)} \;\le\; C_1\left\{\|f\|_{L^p(Q)} + \|u_0\|_{W_p^{2-\frac{2}{p}}(\Omega)} + \|g\|_{W_p^{2-\nu-\frac{1}{p},1-\frac{\nu}{2}-\frac{1}{2p}}(S)}\right\}, \tag{17}$$

where C_1 is a constant depending on β, a_{ij}, a_j, a and k.

(ii) *Let $\beta, a_{ij}, a_j, a \in C^{l,\frac{l}{2}}(Q)$ and $k\in L^{\frac{2}{2-l}}(0,T;C^l(\Omega))$ with some $l\in(0,1)$. Moreover, let $f\in C^{l,\frac{l}{2}}(Q)$, $u_0\in C^{2+l}(\Omega)$, $g\in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$ and in case* I *the consistency conditions $u_0 = g$, $\beta g_t = Au_0 + f$ hold in $\Gamma\times\{0\}$ and in case* II *the consistency condition $\omega\cdot\nabla_x u_0 = g$ hold in $\Gamma\times\{0\}$. Then the solution of (13) belongs to $C^{2+l,1+\frac{l}{2}}(Q)$ and satisfies the estimate*

$$\|u\|_{2+l,1+\frac{l}{2}} \;\le\; C_2\left\{\|f\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}\right\} \tag{18}$$

with some constant C_2 depending on β, a_{ij}, a_j, a and k.

Proof. Let us denote $Q_t = \Omega\times(0,t)$. To prove (i), we need the relation

$$\|k*v\|_{L^p(Q_t)} \;\le\; \int_0^t \|k(\cdot,t-\tau)\|_{L^\infty(\Omega)}\|v\|_{L^p(Q_\tau)}d\tau\,, \quad t\in(0,T)\,, \quad v\in L^p(Q). \tag{19}$$

Denoting $\overline{k}(t) = k(\cdot,t)$, $\overline{v}(t) = v(\cdot,t)$, $\|\overline{k}(t)\| = \|\overline{k}(t)\|_{L^\infty(\Omega)}$, $\|\overline{v}(t)\| = \|\overline{v}(t)\|_{L^p(\Omega)}$, making use of the following property of the Bochner integral: $\|\int_0^s w(s,\tau)d\tau\|_{L^p(\Omega)} \le$

$\int_0^s \|w(s,\tau)\|_{L^p(\Omega)}d\tau$ for functions $w(s,\cdot)\in L^1(0,T;L^p(\Omega))$ and the Hölder's inequality, the sought relation (19) can be deduced by means of the following computations:

$$\begin{aligned}
&\|k*v\|_{L^p(Q_t)} = \left[\int_0^t \left\|\int_0^s \bar{k}(\tau)\bar{v}(s-\tau)d\tau\right\|^p_{L^p(\Omega)} ds\right]^{\frac{1}{p}} \le I, \quad \text{where}\\
&I = \left[\int_0^t \left(\int_0^s \|\bar{k}(\tau)\|\,\|\bar{v}(s-\tau)\|d\tau\right)^p ds\right]^{\frac{1}{p}}\\
&\quad = \left[\int_0^t \|\bar{k}(\tau)\| \int_\tau^t \|\bar{v}(s-\tau)\| \left(\int_0^s \|\bar{k}(y)\|\,\|\bar{v}(s-y)\|dy\right)^{p-1} ds d\tau\right]^{\frac{1}{p}}\\
&\quad \le \left[\int_0^t \|\bar{k}(\tau)\| \left[\int_\tau^t \|\bar{v}(s-\tau)\|^p ds\right]^{\frac{1}{p}} \left[\int_\tau^t \left(\int_0^s \|\bar{k}(y)\|\,\|\bar{v}(s-y)\|dy\right)^p ds\right]^{\frac{p-1}{p}} d\tau\right]^{\frac{1}{p}}\\
&\quad \le \left[\int_0^t \|\bar{k}(\tau)\| \left[\int_0^{t-\tau} \|\bar{v}(s)\|^p ds\right]^{\frac{1}{p}} d\tau\right]^{\frac{1}{p}} \left[\int_0^t \left(\int_0^s \|\bar{k}(y)\|\,\|\bar{v}(s-y)\|dy\right)^p ds\right]^{\frac{p-1}{p^2}}\\
&\quad = \left[\int_0^t \|k(\cdot,t-\tau)\|_{L^\infty(\Omega)}\|v\|_{L^p(Q_\tau)}d\tau\right]^{\frac{1}{p}} \times I^{\frac{p-1}{p}}.
\end{aligned}$$

By Theorem 5.4 in [14], under the assumptions of (i) problem (13) in case $k=0$ has a unique solution $\widehat{u}$ in the space $W_p^{2,1}(Q)$. Thus, (13) for $u\in W_p^{2,1}(Q)$ in case $k\ne 0$ is equivalent to the following problem for the difference $v=u-\widehat{u}\in W_p^{2,1}(Q)$:

$$\beta v_t = Av - \beta(k*(v_t+\widehat{u}_t)) \ \text{ in } Q\,, \quad v=0 \ \text{ in } \Omega\times\{0\}\,, \quad B_1 v = 0 \ \text{ in } S\,. \tag{20}$$

Let F stand for the operator that assigns to a function f the solution of the problem (13) in the case $k=0$, $u_0=0$, $g=0$. By Theorem 5.4 in [14], $F\in\mathscr{L}(L^p(Q),W_p^{2,1}(Q))$. This implies that the problem (20) in $W_p^{2,1}(Q_t)$ is equivalent to the following fixed-point equation:

$$v = \mathcal{F}v + \mathcal{F}\widehat{u}\,, \quad \text{where } \mathcal{F}u = -F(\beta(k*u_t)) \tag{21}$$

and $\mathcal{F}\in\mathscr{L}(W_p^{2,1}(Q))$. Let $t\in(0,T)$ and define the cutting operator P_t by the formula $P_t v = \begin{cases} v & \text{in } Q_t \\ 0 & \text{in } Q\setminus Q_t \end{cases}$. Observing that $Fv = FP_tv$ in Q_t and using (19) we deduce the estimate

$$\begin{aligned}
\|\mathcal{F}v\|_{W_p^{2,1}(Q_t)} &= \|F(\beta(k*v_t))\|_{W_p^{2,1}(Q_t)} = \|FP_t(\beta(k*v_t))\|_{W_p^{2,1}(Q_t)}\\
&\le \|FP_t(\beta(k*v_t))\|_{W_p^{2,1}(Q)} \le \|F\|\,\|P_t(\beta(k*v_t))\|_{L^p(Q)}\\
&= \|F\|\,\|\beta(k*v_t)\|_{L^p(Q_t)} \le C_2\int_0^t \|\bar{k}(t-\tau)\|_{L^\infty(\Omega)}\|v\|_{W_p^{2,1}(Q_\tau)}d\tau
\end{aligned}$$

with $C_2 = \|F\|\,\|\beta\|_{C(\overline{Q})}$ and $\bar{k}(t)=k(\cdot,t)$, as before. Now we define the weighted norms $\|v\|_\sigma = \sup\limits_{0<t<T} e^{-\sigma t}\|v\|_{W_p^{2,1}(Q_t)}$, $\sigma>0$, in the space $W_p^{2,1}(Q)$ and deduce the estimate

$$\begin{aligned}
\|\mathcal{F}v\|_\sigma &\le C_2 \sup_{0<t<T} e^{-\sigma t}\int_0^t \|\bar{k}(t-\tau)\|_{L^\infty(\Omega)}\|v\|_{W_p^{2,1}(Q_\tau)}d\tau\\
&= C_2 \sup_{0<t<T}\int_0^t e^{-\sigma(t-\tau)}\|\bar{k}(t-\tau)\|_{L^\infty(\Omega)} e^{-\sigma\tau}\|v\|_{W_p^{2,1}(Q_\tau)}d\tau \le c_\sigma\|v\|_\sigma
\end{aligned} \tag{22}$$

where $c_\sigma = C_2\int_0^T e^{-\sigma\tau}\|\bar{k}(\tau)\|_{L^\infty(\Omega)}d\tau$. Since $\bar{k}\in L^1(0,T;L^\infty(\Omega))$, there exists $\sigma_0>0$ such that $c_{\sigma_0}<\frac{1}{2}$. Consequently, $\mathcal{F}$ is a contraction and the equation (21), which is equivalent to (13), has a unique solution in $W_p^{2,1}(Q)$. Moreover, from (21) and (22) we deduce $\|v\|_{\sigma_0}\le\|\mathcal{F}v\|_{\sigma_0}+\|\mathcal{F}\widehat{u}\|_{\sigma_0}\le\frac{1}{2}(\|v\|_{\sigma_0}+\|\widehat{u}\|_{\sigma_0})$. This implies $\|v\|_{\sigma_0}\le\|\widehat{u}\|_{\sigma_0}$. Since $\|\widehat{u}\|_{W_p^{2,1}(Q)}$ is bounded by the right-hand side of (17)

(see Theorem 5.4 in [14]) and $\|.\|_\sigma$ is equivalent to the usual norm in $W_p^{2,1}(Q)$, we obtain the estimate (17).

To prove (ii), we need the relation

$$\|k*v\|_{C^{l,\frac{l}{2}}(Q_t)} \le C_3\left[\int_0^t\left\{\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\|v\|_{C^{l,\frac{l}{2}}(Q_\tau)}\right\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}}, \tag{23}$$
$$t\in(0,T),\ v\in C^{l,\frac{l}{2}}(Q)$$

that holds with some constant C_3. It follows by splitting $\|k*v\|_{C^{l,\frac{l}{2}}(Q_t)} = N_1+N_2$, where

$$\begin{aligned}
N_1 &= \sup_{(x,s)\in Q_t}|(k*v)(x,s)| + \sup_{(x_1,s),(x_2,s)\in Q_t}|\langle k*v\rangle_l(x_1,x_1;s)|\\
&\le \sup_{s\in(0,t)}\int_0^s\sup_{x\in\Omega}|k(x,s-\tau)|\sup_{(x,\eta)\in Q_\tau}|v(x,\eta)|d\tau\\
&+ \sup_{s\in(0,t)}\Big[\int_0^s\sup_{x\in\Omega}|k(x,s-\tau)|\sup_{(x_1,\eta),(x_2,\eta)\in Q_\tau}|\langle v\rangle_l(x_1,x_1;\eta)|d\tau\\
&\qquad + \int_0^s\sup_{x_1,x_2\in\Omega}|\langle k\rangle_l(x_1,x_2;s-\tau)|\sup_{(x,\eta)\in Q_\tau}|v(x,\eta)|d\tau\Big]\\
&\le \int_0^t\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\|v\|_{C^{l,\frac{l}{2}}(Q_\tau)}d\tau\\
&\le t^{\frac{l}{2}}\left[\int_0^t\left\{\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\|v\|_{C^{l,\frac{l}{2}}(Q_\tau)}\right\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}},
\end{aligned}$$

$$\begin{aligned}
N_2 &= \sup_{(x,s_1),(x,s_2)\in Q_t}|\langle k*v\rangle_{\frac{l}{2}}(x;s_1,s_2)|\\
&\le \sup_{\substack{0<s_2<s_1<t,\\ x\in\Omega}}\frac{1}{|s_1-s_2|^{\frac{l}{2}}}\int_{s_2}^{s_1}|k(x,s_1-\tau)||v(x,\tau)|d\tau\\
&+ \sup_{\substack{0<s_2<s_1<t,\\ x\in\Omega}}\frac{1}{|s_1-s_2|^{\frac{l}{2}}}\int_{s_1-s_2}^{s_1}|k(x,s_1-\tau)||v(x,\tau)-v(x,s_2-s_1+\tau)|d\tau\\
&\le \sup_{\substack{0<s_2<s_1<t,\\ x\in\Omega}}\left[\int_{s_2}^{s_1}\{|k(x,s_1-\tau)||v(x,\tau)|\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}}\\
&+ \sup_{0<s_2<s_1<t}\int_{s_1-s_2}^{s_1}\sup_{x\in\Omega}|k(x,s_1-\tau)|\sup_{(x,\eta_1),(x,\eta_2)\in Q_\tau}|\langle v\rangle_{\frac{l}{2}}(x;\eta_1,\eta_2)|d\tau\\
&\le C_4\left[\int_0^t\left\{\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\|v\|_{C^{l,\frac{l}{2}}(Q_\tau)}\right\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}}
\end{aligned}$$

with some constant C_4. In these estimations we used the Hölder inequality and the fact that the convolution of a nonnegative and a nonnegative nondecreasing function is nondecreasing. By Theorem 4.9 in[14] under the assumptions (ii) we have $\widehat{u}\in C^{l+2,\frac{l}{2}+1}(Q)$, $F\in\mathscr{L}(C^{l,\frac{l}{2}}(Q),C^{l+2,\frac{l}{2}+1}(Q))$, $\mathcal{F}\in\mathscr{L}(C^{l+2,\frac{l}{2}+1}(Q))$ and the problem (13) for $u\in C^{l+2,\frac{l}{2}+1}(Q)$ in case $k\ne 0$ is equivalent to the fixed-point equation (21) for $v=u-\widehat{u}\in C^{l+2,\frac{l}{2}+1}(Q)$. Let $t\in(0,T)$ and define the operator $\widetilde{P}_t$ such that $\widetilde{P}_tv=v$ in Q_t and $\widetilde{P}_tv(x,\tau)=v(x,t)$ for $\tau\in[t,T]$. Then, $\|\widetilde{P}_tv\|_{C^{l,\frac{l}{2}}(Q)}=\|v\|_{C^{l,\frac{l}{2}}(Q_t)}$ and $Fv=F\widetilde{P}_tv$ in Q_t. Thus, using (23) we obtain

$$\|\mathcal{F}v\|_{C^{l+2,\frac{l}{2}+1}(Q_t)} = \|F\widetilde{P}_t(\beta(k*v_t))\|_{C^{l+2,\frac{l}{2}+1}(Q_t)} \le \|F\widetilde{P}_t(\beta(k*v_t))\|_{C^{l+2,\frac{l}{2}+1}(Q)}$$

$$\le \|F\|\,\|\widetilde{P}_t(\beta(k*v_t))\|_{C^{l,\frac{l}{2}}(Q)} = \|F\|\,\|\beta(k*v_t)\|_{C^{l,\frac{l}{2}}(Q_t)}$$
$$\le C_5\left[\int_0^t\Big\{\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\,\|v\|_{C^{2+l,\frac{l}{2}+1}(Q_\tau)}\Big\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}}.$$

Using the weighted norms $\|w\|_\sigma = \sup\limits_{0<t<T} e^{-\sigma t}\|w\|_{C^{2+l,\frac{l}{2}+1}(Q_t)}$, $\sigma>0$, in the space $C^{l+2,\frac{l}{2}+1}(Q)$, we deduce

$$\|\mathcal{F}v\|_\sigma \le C_5\sup_{0<t<T}\left[\int_0^t\Big\{e^{-\sigma(t-\tau)}\|k(\cdot,t-\tau)\|_{C^l(\Omega)}e^{-\sigma\tau}\|v\|_{C^{2+l,\frac{l}{2}+1}(Q_\tau)}\Big\}^{\frac{2}{2-l}}d\tau\right]^{\frac{2-l}{2}}$$
$$\le C_5\left[\int_0^T\big\{e^{-\sigma\tau}\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\big\}^{\frac{2}{2-l}}\,d\tau\right]^{\frac{2-l}{2}}\|v\|_\sigma.$$

Since $k\in L^{\frac{2}{2-l}}(0,T;C^l(\Omega))$, the coefficient $\left[\int_0^T\big\{e^{-\sigma\tau}\|k(\cdot,t-\tau)\|_{C^l(\Omega)}\big\}^{\frac{2}{2-l}}\,d\tau\right]^{\frac{2-l}{2}}$ is small for large σ. Owing to this, the proof can be finished as in case (i) making use of the fixed-point argument and an estimate for $\|\widehat{a}\|_{C^{2+l,\frac{l}{2}+1}(Q)}$ in Theorem 4.9 of [14]. □

Theorem 2. *Assume* (3), (4), $\beta, a_{ij}, a_j, a\in C(\overline{Q})$ *and*

$$k\in W_1^1(0,T;L^\infty(\Omega)),\ k\ge 0,\ k_t\le 0. \tag{24}$$

Let $u\in W_p^{2,1}(Q)$ *with some* $p\in(1,\infty)$ *solve the problem* (13) *and* $u_0\ge 0$, $g\ge 0$, $f\ge 0$. *Then the following assertions are valid:*

(i) $u\ge 0$;

(ii) *if, in addition,* $\beta, a_{ij}, a_j, a\in C^{l,\frac{l}{2}}(Q)$ *with some* $l\in(0,1)$ *and either* $f\ne 0$ *or* $g\ne 0$, *then* $u(\cdot,T)>0$ *in* Ω *in case* I *and* $u(\cdot,T)>0$ *in* $\overline{\Omega}$ *in case* II.

Proof. It consists of 4 steps.

1. step: proof the assertion (i) in case $u\in C^{2,1}(\overline{Q})$, $a\le 0$ and k satisfies the conditions

$$k\in W_1^1(0,T;C(\overline{\Omega})),\ k\ge 0,\ k_t\le 0. \tag{25}$$

Since $u=u_0\ge 0$ in $\Omega\times\{0\}$, there exists

$$t_0=\sup\{t:\ u(x,\tau)\ge 0\ \text{ for }\ (x,\tau)\in\Omega\times[0,t],\ 0\le t\le T\}.$$

In case the assertion $u\ge 0$ holds in Q, we have $t_0=T$. Suppose on the contrary that $t_0<T$. Then, we fix some $h\in(0,T-t_0]$ and define the sets $V_{t_0,h}=\overline{\Omega}\times(t_0,t_0+h]$ and $\overline{V_{t_0,h}}=\overline{\Omega}\times[t_0,t_0+h]$. By this definition, there exists $(x_h^*,t_h^*)\in V_{t_0,h}$ such that $u(x_h^*,t_h^*)<0$. Let

$$v(x,t)=u(x,t)+\mu_h(t-t_0-h)\,,\quad \mu_h=-\frac{u(x_h^*,t_h^*)}{2h}>0. \tag{26}$$

Then we have

$$u(x,t)-\mu_h h\le v(x,t)\le u(x,t)\ \text{ for }\ (x,t)\in\overline{V_{t_0,h}}. \tag{27}$$

Observing (27) and the inequality $u(x_h^*,t_h^*)<0$ we see that for all $(x,t)\in\overline{V_{t_0,h}}$ such that $u(x,t)\ge 0$, the relations

$$v(x,t)\ge u(x,t)-\mu_h h\ge -\mu_h h=\frac{u(x_h^*,t_h^*)}{2}\ >\ u(x_h^*,t_h^*)\ge v(x_h^*,t_h^*)$$

are valid. They imply that

$$\text{function } v \text{ cannot attain its minimum over } \overline{V_{t_0,h}} \quad \text{in a point } (x,t) \text{ where } u(x,t) \geq 0. \tag{28}$$

In particular, (28) implies that v cannot attain its minimum over $\overline{V_{t_0,h}}$ on the subset $\overline{V_{t_0,h}} \setminus V_{t_0,h} = \overline{\Omega} \times \{t_0\}$, because there $u \geq 0$ in view of the definition of t_0. Therefore,

$$\exists (x_h, t_h) \in V_{t_0,h} \ : \quad v(x_h,t_h) \leq v(x,t) \quad \text{for all} \quad (x,t) \in \overline{V_{t_0,h}}\,.$$

Moreover, $v(x_h,t_h) < 0$, because $v(x_h,t_h) \leq v(x_h^*,t_h^*) \leq u(x_h^*,t_h^*)$ and $u(x_h^*,t_h^*) < 0$.

Let us show that $x = x_h$ is the stationary minimum point of $v(x,t_h)$, i.e.

$$\nabla_x\, v(x_h,t_h) = 0\,. \tag{29}$$

This relation may fail only in case the minimum occurs in the lateral boundary of $V_{t_0,h}$, i.e. when $x_h \in \Gamma$. In case I we have $u = g \geq 0$ for $x \in \Gamma$ and, by statement (28), x_h cannot belong to Γ. Thus, it remains to show (29) for the case II when $x_h \in \Gamma$. In this case $\omega \cdot \nabla_x\, v = \omega \cdot \nabla_x\, u = g$ in Γ. Note that then the inequality $g(x_h,t_h) > 0$ cannot hold, because otherwise v is strictly decreasing in the inner direction $-\omega(x_h)$ at (x_h,t_h) which implies that $v(x_h,t_h)$ cannot be the minimum of v. Consequently, due to the assumption $g \geq 0$, it holds $g(x_h,t_h) = 0$ and we have $\omega(x_h) \cdot \nabla_x\, v(x_h,t_h) = 0$. In addition, we also have $\tau \cdot \nabla_x\, v(x_h,t_h) = 0$, where τ is an arbitrary tangential direction of Γ at x_h, because $x = x_h$ is the minimum point of the x-dependent function $v(x,t_h)$ over the set Γ. These relations yield $\xi \cdot \nabla_x\, v(x_h,t_h) = 0$, where ξ is any direction, hence (29).

Now we are going to estimate the operator $Lu = \beta(u_t + k*u_t) - Au$ of the equation (13) termwise at $(x,t) = (x_h,t_h)$. By (26) we have $u_t(x_h,t_h) = v_t(x_h,t_h) - \mu_h$. Since t_h is the minimum point of the $t-$dependent function $v(x_h,t)$ in the half-interval $(t_0, t_0+h]$, it holds $v_t(x_h,t_h) \leq 0$. Thus, we obtain

$$u_t(x_h,t_h) \leq -\mu_h. \tag{30}$$

By (2) and (26) we have $Au = \sum\limits_{i,j=1}^{N} a_{ij} v_{x_i x_j} + \sum\limits_{j=1}^{n} a_j v_{x_j} + a[v - \mu_h(t - t_0 - h)]$. Since $x = x_h$ is the stationary minimum point of $v(x,t_h)$ and the principal part of A is elliptic (see (3)), it holds $\sum\limits_{i,j=1}^{n} a_{ij}(x_h,t_h) v_{x_i x_j}(x_h,t_h) \geq 0$ and $\sum\limits_{j=1}^{n} a_j(x_h,t_h) v_{x_j}(x_h,t_h) = 0$. Moreover, in view of $a \leq 0$ and $v(x_h,t_h) < 0$ we have $a(x_h,t_h) v(x_h,t_h) \geq 0$. By means of these relations and $t_h \in (t_0, t_0+h]$ we deduce the estimate

$$-Au(x_h,t_h) \leq a(x_h,t_h)\mu_h(t_h - t_0 - h) \leq C_6 \mu_h h \tag{31}$$

where $C_6 = \|a\|_{C(\overline{Q})}$.

It remains to deal with the term $(k * u_t)(x_h,t_h)$ in $Lu(x_h,t_h)$. Integrating by parts we have

$$\int_0^{t_h} k(x_h,t_h-\tau) u_\tau(x_h,\tau) d\tau = k(x_h,0)u(x_h,t_h) - k(x_h,t_h)u_0(x_h) + \int_0^{t_0} k_t(x_h,t_h-\tau)u(x_h,\tau)d\tau + \int_{t_0}^{t_h} k_t(x_h,t_h-\tau)u(x_h,\tau)d\tau.$$

In this relation $-k(x_h,t_h)u_0(x_h) \leq 0$ and $\int_0^{t_0} k_t(x_h,t_h-\tau)u(x_h,\tau)d\tau \leq 0$, because $k \geq 0$, $k_t \leq 0$, $u_0 \geq 0$ and $u(x_h,\tau) \geq 0$ for $\tau \in [0,t_0]$ by the definition of t_0.

Consequently,

$$\int_0^{t_h} k(x_h,t_h-\tau)u_\tau(x_h,\tau)d\tau \le k(x_h,0)u(x_h,t_h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)u(x_h,\tau)d\tau.$$

Substituting here $u(x_h,t)$ by $v(x_h,t)-\mu_h(t-t_0-h)$ (see (26)), we have

$$\begin{aligned}(32)\quad &\int_0^{t_h} k(x_h,t_h-\tau)u_\tau(x_h,\tau)d\tau\\ &\le k(x_h,0)v(x_h,t_h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau-\\ &-\mu_h\left[k(x_h,0)(t_h-t_0-h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right].\end{aligned}$$

In this estimate we analyze separately the term $\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau$. To this end, define the following subsets of $[t_0,t_h]$:

$$U_h^+=\{\tau:\ \tau\in[t_0,t_h],\ \ v(x_h,\tau)\ge 0\},\ \ U_h^-=\{\tau:\ \tau\in[t_0,t_h],\ \ v(x_h,\tau)<0\}.$$

Taking account of $k_t\le 0$ and the fact that $v(x_h,t_h)<0$ is the minimum of $v(x_h,t)$ on the interval $[t_0,t_h]$, we deduce

$$\begin{aligned}&\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\\ &\quad=\int_{U_h^+} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau+\int_{U_h^-} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\\ &\quad\le\int_{U_h^-} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\le\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)d\tau\cdot v(x_h,t_h)\\ &\quad=\left(k(x_h,t_h-t_0)-k(x_h,0)\right)v(x_h,t_h).\end{aligned}$$

Thus, due to $k\ge 0$ we get

$$k(x_h,0)v(x_h,t_h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)v(x_h,\tau)d\tau\le k(x_h,t_h-t_0)v(x_h,t_h)\le 0$$

and from (32) we finally obtain

$$\begin{aligned}(33)\quad &\int_0^{t_h} k(x_h,t_h-\tau)u_\tau(x_h,\tau)d\tau\\ &\le-\mu_h\left[k(x_h,0)(t_h-t_0-h)+\int_{t_0}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right].\end{aligned}$$

Let's return to the operator L. Making use of (30), (31), (33), the assumption (4) and the relation $0<t_h-t_0\le h$ we obtain

$$\begin{aligned}&Lu(x_h,t_h)\le\mu_h\Bigg\{-\beta(x_h)\\ &-\beta(x_h)\left[k(x_h,0)(t_h-t_0-h)+\int_{t_{0,h}}^{t_h} k_t(x_h,t_h-\tau)(\tau-t_0-h)d\tau\right]+C_6h\Bigg\}\\ &\le\ \mu_h\left\{-\beta_0+C_7h\left(k(x_h,0)+\|k_t\|_{L^1(0,T;C(\overline{\Omega}))}+C_6\right)\right\}.\end{aligned}$$

In case $h>0$ is sufficiently small, due to the inequalities $\mu_h>0$ and $\beta_0>0$ the relation $Lu(x_h,t_h)<0$ holds. But this contradicts to the assumption $Lu=f\ge 0$.

Consequently, the supposition $t_0 < T$ was not right. We have $t_0 = T$, which by the definition of t_0 implies $u \geq 0$.

2. step: proof of the assertion (i) in case $u \in C^{2,1}(\overline{Q})$, $u_0 = 0$ and k satisfies (25).

Let us define $\widetilde{u} = e^{-\sigma t}u$, $\widetilde{k}(x,t) = e^{-\sigma t}k(x,t) + \sigma \int_0^t e^{-\sigma\eta}k(x,\eta)d\eta$, $\widetilde{f} = e^{-\sigma t}f$, $\widetilde{g} = e^{-\sigma t}g$, where $\sigma = \frac{\|a\|_{C(\overline{Q})}}{\beta_0}$. Then $\widetilde{u}$ solves the problem

$$\beta(\widetilde{u}_t + \widetilde{k} * \widetilde{u}_t) = \widetilde{A}\widetilde{u} + \widetilde{f} \ \text{ in } Q\,, \ \ \widetilde{u} = u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1\widetilde{u} = \widetilde{g} \ \text{ in } S,$$

where $\widetilde{A}\widetilde{u} = \sum\limits_{i,j=1}^{n} a_{ij}\widetilde{u}_{x_ix_j} + \sum\limits_{j=1}^{n} a_j\widetilde{u}_{x_j} + \widetilde{a}\widetilde{u}$ and

$$\widetilde{a}(x,t) = a(x,t) - \sigma\beta(x,t)\left[1 + \int_0^t e^{-\sigma\eta}k(x,\eta)d\eta\right] \leq 0.$$

The kernel $\widetilde{k}$ satisfies (25) and $\widetilde{g} \geq 0$, $\widetilde{f} \geq 0$. Using part 1 of the proof, we get $\widetilde{u} \geq 0$. This implies $u \geq 0$.

3. step: proof of assertion (i) in the general case.

It is enough to prove this assertion for $p \in (1, \frac{3}{2})$. Let us consider the problems

$$\beta\bar{u}_t = (A - k(\cdot,0)\beta)\bar{u} \ \text{ in } Q\,, \ \ \bar{u} = u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1\bar{u} = g \ \text{ in } S, \tag{34}$$

$$\beta(\tilde{u}_t + k * \tilde{u}_t) = A\tilde{u} + f_{\bar{u}} \ \text{ in } Q\,, \ \ \tilde{u} = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1\tilde{u} = 0 \ \text{ in } S, \tag{35}$$

where $f_{\bar{u}} = f - \beta(k_t * \bar{u}) + \beta k(\cdot,0)u_0$. Since $u \in W_p^{2,1}(Q)$, by embedding theorems [10] we have $u_0 = u|_{\Omega\times\{0\}} \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g = B_1u|_S \in W_p^{2-\frac{1}{p}-\nu,1-\frac{1}{2p}-\frac{\nu}{2}}(S)$. Thus, by Theorem 1 (i), problem (34) has a unique solution $\bar{u} \in W_p^{2,1}(Q)$. Further, due to the assumptions of the theorem and $\bar{u} \in W_p^{2,1}(Q)$, it holds $f_{\bar{u}} \in L^p(Q)$ and by Theorem 1 (i), again, problem (35) has a unique solution $\tilde{u} \in W_p^{2,1}(Q)$. Adding (34) and (35) and integrating by parts in the convolution term in $f_{\bar{u}}$, we see that $\bar{u} + \tilde{u}$ solves (13), hence by the uniqueness, $u = \bar{u} + \tilde{u}$.

By the well-known extremum principle for weak solutions of parabolic equations (e.g. [10] Ch. 3, Theorem 7.2) we obtain $\bar{u} \geq 0$. This together with the assumptions of theorem implies $f_{\bar{u}} \geq 0$.

Further, let us choose some functions $\beta^m, a_{ij}^m, a_j^m, a^m, \hat{f}^m, \hat{k}^m \in C^\infty(\overline{Q})$ such that

$$\begin{aligned} &\|\beta^m - \beta\|_{C(\overline{Q})},\ \|a_{ij}^m - a_{ij}\|_{C(\overline{Q})},\ \|a_j^m - a_j\|_{C(\overline{Q})},\ \|a^m - a\|_{C(\overline{Q})},\\ &\|\hat{f}^m - f_{\bar{u}}\|_{L^p(Q)},\ \|\hat{k}^m - k\|_{W_1^1(0,T;L^\infty(\Omega))} \ \to\ 0 \quad \text{as} \quad m \to \infty \end{aligned} \tag{36}$$

and $\hat{f}^m = 0$ in $\Gamma \times \{0\}$. Then, by virtue of (3) and (4), for sufficiently large m there hold the relations

$$\sum_{i,j=1}^{n} a_{ij}^m\lambda_i\lambda_j \geq \frac{\epsilon}{2}|\lambda|^2 \quad \text{for any } \ \lambda \in \mathbb{R}^n \ \text{ and } \ \beta^m \geq \frac{\beta_0}{2} \ \text{ in } \overline{Q}.$$

In addition, we define $f^m(x,t) = \begin{cases} \hat{f}^m(x,t) & \text{if } \hat{f}^m(x,t) \geq 0 \\ 0 & \text{if } \hat{f}^m(x,t) < 0 \end{cases}$. Then $f^m \in C^{l,\frac{l}{2}}(Q)$ for any $l \in (0,1)$ and $f^m = 0$ in $\Gamma \times \{0\}$. Moreover, due to the inequality $f_{\bar{u}} \geq 0$ it holds $|f^m - f_{\bar{u}}| \leq |\hat{f}^m - f_{\bar{u}}|$ in Q. Therefore, using (36) we obtain

$$\|f^m - f_{\bar{u}}\|_{L^p(Q)} \ \leq\ \|\hat{f}^m - f_{\bar{u}}\|_{L^p(Q)} \ \to\ 0 \quad \text{as} \quad m \to \infty. \tag{37}$$

Finally, we define $k^m(x,t) = \int_T^t \kappa^m(x,\tau)d\tau + q^m(x)$ with

$$\kappa^m(x,t) = \begin{cases} \hat{k}_t^m(x,t) & \text{if } \hat{k}_t^m(x,t) \le 0 \\ 0 & \text{if } \hat{k}_t^m(x,t) > 0 \end{cases}, \quad q^m(x) = \begin{cases} \hat{k}^m(x,T) & \text{if } \hat{k}^m(x,T) \ge 0 \\ 0 & \text{if } \hat{k}^m(x,T) < 0 \end{cases}.$$

Then $k^m \in W_1^1(0,T;C^l(\Omega))$ for any $l \in (0,1)$. Moreover, since $k \ge 0$ and $k_t \le 0$, we obtain $|\kappa^m - k_t| \le |\hat{k}_t^m - k_t|$ in Q and $|q^m - k(\cdot,T)| \le |\hat{k}^m(\cdot,T) - k(\cdot,T)|$ in Ω. Observing these relations and (36) we deduce

$$\begin{aligned} (38)\ & \|k^m - k\|_{L^1(0,T;L^\infty(\Omega))} \\ & \le \int_0^T \operatorname*{ess\,sup}_{x\in\Omega} \left[\int_t^T |\kappa^m(x,\tau) - k_\tau(x,\tau)|d\tau + |q^m(x) - k(x,T)| \right] dt \\ & \le T \left[\|\hat{k}_t^m - k_t\|_{L^1(0,T;L^\infty(\Omega))} + \|\hat{k}^m(\cdot,T) - k(\cdot,T)\|_{L^\infty(\Omega)} \right] \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

Now let us consider the following approximating problems of (35):

$$\begin{aligned} \beta^m(u_t^m + k^m * u_t^m) &= A^m u^m + f^m \ \text{ in } Q, \\ u^m &= 0 \ \text{ in } \Omega \times \{0\}, \ \ B_1 u^m = 0 \ \text{ in } S, \end{aligned} \tag{39}$$

where $A^m v = \sum_{i,j=1}^n a_{ij}^m v_{x_i x_j} + \sum_{j=1}^n a_j^m v_{x_j} + a^m v$. Observing the proved regularity of the data of these problems and Theorem 1 (ii) we conclude that for each integer m, (39) has the unique solution $u^m \in C^{2+l,1+\frac{l}{2}}(Q)$. Subtracting (35) from (39) we obtain the following problems for the differences $v^m = u^m - \tilde{u}$:

$$\beta(v_t^m + k * v_t^m) = Av^m + \phi^m \ \text{ in } Q, \ \ v^m = 0 \ \text{ in } \Omega \times \{0\}, \ \ B_1 v^m = 0 \ \text{ in } S$$

where

$$\begin{aligned} \phi^m = (A^m - A)(v^m + \tilde{u}) - (\beta^m - \beta)[v_t^m + \tilde{u}_t + k * (v_t^m + \tilde{u}_t)] \\ -(\beta + \beta^m - \beta)[(k^m - k) * (v_t^m + \tilde{u}_t)] + f^m - f_{\bar{u}}. \end{aligned}$$

Using Theorem 1 (i) for these problems we deduce

$$\begin{aligned} \|v^m\|_{W_p^{2,1}(Q)} \le\ & C_1 \|\phi^m\|_{L^p(Q)} \le C_1 \|f^m - f_{\bar{u}}\|_{L^p(Q)} \\ & + \widehat{C}_1 \Big[\|a_{ij}^m - a_{ij}\|_{C(\overline{Q})} + \|a_j^m - a_j\|_{C(\overline{Q})} + \|a^m - a\|_{C(\overline{Q})} + \|\beta^m - \beta\|_{C(\overline{Q})} \\ & + (1 + \|\beta^m - \beta\|_{C(\overline{Q})}) \|k^m - k\|_{L^1(0,T;L^\infty(\Omega))} \Big] (\|v^m\|_{W_p^{2,1}(Q)} + \|\tilde{u}\|_{W_p^{2,1}(Q)}) \end{aligned}$$

with some constant $\widehat{C}_1$. Taking here (36) - (38) into account, we get

$$\|v^m\|_{W_p^{2,1}(Q)} = \|u^m - \tilde{u}\|_{W_p^{2,1}(Q)} \to 0 \quad \text{as} \quad m \to \infty. \tag{40}$$

Note that $f^m \ge 0$, $k^m \ge 0$ and $k_t^m \le 0$, by the definitions. Thus, since $u^m = 0$ on $\Omega \times \{0\}$, using step 2 of the proof for the solution of the problem (39), we get $u^m \ge 0$. This with (40) implies $\tilde{u} \ge 0$. Since $\bar{u} \ge 0$, we obtain $u = \bar{u} + \tilde{u} \ge 0$. The assertion (i) is proved.

4. step: proof of assertion (ii).

Firstly, let us consider the case $f \ne 0$. Then, due to (11) and (10), there exists an open ball $U \subseteq Q$ and $\varepsilon > 0$ such that $f \ge \varepsilon$ in U, hence we can choose some

$f^\dagger \in C^\infty(\overline{Q})$ so that $f^\dagger = 0$ in $Q \setminus U$ and $0 < f^\dagger \le \frac{\varepsilon}{2}$ in U. Further, let us define $c_k = \operatorname*{ess\,sup}_{x\in\Omega} k(x,0)$ and consider the following problems:

$$\beta u_t^\dagger = (A - c_k\beta)u^\dagger + f^\dagger \ \text{ in } Q\,,\quad u^\dagger = 0 \ \text{ in } \Omega\times\{0\}\,,\quad B_1 u^\dagger = 0 \ \text{ in } S, \tag{41}$$

$$\beta(\hat u_t + k * \hat u_t) = A\hat u + \hat f_{u^\dagger} \ \text{ in } Q\,,\quad \hat u = u_0 \ \text{ in } \Omega\times\{0\}\,,\quad B_1\hat u = g \ \text{ in } S, \tag{42}$$

where $\hat f_{u^\dagger} = f - f^\dagger - \beta(k_t * u^\dagger) + \beta(c_k - k(\cdot,0))u^\dagger$. By Theorem 1 (ii), problem (41) has a unique solution $u^\dagger \in C^{2+l,1+\frac{l}{2}}(Q)$. Observing that $f^\dagger \ge 0$ and $f^\dagger \ne 0$ and using the well-known strong extremum principles for parabolic equations (see e.g. Theorem 6.1.1 (ii) in [8]), we deduce the relation

$$u^\dagger(\cdot,T) > 0 \ \text{ in } \Omega\ (\overline{\Omega}) \text{ in case I (II)}. \tag{43}$$

Since $\hat f_{u^\dagger} \in L^p(Q)$, $u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g \in W_p^{2-\frac{1}{p}-\nu,1-\frac{1}{2p}-\frac{\nu}{2}}(S)$, by Theorem 1 (i) the problem (42) has a unique solution $\hat u \in W_p^{2,1}(Q)$. Adding (41) and (42) we see that $u^\dagger + \hat u$ solves (13), hence $u = u^\dagger + \hat u$. Observing the assumptions of theorem and the relations $f - f^\dagger \ge 0$ and $c_k - k(\cdot,0) \ge 0$, following from the definitions of $f^\dagger$ and c_k, we get $\hat f_{u^\dagger} \ge 0$. Since, in addition, $u_0 \ge 0$ and $g \ge 0$, assertion (i) implies $\hat u \ge 0$. Finally, using (43) we prove assertion (ii) for $u = u^\dagger + \hat u$.

Secondly, in case $g \ne 0$ we define a function $g^\dagger \in C^\infty(\overline{S})$ such that $g^\dagger \ge 0$, $g^\dagger \ne 0$ and $g^\dagger \le g$. Then we set $f^\dagger = 0$ and replace the boundary conditions of the problems (41) and (42) by $B_1 u^\dagger = g^\dagger$ and $B_1\hat u = g - g^\dagger$, respectively. The other arguments concerning the solutions of these systems are almost the same as in the case $f \ne 0$. The proof is complete. □

From now on, let the given functions β, a_{ij}, a_j and k have again the domains specified in Section 2, i.e. *β, a_{ij}, a_j depend only on x and k depends only on t.*

Remark 1. For k, depending only on t, the assumptions (24) read

$$k \in W_1^1(0,T)\,,\quad k \ge 0\,,\quad k' \le 0. \tag{44}$$

Let us deduce sufficient conditions for these assumptions in terms of the original kernel m. Provided $m \in W_1^1(0,T)$, the solution k of (12) belongs to $W_1^1(0,T)$ and has the following representation in the form of Neumann series:

$$k = \sum_{i=0}^{\infty} m\,(*\,m)^i\,,\quad k' = \sum_{i=0}^{\infty}[m(0)m + m'\,](*\,m)^i.$$

Thus, the sufficient conditions for (44) are

$$m \in W_1^1(0,T),\ \ m \ge 0\,,\quad m'(t) \le -m(0)m(t). \tag{45}$$

For instance, the widely used exponential kernels $m(t) = \sum\limits_{i=1}^{N} \alpha_i e^{-\beta_i t}$ satisfy the conditions (45) provided $\beta_i \ge \alpha_i \ge 0$.

Remark 2. Let us return to the direct problem for u in the original form (1), (5). Observing Lemma 1 and Theorem 2, we see that the assumptions for the concentration (or temperature) u to be nonnegative are $f = \chi + k * \chi \ge 0$, $u_0 \ge 0$ and $g = b \ge 0$ in case I and $g = b + k*b \ge 0$ in case II. In case of non-vanishing $k \ge 0$ the conditions $\chi + k * \chi \ge 0$ and $b + k * b \ge 0$ are weaker than the conditions $\chi \ge 0$ and $b \ge 0$ that occur in the case of vanishing k. (The latter case corresponds to the usual parabolic problem without the integral term.) Physically, this phenomenon

can be explained by a certain inertia of the medium with memory. For instance, choosing the following source term

$$\chi = 1 \ \text{ in } \ \Omega \times (0, T-\delta) \quad \text{and} \quad \chi = -\epsilon < 0 \ \text{ in } \ \Omega \times (T-\delta, T),$$

where the numbers $\epsilon > 0$ and $\delta > 0$ are sufficiently small, so that

$$\int_0^{T-\delta} k(t-\tau)d\tau \geq \epsilon \left(1 + \int_{T-\delta}^{t} k(t-\tau)d\tau\right) \quad \text{for any} \quad t \in (T-\delta, T)$$

we have $\chi + k * \chi \geq 0$ in Q that implies $u \geq 0$ in Q.

4. **Results for IP1.** We start by proving a technical result.

Lemma 2. *Let (3), (4) hold. Assume $\beta \in C^l(\Omega)$ with some $l \in (0,1)$, $a_{ij}, a_j \in C(\overline{\Omega})$, $a \in C(\overline{Q})$, $a_t \in L^p(Q)$, $k \in L^p(0,T)$ with some $p \in (1, \min\{\frac{3}{2}, \frac{2}{2-l}\})$ and the problem (13) has a solution $u \in W_p^{2,1}(Q)$ such that $B_1 u$ is continuous in a neighborhood of $\Gamma \times \{0\}$. If $f_t \in L^p(Q)$, $u_0 \in W_p^2(\Omega)$, $A(0)u_0 + f(\cdot,0) \in W_p^{2-\frac{2}{p}}(\Omega)$ and $g_t \in W_p^{2-\frac{1}{p}-\nu, 1-\frac{1}{2p}-\frac{\nu}{2}}(S)$ then $u_t \in W_p^{2,1}(Q)$.*

Proof. Let us consider the problem

$$\begin{aligned} &\beta(v_t + k * v_t) = Av + f_t - k(A(0)u_0 + f(\cdot,0)) + a_t u \ \text{ in } Q\,, \\ &v = \frac{1}{\beta}(A(0)u_0 + f(\cdot,0)) \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 v = g_t \ \text{ in } S. \end{aligned} \tag{46}$$

Due to the assumptions imposed on β, $A(0)u_0 + f(\cdot,0)$ and the relation $p < \frac{2}{2-l}$ we have $\frac{1}{\beta}(A(0)u_0 + f(\cdot,0)) \in W_p^{2-\frac{2}{p}}(\Omega)$. Moreover, $f_t - k(A(0)u_0 + f(\cdot,0)) + a_t u \in L^p(Q)$. Therefore, by Theorem 1 (i), the problem (46) has a unique solution v in $W_p^{2,1}(Q)$. Let us define $w(x,t) = \int_0^t v(x,\tau)d\tau + u_0(x) \in W_p^{2,1}(Q)$ and integrate the integro-differential equation and boundary condition in (46) with respect time from 0 to t. Taking into account the initial values of v and w and the consistency condition $B_1 u_0 = g(\cdot,0)$, that holds in view of the continuity assumption of $B_1 u$, we reach the following relations:

$$\beta(w_t + k * w_t) = Aw + f + \hat{f} \ \text{ in } Q\,, \ \ w = u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 w = g \ \text{ in } S, \tag{47}$$

where $\hat{f}(x,t) = \int_0^t (a(x,\tau) - a(x,t))(u_\tau(x,\tau) - w_\tau(x,\tau))d\tau$. Subtracting (47) from (13) we obtain the problem for $\omega = u - w$

$$\beta(\omega_t + k * \omega_t) = A\omega + \hat{f} \ \text{ in } Q\,, \ \ \omega = 0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 \omega = 0 \ \text{ in } S.$$

Using the estimate (17) for the solution of this problem, we obtain $\|\omega\|_{W_p^{2,1}(Q_t)} \leq C_1 \|\hat{f}\|_{L^p(Q_t)}$ for any $t \in (0,T)$, where $Q_t = \Omega \times (0,t)$, as before. Further, observing the continuity of a and the inequality (19) with $k = 1$, we deduce

$$\begin{aligned} \|\hat{f}\|_{L^p(Q_t)} &\leq 2\|a\|_{C(\overline{Q})} \|1 * \omega_t\|_{L^p(Q_t)} \\ &\leq 2\|a\|_{C(\overline{Q})} \int_0^t \|\omega_t\|_{L^p(Q_\tau)} d\tau \leq 2\|a\|_{C(\overline{Q})} \int_0^t \|\omega\|_{W_p^{2,1}(Q_\tau)} d\tau \end{aligned}$$

for any $t \in (0,T)$. Thus, we obtain the following integral inequality:

$$\|\omega\|_{W_p^{2,1}(Q_t)} \leq \text{Const} \int_0^t \|\omega\|_{W_p^{2,1}(Q_\tau)} d\tau\,, \quad t \in (0,T).$$

By Gronwall's lemma, the solution $\|\omega\|_{W_p^{2,1}(Q_t)}$ of this inequality is zero. Consequently, $\omega = u - w = 0$. Since $v = w_t \in W_p^{2,1}(Q)$, we prove the assertion $u_t \in W_p^{2,1}(Q)$. □

Due to Lemma 1, IP1 is equivalent to the following problem for (z, u):

$$\beta(u_t + k * u_t) = Au + zr + f_0 \ \text{ in } Q\,, \quad u = u_0 \ \text{ in } \Omega \times \{0\}\,, \quad B_1 u = g \ \text{ in } S, \tag{48}$$

$$u = u_T \ \text{ in } \Omega \times \{T\}, \tag{49}$$

where B_1, g are given by (15), (16) and

$$r = \phi + k * \phi\,, \quad f_0 = \chi_0 + k * \chi_0. \tag{50}$$

Let us prove the following uniqueness result for the problem (48), (49):

Theorem 3. *Let* (3), (4), (44) *hold and* $\beta, a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ *with some* $l \in (0,1), p \in (1,\infty)$. *Moreover, let* $a_t \geq 0$ *in* Q, $r \in C^{l,\frac{l}{2}}(Q), r_t \in L^p(Q)$,

$$r \geq 0\,, \quad r_t + k * r_t - \theta r \geq 0 \quad \text{in } Q \tag{51}$$

and

$$\text{for any } U \subseteq \Omega, \operatorname{meas} U > 0, \ \text{ it holds } \ r_t + k * r_t - \theta r \neq 0 \ \text{ in } U \times (0,T). \tag{52}$$

Here

$$\theta = \sup_{x \in \Omega} \frac{a(x,T)}{\beta(x)}.$$

If $(z,u) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *solves* (48), (49) *and* $f_0, u_0, g, u_T = 0$ *then* $z = 0$, $u = 0$.

Proof. Note that (52) immediately implies that

$$\text{for any } U \subseteq \Omega, \operatorname{meas} U > 0, \ \text{ it holds } \ r \neq 0 \ \text{ in } U \times (0,T). \tag{53}$$

Suppose contrary that $z \neq 0$ and denote $z^+ = \frac{|z|+z}{2}$, $z^- = \frac{|z|-z}{2}$.

Firstly, we show that

$$z^+ \neq 0 \ \text{ and } \ z^- \neq 0. \tag{54}$$

Indeed, let $z^- = 0$. Then $zr = z^+ r \geq 0$ and due to $z \neq 0$ and (53) it holds $zr \neq 0$. Observing the assumptions $f_0, u_0, g = 0$ and applying Theorem 2 to the solution u of the problem (48) we get $u(x,T) > 0$, $x \in \Omega$. But this contradicts to the assumption $u_T = 0$. Similarly, we reach the contradiction in case $z^+ = 0$ making use of Theorem 2 for $-u$.

Further, let us consider the following problems for $u^\pm$:

$$\beta(u_t^\pm + k * u_t^\pm) = Au^\pm + z^\pm r \ \text{ in } Q\,, \quad u^\pm = 0 \ \text{ in } \Omega \times \{0\}\,, \quad B_1 u^\pm = 0 \ \text{ in } S. \tag{55}$$

By the assumptions of theorem, it holds $z^\pm r \in C^{l,\frac{l}{2}}(Q)$. Moreover, in case I the assumption $u \in C^{2+l,1+\frac{l}{2}}(Q)$ implies the consistency condition $zr = 0$ in $\Gamma \times \{0\}$ for the right-hand side of (48). This yields $z^\pm r = 0$ in $\Gamma \times \{0\}$, too. Consequently, assumptions of Theorem 1 (ii) are satisfied for problems (55). Hence, they have the unique solutions $u^\pm \in C^{2+l,1+\frac{l}{2}}(Q)$. Let us prove the following inequalities:

$$u^\pm \geq 0\,, \quad u^\pm(\cdot,T) > 0 \ \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II)}, \tag{56}$$

$$u_t^\pm + k * u_t^\pm - \theta u^\pm \geq 0\,, \quad (u_t^\pm + k * u_t^\pm - \theta u^\pm)(\cdot,T) > 0 \ \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II)}. \tag{57}$$

By (51) and $z^{\pm} \geq 0$ we have $z^{\pm} r \geq 0$. Moreover, by (53) and (54) we get $z^{\pm} r \neq 0$. Using Theorem 2 for solutions of problems (55) we immediately obtain (56). Let us prove (57). Assume without restriction of generality that $p \in (1, \min\{\frac{3}{2}; \frac{2}{2-l}\})$. Then the assumptions of Lemma 2 are satisfied for the solutions $u^{\pm}$ of the problems (55). Thus, we obtain $u_t^{\pm} \in W_p^{2,1}(Q)$. From (55) we deduce the following problems for the functions $v^{\pm} = u_t^{\pm} + k * u_t^{\pm} - \theta u^{\pm} \in W_p^{2,1}(Q) \cap C^{l,\frac{l}{2}}(Q)$:

$$
\begin{aligned}
&\beta(v_t^{\pm} + k * v_t^{\pm}) = A v^{\pm} + z^{\pm}[r_t + k * r_t - \theta r] + f_1^{\pm} \ \text{ in } Q\,,\\
&v^{\pm} = \frac{1}{\beta} z^{\pm} r \ \text{ in } \Omega \times \{0\}\,, \quad B_1 v^{\pm} = 0 \ \text{ in } S,
\end{aligned}
\tag{58}
$$

where

$$
f_1^{\pm}(x,t) = a_t(x,t) u^{\pm}(x,t) + \int_0^t k'(t-\tau)(a(x,\tau) - a(x,t)) u^{\pm}(x,\tau) d\tau.
$$

By the assumptions of theorem and (56), (54) we have $z^{\pm}[r_t + k * r_t - \theta r] + f_1^{\pm} \geq 0$ and $z^{\pm}[r_t + k * r_t - \theta r] + f_1^{\pm} \neq 0$. Consequently, Theorem 2 implies (57).

Let $x^* \in \overline{\Omega}$ be such that

$$
u^+(x,T) \leq u^+(x^*,T) \ \text{ for any } \ x \in \overline{\Omega}. \tag{59}
$$

Since $u = u^+ - u^-$ and $u(x,T) = u_T(x) \equiv 0$, we have $u^+(x,T) = u^-(x,T)$, $x \in \overline{\Omega}$, and (59) implies

$$
u^-(x,T) \leq u^-(x^*,T) \ \text{ for any } \ x \in \overline{\Omega}. \tag{60}
$$

Let us show that the point x^* is the stationary maximum of $u^{\pm}(x,T)$, i.e.

$$
\nabla_x u^{\pm}(x^*,T) = 0. \tag{61}
$$

The equality (61) may fail only when $x^* \in \Gamma$. In case I we have the boundary condition $u = 0$ in Γ, hence in view of (56) the function $u^+(x,T)$ cannot achieve its maximum on Γ, and we automatically get (61). It remains to show (61) for the case II when $x^* \in \Gamma$. In this case $\omega(x^*) \cdot \nabla_x u^+(x^*,T) = 0$, where $\omega(x^*)$ is an outer direction at x^*. Furthermore, since $u^*(x,T)$ achieves its maximum over Γ in the point $x = x^*$, we have $\tau \cdot \nabla_x u^+(x^*,T) = 0$, where τ is any tangential direction at x^*. Summing up, we get $\xi \cdot \nabla_x u^+(x^*,T) = 0$, where ξ is any direction. This yields (61).

By the definition of z^+ and z^-, it holds either $z^+(x^*) = 0$ or $z^-(x^*) = 0$. In case $z^+(x^*) = 0$ we have $(z^+ r)(x^*,T) = 0$ and from the equation (55) we obtain

$$
\left[\beta(u_t^+ + k * u_t^+) - a u^+\right](x^*,T) = A_0 u^+(x^*,T)\,, \quad A_0 = A - a. \tag{62}
$$

The left-hand side of (62) is strictly positive due to the inequalities (4), (56) and (57). Indeed:

$$
\begin{aligned}
\left[\beta(u_t^+ + k * u_t^+) - a u^+\right](x^*,T) &= \beta(x^*)\left[u_t^+ + k * u_t^+ - \frac{a}{\beta} u^+\right](x^*,T)\\
&\geq \beta_0 \left[u_t^+ + k * u_t^+ - \theta u^+\right](x^*,T) > 0.
\end{aligned}
$$

Therefore,

$$
A u^+(x^*,T) > 0. \tag{63}
$$

On the other hand, since $x = x^*$ is the stationary maximum point of $u^+(x,T)$ and the principal part of A_0 is elliptic (see (3)), we obtain

$$A_0 u^+(x^*,T) = \sum_{i,j=1}^{n} a_{ij}(x^*) u^+_{x_i x_j}(x^*,T) + \sum_{j=1}^{n} a_j(x^*) u^+_{x_j}(x^*,T) \le 0.$$

This contradicts (63). Analogously we come to a contradiction in case $z^-(x^*) = 0$. Hence, the assumption $z \neq 0$ was uncorrect. We have $z = 0$. Finally, whereas $f_0, u_0, g = 0$ by assumption, problem (48) is homogeneous. Thus, by Theorem 1 it holds $u = 0$. Proof is complete. □

Remark 3. We note that in case k is non-vanishing and satisfies (44) the conditions (51) with (50) for the coefficient ϕ are weaker than in case $k = 0$ when the usual parabolic problem occurs.

To deal with the existence and stability issues, we have to impose additional assumptions on r:

$$\begin{aligned} &r \ge \delta \ \text{ in } \overline{\Omega} \times (T-\delta, T) \quad \text{with some } \delta \in (0, \tfrac{T}{2}) \text{ and} \\ &\text{either } \ r \ge \delta \ \text{ in } \overline{\Omega} \times (0,\delta) \ \ (\textit{case } (1)) \ \text{ or } \ r = 0 \ \text{ in } \overline{\Omega} \times (0,\delta) \ \ (\textit{case } (2)). \end{aligned} \tag{64}$$

Note that in case I & (1) the following explicit relation for boundary values of z from (48) can be obtained:

$$z \ = \ r^{-1}(g_t - Au_0 - f) \quad \text{in } \Gamma \times \{0\}. \tag{65}$$

In this case it is possible to reduce the inverse problem to the case when $z = 0$ in Γ. Indeed, let us represent the term zr in (48) in the form $zr = (z - \tilde{z})r + \tilde{z}r$, where $\tilde{z}$ is an arbitrary function satisfying the condition $\tilde{z} = r^{-1}(g_t - Au_0 - f)$ in $\Gamma \times \{0\}$, shift the addend $\tilde{z}r$ into f_0 and consider the inverse problem for the unknown $z - \tilde{z}$ in place of z. Then this new unknown is zero in Γ.

Now we are going to prove an existence and stability theorem for (48), (49). The Fredholm-type result of this theorem (i.e. the assertion (i) below) was already obtained in [11], but under different assumptions. Namely, in [11] certain positivity conditions on the original kernel m where imposed. We do not need such assumptions in the assertion (i).

Theorem 4. *Let (3), (4) hold and $\beta, a_{ij}, a_j, \beta \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$, with some $l \in (0,1), p \in (1,\infty)$. Moreover, let $a_t \ge 0$, $r \in C^{l,\frac{l}{2}}(Q)$, $r_t \in L^p(Q)$ and (64) hold. In addition, let $f \in C^{l,\frac{l}{2}}(Q)$, $u_0 \in C^{2+l}(\Omega)$, $g \in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $u_T \in C^{2+l}(\Omega)$ and the consistency conditions*

$$\begin{aligned} &u_0 = g\,, \ \ \beta g_t = Au_0 + f_0 \quad \textit{in case I}, \quad \omega \cdot \nabla_x u_0 = g \quad \textit{in case II} \quad \textit{in } \Gamma \times \{0\} \\ &u_T = g \ \textit{ in case I}, \quad \omega \cdot \nabla_x u_T = g \quad \textit{in case II} \quad \textit{in } \Gamma \times \{T\} \\ &Au_T = 0 \quad \textit{in case I \& (1)} \quad \textit{in } \Gamma \times \{T\} \end{aligned} \tag{66}$$

be satisfied. Then the following assertions are valid.

(i) *If $k \in W^1_{\frac{2}{2-l}}(0,T)$, $r \ge 0$, $r_t - \theta r \ge 0$,*

$$\textit{for any } U \subseteq \Omega, \ \operatorname{meas} U > 0, \quad \textit{it holds} \quad r_t - \theta r \neq 0 \ \textit{ in } U \times (0,T) \tag{67}$$

and the homogeneous inverse problem, i.e.

$$\begin{aligned} &\beta(v_t^0 + k * v_t^0) = Av^0 + q^0 r \ \text{ in } Q, \\ &v^0 = 0 \ \text{ in } \Omega\times\{0\}, \ \ B_1 v^0 = 0 \ \text{ in } S, \ \ v^0 = 0 \ \text{ in } \Omega\times\{T\} \end{aligned} \tag{68}$$

has in $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *only the trivial solution* $q^0 = 0$, $v^0 = 0$ *then the inverse problem* (48), (49) *has a unique solution* (z, u) *in the space* $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *and in case* I & (1) *it holds* $z = 0$ *in* Γ. *Moreover, the solution* (z, u) *satisfies the estimate*

$$\begin{aligned} \|z\|_l + \|u\|_{2+l,1+\frac{l}{2}} \le{}& \Lambda(\beta, a_{ij}, a_j, a, k, r) \\ &\times \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|u_T\|_{2+l} \right\} \end{aligned} \tag{69}$$

with some constant Λ *depending on the quantities shown in brackets.*

(ii) *If*

$$k \in W^1_{\frac{2}{2-l}}(0,T), \quad k \ge 0, \quad k' \le 0 \tag{70}$$

and r *satisfies* (51), (52) *then the inverse problem* (48), (49) *has a unique solution* (z, u) *in the space* $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ *and in case* I & (1) *it holds* $z = 0$ *in* Γ. *The solution satisfies the estimate* (69).

Proof. Evidently, the assertion (ii) follows from the assertion (i) and Theorem 3. Therefore, let us prove (i). We note that, due the assumptions of the theorem, the inverse problem in case $k = 0$, i.e. the problem

$$\begin{aligned} &\beta u_t^1 = Au^1 + z^1 r + f_0 \text{ in } Q, \\ &u^1 = u_0 \text{ in } \Omega\times\{0\}, \ \ B_1 u^1 = g \text{ in } S, \ \ u^1 = u_T \text{ in } \Omega\times\{T\} \end{aligned} \tag{71}$$

has a unique solution $(z^1, u^1) \in \mathcal{X}^l = C^l_*(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, where

$$C^l_*(\Omega) = \{z : z \in C^l(\Omega), \ \ z = 0 \text{ in } \Gamma \text{ in case I \& (1)}\},$$

and this solution satisfies the estimate

$$\begin{aligned} \|z^1\|_l + \|u^1\|_{2+l,1+\frac{l}{2}} \le{}& \Lambda^1(\beta, a_{ij}, a_j, a, r) \\ &\times \left\{ \|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|u_T\|_{2+l} \right\}. \end{aligned} \tag{72}$$

This assertion follows from a known Fredholm-type result for the problem (71) (Theorem 1.2 in [7]) and the related uniqueness result (Theorem 3 of the present paper with $k = 0$)[1].

Let us denote $q = z - z^1$ and $v = u - u^1$. Then the inverse problem (48), (49) for $(z, u) \in \mathcal{X}^l$ is equivalent to the following inverse problem for the pair $X = (q, v) \in \mathcal{X}^l$:

$$\begin{aligned} &\beta v_t = Av + qr - \beta k * (u_t^1 + v_t) \text{ in } Q, \\ &v = 0 \text{ in } \Omega\times\{0\}, \ B_1 v = 0 \text{ in } S, \ v = 0 \text{ in } \Omega\times\{T\}. \end{aligned} \tag{73}$$

Let $\mathcal{P}$ stand for the operator that assigns to a given right-hand side f_0 the solution of the problem (71) with $u_0, g, u_T = 0$. Then the problem (73) is in $\mathcal{X}^l$ equivalent to the operator equation

$$X = \mathcal{T}X + \Psi \tag{74}$$

[1] The uniqueness for (71) was proved also in [7], but under different assumptions.

where $\mathcal{T}(q,v) = \mathcal{P}(-\beta k * v_t) = \mathcal{P}(-\beta k' * v - \beta k(0)v)$ and $\Psi = \mathcal{P}(-\beta k * u_t^1)$. Observing the assumptions of theorem and the implication $k' \in L^{\frac{2}{2-l}}(0,T), w \in C^{l,\frac{l}{2}}(Q) \Rightarrow k' * w \in C^{l,\frac{l}{2}}(Q)$, following from (23), we see that $\mathcal{T} \in \mathscr{L}(\mathcal{U}^{l,l'}, \mathcal{X}^l)$ and $\Psi \in \mathcal{X}^l$, where $\mathcal{U}^{l,l'} = C_*^{l'}(\Omega) \times C^{l,\frac{l}{2}}(Q)$ and l' is an arbitrary number in the interval $(0,l)$. Since $\mathcal{X}^l$ is compactly embedded in $\mathcal{U}^{l,l'}$, the operator $\mathcal{T}$ is compact in $\mathcal{U}^{l,l'}$. Moreover, 1 is not an eigenvalue of $\mathcal{T}$, because the equation $X^0 = \mathcal{T}X^0$ is in $\mathcal{X}^l$ equivalent to the problem (68), whose solution $X^0 = (q^0, v^0)$ is zero by the assumption. Consequently, by the Fredholm's alternative, the equation (74) has a unique solution in $\mathcal{X}^l$. This proves the existence assertion of theorem.

It remains to prove (69). Since 1 belongs to the resolvent set of $\mathcal{T}$, the operator $\mathcal{T}$ depends on $\beta, a_{ij}, a_j, a, k, r$ and the inequality $\|\cdot\|_{\mathcal{U}^{l,l'}} \le \mathrm{Const}\,\|\cdot\|_{\mathcal{X}^l}$ holds, we get

$$\|X\|_{\mathcal{U}^{l,l'}} \le \Lambda^2(\beta, a_{ij}, a_j, a, k, r)\|\Psi\|_{\mathcal{U}^{l,l'}} \le \Lambda^3(\beta, a_{ij}, a_j, a, k, r)\|\Psi\|_{\mathcal{X}^l}. \tag{75}$$

Let us estimate the equation (74) taking the relations (75), (23) and (72) into account:

$$\begin{aligned}\|q\|_l + \|v\|_{2+l,1+\frac{l}{2}} = \|X\|_{\mathcal{X}^l} &\le \|\mathcal{T}\|\|X\|_{\mathcal{U}^{l,l'}} + \|\Psi\|_{\mathcal{X}^l} \\ &\le \left\{\|\mathcal{T}\|\Lambda^3(\beta, a_{ij}, a_j, a, k, r) + 1\right\}\|\Psi\|_{\mathcal{X}^l} \le \Lambda^4(\beta, a_{ij}, a_j, a, k, r)\|\beta k * u_t^1\|_{l,\frac{l}{2}} \\ &\le \Lambda^5(\beta, a_{ij}, a_j, a, k, r)\left\{\|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|u_T\|_{2+l}\right\}.\end{aligned} \tag{76}$$

Since $z = z^1 + q$ and $u = u^1 + v$, the estimate (69) follows from (72) and (76). □

Let us denote by $\hat{F}_{\beta,a,r}$ the operator that assigns to the vector $d = (f_0, u_0, g, u_T)$ the solution of the linear inverse problem (48), (49). Provided β, a, r and also a_{ij}, a_j, k satisfy the assumptions of Theorem 4 (incl. the additional assumptions of the assertion (ii) of Theorem 4), the operator $\hat{F}_{\beta,a,r}$ is well-defined from the space

$$\begin{aligned}\mathscr{D} = \{d :\ & d \in C^{l,\frac{l}{2}}(Q) \times C^{2+l}(\Omega) \times C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S) \times C^{2+l}(\Omega), \\ & d \text{ satisfy the consistency conditions (66)}\}\end{aligned}$$

to the space $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and satisfies the estimate

$$\begin{aligned}\|\hat{F}_{\beta,a,r}(f_0, u_0, g, u_T)\|_{C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q)} \le\ & \Lambda(\beta, a_{ij}, a_j, a, k, r) \\ & \times \left\{\|f_0\|_{l,\frac{l}{2}} + \|u_0\|_{2+l} + \|g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|u_T\|_{2+l}\right\}.\end{aligned} \tag{77}$$

5. **Results for IP2 and IP3.** By Lemma 1, IP2 is equivalent to the following problem for (a,u):

$$\beta(u_t + k * u_t) = A_0 u + au + f \ \text{ in } Q, \quad u = u_0 \ \text{ in } \Omega \times \{0\}, \quad B_1 u = g \ \text{ in } S, \tag{78}$$

$$u = u_T \ \text{ in } \Omega \times \{T\}, \tag{79}$$

where f, B_1 and g are given by (14) - (16) and $A_0 u = \sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{j=1}^n a_j u_{x_j}$.

Let us define the following set of the coefficients a that depends on $\theta \in \mathbb{R}$:

$$\mathcal{A}^l_{\beta,\theta} = \{a \in C^l(\Omega) :\ \sup_{x\in\Omega} \frac{a(x)}{\beta(x)} \le \theta\}.$$

We will prove a theorem that comprises global uniqueness and local conditional existence and stability.

Theorem 5. *Let (3), (4) hold, $\beta, a_{ij}, a_j \in C^l(\Omega)$ with some $l \in (0,1)$ and $\theta \in \mathbb{R}$. Then the following assertions are valid.*

(i) *If k satisfies (44) and the problem (78), (79) has the solutions $(a_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$, $(a_2, u_2) \in \mathcal{A}^l_{\beta,\theta} \times C^{2+l,1+\frac{l}{2}}(Q)$, where $u = u_1$ satisfies the conditions*

$$\begin{aligned} &u \geq 0\,, \quad u_t + k * u_t - \theta u \geq 0\,, \\ &\text{for any } U \subseteq \Omega,\ \operatorname{meas} U > 0, \quad \text{it holds } u_t + k * u_t - \theta u \neq 0 \ \text{ in } U \times (0,T), \end{aligned} \tag{80}$$

then $a_1 = a_2$ and $u_1 = u_2$.

(ii) *If k satisfies (70) and (78), (79) has a solution $(a, u) \in \mathcal{A}^l_{\beta,\theta} \times C^{2+l,1+\frac{l}{2}}(Q)$ such that u fulfills (80),*

$$u \geq \delta \ \text{ in } \overline{\Omega} \times (T-\delta, T) \quad \text{and} \quad u = 0 \ \text{ in } \overline{\Omega} \times (0, \delta) \ \text{ with some } \delta \in (0, \tfrac{T}{2}), \tag{81}$$

then for any $\widetilde{f}, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ such that

$$D := \|\widetilde{f} - f\|_{l,\frac{l}{2}} + \|\widetilde{u}_0 - u_0\|_{2+l} + \|\widetilde{g} - g\|_{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}} + \|\widetilde{u}_T - u_T\|_{2+l} < \frac{1}{2\lambda^2} \tag{82}$$

where $\lambda = \Lambda(\beta, a_{ij}, a_j, a, k, u)$,

$$\begin{aligned} &\widetilde{u}_0 = \widetilde{g}\,, \quad \beta \widetilde{g}_t = (A_0 + a)\widetilde{u}_0 + \widetilde{f} \quad \text{in case I}, \\ &\qquad\qquad \omega \cdot \nabla_x \widetilde{u}_0 = \widetilde{g} \quad \text{in case II} \quad \text{in } \Gamma \times \{0\}, \end{aligned} \tag{83}$$

$$\widetilde{u}_T = \widetilde{g} \quad \text{in case I}, \quad \omega \cdot \nabla_x \widetilde{u}_T = \widetilde{g} \quad \text{in case II} \quad \text{in } \Gamma \times \{T\}, \tag{84}$$

and $\widetilde{u}_0 = u_0$ in case I in Γ, the problem (78), (79) with f_0, u_0, g, u_T replaced by $\widetilde{f}_0, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ has a unique solution $(\widetilde{a}, \widetilde{u})$ in the ball

$$\mathcal{U} = \left\{ (\widetilde{a}, \widetilde{u}) : \|\widetilde{a} - a\|_l + \|\widetilde{u} - u\|_{2+l,1+\frac{l}{2}} \leq \frac{1}{\lambda} \left(1 - \sqrt{1 - 2\lambda^2 D}\right) \right\}. \tag{85}$$

(iii) *If k satisfies (44), $a \in \mathcal{A}^l_{\beta,\theta}$, $u_0 \in C^{2+l}(\Omega)$, $A_0 u_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $f \in C^{l,\frac{l}{2}}(Q)$, $f_t \in L^p(Q)$, $g \in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $g_t \in W_p^{2-\nu-\frac{1}{p},1-\frac{\nu}{2}-\frac{1}{2p}}(S)$ with some $p \in (1,\infty)$,*

$$\begin{aligned} &u_0 = g, \ \beta g_t + A_0 u_0 + a u_0 = f \quad \text{in case I}, \\ &\qquad\qquad \omega \cdot \nabla_x u_0 = g \quad \text{in case II} \quad \text{in } \Gamma \times \{0\}, \end{aligned} \tag{86}$$

$$\begin{aligned} &u_0 \geq 0,\ f \geq 0,\ g \geq 0,\ f_t + k * f_t - \theta f \geq 0,\ g_t + k * g_t - \theta g \geq 0, \\ &f_t + k * f_t - \theta f \neq 0 \quad \text{or} \quad g_t + k * g_t - \theta g \neq 0 \end{aligned} \tag{87}$$

and

$$(\theta\beta - a)u_0 \leq A_0 u_0 + f(\cdot, 0) \tag{88}$$

then the solution u of the direct problem (78) belongs to $C^{2+l,1+\frac{l}{2}}(Q)$ and satisfies (80). If, in addition, $f(\cdot,t) = 0$ and $g(\cdot,t) = 0$ for $t \in (0,\delta_0)$ with some $\delta_0 \in (0, \frac{T}{2})$, $u_0 = 0$ and $g > 0$ in $\Gamma \times \{T\}$ in case I, then u satisfies (81), too.

Remark 4.

1. Since $\frac{1}{\lambda}(1 - \sqrt{1 - 2\lambda^2 D}) \sim \lambda D$ as $D \to 0^+$, the relation (85) implies that the solution operator of the problem (78), (79) is locally Lipschitz-continuous in the neighborhood of (a, u).

2. In case $u_0 = 0$ the assumptions of (iii) do not contain a, except for the condition $a \in \mathcal{A}^l_{\beta,\theta}$. Namely, in this case (88) is dropped and the consistency conditions (86) read $g = 0$, $\beta g_t = f$ in case I and $g = 0$ in case II in $\Gamma \times \{0\}$.

Proof of Theorem 5. Subtracting the problems (78), (79) for the pairs (a_1, u_1) and (a_2, u_2) we obtain the problem (48), (49) for the difference $z = a_1 - a_2$, $u = u_1 - u_2$ that has the zero free term f_0, zero initial, boundary and final conditions and contains a_2 and u_1 instead of a and r, respectively. Note that the conditions (80) hold with θ replaced by $\theta_2 = \sup\limits_{x\in\Omega} \frac{a_2(x)}{\beta(x)} \le \theta$, too. Thus, we can apply Theorem 3 to the problem for z and u to obtain (i).

Let us prove (ii). Firstly, we note that the problem for $(\widetilde{a}, \widetilde{u})$ is equivalent to the following problem for the differences $z = \widetilde{a} - a$, $w = \widetilde{u} - u$:

$$\begin{aligned} &\beta(w_t + k * w_t) = (A_0 + a)w + zu + z\widetilde{w} + \widetilde{f} - f \ \text{ in } Q\,, \\ &w = \widetilde{u}_0 - u_0 \ \text{ in } \Omega \times \{0\}\,, \ \ B_1 w = \widetilde{g} - g \ \text{ in } S\,, \ \ w = \widetilde{u}_T - u_T \ \text{ in } \Omega \times \{T\}\,. \end{aligned} \tag{89}$$

Further, owing to the properties of β, a, u and also a_{ij}, a_j, k, the operator $\hat{F}_{\beta,a,u}$ is well-defined from $\mathscr{D}$ to $C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ (definitions of $\hat{F}$ and $\mathscr{D}$ can be found at the end of Section 4). Since $(zw + \widetilde{f} - f, \widetilde{u}_0 - u_0, \widetilde{g} - g, \widetilde{u}_T - u_T) \in \mathscr{D}$ for any $S = (z, w) \in \mathscr{S}$ where

$$\mathscr{S} = \{S : S \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q), \ \ w = 0 \text{ in case I in } \Gamma \times \{0\}\},$$

we can define the operator $\hat{\mathcal{F}}(S) = \hat{F}_{\beta,a,u}(zw + \widetilde{f} - f, \widetilde{u}_0 - u_0, \widetilde{g} - g, \widetilde{u}_T - u_T)$ for any $S \in \mathscr{S}$. Moreover, it holds $\hat{\mathcal{F}}(\mathscr{S}) \subseteq \mathscr{S}$. Now we can immediately check that the problem (89) is in the space $\mathscr{S}$ equivalent to the fixed-point equation $S = \hat{\mathcal{F}}(S)$. Defining $\|S\| = \|z\|_l + \|w\|_{2+l,1+\frac{l}{2}}$ and using the formula (77) and the definitions of D and λ in (82) we estimate:

$$\|\hat{\mathcal{F}}(S)\| \le \lambda \left\{ \|zw\|_{l,\frac{l}{2}} + D \right\} \le \lambda \left\{ \|z\|_l \|w\|_{2+l,1+\frac{l}{2}} + D \right\} \le \lambda \left\{ \frac{1}{2}\|S\|^2 + D \right\}.$$

Similarly, for $S^j = (z^j w^j)$, $j = 1, 2$, in view of the relation

$$z^1 w^1 - z^2 w^2 = \frac{z^1 + z^2}{2}(w^1 - w^2) + (z_1 - z_2)\frac{w^1 + w^2}{2}$$

we obtain

$$\|\hat{\mathcal{F}}(S^1) - \hat{\mathcal{F}}(S^2)\| = \|\hat{F}_{\beta,a,u}(z^1 w^1 - z^2 w^2, 0, 0, 0)\| \le \lambda \left\| \frac{S^1 + S^2}{2} \right\| \|S^1 - S^2\|.$$

Thus, by virtue of the assumed inequality $D < \frac{1}{2\lambda^2}$, for any $S, S^1, S^2 \in \mathcal{U}_0$, where

$$\mathcal{U}_0 = \left\{ S : S \in \mathscr{S}, \ \|S\| \le \varrho := \frac{1}{\lambda}\left(1 - \sqrt{1 - 2\lambda^2 D}\right) \right\},$$

we deduce the estimate $\|\hat{\mathcal{F}}(S)\| \le \lambda\left\{\frac{1}{2}\varrho^2 + D\right\} = \varrho$ that implies the relation $\hat{\mathcal{F}}(\mathcal{U}_0) \subseteq \mathcal{U}_0$ and the inequality $\|\hat{\mathcal{F}}(S^1) - \hat{\mathcal{F}}(S^2)\| \le q\|S^1 - S^2\|$, where $q = \lambda\varrho < 1$. Therefore, by the contraction principle, the equation $S = \hat{\mathcal{F}}(S)$ has a unique solution in the ball $\mathcal{U}_0$. This proves (ii).

It remains to prove (iii). Applying Theorems 1 and 2 to (78) we immediately get

$$u \in C^{2+l,1+\frac{l}{2}}(Q)\,, \ \ u \ge 0\,, \ \ u(\cdot, T) > 0 \ \text{ in } \Omega \ (\overline{\Omega}) \text{ in case I (II).} \tag{90}$$

Further, let us assume without restricting the generality that $p \in (1, \min\{\frac{3}{2}; \frac{2}{2-l}\})$. Then the assumptions of Lemma 2 are satisfied for the solution u of (78). Applying this lemma, we obtain $u_t \in W_p^{2,1}(Q)$. Thus, the function $v = u_t + k*u_t - \theta u$ belongs to $C^{l,\frac{l}{2}}(Q) \times W_p^{2,1}(Q)$. One can immediately check that v solves the problem

$$\beta(v_t + k * v_t) = A_0 v + av + f_t + k * f_t - \theta f \ \text{ in } Q,$$
$$v = \frac{1}{\beta}[A_0 u_0 + f + (a - \theta\beta)u_0] \ \text{ in } \Omega \times \{0\}, \ \ B_1 v = g_t + k * g_t - \theta g \ \text{ in } S.$$

Observing (87), (88) and Theorem 2 we obtain $v \geq 0$, $v(x,T) > 0$ for $x \in \Omega$. These relations with (90) imply (80). Finally, the relations (90) with the additional assumptions $g > 0$ in $\Gamma \times \{T\}$ in case I and $f(\cdot,t) = 0$, $g(\cdot,t) = 0$ for $t \in (0, \delta_0)$ with some $\delta_0 \in (0, \frac{T}{2})$ and $u_0 = 0$ imply (81) with some $\delta \leq \delta_0$. □

Finally, we study IP3. Due to Lemma 1, IP3 is equivalent to the following problem for (β, u):

$$\beta(u_t + k * u_t) = Au + f \ \text{ in } Q, \quad u = u_0 \ \text{ in } \Omega \times \{0\}, \ \ B_1 u = g \ \text{ in } S, \tag{91}$$

$$u = u_T \ \text{ in } \Omega \times \{T\}, \tag{92}$$

where f, B_1 and g are given by (14) - (16).

Let us introduce the following set for the coefficients β that depends on $\beta_0 > 0$:

$$\mathcal{B}^l_{\beta_0} = \{\beta \in C^l(\Omega) : \inf_{x \in \Omega} \beta(x) \geq \beta_0\}$$

and define $\theta_{\beta_0} = \max\{0; \frac{1}{\beta_0} \sup_{x \in \Omega} a(x,T)\}$. Then we have $\sup_{x \in \Omega} \frac{a(x,T)}{\beta(x)} \leq \theta_{\beta_0}$ for any $\beta \in \mathcal{B}^l_{\beta_0}$.

Theorem 6. *Let (3) hold, $a_{ij}, a_j \in C^l(\Omega)$, $a \in C^{l,\frac{l}{2}}(Q)$, $a_t \in L^p(Q)$ with some $l \in (0,1)$, $p \in (1,\infty)$, $a_t \geq 0$ and $\beta_0 > 0$. Then the following assertions are valid.*

(i) *If k satisfies (44), the problem (91), (92) has the solutions $(\beta_1, u_1) \in C^l(\Omega) \times C^{2+l,1+\frac{l}{2}}(Q)$ and $(\beta_2, u_2) \in \mathcal{B}^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)$ where $u = u_1$ satisfies the conditions*

$$\begin{aligned} &u_{tt} \in L^p(\Omega), \quad u_t + k * u_t \geq 0, \\ &\hat{u} := (u_t + k * u_t)_t + k * (u_t + k * u_t)_t - \theta_{\beta_0}(u_t + k * u_t) \geq 0, \\ &\text{for any } U \subseteq \Omega, \ \operatorname{meas} U > 0, \ \text{ it holds } \hat{u} \not\equiv 0 \ \text{ in } U \times (0,T), \end{aligned} \tag{93}$$

then $\beta_1 = \beta_2$ and $u_1 = u_2$.

(ii) *If k satisfies (70), $Au_0 - f = 0$ in case I in $\Gamma \times \{0\}$ and the problem (91), (92) has a solution $(\beta, u) \in \mathcal{B}^l_{\beta_0} \times C^{2+l,1+\frac{l}{2}}(Q)$ such that u fulfills (93),*

$$\begin{aligned} &u_t + k * u_t \geq \delta \ \text{ in } \overline{\Omega} \times (T - \delta, T) \ \text{ and} \\ &u_t = 0 \ \text{ in } \overline{\Omega} \times (0, \delta) \ \text{ with some } \delta \in (0, \tfrac{T}{2}), \end{aligned} \tag{94}$$

then for any $\widetilde{f}, \widetilde{u}_0, \widetilde{g}, \widetilde{u}_T$ such that

$$D < \frac{1}{2\bar{\lambda}^2(1 + \|k\|)} \quad \text{where} \quad \bar{\lambda} = \Lambda(\beta, a_{ij}, a_j, a, k, u_t + k * u_t), \tag{95}$$
$$\|k\| = \|k\|_{C[0,T]} \quad \text{and } D \text{ is defined by (82)},$$

$$\begin{aligned} &\widetilde{u}_0=\widetilde{g}\,,\ \ \beta\widetilde{g}_t=A\widetilde{u}_0+\widetilde{f}=0 \quad \textit{in case I},\\ &\omega\cdot\nabla_x\widetilde{u}_0=\widetilde{g} \quad \textit{in case II} \quad \textit{in } \Gamma\times\{0\}, \end{aligned} \tag{96}$$

and (84) *holds, the problem* (91), (92) *with* f_0,u_0,g,u_T *replaced by* $\widetilde{f}_0,\widetilde{u}_0,\widetilde{g},\widetilde{u}_T$ *has a unique solution* $(\widetilde{\beta},\widetilde{u})$ *in the ball*

$$\begin{aligned} \bar{\mathcal{U}} \,=\, \Big\{(\widetilde{\beta},\widetilde{u})\,:\,&\|\widetilde{\beta}-\beta\|_l+\|\widetilde{u}-u\|_{2+l,1+\frac{l}{2}}\\ &\le\frac{1}{\bar{\lambda}(1+\|k\|)}\left(1-\sqrt{1-2\bar{\lambda}^2(1+\|k\|)D}\right)\Big\}. \end{aligned} \tag{97}$$

(iii) *If* k *satisfies* (44), $\beta\in\mathcal{B}^l_{\beta_0}$, $a_t=0$, $u_0\in C^{2+l}(\Omega)$, $A(0)u_0\in W_p^{2-\frac{2}{p}}(\Omega)$, $f\in C^{l,\frac{l}{2}}(Q)$, $f_t,f_{tt}\in L^p(Q)$, $f_t(\cdot,0)\in W_p^{2-\frac{2}{p}}(\Omega)$, $g\in C^{2+l-\nu,1+\frac{l}{2}-\frac{\nu}{2}}(S)$, $g_t,g_{tt}\in W_p^{2-\nu-\frac{1}{p},1-\frac{\nu}{2}-\frac{1}{2p}}(S)$,

$$\begin{aligned} &r_f:=f_t+k*f_t\ge 0,\ r_g:=g_t+k*g_t\ge 0,\\ &\hat{r}_f:=r_{f,t}+k*r_{f,t}-\theta_{\beta_0}r_f\ge 0,\ \hat{r}_g:=r_{g,t}+k*r_{g,t}-\theta_{\beta_0}r_g\ge 0,\\ &\hat{r}_f\ne 0 \ \textit{or}\ \hat{r}_g\ne 0, \end{aligned} \tag{98}$$

the consistency conditions (86) *and the relations*

$$\begin{aligned} &\frac{1}{\beta}(A(0)u_0+f(\cdot,0))\in W_p^2(\Omega),\ A(0)\Big[\frac{1}{\beta}(A(0)u_0+f(\cdot,0))\Big]\in W_p^{2-\frac{2}{p}}(\Omega),\\ &A(0)u_0+f(\cdot,0)\ge 0,\\ &A(0)\Big[\frac{1}{\beta}(A(0)u_0+f(\cdot,0))\Big]-\theta_{\beta_0}A(0)u_0+f_t(\cdot,0)-\theta_{\beta_0}f(\cdot,0)\ge 0 \end{aligned} \tag{99}$$

hold, then the solution u *of the direct problem* (91) *belongs to* $C^{2+l,1+\frac{l}{2}}(Q)$ *and satisfies* (93). *If, in addition,*

$$\begin{aligned} &f_t(\cdot,t)=0,\ g_t(\cdot,t)=0 \ \textit{for}\ t\in(0,\delta_0) \ \textit{with some}\ \delta_0\in(0,\tfrac{T}{2}),\\ &A(0)u_0+f(\cdot,0)=0 \end{aligned} \tag{100}$$

and $r_g>0$ *in* $\Gamma\times\{T\}$ *in case* I, *then* u *satisfies* (94), *too.*

Remark 5.

1. The relation (97) implies that the solution operator of the problem (91), (92) is locally Lipschitz-continuous in the neighborhood of (β,u).
2. If $u_0=0$ and $f(\cdot,0)=0$ then the assumptions of (iii) do not contain β, except for $\beta\in\mathcal{B}^l_{\beta_0}$. Namely, the assumptions (99) are dropped and the consistency conditions (86) read $g(\cdot,0)=0$, $g_t(\cdot,0)=0$ in case I and $g(\cdot,0)=0$ in case II.

Proof of Theorem 6. Subtracting the problems (91), (92) with the pairs (β_1,u_1) and (β_2,u_2) we obtain the problem (48), (49) for the difference $z=\beta_1-\beta_2$, $u=u_1-u_2$ that has the zero free term f_0, zero initial, boundary and final conditions and contains β_2 and $u_{1,t}+k*u_{1,t}$ instead of β and r, respectively. Applying Theorem 3 to this problem, we immediately deduce (i).

Let us prove (ii). The problem for $(\widetilde{\beta},\widetilde{u})$ is equivalent to the following problem for the differences $z=\beta-\widetilde{\beta}$, $w=\widetilde{u}-u$:

$$\begin{aligned} &\beta(w_t+k*w_t)=Aw+z(u_t+k*u_t)+z(w_t+k*w_t)+\widetilde{f}-f \ \text{ in } Q\,,\\ &w=\widetilde{u}_0-u_0 \ \text{ in } \Omega\times\{0\}\,,\ \ B_1w=\widetilde{g}-g \ \text{ in } S\,,\ \ w=\widetilde{u}_T-u_T \ \text{ in } \Omega\times\{T\}\,. \end{aligned} \tag{101}$$

Further, denoting $S=(z,w)$, the problem (101) is in the space

$$\bar{\mathscr{S}}=\{S\,:\,S\in C^l(\Omega)\times C^{2+l,1+\frac{l}{2}}(Q),\ \ w_t=0 \text{ in case I in } \Gamma\times\{0\}\},$$

equivalent to the fixed-point equation $S=\bar{\mathcal{F}}(S)$, where

$$\bar{\mathcal{F}}(S)=\hat{F}_{\beta,a,u_t+k*u_t}(z(w_t+k*w_t)+\widetilde{f}-f,\widetilde{u}_0-u_0,\widetilde{g}-g,\widetilde{u}_T-u_T).$$

Let, as before, $\|S\|=\|z\|_l+\|w\|_{2+l,1+\frac{l}{2}}$. Using (77) we deduce for $S=(\beta,w)$ and $S^j=(\beta^j,w^j)$, $j=1,2$, the relations

$$\|\bar{\mathcal{F}}(S)\|\le\bar{\lambda}\left\{\frac{1+\|k\|}{2}\|S\|^2+D\right\},$$

$$\|\bar{\mathcal{F}}(S^1)-\bar{\mathcal{F}}(S^2)\|\le\bar{\lambda}(1+\|k\|)\Big\|\frac{S^1+S^2}{2}\Big\|\,\|S^1-S^2\|.$$

Like in the proof of Theorem 5, these relations imply that $\bar{\mathcal{F}}$ is the contraction in the ball

$$\bar{\mathcal{U}}_0=\Big\{S\,:\,S\in\bar{\mathscr{S}},\ \|S\|\le\frac{1}{\bar{\lambda}(1+\|k\|)}\left(1-\sqrt{1-2\bar{\lambda}^2(1+\|k\|)D}\right)\Big\}.$$

This implies (ii).

It remains to prove (iii). Assume $p\in(1,\min\{\frac{3}{2};\frac{2}{2-l}\})$. Theorem 1 immediately implies $u\in C^{2+l,1+\frac{l}{2}}(Q)$ for the solution of (91) and Lemma 2 yields $u_t\in W_p^{2,1}(\Omega)$. Define $v=u_t+k*u_t$. Then v belongs to $C^{l,\frac{l}{2}}(Q)\cap W_p^{2,1}(Q)$ and solves the problem

$$\begin{aligned}&\beta(v_t+k*v_t)=Av+f_t+k*f_t\ \text{ in } Q,\\ &v=\frac{1}{\beta}(Au_0+f)\ \text{ in } \Omega\times\{0\},\ \ B_1v=g_t+k*g_t\ \text{ in } S.\end{aligned}\tag{102}$$

Observing (98), (99) and Theorem 2 we obtain

$$v\ge0\,,\quad v(\cdot,T)>0\ \text{ in } \Omega\ (\overline{\Omega}) \text{ in case I (II)}.\tag{103}$$

Further, observing the assumptions of (iii), we see that Lemma 2 holds for the solution v of the problem (102), too. Consequently, $v_t\in W_p^{2,1}(Q)\Rightarrow u_{tt}\in W_p^{2,1}(Q)$. The function $\hat{u}=v_t+k*v_t-\theta_{\beta_0}v$ satisfies the following problem:

$$\begin{aligned}&\beta(\hat{u}_t+k*\hat{u}_t)=A\hat{u}+\hat{r}_f\ \text{ in } Q,\\ &\hat{u}=\frac{1}{\beta}\Big\{A\Big[\frac{1}{\beta}(Au_0+f)\Big]+f_t-\theta_{\beta_0}(Au_0+f)\Big\}\ \text{ in } \Omega\times\{0\},\ \ B_1v=\hat{r}_g\ \text{ in } S.\end{aligned}$$

Observing the assumptions of (iii) and applying Theorem 2 we obtain $\hat{u}\ge0$, $\hat{u}(x,T)>0$ for $x\in\Omega$. These relations with (103) imply the inequalities in (93). If, additionally, $r_g>0$ in $\Gamma\times\{T\}$ in case I, then in view of (103) we get the inequality $v(\cdot,T)>0$ in $\overline{\Omega}$ both in cases I and II. This yields the assertion $u_t+k*u_t\ge\delta$ in $\overline{\Omega}\times(T-\delta,T)$ of (94). Finally, in case (100) the initial condition, the right-hand side and boundary condition of the problem of u_t

$$\begin{aligned}&\beta(u_{tt}+k*u_{tt})=Au_t+f_t-k(A(0)u_0+f(\cdot,0))\ \text{ in } Q,\\ &u_t=\frac{1}{\beta}(Au_0+f)\ \text{ in } \Omega\times\{0\},\ \ B_1u_t=g_t\ \text{ in } S\end{aligned}$$

are zero for $t\in(0,\delta_0)$. This implies the assertion $u_t=0$ in $\overline{\Omega}\times(0,\delta)$ of (94). □

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Research Article

Inverse Problems for a Parabolic Integrodifferential Equation in a Convolutional Weak Form

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We deduce formulas for the Fréchet derivatives of cost functionals of several inverse problems for a parabolic integrodifferential equation in a weak formulation. The method consists in the application of an integrated convolutional form of the weak problem and all computations are implemented in regular Sobolev spaces.

1. Introduction

Many methods to solve inverse problems (e.g., the Landweber iteration, conjugate gradient method) use the Fréchet derivatives of the cost functionals of these problems [1]. The explicit formula for the Fréchet derivative in terms of the variation of the unknowns of the inverse problem contains the solution of an adjoint problem.

The derivation of the explicit formula for such a Fréchet derivative includes testing the direct problem with the solution of the adjoint problem and vice versa: testing the adjoint problem with the solution of the direct problem. In the case of the parabolic weak problem, such a procedure is cumbersome, because of the asymmetry of the properties of the solution and the test function. In the classical formulation of the parabolic weak problem (see, e.g., [2] and also (19) below), the test function must have higher time regularity than the weak solution. This means that in case of nonsmooth coefficients neither the solution of the direct problem nor the solution of the adjoint problem can be used as a test function. Another formulation of the weak parabolic problem consists in reducing the problem to an abstract Cauchy problem over the time variable (see, e.g., [3]). In such a case, a partial integration over the time has to be implemented within singular distributions in the derivation procedure.

In this paper, we present a new method that enables the deduction of the formulas for the Fréchet derivatives for cost functionals of inverse problems related to weak solutions of parabolic problems. The method is based on an integrated convolutional form of the weak direct problem. The requirements to the test function are weaker than in the classical case and coincide with the properties of the solution of the direct problem. All computations in the deduction procedure can be implemented within usual regular Sobolev spaces.

More precisely, we will consider inverse problems related to a parabolic integrodifferential equation that occur in heat flow with memory [4–6]. This equation contains a time convolution. Therefore, the convolutional form of the weak problem is especially suitable. Supposedly, the proposed method can be generalised to parabolic systems, as well.

2. Formal Direct Problem: Notation

Let Ω be an n-dimensional domain, where $n \geq 1$, and Γ be the boundary of Ω. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ where either Γ_1 or Γ_2 is allowed to be an empty set. In case $n \geq 2$, we assume that Γ is sufficiently smooth, meas $\Gamma_1 \cap \Gamma_2 = 0$, and for any $j \in \{1; 2\}$ it holds either $\Gamma_j = \emptyset$ or meas $\Gamma_j > 0$. Denote

$$\Omega_t = \Omega \times (0,t), \qquad \Gamma_{1,t} = \Gamma_1 \times (0,t), \qquad \Gamma_{2,t} = \Gamma_2 \times (0,t), \tag{1}$$

for $t \geq 0$. Consider the problem (direct problem) to find $u(x,t) : \Omega_T \to \mathbb{R}$ such that

$$u_t = Au - m * Au + f + \nabla \cdot \phi \quad \text{in } \Omega_T, \tag{2}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}, \tag{3}$$

$$u = g \quad \text{in } \Gamma_{1,T}, \tag{4}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = \vartheta u + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \tag{5}$$

where $T > 0$ is a fixed number,

$$\begin{aligned} Av &= \sum_{i,j=1}^{n} \left(a_{ij} v_{x_j}\right)_{x_i} + av, \\ \nu_A &= \left(\sum_{j=1}^{n} a_{ij} \nu_j|_{i=1,\dots,n}\right), \\ \nu &= (\nu_1, \dots, \nu_n) - \text{outer normal of } \Gamma_2, \end{aligned} \tag{6}$$

$a_{ij}, a, u_0 : \Omega \to \mathbb{R}$, $f : \Omega_T \to \mathbb{R}$, $\phi : \Omega_T \to \mathbb{R}^n$, $g : \Gamma_{1,T} \to \mathbb{R}$, $\vartheta : \Gamma_2 \to \mathbb{R}$, $h : \Gamma_{2,T} \to \mathbb{R}$, $m : (0,T) \to \mathbb{R}$ are given functions, the subscripts t, x_j, x_i denote the partial derivatives and

$$m * w(t) = \int_0^t m(t-\tau)\, w(\tau)\, d\tau \tag{7}$$

denotes the time convolution. In case $\Gamma_1 = \emptyset$ ($\Gamma_2 = \emptyset$), the boundary condition (4) and (5) is dropped.

The problem (2)–(5) describes the heat flow in the body Ω with the thermal memory. Concerning the physical background, we refer the reader to [4, 6, 7]. The solution u is the temperature of the body and m is the heat flux relaxation (or memory) kernel. The boundary condition (5) is of the third kind where the term $-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u$ equals the heat flux in the direction of the conormal vector.

Let us introduce some additional notations. Let $t > 0$. We use the Sobolev spaces

$$\begin{aligned} W_2^l(\Omega) = \Bigg\{ & v : \Omega \longrightarrow \mathbb{R} : \|v\|_{W_2^l(\Omega)} \\ & := \left[\sum_{|\alpha| \leq l} \left\|D_x^\alpha v\right\|^2_{L^2(\Omega)}\right]^{1/2} < \infty \Bigg\}. \end{aligned} \tag{8}$$

Here, $l = 0, 1, 2, \dots$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is the multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D_x^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. Further, let X be a Banach space. We denote by $C([0,t]; X)$ the space of abstract continuous functions from $[0,t]$ to X endowed with the usual maximum norm $\|v\|_{C([0,t];X)} := \max_{s \in [0,t]} \|v(s)\|$. Moreover, let

$$\begin{aligned} L^p((0,t);X) := \Bigg\{ & v : (0,t) \longrightarrow X : \|v\|_{L^p((0,t);X)} \\ & \geq \left[\int_0^t \|v(s)\|^p ds\right]^{1/p} < \infty \Bigg\} \\ & \text{for } 1 < p < \infty, \\ L^\infty((0,t);X) := \Bigg\{ & v : (0,t) \longrightarrow X : \|v\|_{L^\infty((0,t);X)} \\ & = \operatorname*{ess\,sup}_{t \in (0,T)} \|v(s)\| < \infty \Bigg\}. \end{aligned} \tag{9}$$

By means of these spaces, we define the following important functional spaces:

$$\begin{aligned} \mathcal{U}(\Omega_t) &= C\left([0,t]; L^2(\Omega)\right) \cap L^2\left((0,t); W_2^1(\Omega)\right), \\ \mathcal{U}_0(\Omega_t) &= \left\{\eta \in \mathcal{U}(\Omega_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset\right\}. \end{aligned} \tag{10}$$

Convention. In case $n = 1$, the integrals $\int_{\Gamma_j} v(x) d\Gamma$, $j = 1, 2$ are equal to $\sum_{k=1}^{K} v(x_k)$, where $x_k \in \Gamma_j$ and K is the number of points in Γ_j, and $L_p(\Gamma_j)$ is simply $\mathbb{R}^K$.

3. Weak Direct Problem and Its Convolutional Form

Let us return to the direct problem (2)–(5). Throughout the paper we assume the following basic regularity conditions on the coefficients, the kernel, and the initial and boundary functions:

$$a_{ij} \in L^\infty(\Omega), \qquad a_{ij} = a_{ji}, \qquad \vartheta \in C\left(\overline{\Gamma}_2\right), \quad \vartheta \geq 0, \tag{11}$$

$$a \in L^{q_1}(\Omega), \quad \text{where } q_1 = 1 \text{ if } n = 1,\ q_1 > \frac{n}{2} \text{ if } n \geq 2 \tag{12}$$

$$\begin{aligned} m \in L^1(0,T), \qquad & g \in L^2\left((0,T); W_2^1(\Omega)\right), \\ & g_t \in L^2(\Omega_T), \end{aligned} \tag{13}$$

$$\begin{aligned} f \in L^2\left((0,T); L^{q_2}(\Omega)\right), \quad & \text{where } q_2 = 1 \text{ if } n = 1, \\ q_2 \in (1, q_1) \text{ if } n = 2, \qquad & q_2 = \frac{2n}{n+2} \text{ if } n \geq 3, \end{aligned} \tag{14}$$

$$\phi = (\phi_1, \dots, \phi_n) \in \left(L^2(\Omega_T)\right)^n \tag{15}$$

$$u_0 \in L^2(\Omega), \qquad h \in L^2\left(\Gamma_{2,T}\right) \tag{16}$$

and the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_i \geq \epsilon|\lambda|^2, \quad x \in \overline{\Omega},\ \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0, \tag{17}$$

(for the sake of simplicity we introduced an assumption for the extension of g onto Ω_T).

The first aim is to reformulate the problem (2)–(5) in a weak form. Let us suppose that $a_{ij} \in W_2^1(\Omega)$, $(\partial/\partial x_i)\phi_i \in L^2(\Omega_T)$, $i = 1, \ldots, n$ and (2)–(5) has a classical solution $u \in W_2^{2,1}(\Omega_T)$. Then, we multiply (2) with a test function η from the space

$$\begin{aligned}\mathcal{T}(\Omega_T) = \Big\{\eta \in L^2\left((0,T); W_2^1(\Omega)\right) :\\ \eta_t \in L^2\left((0,T); L^2(\Omega)\right),\\ \eta|_{\Gamma_{1,T}} = 0 \text{ in case } \Gamma_1 \neq \emptyset\Big\}\end{aligned} \tag{18}$$

and integrate by parts with respect to time and space variables. We obtain the following relation:

$$\begin{aligned}0 = &\int_\Omega \left[u(x,T)\eta(x,T) - u_0(x)\eta(x,0)\right]dx - \iint_{\Omega_T} u\eta_t\,dxdt\\ &+ \iint_{\Omega_T}\left[\sum_{i,j=1}^{n} a_{ij}\left(u_{x_j} - m * u_{x_j}\right)\eta_{x_i} - a(u - m*u)\eta\right]dxdt\\ &+ \iint_{\Gamma_{2,T}} (\vartheta u + h)\eta d\Gamma dt - \iint_{\Omega_T}(f\eta - \phi\cdot\nabla\eta)\,dxdt.\end{aligned} \tag{19}$$

This relation makes sense also in a more general case when a_{ij}, ϕ satisfies only (11) and (15) and u does not have regular first-order time and second-order spatial derivatives.

We call a *weak solution* of the problem (2)–(5) a function from the space $\mathcal{U}(\Omega_T)$ that satisfies the relation (19) for any $\eta \in \mathcal{T}(\Omega_T)$ and in case $\Gamma_1 \neq \emptyset$ fulfills the boundary condition (4).

Lemma 1. *The following assertions are valid.*

(i) $\mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T); L^{q_3}(\Omega))$ *where* $q_3 = \infty$ *if* $n = 1$, $q_3 \in (q_1q_2/(q_1 - q_2), \infty)$ *if* $n = 2$ *and* $q_3 = 2n/(n-2)$ *if* $n > 2$, *where* q_1 *and* q_2 *are given in* (12) *and* (14), *respectively;*

(ii) for any $u \in L^2((0,T); L^{q_3}(\Omega))$ *it holds* $au \in L^2((0,T); L^{q_2}(\Omega))$ *and* $\|au\|_{L^2((0,T);L^{q_2}(\Omega))} \leq C\|a\|_{L^{q_1}(\Omega)}\|u\|_{L^2((0,T);L^{q_3}(\Omega))}$, *where* C *is a constant.*

Proof. Since $\mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T); W_2^1(\Omega))$, the assertion (i) follows from the continuous embedding of $W_2^1(\Omega)$ in $L^{q_3}(\Omega)$. The assertion (ii) can be proved by means of Hölder's inequality. □

Theorem 2. *The problem* (2)–(5) *has a unique weak solution. This solution satisfies the estimate*

$$\begin{aligned}&\|u\|_{\mathcal{U}(\Omega_T)}\\ &\leq C_0\Big[\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^{q_2}(\Omega))} + \|\phi\|_{(L^2(\Omega_T))^n}\\ &\quad + \theta_1\left\{\|g\|_{L^2((0,T);W_2^1(\Omega))} + \|g_t\|_{L^2(\Omega_T)}\right\} + \theta_2\|h\|_{L^2(\Gamma_{2,T})}\Big],\end{aligned} \tag{20}$$

where $\theta_1 = 0$ *in case* $\Gamma_1 = \emptyset$, $\theta_2 = 0$ *in case* $\Gamma_2 = \emptyset$ *and* C_0 *is a constant depending on* Ω, Γ_j, a_{ij}, a, ϑ *and* m.

Proof. The assertion of the theorem in case $m = 0$ is well known from the theory of parabolic equations (see, e.g., [2]). Let $\mathcal{F}$ be the operator that assigns to the data vector $d := (u_0, f, \phi, g, h)$ the weak solution of the problem (2)–(5) in case $m = 0$. Then it holds

$$\|\mathcal{F}d\|_{\mathcal{U}(\Omega_T)} \leq \text{RHS}, \tag{21}$$

where RHS is the right-hand side of (20).

Further, let us formulate the problem for the difference $v = u - \mathcal{F}d$. Introducing the linear operator $\mathcal{A}$ by the formula

$$\mathcal{A}w = \mathcal{F}\left(0, -am*w, -\sum_{j=1}^{n} a_{ij}m*w_{x_j}, 0, 0\right), \tag{22}$$

the weak problem (2)–(5) for the solution $u \in \mathcal{U}(\Omega_T)$ equivalent to the following operator equation for the quantity v:

$$v = \mathcal{A}v + \mathcal{A}\mathcal{F}d. \tag{23}$$

We have to estimate $\mathcal{A}$. For this purpose, we firstly prove the following auxiliary inequality:

$$\|m*y\|_{L^2((0,t);L^p(\Omega))} \leq \int_0^t |m(t-\tau)|\,\|y\|_{L^2((0,\tau);L^p(\Omega))}d\tau, \quad t \in [0,T], \tag{24}$$

for any $p \geq 1$ and $y \in L^2((0,t); L^p(\Omega))$.

Denoting $\overline{y}(t) = y(\cdot,t)$, $\|\overline{y}(t)\| = \|\overline{y}(t)\|_{L^p(\Omega)}$, making use of the following property of the Bochner integral: $\|\int_0^s w(s,\tau)d\tau\|_{L^p(\Omega)} \leq \int_0^s \|w(s,\tau)\|_{L^p(\Omega)}d\tau$ for functions $w \in L^1((0,T); L^p(\Omega))$ and the Cauchy's inequality, the relation

(24) can be deduced by means of the following computations:

$$\begin{aligned}
&\|m * y\|_{L^2((0,t);L^p(\Omega))} \\
&\quad = \left[\int_0^t \left\| \int_0^s m(\tau)\,\overline{y}(s-\tau)\,d\tau \right\|^2_{L^p(\Omega)} ds\right]^{1/2} \le I, \ \text{where} \\
&I = \left[\int_0^t \left(\int_0^s |m(\tau)|\ \|\overline{y}(s-\tau)\|\,d\tau\right)^2 ds\right]^{1/2} \\
&\quad = \left[\int_0^t |m(\tau)| \int_\tau^t \|\overline{y}(s-\tau)\| \int_0^s |m(z)|\,\|\overline{y}(s-z)\|\,dz ds d\tau\right]^{1/2} \\
&\quad \le \left[\int_0^t |m(\tau)| \left[\int_\tau^t \|\overline{y}(s-\tau)\|^2 ds\right]^{1/2}\right. \\
&\qquad \left. \times \left[\int_\tau^t \left(\int_0^s |m(z)|\,\|\overline{y}(s-z)\|\,dz\right)^2 ds\right]^{1/2} d\tau\right]^{1/2} \\
&\quad \le \left[\int_0^t |m(\tau)| \left[\int_0^{t-\tau} \|\overline{y}(s)\|^2 ds\right]^{1/2} d\tau\right]^{1/2} \\
&\qquad \times \left[\int_0^t \left(\int_0^s |m(z)|\,\|\overline{y}(s-z)\|\,dz\right)^2 ds\right]^{1/4} \\
&\quad = \left[\int_0^t |m(t-\tau)|\,\|y\|_{L^2((0,\tau);L^p(\Omega))} d\tau\right]^{1/2} \times I^{1/2}.
\end{aligned} \tag{25}$$

Next, let $t \in [0,T]$ and introduce the operator

$$P_t w = \begin{cases} w & \text{in } \Omega_t \\ 0 & \text{in } \Omega_T \setminus \Omega_t. \end{cases} \tag{26}$$

Due to the causality we have $\mathscr{F}(0, P_t f, P_t\phi, 0, 0)(x,t) = \mathscr{F}(0, f, \phi, 0, 0)(x,t)$ for any $(x,t) \in \Omega_t$. Using these relations, the continuity of the linear operator $\mathscr{F}$, the inequality (24), and the boundedness of a_{ij}, we compute the following:

$$\begin{aligned}
&\|\mathscr{A}w\|_{\mathscr{U}(\Omega_t)} \\
&= \left\| \mathscr{F}\left(0, -am * w, -\sum_{j=1}^n a_{ij} m * w_{x_j}, 0, 0\right)\right\|_{\mathscr{U}(\Omega_t)} \\
&= \left\| \mathscr{F}\left(0, -P_t\,[am * w], -P_t\left[\sum_{j=1}^n a_{ij} m * w_{x_j}\right], 0, 0\right)\right\|_{\mathscr{U}(\Omega_t)} \\
&\le \left\| \mathscr{F}\left(0, -P_t\,[am * w], -P_t\left[\sum_{j=1}^n a_{ij} m * w_{x_j}\right], 0, 0\right)\right\|_{\mathscr{U}(\Omega_T)} \\
&\le \widehat{C}\left[\|P_t\,[am * w]\|_{L^2((0,T);L^{q_2}(\Omega))} + \sum_{i=1}^n \left\|P_t\left[a_{ij} m * w_{x_j}\right]\right\|_{L^2(\Omega_T)}\right] \\
&= \widehat{C}\left[\|am * w\|_{L^2((0,t);L^{q_2}(\Omega))} + \sum_{i=1}^n \left\|a_{ij} m * w_{x_j}\right\|_{L^2(\Omega_t)}\right] \\
&\le \widetilde{C} \int_0^t |m(t-\tau)| \left(\|aw\|_{L^2((0,\tau);L^{q_2}(\Omega))} + \|\nabla w\|_{L^2(\Omega_\tau)}\right) d\tau,
\end{aligned} \tag{27}$$

with some constants $\widehat{C}$ and $\widetilde{C}$ depending on Ω, Γ_j, a_{ij}, a, ϑ. Using Lemma 1, we obtain

$$\|aw\|_{L^2((0,\tau);L^{q_2}(\Omega))} \le C_1 \|a\|_{L^{q_1}(\Omega)} \|w\|_{L^2((0,\tau);W_2^1(\Omega))}. \tag{28}$$

Using this relation in (27), we arrive at the following basic estimate for $\mathscr{A}$:

$$\|\mathscr{A}w\|_{\mathscr{U}(\Omega_t)} \le C_2 \int_0^t |m(t-\tau)|\,\|w\|_{\mathscr{U}(\Omega_\tau)} d\tau, \quad t \in [0,T], \tag{29}$$

where C_2 is a constant depending on Ω, Γ_j, a_{ij}, a, ϑ. Let us define the weighted norms in $\mathscr{U}(\Omega_T)$: $\|v\|_\sigma = \sup_{0<t<T} e^{-\sigma t}\|v\|_{\mathscr{U}(\Omega_t)}$ where $\sigma \ge 0$. The estimate (29) implies the further estimate

$$\begin{aligned}
\|\mathscr{A}w\|_\sigma &\le C_2 \sup_{0<t<T} e^{-\sigma t} \int_0^t |m(t-\tau)|\,\|w\|_{\mathscr{U}(\Omega_\tau)} d\tau \\
&= C_2 \sup_{0<t<T} \int_0^t e^{-\sigma(t-\tau)} |m(t-\tau)|\, e^{-\sigma\tau} \|w\|_{\mathscr{U}(\Omega_\tau)} d\tau \\
&\le C_2 \int_0^T e^{-\sigma s} |m(s)|\, ds \|w\|_\sigma.
\end{aligned} \tag{30}$$

Since $\int_0^T e^{-\sigma s}|m(s)|ds \to 0$ as $\sigma \to \infty$, there exists σ_0, depending on C_2 and m, such that $C_2 \int_0^T e^{-\sigma_0 s}|m(s)|ds \le 1/2$. Thus, $\|\mathscr{A}w\|_{\sigma_0} \le (1/2)\|w\|_{\sigma_0}$. The operator $\mathscr{A}$ is a contraction in $\mathscr{U}(\Omega_T)$. This implies the existence and uniqueness assertions of the theorem.

To prove the estimate (20), we firstly deduce from (23) the inequality $\|v\|_{\sigma_0} \le \|\mathscr{A}v\|_{\sigma_0} + \|\mathscr{A}\mathscr{F}d\|_{\sigma_0} \le (1/2)[\|v\|_{\sigma_0} + \|\mathscr{F}d\|_{\sigma_0}]$. This implies $\|v\|_{\sigma_0} \le \|\mathscr{F}d\|_{\sigma_0}$. Using the equivalence relations $e^{-\sigma_0 T}\|\cdot\|_0 \le \|\cdot\|_{\sigma_0} \le \|\cdot\|_0$, where $\|\cdot\|_0 = \|\cdot\|_{\mathscr{U}(\Omega_T)}$ and (21), we reach (20). □

We note the upper integration bound T in (19) can be released to be any number t from the interval $[0,T]$. Indeed, (19) is equivalent to the following problem:

$$\begin{aligned}
0 = &\int_\Omega \left[u(x,t)\,\eta(x,t) - u_0(x)\,\eta(x,0)\right] dx - \iint_{\Omega_t} u\eta_t dx dt \\
&+ \iint_{\Omega_t} \left[\sum_{i,j=1}^n a_{ij}\left(u_{x_j} - m * u_{x_j}\right)\eta_{x_i}\right. \\
&\left. - a(u - m * u)\,\eta\right] dx dt
\end{aligned}$$

$$+\iint_{\Gamma_{2,t}}(\vartheta u+h)\,\eta\,d\Gamma dt$$
$$-\iint_{\Omega_t}(f\eta-\phi\cdot\nabla\eta)\,dxdt,\qquad t\in[0,T],\tag{31}$$

for any $\eta\in\mathcal{T}(\Omega_T)$. This assertion can be proved using the standard technique defining the test function as follows:

$$\eta^{\epsilon}(x,t)=\begin{cases}\eta(x,\tau) & \text{for } \tau\in[0,t],\\ \eta(x,\tau)\left(1-\dfrac{\tau-t}{\epsilon}\right) & \text{for } \tau\in(t,t+\epsilon),\\ 0 & \text{for } \tau\in[t+\epsilon,T],\end{cases}\tag{32}$$

and letting the parameter ϵ to approach 0.

Next we transform the weak direct problem (31) to a form that does not contain a time derivative of the test function η. This form enables the extension of the test space. This is useful for treatment of problems for adjoint states of quasisolutions of inverse problems in next sections.

Theorem 3. *The function $u\in\mathcal{U}(\Omega_T)$ satisfies the relation (19) for any $\eta\in\mathcal{T}(\Omega_T)$ if and only if it satisfies the following relation:*

$$0=\int_\Omega u*\eta dx-\int_\Omega\int_0^t u_0(x)\,\eta(x,\tau)\,d\tau dx$$
$$+\int_\Omega 1*\left[\sum_{i,j=1}^{n}a_{ij}\left(u_{x_j}-m*u_{x_j}\right)*\eta_{x_i}\right.$$
$$\left.-a(u-m*u)*\eta\right]dx\tag{33}$$
$$+\int_{\Gamma_2}1*(\vartheta u+h)*\eta d\Gamma$$
$$-\int_\Omega 1*\left(f*\eta-\sum_{i=1}^{n}\phi_i*\eta_{x_i}\right)dx,\quad t\in[0,T],$$

for any $\eta\in\mathcal{U}_0(\Omega_T)$.

Here, according to the definition of the time convolution in the previous section, $1*w(t)=\int_0^t w(\tau)d\tau$.

Proof. It is sufficient to prove that $u\in\mathcal{U}(\Omega_T)$ satisfies (31) for any $\eta\in\mathcal{T}(\Omega_T)$ if and only if it satisfies (33) for any $\eta\in\mathcal{U}_0(\Omega_T)$. Suppose that $u\in\mathcal{U}(\Omega_T)$ satisfies (31) and choose an arbitrary $\eta\in\mathcal{T}(\Omega_T)$. Let t_1 be an arbitrary number on the interval $[0,T]$ and choose some function $\xi^{t_1}\in\mathcal{T}(\Omega_T)$ such that the relation

$$\xi^{t_1}(x,t)=\eta(x,t_1-t)\quad\text{for } t\in[0,t_1]\tag{34}$$

is valid. For instance, it is possible to define ξ^{t_1} as a periodic function with respect to t, that is, $\xi^{t_1}(x,t)=\eta(x,t_1-t)$ for $t\in[0,t_1]$, $\xi^{t_1}(x,t)=\eta(x,t-t_1)$ for $t\in[t_1,2t_1]$, $\xi^{t_1}(x,t)=\eta(x,3t_1-t)$ for $t\in[2t_1,3t_1]$ and so on. Using the relation (31) with η replaced by ξ^{t_1} and setting there $t=t_1$ we obtain the equality

$$0=K_1(t_1)+K_2(t_1),\tag{35}$$

where

$$K_1(t)=\int_\Omega\left[u(x,t)\,\eta(x,0)-u_0(x)\,\eta(x,t)\right]dx$$
$$+\int_\Omega\int_0^t u(x,\tau)\,\eta_t(x,t-\tau)\,d\tau dx,\tag{36}$$

$$K_2(t)=\int_\Omega\left[\sum_{i,j=1}^{n}a_{ij}\left(u_{x_j}-m*u_{x_j}\right)*\eta_{x_i}\right.$$
$$\left.-a(u-m*u)*\eta\right]dx$$
$$+\int_{\Gamma_2}(\vartheta u+h)*\eta d\Gamma\tag{37}$$
$$-\int_\Omega\left(f*\eta-\sum_{i=1}^{n}\phi_i*\eta_{x_i}\right)dx.$$

Note that the time derivative of η can be removed from K_1 by integration. Indeed, let $t_2\in[0,T]$. Then

$$\int_0^{t_2}K_1(t_1)\,dt_1=\int_0^{t_2}\int_\Omega u(x,t_1)\,\eta(x,0)\,dxdt_1$$
$$-\int_0^{t_2}\int_\Omega u_0(x)\,\eta(x,t_1)\,dxdt_1$$
$$+\int_0^{t_2}\int_\Omega\int_0^{t_1}u(x,\tau)\,\eta_{t_1}(x,t_1-\tau)\,d\tau dxdt_1.\tag{38}$$

Changing the order of the integrals over τ and t_1 in the last term, we easily obtain

$$\int_0^{t_2}K_1(t_1)\,dt_1$$
$$=\int_\Omega\int_0^{t_2}u(x,\tau)\,\eta(x,t_2-\tau)\,d\tau dx\tag{39}$$
$$-\int_0^{t_2}\int_\Omega u_0(x)\,\eta(x,t_1)\,dxdt_1.$$

Integrating now the whole equality (35) over t_1 from 0 to t_2, observing (37) and (39), and finally redenoting t_2 by t, we reach the desired relation (33). Summing up, we have proved that (33) holds for any $\eta\in\mathcal{T}(\Omega_T)$. But all terms in the right-hand side of (33) are well defined for $\eta\in\mathcal{U}_0(\Omega_T)$, too. Since $\mathcal{T}(\Omega_T)$ is densely embedded in $\mathcal{U}_0(\Omega_T)$, we conclude that (33) holds for any $\eta\in\mathcal{U}_0(\Omega_T)$.

It remains to show that (33) implies (31). Suppose that $u\in\mathcal{U}(\Omega_T)$ satisfies (33) and choose an arbitrary $\eta\in\mathcal{T}(\Omega_T)$

and $t_1 \in [0,T]$. Again, let ξ^{t_1} be a function from $\mathcal{T}(\Omega_T)$ such (34) is valid. Inserting ξ^{t_1} instead of η into (33), differentiating with respect to t and setting $t = t_1$ we come to the relation (31). Theorem is proved. □

Corollary 4. *A function $u \in \mathcal{U}(\Omega_T)$ is a weak solution of* (2)–(5) *if and only if it satisfies the relation* (33) *for any $\eta \in \mathcal{U}_0(\Omega_T)$ and in case $\Gamma_1 \neq \emptyset$ fulfills the boundary condition* (4).

4. Inverse Problems and Quasisolutions

In the sequel, let us pose some inverse problems for the weak solution of (2)–(5). These problems are selected in order to demonstrate the wide possibilities of the method that we will introduce in Section 5.

Firstly, we suppose that (2)–(5) has the following specific form:

$$\begin{gathered} u_t = Au - m * Au + f_0 + \nabla \cdot \phi + \sum_{j=1}^{N} \gamma_j(t)\,\omega_j(x) \quad \text{in } \Omega_T, \\ u = u_0 \quad \text{in } \Omega \times \{0\}, \\ u = g \quad \text{in } \Gamma_{1,T}, \\ -\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = \vartheta u + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \end{gathered} \tag{40}$$

where $\omega = (\omega_1, \ldots, \omega_N)$ is unknown. The coefficients and other given functions f_0, ϕ, u_0, g, h are assumed to satisfy (11)–(17). Moreover, $\gamma \in (L^2(0,T))^N$ is prescribed.

IP1. Find the vector $\omega \in (L^2(\Omega))^N$ such that the weak solution of (40) satisfies the following instant additional conditions:

$$u(x,T_i) = u_{T_i}(x), \quad x \in \Omega,\ i = 1,2,\ldots,N, \tag{41}$$

where $0 < T_1 < T_2 < \cdots < T_N \leq T$ and $u_{T_i} \in L^2(\Omega)$, $i = 1,\ldots,N$ are given functions (observations of u).

Since $\sum_{i=1}^{N} \gamma_j \omega_j \in L^2((0,T);L^2(\Omega)) \subset L^2((0,T);L^{q_2}(\Omega))$ for $\omega \in (L^2(\Omega))^N$, the weak solution u of (40) exists in $\mathcal{U}(\Omega_T)$; hence, it has traces $u(\cdot,T_i) \in L^2(\Omega)$, $i = 1,\ldots,N$. In practice, the term $\sum_{j=1}^{N} \gamma_j \omega_j$ may represent an approximation of a more general function $F(x,t) \in L^2(\Omega_T)$, where γ_j, $j = 1,2,\ldots$ form a basis in $L^2(0,T)$.

Further, let u_0 also be unknown.

IP2. Find the vector $\omega \in (L^2(\Omega))^N$ and $u_0 \in L^2(\Omega)$ such that the weak solution of (40) satisfies the following integral additional conditions:

$$\int_0^T \kappa_i(x,t)\,u(x,t)\,dt = v_i(x), \quad x \in \Omega,\ i = 1,2,\ldots,N+1, \tag{42}$$

where $v_i \in L^2(\Omega)$, $i = 1,\ldots,N+1$ are given observation functions and κ_i, $i = 1,\ldots,N$ are given weights that satisfy the following condition:

$$\begin{gathered} |\kappa_i(x,t)| \leq \widehat{\kappa}(t) \quad \text{in } \Omega_T,\ i = 1,\ldots,N+1 \\ \text{with some } \widehat{\kappa} \in L^2(0,T). \end{gathered} \tag{43}$$

Note that the integral $\int_0^T \kappa_i(\cdot,t)u(\cdot,t)dt$ in (42) belongs to $L^2(\Omega)$ for any $\omega \in (L^2(\Omega))^N$ and $u_0 \in L^2(\Omega)$. Indeed, for such ω and u_0 it holds $u \in \mathcal{U}(\Omega_T) \subset L^2(\Omega_T)$, which implies

$$\left\| \int_0^T \kappa_i(x,t)\,u(x,t)\,dt \right\|_{L^2(\Omega)} \leq \|\widehat{\kappa}\|_{L^2(0,T)} \|u\|_{L^2(\Omega_T)} < \infty. \tag{44}$$

In practice, the weights κ_i are usually concentrated in neighborhoods of some fixed values of time $t = T_i$.

Finally, let us pose a nonlinear inverse problem for the coefficient a and the kernel m. Assume that $n \in \{1;2;3\}$. Then any coefficient a that belongs to $L^2(\Omega)$ satisfies (12). Moreover, let us set $q_1 = 2$ if $n = 2$ and $\Gamma_2 \neq \emptyset$. The other coefficients and the given functions u_0, f, ϕ, g, h are assumed to satisfy (11)–(17).

IP3. Find $a \in L^2(\Omega)$ and $m \in L^1(0,T)$ such that the weak solution of (2)–(5) satisfies the following integral additional conditions:

$$\begin{gathered} u(x,T) = u_T(x), \quad x \in \Omega, \\ \int_{\Gamma_2} \kappa(x,t)\,u(x,t)\,d\Gamma = v(t), \quad t \in (0,T), \end{gathered} \tag{45}$$

where $u_T \in L^2(\Omega)$, $v \in L^2(0,T)$ are given observation functions and κ is a given weight function such that $\kappa \in L^\infty((0,T);L^2(\Gamma_2))$.

As in IP1, we can show that the trace $u(\cdot,T)$ belongs to $L^2(\Omega)$. Moreover, using the property $u \in \mathcal{U}(\Omega_T)$, the embedding of $W_2^1(\Omega)$ in $L^2(\Gamma_2)$ and Hölder's inequality, one can immediately check that the term $\int_{\Gamma_2} \kappa(x,\cdot)u(x,\cdot)d\Gamma$ in (45) belongs to $L^2(0,T)$.

Available existence, uniqueness, and stability results for IP1–IP3 require stronger smoothness of the data than imposed in the present paper. Let us cite some of these results.

In case $N = 1$, the well posedness of IP1 was proved in [8]. Partial results were deduced earlier in [9]. A more general problem involving both IP1 and IP2 without the unknown u_0 in case $N = 1$ was studied in [10] by means of different techniques. IP1 and IP2 in case $m = 0$ and $N = 1$ were treated in many papers, for example, [11–14]. The case $N > 1$ is open even if $m = 0$. Inverse problems to determine m with given a were studied in a number of papers, for example, [7, 15–23]. The problem for a with given m was treated in [8].

In the present paper, we will deal with quasisolutions of IP1–IP3 and related cost functionals. Denote $\mathcal{Z}_1 = (L^2(\Omega))^N$.

Let $M \subseteq \mathscr{F}_1$. The quasi-solution of IP1 in the set M is an element $\omega^* \in \arg\min_{\omega\in M} J_1(\omega)$, where J_1 is the following cost functional

$$J_1(\omega) = \sum_{i=1}^{N} \left\| u(x,T_i;\omega) - u_{T_i}(x) \right\|^2_{L^2(\Omega)}, \tag{46}$$

and $u(x,t;\omega)$ is the solution of (40) that corresponds to a fixed element ω.

Similarly, let $M \subseteq \mathscr{F}_2 := (L^2(\Omega))^{N+1}$. The quasi-solution of IP2 in the set M is $z^* \in \arg\min_{z\in M} J_2(z)$, where J_2 is the cost functional

$$J_2(z) = \sum_{i=1}^{N+1} \left\| \int_0^T \kappa_i(x,t)\, u(x,t;z)\, dt - \upsilon_i(x) \right\|^2_{L^2(\Omega)}, \tag{47}$$

and $u(x,t;z)$ is the weak solution of (40) that corresponds to a given vector $z = (\omega, u_0)$.

Finally, defining $M \subseteq \mathscr{F}_3 := L^2(\Omega) \times L^2(0,T)$, the quasi-solution of IP3 in M is an element $z^* \in \arg\min_{z\in M} J_3(z)$, where J_3 is the cost functional

$$\begin{aligned} J_3(z) = {} & \left\| u(x,T;z) - u_T(x) \right\|^2_{L^2(\Omega)} \\ & + \left\| \int_{\Gamma_2} \kappa(x,t)\, u(x,t;z)\, d\Gamma - \upsilon(t) \right\|^2_{L^2(0,T)}, \end{aligned} \tag{48}$$

and $u(x,t;z)$ is the weak solution of the direct problem (2)–(5) corresponding to given $z = (a,m)$. Here, we restricted the space for the unknown m to $L^2(0,T)$, because we will seek for the Fréchet derivative of J_3 in a Hilbert space. Moreover, the kernel of the second addend corresponding to m in the representation formula of J_3' (90) is an element of $L^2(0,T)$ and in general does not belong to the adjoint space $L^\infty(0,T)$.

According to the above-mentioned arguments, the functionals J_k, $k = 1,2,3$, are well-defined in $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$, respectively.

5. The Fréchet Derivatives of Cost Functionals of Inverse Problems

5.1. General Procedure. Suppose that the solution u of the direct problem depends on a vector of parameters p that has to be determined in an inverse problem making use of certain measurements of u. Let the quasi-solution of the inverse problem be sought by a method involving the Fréchet derivative of the cost functional (i.e., some gradient-type method). Usually in practice, a solution of a proper adjoint problem is used to represent the Fréchet derivative.

We introduce a general procedure to deduce such adjoint problems. Assume that Δu is the difference of solutions of the direct problem corresponding to a difference of the vector of the parameters Δp. More precisely, we suppose that Δu is a solution of the following problem:

$$\Delta u_t = A\Delta u - m * A\Delta u + f^\dagger + \nabla\cdot\phi^\dagger \quad \text{in } \Omega_T, \tag{49}$$

$$\Delta u = \Delta u_0 \quad \text{in } \Omega\times\{0\}, \tag{50}$$

$$\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \tag{51}$$

$$-\nu_A\cdot\nabla\Delta u + m * \nu_A\cdot\nabla\Delta u = \vartheta\Delta u + h^\dagger + \nu\cdot\phi^\dagger \quad \text{in } \Gamma_{2,T}, \tag{52}$$

with some data $f^\dagger, \phi^\dagger, \Delta u_0, h^\dagger$ depending on Δp. We restrict ourselves to the case when the Dirichlet boundary condition of the state u is independent of p. Therefore, the condition (51) for Δu is homogeneous.

In practice, the adjoint parabolic problems are usually formulated as backward problems. In our context, it is better to pose adjoint problems in the forward form. The involved memory term with m is defined via a forward convolution and from the practical viewpoint, it is preferable to have the direct and adjoint problems in a similar form (e.g., to simplify parallelisation of computations).

More precisely, let the adjoint state ψ be a solution of the following problem:

$$\begin{aligned} \psi_t &= A\psi - m * A\psi + f^\circ + \nabla\cdot\phi^\circ \quad \text{in } \Omega_T, \\ \psi &= u^\circ \quad \text{in } \Omega\times\{0\}, \\ \psi &= 0 \quad \text{in } \Gamma_{1,T}, \\ -\nu_A\cdot\nabla\psi + m * \nu_A\cdot\nabla\psi &= \vartheta\psi + h^\circ + \nu\cdot\phi^\circ \quad \text{in } \Gamma_{2,T}, \end{aligned} \tag{53}$$

where $f^\circ, \phi^\circ, u^\circ$, and h° are some data depending on Δu and the cost functional under consideration.

Assume that the quadruplets $f^\dagger, \phi^\dagger, \Delta u_0, h^\dagger$, and $f^\circ, \phi^\circ, u^\circ, h^\circ$ satisfy the conditions (14)–(16). Then, due to Theorem 2, the problems (49)–(52) and (53) have unique weak solutions in the space $\mathscr{U}(\Omega_T)$. Actually, we have $\Delta u, \psi \in \mathscr{U}_0(\Omega_T)$ because of the homogeneous boundary conditions on $\Gamma_{1,T}$.

Let us write the relation (33) for Δu and use the test function $\eta = \psi$. Then we obtain for any $t \in [0,T]$

$$\begin{aligned} 0 = {} & \int_\Omega \Delta u * \psi dx - \int_\Omega \int_0^t \Delta u_0(x)\, \psi(x,\tau)\, d\tau dx \\ & + \int_\Omega 1 * \left[\sum_{i,j=1}^{n} a_{ij} \left(\Delta u_{x_j} - m * \Delta u_{x_j} \right) * \psi_{x_i} \right. \\ & \qquad \left. - a(\Delta u - m * \Delta u) * \psi\eta \right] dx \\ & + \int_{\Gamma_2} 1 * \left(\vartheta\Delta u + h^\dagger \right) * \psi d\Gamma \\ & - \int_\Omega 1 * \left(f^\dagger * \psi - \sum_{i=1}^{n} \phi_i^\dagger * \psi_{x_i} \right) dx. \end{aligned} \tag{54}$$

Secondly, let us write this relation for ψ and use the test function $\eta = \Delta u$. Then we have for any $t \in [0,T]$

$$\begin{aligned}0 &= \int_\Omega \psi * \Delta u dx - \int_\Omega \int_0^t u^\circ(x)\,\Delta u(x,\tau)\,d\tau dx \\ &+ \int_\Omega 1 * \left[\sum_{i,j=1}^n a_{ij}\left(\psi_{x_j} - m * \psi_{x_j}\right) * \Delta u_{x_i} \right. \\ &\qquad \left. - a\left(\psi - m * \psi\right) * \Delta u \right] dx \\ &+ \int_{\Gamma_2} 1 * \left(\vartheta\psi + h^\circ\right) * \Delta u d\Gamma \\ &- \int_\Omega 1 * \left(f^\circ * \Delta u - \sum_{i=1}^n \phi_i^\circ * \Delta u_{x_i} \right) dx.\end{aligned} \tag{55}$$

Subtracting (54) from (55), using the commutativity of the convolution, the symmetricity relations $a_{ij} = a_{ji}$ and differentiating with respect to t, we arrive at the following basic equality that can be used in various inverse problems:

$$\begin{aligned}&\int_\Omega u^\circ(x)\,\Delta u(x,t)\,dx - \int_{\Gamma_2} h^\circ * \Delta u d\Gamma \\ &+ \int_\Omega \left(f^\circ * \Delta u - \sum_{i=1}^n \phi_i^\circ * \Delta u_{x_i} \right) dx \\ &= \int_\Omega \Delta u_0(x)\,\psi(x,t)\,dx - \int_{\Gamma_2} h^\dagger * \psi d\Gamma \\ &+ \int_\Omega \left(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i} \right) dx, \quad t \in [0,T].\end{aligned} \tag{56}$$

5.2. Derivative of J_1

Theorem 5. *The functional J_1 is the Fréchet differentiable in $\left(L^2(\Omega)\right)^N$ and*

$$\begin{aligned}&J_1'(\omega)\,\Delta\omega \\ &= \sum_{j=1}^N \int_\Omega \sum_{i=1}^N \int_0^{T_i} \gamma_j(t)\,\psi_i(x, T_i - t; \omega)\,dt\,\Delta\omega_j(x)\,dx,\end{aligned} \tag{57}$$

where $\psi_i \in \mathcal{U}(\Omega_{T_i})$, $i = 1,\dots,N$, are the unique ω-dependent weak solutions of the following problems:

$$\begin{aligned}&\psi_{i,t} = A\psi_i - m * A\psi_i \quad in\ \Omega_{T_i}, \\ &\psi_i = 2\left[u(x,T_i;\omega) - u_{T_i}(x)\right] \quad in\ \Omega \times \{0\}, \\ &\psi_i = 0 \quad in\ \Gamma_{1,T_i}, \\ &-\nu_A \cdot \nabla\psi_i + m * \nu_A \cdot \nabla\psi_i = \vartheta\psi_i \quad in\ \Gamma_{2,T_i},\end{aligned} \tag{58}$$

$i = 1,\dots,N$.

Proof. Let us fix some $\omega, \Delta\omega \in \left(L^2(\Omega)\right)^N$. One can immediately check that it holds

$$\begin{aligned}&J_1(\omega + \Delta\omega) - J_1(\omega) \\ &= 2\sum_{i=1}^N \int_\Omega \left[u(x,T_i;\omega) - u_{T_i}(x)\right] \Delta u(x,T_i;\omega)\,dx \\ &+ \sum_{i=1}^N \int_\Omega \Delta u(x,T_i;\omega)^2 dx,\end{aligned} \tag{59}$$

where $\Delta u(x,t;\omega) = u(x,t;\omega+\Delta\omega) - u(x,t;\omega) \in \mathcal{U}_0(\Omega_T)$ is the weak solution of the following problem:

$$\begin{aligned}&\Delta u_t = A\Delta u - m * A\Delta u + \sum_{j=1}^N \gamma_j \Delta\omega_j \quad in\ \Omega_T, \\ &\Delta u = 0 \quad in\ \Omega \times \{0\}, \\ &\Delta u = 0 \quad in\ \Gamma_{1,T}, \\ &-\nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u = \vartheta\Delta u \quad in\ \Gamma_{2,T}.\end{aligned} \tag{60}$$

Applying the estimate (20) to the solution of this problem we deduce the following estimate for the second term in the right-hand side of (59):

$$\begin{aligned}\sum_{i=1}^N \int_\Omega \Delta u(x,T_i;\omega)^2 dx &\le n\|\Delta u\|^2_{\mathcal{U}(\Omega_T)} \\ &\le C_0^2 n \left\| \sum_{j=1}^N \gamma_j \Delta\omega_j \right\|^2_{L^2((0,T);L^{q_2}(\Omega))} \le C_4 \|\omega\|^2_{\left(L^2(\Omega)\right)^N},\end{aligned} \tag{61}$$

with some constant C_4. This implies that J_1 is the Fréchet differentiable and the first term in the right-hand side of (59) represents the Fréchet derivative, that is,

$$\begin{aligned}&J_1'(\omega)\,\Delta\omega = \sum_{i=1}^N \sigma_i \\ &\text{with } \sigma_i = 2\int_\Omega \left[u(x,T_i;\omega) - u_{T_i}(x)\right] \Delta u(x,T_i;\omega)\,dx.\end{aligned} \tag{62}$$

Further, let us use the method presented in Section 5.1 to deduce the proper adjoint problems. Comparing (60) with (49)–(52) we see that $f^\dagger = \sum_{j=1}^N \gamma_j \Delta\omega_j$, $\phi^\dagger = \Delta u_0 = h^\dagger = 0$. Therefore, the relation (56) has the form

$$\begin{aligned}&\int_\Omega u^\circ(x)\,\Delta u(x,t)\,dx - \int_{\Gamma_2} h^\circ * \Delta u d\Gamma \\ &+ \int_\Omega \left(f^\circ * \Delta u - \sum_{i=1}^n \phi_i^\circ * \Delta u_{x_i} \right) dx \\ &= \sum_{j=1}^N \int_\Omega \gamma_j \Delta\omega_j * \psi dx, \quad t \in [0,T].\end{aligned} \tag{63}$$

In order to deduce a formula for the component σ_i in the quantity $J_1'(\omega)\Delta\omega$, we set $u^\circ = u_i^\circ = 2[u(x,T_i;\omega) - u_{T_i}(x)]$, $h^\circ = f^\circ = \phi^\circ = 0$ and $t = T_i$ in (63). Then we immediately have

$$\sigma_i = \sum_{j=1}^{N} \int_\Omega \gamma_j \Delta\omega_j * \psi_i dx|_{t=T_i}, \tag{64}$$

where according to (53) and the definition of $u_i^\circ, h^\circ, \phi^\circ$, the function ψ_i is the weak solution of the problem (58) in the domain Ω_T instead of Ω_{T_i}. Due to Theorem 2, this problem has a unique solution. From (62) and (64) we obtain (57). The latter formula contains the values of ψ_i in Ω_{T_i}. Therefore, we can restrict the problem (58) from Ω_T to Ω_{T_i}. □

To use the formula (57) one has to solve N weak problems for the functions ψ_i in domains Ω_{T_i}. In the following theorem, we will show that computational work related to the evaluation of the Fréchet derivative can be considerably reduced. Actually, it is sufficient to solve N weak problems in the smaller domains $\Omega_{T_i - T_{i-1}}$, $i = 1, \ldots, N$. Here, $T_0 = 0$.

Theorem 6. *The Fréchet derivative of the functional J_1 can also be written in the form*

$$J_1'(\omega)\Delta\omega = \sum_{j=1}^{N} \int_\Omega \sum_{l=1}^{N} \int_{T_{l-1}}^{T_l} \gamma_j(t)\, \beta_l(x, T_l - t, \omega)\, dt \Delta\omega_j(x)\, dx, \tag{65}$$

where $\beta_l \in \mathcal{U}_0(\Omega_{T_l - T_{l-1}})$ are the unique ω-dependent weak solutions of the following sequence of recursive problems in the domains $\Omega_{T_l - T_{l-1}}$:

$$\begin{aligned} &\beta_{l,t} = A\beta_l - m * A\beta_l - af^l - \nabla\cdot\Phi^l \quad \text{in } \Omega_{T_l - T_{l-1}}, \\ &\beta_l = u_0^l \quad \text{in } \Omega \times \{0\}, \\ &\beta_l = 0 \quad \text{in } \Gamma_{1, T_l - T_{l-1}}, \\ &-\nu_A \cdot \nabla\beta_l + m * \nu_A \cdot \nabla\beta_l = \vartheta\beta_l - \nu\cdot\Phi^l \quad \text{in } \Gamma_{2, T_l - T_{l-1}}, \end{aligned} \tag{66}$$

where $l = N, N-1, \ldots, 2, 1$. Here,

$$u_0^l(x) = 2\left[u(x,T_l;\omega) - u_{T_l}(x)\right] + \Theta_l \beta_{l+1}(x, T_{l+1} - T_l) \tag{67}$$

and the function f^l and the vector Φ^l are defined via $\beta_N, \beta_{N-1}, \ldots, \beta_{l+1}$ as follows:

$$\begin{aligned} f^l = \Theta_l \sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} & m(T_k - T_l + t + \tau)\, \beta_{k+1} \\ & \times (x, T_{k+1} - T_k - \tau)\, d\tau, \end{aligned} \tag{68}$$

$\Phi^l = (\Phi_1^l, \ldots, \Phi_n^l)$, $\Phi_i^l = \sum_{j=1}^n a_{ij}(\partial/\partial x_j) f^l$ *and* $\Theta_N = 0$, $\Theta_l = 1$ *for* $l < N$.

Proof. Firstly, let us check that (66) indeed have unique weak solutions β_l in $\mathcal{U}_0(\Omega_{T_l - T_{l-1}})$. To this end we can use Theorem 2. For the problem β_N this is immediate, because the initial condition of the problem for β_N belongs to $L^2(\Omega)$ and other equations in this problem are homogeneous. Further, we use the induction. Choose some l in the range $N > l \geq 1$ and suppose that $\beta_{k+1} \in \mathcal{U}_0(\Omega_{T_{k+1} - T_k})$ for all k such that $N - 1 \geq k \geq l$. The aim is to us to show that then the problem for β_l has a unique weak solution in $\mathcal{U}_0(\Omega_{T_l - T_{l-1}})$. Let us represent the kth addend in (68) in the form

$$\begin{aligned} I_k &= \int_0^{T_{k+1}-T_k} m(T_k - T_l + t + \tau)\, \beta_{k+1}(x, T_{k+1} - T_k - \tau)\, d\tau \\ &= \int_t^{T_{k+1}-T_k+t} m(T_k - T_l + \tau)\, \beta_{k+1}(x, T_{k+1} - T_k + t - \tau)\, d\tau. \end{aligned} \tag{69}$$

For any k in the range $N - 1 \geq k \geq l$ we have

$$\begin{aligned} &\|I_k\|^2_{L^2((0,T_l - T_{l-1}); W_2^1(\Omega))} \\ &\leq \sum_{|\alpha|\leq 1} \int_0^{T_l - T_{l-1}} \left[\int_t^{T_{k+1}-T_k+t} \left| m(T_k - T_l + \tau) \right| \right. \\ &\qquad \left. \cdot \left\| D_x^\alpha \beta_{k+1}(\cdot, T_{k+1} - T_k + t - \tau) \right\|_{L^2(\Omega)} d\tau \right]^2 dt \\ &= \sum_{|\alpha|\leq 1} \int_0^{T_l - T_{l-1}} \left[\int_t^{T_{k+1}-T_k+t} \int_0^{T_{k+1}-T_k+t} \left| m(T_k - T_l + \tau) \right| z_{k,\alpha} \right. \\ &\qquad \left. \cdot (T_{k+1} - T_k + t - \tau)\, d\tau \right]^2 dt \\ &= \sum_{|\alpha|\leq 1} \int_{T_{k+1}-T_k}^{T_l - T_{l-1} + T_{k+1} - T_k} \left[\int_0^t m_k(\tau)\, z_{k,\alpha}(t - \tau)\, d\tau \right]^2 dt, \end{aligned} \tag{70}$$

where $z_{k,\alpha}(t) = \|D_x^\alpha \beta_{k+1}(\cdot, t)\|_{L^2(\Omega)}$ for $t \in [0, T_{k+1} - T_k]$, $z_{k,\alpha}(t) = 0$ for $t \notin [0, T_{k+1} - T_k]$ and $m_k(t) = |m(T_k - T_l + t)|$. Since $m \in L^1(0,T)$ and $\beta_{k+1} \in L^2((0, T_{k+1} - T_k); W_2^1(\Omega))$, we have $m_k \in L^1(0, T_l - T_{l-1} + T_{k+1} - T_k)$ and $z_{k,\alpha} \in L^2(0, T_l - T_{l-1} + T_{k+1} - T_k)$. Due to the Young's theorem for convolutions, we get $m_k * z_{k,\alpha} \in L^2(0, T_l - T_{l-1} + T_{k+1} - T_k)$. Therefore, $\|I_k\|^2_{L^2((0,T_l - T_{l-1}); W_2^1(\Omega))} < \infty$. This implies that f_l belongs to $L^2((0, T_l - T_{l-1}); W_2^1(\Omega))$. From the latter relation and $a_{ij} \in L^\infty(\Omega)$ we immediately have $\Phi^l \in (L^2(\Omega_{T_l - T_{l-1}}))^n$. Using the embedding theorem and Lemma 1 we see that af^l satisfies the property (14). Finally, the initial condition u_0^l belongs to $L^2(\Omega)$, because $u - u_{T_l}$, $\beta_{l+1} \in C([0, T_{l+1} - T_l], L^2(\Omega))$. All assumptions of Theorem 2 are satisfied for the problem for β_l. Consequently, it possesses a unique weak solution in $\mathcal{U}_0(\Omega_{T_l - T_{l-1}})$.

Secondly, let us define the functions

$$\beta_l^*(x,t) = \sum_{i=l}^{N} \psi_i(x, T_i - T_l + t) \quad \text{for } (x,t) \in \Omega_{T_l - T_{l-1}}, \tag{71}$$

where $l = 1 \ldots, N$ and ψ_i are the solutions of (58). We are going to show that $\beta_l = \beta_l^*$, $l = 1, \ldots, N$. From the definition of β_l^* using the value of $\psi_l(x,0)$ and simple computations, we immediately get

$$\beta_l^*(x,0) = 2\left[u(x,T_l;\omega) - u_{T_l}(x)\right] + \Theta_l \beta_{l+1}^*(x, T_{l+1} - T_l), \quad l = 1, \ldots, N. \tag{72}$$

Let us fix $l = 1, \ldots, N$ and choose some $\eta \in \mathcal{T}(\Omega_{T_l - T_{l-1}})$. We continue η by the formulae $\eta(x,t) = \eta(x, T_l - T_{l-1})$ for $t > T_l - T_{l-1}$ and $\eta(x,t) = \eta(x,0)$ for $t < 0$. Further, let us define $\eta_i(x,t) = \eta(x, T_l - T_i + t)$ where $i = l, \ldots, N$. By the definition, it holds $\eta_i \in \mathcal{T}(\Omega_{T_i})$.

Let us write down the weak form (31) for the problem for ψ_i (58) with the test function η_i. We fix some $t \in [0, T_l - T_{l-1}]$ and compute the difference of this weak problem with t replaced by $T_i - T_l + t$ and t replaced by $T_i - T_l$ and take the sum over $i = l, \ldots, N$. This results in the following expression:

$$0 = Z_1 + Z_2 + Z_3 + Z_4, \tag{73}$$

where

$$\begin{aligned}
Z_1 &= \sum_{i=l}^{N} \int_{\Omega} \left[\psi_i(x, T_i - T_l + t)\eta_i(x, T_i - T_l + t) - \psi_i(x, T_i - T_l)\eta_i(x, T_i - T_l)\right] dx,\\
Z_2 &= \sum_{i=l}^{N} \int\int_{\Omega_{T_i - T_l + t} \setminus \Omega_{T_i - T_l}} \left[-\psi_i \eta_{i,t} + \sum_{s,j=1}^{n} a_{sj} \psi_{i,x_j} \eta_{i,x_s} - a\psi_i \eta_i \right] dx dt\\
&\quad + \sum_{i=l}^{N} \int\int_{\Gamma_{2,T_i - T_l + t} \setminus \Gamma_{2,T_i - T_l}} \vartheta \psi_i \eta_i d\Gamma dt,\\
Z_3 &= \sum_{i=l}^{N} \int\int_{\Omega_{T_i - T_l + t} \setminus \Omega_{T_i - T_l}} am * \psi_i \eta_i dx dt,\\
Z_4 &= -\sum_{i=l}^{N} \int\int_{\Omega_{T_i - T_l + t} \setminus \Omega_{T_i - T_l}} \sum_{s,j=1}^{n} a_{sj} m * \psi_{i,x_j} \eta_{i,x_s} dx dt.
\end{aligned} \tag{74}$$

Using the definitions of η and β_l^* and the formula (72), we have

$$\begin{aligned}
Z_1 &= \int_{\Omega} \left[\beta_l^*(x,t)\eta(x,t) - \beta_l^*(x,0)\eta(x,0)\right] dx\\
&= \int_{\Omega} \Big[\beta_l^*(x,t)\eta(x,t) - \left\{2\left[u(x,T_l;\omega) - u_{T_l}(x)\right] + \Theta_l \beta_{l+1}^*(x, T_{l+1} - T_l)\right\} \times \eta(x,0)\Big] dx.
\end{aligned} \tag{75}$$

Similarly, using the definitions of η and β_l^* and changing the variable of integration in Z_2, we deduce

$$\begin{aligned}
Z_2 = &\int\int_{\Omega_t} \left[-\beta_l^* \eta_t + \sum_{s,j=1}^{n} a_{sj} \beta_l^* \eta_{x_s} - a\beta_l^* \eta \right] dx dt\\
&+ \int\int_{\Gamma_{2,t}} \vartheta \beta_l^* \eta d\Gamma dt.
\end{aligned} \tag{76}$$

By the change of variable, the quantity Z_3 is transformed to

$$Z_3 = \int\int_{\Omega_t} a(x) \sum_{i=l}^{N} (m * \psi_i)(x, T_i - T_l + t)\eta(x,t)\, dx dt. \tag{77}$$

Let us consider the term $\sum_{i=l}^{N} (m * \psi_i)(x, T_i - T_l + t)$ in the latter formula. We compute

$$\begin{aligned}
&\sum_{i=l}^{N} (m * \psi_i)(x, T_i - T_l + t)\\
&= \sum_{i=l}^{N} \int_0^{T_i - T_l + t} m(\tau)\psi_i(x, T_i - T_l + t - \tau) d\tau\\
&= \int_0^t m(\tau) \sum_{i=l}^{N} \psi_i(x, T_i - T_l + t - \tau) d\tau\\
&\quad + \sum_{i=l}^{N} \int_t^{T_i - T_l + t} m(\tau)\psi_i(x, T_i - T_l + t - \tau) d\tau\\
&= \int_0^t m(\tau)\beta_l^*(x, t - \tau) d\tau\\
&\quad + \sum_{i=l}^{N} \sum_{k=l}^{i-1} \int_0^{T_{k+1} - T_k} m(T_k - T_l + t + \tau)\psi_i(x, T_i - T_k - \tau) d\tau\\
&= (m * \beta_l^*)(x,t)\\
&\quad + \sum_{k=l}^{N-1} \int_0^{T_{k+1} - T_k} m(T_k - T_l + t + \tau)\\
&\quad \times \sum_{i=k+1}^{N} \psi_i(x, T_i - T_k - \tau) d\tau
\end{aligned}$$

$$= \left(m * \beta_l^*\right)(x,t) + \sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} m\left(T_k - T_l + t + \tau\right) \times \beta_{k+1}^*\left(x, T_{k+1} - T_k - \tau\right) d\tau. \tag{78}$$

Thus, (77) reads

$$Z_3 = \int\int_{\Omega_t} am * \beta_l^* \eta dx dt + \int\int_{\Omega_t} a \left[\sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} m\left(T_k - T_l + t + \tau\right) \times \beta_{k+1}^*\left(x, T_{k+1} - T_k - \tau\right) d\tau \right] \eta dx dt. \tag{79}$$

Using similar computations, we obtain

$$Z_4 = -\int\int_{\Omega_t} \sum_{s,j=1}^{n} a_{sj} m * \beta_{l,x_j}^* \eta_{x_s} dx dt - \int\int_{\Omega_t} \sum_{s,j=1}^{n} a_{sj} \times \left[\sum_{k=l}^{N-1} \int_0^{T_{k+1}-T_k} m\left(T_k - T_l + t + \tau\right) \times \beta_{k+1,x_j}^*\left(x, T_{k+1} - T_k - \tau\right) d\tau \right] \eta_{x_s} dx dt. \tag{80}$$

Plugging (75), (76), (79), and (80) into (73), we arrive at a certain weak problem for β_l^* that coincides with the weak problem for β_l. Moreover, since $\psi_i \in \mathcal{U}_0(\Omega_{T_i})$, from (71) we see that $\beta_l^* \in \mathcal{U}_0(\Omega_{T_l - T_{l-1}})$. But we have shown the uniqueness of the weak solutions of the problems for β_l in $\mathcal{U}_0(\Omega_{T_l - T_{l-1}})$. This implies $\beta_l^* = \beta_l$.

Finally, from (57), we have

$$J_1'(\omega)\Delta\omega = \sum_{j=1}^{N} \int_\Omega \sum_{i=1}^{N} \int_0^{T_i} \gamma_j(t)\psi_i\left(x, T_i - t\right) dt \Delta\omega_j(x) dx = \sum_{j=1}^{N} \int_\Omega \sum_{i=1}^{N} \sum_{l=1}^{i} \int_{T_{l-1}}^{T_l} \gamma_j(t)\psi_i\left(x, T_i - t\right) dt \Delta\omega_j(x) dx. \tag{81}$$

Changing here the order of sums over i and l and observing (71) with β_l^* replaced by β_l, we obtain (65). The proof is complete. □

5.3. Derivative of J_2

Theorem 7. *The functional J_2 is the Fréchet differentiable in $\left(L^2(\Omega)\right)^{N+1}$ and*

$$J_2'(z)\Delta z = \sum_{j=1}^{N} \int_\Omega \int_0^T \gamma_j(t)\psi(x, T-t; z)\, dt\ \Delta\omega_j(x) dx + \int_\Omega \psi(x, T; z)\Delta u_0(x) dx, \tag{82}$$

where $\psi \in \mathcal{U}(\Omega_T)$ is the unique z-dependent weak solution of the following problem:

$$\begin{aligned} \psi_t &= A\psi - m * A\psi + 2\sum_{i=1}^{N+1} \kappa_i(x, T-t) \times \left[\int_0^T \kappa_i(x,\tau) u(x,\tau; z) d\tau - \upsilon_i(x) \right] \quad \text{in } \Omega_T, \\ \psi &= 0 \quad \text{in } \Omega \times \{0\}, \\ \psi &= 0 \quad \text{in } \Gamma_{1,T}, \\ -\nu_A \cdot \nabla\psi + m * \nu_A \cdot \nabla\psi &= \vartheta\psi \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{83}$$

Proof. Let us fix some $\Delta z = (\Delta\omega, \Delta u_0) \in \left(L^2(\Omega)\right)^{N+1}$. It holds

$$\begin{aligned} J_2(z + \Delta z) - J_2(z) &= 2\sum_{i=1}^{N+1} \int_\Omega \int_0^T \kappa_i(x,t) \times \left[\int_0^T \kappa_i(x,\tau) u(x,\tau; z) d\tau - \upsilon_i(x) \right] \times \Delta u(x,t; z) dt dx \\ &+ \sum_{i=1}^{N+1} \int_\Omega \left[\int_0^T \kappa_i(x,t) \Delta u(x,t; z) dt \right]^2 dx, \end{aligned} \tag{84}$$

where $\Delta u(x,t;z) = u(x,t;z+\Delta z) - u(x,t;z) \in \mathcal{U}_0(\Omega_T)$ is the weak solution of the following problem:

$$\begin{aligned} \Delta u_t &= A\Delta u - m * A\Delta u + \sum_{j=1}^{N} \gamma_j \Delta\omega_j \quad \text{in } \Omega_T, \\ \Delta u &= \Delta u_0 \quad \text{in } \Omega \times \{0\}, \\ \Delta u &= 0 \quad \text{in } \Gamma_{1,T}, \\ -\nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u &= \vartheta\Delta u \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{85}$$

Using (43), the Cauchy inequality and estimate (20) from Theorem 2 for the problem of $\Delta u(x,t;z)$, we come to the estimate

$$\begin{aligned}&\left|\sum_{i=1}^{N+1}\int_\Omega\left[\int_0^T\kappa_i(x,t)\,\Delta u(x,t;z)\,dt\right]^2dx\right|\\&\quad\le(N+1)\,\|\widehat{\kappa}\|^2_{L^2(0,T)}\|\Delta u\|^2_{L^2(\Omega_T)}\\&\quad\le C_5(N+1)\,\|\widehat{\kappa}\|^2_{L^2(0,T)}\|\Delta u\|^2_{\mathscr{U}(\Omega_T)}\le C_6\|\Delta z\|^2_{\left(L^2(\Omega)\right)^{N+1}},\end{aligned}\tag{86}$$

with some constants C_5 and C_6. Therefore, J_2 is the Fréchet differentiable and the first term in the right-hand side of (84) represents the Fréchet derivative, that is,

$$\begin{aligned}J_2'(z)\,\Delta z=2\sum_{i=1}^{N+1}&\int_\Omega\int_0^T\kappa_i(x,t)\\&\times\left[\int_0^T\kappa_i(x,\tau)\,u(x,\tau;z)\,d\tau-\upsilon_i(x)\right]\\&\times\Delta u(x,t;z)\,dtdx.\end{aligned}\tag{87}$$

Comparing (85) with (49)–(52), we see that $f^\dagger=\sum_{j=1}^N\gamma_j\Delta\omega_j$, $\phi^\dagger=h^\dagger=0$. Consequently, the relation (56) has the form

$$\begin{aligned}&\int_\Omega u^\circ(x)\,\Delta u(x,t)\,dx-\int_{\Gamma_2}h^\circ*\Delta u d\Gamma\\&\quad+\int_\Omega\left(f^\circ*\Delta u-\sum_{i=1}^n\phi_i^\circ*\Delta u_{x_i}\right)dx\\&\quad=\int_\Omega\Delta u_0(x)\,\psi(x,t)\,dx\\&\quad+\sum_{j=1}^N\int_\Omega\gamma_j\Delta\omega_j*\psi dx,\quad t\in[0,T].\end{aligned}\tag{88}$$

To deduce a formula for $J_2'(z)\Delta z$, we define

$$f^\circ=2\sum_{i=1}^{N+1}\kappa_i(x,T-t)\left[\int_0^T\kappa_i(x,\tau)\,u(x,\tau;z)\,d\tau-\upsilon_i(x)\right],\tag{89}$$

$u^\circ=h^\circ=\phi^\circ=0$ and $t=T$ in (88). Then from (87) and (88), we obtain (82), where due to (53), ψ_i is the weak solution of the problem (83). In view of Theorem 2, this problem has a unique solution in $\mathscr{U}(\Omega_T)$. □

5.4. Derivative of J_3

Theorem 8. *The functional J_3 is the Fréchet differentiable in $L^2(\Omega)\times L^2(0,T)$ and*

$$\begin{aligned}J_3'(z)\,\Delta z=&\int_\Omega\left[(u-m*u)*\psi\right](x,T;z)\,\Delta a(x)\,dx\\&+\int_0^T\int_\Omega\left[\sum_{i,j=1}^n a_{ij}u_{x_j}*\psi_{x_i}-au*\psi\right]\\&\times(x,T-t;z)\,dx\Delta m(t)\,dt,\end{aligned}\tag{90}$$

where $\psi\in\mathscr{U}_0(\Omega_T)$ is the unique z-dependent weak solution of the problem

$$\begin{aligned}&\psi_t=A\psi-m*A\psi\quad in\ \Omega_T,\\&\psi=2\left[u(x,T;z)-u_T(x)\right]\quad in\ \Omega\times\{0\},\\&\psi=0\quad in\ \Gamma_{1,T},\\&-\nu_A\cdot\nabla\psi+m*\nu_A\cdot\nabla\psi=\vartheta\psi\\&\quad-2\kappa(x,T-t)\\&\quad\times\left[\int_{\Gamma_2}\kappa(y,T-t)\,u(y,T-t;z)\,d\Gamma-\upsilon(T-t)\right]\quad in\ \Gamma_{2,T}.\end{aligned}\tag{91}$$

Proof. Due to $u(x,t;z)\in\mathscr{U}(\Omega_T)$, $\kappa\in L^\infty((0,T);L^2(\Gamma_2))$, $\upsilon\in L^2(0,T)$, and $u_T\in L^2(\Omega)$, the problem (91) satisfies the assumptions of Theorem 2. Therefore, it has a unique weak solution in $\mathscr{U}_0(\Omega_T)$.

Let $\Delta z=(\Delta a,\Delta m)\in L^2(\Omega)\times L^2(0,T)$ and define $\widetilde{\Delta}u=u(x,t;z+\Delta z)-u(x,t;z)$. We split $\widetilde{\Delta}u$ as follows: $\widetilde{\Delta}u=\Delta u+\widehat{\Delta}u$, where Δu is the weak solution of the following problem:

$$\begin{aligned}\Delta u_t=&A\Delta u-m*A\Delta u+\Delta a\,[u-m*u]-\Delta m*au\\&-\nabla\cdot\left[\Delta m*\sum_{j=1}^n a_{ij}u_{x_j}\right]\quad\text{in }\Omega_T,\\&\Delta u=0\quad\text{in }\Omega\times\{0\},\\&\Delta u=0\quad\text{in }\Gamma_{1,T},\\&-\nu_A\cdot\nabla\Delta u+m*\nu_A\cdot\nabla\Delta u\\&=\vartheta\Delta u-\nu\cdot\left[\Delta m*\sum_{j=1}^n a_{ij}u_{x_j}\right]\quad\text{in }\Gamma_{2,T}.\end{aligned}\tag{92}$$

In view of Lemma 1(i), $u\in\mathscr{U}(\Omega_T)$, $m\in L^1(0,T)$, and the Young's theorem, it holds $u-m*u\in L^2((0,T);L^{q_3}(\Omega))$.

Therefore, Lemma 1(ii) implies

$$\begin{aligned}\|\Delta a\,[u - m * u]\|_{L^2((0,T);L^{q_2}(\Omega))} &\le C_8\,(u,m)\,\|\Delta a\|_{L^{q_1}(\Omega)}\\ &\le C_9\,(u,m)\,\|\Delta a\|_{L^2(\Omega)},\end{aligned} \tag{93}$$

where C_8 and C_9 are some constants depending on u, m. Moreover, since $u \in L^2((0,T);W_2^1(\Omega))$, by Young's inequality we have also

$$\begin{aligned}\left\|\Delta m * \sum_{j=1}^{n} a_{ij} u_{x_j}\right\|_{(L^2(\Omega_T))^n} &\le C_{10}\,(u)\,\|\Delta m\|_{L^1(0,T)}\\ &\le C_{11}\,(u)\,\|\Delta m\|_{L^2(0,T)},\end{aligned} \tag{94}$$

with some constants C_{10} and C_{11} depending on u. The obtained estimates show that assumptions of Theorem 2 are satisfied for the problem (92) and it indeed has a unique weak solution $\Delta u \in \mathscr{U}(\Omega_T)$. Moreover, applying the relation (20) from Theorem 2, we get

$$\begin{aligned}\|\Delta u\|_{\mathscr{U}(\Omega_T)} &\le C_{12}\,(m,u)\left[\|\Delta a\|_{L^2(\Omega)} + \|\Delta m\|_{L^2(0,T)}\right]\\ &= C_{12}\,(m,u)\,\|\Delta z\|,\end{aligned} \tag{95}$$

where $C_{12}(m,u)$ is a constant depending on m, u.

Further, writing the problem for $\widetilde{\Delta} u$ and subtracting the problem for Δu, we obtain the following problem for $\widehat{\Delta} u$:

$$\begin{aligned}\widehat{\Delta} u_t &= A\widehat{\Delta} u - m * A\widehat{\Delta} u + f + \widehat{f} + \nabla\cdot\phi + \nabla\cdot\widehat{\phi} \quad \text{in } \Omega_T,\\ \widehat{\Delta} u &= 0 \quad \text{in } \Omega\times\{0\},\\ \widehat{\Delta} u &= 0 \quad \text{in } \Gamma_{1,T},\\ -\nu_A\cdot\nabla\widehat{\Delta} u + m * \nu_A\cdot\nabla\widehat{\Delta} u &= \vartheta\widehat{\Delta} u + \nu\cdot\phi + \nu\cdot\widehat{\phi} \quad \text{in } \Gamma_{2,T},\end{aligned} \tag{96}$$

where

$$\begin{aligned}f &= \Delta a\Delta u - (m+\Delta m) * \Delta a\Delta u - \Delta m * \Delta a u - \Delta m * a\Delta u,\\ \widehat{f} &= \Delta a\widehat{\Delta} u - (m+\Delta m) * \Delta a\widehat{\Delta} u - \Delta m * a\widehat{\Delta} u,\\ \phi &= -\Delta m * \sum_{j=1}^{n} a_{ij}\Delta u_{x_j}, \quad \widehat{\phi} = -\Delta m * \sum_{j=1}^{n} a_{ij}\widehat{\Delta} u_{x_j}.\end{aligned} \tag{97}$$

Using again Lemma 1 and the Young's inequality, we deduce the estimates

$$\begin{aligned}&\|f\|_{L^2((0,T);L^{q_2}(\Omega))}\\ &\quad\le C_{13}\left\{\left[\|\Delta a\|_{L^2(\Omega)} + \left\{\|m\|_{L^2(0,T)} + \|\Delta m\|_{L^2(0,T)}\right\}\|\Delta a\|_{L^2(\Omega)}\right.\right.\\ &\qquad\left.+\|\Delta m\|_{L^2(0,T)}\|a\|_{L^2(\Omega)}\right]\|\Delta u\|_{\mathscr{U}(\Omega_T)}\\ &\qquad\left.+\|u\|_{\mathscr{U}(\Omega_T)}\|\Delta a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\right\}\\ &\quad\le C_{14}\,(z,u)\left\{\left[\|\Delta z\| + \|\Delta z\|^2\right]\|\Delta u\|_{\mathscr{U}(\Omega_T)} + \|\Delta z\|^2\right\},\\ &\left\|\widehat{f}\right\|_{L^2((0,T);L^{q_2}(\Omega))} \le C_{15}\,(z)\left[\|\Delta z\| + \|\Delta z\|^2\right]\left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)},\\ &\|\phi\|_{(L^2(\Omega_T))^n} \le C_{16}\|\Delta m\|_{L^2(0,T)}\|\Delta u\|_{\mathscr{U}(\Omega_T)}\\ &\qquad\le C_{16}\,\|\Delta z\|\,\|\Delta u\|_{\mathscr{U}(\Omega_T)},\\ &\left\|\widehat{\phi}\right\|_{(L^2(\Omega_T))^n} \le C_{17}\,\|\Delta z\|\left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)},\end{aligned} \tag{98}$$

with some constants $C_{13}\cdots C_{17}$. Therefore, applying the relation (20) to the solution of the problem (96) we obtain

$$\begin{aligned}&\left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)}\\ &\le C_{18}\,(z,u)\\ &\quad\times\left\{\left[\|\Delta z\| + \|\Delta z\|^2\right]\left\{\|\Delta u\|_{\mathscr{U}(\Omega_T)} + \left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)}\right\} + \|\Delta z\|^2\right\},\end{aligned} \tag{99}$$

with some constant C_{18}. In case $\|\Delta z\|$ is small enough, that is,

$$\|\Delta z\| + \|\Delta z\|^2 \le \frac{1}{2C_{18}\,(z,u)}, \tag{100}$$

we have

$$\begin{aligned}\left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)} &\le 2C_{18}\,(z,u)\\ &\quad\times\left\{\left[\|\Delta z\| + \|\Delta z\|^2\right]\|\Delta u\|_{\mathscr{U}(\Omega_T)} + \|\Delta z\|^2\right\}.\end{aligned} \tag{101}$$

In view of (95), this implies

$$\left\|\widehat{\Delta} u\right\|_{\mathscr{U}(\Omega_T)} \le C_{19}\,(z,u)\left[\|\Delta z\|^2 + \|\Delta z\|^3\right], \tag{102}$$

with a constant C_{19}.

Similarly, for the solution of the problem (92), we deduce the estimate

$$\|\Delta u\|_{\mathscr{U}(\Omega_T)} \le C_{20}\,(z,u)\,\|\Delta z\|, \tag{103}$$

with a constant C_{20}.

Now, we write the difference of J_3 in the following form:

$$\begin{aligned} &J_3(z+\Delta z)-J_3(z)\\ &=2\int_\Omega \left[u(x,T;z)-u_T(x)\right]\Delta u(x,T;z)\,dx\\ &\quad+2\int_0^T\int_{\Gamma_2}\kappa(x,t)\left[\int_{\Gamma_2}\kappa(y,t)\,u(y,t;z)\,d\Gamma-v(t)\right]\\ &\quad\times\Delta u(x,t;z)\,d\Gamma dt+\Theta, \end{aligned} \tag{104}$$

where

$$\begin{aligned} \Theta&=2\int_\Omega\left[u(x,T;z)-u_T(x)\right]\hat{\Delta}u(x,T;z)\,dx\\ &\quad+2\int_0^T\int_{\Gamma_2}\kappa(x,t)\left[\int_{\Gamma_2}\kappa(y,t)\,u(y,t;z)\,d\Gamma-v(t)\right]\\ &\quad\times\hat{\Delta}u(x,t;z)\,d\Gamma dt\\ &\quad+\int_\Omega\left\{\left(\Delta u+\hat{\Delta}u\right)(x,T;z)\right\}^2dx\\ &\quad+\int_0^T\left\{\int_{\Gamma_2}\kappa(x,t)\left(\Delta u+\hat{\Delta}u\right)(x,t;z)\,d\Gamma\right\}^2dt. \end{aligned} \tag{105}$$

Using (102), (103), and the property $\kappa\in L^\infty((0,T);L^2(\Gamma_2))$, we obtain the estimate $|\Theta|\le C_{21}(z,u)\sum_{j=2}^6\|\Delta z\|^j$ in case (100). This shows that J_3 is the Fréchet differentiable and

$$\begin{aligned} J_3'(z)\,\Delta z&=2\int_\Omega\left[u(x,T;z)-u_T(x)\right]\Delta u(x,T;z)\,dx\\ &\quad+2\int_0^T\int_{\Gamma_2}\kappa(x,t)\\ &\quad\times\left[\int_{\Gamma_2}\kappa(y,t)\,u(y,t;z)\,d\Gamma-v(t)\right]\\ &\quad\times\Delta u(x,t;z)\,d\Gamma dt. \end{aligned} \tag{106}$$

Finally, let us prove (90) and (91). Comparing (92) with (49)–(52), we see that $f^\dagger=\Delta a[u-m*u]-\Delta m*au$, $\phi_i^\dagger=-\Delta m*\sum_{j=1}^n a_{ij}u_{x_j}$ and $\Delta u_0=h^\dagger=0$. Thus, (56) reads

$$\begin{aligned} &\int_\Omega u^\circ(x)\,\Delta u(x,t)\,dx-\int_{\Gamma_2}h^\circ*\Delta u d\Gamma\\ &\quad+\int_\Omega\left(f^\circ*\Delta u-\sum_{i=1}^n\phi_i^\circ*\Delta u_{x_i}\right)dx\\ &=\int_\Omega\Bigg(\{\Delta a\,[u-m*u]-\Delta m*au\}*\psi\\ &\qquad+\Delta m*\sum_{i,j=1}^n a_{ij}u_{x_j}*\psi_{x_i}\Bigg)dx,\quad t\in[0,T]. \end{aligned} \tag{107}$$

In order to obtain a formula for the right-hand side in (106), we set $u^\circ=2[u(x,T;z)-u_T(x)]$,

$$\begin{aligned} h^\circ(x,t)&=-2\kappa(x,T-t)\\ &\quad\times\left[\int_{\Gamma_2}\kappa(y,T-t)\,u(y,T-t;z)\,d\Gamma-v(T-t)\right], \end{aligned} \tag{108}$$

$f^\circ=\phi^\circ=0$ and $t=T$. Then, we obtain (90), where in view of (53) the function ψ is the weak solution of (91). □

6. Further Aspects of Minimisation

6.1. Existence of Quasisolutions. For the convenience, we will use also the symbol z to denote the argument ω of J_1.

Theorem 9. *(i) Let $k\in\{1;2\}$ and $M\subset\mathscr{Z}_k$ be bounded, closed, and convex. Then, IPk has a quasi-solution in M. The set of quasisolutions is closed and convex.*

(ii) Let $k\in\{1;2;3\}$ and $M\subset\mathscr{Z}_3$ be compact. Then IPk has a quasi-solution in M.

Proof. Let us prove (i). The existence assertion follows from Weierstrass existence theorem (see [24, Section 2.5, Theorem 2D]) once we have proved that J_k is weakly sequentially lower semicontinuous in $\mathscr{F}$, that is,

$$J_k(z)\le\liminf_{n\to\infty}J_k(z_n)\quad\text{as } z_n\rightharpoonup z\text{ in }\mathscr{Z}_k. \tag{109}$$

But (109) follows from the continuity and convexity of J_k [24]. The convexity of J_k can be immediately deduced making use of the linearity of the ingredient $u(x,t;z)$ with respect to z inside the quadratic functional J_k (for similar computations see [25, Theorem 2]). The closedness of the set of quasisolutions is again a direct consequence of the continuity of J_k. The convexity of the set of solutions follows from the convexity of J_k.

Next, we prove (ii). Let $m=\inf_{z\in M}J_k(z)$ and $z_l\in M$ be the minimising sequence, that is, $\lim J_k(z_l)=m$. By the compactness, there exists a subsequence $z_{l_j}\in M$ such that $\lim z_{l_j}=z^*\in M$. Due to the continuity of J_k we have $\lim J_k(z_{l_j})=J_k(z^*)$. Thus, $J_k(z^*)=m$. This proves (ii). □

In practice, the compact set M may be a bounded and closed finite-dimensional subset of $\mathscr{Z}_k$. The proof of weak lower semicontinuity of J_3 may be harder because this functional is not convex.

6.2. Discretisation and Minimisation. Let us consider the penalised discrete problems

$$z^\dagger\in\arg\min_{z\in\mathscr{Z}_{k,L}}\Phi_{k,L}(z),\qquad\Phi_{k,L}=\Pi_L(z)+J_k(z), \tag{110}$$

where $k\in\{1;2;3\}$, $\mathscr{Z}_{k,L}$ is an L-dimensional subspace of Z_k ($L\in\{1,2,\ldots\}$) and Π_L is a penalty function related to

the set $M_L = P_L M$ with P_L being the orthogonal projection onto $\mathcal{F}_{k,L}$. The general assumptions for Π_L are

$$\begin{aligned} &\Pi_L\text{-accretive, convex, Fréchet differentiable,}\\ &\Pi_L'\text{-uniformly Lipschitz continuous in } \mathcal{F}_{k,L}. \end{aligned} \tag{111}$$

Theorem 10. *The problem* (110) *has a solution.*

Proof. The proof repeats the proof of the statement (ii) of Theorem 9, because in view of the accretivity of $\Phi_{k,L}$, a minimizing sequence is bounded and in a finite-dimensional space any bounded sequence is compact. □

The Fréchet derivative of $\Phi_{k,L}$, that is, $\Phi_{k,L}'(z) = \Pi_L'(z) + J_k'(z) \in \mathcal{F}_{k,L}^* = \mathcal{F}_{k,L}$ can be identified by a certain element in $\mathcal{F}_{k,L}$, that is,

$$\Phi_{k,L}'(z)\,\Delta z = \left\langle \Phi_{k,L}'(z), \Delta z \right\rangle_{\mathcal{F}_k} \quad \forall \Delta z \in \mathcal{F}_{k,L}, \tag{112}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{F}_k}$ is the inner product of $\mathcal{F}_k$. In particular, the addend $J_k'(z)$ is identical to the element $P_L w_k(z)$ where $w_k(z)$ is the kernel of the functional $J_k'(z)$. Thus, by virtue of (57), (65), (82), and (90), it holds

$$\begin{aligned} w_1(z) &= \left(\sum_{i=1}^{N} \int_0^{T_i} \gamma_j(t)\, \psi_i(\cdot, T_i - t; z)\, dt\big|_{j=1,\dots,N} \right)\\ &= \left(\sum_{l=1}^{N} \int_{T_{l-1}}^{T_l} \gamma_j(t)\, \beta_l(\cdot, T_l - t; z)\, dt\big|_{j=1,\dots,N} \right),\\ w_2(z) &= \left(\int_0^T \gamma_j(t)\, \psi(\cdot, T-t; z)\, dt\big|_{j=1,\dots,N}, \psi(\cdot, T; z) \right),\\ w_3(z) &= \Bigg((u - m * u) * \psi(\cdot, T),\\ &\qquad \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij} u_{x_j} * \psi_{x_i} - a u * \psi \right](x, T - \cdot)\, dx \Bigg). \end{aligned} \tag{113}$$

In w_1, the functions ψ_i and β_l are the z- (or, equivalently, ω-) dependent weak solutions of the problems (58) and (66), respectively. In w_2 the function ψ is the weak solution of (83) and in w_3 the functions u and ψ are the z-dependent weak solutions of (2)–(5) and (91), respectively.

Example 11. Consider the case $k = 1$. Let $M = \{z \in \mathcal{F}_1 : \|z\| \le \rho\}$, where $\rho > 0$. Further, let ξ_j, $j = 1, 2, \dots$, be an orthonormal basis in $L^2(\Omega)$ and $\mathcal{F}_{1,L} = (\mathrm{span}(\xi_1, \dots, \xi_L))^N$. Then $J_1'(z)$ is in $\mathcal{F}_{1,L}$ identical to the element

$$\begin{aligned} P_L w_1(z) = \Bigg(&\sum_{i=1}^{L} \int_\Omega \sum_{l=1}^{N} \int_{T_{l-1}}^{T_l} \gamma_j(t)\, \beta_l(x, T_l - t; z)\, dt\\ &\times \xi_i(x)\, dx\ \xi_i\big|_{j=1,\dots,N} \Bigg). \end{aligned} \tag{114}$$

Moreover, it holds $M_L = \{z \in \mathcal{F}_{1,L} : \|z\| \le \rho\}$. Define a convex penalty function $\Pi_L \in C^\infty[0,\infty)$ such that $\Pi_L(z) = 0$ for $\|z\| \le \rho$ and $\Pi_L(z) = d(\|z\|^2 - \rho^2)$ for $\|z\| \ge \rho + \varepsilon$ with some $d, \varepsilon > 0$. Then Π_L satisfies (111).

Let $k \in \{1; 2; 3\}$. Choose some initial guess $z_0 \in \mathcal{F}_{k,L}$. Compute the approximate solutions by the gradient method

$$z_{s+1} = z_s - c_s \Phi_{k,L}'(z_s), \tag{115}$$

where $s = 0, 1, 2, \dots$ and $c_s > 0$.

Theorem 12. *Let $k \in \{1; 2\}$ and c_s be chosen by the rule*

$$\begin{aligned} \inf_{c>0} \Phi_{k,L}\left(z_s - c\Phi_{k,L}'(z_s)\right) &\le \Phi_{k,L}\left(z_s - c_s \Phi_{k,L}'(z_s)\right)\\ &\le \inf_{c>0} \Phi_{k,L}\left(z_s - c\Phi_{k,L}'(z_s)\right) + \delta_s, \end{aligned} \tag{116}$$

where $\delta_s \ge 0$, $\sum_{s=0}^{\infty} \delta_s =: \delta < \infty$. Then it holds dist$(z_s, S) \to 0$ as $s \to \infty$, where S is the set of solutions of (110).

Proof. The assertion follows from Theorem 5.1.2 of [26] once we have proved that $\Phi_{k,L}'$ is uniformly Lipschitz continuous, $\Phi_{k,L}$ is convex, and the set $M(z_0) = \{z \in \mathcal{F}_{k,L} : \Phi_{k,L}(z) \le \Phi_{k,L}(z_0) + \delta\}$ is bounded. The convexity of $\Phi_{k,L}$ follows from the convexity of its addends Π_L and J_k. The boundedness of $M(z_0)$ is a direct consequence of the accretivity of $\Phi_{k,L}$ following from the accretivity of the addend Π_L.

It remains to show the uniform Lipschitz continuity of J_k' in $\mathcal{F}_{k,L}$ (such a property for Π_L' is assumed in (111)). Let $k = 1$. Then by (113) and $J_k'(z) = P_L w_1(z)$ for any $z, \tilde{z} \in \mathcal{F}_{k,L}$, we have

$$\begin{aligned} \left\| J_1'(\tilde{z}) - J_1'(z) \right\| &\le \|P_L\|\, \|w_1(\tilde{z}) - w_1(z)\|\\ &\le C_{22} \sum_{i=1}^{N} \left\| \psi_i(\cdot, \cdot; \tilde{z}) - \psi_i(\cdot, \cdot; z) \right\|_{\mathcal{U}(\Omega_T)}, \end{aligned} \tag{117}$$

where C_{22} is a constant independent of z and $\tilde{z}$. Further, observing (58) and (40), the estimate (20) of Theorem 2 and $z = \omega$, we deduce

$$\begin{aligned} \left\| J_1'(\tilde{z}) - J_1'(z) \right\| &\le 2C_{22}C_0 \sum_{i=1}^{N} \left\| u(\cdot, T_i; \tilde{z}) - u(\cdot, T_i; z) \right\|_{L^2(\Omega)}\\ &\le C_{23} \|u(\cdot, \cdot; \tilde{z}) - u(\cdot, \cdot; z)\|_{\mathcal{U}(\Omega_T)}\\ &\le C_{24} \|\tilde{z} - z\|, \end{aligned} \tag{118}$$

where C_{23}, C_{24} are independent of z and $\tilde{z}$. This proves the uniform Lipschitz continuity of J_1'. Such a property of J_2' can be proved in a similar manner. □

The convergence of z_s in case $k = 3$ is an open issue. This case is more complex because IP3 is nonlinear and the Fréchet derivative of J_3 is not uniformly Lipschitz continuous.

The quasisolutions of IP1–IP3 are not expected to be stable with respect to the noise of the data, that is, the problems

under consideration may be ill posed. Nevertheless, from the intuitive viewpoint, a discretisation should regularise an ill-posed problem. Such a property of the discretisation has been proved in many cases [27, 28]. Alternatively, the index s of the gradient method could be used as a regularization parameter (see [29, 30]). Moreover, the addend Π_L can be defined to be the stabilizing term of the Tikhonov's method instead of the penalty function, that is, $\Pi_L = \alpha\|z\|^2$, where $\alpha > 0$ is the regularisation parameter. Such a Π_L satisfies (111).

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WEAK INVERSE PROBLEMS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS CONTAINING TWO KERNELS

KAIRI KASEMETS, JAAN JANNO

Abstract. An inverse problem to determine a coefficient and two kernels in a parabolic integro-differential equation is considered. A corresponding direct problem is supposed to be in the weak form. Existence of the quasi-solution is proved and issues related to Fréchet differentiation of the cost functional are treated.

1. Introduction

Inverse problems to determine coefficients and kernels in integro-differential heat equations are well-studied in the smooth case when the medium is continuous and corresponding direct problems hold in the classical sense (selection of references: [2, 4, 5, 9, 10, 12, 13, 15, 16, 17, 19]). For instance, in [10] problems to determine space-dependent coefficients by means of final over-determination of the solution of the direct problem are dealt with. This paper exploits and generalizes methods developed earlier in the usual parabolic case [3, 7].

Results are known for particular non-smooth cases, as well. For instance, identification problems for parabolic transmission problems are considered in [11] under additional smoothness assumptions in neighbourhoods of observation areas. Several papers deal with degenerate cases (see [8] and references therein). In [14] problems to reconstruct free terms and coefficients in a weak parabolic problem containing a single kernel (heat flux relaxation kernel) are considered. In particular, a new method that enables to deduce formulas for Fréchet derivatives for cost functionals of inverse problems is proposed.

In the present article we consider the inverse problem of determining two kernels and a coefficient in a parabolic integro-differential equation. The corresponding direct problem is posed in the weak form. We prove the Fréchet differentiability of the cost functional related to the inverse problem and deduce a suitable form for the Fréchet derivative in terms of an adjoint problem. In this connection we use an integrated convolutional form of the weak direct problem that enables to use test functions without classical time derivatives. Finally, we prove the existence of the quasi-solution of the inverse problem under certain restrictions.

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Inverse problems for smooth models with two kernels were formerly considered in [5, 12, 19].

2. Formal statement of problems

Let Ω be a n-dimensional domain, where $n \geq 1$ and $\Gamma = \partial\Omega$. Further, let $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\operatorname{meas}\Gamma_1 \cap \Gamma_2 = 0$, $\operatorname{meas}\Gamma_2 > 0$ and either $\Gamma_1 = \emptyset$ or $\operatorname{meas}\Gamma_1 > 0$. In case $n \geq 2$ we assume Γ to be sufficiently smooth. Define

$$\Omega_t = \Omega \times (0,t), \quad \Gamma_{1,t} = \Gamma_1 \times (0,t), \quad \Gamma_{2,t} = \Gamma_2 \times (0,t)$$

for $t \geq 0$.

Let $T > 0$. We pose the formal direct problem: find $u(x,t) : \Omega_T \to \mathbb{R}$ such that

$$u_t + (\mu * u)_t = Au - m * Au + f + \nabla \cdot \phi + \varphi_t \quad \text{in } \Omega_T, \tag{2.1}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}, \tag{2.2}$$

$$u = g \quad \text{in } \Gamma_{1,T}, \tag{2.3}$$

$$-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u = \vartheta u + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \tag{2.4}$$

where

$$Av = \sum_{i,j=1}^{n} \left(a_{ij} v_{x_j}\right)_{x_i} + av, \quad \nu_A = \Big(\sum_{j=1}^{n} a_{ij}\nu_j\big|_{i=1,\dots,n}\Big),$$

$\nu = (\nu_1, \dots, \nu_n)$ is the outer normal of Γ_2, $a_{ij}, a, u_0 : \Omega \to \mathbb{R}$, $f, \varphi : \Omega_T \to \mathbb{R}$, $\phi : \Omega_T \to \mathbb{R}^n$, $g : \Omega_T \to \mathbb{R}$, $\vartheta : \Gamma_2 \to \mathbb{R}$, $h : \Gamma_{2,T} \to \mathbb{R}$, $\mu, m : (0,T) \to \mathbb{R}$ are given functions and

$$z * w(t) = \int_0^t z(t-\tau) w(\tau) d\tau$$

is the convolution with respect to the variable t. In the case $\Gamma_1 = \emptyset$, the boundary condition (2.3) is omitted. The second and third addend of the free term of the equation (2.1); i.e., $\nabla \cdot \phi$ and φ_t may be singular distributions.

The problem (2.1)–(2.4) governs the heat conduction in the body Ω filled with material with memory, where μ and m are the relaxation kernels of the internal energy and the heat flux, respectively and u is the temperature [1, 4, 5, 18]. Then the condition (2.4) corresponds to the third kind boundary condition, namely it contains the heat flux to the co-normal direction $-\nu_A \cdot \nabla u + m * \nu_A \cdot \nabla u$.

Let us formulate the inverse problem:

IP. Find a, m and μ such that the solution of (2.1)–(2.4) satisfies the following final and integral additional conditions:

$$u = u_T \quad \text{in } \Omega \times \{T\}, \tag{2.5}$$

$$\int_{\Gamma_2} \kappa_j(x,\cdot) u(x,\cdot) d\Gamma = v_j \quad \text{in } (0,T),\ j = 1,2, \tag{2.6}$$

where $u_T : \Omega \to \mathbb{R}$, $\kappa_j : \Gamma_{2,T} \to \mathbb{R}$ and $v_j : (0,T) \to \mathbb{R}$ are prescribed functions.

Remark 2.1. In the case $n = 1$ and $\Omega = (c,d)$, the integral $\int_{\Gamma_2} z(x) d\Gamma$ is merely the sum $\sum_{l=1}^{L} z(x_l)$, where $x_l \in \Gamma_2 \subseteq \{c; d\}$ and L is the number of points in Γ_2 (i.e $L \in \{1; 2\}$). Then the conditions (2.6) read

$$\sum_{l=1}^{L} \kappa_j(x_l,\cdot) u(x_l,\cdot) = v_j \quad \text{in } (0,T), \quad j = 1,2. \tag{2.7}$$

3. Results concerning direct problem

Let us start by a rigorous mathematical formulation of the direct problem. Define the following functional spaces:

$$\mathcal{U}(\Omega_t) = C([0,t]; L^2(\Omega)) \cap L^2((0,t); W_2^1(\Omega)),$$
$$\mathcal{U}_0(\Omega_t) = \Big\{\eta \in \mathcal{U}(\Omega_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset\Big\},$$
$$\mathcal{T}(\Omega_t) = \big\{\eta \in L^2((0,t); W_2^1(\Omega)) : \eta_t \in L^2((0,t); L^2(\Omega))\big\},$$
$$\mathcal{T}_0(\Omega_t) = \Big\{\eta \in \mathcal{T}(\Omega_t) : \eta|_{\Gamma_{1,t}} = 0 \text{ in case } \Gamma_1 \neq \emptyset\Big\}$$

and introduce the following basic assumptions on the data of the direct problem:

$$a_{ij} \in L^\infty(\Omega), \quad a_{ij} = a_{ji}, \quad \vartheta \in C(\overline{\Gamma}_2), \quad \vartheta \geq 0, \tag{3.1}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_i \geq \epsilon|\lambda|^2, \quad x \in \Omega,\ \lambda \in \mathbb{R}^n \text{ with some } \epsilon > 0, \tag{3.2}$$

$$a \in L^{q_1}(\Omega), \quad \text{where } q_1 = 1 \text{ if } n = 1,\ q_1 > \frac{n}{2} \text{ if } n \geq 2, \tag{3.3}$$

$$\mu \in L^2(0,T), \quad m \in L^1(0,T)\,, \tag{3.4}$$

$$u_0 \in L^2(\Omega), \quad g \in \mathcal{T}(\Omega_T), \quad h \in L^2(\Gamma_{2,T}), \tag{3.5}$$

$$\begin{gathered} f \in L^2((0,T); L^{q_2}(\Omega)), \text{ where } q_2 = 1 \text{ if } n = 1, \\ q_2 \in (1, q_1) \text{ if } n = 2,\ \ q_2 = \frac{2n}{n+2} \text{ if } n \geq 3, \end{gathered} \tag{3.6}$$

$$\phi = (\phi_1, \dots, \phi_n) \in (L^2(\Omega_T))^n\,, \tag{3.7}$$

$$\varphi \in \mathcal{U}(\Omega_T), \text{ in case } \Gamma_1 \neq \emptyset \tag{3.8}$$

$$; \exists g_\varphi \in \mathcal{T}(\Omega_T) : \varphi = g_\varphi \text{ in } \Gamma_{1,T}. \tag{3.9}$$

If we assume additional conditions $a_{ij} \in W_2^1(\Omega)$, $\frac{\partial}{\partial x_i}\phi_i \in L^2(\Omega_T)$, $i = 1, \dots, n$, $\varphi_t \in L^2(\Omega_T)$ and suppose that (2.1)–(2.4) has a classical solution $u \in L^2(\Omega_T)$ such that $u_t, u_{x_i}, u_{x_ix_j} \in L^2(\Omega_T)$, $i, j = 1, \dots, n$, then multiplying (2.1) with a test function $\eta \in \mathcal{T}_0(\Omega_T)$ and integrating by parts we come to the relation

$$\begin{aligned} 0 = &\int_\Omega [(u + \mu * u)(x,T)\eta(x,T) - u_0(x)\eta(x,0)]\,dx - \iint_{\Omega_T} (u + \mu * u)\eta_t\,dx\,dt \\ &+ \iint_{\Omega_T} \Big[\sum_{i,j=1}^{n} a_{ij}(u_{x_j} - m * u_{x_j})\eta_{x_i} - a(u - m * u)\eta\Big]\,dx\,dt \\ &+ \iint_{\Gamma_{2,T}} (\vartheta u + h)\eta\,d\Gamma dt - \iint_{\Omega_T} (f\eta - \phi \cdot \nabla\eta)\,dx\,dt \\ &- \int_\Omega [\varphi(x,T)\eta(x,T) - \varphi(x,0)\eta(x,0)]dx + \iint_{\Omega_T} \varphi\eta_t\,dx\,dt. \end{aligned} \tag{3.10}$$

This relation makes sense also in a more general case when a_{ij}, ϕ, φ satisfy (3.1), (3.7), (3.8) and $u \in \mathcal{U}(\Omega_T)$.

We call a *weak solution* of the problem (2.1)–(2.4) a function belonging to $\mathcal{U}(\Omega_T)$ that satisfies the relation (3.10) for any $\eta \in \mathcal{T}_0(\Omega_T)$ and, in case $\Gamma_1 \neq \emptyset$, that fulfills the boundary condition (2.3).

Theorem 3.1. *Problem* (2.1)–(2.4) *has a unique weak solution. This solution satisfies the estimate*

$$\begin{aligned}\|u\|_{\mathcal{U}(\Omega_T)} \le C_0\Big[&\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^{q_2}(\Omega))} + \|\phi\|_{(L^2(\Omega_T))^n}\\ &+ \|\varphi\|_{\mathcal{U}(\Omega_T)} + \theta\{\|g\|_{\mathcal{T}(\Omega_T)} + \|g_\varphi\|_{\mathcal{T}(\Omega_T)}\} + \|h\|_{L^2(\Gamma_{2,T})}\Big],\end{aligned} \tag{3.11}$$

where $\theta = 0$ *in case* $\Gamma_1 = \emptyset$ *and* C_0 *is a constant independent of* $u_0, f, \phi, \varphi, g, h$.

Proof. Since $\mu \in L^2(0,T)$, the Volterra equation of the second kind

$$\widehat{\mu} + \mu * \widehat{\mu} = -\mu \quad \text{in } (0,T). \tag{3.12}$$

has a unique solution $\widehat{\mu} \in L^2(0,T)$ [6]. We call $\widehat{\mu}$ the resolvent kernel of μ. Further, let us consider the following problem:

$$\widehat{u}_t = A\widehat{u} - \widehat{m} * A\widehat{u} + \widehat{f} + \nabla\cdot\widehat{\phi} \quad \text{in } \Omega_T, \tag{3.13}$$

$$\widehat{u} = \widehat{u}_0 \quad \text{in } \Omega \times \{0\}, \tag{3.14}$$

$$\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T}, \tag{3.15}$$

$$-\nu_A\cdot\nabla\widehat{u} + \widehat{m} * \nu_A\cdot\nabla\widehat{u} = \vartheta\widehat{u} + \vartheta\widehat{\mu} * \widehat{u} + \widehat{h} + \nu\cdot\widehat{\phi} \quad \text{in } \Gamma_{2,T}, \tag{3.16}$$

where

$$\widehat{m} = m - \widehat{\mu} + m * \widehat{\mu}\,, \quad \widehat{f} = f + a\varphi - \widehat{m} * a\varphi,$$

$$\widehat{\phi}_i = \phi_i + \sum_{j=1}^n a_{ij}\varphi_{x_j} - \widehat{m} * \sum_{j=1}^n a_{ij}\varphi_{x_j}\,,$$

$$\widehat{h} = h + \vartheta\varphi + \vartheta\widehat{\mu} * \varphi\,, \quad \widehat{g} = g + \mu * g - g_\varphi\,, \quad \widehat{u}_0 = u_0 - \varphi(\cdot,0).$$

By the properties of m and $\widehat{\mu}$ we have $\widehat{m} \in L^1(0,T)$. Further, [14, Lemma 1] yields

$$\begin{gathered}\mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T);L^{q_3}(\Omega))\,, \quad \text{where } q_3 = \infty \text{ if } n = 1,\\ q_3 > \frac{q_1 q_2}{q_1 - q_2} \text{ if } n = 2,\ q_3 = \frac{2n}{n-2} \text{ if } n \ge 3\end{gathered} \tag{3.17}$$

and

$$\begin{gathered}av \in L^2((0,T);L^{q_2}(\Omega)) \text{ if } a \in L^{q_1}(\Omega),\ v \in L^2((0,T);L^{q_3}(\Omega)),\\ \|av\|_{L^2((0,T);L^{q_2}(\Omega))} \le C\|a\|_{L^{q_1}(\Omega)}\|v\|_{L^2((0,T);L^{q_3}(\Omega))},\end{gathered} \tag{3.18}$$

where C is a constant. Using the relations (3.17), (3.18), the properties of $\widehat{m}$, $\widehat{\mu}$, the assumptions (3.1)–(3.8), trace theorems and the Young theorem for convolutions we obtain

$$\begin{aligned}\widehat{d} &:= (\widehat{f}, \widehat{\phi}, \widehat{u}_0, \widehat{g}, \widehat{h}) \in \mathcal{X}\\ &:= L^2((0,T);L^{q_2}(\Omega)) \times (L^2(\Omega_T))^n \times L^2(\Omega) \times \mathcal{T}(\Omega_T) \times L^2(\Gamma_{2,T}),\end{aligned}$$

$$\|\widehat{d}\|_{\mathcal{X}} \le \bar{C}\|d\|_{\bar{\mathcal{X}}} \tag{3.19}$$

where $d = (f, \phi, u_0, g, h, \varphi, g_\varphi)$ and

$$\bar{\mathcal{X}} = L^2((0,T);L^{q_2}(\Omega)) \times (L^2(\Omega_T))^n \times L^2(\Omega) \times \mathcal{T}(\Omega_T) \times L^2(\Gamma_{2,T}) \times \mathcal{U}(\Omega_T) \times \mathcal{T}(\Omega_T)$$

and $\bar{C}$ is a constant. It was proved in [14, Theorem 1] that problem (2.1)–(2.4) in case $\mu = 0$ and $\varphi = 0$ has for any $(f, \phi, u_0, g, h) \in \mathcal{X}$ a unique weak solution and the corresponding solution operator $\mathcal{B}$ belongs to $\mathcal{L}(\mathcal{X};\mathcal{U}(\Omega_T))$. (Here $\mathcal{L}(X,Y)$ stands for the space of linear bounded operators from a Banach space X to a Banach space

Y.) This implies that problem (3.13)–(3.16) is equivalent in $\mathcal{U}(\Omega_T)$ to the following operator equation:

$$\widehat{u} = \mathcal{Q}\widehat{u} \quad \text{with} \quad \mathcal{Q}\widehat{u} = \mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * \widehat{u}) + \mathcal{B}\widehat{d}. \tag{3.20}$$

To study this equation, we will use the inequality

$$\|\widehat{\mu} * y\|_{L^2(\Omega_t)} \le \int_0^t |\widehat{\mu}(t-\tau)| \, \|y\|_{L^2(\Omega_\tau)} d\tau \,, \quad t \in [0,T] \tag{3.21}$$

that holds for any $y \in L^2(\Omega_T)$. This was proved in [14, inequality (3.12)].

Let $\widehat{u}^1, \widehat{u}^2 \in \mathcal{U}(\Omega_T)$, denote $v = \widehat{u}^1 - \widehat{u}^2$ and estimate $\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2 = \mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * v)$. To this end, fix $t \in [0,T]$ and define

$$P_t w = \begin{cases} w & \text{in } \Gamma_{2,t} \\ 0 & \text{in } \Gamma_{2,T} \setminus \Gamma_{2,t} \end{cases} \tag{3.22}$$

for $w : \Gamma_{2,T} \to \mathbb{R}$. Due to the causality, we have $\mathcal{B}(0,0,0,P_t\vartheta\widehat{\mu} * v)(x,\tau) = \mathcal{B}(0,0,0,\vartheta\widehat{\mu} * v)(x,\tau)$ for any $(x,\tau) \in \Omega_t$. Since $\mathcal{B} \in \mathcal{L}(\mathcal{X};\mathcal{U}(\Omega_T))$, the continuity of ϑ, the trace theorem and the inequality (3.21) with $y = v, v_{x_i}$, $i = 1,\dots,n$, it follows that

$$\begin{aligned} \|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_{\mathcal{U}(\Omega_t)} &= \|\mathcal{B}(0,0,0,0,\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_t)} \\ &= \|\mathcal{B}(0,0,0,0,P_t\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_t)} \\ &\le \|\mathcal{B}(0,0,0,0,P_t\vartheta\widehat{\mu} * v)\|_{\mathcal{U}(\Omega_T)} \\ &\le \|\mathcal{B}\|\|P_t\vartheta\widehat{\mu} * v\|_{L^2(\Gamma_{2,T})} = \|\mathcal{B}\|\|\vartheta\widehat{\mu} * v\|_{L^2(\Gamma_{2,t})} \\ &\le C_1\|\widehat{\mu} * v\|_{L^2((0,t);W_2^1(\Omega))} \\ &\le C_2 \int_0^t |\widehat{\mu}(t-\tau)| \, \|v\|_{L^2((0,\tau);W_2^1(\Omega))} d\tau \end{aligned} \tag{3.23}$$

with some constants C_1 and C_2. Let us define the weighted norm in $\mathcal{U}(\Omega_T)$: $\|v\|_\sigma = \sup_{0<t<T} e^{-\sigma t}\|v\|_{\mathcal{U}(\Omega_t)}$ where $\sigma \ge 0$. In view of (3.23) and $\mathcal{U}(\Omega_t) \hookrightarrow L^2((0,t);W_2^1(\Omega))$ we get

$$\begin{aligned} \|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_\sigma &\le C_3 \sup_{0<t<T} e^{-\sigma t} \int_0^t |\widehat{\mu}(t-\tau)| \, \|v\|_{\mathcal{U}(\Omega_\tau)} d\tau \\ &= C_3 \sup_{0<t<T} \int_0^t e^{-\sigma(t-\tau)} |\widehat{\mu}(t-\tau)| \, e^{-\sigma\tau} \, \|v\|_{\mathcal{U}(\Omega_\tau)} d\tau \\ &\le C_3 \int_0^T e^{-\sigma s} |\widehat{\mu}(s)| ds \sup_{0<\tau<T} e^{-\sigma\tau} \, \|v\|_{\mathcal{U}(\Omega_\tau)} \\ &= C_2 \int_0^T e^{-\sigma s} |\widehat{\mu}(s)| ds \, \|v\|_\sigma \end{aligned}$$

with some constant C_3. By the dominated convergence theorem, $\int_0^T e^{-\sigma s}|\widehat{\mu}(s)|ds \to 0$ as $\sigma \to \infty$. Thus, there exists σ_0 such that

$$C_3 \int_0^T e^{-\sigma_0 s} |\widehat{\mu}(s)| ds \le \frac{1}{2}.$$

Therefore, $\|\mathcal{Q}\widehat{u}^1 - \mathcal{Q}\widehat{u}^2\|_{\sigma_0} \le \frac{1}{2}\|\widehat{u}^1 - \widehat{u}^2\|_{\sigma_0}$. The operator $\mathcal{Q}$ is a contraction in $\mathcal{U}(\Omega_T)$. This implies that (3.13)–(3.16) has a unique weak solution in $\mathcal{U}(\Omega_T)$.

Moreover, observing (3.20) and the relation $\mathcal{Q}0 = \mathcal{B}\widehat{d}$, for the solution of (3.13)–(3.16) we obtain the estimate

$$\|\widehat{u}\|_{\sigma_0} = \|\mathcal{Q}\widehat{u} - \mathcal{Q}0 + \mathcal{Q}0\|_{\sigma_0} \le \|\mathcal{Q}\widehat{u} - \mathcal{Q}0\|_{\sigma_0} + \|\mathcal{B}\widehat{d}\|_{\sigma_0} \le \frac{1}{2}\|\widehat{u}\|_{\sigma_0} + \|\mathcal{B}\widehat{d}\|_{\sigma_0}$$

which implies

$$\|\widehat{u}\|_{\sigma_0} \le 2\|\mathcal{B}\widehat{d}\|_{\sigma_0} \le 2\|\mathcal{B}\widehat{d}\|_{\mathcal{U}(\Omega_T)} \le 2\|\mathcal{B}\|\|\widehat{d}\|_{\mathcal{X}}.$$

Observing the relation $e^{-\sigma_0 T}\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} \le \|\widehat{u}\|_{\sigma_0}$ and (3.19) we arrive at the estimate

$$\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} \le C_4\|d\|_{\bar{\mathcal{X}}} \tag{3.24}$$

with a constant C_4.

Further, let us define

$$u = \widehat{u} + \varphi + \widehat{\mu} * (\widehat{u} + \varphi). \tag{3.25}$$

Then $\widehat{u}$ is expressed in terms of u as

$$\widehat{u} = u + \mu * u - \varphi. \tag{3.26}$$

One can immediately check that the implications $u \in \mathcal{U}(\Omega_T) \Leftrightarrow \widehat{u} \in \mathcal{U}(\Omega_T)$ are valid. Moreover, it is easy to see that $\widehat{u}$ is a weak solution of (3.13)–(3.16) if and only if u is weak solution of (2.1)–(2.4). In view of the above-presented arguments we can conclude that (2.1)–(2.4) has a unique weak solution. From (3.25) we obtain

$$\|u\|_{\mathcal{U}(\Omega_T)} \le C_4(\|\widehat{u}\|_{\mathcal{U}(\Omega_T)} + \|\varphi\|_{\mathcal{U}(\Omega_T)})$$

with a constant C_5. This with (3.24) implies (3.11). The proof is complete. □

It is possible to give an equivalent form to the relation (3.10) that does not contain the derivative of the test function with respect to t. Namely, the following theorem holds.

Theorem 3.2. *The function $u \in \mathcal{U}(\Omega_T)$ satisfies the relation (3.10) for any $\eta \in \mathcal{T}_0(\Omega_T)$ if and only if it satisfies the relation*

$$\begin{aligned} 0 = &\int_\Omega (u + \mu * u - \varphi) * \eta \, dx - \int_\Omega \int_0^t (u_0(x) - \varphi(x,0))\eta(x,\tau)\, d\tau \, dx \\ &+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(u_{x_j} - m * u_{x_j}) * \eta_{x_i} - a(u - m * u) * \eta \Big] dx \\ &+ \int_{\Gamma_2} 1 * (\vartheta u + h) * \eta \, d\Gamma - \int_\Omega 1 * \Big(f * \eta - \sum_{i=1}^n \phi_i * \eta_{x_i} \Big) \, dx, \end{aligned} \tag{3.27}$$

for any $t \in [0,T]$ and $\eta \in \mathcal{U}_0(\Omega_T)$.

Proof. It is analogous to the proof of [14, Theorem 2] that considers the case $\varphi = 0$, $\mu = 0$. We have only to replace u by $\widehat{u} = u + \mu * u - \varphi$ in the term $K_1(t)$ appearing in formulas [14, (3.19), (3.20)] to get the desired result. □

Remark 3.3. Theorems 3.1 and 3.2 remain valid also in the case $\Gamma_2 = \emptyset$. In this case the terms $\|h\|_{\Gamma_{2,T}}$ and $\int_{\Gamma_2} 1 * (\vartheta u + h) * \eta \, d\Gamma$ are missing in (3.11) and (3.27), respectively.

4. Quasi-solution of IP. Fréchet derivative of cost functional

Assume that $n \in \{1;2;3\}$. Moreover, let us set $q_1 = 2$ if $n = 2$. Then any coefficient a that belongs to $L^2(\Omega)$ satisfies (3.3). For the weight functions κ_j we assume that

$$\kappa_j \in L^\infty((0,T);L^2(\Gamma_2)), \quad j = 1,2. \tag{4.1}$$

In the case $n = 1$ this assumption is simply $\kappa(x_l,\cdot) \in L^\infty(0,T)$, $x_l \in \Gamma_2 \subseteq \{c;d\}$. According to Theorem 3.1, $u \in \mathcal{U}(\Omega_T)$, thus $u(\cdot,T) \in L^2(\Omega)$, and the condition (2.5) is well-defined for $u_T \in L^2(\Omega)$. Moreover, by a trace theorem we have $u \in L^2(\Gamma_{2,T})$. This implies that $\int_{\Gamma_2} \kappa_j(x,\cdot)u(x,\cdot)d\Gamma \in L^2(0,T)$, $j = 1,2$, hence the condition (2.6) is well-defined for $v_j \in L^2(0,T)$, $j = 1,2$.

Let $M \subseteq \mathcal{Z} := L^2(\Omega) \times (L^2(0,T))^2$. We call the *quasi-solution* of IP in the set M an element $z^* \in \arg\min_{z\in M} J(z)$, where J is the cost functional

$$J(z) = \|u(\cdot,T;z) - u_T\|^2_{L^2(\Omega)} + \sum_{j=1}^{2} \| \int_{\Gamma_2} \kappa_j(x,\cdot)u(x,\cdot;z)d\Gamma - v_j\|^2_{L^2(0,T)}$$

and $u(x,t;z)$ is the weak solution of the direct problem (2.1)–(2.4) corresponding to given $z = (a,m,\mu)$. In case $n = 1$ the integral $\int_{\Gamma_2} \kappa_j(x,t)u(x,t;z)d\Gamma$ in the definition of J is replaced by $\sum_{l=1}^{L} \kappa_j(x_l,t)u(x_l,t;z)$.

Theorem 4.1. *The functional J is Fréchet differentiable in $\mathcal{Z}$ and*

$$\begin{aligned} &J'(z)\Delta z \\ &= 2\int_\Omega [u(x,T;z) - u_T(x)]\,\Delta u(x,T)dx \\ &\quad + 2\sum_{j=1}^{2}\int_0^T \Big[\int_{\Gamma_2} \kappa_j(y,t)u(y,t;z)d\Gamma - v_j(t)\Big] \int_{\Gamma_2} \kappa_j(x,t)\Delta u(x,t)d\Gamma dt, \end{aligned} \tag{4.2}$$

where $\Delta z = (\Delta a, \Delta m, \Delta\mu) \in \mathcal{Z}$ and $\Delta u \in \mathcal{U}(\Omega_T)$ is the z- and Δz-dependent weak solution of the following problem:

$$\begin{aligned} \Delta u_t + (\mu * \Delta u)_t = A\Delta u - m * A\Delta u + \Delta a[u - m * u] - \Delta m * au \\ - \nabla\cdot\Big[\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\Big] - (\Delta\mu * u)_t \quad \text{in } \Omega_T, \end{aligned} \tag{4.3}$$

$$\Delta u = 0 \quad \text{in } \Omega \times \{0\}, \tag{4.4}$$

$$\Delta u = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.5}$$

$$\begin{aligned} &- \nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u \\ &= \vartheta\Delta u - \nu\cdot\Big[\Delta m * \sum_{j=1}^{n} a_{ij}u_{x_j}\Big] \quad \text{in } \Gamma_{2,T}. \end{aligned} \tag{4.6}$$

Proof. Denote $\widetilde{\Delta}u = u(x,t;z+\Delta z) - u(x,t;z)$ and define $\widehat{\Delta}u = \widetilde{\Delta}u - \Delta u$. Then we can represent the difference of J as follows:

$$J(z+\Delta z) - J(z) = \mathrm{RHS} + \Theta, \tag{4.7}$$

where RHS is the right-hand side of the equality (4.2) and

$$\Theta = 2\int_\Omega [u(x,T) - u_T(x)]\widehat{\Delta}u(x,T)dx$$

$$
\begin{aligned}
&+2\sum_{j=1}^{2}\int_0^T\Big[\int_{\Gamma_2}\kappa_j(y,t)u(y,t)d\Gamma-v_j(t)\Big]\int_{\Gamma_2}\kappa_j(x,t)\widehat{\Delta}u(x,t)d\Gamma dt\\
&+\int_\Omega\Big\{(\Delta u+\widehat{\Delta}u)(x,T)\Big\}^2dx\\
&+\sum_{j=1}^{2}\int_0^T\Big\{\int_{\Gamma_2}\kappa_j(x,t)(\Delta u+\widehat{\Delta}u)(x,t)d\Gamma\Big\}^2dt.
\end{aligned}
$$

Let us study problem (4.3)–(4.6). To this end we estimate the terms in the right-hand side of (4.3). Observing the relations $u\in\mathcal{U}(\Omega_T)$, (3.17), (3.18), $L^2(\Omega)\hookrightarrow L^{q_1}(\Omega)$ and using the Young and Cauchy inequalities we deduce

$$
\begin{aligned}
&\|\Delta a[u-m*u]-\Delta m*au\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\le c_1\|u\|_{\mathcal{U}(\Omega_T)}\left[(1+\|m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)}+\|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\right]\\
&\le c_2(z,u)\|\Delta z\|,
\end{aligned}
\tag{4.8}
$$

where c_1 is a constant, c_2 is a coefficient depending on $z=(a,m,\mu)$, u and $\|\cdot\|$ denotes the norm in $\mathcal{Z}$. Taking the boundendness of a_{ij} into account we similarly get

$$
\|\Delta m*\sum_{j=1}^{n}a_{ij}u_{x_j}\|_{(L^2(\Omega_T))^n}\le c_3\|u\|_{\mathcal{U}(\Omega_T)}\|\Delta m\|_{L^2(0,T)}
\tag{4.9}
$$

with a constant c_3. Next let us estimate the term $\Delta\mu*u$ at the right-hand side of (4.3). Since $u\in C([0,T];L^2(\Omega))$ and $\Delta\mu\in L^2(0,T)$, it is easy to check that $\Delta\mu*u\in C([0,T];L^2(\Omega))$ and $\|\Delta\mu*u\|_{C([0,T];L^2(\Omega))}\le T^{1/2}\|u\|_{C([0,T];L^2(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Similarly, $\|\Delta\mu*u\|_{L^2((0,T);W_2^1(\Omega))}\le T^{1/2}\|u\|_{L^2((0,T);W_2^1(\Omega))}\|\Delta\mu\|_{L^2(0,T)}$. Taking these estimates together, we have

$$
\|\Delta\mu*u\|_{\mathcal{U}(\Omega_T)}\le T^{1/2}\|u\|_{\mathcal{U}(\Omega_T)}\|\Delta\mu\|_{L^2(0,T)}.
\tag{4.10}
$$

Since $u=g$ in $\Gamma_{1,T}$, we find that

$$
\Delta\mu*u=\Delta\mu*g\quad\text{in }\Gamma_{1,T}.
\tag{4.11}
$$

Using the assumption $g\in\mathcal{T}(\Omega_T)$ and the Young and Cauchy inequalities again, we obtain

$$
\begin{aligned}
\|\Delta\mu*g\|_{\mathcal{T}(\Omega_T)}&=\|\Delta\mu*g\|_{L^2((0,T);W_2^1(\Omega))}+\|(\Delta\mu*g)_t\|_{L^2((0,T);L^2(\Omega))}\\
&=\|\Delta\mu*g\|_{L^2((0,T);W_2^1(\Omega))}+\|\Delta\mu*g_t\|_{L^2((0,T);L^2(\Omega))}\\
&\quad+\|\Delta\mu\,g(\cdot,0)\|_{L^2((0,T);L^2(\Omega))}\\
&\le c_4\|\Delta\mu\|_{L^2(0,T)}
\end{aligned}
\tag{4.12}
$$

with a constant c_4. Relations (4.8)–(4.12) show that Theorem 3.1 holds for problem (4.3)–(4.6), hence it has a unique weak solution $\Delta u\in\mathcal{U}(\Omega_T)$. Using the estimate (3.11) for the solution of this problem we obtain

$$
\begin{aligned}
&\|\Delta u\|_{\mathcal{U}(\Omega_T)}\\
&\le C_0\Big[\|\Delta a[u-m*u]+\Delta m*au\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\quad+\|\Delta m*\sum_{j=1}^{n}a_{ij}u_{x_j}\|_{(L^2(\Omega_T))^n}+\|\Delta\mu*u\|_{\mathcal{U}(\Omega_T)}+\theta\|\Delta\mu*g\|_{\mathcal{T}(\Omega_T)}\Big]\\
&\le c_5(z,u)\|\Delta z\|
\end{aligned}
\tag{4.13}
$$

with a coefficient c_5 depending on z, u.

The function $\widetilde{\Delta}u$ satisfies the problem

$$\widehat{\Delta}u_t + (\mu * \widehat{\Delta}u)_t = A\widehat{\Delta}u - m * A\widehat{\Delta}u + f + \widehat{f} + \nabla\cdot\phi + \nabla\cdot\widehat{\phi} + \varphi_t + \widehat{\varphi}_t \quad \text{in } \Omega_T, \tag{4.14}$$

$$\widehat{\Delta}u = 0 \quad \text{in } \Omega\times\{0\}, \tag{4.15}$$

$$\widehat{\Delta}u = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.16}$$

$$-\nu_A\cdot\nabla\widehat{\Delta}u + m * \nu_A\cdot\nabla\widehat{\Delta}u = \vartheta\widehat{\Delta}u + \nu\cdot\phi + \nu\cdot\widehat{\phi} \quad \text{in } \Gamma_{2,T}, \tag{4.17}$$

where

$$f = \Delta a\Delta u - (m+\Delta m) * \Delta a\Delta u - \Delta m * \Delta a u - \Delta m * a\Delta u,$$
$$\widehat{f} = \Delta a\widehat{\Delta}u - (m+\Delta m) * \Delta a\widehat{\Delta}u - \Delta m * a\widehat{\Delta}u,$$
$$\phi = -\Delta m * \sum_{j=1}^{n} a_{ij}\Delta u_{x_j}, \quad \widehat{\phi} = -\Delta m * \sum_{j=1}^{n} a_{ij}\widehat{\Delta}u_{x_j},$$
$$\varphi = -\Delta\mu * \Delta u, \quad \widehat{\varphi} = -\Delta\mu * \widehat{\Delta}u.$$

Similarly to (4.8)–(4.10) we deduce the following estimates:

$$\begin{aligned}
&\|f\|_{L^2((0,T);L^{q_2}(\Omega))}\\
&\le c_6\Big\{(1 + \|m\|_{L^2(0,T)} + \|\Delta m\|_{L^2(0,T)})\|\Delta a\|_{L^2(\Omega)}\|\Delta u\|_{\mathcal{U}(\Omega_T)}\\
&\quad + \|u\|_{\mathcal{U}(\Omega_T)}\|\Delta m\|_{L^2(0,T)}\|\Delta a\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}\|\Delta m\|_{L^2(0,T)}\|\Delta u\|_{\mathcal{U}(\Omega_T)}\Big\}\\
&\le c_7(z,u)\Big\{\big[\|\Delta z\| + \|\Delta z\|^2\big]\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\Delta z\|^2\Big\},\\
&\|\widehat{f}\|_{L^2(0,T;L^{q_2}(\Omega))} \le c_8(z)\left[\|\Delta z\| + \|\Delta z\|^2\right]\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)},\\
&\|\phi\|_{(L^2(\Omega_T))^n} \le c_9\|\Delta z\|\|\Delta u\|_{\mathcal{U}(\Omega_T)},\\
&\|\widehat{\phi}\|_{(L^2(\Omega_T))^n} \le c_9\|\Delta z\|\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)},\\
&\|\varphi\|_{\mathcal{U}(\Omega_T)} \le T^{1/2}\|\Delta z\|\|\Delta u\|_{\mathcal{U}(\Omega_T)},\\
&\|\widehat{\varphi}\|_{\mathcal{U}(\Omega_T)} \le T^{1/2}\|\Delta z\|\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)}
\end{aligned}$$

with some coefficients $c_6,\dots,c_9$. Moreover, since $\Delta u = \widetilde{\Delta}u = 0$ in $\Gamma_{1,T}$, we have $\varphi = \widehat{\varphi} = 0$ in $\Gamma_{1,T}$. Applying the estimate (3.11) to the solution of the problem (4.14)–(4.17) we get

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le c_{10}(z,u)\Big\{\left[\|\Delta z\| + \|\Delta z\|^2\right]\Big\{\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)}\Big\} + \|\Delta z\|^2\Big\}$$

with a coefficient c_{10}. Provided $\|\Delta z\|$ is sufficiently small; i.e., $\|\Delta z\| + \|\Delta z\|^2 \le \frac{1}{2c_{10}(z,u)}$, we have

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le 2c_{10}(z,u)\Big\{\left[\|\Delta z\| + \|\Delta z\|^2\right]\|\Delta u\|_{\mathcal{U}(\Omega_T)} + \|\Delta z\|^2\Big\}.$$

Due to (4.13), this yields

$$\|\widehat{\Delta}u\|_{\mathcal{U}(\Omega_T)} \le c_{11}(z,u)\left[\|\Delta z\|^2 + \|\Delta z\|^3\right] \tag{4.18}$$

with a coefficient c_{11}.

In view of (4.13), (4.18) and the assumption $\kappa_j \in L^\infty((0,T);L^2(\Gamma_2))$ the right-hand side of (4.2) RHS and the quantity Θ satisfy the estimates

$$st|\mathrm{RHS}| \le c_{12}(z,u)\|\Delta z\|, \quad |\Theta| \le c_{13}(z,u)\sum_{l=2}^{6}\|\Delta z\|^l, \tag{4.19}$$

st where c_{12} and c_{13} are some coefficients. Moreover, RHS is linear with respect to Δz. This with (4.7) shows that J is Fréchet differentiable in $\mathcal{Z}$ and $J'(z)\Delta z$ equals RHS. □

Theorem 4.2. *Assume $g = 0$. Then the Fréchet derivative of J admits the form*

$$J'(z)\Delta z = \int_\Omega \gamma_1(x)\Delta a(x)dx + \int_0^T \gamma_2(t)\Delta m(t)dt + \int_0^T \gamma_3(t)\Delta\mu(t)dt, \tag{4.20}$$

where

$$\gamma_1(x) = [(u - m*u)*\psi](x,T), \tag{4.21}$$

$$\gamma_2(t) = -\int_\Omega \Big[au*\psi + \sum_{i,j=1}^n a_{ij}\psi_{x_i}*u_{x_j}\Big](x,T-t)dx, \tag{4.22}$$

$$\begin{aligned}
&\gamma_3(t)\\
&= -\int_\Omega \Big[au*\psi + au*\psi*[\widehat{\mu} - m - m*\widehat{\mu}]\\
&\quad + \sum_{i,j=1}^n a_{ij}\psi_{x_i}*u_{x_j} + \sum_{i,j=1}^n a_{ij}\psi_{x_i}*u_{x_j}*[\widehat{\mu} - m - m*\widehat{\mu}]\Big](x,T-t)dx\\
&\quad - \int_{\Gamma_2}[\vartheta(u + \widehat{\mu}*u)*\psi](x,T-t)d\Gamma\\
&\quad - 2\int_\Omega\{u(x,T) - u_T(x)\}[u + \widehat{\mu}*u](x,T-t)dx\\
&\quad - 2\sum_{j=1}^2\int_t^T\Big[\int_{\Gamma_2}\kappa_j(y,\tau)u(y,\tau)d\Gamma - v_j(\tau)\Big]\int_{\Gamma_2}\kappa_j(x,\tau)\big[u\\
&\quad + \widehat{\mu}*u\big](x,\tau-t)d\Gamma d\tau,
\end{aligned} \tag{4.23}$$

where $\widehat{\mu}$ is the solution of (3.12), *$u(x,t) = u(x,t;z)$ and $\psi \in \mathcal{U}(\Omega_T)$ is the z-dependent weak solution of the following "adjoint" problem:*

$$\Delta\psi_t + (\mu*\Delta\psi)_t = A\Delta\psi - m*A\Delta\psi \quad \text{in } \Omega_T, \tag{4.24}$$

$$\Delta\psi = 2[u(\cdot,T) - u_T] \quad \text{in } \Omega\times\{0\}, \tag{4.25}$$

$$\Delta\psi = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.26}$$

$$-\nu_A\cdot\nabla\Delta\psi + m*\nu_A\cdot\nabla\Delta\psi = \vartheta\Delta\psi + h^\circ \quad \text{in } \Gamma_{2,T}, \tag{4.27}$$

where

$$\begin{aligned}
&h^\circ(x,t)\\
&= -2\sum_{j=1}^2\kappa_j(x,T-t)\Big[\int_{\Gamma_2}\kappa_j(y,T-t)u(y,T-t)d\Gamma - v_j(T-t)\Big].
\end{aligned} \tag{4.28}$$

Proof. Define $\Delta w = \Delta u + \Delta\mu * u + \widehat{\mu} * \Delta\mu * u$. Since $u, \Delta u \in \mathcal{U}(\Omega_T)$, we have $\Delta w \in \mathcal{U}(\Omega_T)$. Moreover, using (3.12) it is easy to see that $\Delta u + \mu * \Delta u + \Delta\mu * u = \Delta w + \mu * \Delta w$. Using this relation for the time derivatives in (4.3) and the equality $\Delta u = \Delta w - \Delta\mu * u - \widehat{\mu} * \Delta\mu * u$ for other terms containing Δu in (4.3)–(4.6) we see that Δw is the weak solution of the problem

$$\Delta w_t + (\mu * \Delta w)_t = A\Delta w - m * A\Delta w + f^\dagger + \nabla \cdot \phi^\dagger \quad \text{in } \Omega_T, \tag{4.29}$$

$$\Delta w = 0 \quad \text{in } \Omega \times \{0\}, \tag{4.30}$$

$$\Delta w = 0 \quad \text{in } \Gamma_{1,T}, \tag{4.31}$$

$$-\nu_A \cdot \nabla\Delta u + m * \nu_A \cdot \nabla\Delta u = \vartheta\Delta u + h^\dagger + \nu \cdot \phi^\dagger \quad \text{in } \Gamma_{2,T}, \tag{4.32}$$

where

$$f^\dagger = \Delta a[u - m * u] - a\Delta m * u - a\Delta\mu * u - a\Delta\mu * u * [\widehat{\mu} - m - m * \widehat{\mu}], \tag{4.33}$$

$$\phi^\dagger = (\phi_1^\dagger, \dots, \phi_n^\dagger),$$

$$\phi_i^\dagger = -\Delta m * \sum_{j=1}^n a_{ij} u_{x_j} - \Delta\mu * \sum_{j=1}^n a_{ij} u_{x_j} - \Delta\mu * \sum_{j=1}^n a_{ij} u_{x_j} * [\widehat{\mu} - m - m * \widehat{\mu}], \tag{4.34}$$

$$h^\dagger = -\vartheta\Delta\mu * [u + \widehat{\mu} * u]. \tag{4.35}$$

Let us write the weak form (3.27) for the problem for Δw and use the test function $\eta = \psi$. Then we obtain

$$\begin{aligned} 0 = &\int_\Omega (\Delta w + \mu * \Delta w) * \psi \, dx + \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(\Delta w_{x_j} - m * \Delta w_{x_j}) * \psi_{x_i} \\ &- a(\Delta w - m * \Delta w) * \psi \Big] dx + \int_{\Gamma_2} 1 * (\vartheta \Delta w + h^\dagger) * \psi \, d\Gamma \\ &- \int_\Omega 1 * \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i} \Big) dx. \end{aligned} \tag{4.36}$$

Next we write the weak form (3.27) for the problem for ψ and use the test function $\eta = \Delta w$ to get

$$\begin{aligned} 0 = &\int_\Omega (\psi + \mu * \psi) * \Delta w \, dx - 2 \int_\Omega \int_0^t u(x,T) - u_T(x)]\Delta w(x,\tau) d\tau dx \\ &+ \int_\Omega 1 * \Big[\sum_{i,j=1}^n a_{ij}(\psi_{x_j} - m * \psi_{x_j}) * \Delta w_{x_i} - a(\psi - m * \psi) * \Delta w \Big] dx \\ &+ \int_{\Gamma_2} 1 * (\vartheta\psi + h^\circ) * \Delta w \, d\Gamma. \end{aligned} \tag{4.37}$$

Subtracting (4.36) from (4.37), differentiating with respect to t and setting $t = T$ we have

$$\begin{aligned} &2 \int_\Omega [u(x,T) - u_T(x)]\Delta w(x,T) d\tau dx - \int_{\Gamma_2} (h^\circ * \Delta w)(x,T) \, d\Gamma \\ &= \int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i} \Big)(x,T) \, dx - \int_{\Gamma_2} (h^\dagger * \psi)(x,T) \, d\Gamma. \end{aligned}$$

Observing the relations $\Delta w = \Delta u + \Delta\mu * u + \widehat{\mu} * \Delta\mu * u$, (4.28) and (4.2) we obtain the formula

$$\begin{aligned} J'(z)\Delta z = &\int_\Omega \Big(f^\dagger * \psi - \sum_{i=1}^n \phi_i^\dagger * \psi_{x_i}\Big)(x,T)dx - \int_{\Gamma_2} (h^\dagger * \psi)(x,T)\, d\Gamma \\ &- 2\int_\Omega [u(x,T) - u_T(x)]\big\{(\Delta\mu + \widehat{\mu} * \Delta\mu) * u\big\}(x,T)dx \\ &- 2\sum_{j=1}^2 \int_0^T \Big[\int_{\Gamma_2} \kappa_j(y,t)u(y,t;z)d\Gamma - v_j(t)\Big] \\ &\times \int_{\Gamma_2} \kappa_j(x,t)\big\{(\Delta\mu + \widehat{\mu} * \Delta\mu) * u\big\}(x,t)d\Gamma dt. \end{aligned}$$

Rearranging the terms yields (4.20) with (4.21)-(4.23). □

The formula (4.20) shows that the vector $(\gamma_1, \gamma_2, \gamma_3)$ is a representation of $J'(z)$ in the space $\mathcal{Z}$. It can be used in gradient-type minimization algorithms (cf. [13, 14]).

5. Existence of quasi-solutions

Theorem 5.1. *Let M be compact. Then IP has a quasi-solution in M.*

Proof. It coincides with the proof of [14, Theorem 7 (ii)]. We use the continuity of J that is a consequence of the Fréchet differentiability of J proved in the previous section. □

Theorem 5.2. *Let $n = 1$, $\Omega = (c,d)$, $\varphi = g_\varphi = 0$, $g(x,0) = 0$ and M be bounded, closed and convex. Then IP has a quasi-solution in M.*

Proof. This theorem follows from Weierstrass existence theorem [20] provided we are able to show that J is weakly sequentially lower semi-continuous in M. We will prove that J is in fact weakly sequentially continuous in M.

Let us choose some sequence $z_k = (a_k, m_k, \mu_k) \in M$ such that $z_k \rightharpoonup z = (a, m, \mu) \in M$. Then it is easy to see that $a_k \rightharpoonup a$ in $L^2(c,d)$ and $m_k \rightharpoonup m$, $\mu_k \rightharpoonup \mu$ in $L^2(0,T)$. As in the proof of Theorem 3.1, let $\widehat{\mu} \in L^2(0,T)$ be the solution of (3.12). Similarly, let $\widehat{\mu}_k \in L^2(0,T)$ be the solution of the equation $\widehat{\mu}_k + \mu_k * \widehat{\mu}_k = -\mu_k$ in $(0,T)$. Let us show that $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ in $L^2(0,T)$. To this end we firstly verify the boundedness of the sequence $\widehat{\mu}_k$. Multiplying the equation of $\widehat{\mu}_k$ by $e^{-\sigma t}$, $\sigma > 0$, and estimating by means of the Young and Cauchy inequalities we obtain

$$\begin{aligned} \|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} &\le \|e^{-\sigma t}\mu_k * e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\ &\le \|e^{-\sigma t}\mu_k\|_{L^1(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)} \\ &\le \|e^{-\sigma t}\|_{L^2(0,T)}\|\mu_k\|_{L^2(0,T)}\|e^{-\sigma t}\widehat{\mu}_k\|_{L^2(0,T)} + \|e^{-\sigma t}\mu_k\|_{L^2(0,T)}. \end{aligned}$$

Observing that $\|e^{-\sigma t}\|_{L^2(0,T)} \le 1/\sqrt{2\sigma}$ and choosing $\sigma = \sigma_1 = 2[\sup\|\mu_k\|_{L^2(0,T)}]^2$ we get

$$\|e^{-\sigma_1 t}\widehat{\mu}_k\|_{L^2(0,T)} \le 2\|e^{-\sigma_1 t}\mu_k\|_{L^2(0,T)} \;\Rightarrow\; \|\widehat{\mu}_k\|_{L^2(0,T)} \le 2e^{\sigma_1 T}\sup\|\mu_k\|_{L^2(0,T)}.$$

This shows that the sequence $\widehat{\mu}_k$ is bounded. The difference $\widehat{\mu}_k - \widehat{\mu}$ can be expressed as

$$\widehat{\mu}_k - \widehat{\mu} = -(\mu_k - \mu) - v_k * (\mu_k - \mu),$$

where $v_k = \widehat{\mu} + \widehat{\mu}_k + \widehat{\mu} * \widehat{\mu}_k$ is a bounded sequence in $L^2(0,T)$. Denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0,T)$. With an arbitrary $\zeta \in L^2(0,T)$ we have

$$\langle \widehat{\mu}_k - \widehat{\mu}, \zeta \rangle = -\langle \mu_k - \mu, \zeta \rangle - N_k, \; N_k = \int_0^T v_k(\tau) \int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau+s) ds d\tau. \quad (5.1)$$

Since $\zeta(\tau + \cdot) \in L^2(0, T-\tau)$ for $\tau \in (0,T)$, it holds $\int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau+s)ds \to 0$ for $\tau \in (0,T)$. Moreover, since μ_k is bounded in $L^2(0,T)$, the sequence of τ-dependent functions $| \int_0^{T-\tau} (\mu_k - \mu)(s)\zeta(\tau+s)ds|$ is bounded by a constant. By the Cauchy inequality and the dominated convergence theorem, we find

$$|N_k| \le \|v_k\|_{L^2(0,T)} \| \int_0^{T-\cdot} (\mu_k - \mu)(s)\zeta(\cdot + s) ds\|_{L^2(0,T)} \to 0.$$

Thus, from (5.1), in view of $\mu_k \rightharpoonup \mu$, we obtain $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$.

Let us define

$$\widehat{u} = u + \mu * u\,, \quad \widehat{u}_k = u_k + \mu_k * u_k,$$

where $u = u(x,t;z)$ and $u_k = u(x,t;z_k)$ are the weak solutions of (2.1)–(2.4) corresponding to the vectors z and z_k, respectively. The relations $u, u_k \in \mathcal{U}(\Omega_T)$ and $\mu, \mu_k \in L^2(0,T)$ imply $\widehat{u}, \widehat{u}_k \in \mathcal{U}(\Omega_T)$. Observing the definitions of the resolvent kernels $\widehat{\mu}$ and $\widehat{\mu}_k$ we deduce

$$\begin{gathered} u = \widehat{u} + \widehat{\mu} * \widehat{u}\,, \quad u_k = \widehat{u}_k + \widehat{\mu}_k * \widehat{u}_k, \\ u_k - u = \widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}. \end{gathered}$$

In view of the latter relation we express the difference of values of the functional J as follows:

$$\begin{aligned} &J(z_k) - J(z) \\ &= \int_c^d (u_k - u)^2(x,T)dx + 2\int_c^d [u(x,T) - u_T(x)](u_k - u)(x,T)dx \\ &\quad + \sum_{j=1}^2 \int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)(u_k - u)(x_l,t)\Big]^2 dt \\ &\quad + 2\sum_{j=1}^2 \int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)u(x_l,t) - v_j(t)\Big]\Big[\sum_{l=1}^L \kappa_j(x_l,t)(u_k - u)(x_l,t)\Big] dt \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} I_k^1 &= \int_c^d \Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)^2 (x,T)dx, \\ I_k^2 &= 2\int_c^d [u(x,T) - u_T(x)]\Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x,T)dx, \\ I_k^3 &= \sum_{j=1}^2 \int_0^T \Big[\sum_{l=1}^L \kappa_j(x_l,t)\Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x_l,t)\Big]^2 dt, \end{aligned}$$

$$I_k^4 = 2\sum_{j=1}^{2}\int_0^T \Big[\sum_{l=1}^{L}\kappa_j(x_l,t)u(x_l,t) - v_j(t)\Big] \times \Big[\sum_{l=1}^{L}\kappa_j(x_l,t)\Big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u}) + (\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\Big)(x_l,t)\Big]dt.$$

Using the Cauchy inequality, $\widehat{\mu} \in L^2(0,T)$, $\widehat{u}_k, \widehat{u} \in \mathcal{U}(\Omega_T)$ and the boundedness of the sequence $\widehat{\mu}_k$ in $L^2(0,T)$ we obtain

$$\begin{aligned}|I_k^1| &\le \|\big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|^2_{L^2(c,d)} \\ &\quad + 2\|\big((\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\big)(\cdot,T)\|_{L^2(c,d)}\|\big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|_{L^2(c,d)} + R_k^1 \\ &\le \tilde{C}_1\big(\|\widehat{u}_k - \widehat{u}\|^2_{\mathcal{U}(\Omega_T)} + \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)}\big) + R_k^1\end{aligned}$$

with a constant $\tilde{C}_1$ and

$$R_k^1 = \int_c^d \Big[\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big]^2 dx.$$

Since $\widehat{u} \in \mathcal{U}(\Omega_T) \subset L^2(\Omega_T)$, by Tonelli's theorem it holds $\widehat{u}(x,\cdot) \in L^2(0,T)$ a.e. $x \in (c,d) \Rightarrow \widehat{u}(x,T-\cdot) \in L^2(0,T)$ a.e. $x \in (c,d)$. Thus, in view of $\mu_k \rightharpoonup \widehat{\mu}$ in $L^2(0,T)$ we have $\int_0^T(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau \to 0$ a.e. $x \in (c,d)$. Moreover, $\Big[\int_0^T(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau\Big]^2 \le \tilde{C}_{11}\int_0^T[\widehat{u}(x,\tau)]^2 d\tau \in L^1(c,d)$ with a constant $\tilde{C}_{11}$, because the sequence $\widehat{\mu}_k$ is bounded in $L^2(0,T)$. Therefore, by the dominated convergence theorem we obtain $R_k^1 \to 0$. Similarly for I_k^2 we get

$$\begin{aligned}|I_k^2| &\le 2\|u(\cdot,T) - u_T\|_{L^2(c,d)}\|\big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(\cdot,T)\|_{L^2(c,d)} + R_k^2 \\ &\le \tilde{C}_2\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} + R_k^2,\end{aligned}$$

where $\tilde{C}_2$ is a constant and

$$stR_k^2 = \int_c^d [u(x,T) - u_T(x)]\int_0^T (\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x,T-\tau)d\tau dx. \tag{5.3}$$

st By the same reasons as above, it holds $R_k^2 \to 0$. Next, let us estimate I_k^3:

$$\begin{aligned}|I_k^3| &\le L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big[\|\kappa_j(x_l,\cdot)\|^2_{L^\infty(0,T)}\|\big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(x_l,\cdot)\|^2_{L^2(0,T)}\Big] \\ &\quad + 2L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big[\|\kappa_j(x_l,\cdot)\|^2_{L^\infty(0,T)}\|\big((\widehat{\mu}_k - \widehat{\mu}) * \widehat{u}\big)(x_l,\cdot)\|_{L^2(0,T)} \\ &\quad \times \|\big(\widehat{u}_k - \widehat{u} + \widehat{\mu}_k * (\widehat{u}_k - \widehat{u})\big)(x_l,\cdot)\|_{L^2(0,T)}\Big] + R_k^3 \\ &\le \tilde{C}_3\big(\|\widehat{u}_k - \widehat{u}\|^2_{\mathcal{U}(\Omega_T)} + \|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)}\big) + R_k^3,\end{aligned}$$

where $\tilde{C}_3$ is a constant and

$$R_k^3 = L^2\sum_{j=1}^{2}\max_{1\le l\le L}\Big\{\|\kappa_j(x_l,\cdot)\|^2_{L^\infty(0,T)}\int_0^T\Big[\int_0^t(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l,t-\tau)d\tau\Big]^2 dt\Big\}.$$

Here we also used the embedding $\mathcal{U}(\Omega_T) \hookrightarrow L^2((0,T);C[c,d])$ that holds in the case $n=1$. Since $\widehat{u}(x_l,t-\cdot) \in L^2(0,t)$ for all $t \in (0,T)$ we get $\int_0^t(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l,t-$

$\tau)d\tau \to 0$ for all $t \in (0,T)$. Moreover, the sequence $|\int_0^t(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l, t-\tau)d\tau|$ is bounded by a constant. Consequently, $R_k^3 \to 0$. Analogously we deduce the estimate

$$|I_k^4| \le \tilde{C}_4\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} + R_k^4, \quad \text{where } \tilde{C}_4 \text{ is a constant},$$

$$R_k^4 = 2L\sum_{j=1}^{2}\Big\|\sum_{l=1}^{L}\kappa_j(x_l,t)u(x_l,\cdot) - v_j\Big\|_{L^2(0,T)}\max_{1\le l\le L}\Big\{\|\kappa_j(x_l,\cdot)\|_{L^\infty(0,T)} \times \Big[\int_0^T\Big[\int_0^t(\widehat{\mu}_k - \widehat{\mu})(\tau)\widehat{u}(x_l,t-\tau)d\tau\Big]^2 dt\Big]^{1/2}\Big\},$$

where $R_k^4 \to 0$.

Note that if we manage to show that $\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} \to 0$ then the proof is complete. Indeed, in this case by virtue of $R_k^i \to 0$, $i = 1,2,3,4$, from the estimates of I_k^i we get $I_k^i \to 0$, $i = 1,2,3,4$ and due to (5.2) we obtain $J(z_k) \to J(z)$, which implies the statement of the theorem.

As in the proof of Theorem 3.1 we can show that $\widehat{u}$ and $\widehat{u}_k$ are the weak solutions of the following problems:

$$\widehat{u}_t = A\widehat{u} - \widehat{m} * A\widehat{u} + f + \phi_x \quad \text{in } \Omega_T, \tag{5.4}$$

$$\widehat{u} = u_0 \quad \text{in } \Omega \times \{0\}, \tag{5.5}$$

$$\widehat{u} = \widehat{g} \quad \text{in } \Gamma_{1,T}, \tag{5.6}$$

$$-\nu_A \cdot \nabla\widehat{u} + \widehat{m} * \nu_A \cdot \nabla\widehat{u} = \vartheta\widehat{u} + \vartheta\widehat{\mu} * \widehat{u} + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \tag{5.7}$$

$$\widehat{u}_{k,t} = A_k\widehat{u}_k - \widehat{m}_k * A_k\widehat{u}_k + f + \phi_x \quad \text{in } \Omega_T, \tag{5.8}$$

$$\widehat{u}_k = u_0 \quad \text{in } \Omega \times \{0\}, \tag{5.9}$$

$$\widehat{u}_k = \widehat{g}_k \quad \text{in } \Gamma_{1,T}, \tag{5.10}$$

$$-\nu_A \cdot \nabla\widehat{u}_k + \widehat{m}_k * \nu_A \cdot \nabla\widehat{u}_k = \vartheta\widehat{u}_k + \vartheta\widehat{\mu}_k * \widehat{u}_k + h + \nu \cdot \phi \quad \text{in } \Gamma_{2,T}, \tag{5.11}$$

where $A_k v = (a_{11}v_x)_x + a_k v$,

$$\widehat{m} = m - \widehat{\mu} + m * \widehat{\mu}\,, \quad \widehat{m}_k = m_k - \widehat{\mu}_k + m_k * \widehat{\mu}_k,$$
$$\widehat{g} = g + \mu * g\,, \quad \widehat{g}_k = g + \mu_k * g.$$

We now show that $\widehat{m}_k \rightharpoonup \widehat{m}$. With any $\zeta \in L^2(0,T)$ we compute

$$\langle\widehat{m}_k - \widehat{m}, \zeta\rangle = \langle m_k - m, \zeta\rangle - \langle\widehat{\mu}_k - \widehat{\mu}, \zeta\rangle + N_k^1,$$
$$N_k^1 = \int_0^T \widehat{\mu}_k(\tau)\int_0^{T-\tau}(m_k - m)(s)\zeta(\tau+s)ds d\tau + \int_0^T m(\tau)\int_0^{T-\tau}(\widehat{\mu}_k - \widehat{\mu})(s)\zeta(\tau+s)ds d\tau.$$

We use the relations $m_k \rightharpoonup m$, $\widehat{\mu}_k \rightharpoonup \widehat{\mu}$ and treat the term N_k^1 similarly to the term N_k in (5.1) to get $N_k^1 \to 0$. As a result we get $\langle\widehat{m}_k - \widehat{m}, \zeta\rangle \to 0$, hence $\widehat{m}_k \rightharpoonup \widehat{m}$.

Subtracting the problem of $\widehat{u}$ from the problem of $\widehat{u}_k$ we see that $w_k := \widehat{u}_k - \widehat{u}$ is a weak solution of the problem

$$w_{k,t} = Aw_k - \widehat{m} * Aw_k + \tilde{f}_k + \tilde{\phi}_{k,x} \quad \text{in } \Omega_T, \tag{5.12}$$

$$\widehat{u} = 0 \quad \text{in } \Omega \times \{0\}, \tag{5.13}$$

$$\widehat{u} = \tilde{g}_k \quad \text{in } \Gamma_{1,T}, \tag{5.14}$$

$$-\nu_A\cdot\nabla w_k+\widehat{m}*\nu_A\cdot\nabla w_k=\vartheta w_k+\tilde{h}_k+\nu\cdot\tilde{\phi}_k\quad\text{in }\Gamma_{2,T},\tag{5.15}$$

where

$$\tilde{f}_k=(a_k-a)(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k)-a(\widehat{m}_k-\widehat{m})*\widehat{u}_k,$$
$$\tilde{\phi}_k=-a_{11}(\widehat{m}_k-\widehat{m})*\widehat{u}_{k,x},\quad \tilde{g}_k=(\mu_k-\mu)*g,$$
$$\tilde{h}_k=\vartheta[\widehat{\mu}_k*w_k+(\widehat{\mu}_k-\widehat{\mu})*\widehat{u}].$$

To use the weak convergence $a_k\rightharpoonup a$ in forthcoming estimations we have to introduce the functions $\rho_k\in W_2^2(c,d)$ being the solutions of the following Neumann problems:

$$\rho_k''-\rho_k=a_k-a\quad\text{in }(c,d)\,,\qquad \rho_k'(c)=\rho_k'(d)=0.$$

Then $\rho_k(x)=\int_c^d G(x,y)(a_k-a)(y)dy$, $x\in(c,d)$, where

$$G(x,y)=\frac{1}{2(e^{c-d}-e^{d-c})}\begin{cases}(e^{c-y}+e^{y-c})(e^{d-x}+e^{x-d}) & \text{for } y<x\\ (e^{c-x}+e^{x-c})(e^{d-y}+e^{y-d}) & \text{for } y>x\end{cases}$$

is a Green function that satisfies the properties $G,G_x\in L^\infty(\Omega_T)$. The weak convergence $a_k\rightharpoonup a$ in $L^2(c,d)$ implies

$$\|\rho_k\|_{W_2^1(c,d)}\to 0.\tag{5.16}$$

Using ρ_k we rewrite the term $(a_k-a)(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k)$ in $\tilde{f}_k$ as follows:

$$\begin{aligned}&(a_k-a)(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k)\\ &=[\rho_k'(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k)]_x-\rho_k'(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k)_x-\rho_k(\widehat{u}_k-\widehat{m}_k*\widehat{u}_k).\end{aligned}$$

According to this relation we change the form of the problem for w_k as follows:

$$w_{k,t}=Aw_k-\widehat{m}*Aw_k+\overline{f}_k+\overline{\phi}_{k,x}\quad\text{in }\Omega_T,\tag{5.17}$$
$$\widehat{u}=0\quad\text{in }\Omega\times\{0\},\tag{5.18}$$
$$\widehat{u}=\tilde{g}_k\quad\text{in }\Gamma_{1,T},\tag{5.19}$$
$$-\nu_A\cdot\nabla w_k+\widehat{m}*\nu_A\cdot\nabla w_k=\vartheta w_k+\tilde{h}_k+\nu\cdot\overline{\phi}_k\quad\text{in }\Gamma_{2,T},\tag{5.20}$$

where

$$\overline{f}_k=-\rho_k'(\widehat{u}_k+\widehat{m}_k*\widehat{u}_k)_x-\rho_k(\widehat{u}_k+\widehat{m}_k*\widehat{u}_k)-a(\widehat{m}_k-\widehat{m})*\widehat{u}_k,$$
$$\overline{\phi}_k=\rho_k'(\widehat{u}_k+\widehat{m}_k*\widehat{u}_k)-a_{11}(\widehat{m}_k-\widehat{m})*\widehat{u}_{k,x}.$$

Let t be an arbitrary number in $[0,T]$. To estimate w_k we will use the projection operators P_t, defined in (3.22), and $\overline{P}_tw=\begin{cases}w & \text{in }\Omega_t\\ 0 & \text{in }\Omega_T\setminus\Omega_t\end{cases}$. for $w:\Omega_T\to\mathbb{R}$. Let w_k^t stand for the weak solution of problem (5.17)–(5.20) with $\overline{f}_k$, $\overline{\phi}_k$ and $\tilde{h}_k$ replaced by $\overline{P}_t\overline{f}_k$, $\overline{P}_t\overline{\phi}_k$ and $P_t\tilde{h}_k$, respectively. Then, due to the causality $w_k^t=w_k$ in Ω_t. Applying (3.11) for w_k^t we obtain

$$\begin{aligned}\|w_k\|_{\mathcal{U}(\Omega_t)}&=\|w_k^t\|_{\mathcal{U}(\Omega_t)}\le\|w_k^t\|_{\mathcal{U}(\Omega_T)}\le\overline{C}_0\Big[\|\overline{P}_t\overline{f}_k\|_{L^2((0,T);L^1(c,d))}\\ &\quad+\|\overline{P}_t\overline{\phi}_k\|_{(L^2(\Omega_T))^n}+\theta\|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)}+\|P_t\tilde{h}_k\|_{L^2(\Gamma_{2,T})}\Big]\\ &=\overline{C}_0\Big[\|\overline{f}_k\|_{L^2((0,t);L^1(c,d))}+\|\overline{\phi}_k\|_{L^2(\Omega_t)}+\theta\|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)}+\|\tilde{h}_k\|_{L^2(\Gamma_{2,t})}\Big]\end{aligned}\tag{5.21}$$

with a constant $\overline{C}_0$. Using the relation $a \in L^2(c,d)$, Cauchy inequality, the inequality (3.21), $g(x,0)=0$, the embedding $W_2^1(c,d) \hookrightarrow C[c,d]$ and $\widehat{u}_k = w_k + \widehat{u}$ we estimate:

$$\begin{aligned}
\|\overline{f}_k\|_{L^2((0,t);L^1(c,d))} &\le \overline{C}_1\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}_k\|_{L^2((0,t);L^2(c,d))} \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}_k\|_{\mathcal{U}(\Omega_t)}\Big] \\
&\le \overline{C}_1\Big[\int_0^t |(\widehat{m}_k-\widehat{m})(t-\tau)|\,\|w_k\|_{L^2(\Omega_\tau)}d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|w_k\|_{\mathcal{U}(\Omega_t)}\Big] + \overline{R}_k^1,
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
\|\overline{\phi}\|_{L^2(\Omega_t)} &\le \overline{C}_2\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}_{k,x}\|_{L^2(\Omega_t)} \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}_k\|_{L^2((0,T);C[c,d])}\Big] \\
&\le \overline{C}_2\Big[\int_0^t |(\widehat{m}_k-\widehat{m})(t-\tau)|\,\|w_{k,x}\|_{L^2(\Omega_\tau)}d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|w_k\|_{L^2((0,t);C[c,d])}\Big] + \overline{R}_k^2,
\end{aligned} \tag{5.23}$$

$$\|\tilde{g}_k\|_{\mathcal{T}(\Omega_T)} \le \overline{R}_k^3, \tag{5.24}$$

$$\begin{aligned}
\|\tilde{h}_k\|_{L^2(\Gamma_{2,t})} &\le \overline{C}_4\Big[\|\widehat{\mu}_k * w_k\|_{L^2((0,t);W_2^1(c,d))} + \|(\widehat{\mu}_k-\widehat{\mu})*\widehat{u}\|_{L^2((0,t);W_2^1(c,d))}\Big] \\
&\le \overline{C}_4\int_0^t |\widehat{\mu}_k(t-\tau)|\,\|w_k\|_{L^2((0,\tau);W_2^1(c,d))}d\tau + \overline{R}_k^4,
\end{aligned} \tag{5.25}$$

where $\overline{C}_1, \overline{C}_2, \overline{C}_4$ are constants and

$$\begin{aligned}
\overline{R}_k^1 &= \overline{C}_1\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}\|_{L^2(\Omega_T)} + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|\widehat{u}\|_{\mathcal{U}(\Omega_T)}\Big], \\
\overline{R}_k^2 &= \overline{C}_2\Big[\|(\widehat{m}_k-\widehat{m})*\widehat{u}_x\|_{L^2(\Omega_T)} + \|\rho_k\|_{W_2^1(c,d)}\big(1 \\
&\quad + \|\widehat{m}_k\|_{L^2(0,T)}\big)\|\widehat{u}\|_{L^2((0,T);C[c,d])}\Big], \\
\overline{R}_k^3 &= \|(\mu_k-\mu)*g\|_{L^2(\Omega_T)} + \|(\mu_k-\mu)*g_x\|_{L^2(\Omega_T)} + \|(\mu_k-\mu)*g_t\|_{L^2(\Omega_T)}, \\
\overline{R}_k^4 &= \overline{C}_4\|(\widehat{\mu}_k-\widehat{\mu})*\widehat{u}\|_{L^2((0,T);W_2^1(c,d))}.
\end{aligned}$$

By the weak convergence $\widehat{m}_k \rightharpoonup \widehat{m}$, $\mu_k \rightharpoonup \mu$, $\widehat{\mu}_k \rightharpoonup \widehat{u}$ in $L^2(0,T)$ and the relation $\|\rho_k\|_{W_2^1(c,d)} \to 0$ it holds

$$\overline{R}_k^j \to 0, \quad j=1,2,3,4. \tag{5.26}$$

Indeed, to prove that $\|z_k * \hat{v}\|_{L^2(\Omega_T)} \to 0$, where z_k is one of the functions $\widehat{m}_k - \widehat{m}$, $\mu_k - \mu$ or $\widehat{\mu}_k - \widehat{\mu}$ and $\hat{v} \in L^2(\Omega_T)$ is one of the functions $\widehat{u}$, $\widehat{u}_x$, g, g_x or g_t, it is possible to use the dominated convergence theorem, again. More precisely,

$$\|z_k * \hat{v}\|_{L^2(\Omega_T)} = \Big\{\int_0^T\int_c^d\Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2 dx\,dt\Big\}^{1/2},$$

where the component $\Big[\int_0^t z_k(\tau)\hat{v}(x,t-\tau)d\tau\Big]^2$ is bounded by an integrable in $x \in (c,d)$ function $\sup\|z_k\|^2_{L^2(0,T)}\|\hat{v}(x,\cdot)\|^2_{L^2(0,T)}$ and tends to zero for all $t \in (0,T)$ and

a.e. $x \in (c,d)$, because $z_k \rightharpoonup 0$ and $\hat{v}(x,t-\cdot) \in L^2(0,T)$ for all $t \in (0,T)$ and a.e. $x \in (c,d)$. (The latter relation follows from $\hat{v} \in L^2(\Omega_T)$ and Tonelli's theorem.) Thus, $\|z_k * \hat{v}\|_{L^2(\Omega_T)} \to 0$.

As in proof of Theorem 3.1, we use the norms $\|w\|_\sigma = \sup_{0<t<T} e^{-\sigma t}\|w\|_{\mathcal{U}(\Omega_t)}$ with the weights $\sigma \geq 0$ in the space $\mathcal{U}(\Omega_T)$. Then in view of (5.22)–(5.25) from (5.21) we deduce

$$\begin{aligned}
\|w_k\|_\sigma &\leq \overline{C}_5\Big[\sup_{0<t<T}\int_0^t e^{-\sigma(t-\tau)}r_k(t-\tau)\,e^{-\sigma\tau}\|w_k\|_{\mathcal{U}(\Omega_\tau)}d\tau \\
&\quad + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\|w_k\|_\sigma + \sum_{j=1}^4 \overline{R}_k^j\Big] \\
&\leq \overline{C}_5\Big[\Big\{\|e^{-\sigma t}\|_{L^2(0,T)}\|r_k\|_{L^2(0,T)} + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)})\Big\}\|w_k\|_\sigma \\
&\quad + \sum_{j=1}^4 \overline{R}_k^j\Big],
\end{aligned}$$

where $\overline{C}_5$ is a constant and $r_k = |\widehat{m}_k - \widehat{m}| + |\widehat{\mu}_k|$. Since $\|e^{-\sigma t}\|_{L^2(0,T)} \to 0$ as $\sigma \to \infty$, $\|\rho_k\|_{W_2^1(c,d)} \to 0$ and the sequences $\|r_k\|_{L^2(0,T)}$, $\|\widehat{m}_k\|_{L^2(0,T)}$ are bounded, there exist $\sigma_2 > 0$ and $K_2 \in \mathbb{N}$ such that

$$\|e^{-\sigma_2 t}\|_{L^2(0,T)}\|r_k\|_{L^2(0,T)} + \|\rho_k\|_{W_2^1(c,d)}(1+\|\widehat{m}_k\|_{L^2(0,T)}) \leq \frac{1}{2\overline{C}_5}$$

for $k \geq K_2$. This, along with the previous inequality, implies

$$\|w_k\|_{\sigma_2} \leq 2\overline{C}_5\sum_{j=1}^4 \overline{R}_k^j \quad \text{and hence} \quad \|w_k\|_{\mathcal{U}(\Omega_T)} \leq 2e^{\sigma_2 T}\overline{C}_5\sum_{j=1}^4 \overline{R}_k^j$$

for $k \geq K_2$. Taking (5.26) into account we obtain the desired convergence: $\|\widehat{u}_k - \widehat{u}\|_{\mathcal{U}(\Omega_T)} = \|w_k\|_{\mathcal{U}(\Omega_T)} \to 0$. The theorem is proved. □

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