DOCTORAL THESIS

Inverse Scattering of Acoustic and Electromagnetic Waves from flat Screens and Properties of Integral Transforms on a Half Axis

Sadia Sadique

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology, has not been submitted for any academic degree elsewhere.

Sadia Sadique

signature

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Akustiliste ja elektromagnetlainete pöördhajumine lameekraanilt ja integraalteisenduste omadused poolteljel

SADIA SADIQUE

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List of Publications

I Blåsten. E, Päivärinta. L, Sadique. S, Unique Determination of the Shape of a Scattering Screen from a Passive Measurement, *Mathematics*, 8(7), 1156, 2020.

II Ola. P, Päivärinta. L, and Sadique. S, Unique Determination of a Planar Screen in Electromagnetic Inverse Scattering, *Mathematics*,§ 11(22), 4655, 2023.

III Blåsten. E, Päivärinta. L, Sadique. S, The Fourier, Hilbert, and Mellin Transforms on a Half-Line, *SIAM Journal on Mathematical Analysis*, 55(6), 7529-7548, 2023.

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Author's Contributions to the Publications

I In I publication I was one of the three equal authors. I was involved in formulating the problem and finding the logical idea of the proof of the main theorem.

II In II publication I was one of the three equal authors. I worked on problem formulation and was involved in discussions that led to the solution of the problem that is solved in this work.

III In III publication I was the corresponding author, analyzed the result, and formulated the research problem.

Approbation

I presented the results of the thesis at the following conferences:

1. Sadique. S, *Scattering of acoustic and electromagnetic waves by flat screen*, 28–30 May, Taltech, 24th International Conference on Mathematical Modelling and Analysis, 2019.
2. Sadique. S, *single far-field pattern determines the shape of the scattering screen*, Vilnius Gediminas Technical University, International Conference on Mathematical Modelling and Analysis, 30th May–2nd June, 2022.
3. Sadique.S, *Does a single far-field determine the shape of the scattering screen*, XIV Science Conference of the School of Science, Tallinn university of technology, 30th November 2022.
4. Sadique.S, *Inverse scattering of electromagnetic waves by planner screen*, 26th International Conference Mathematical Modelling and Analysis, Jurmala, Latvia, May 30–June 2, 2023.

Abbreviations

E.M	Electromagnetic
L.C.A	Local Compact Abelian

1 Introduction

In this doctoral thesis, I studied the inverse scattering problem of acoustic and EM waves interacting with flat screens and investigate the singular behavior caused by curved-shaped scatterers.

In antenna theory, the inverse problem of wave scattering for large and thin objects is an important area. This area of study involves determining the characteristics of an object through the analysis of its scattering features. This is beneficial in the analysis of radar and other imaging systems and in the design and optimization of antennas. The study was initiated when the Prussian Academy declared an open competition in 1879 to see who could demonstrate the existence and non-existence of EM waves. In his pioneering work [43] James Clerk Maxwell predicted such kind of wave. In favor of Maxwell's theory, this competition was won in 1882 by Heinrich Hertz.

This begin by outlining the inverse scattering problem. According to [16] direct scattering problem has been inquired and a considerable size of knowledge is obtainable regarding its solution. Conversely, the inverse scattering problem has only progressed since 1980 from a small collection of specific approaches with a strict mathematical base to an area of vigorous activity with a stable mathematical foundation. The inverse scattering problem, as viewed through numerical computations, is inherently nonlinear and has been improperly posed. Despite that, it has significant applications in areas such as radar, sonar, medical imaging, geophysics, and non-destructive testing. Indeed, it is worth noting that the inverse problem has acquired a similar interest as the direct problem.

The research in Article I studies the question of whether an acoustic screen can be determined by using only one far field measurement. This means we have only one transmitter but the scattering field is measured in all directions. Such a measurement is called passive. The main result of this article is the mathematically rigorous proof that this is true for two-dimensional flat screens.

The research applies to issues such as bringing down echo in an office space or directing acoustic vibrations. By resolving the problem of finding a screen that can generate a specific far-field pattern, the study contributes to the understanding of passive sonars and their effect on the sound pattern. The research focuses on the question that the shape and location of a passive sonar can be determined by its sound reflection. This is a quite difficult problem. The research showed however that a single input-output pair of sound waves uniquely specifies the shape of a flat acoustic screen. Specifically, the research objective was to demonstrate that flat screens can be uniquely determined by a single input-output pair of sound waves.

In literature [16] the shape determination problem is known as Schiffer's problem. Schiffer proved that a sound-soft obstacle with a non-empty interior can be uniquely determined by infinitely many far-field patterns. The proof was published after a private communication in the monograph by Lax and Phillips [36]. Research in this area got significant interest and a very active research community. For the shape determination, and result from a numerical point of view, linear sampling method [13] and factorization method [31] were developed, and they are well suited in this area. These methods were applied in the context of the curved screens in acoustic [4] and electromagnetic scattering [11] to determine the shape and location of the screen. Colton and Sleeman in [15] reduced the requirements to finitely many far-field patterns in [16, 28]. It is widely conjectured that the uniqueness of Schiffer's problem follows from a single far-field pattern. Various authors showed in [2, 12, 20, 38, 39, 40, 57], that polyhedral sound-soft obstacles are uniquely determined by a single far-field pattern in several setting. The method for determining the shape of a flat-screen using a single incident plane-wave was described in [4]. However, the proof of this method requires that the incident wave has some non-vanishing properties everywhere on the plane where the screen is located. So far, there is no evidence supporting the unique determination of an obstacle's shape by a single far-field pattern without any restricted a priori assumptions.

Here, I considered the scattering of a two-dimensional sound-soft and flat obstacle Ω in three-dimensional space. It was assumed that Ω is an open subset of $\{x_3 = 0\} = \mathbb{R}^2 \subset \mathbb{R}^3$. Defining the direct scattering problem for screen Ω as follows. Given an incident wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ and a screen Ω, the scattering problem has a solution if there is $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ that satisfies the (10), (11), and (12).

The methods for acoustic waves are based on ideas that are partially motivated by the study of certain integral operators [50] on the screen. As in [4] it was first established that the far-field is the restriction to a ball of radius k of the two-dimensional Fourier transform of a function supported on the screen. Next, I demonstrated that the screen's design precisely supports that function, given that the incident wave might vanish on a portion of the screen. The latter part involves a delicate analysis of Taylor coefficients of the scattered wave at the screen, but it ultimately leads to our main theorem: that for incident waves that cause scattering, Schiffer's problem is uniquely solvable for the flat screen on any plane in three dimensions.

Furthermore, in my thesis, in Article II I proceed to solve the distinctive identification of inverse EM scattering in planar screen. I highlighted the shape determination of a screen of electromagnetic waves with a single measurement. I demonstrated that the far-field pattern of a scattered electromagnetic field relative to a single incoming plane wave uniquely determines a bounded superconductive planar screen. This work is the generalization of our previous work [8]. The proof arises from the representation formula for the exterior solution of Maxwell's equations. The approach we used is based on the concepts of certain integral operators [49, 50]. Similar to [8] the important part of our argument is that the shape of the screen is precisely the support of the jump of the tangential component of the scattered magnetic field.

The proof of Schiffer is presented in [36] and it gives the basic uniqueness result for the case of the Dirichlet obstacle problem. Schiffer's uniqueness theorem needs scattering data from an infinite number of incoming waves. Schiffer's uniqueness result has a wide range of study in to the behavior of sound soft obstacles subjected to countable number of incident plane waves [16, 36] in the direction , including [35, 23, 40, 56, 59, 57] the result of uniqueness in the general domain, [12, 20, 19] for polyhedral scatterers, [37, 41] for ball or disc, [33, 34, 46] and for smooth planer curves.
The work has been done on inverse electromagnetic scattering problem In [26] in the TE polarization case. They demonstrated that the knowledge of the electric far-field pattern for a single incoming wave is suitable to determine the shape of a rectangular penetrable scatterer uniquely. Liu and Zou [42] emphasized in recent progress on the unique determination of general polyhedral scatterers by the far field data corresponding to one or several incident fields. For recent results in the time-harmonic inverse EM-scattering see the short review by Rainer Kress [35].

The Article III in my thesis is the most important part. The motivation for this comes from studying the scattering of acoustic waves from a crack in a two-dimensional domain. In this research, incident wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ reacts with a screen S. By screen S we mean a $n-1$ dimensional surface in a n dimensional spaces around, $n =$

2 or $n=3$. Mathematically I define it as follows:

$$(\Delta+k^2)u_s=0, \qquad \mathbb{R}^3\setminus\overline{S}, \tag{1}$$

$$u_i(x)+u_s(x)=0, \qquad x\in S, \tag{2}$$

$$r\Big(\frac{\partial}{\partial r}-ik\Big)u_s=0, \qquad r\to\infty, \tag{3}$$

where $r=|x|$ and the limit is uniform over all directions $\hat{x}=x/r$ as $r\to\infty$. Our problem is whether the *far-field pattern* of u_s uniquely determines the shape of screen S. Research of this problem leads to studying the support of a generalized function ρ which satisfies an integral equation of the form

$$-\int_S \Phi(x-y)\rho(y)d\sigma(y)=u_i(x), \tag{4}$$

where Φ is the Green's function for $\Delta+k^2$. Notice how it is analogous to (6). As for the flat scatterers integral equation method can be used. For a more general object, it's worthwhile to study the singular behavior of solutions to inhomogeneous integral equation [3, 22]. The acoustic equation has yet to be resolved. To start with this research we study the so-called one-sided Hilbert transform.

$$\mathscr{H}f(x)=p.v\frac{1}{\pi}\int_0^{+\infty}\frac{f(y)}{x-y}dy. \tag{5}$$

Different terminology can be used for this transform semi-infinite Hilbert transform, half Hilbert transform, or the reduced Hilbert transform [30]. Our interest is to understand the existence, uniqueness and behaviour at the origin of the solution ρ to the inhomogenous equations

$$\mathscr{H}\rho=e \tag{6}$$

for a given e. The equation (6) has been studied earlier in [17, 48, 53, 54] when e and ρ are classically smooth or Lebesgue integrable. These sources have a practical point of view, which focuses on computations or asymptotic expansion.

The scientific novelty of my thesis is explained by the following results.

- Unique determination of the shape of flat screen Ω by inverse scattering of acoustic waves.
- Unique identification of a Planar Screen in EM Inverse Scattering.
- Finding the singular behavior at the origin of solutions to the equation $\mathscr{H}\rho=e$ on a half-axis by using the Fourier, Mellin, and Hilbert transform and by explaining the connections between these three transforms.

The analysis began by addressing the mathematical challenges. Complex plane waves offer clarity in many scattering scenarios due to their explicit form and non-zero values throughout space. This often simplifies the nonlinear inverse scattering problem, transforming it into a linear inverse source problem with appropriate interpretation. This contrast can be observed by comparing [9, 55, 27] with [6, 7, 10]. This type of measurement where the incident field is unchanged is called passive because we don't need to change the transmitter direction. This is relevant when one wishes to uniquely determine a screen where space contains other scatterers that are known. On the other hand, from the applied perspective, solving the inverse problems has become essential regardless of the incident field's control. This implies that even if we do not have control to manipulate the incident field, or cannot afford to control it, we can still attain the unique determination of the scatterer's shape. Similarly, the goal of electromagnetic inverse scattering work is to find a unique determination of supporting hyperplanes corresponding to the single measurement having non-vanishing far-fields.

The basic motivation is to understand the singular behavior of the solution

$$\mathscr{H}\rho = e$$

also in cases where the right-hand side is not smooth, or integrable in the classical sense. Our first step in this study is to understand the singular behavior of waves near the end-points of cracks in the acoustic medium by simplifying the applied problem leads to the study of this equation $\mathscr{H}\rho = e$ on the half line in a class of generalized functions. Our approach is to use the Mellin transform defined for generalized functions

$$\mathscr{M}[f](s) = \int_0^\infty f(t)t^{s-1}dt \tag{7}$$

It has proven that ρ has a singularity of the form $\mathscr{M}[e](1/2)\frac{1}{\sqrt{t}}$ where $\mathscr{M}$ is the Mellin transform. For this, I am using specially built function spaces $\mathscr{M}'(a,b)$ by Zemanian [32]. These spaces enable me to investigate the relationship between the Mellin and Hilbert transform. Here, Fourier transforms also play a significant role. Since the Mellin transform is simply the Fourier transform on the locally compact Abelian multiplicative group of the half-line.

A more natural way of thinking of these spaces is that $u \in \mathscr{M}'(a,b)$ if informally

$$\begin{aligned} u(t) &= O(t^{-a}), \qquad t \to 0, \\ u(t) &= O(t^{-b}), \qquad t \to \infty. \end{aligned} \tag{8}$$

More exactly, $u \in \mathscr{M}'(a,b)$ if the Mellin transform $\mathscr{M}[u](s)$ is holomorphic in the vertical strip $s \in S(a,b)$ defined by $a < \Re(s) < b$ and has polynomial growth on vertical lines.

Below I am providing a content overview of my thesis.
In **Chapter 2** I established the foundational concepts necessary for fundamental information in the subsequent discussions of Chapters 3 and 4. The initial concepts focus on screens, direct and inverse scattering problems, the far-field representation of electric and magnetic fields, Maxwell's equations, and the Silver-Müller radiation condition. Furthermore, I discussed some tools that are essential for the analysis of Hilbert, Mellin and Fourier transform. We considered Mellin transform in distribution, local compact Abelian group, haar measure, test function, and inversion formula for Mellin transform.

In **Chapter 3** I analyzed the interesting area of acoustic scattering. A crucial aspect of our investigation is the solution to the direct scattering problem, which elegantly satisfies equation (37). In Section 1 I computed the fundamental solution of the Helmholtz equation and radiating boundary condition which captures the scattered wave at point x outside of the screen. The far-field representation, derived from Equation (37), assumes that ρ is a function. However, to expand the applicability of our findings, I extended the concept of ρ to be a generalized function in the Sobolev spaces $\mathscr{H}^{-\frac{1}{2}}(\mathbb{R}^2)$. The uniqueness theorem is essential that shows that the scattering caused by flat screens determines the shape uniquely as long as u_i is not antisymmetric concerning $\mathbb{R}^2 \times 0$. In Section 2 I introduced the formulation of the inverse problem, where I showed that a single farfield determines the shape of the screen uniquely. Moreover, I showed that the shape of screen contained in support of ρ provides insight how scattered wave behaves in the area of the screen.

Additionally, in Section 3 of this chapter, I studied EM scattering solutions outside of the planner screen. I presented a brief overview of a perfectly conducting screen. The representation theorem has been discussed. Some important propositions and lemmas for far-field patterns and layer potential in Sobolev spaces have been considered. I also explained how our function belongs to the relevant Sobolev spaces. Integral equations and representation formulas for scattered fields have been introduced. In Section 4 I analyzed that the far-field pattern of scattered electromagnetic waves uniquely determines the shape of the screen corresponding to the single incoming waves. In Section 5 solution of the inverse problem is demonstrated. I formulated the proposition for the unique determination of the planner screen. Moreover, I discussed some lemmas: about uniqueness when supporting hyperplane is known.

Chapter 4 explores the essential properties, applications, and connections of Fourier, Hilbert, and Mellin transforms on a half-line. The Mellin transform is established as a Fourier transform in an LCA group $(\mathbb{R}_+, \frac{dt}{t})$. Furthermore, the Chapter examines the relationship between the Hilbert transform and the Mellin transform. Various properties of these transforms are studied, which demonstrates their significance. An important aspect of this chapter is to find unique solutions to the equation $\mathscr{H}\rho = e$ on a half-line, with a particular focus on the singular behavior at 0. It explores mathematical techniques and formulations to characterize and understand the singular properties of the solution ρ at this critical point and shows that the main asymptotic is of the form $ct^{-1/2}$.

2 Preliminary Concepts

2.0.1 Scattering theory

Scattering theory is a branch of physics that deals with the results of an inhomogeneous medium having incident waves or particles. Classical scattering theory addresses two basic problems.

Scattering of time-harmonic acoustic or EM waves with inhomogeneous medium characterized by compact support and with impenetrable bounded obstacles. Here we are considering the case of acoustic waves, the incident field is assumed to be a time-harmonic plane wave. i.e

$$u_i(x,t) = e^{i(kx\cdot d - \omega t)} \tag{9}$$

where $k = \omega/c_0$ the wave number, ω the frequency, c_0 the speed of sound, and d is direction of propagation.

2.0.2 Direct and inverse scattering problem

According to [16] the total field of scattered wave is the sum of the incident field u_i and scattered field u_s, then the direct scattering problem is to determine u_s with the knowledge of u_i and differential equation conducting the wave motion. In contrast, the Inverse scattering problem is to determine the shape of the scatterer with the knowledge of the asymptotic behavior of u_s.

2.0.3 Definition of the screen

We call a set $\Omega \subset \mathbb{R}^3$ *a screen*, if $\Omega = \Omega_0 \times \{0\}$ for some simply connected bounded domain $\Omega_0 \subset \mathbb{R}^2$ whose boundary is smooth, and which we call its *shape*.

2.0.4 Direct scattering problem for the screen Ω

The direct scattering problem for screen Ω can be defined as follows.

Given an incident wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$. The direct scattering problem has a solution in distributional function if there is $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ that satisfies the following conditions.

$$(\Delta + k^2)u_s = 0, \qquad \mathbb{R}^3 \setminus \overline{\Omega}, \tag{10}$$

$$u_i(x) + u_s(x) = 0, \qquad x \in \Omega, \tag{11}$$

$$r\Big(\frac{\partial}{\partial r} - ik\Big)u_s = 0, \qquad r \to \infty, \tag{12}$$

where $r = |x|$, and the limit is uniform across all directions with $\hat{x} = x/r \in \mathbb{R}^2$ as $r \to \infty$.

2.0.5 Scattering from a sound-soft screen

The mathematical description of the scattering of time-harmonic waves by screen Ω leads to boundary-value problems of the Helmholtz equation. Prescribing the values of u on the boundary of the Ω (i.e., the Dirichlet problem) physically corresponds to prescribing the pressure of the acoustic. If u_i is the incoming acoustic wave, then the total wave is the form of $u = u_i + u_s$ where u_s denotes the scattered wave. For a sound-soft obstacle, the total pressure must vanish on the boundary. In this case, $u_s = -u_i$ on the boundary.

2.0.6 Sobolev spaces

Sobolev spaces are a family of function spaces, that play a significant role in the theory of partial differential equations and related areas of mathematics. They were introduced by

Sergei Sobolev in the 1930s as a way to study the regularity of solutions to certain types of partial differential equations. Weak derivatives are the basis of the general definition of Sobolev spaces. [21] defines certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces. Mathematically, it can be represented as:

$$W^{k,p}(\Omega)$$

where Ω is domain in R^n, $1 \leq p \leq \infty$ is a positive real number representing the integrability of the weak derivatives, and k is nonnegative integer consists of all locally summable functions $u : \Omega \to R$ such that for each multi-index α with $|a| \leq k$, $D^\alpha u$ exist in the weak sense and belongs to $L^p(U)$. The norm in $W^{k,p}(\Omega)$ is given by:

$$|u|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^p dx \right)^{1/p}$$

where α is a multi-index with $|\alpha| = k$, ∂^α is the corresponding weak derivative operator, and dx denotes the Lebesgue measure in $\mathbb{R}^n$.

2.0.7 Traces of Sobolev spaces

For evaluating functions at the boundary $\partial\Omega$ of the domain, I need to define traces of spaces. [21] describes $\partial\Omega$ to a function $u \in W^{1,p}(\Omega)$ assuming that $\partial\Omega$ is C^1. The function u has value on $\partial\Omega$ in usual sense if $u \in C(\overline{\Omega})$. I use trace operator denoted by $u|_{\partial\Omega}$ when the function $u \in W^{1,p}(\Omega)$ is not general continuous. Sobolev trace theorem follows by trace operator.
I define here a bounded linear operator T assuming that Ω is bounded and $\partial\Omega$ t is C^1.

$$T : W^{I,p}(\Omega) \to L^p(\partial\Omega)$$

such that

1. If $u \in W^{I,p}(\Omega) \cap C(\overline{\Omega})$, then $Tu = u|_{\partial\Omega}$.

2. For every $u \in W^{I,p}$ the inequality $||Tu||_{L^p(\partial\Omega)} \leq C||u||_{W^{I,p}(\Omega)}$,

where the constant C depends only on p and Ω. Here u is the function space and T is the operator.

2.0.8 Far-field patterns

Far-field pattern is also called scattering amplitude. According to [16] if u_s satisfy the Sommerfeld radiation condition (12) and the Helmholtz equation $(\Delta + k^2)u_s = 0$, We say that u_s^∞ is the far-field of u_s if

$$u_s(x) = \frac{e^{ik|x|}}{|x|} \left(u_s^\infty(\hat{x}) + \mathscr{O}\left(\frac{1}{|x|}\right)\right), \quad |x| \to \infty \tag{13}$$

uniformly across all direction $\hat{x} = x/|x|$.
This equation expresses how the solution $u_s(x)$ behaves as you move to points at an infinite distance from the source.

2.0.9 Time Harmonic Maxwell's equation in the exterior of screen

$$curlE - ikH = 0, \qquad curlH + ikE = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \tag{14}$$

$$E(x) = \frac{i}{k} curlcurl pe^{ikx\cdot d} + E^s(x), \qquad H(x) = \frac{i}{k} curlcurl pe^{ikx\cdot d} + H^s(x). \tag{15}$$

Equation (14) represents Maxwell's equations, which are fundamental in describing electromagnetic phenomena. Equation (15), on the other hand, represents the combined effect of both the incoming wave $e^{ikx\cdot d}$ and the scattered wave, where vector d indicates the direction of wave propagation, and p denote the polarization of the waves.

2.0.10 C^k screen

A C^k-screen, $k = 1, 2....\infty$ is a compact, connected submanifold of a two-dimensional hyperplane in three-dimensional space $\mathbb{R}^3$. The term C^k indicates that the screen is smooth up to k^{th} derivatives, meaning that it is differentiable up to order k in its local coordinates charts. The supporting hyperplane is called the affine hyperplane of the screen.

2.0.11 Perfectly conducting screen

The direct scattering problem for the perfectly conducting screen S is a specific electromagnetic scattering problem where the screen S is assumed to be a perfectly conducting surface. This means that the total electric field vanishes on the screen. Mathematically it can be written as

$$\nu \times (E_s + E_0) = 0 \quad \text{on } S. \tag{16}$$

The symbol ν represents the outward unit normal vector to the boundary ∂D.

2.0.12 Silver–Müller–radiation conditions

In the context of Maxwell equations, a solution E, H [16] whose domain contains the exterior of some sphere, is termed as a "radiating solution" if its fulfills one of the Silver–Müller–radiation conditions. These conditions are designed to ensure that the solution behaves like outgoing waves as the distance from the sphere tends to infinity.

$$\lim_{r\to\infty}(H \times x - rE) = 0 \qquad or \tag{17}$$

$$\lim_{r\to\infty}(E \times x - rH) = 0. \tag{18}$$

In both cases, $r = |x|$ represents the magnitude of position vector. These limits are expected to be uniform across all directions $x/|x|$.

2.0.13 EM-Plane Waves with Directional Propagation and Polarization Variations

The provided equations define EM-plane waves. These are special types of electromagnetic waves characterized by specific propagation directions θ and polarizations direction $p \times q$.

$$E(\theta; p, q) = \mu^{1/2}(p \times \theta)e^{ik\langle\theta,x\rangle}, \qquad H(\theta; p, q) = \varepsilon^{1/2}(q \times \theta)e^{ik\langle\theta,x\rangle}$$

. The expressions $E(\theta; p, q)$ and $H(\theta; p, q)$ represent the electric and magnetic fields of these EM plane waves, respectively.

It is apparent that these fields satisfy the time-harmonic Maxwell's equation

$$\nabla \times E(\theta; p, q) = i\omega\mu H(\theta; p, q), \quad \nabla \times H(\theta; p, q) = -i\omega\varepsilon E(\theta; p, q).$$

Here, ε is the electric permittivity, and μ is the magnetic permeability.

2.0.14 Far–field pattern of electric and magnetic field

According to the [16], every solution to the Maxwell equations involving radiating field denoted by E,H exhibits a characteristic asymptotic behavior.
For every $|x| \to \infty$, the expression $E(x)$ and $H(x)$ is given by

$$\begin{aligned} E(x) =& \frac{e^{ik|x|}}{|x|}\{E_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\} \\ H(x) =& \frac{e^{ik|x|}}{|x|}\{H_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\} \end{aligned} \tag{19}$$

uniform across all directions $\hat{x} = \frac{x}{|x|}$, where vector field E_∞ and H_∞ are the electric far field and magnetic far field pattern on the unit sphere S^2 .

2.0.15 Hilbert transform

[51] define the classical Hilbert transform as follows:

Consider the function defined on the real line, then Hilbert transform of f is function $\mathscr{H}f(x)$ is defined by following formula

$$\begin{aligned} \mathscr{H}f(x) =& \frac{1}{\pi} p.v \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt \\ =& \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \left(\int_{\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \right) \frac{f(t)}{x-t} dt \end{aligned} \tag{20}$$

provided that the limit exists.

In addition, I define the half-line Hilbert transform. Other terminology for this transform are the reduced Hilbert transform, the half Hilbert transform and the semi-Hilbert transform [[30] section12.7].

$$\mathscr{H}f(x) = \frac{1}{\pi} \text{p.v.} \int_0^{\infty} \frac{f(t)}{x-t} dt.$$

Correspondingly, the connection of the Hilbert transform to the Fourier transform $\mathscr{F}$ is well-known.

$$\mathscr{F}(\mathscr{H}f)(\xi) = i \operatorname{sgn} \xi \widehat{f}(\xi) \tag{21}$$

where $\widehat{f} = \mathscr{F}f$, for more explanation see [30, 61, 60].

2.0.16 Mellin transform

Finnish mathematician R. H. Mellin (1854-1933) was the first who introduce the Mellin transform and presented a systematic expression of the transformation along its corresponding inverse. During his work, he established the applications to the solution of the hypergeometric, and differential equations, as well as in the derivation of asymptotic expansions with the effect of special functions theory.
I can define it as follows:
According to [5], consider a function denoted by $f(t)$ defined on the positive real axis $0 < t < \infty$. The Mellin transform denoted as $\mathscr{M}$ is the operation that maps the function f into another function denoted by F defined on the complex plane as follows:

$$\mathscr{M}[f;s] = F(s) = \int_0^{\infty} f(t) t^{s-1} dt. \tag{22}$$

The function $F(s)$ is defined as the Mellin transform of f. In general the integral defining Mellin transform is valid only for complex values specifically for $s = a + ib$ where $a_1 < a < a_2$. The values of a_1 and a_2 depend on the function $f(t)$ to transform. This introduces, what is known as the "strip definition" of Mellin transform represented as $S(a_1, a_2)$. In some situations, this strip may extend to a half-plane such as ($a_1 = 0$; or $a_2 = +\infty$) or even to the whole complex s-plane as in the case ($a_1 = -\infty$; or $a_2 = \infty$).

2.0.17 Local compact abelian group(LCA) and Haar measure

Let $G = (X, \cdot)$ be an LCA group, having group operation multiplication. Usually, The group operation is denoted by by addition [58] and the identity element is 0. Since our interest is multiplicative group $G_+ = (\mathbb{R}_+, \cdot)$. Here the group operation is denoted by product and the identity element is 1. There exists a Haar measure m defined on X which is unchanged under the group action. This measure has property $m(xE) = m(E)$ for all $x \in X$ and Borel set E. The uniqueness of the Haar measures is up to a positive constant. In the specific case $G_+ = (\mathbb{R}_+, \cdot)$, the Haar measure is dt/t. Mathematically it can be written as

$$m(E) = \int_E \frac{dt}{t} \tag{23}$$

for any borel set $(\mathbb{R}_+)$.

The Haar measure m on G is also used to define function spaces such as $L^p(G)$. We present by $L^p(G)$ instead of $L^p(m)$.

$$\|f\|_{L^p(G)} = \left(\int_X |f(x)|^p dm(x) \right)^{1/p}. \tag{24}$$

The norm $\|f\|_{L^p(G)}$ is shown to be scaling invariant, illustrated through an example with $G+$.
If $f_x(y) = f(yx^{-1})$ then $\|f_x\|_{L^p(G)} = \|f\|_{L^p(G)}$.

In particular for G_+ we have $f_t(s) = f(s/t)$ and

$$\int_{\mathbb{R}_+} |f_t(s)|^p \frac{ds}{s} = \int_{\mathbb{R}_+} |f(s)|^p \frac{ds}{s}. \tag{25}$$

2.0.18 Fourier transform

Fourier transform is an integral transform of a function of time $f(t)$, in to function of frequency $g(x)$. [45] describes the notation and terminology of Fourier transform as follows:

$$g(x) = \mathscr{F}_x(t) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi xt} dt \tag{26}$$

where $\mathscr{F}$ denotes the Fourier transformation, and $\mathscr{F}_x(t)$ denotes the Fourier transform of f. We also define

$$\mathscr{F}_t^{-1} g(x) = \int_{-\infty}^{+\infty} g(x) e^{2\pi xt} dx \tag{27}$$

where the notation $\mathscr{F}_t^{-1}$ denotes the inverse Fourier transformation. $\mathscr{F}_t^{-1} g(x)$ is called the inverse Fourier transform of $g(x)$.

2.0.19 Fourier transforms in a locally compact Abelian group

Suppose $G = (X, \cdot)$ is a LCA and a function $\gamma : X \to \mathbb{C}$ is called a character if $|\gamma(x)| = 1$ for all $x \in X$ and $\gamma(xy) = \gamma(x)\gamma(y)$ for every $x, y \in X$.

$$\gamma(xy) = \gamma(x)\gamma(y) \tag{28}$$

for every $x, y \in X$. Here character G is homomorphism from $G \to T$ where T is rotation group in a unit circle in a complex plane. The set of characters on LCA is denoted by Γ.

$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x) \tag{29}$$

for $x \in X$. Γ is dual group of G. The Fourier transform of a function $f \in L^1(G)$ is defined as

$$\widehat{f}(\gamma) = \int_X f(x)\gamma(-x)dx \tag{30}$$

for $\gamma \in \Gamma$.

2.0.20 Test function on $\mathscr{M}_{a_1,a_2}$

The space $\mathscr{M} a_1, a_2$ where $a_1 < a_2$ contains smooth functions $\phi : \mathbb{R}+ \to \mathbb{C}$ satisfying, for every $k \in \mathbb{N}$. Here $|\phi|_{a_1,a_2,k} < \infty$. The norm is defined as

$$\|\phi\|_{a_1,a_2,k} = \sup_{0<t<\infty} \zeta_{a_1,a_2}(t)t^{k+1}\left|\frac{d^k}{dt^k}\phi(t)\right|, \tag{31}$$

$$\zeta_{a_1,a_2}(t) = \begin{cases} t^{-a_1}, & 0 < x \le 1, \\ t^{-a_2}, & 1 < x < \infty. \end{cases} \tag{32}$$

A sequence $(\phi_j)_{j=1}^{\infty} \subset \mathscr{M}_{a_1,a_2}$ converges to $\phi \in \mathscr{M}_{a_1,a_2}$ if

$$\left\|\phi_j - \phi\right\|_{a_1,a_2,k} \to 0 \tag{33}$$

as $j \to \infty$ for each $k = 0, 1, 2, \ldots$.

For $a_1 < a_2$ (real or $\pm\infty$), $\mathscr{M}_(a_1, a_2)$ is defined. A function ϕ belongs to $\mathscr{M}_(a_1, a_2)$ if $\phi \in \mathscr{M}_{a,b}$ for some $a_1 < a < b < a_2$. Convergence of a sequence $(\phi_j)_{j=1}^{\infty} \subset \mathscr{M}_(a_1, a_2)$ is specified through its tail converging to ϕ in a fixed space $\mathscr{M}_{a,b}$ with $a_1 < a < b < a_2$.

2.0.21 Convolution in LCA group

The convolution of $f \in L^1(G)$ and $g \in L^p(G)$, where $1 \le p < \infty$, is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(xy^{-1})g(y)\,dy,$$

and the convolution theorem

$$\widehat{(f * g)}(\gamma) = \widehat{f}(\gamma) \cdot \widehat{g}(\gamma)$$

holds in any locally compact Abelian group G.

2.0.22 Strip of holomorphicity in Mellin transform

When we say $\mathscr{M} f$ has strip of holomorphicity S (or S_f) we mean that

$$S = \{s \in \mathbb{C} \mid a_1 < \Re(s) < a_2\} \tag{34}$$

for some $a_1 < a_2$ and $\mathscr{M} f$ is holomorphic on S. If $f \in \mathscr{M}'(a_1, a_2)$ with S as above, we write $f \in \mathscr{M}'_S$ or $f \in \mathscr{M}'_{S_f}$. Also, given $a_1, a_2 \in \mathbb{R} \cup \{-\infty, +\infty\}$, we denote

$$S(a_1, a_2) = \{s \in \mathbb{C} \mid a_1 < \Re(s) < a_2\}. \tag{35}$$

2.0.23 Inversion formula for Mellin transform

If $F : S(a_1, a_2) \to \mathbb{C}$ is holomorphic and satisfies $|F(s)| \leq K|s|^{-2}$ for some finite constant K, and we set

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) t^{-s} \, ds, \tag{36}$$

for a fixed $\sigma \in (a_1, a_2)$, then $f : \mathbb{R}_+ \to \mathbb{C}$ is continuous, does not depend on the choice of σ and is in $\mathscr{M}'(a_1, a_2)$. Furthermore $\mathscr{M} f = F$ on $S(a_1, a_2)$.

3 Scattering Analysis of Acoustic and Electromagnetic Wave

I begin this chapter with some discussion and results of inverse acoustic and EM waves. The direct scattering problem is a fundamental aspect of scattering theory. Here I am recalling that the direct scattering problem gives knowledge about the boundary of the scatterer and the nature of the imposed boundary condition. Through this problem, we determine the scattered wave and behavior of the scattered wave at large distances which we call far field. The inverse problem begins with the query of the direct problem. The knowledge of the far field pattern and nature of the scatterer is determined by the inverse problem.

The scattering of waves/or particles is a universal phenomenon that has a variety of applications across many scientific disciplines. Particularly, the scattering of plane waves from spheres is the simplest situation [1] to study mathematically and it is quite an old subject, however, it has an extremely enriched field. This field is still developing and has a variety of connections in many applied areas, such as seismology, optics, imaging, acoustics, quantum mechanics, nuclear physics, and many more. The practitioner of this study has enabled us to discover much more worldwide, especially in direct and inverse problems. However, It is safe to say that, whatever we hear and see irrespectively, is a consequence of acoustic and EM scattering from various objects.

The problem of inverse scattering with reduced measurement has gained a lot of interest lately. The motivation for the study of wave scattering comes from antenna theory. Our aim is to derive general principles for antenna structures instead of relying upon traditional antenna structures having frequency independent features. To know about the shape of the antenna's design inverse scattering problem strategy we use to solve the input-output pair of waves. In acoustics scattering surface or screen is considered as sonar which is different from formal antennas. Depending on the sonar role they are categorized as active or passive which behaves like a sound source or receiver. The study focuses on acoustic scattering from screens, lying between active and passive sonars. More appropriately, we call these sonars as passive sonars as they don't have an energy source but they are active in the sense of a sound source which is more significant.

I am considering here a two-dimensional sound soft screen in three-dimensional space i.e $\Omega \subset \mathbb{R}^3$. The study of acoustic wave leads to the partial differential Helmholtz equation $(\nabla + k^2)u = 0$, k is the wave number which is equal to $\frac{\omega}{c}$, where ω is the angular frequency and c is the speed of sound. In the context of a sound soft screen, the total pressure vanishes at the boundary of the screen. The total wave is expressed as the combination of incident wave and scattered wave.

After an acoustic scattering discussion, I extended our result for a planar screen of time harmonic EM waves. According to analysis of acoustic scattering, I began formulation with the idea of the direct scattering problem. Screen is perfectly conducting C^k–screen, $k = 1, \ldots, \infty$, in $\mathbb{R}^3$ is a compact, connected C^k–submanifold of an affine hyperplane $L \subset \mathbb{R}^3$. Similarly, the scattering of EM waves by planer screen leads to the study of Maxwell's equations. This generalizes the scalar result of the Helmholtz equation. In EM inverse scattering the main study reflected here is to determine the planar screen by using a single far field. This means that we have one fixed transmitter wave and the resulting scattered field is measured for all directions in the far field. We show that the far field of a scattered EM field corresponding to a single incoming plane wave always uniquely determines a bounded super-conductive planar screen.

3.0.1 Solution of Direct Scattering Problem

The solution to the direct scattering problem is formulated by the Helmholtz equation and Sommerfeld radiation condition. The scattered field u_s is denoted by an integral equation involving the potential function and distribution paring defined on screen. Mathematically we can write as

$$u_s(x) = \int_{R^2} \Phi(x, y^0)\rho(y')dy' \tag{37}$$

for all $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, where $\rho(y')$ is defined by the difference between the third derivative of the scattered field on the positive and negative sides of the screen. i.e

$$\rho(y') = \partial_3 u_s^+(y^0) - \partial_3 u_s^-(y^0). \tag{38}$$

Here, it can be seen that distribution $\rho(y')$ is an element of $\widetilde{H}^{-1/2}(\Omega_0)$. By taking the trace $x \to \Omega$ in (37) and considering that the total field at the boundary of the screen is zero (11) i.e $u_s = -u_i$ on Ω the sense of traces.

$$u_i(x) = -\int_{\mathbb{R}^2} \Phi(x, y^0)\rho(y')dy'. \tag{39}$$

The potential solution of u_s belongs to localized in $\mathbb{R}^3 \setminus \overline{\Omega}$. The direct problem can be solved if and only id ρ has a solution in $\widetilde{H}^{-1/2}(\Omega_0)$. If I elobrate more, I can say that for given ρ in integral equation, I can define $u_s$by (37). This was shown in Theorem 2.5 in [62]. Theorem 2.7 in the same source, proves that (39) has a unique solution $\rho \in \widetilde{H}^{-1/2}(\Omega_0)$ given any $u_i \in H^{1/2}(\Omega_0)$.

3.0.2 Fundamentals solution for the Helmholtz equation and radiating boundary conditions

$\Phi(x, y)$ is the fundamental solution of the Helmholtz equation in three-dimensional space $\mathbb{R}^3$ with a given wave number k. It connects the behavior of a function φ within a bounded domain D to its behavior on the boundary ∂D.

Let $D \subset \mathbb{R}^3$ be a bounded domain whose boundary is piecewise of class C^1 and $k \in \mathbb{R}_+$. Here,

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \tag{40}$$

for $x, y \in \mathbb{R}^3$, $x \neq y$. Then for any $\varphi \in C^2(\overline{D})$ and $x \in \mathbb{R}^3 \setminus \partial D$ we have

$$\int_D \Phi(x, y)(\Delta + k^2)\varphi(y)dy = \int_{\partial D} \big(\Phi(x, y)\partial_\nu \varphi(y) - \varphi(y)\partial_\nu \Phi(x, y)\big)ds(y) \tag{41}$$

This equation holds for a point x with both inside and outside of D having appropriate boundary conditions.

On building upon these formulations, the next step in our analysis focuses on characterizing solutions that satisfy both the Helmholtz equation and the Sommerfeld radiation condition.

Let u_s belong to the local Sobolev-spaces such that
$u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$.
If $(\Delta + k^2)u_s = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and it satisfies the Sommerfeld radiation condition, then the following representation arises

$$u_s(x) = \int_{\mathbb{R}^2} \Phi(x, y^0)(\partial_3 u_s^+ - \partial_3 u_s^-)(y^0) dy' \tag{42}$$

This representation effectively captures the behavior of the solution u_s at a point x outside the screen Ω. Here, $(\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)$ denotes a function that contains the difference in normal derivatives of u_s at the point y^0 on the screen. Importantly, this function is part of the space $\widetilde{H}^{-1/2}(\Omega_0)$.

3.0.3 Asymptotic convergence of the fundamental solution

Asymptotic convergence which highlights our focus on the behavior of solutions to the Helmholtz equation as they propagate away from their sources.
Here $K \subset \mathbb{R}^3$ is a nonempty compact set and positive constant k then the following expression tends to zero as $\lim_{r\to\infty}$. Mathematically expression as follows:

$$\lim_{r\to\infty} \sup_{|x|=r} \sup_{y\in K} |x| \left| \partial_y^\alpha \left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x}\cdot y} \right) \right| = 0$$

for any multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 1$. Recall that $\hat{x} = x/|x|$.
The proof of this expression involves differentiation rules and convergence is illustrate for multi-indices with $|\alpha| \leq 1$
This can be proven by splitting the terms in the above expression and showing their uniformly individual convergence to zero. I used the differentiation rules for ∇_y and estimate the fundamental solution. The results of these terms uniformly go to zero as $r \to \infty$. The results hold for both $|\alpha| = 0$, and $|\alpha| = 1$, and establish the desired asymptotic convergence.

3.0.4 Far-Field Representation of Scattered Waves on Screens

Consider $\Omega \subset \mathbb{R}^3$ is a screen and our scattered field u_s satisfy the direct scattering problem having an incident field u_i on screen Ω. Then $u_s^\infty(\hat{x})$ is the far field of scattered field is denoted by following expression

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi} \left\langle \left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0), e^{-ik\hat{x}\cdot y^0} \right\rangle_{y'} \tag{43}$$

where $\hat{x}$ represent point on the unit sphere.
If the normal derivative of the scattered field is integrable over the screen, then the above expression is equivalent to the following integral formulation

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y^0} \left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0) dy'.$$

I can start its proof by definition 2.0.8 of far-field representation, which states that the difference between far-field $u_s^\infty(\hat{x})$ and involving certain expression $u_s(x)$.

The primary step involves applying Lemma 3.0.3, which ensures that certain terms in the representation tend to zero uniformly. This relies on the convergence of specific functions in the restricted to any compact set

$$u_s^\infty(\hat{x}) = \lim_{|x|\to\infty} |x| e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle_{y'}$$

should the limit exists. The distribution pairing is over $y' \in \mathbb{R}^2$, We can rewrite

$$|x| e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle = \left\langle \rho(y'), |x| e^{-ik|x|} \Phi(x, y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi} \right\rangle_{y'} + \frac{1}{4\pi} \langle \rho(y'), e^{-ik\hat{x}\cdot y^0} \rangle_{y'}.$$

I can write the C^1-test function in the first pairing on the right as

$$|x|e^{-ik|x|}\Phi(x,y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi} = \frac{e^{-ik|x|}|x|}{4\pi}\left(\frac{e^{ik|x-y^0|}}{|x-y^0|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y^0}\right)$$

which converge to zero in the C^1 topology over y', and a fortiori y^0, restricted to any compact set by 3.0.3. Note that the C^1-seminorms are taken with respect to the y'-variable, and the absolute value makes the $e^{-ik|x|}$ that doesn't appear in the lemma disappear. Hence the application of the lemma is allowed. Elements of $\widetilde{H}^{-1/2}(\Omega_0)$ acts well on C^1 functions, so the distribution pairing with ρ and the test function tends to zero. Thus

$$\lim_{|x|\to\infty} |x|e^{-ik|x|}\langle\rho(y'),\Phi(x,y^0)\rangle_{y'} = \frac{1}{4\pi}\left\langle\rho(y'),e^{-ik\hat{x}\cdot y^0}\right\rangle_{y'}$$

as claimed.

3.0.5 Uniqueness of the scattered waves on screens

We have considered two screens $\Omega, \tilde{\Omega} \subset \mathbb{R}^3$ and $k \in \mathbb{R}_+$. The incident wave u_i and $u_s, \tilde{u}_s$ be scattered waves that satisfy the direct scattering problem for screens on $\Omega, \tilde{\Omega}$, respectively.

Here two conditions arise.
If u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ and $u_s^\infty = \tilde{u}_s^\infty$, then $\Omega = \tilde{\Omega}$. If u_i is antisymmetric then $u_s^\infty = \tilde{u}_s^\infty = 0$ for any screens $\Omega, \tilde{\Omega}$.
For more explanation I am using as a reference Theorem 3.0.4 and Lemma 3.1.
When incident wave u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ then it leads to the equality of $\rho = \tilde{\rho}$ when far field scattered wave u_s^∞ and $\tilde{u}_s^\infty$ are equal.
This can be expressed by the relationship

$$\overline{\Omega_0} = \operatorname{supp}\rho = \operatorname{supp}\tilde{\rho} = \overline{\tilde{\Omega}_0}$$

by Lemma 3.2. Because Ω_0 is a smooth domain, we have $\Omega_0 = \operatorname{int}\overline{\Omega_0}$, and similarly for $\tilde{\Omega}_0$. Thus the equation above implies $\Omega_0 = \tilde{\Omega}_0$ and by lifting, $\Omega = \tilde{\Omega}$.

If incident wave u_i antisymmetric it necessitates that $u_i = 0$ everywhere on $\mathbb{R}^2 \times \{0\}$ and as a result $u_s = 0$ satisfies all conditions of the direct scattering problem. Due to uniqueness solutions to the direct scattering problem established by (2.0.4) are unique by [62, Thms 2.5–2.7], this is the only solution that is $u_s = \tilde{u}_s = 0$ and the same holds for their far-fields. Note that this result is independent of the shape of $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$.

3.1 The inverse scattering problem for screens

The solution to the inverse problem, determining a screen Ω from the knowledge of a single incident wave u_i, and the corresponding far-field u_s^∞ scattered wave from the screen contains two steps: The first step determines the density function ρ from the far field and then determines the screen from ρ. There is an interesting observation that the problem is only solvable for the screen if incident waves u_i are not antisymmetric. One can see that only antisymmetric is not a deciding factor, whether u_i is identically zero rather than antisymmetry. If u_i is zero on a non-empty open subset $\mathbb{R}^2 \times \{0\}$ then it implies the following symmetry property.

$$u_i(x', x_3) = -u_i(x', -x_3)$$

for all $x \in \mathbb{R}^3$. It is interesting to see that partial invisibility is achieved inside thickened screens as long as the incident plane wave comes from a direction almost parallel to the screen's normal [18]. The direction of incident waves seems very important in scattering from objects that are thin in one direction.

3.1.1 Formulation of inverse problem

I am going to solve the context of the inverse problem by following lemma. The lemma establishes the relationship between the far-field scattered wave $u_s^\infty(\hat{x})$ and ρ in two dimensions.

Lemma 3.1 *Let $k \in \mathbb{R}_+$ and $\rho \in \mathscr{E}'(\mathbb{R}^2)$ be a distribution of compact support,*

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\left\langle \rho, e^{-ik\hat{x}\cdot y^0} \right\rangle. \tag{44}$$

Here, $\hat{x} \in \mathbb{S}^2$ belongs to the unit sphere, and the distribution pairing in variable $y' = (y_1, y_2) \in \mathbb{R}^2$. This uniquely determines ρ by u_s^∞. It establishes a direct connection between the far-field scattered wave and the distribution of compact support, enabling the unique determination of ρ based on the observed far-field behavior.

Proof 1 *The proof begins that the operator mapping $\rho \mapsto u_s^\infty$ is bounded and linear $\mathscr{E}'(\mathbb{R}^2) \to C^0(\mathbb{S}^2)$. This is because $\hat{x} \mapsto \left(y' \mapsto \exp(-ik\hat{x}\cdot y^0)\right)$ is continuous $\mathbb{S}^2 \to \mathscr{E}(\mathbb{R}^2)$.*

So it is enough to show that if $u_s^\infty = 0$ then $\rho = 0$.

Let us assume the former and suppose that $u_s^\infty = 0$. Similar formula as in (44)*, if $u_s^\infty = 0$ we have*

$$\hat{\rho}(\xi') = \frac{1}{2\pi}\left\langle \rho, e^{-i\xi'\cdot y'} \right\rangle$$

where the distribution pairing is over the variable $y' \in \mathbb{R}^2$.

Rewriting the expression involving $k\hat{x}\cdot y$, the proof derives following expression

$$u_s^\infty(\hat{x}) = \frac{1}{2}\hat{\rho}(k\hat{x}_1, k\hat{x}_2). \tag{45}$$

The proof shows that if u_s^∞ is zero for all $\hat{x} \in \mathbb{S}^2$, this implies $\hat{\rho}(\xi') = 0$ for all $|\xi'| \leq k$. Utilizing the fact that $\hat{\rho}$ can be extended to an entire function on $\mathbb{C}^2$, and it vanish on an open subset of $\mathbb{R}^2$. From this we conclude that $\hat{\rho}$ must be zero, which implies that $u_s^\infty = 0$ and hence $\rho = 0$.

Below is presented a lemma exploring a significant connection between the behavior of scattered waves and the shape of a screen in the context of wave scattering.

Lemma 3.2 *Consider a Helmholtz equation $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a screen and u_s satisfy the direct scattering problem Denote*

$$\rho(x') = \partial_3 u_s^+(x^0) - \partial_3 u_s^-(x^0)$$

for $x' \in \mathbb{R}^2$. If $u_i(x', x_3) \neq -u_i(x', -x_3)$ for some $x \in \mathbb{R}^3$ then

$$\overline{\Omega_0} = \operatorname{supp} \rho \tag{46}$$

for the shape Ω_0 of the screen Ω.

3.2 Solution to the Maxwell's equations in $\mathbb{R}^3/S$

A system of Maxwell's equations is considered

$$\nabla \times e = i\omega\mu\, h \tag{47}$$
$$\nabla \times h = -i\omega\varepsilon\, e \quad \text{in } \mathbb{R}^3 \setminus S$$

and it satisfies the Silver–Müller–radiation condition

$$\hat{r} \times e + \sqrt{\frac{\varepsilon}{\mu}} h = o(|x|^{-1}) \tag{48}$$
$$\hat{r} \times h - \sqrt{\frac{\mu}{\varepsilon}} e = o(|x|^{-1}).$$

As $|x| \to \infty$, where $\hat{r} = \frac{x}{|x|} > 0$.
Then, for all $\mathbf{x}$ in $\mathbb{R}^3$:

$$\begin{aligned} e(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y) \\ &- \frac{1}{i\omega\varepsilon} (\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \end{aligned} \tag{49}$$

$$\begin{aligned} h(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \\ &+ \frac{1}{i\omega\mu} (\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y) \end{aligned} \tag{50}$$

The proof involves constructing collar neighborhoods $\delta > 0$. Let $\mathscr{O}_\delta = \{x \pm t\nu(x); x \in S, 0 \le t < \delta\}$ be a collar neighbourhood of the scattering surface S, and applying standard representation formulas [14]. For small δ, $\mathscr{O}_\delta$ is a bounded piecewise analytic domain. The outgoing fundamental solution Φ of the Helmholtz operator $\Delta + k^2$ and the exterior unit normal ν_δ of $\partial\mathscr{O}_\delta$ are used.
By utilizing the standard representation formulas, fields e and h can be represented outside $\mathscr{O}_\delta$ in terms of $\partial\mathscr{O}_\delta$. The expression converge to the corresponding integral as $\delta \to 0$ over the surface S.

3.2.1 Exploring Sobolev–spaces in EM scattering

In the given context, a space of functions denoted as $L^2_{\text{loc}}(\mathbb{R}^3 \setminus S)$ [47, 24] is considered. This space consists of measurable functions defined on $\mathbb{R}^3 \setminus S$, and these functions are required to be square integrable on compact subsets of $\mathbb{R}^3 \setminus S$. This becomes equipped with semi–norms

$$\|f\|_R = \|f\|_{L^2(\mathbb{R}^3 \setminus S) \cap B_R(0))}, \quad R > R_0,$$

where R_0 is so large that $S \subset B_{R_0}(0)$.

Next, its defined the curl and divergence operators in the square-integrable function space:

$$L^2_{\text{loc, curl}}(\mathbb{R}^3 \setminus S) = \{u \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus S); \nabla \times u \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus S)\},$$

$$L^2_{\text{loc, div}}(\mathbb{R}^3 \setminus S) = \{u \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus S); \nabla \cdot u \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus S)\},$$

with seminorms we defines as

$$\|f\|_{R,\text{curl}} = (\|f\|_R^2 + \|\nabla \times f\|_R^2)^{1/2}, \quad \|f\|_{R,\text{div}} = (\|f\|_R^2 + \|\nabla \cdot f\|_R^2)^{1/2}.$$

These seminorms are used to quantify the "size" or "magnitude" of the functions in these spaces, constructed based on the L^2 norms of the functions themselves and the norms of their curl and divergence.

In this definition of electromagnetic phenomena on a surface denoted as S, we focus on specific functions that possess $H^{-1/2}$ regularity. These functions align with the surface's curvature and have a certain level of smoothness. We quantify their properties by equipping the space of these tangential $H^{-1/2}$ fields on S with a norm that reflects their size and behavior on the surface.

$$TH^{-1/2}(S) = \{u \in H^{-1/2}(S)^3; \langle \nu, u \rangle = 0\}$$

Here, the concept of surface divergence becomes significant. To extend this notion, I introduced the space $TH^{-1/2}_{\text{Div}}(S)$ comprising functions from $TH^{-1/2}(S)$ that satisfies a condition involving their surface divergence.

$$TH^{-1/2}_{\text{Div}}(S) = \{u \in TH^{-1/2}(S); \text{Div(u)} \in \text{H}^{-1/2}(\text{S})\}$$

and equip it with the Hilbert–norm defined by

$$\|u\|^2_{TH^{-1/2}_{\text{Div}}(S)} = \|u\|^2_{TH^{-1/2}(S)} + \left\|\text{Div(u)}^2\right\|_{H^{-1/2}(S)}.$$

3.2.2 Layer potentials in Sobolev–spaces

Various layer potential operators associated with the electromagnetic field are introduced and defined within the context of Sobolev spaces. For x belonging to the complement of S in R^3 and u in $C_0^\infty(S)^3$, I define the single-layer potential of u as follows:

$$V_{\mathbb{R}^3\setminus S}(u)(x) = \int_S \Phi(x-y)u(y), ds(y)$$

where ϕ is a fundamental solution of the vector Helmholtz equation.

I also introduce the electromagnetic layer operators: $K_{\mathbb{R}^3\setminus S}$, $N_{\mathbb{R}^3\setminus S}$

$$K_{\mathbb{R}^3\setminus S}(u)(x) = \nabla \times V_{\mathbb{R}^3\setminus S}(u)(x),$$

and

$$N_{\mathbb{R}^3\setminus S}(u)(x) = (\nabla\times)^2 V_{\mathbb{R}^3\setminus S}(u)(x).$$

3.2.3 Extension and Radiation Conditions for Electromagnetic Potentials on bounded domains

Assuming that the surface S can be extended to a well-defined boundary ∂U where U represents a bounded domain with C^2 characteristics. A link has established between surface properties and electromagnetic behavior. The concept of the single-layer potential $V_{\mathbb{R}^3\setminus S}$ which describes the interaction of a vector field with a surface, can be extended to yield a meaningful mapping. This extension takes place within distributional spaces, particularly the $H^{-1/2}(S)$ space.

$$V_{\mathbb{R}^3\setminus S}: H^{-1/2}(S) \to H^1_{\text{loc}}(\mathbb{R}^3 \setminus S)$$

and $V_{\mathbb{R}^3\setminus S}(u)$ satisfies the Sommerfeld–radiation condition.

Additionally, the electromagnetic layer operators $K_{\mathbb{R}^3\setminus S},\ N_{\mathbb{R}^3\setminus S}$ which originate from the single layer potential, also undergo extensions.

$$K_{\mathbb{R}^3\setminus S},\ N_{\mathbb{R}^3\setminus S}: T\dot{H}^{-1/2}_{\text{Div}}(S) \to L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S)$$

Importantly, these operators on functions belonging to the space $TH^{-1/2}_{\text{Div}}(S)$ which takes the tangential traces of vector fields on the surface S. When these electromagnetic layer operators are applied to functions u within $TH^{-1/2}_{\text{Div}}(S)$, the resulting electromagnetic fields satisfy the Sommerfeld–radiation conditions.

3.2.4 Representation theorem for Electromagnetic Solutions on screens

A proposition states the representation of electromagnetic solutions in the context of screens. A screen S in three-dimensional space is considered with a C^1 smoothness condition. Assuming the pairs $(e,h) \in L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S)$ and satisfy a set of Maxwell's equations involving the curl operator, a specific radiation condition known as the Silver-Müller condition.

Proposition 3.3 *Let $S \subset \mathbb{R}^3$ be a C^1–screen. Assume $(e,h) \in L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S)$ solves*

$$\nabla \times e = i\omega\mu\, h, \quad \nabla \times h = -i\omega\varepsilon\, e \quad \text{in} \quad \mathbb{R}^3 \setminus S,$$

and the Silver–Müller–radiation condition

$$\hat{r} \times e + \sqrt{\frac{\varepsilon}{\mu}} h = o(|x|^{-1}), \quad \hat{r} \times h - \sqrt{\frac{\mu}{\varepsilon}} e = o(|x|^{-1}), \quad \text{as } |x| \to \infty, \quad \hat{r} = x/|x| > 0.$$

If $\nu \times [e],\ \nu \times [h] \in TH^{-1/2}_{\text{Div}}(S)$, then in $\mathbb{R}^3 \setminus \overline{S}$,

$$e = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [e]) - \frac{1}{i\omega\varepsilon} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [h]),$$

and

$$h = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [h]) + \frac{1}{i\omega\mu} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [e]).$$

Here ν is the specified unit normal of S. □

3.2.5 Representation Formulas for the Scattered Field

In the following, the proposition is presented into the argument of electromagnetic scattering phenomena by considering a perfectly conducting C^2-screen. In this scenario, the scattered fields (E_{sc}, H_{sc}) associated with the (E_0, H_0) are explored. Specifically, these scattered fields belong to the spaces of $L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S)$.

Proposition 3.4 *Let S be a perfectly conducting C^2–screen, and let*

$$(E_{sc}, H_{sc}) \in L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\text{loc,curl}}(\mathbb{R}^3 \setminus S)$$

be the scattered field corresponding to an incoming field (E_0, H_0). *Then in* $\mathbb{R}^3 \setminus \overline{S}$ *one has*

$$E_{\mathrm{sc}} = -\frac{1}{i\omega\varepsilon} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [H_{\mathrm{sc}}])$$

and

$$H_{\mathrm{sc}} = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [H_{\mathrm{sc}}]).$$

These fields have the following asymptotic behaviour as $|x| \to \infty$*:*

$$E_{\mathrm{sc}}(x) = -\hat{x} \times \left(\hat{x} \times \frac{e^{ik|x|}}{4\pi i\omega\varepsilon|x|} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\mathrm{sc}}])(y)\, ds(y) \right) + O(|x|^{-2}),$$

$$H_{\mathrm{sc}}(x) = \hat{x} \times \frac{e^{ik|x|}}{4\pi i\omega\mu|x|} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\mathrm{sc}}])(y)\, ds(y) + O(|x|^{-2}),$$

where $\hat{x} = x/|x|, x \neq 0$.

3.3 Inverse problem for EM scattering by screens

This section determines unique aspects when the supporting hyperplane is predetermined. This holds for planar screens, where the uniqueness of the tangential density in the far-field pattern is explained by a key lemma. This lemma highlights the uniqueness of the tangential distribution, offering valuable insights into electromagnetic scattering.

Lemma 3.5 *Consider a* ρ *signifies a tangential distributional density defined on a hyperplane* L. $\rho^\infty(\hat{x})$ *defined for each* $\hat{x}$ *within the unit sphere* S^2.

$$\rho^\infty(\hat{x}) = \hat{x} \times \langle \rho, \exp\{-ik\langle \hat{x}, \cdot\rangle\}\rangle, \quad \hat{x} \in S^2.$$

Then the map $\rho \mapsto \rho^\infty$ *is injective.*

Proof 2 *The proof begins by selecting suitable coordinates to define the hyperplane* L *as* $\{x; x_3 = 0\}$. *Introducing a distributional density* ρ *on* L, *represented as* $\rho = a d\sigma$, *where* $a = (a_1, a_2) \in E'(\mathbb{R}^2)$ *and* $d\sigma$ *is the surface measure on the hyperplane* L. *Then* $\rho^\infty = 0$ *is equivalent with*

$$\xi \times (\hat{a}_1(\xi'), \hat{a}_2(\xi'), 0) = 0, \xi = (\xi', (k^2 - |\xi'|^2)^{1/2}), |\xi'| < k,$$

and hence $\hat{a}_1$ *and* $\hat{a}_2$ *vanish in the unit ball of* $\mathbb{R}^2$ *and since they are entire functions they are identically zero.*

3.4 Unique Determination of a Planar Screen

Proposition 3.6 *The unique determination of the supporting hyperplane from the far-field behavior of a single scattering solution is established. Assume* S_1 *and* S_2 *are two planar screens contained in supporting hyperplanes* π_1 *and* π_2 *respectively. Assume* $u_1 = (e_1, h_2)$ *and* $u_2 = (e_2, h_2)$ *scattering solutions for the screens* S_1 *and* S_2 *corresponding to the same initial field and having equal non–vanishing far fields. Then* $\pi_1 = \pi_2$.

Proof 3 *The proof revolves around understanding the behavior of scattered waves in relation to the determination of supporting hyperplanes. Beginning with the jumps across screens* S_1 *and* S_2 *denoted by* ρ_1 *and* ρ_2 *respectively. These jumps capture the discontinuities in the cross products* $\nu_1 \times h_1$ *and* $\nu_2 \times h_2$, *where* ν_i *represent the unit normal across*

S_i. Since u_1 and u_2 have equal far fields and $\mathbb{R}^3 \setminus (S_1 \cup S_2)$ is connected, we must have $u_1 = u_2$ there. Hence both fields must be smooth across $(S_1 \cup S_2) \setminus (S_1 \cap S_2)$ i.e both densities ρ_1 and ρ_2 are supported in the intersection $S_1 \cap S_2$. If the planes π_1 and π_2 intersect transversally, the jumps are supported on a codimension 2 subspace, and since they belong $\mathrm{T\dot{H}}^{-1/2}(S_1 \cup S_2)$ they must vanis in [25]. If the intersection is transversal so also the far fields vanish.

4 The role of Mellin, Fourier, and Hilbert transform in scattering

This chapter is about the connection of Mellin, Fourier, and Hilbert transform on the half-axis. The study analysis how sound waves behave and present peculiar characteristics near precise geometric features like cracks or screens. Such type of consideration plays an important role in understanding the irregularities or discontinuities in the propagation of acoustic waves. This type of study has potential applications in materials science, structural engineering, and medical imaging.

Mellin transform plays an important role in the asymptotic expansion of the theory of special function. It plays a significant character in the singular behavior of the solution of the equation $\mathscr{H}\rho = e$ where the right-hand side might not be smooth or integrable in the classical sense. The integral of Mellin transform exists only for complex value functions specifically, $s = a + ib$ where $a_1 < a < a_2$. The value of a_1 and a_2 depends on the function $f(t)$. We initiate our problem with the help of Mellin transform for generalized function. For function spaces we follow the approach of [64] where they allow us to build a relationship between Mellin and Hilbert transform. In an intuitive sense, the function u belongs to the space $\mathscr{M}\prime(a_1, a_2)$ class of distributions where $a_1, a_2 \in \mathbb{R}$, and its behavior satisfies certain growth conditions (8). The Mellin transform $\mathscr{M}[u](s)$ these distributions will be functions that are holomorphic on a vertical strip in the complex plane and also polynomially bounded as the imaginary part of the argument grows.

After studying and establishing the connection between the Mellin and Hilbert transform on the positive real axis, we begin to review some facts about Fourier transforms on locally compact Abelian groups. Particularly, the discussion focuses that in the case of the multiplicative group $(\mathbb{R}_+ \dot{.})$ Mellin transform is exactly the Fourier transform in LCA.

4.0.1 Illustrations of the Fourier transform in different structures

1. For $G = (\mathbb{R}, +)$ and character are given

$$\gamma_\xi(x) = e^{ix\xi}$$

is a character and by denoting γ_ξ simply by ξ, the Fourier transform is exprssed as:

$$\widehat{f}(\xi) = \int_0^\infty f(x)e^{-ix\xi}dx.$$

$\xi \in \mathbb{R}$. This implies that the dual group of $(\mathbb{R}, +)$ is $(\mathbb{R}, +)$ itself.

2. If $G = T$, the dual group is $(\mathbb{Z}, +)$ and the Fourier transform is given by

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta.$$

3. By Pontryagin Duality Theorem states that the dual group of $\mathbb{Z}$ is T and the Fourier transform in this context is given by

$$\widehat{f}(e^{ix}) = \int_{n=-\infty}^{\infty} f(n)e^{inx}.$$

.

4.0.2 Mellin transform is a Fourier transform in $\mathbb{R}_+$.

The content describes the Mellin transform as a Fourier transform in the multiplicative group on $\mathbb{R}_+$. The dual group of G_+ is identified as the additive imaginary axis of the complex plane. To find Fourier transform in the group $G_+ = (\mathbb{R}_+, \cdot)$ we need to find its dual space Γ. For $z = ix$, $x \in \mathbb{R}$, define

$$\gamma_z(t) = t^z = t^{ix}, \qquad t \in \mathbb{R}_+. \tag{51}$$

Clearly, this is a character in G_+ since

$$\gamma_z(ts) = (ts)^{ix} = t^{ix}s^{ix}$$

for $s,t \in \mathbb{R}_+$.

I can see in ([58], Section 2.2) that there are no other characters. Hence we can interpret that the dual group of G_+ is the additive imaginary axis of the complex plane and the Fourier transform is given by

$$\widehat{f}(z) = \int_0^\infty t^z f(t) \frac{dt}{t} \tag{52}$$

for $f \in L^1(G_+)$ and $z \in i\mathbb{R}$. But this is exactly the definition of the Mellin transform [44, 63], whenever the right-hand side is integrable. Thus we have shown that the Mellin transform is nothing else than the Fourier transform in the multiplicative group on $\mathbb{R}_+$.

4.0.3 The connection of Mellin transform with Hilbert transform

The connection of Mellin transform with Hilbert transform is represented by the formula

$$\mathscr{H} f(t) = \mathrm{p.v.} \int_0^\infty \frac{1}{1-t/s} f(s) \frac{ds}{s} = h \vee f(t) \tag{53}$$

where $h = \mathrm{p.v.}\, \frac{1}{1-t}$ and $\vee$ stands for the Mellin convolution is $(\mathbb{R}_+, \cdot)$.

The convolution theorem suggests that (53) implies that the Mellin transform of $\mathscr{H} f$ is

$$\mathscr{M}\mathscr{H} f(z) = \widehat{h}(z)\widehat{f}(z) = \cot(\pi z)\widehat{f}(z) \tag{54}$$

where $\widehat{\ }$ is the Fourier transform on the LCA $(\mathbb{R}_+, \cdot)$, or in other words, the Mellin transform. The second equality follows from Example 8.24.II in [52],

$$\mathrm{p.v.} \int_0^\infty t^z \frac{1}{1-t} \frac{dt}{t} = \pi \cot(\pi z). \tag{55}$$

4.1 Space of Mellin transform in distributions

In this section, I define a class of distributions on the positive real axis which we call Mellin transformable distribution. These distributions are denoted by $\mathscr{M}'(a_1, a_2)$ where $a_1, a_2 \in \mathbb{R}$. The Mellin transform of these distributions will be holomorphic functions on a vertical strip in the complex plane and polynomially bounded as the imaginary part of the argument grows. To construct this class, spaces of ordinary smooth test functions on $\mathscr{D}(\mathbb{R}_+)$ is defined, which is compactly supported and also functions of the form t^{s-1} for some complex numbers s. The dual of these spaces become the Mellin transform of a class of distributions. There is precise detail in both [64] and [5] related to all these distributions.

Here we have longer discussion about Mellin transform on space of distributions, which shows that $\mathscr{M}(a_1, a_2)$, $a_1 < a_2$ are non-trivial. As a consequence of the following, we see that the linear functionals that we are building are in fact distributions $\mathscr{D}'(\mathbb{R}_+)$. It is worth noting that they allow exponential growth, so cannot be interpreted as tempered distributions. The result of Mellin transform distribution is represented by following two points.

- Let a_1, a_2 be real numbers and $s \in \mathbb{C}$. Let $\phi(t) = t^{s-1}$ for $t > 0$. Then $\phi \in \mathscr{M}_{a_1,a_2}$ if and only if $a_1 \leq \Re(s) \leq a_2$. As a consequence $\phi \in \mathscr{M}(a_1,a_2)$ if and only if $a_1 < \Re(s) < a_2$.
- Let $\mathscr{D}(\mathbb{R}_+)$ be the space of compactly supported smooth test functions on $\mathbb{R}_+$ with the usual topology. Then $\mathscr{D}(\mathbb{R}_+) \subset \mathscr{M}(a_1,a_2)$ continuously for any $a_1 < a_2$ real or infinite. The inclusion is dense.

4.1.1 Mellin transform for distributions

Before defining Mellin transform for distribution I am recalling some formulations of measurable function $\phi \in \mathscr{M}(a_1,a_2)$ and $f_u : \mathbb{R}_+ \to \mathbb{C}$,

$$\langle u, \phi \rangle = \int_0^\infty f_u(t)\phi(t)\,dt, \qquad \phi \in \mathscr{M}(a_1,a_2). \tag{56}$$

Recalling that the Mellin transform of a measurable function $f : \mathbb{R}_+ \to \mathbb{C}$ is given by

$$\mathscr{M}f(s) = \widetilde{f}(s) = \int_0^\infty f(t)t^{s-1}\,dt \tag{57}$$

for that $s \in \mathbb{C}$ for which the integral converges in the sense of Lebesgue. Inspired by these two observations I define the Mellin transform of distributions Let $a_1, a_2 \in \{-\infty, +\infty\} \cup \mathbb{R}$ with $a_1 < a_2$ and let $u \in \mathscr{M}'(a_1,a_2)$. Then the Mellin transform of u is

$$\mathscr{M}u(s) = \widetilde{u}(s) = \langle u, t^{s-1} \rangle \tag{58}$$

for $s \in \mathbb{C}$, $a_1 < \Re(s) < a_2$.

4.2 The Hilbert transform

The following subsection discusses the Mellin transform of the Hilbert transform distribution. Mellin transform of the distribution

$$\langle H, \phi \rangle = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty \right) \frac{\phi(t)}{1-t}dt, \tag{59}$$

where in the principal value sense i$H = \pi^{-1}/(1-t)$ is almost the kernel of the Hilbert transform of a function vanishing on $\mathbb{R}_-$.

$$\mathscr{H}f(x) = \frac{1}{\pi}\,\mathrm{p.v.} \int_0^\infty \frac{f(y)}{x-y}\,dt. \tag{60}$$

By changing the variables $y = x/t$, $dy = -x\,dt/t^2$ in (60) we get the following:

$$\mathscr{H}f(x) = -\frac{1}{\pi}\,\mathrm{p.v.} \int_0^\infty \frac{1}{1-t} f\left(\frac{x}{t}\right)\frac{dt}{t} = -(H \vee f)(x). \tag{61}$$

where $\vee$ can be defined as

$$(f \vee g)(\tau) = \int_0^\infty f(t)g\left(\frac{\tau}{t}\right)\frac{dt}{t}, \qquad \tau > 0 \tag{62}$$

if f and g are integrable functions and otherwise

$$\langle f \vee g, \theta \rangle = \langle f, \psi \rangle, \qquad \psi(t) = \langle g, \theta_t \rangle \tag{63}$$

for $\theta \in \mathscr{M}(a_1,a_2)$, $t > 0$ and $\theta_t(\tau) = \theta(t\tau)$.

4.2.1 Distribution of $1/(1-t)$ belong to $\mathscr{M}'(0,1)$

The distribution $1/(1-t)$ in the principal value sense belongs to $\mathscr{M}'(0,1)$. Moreover, it can be written as

$$\left\langle \frac{1}{1-t}, \phi \right\rangle = \left(\int_0^{1/2} + \int_{3/2}^{\infty} \right) \frac{\phi(t)}{1-t} dt - \int_{1/2}^{3/2} \frac{\phi(t) - \phi(1)}{t-1} dt \tag{64}$$

where $1/(1-t)$ is interpreted as a pointwise function on the right-hand side. Lastly, there is a finite C such that $|\langle 1/(1-t), \phi \rangle| \leq C(\|\phi\|_{0,1,0} + \|\phi\|_{0,1,1})$.

I analyze here how distributions will satisfy their results. Let us denote $u = 1/(1-t)$, and the proof demonstrates that the distribution pairing $\langle u, \phi \rangle \in \mathbb{C}$ is well-defined on the complex number for $\phi \in \mathscr{M}(0,1)$ where $\mathscr{M}(0,1)$ is space of test function with certain smoothness properties. Suppose a function $h(t)$ which is used for simplifying the expression, where $h(t) = 1$ for $1/2 < t < 3/2$ and $h(t) = 0$ otherwise. The limit of integrals involving a test function ϕ and cut off function $h(t)$ is represented as:

$$\langle u, \phi \rangle = \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty} \right) \left(\frac{\phi(t) - \phi(1)h(t)}{1-t} + \frac{\phi(1)h(t)}{1-t} \right) dt \tag{65}$$

with $s = 2 - t$.

The representation involves a combination of integrals over different intervals and terms related to the smoothness of ϕ is employed to simplify the expression. The use of the secant function ensures the continuity of integrand.

$$\begin{aligned} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty} \right) \frac{\phi(t) - \phi(1)h(t)}{1-t} dt &= \left(\int_0^{1/2} + \int_{3/2}^{\infty} \right) \frac{\phi(t)}{1-t} dt \\ &+ \left(\int_{1/2}^{1-\varepsilon} + \int_{1+\varepsilon}^{3/2} \right) \frac{\phi(t) - \phi(1)}{1-t} dt. \end{aligned} \tag{66}$$

The first integrand is continuous on $(0,1/2) \cup (3/2,\infty)$ because the secant function ensures the continuity of integrand. It is also integrable since

$$\int_0^{1/2} \left| \frac{\phi(t)}{1-t} \right| dt + \int_{3/2}^{\infty} \left| \frac{\phi(t)}{1-t} \right| dt < \infty$$

. We can establish the continuity of the distribution pairing by considering the limit of the distribution pairing for a sequence of test functions converging to zero, as

$$\left| \langle u, \phi_j \rangle \right| \leq C \left(\left\| \phi_j \right\|_{a,b,0} + \left\| \phi_j \right\|_{a,b,1} \right) \to 0$$

and continuity is proven. The linearity property is trivial. Hence $u \in \mathscr{M}'(0,1)$. These results shows that the distribution $u = 1/(1-t)$ behaves pretty well within the wide class of test functions and making it a valid element of the distribution space $\mathscr{M}'(0,1)$ when principal value pairings are considered.

4.2.2 Mellin transform of $[1/(1-t)](s) = \pi \cot(\pi s)$

We have $\mathscr{M}[1/(1-t)](s) = \pi \cot(\pi s)$ in the principal value sense for $0 < \Re(s) < 1$. The distribution is in $\mathscr{M}(0,1)$. We need to calculate

$$\text{p.v.} \int_0^{\infty} \frac{t^{s-1}}{1-t} dt. \tag{67}$$

Refer to Example 8.24.II in [52], especially pages 219–220 for the calculations.

4.3 Mellin transform of the Hilbert transform:

I established the Hilbert transform for both test functions and distributions. The principal value integrals satisfy that the Hilbert transform is well-defined for a wider class of distributions and functions.

For a test function $f \in \mathscr{D}(\mathbb{R}_+)$, the Hilbert transform can be expressed as involving the Cauchy Principle value

$$\mathscr{H} f(x) = \mathrm{p.v.}\, \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x-y} dy = \frac{1}{\pi} \int_0^\infty \frac{f(x/t)}{1-t} \frac{dt}{t}. \tag{68}$$

If we have distribution $u \in \mathscr{M}'(a,b)$ with $0 \leq a < b \leq 1$, it is an element of $\mathscr{M}'(a,b)$ then Hilbert transform is defined as

$$\mathscr{H} u = -H \vee u$$

with

$$\langle H, \phi \rangle = \mathrm{p.v.}\, \frac{1}{\pi} \int_0^\infty \frac{\phi(t)}{1-t} dt \tag{69}$$

$$\langle H \vee u, \theta \rangle = \langle H, \psi \rangle, \qquad \psi(t) = \langle u, \theta_t \rangle,\, \theta_t(s) = \theta(ts) \tag{70}$$

for $\theta \in \mathscr{M}(a,b)$. Here,

$$\mathscr{M}[f \vee g](s) = \mathscr{M} f(s) \mathscr{M} g(s), \quad a_1 < \mathbb{R}(s) < a_2 \tag{71}$$

Lastly, if u is distribution $\mathscr{M}'(a,b)$ with $0 \leq a < b$, then Millen transform of $\mathscr{H} u$ is represented by the equation

$$\mathscr{M}[\mathscr{H} u](s) = -\cot(\pi s) \mathscr{M}[u](s) \tag{72}$$

for $a < \Re(s) < b$.

4.4 Inhomogenous Hilbert transform on a half-line

In this section, I will prove that the solution ρ to the equation

$$\mathscr{H} \rho = e, \qquad \mathbb{R}_+ \tag{73}$$

has a blow-up singularity at $x = 0$ when e is general but in a suitable function space. In this section, the inhomogeneous Hilbert transform on a half-line is investigated and established a significant result regarding the behavior of the solution to the equation $\mathscr{H} \rho = e$. Through rigorous proof, we demonstrate that the blow-up singularity occurs for a general e within the appropriate function space. This result holds significant implications for practical applications and theoretical analysis, as it sheds light on the behavior of the inhomogeneous Hilbert transform in specific settings.

4.4.1 Mellin transform at 1/2

Here it discusses the Mellin transform at the point $\mathscr{M}[e](1/2) = 0$. Suppose that $e \in \mathscr{M}'(a,b)$, $\rho \in \mathscr{M}'(\alpha,\beta)$ belong to certain function spaces where $0 \leq a \leq \alpha < \beta \leq b \leq 1$. It also assumes the validity of an equation (73) within $\mathscr{M}'(\alpha,\beta)$ By applying the Mellin transform of (73) and using properties from the previous result 4.3 we have

$$-\cot(\pi s) \mathscr{M}[\rho](s) = \mathscr{M}[e](s)$$

A specific case where $1/2$ lies within the interval (α, β). If this is true then $\mathscr{M}[e](1/2)$ and $\mathscr{M}[\rho](1/2)$ are well defined complex numbers. Since Mellin transform of $-\cot(\pi/2) = 0$. The equation simplifies that $\mathscr{M}[e](1/2) = 0$. The implies result concludes that Mellin transform of the distribution e at $s = 1/2$ is zero.

The solution to equation $\mathscr{H}\rho = e$ in suitable function spaces. Two cases arise here.

- Case 1 : Unique solution exist when $\mathscr{M}[e](1/2) = 0$.

- Case 2: Two solution exist $\mathscr{M}[e](1/2) \neq 0$.

By employing the Mellin transform inversion formula and certain estimates for the function $F(s) = -\tan(\pi s)\mathscr{M}e$. The behavior of $F(s)$ and $\mathscr{M}e$ in different vertical strips allows the conclusion of existence and representation of solutions for the inhomogeneous Hilbert transform on a half-line.

4.4.2 Existence of solution

Lemma 4.1 *Let $e \in \mathscr{M}'(a,b)$ for some $0 \leq a < b \leq 1$. If*

$$a < b \leq 1/2, \qquad \text{or} \qquad 1/2 \leq a < b, \qquad \text{or} \qquad \mathscr{M}[e](1/2) = 0$$

then there is $\rho \in \mathscr{M}'(a,b)$ satisfying $\mathscr{H}\rho = e$. Furthermore, for any α, β, c with $a < \alpha < c < \beta < b$ for this ρ it holds that

$$\rho(t) = \frac{-1}{2\pi i}(-t\,d/dt)^{m+2} \int_{c-i\infty}^{c+i\infty} s^{-m-2} \tan(\pi s)\mathscr{M}[e](s)t^{-s} ds \tag{74}$$

in $\mathscr{M}'(\alpha, \beta)$. Here $m \in \mathbb{N}$ can be any number for which there is a polynomial P of degree m such that $|\mathscr{M}[e](s)| \leq P(|x|)$ on $S(\alpha, \beta)$.

In the case where

$$a < 1/2 < b, \qquad \text{and} \qquad \mathscr{M}[e](1/2) \neq 0$$

there are no solutions in any $\mathscr{M}'(\alpha, \beta)$ with $\alpha < 1/2 < \beta$. Instead there is $\rho_- \in \mathscr{M}'(a, 1/2)$ and $\rho_+ \in \mathscr{M}'(1/2, b)$ such that $\mathscr{H}\rho_\pm = e$ in $\mathscr{M}'(a, 1/2)$ and $\mathscr{M}'(1/2, b)$, respectively. They satisfy

$$\rho_-(t) = \frac{-1}{2\pi i}(-t\,d/dt)^{m+2} \int_{c_- - i\infty}^{c_- + i\infty} s^{-m-2} \tan(\pi s)\mathscr{M}[e](s)t^{-s} ds, \tag{75}$$

$$\rho_+(t) = \frac{-1}{2\pi i}(-t\,d/dt)^{m+2} \int_{c_+ - i\infty}^{c_+ + i\infty} s^{-m-2} \tan(\pi s)\mathscr{M}[e](s)t^{-s} ds \tag{76}$$

in $\mathscr{M}'(\alpha_-, \beta_-)$ and $\mathscr{M}'(\alpha_+, \beta_+)$, respectively, for any $a < \alpha_- < c_- < \beta_- < 1/2$ and $1/2 < \alpha_+ < c_+ < \beta_+ < b$. Here $m \in \mathbb{N}$ can be any number for which there is a polynomial P of degree m such that $|\mathscr{M}[e](s)| \leq P(|x|)$ on $S(\alpha_-, \beta_+)$.

4.4.3 Unique solution

If two distributions, denoted as ρ_1 and ρ_2 belongs to the space of tempered distributions $\mathscr{M}'(a,b)$ and the same Hilbert transform, i.e., $\mathscr{H}\rho_1 = \mathscr{H}\rho_2$, then they are identical same. Uniqueness can be shown by taking Mellin transform of the equation and using the Properties of the Hilbert transform This leads to the expression

$$-\cot(\pi s)\mathscr{M}[\rho_1](s) = -\cot(\pi s)\mathscr{M}[\rho_2](s)$$

for $s \in S(a,b)$. When $s \neq 1/2$ we can divide by the cotangent and get

$$\mathscr{M}[\rho_1](s) = \mathscr{M}[\rho_2](s)$$

Since the difference of the Mellin transforms is holomorphic in the strip (a,b), the two distributions are equal throughout this region.
The properties of the Mellin transform are referred to conclude that

$$\rho_1 = \rho_2 \text{ in } \mathscr{M}'(a,b)$$

4.4.4 Cauchy Integral applications in distribution theory

Cauchy Integral plays an important role in understanding the behavior of these distributions in the context of integral transforms. It provides a mathematical tool to express distributions in terms of integrals involving a holomorphic function with certain growth conditions.

Let us suppose $0 < \alpha < 1/2 < \beta < 1$ and $f : S(\alpha,\beta) \to \mathbb{C}$ be holomorphic with $|f(s)| \leq Cs^m$ for some $m \in \mathbb{N}$. For $\alpha < c_- < 1/2 < c_+ < \beta$ define

$$\bar{\rho}_-(t) = \frac{-1}{2\pi i}\int_{c_- - i\infty}^{c_- + i\infty} s^{-m-2}\tan(\pi s) f(s) t^{-s} ds,$$
$$\bar{\rho}_+(t) = \frac{-1}{2\pi i}\int_{c_+ - i\infty}^{c_+ + i\infty} s^{-m-2}\tan(\pi s) f(s) t^{-s} ds.$$

Then

$$\bar{\rho}_+(t) = \frac{2^{m+2}}{\pi} f(\tfrac{1}{2}) t^{-1/2} + \bar{\rho}_-(t) \tag{77}$$

for all $t \in \mathbb{R}_+$. The function f is holomorphic in the strip $S(\alpha,\beta)$, and shows controlled growth, precisely $|f(s)| \leq Cs^m$, where C is a constant and m is a natural number. The integrands in ρ_+, ρ_- are holomorphic in $S(a,b) \setminus \{1/2\}$. Since f is holomorphic in $S(a,b)$. The estimates for the tangent function of imply that

$$|\tan(\pi s)| \leq C_{c_+}$$

when $\Re s = c_+$. This is because c_+ is fixed and away from half-integers, and the estimate for f in the assumptions give

$$\left|s^{-m-2}\tan(\pi s) f(s)\right| \leq K|s|^{-2} \tag{78}$$

for $\Re s = c_+$. Since $|s|^{-2}$ is integrable on $\{c_+ + it \mid t \in \mathbb{R}\}$ we get

$$\bar{\rho}_+(t) = \lim_{M\to\infty} \frac{-1}{2\pi i}\int_{c_+ - iM}^{c_+ + iM} s^{-m-2}\tan(\pi s) f(s) t^{-s} ds. \tag{79}$$

The integrals over specific paths play an important role in the analysis, particularly in the calculation of integrals and residues. They are used to design a counterclockwise rectangle with the point $s = \frac{1}{2}$ in the interior. The integrand in (79) is holormorphic in a neighbourhood of this rectangle as long as the neighbourhood is small enough to not reach $s = 1/2$.

$$\begin{cases} P_{+-} = c_+ - iM \\ P_{++} = c_+ + iM \\ P_{-+} = c_- + iM \\ P_{--} = c_- - iM \end{cases} \qquad \begin{cases} \gamma_{+-}(r) = (1-r)P_{+-} + rP_{++} \\ \gamma_{++}(r) = (1-r)P_{++} + rP_{-+} \\ \gamma_{-+}(r) = (1-r)P_{-+} + rP_{--} \\ \gamma_{--}(r) = (1-r)P_{--} + rP_{+-} \end{cases} \tag{80}$$

By Cauchy's residue theorem

$$\frac{-1}{2\pi i}\left(\int_{\gamma_{+-}} + \int_{\gamma_{++}} + \int_{\gamma_{-+}} + \int_{\gamma_{--}}\right) I_t(s)ds = -\operatorname{Res}(I_t, 1/2). \tag{81}$$

Now let us consider the behavior of Integrand on horizontal path $\gamma_{++}(r), \gamma_{--}(r)$. The estimates for integration are given by

$$|I_t(s)| = \left|s^{-m-2}\tan(\pi s) f(s) t^{-s}\right| \leq C t^{-a} M^{-2}$$

As M approaches infinity and the term M^{-2} diminishes, concluding that the integrands vanish in the limit.

Now let us investigate what happens when it goes to vertical path.

The integral over γ_{+-} multiplied by the constant in front of it in 81 equals $\bar{\rho}_+(t)$, as we saw above in 79 when we passed the integral limits to infinity. Lastly, just as at the beginning of this proof, we can let $M \to \infty$ in the integral over γ_{-+} and get $-\bar{\rho}_-(t)$.

4.5 Solution of $\mathscr{H}\rho = e$

In the context of equation $\mathscr{H}\rho = e$, where $\mathscr{H}$ represents the Hilbert transform. Before starting the theorems, it's necessary to clarify the notation used in these spaces. The spaces under consideration are denoted by $u \in \mathscr{M}'(a,b)$ if informally

$$u(t) = O(t^{-a}), \qquad t \to 0,$$
$$u(t) = O(t^{-b}), \qquad t \to \infty.$$

A more exact understanding of these spaces $u \in \mathscr{M}'(a,b)$ involves the Mellin transform $\mathscr{M}[u](s)$ which is expected to be holomorphic in the vertical strip $s \in S(a,b)$ is defined by $a < \Re(s) < b$ and has polynomial growth on vertical lines.

Following theorems are presented that are related to the solution of an equation involving a linear operator $\mathscr{H}$ acting on a function ρ resulting in a prescribed function e. These theorems characterize a solution's uniqueness, existence, and properties to a particular partial differential equation.

Theorem 4.2 *Let $e \in \mathscr{M}'(a,b)$ with $0 \leq a < b \leq 1$. If $b \leq 1/2$ or $1/2 \leq a$ or $a < 1/2 < b$ and $\mathscr{M}[e](1/2) = 0$ the equation*

$$\mathscr{H}\rho = e$$

has a unique solution $\rho = \rho_0 \in \mathscr{M}'(a,b)$. Furthermore if $\rho' \in \mathscr{M}'(a',b')$ is another solution with $S(a',b') \subset S(a,b)$ then $\rho' = \rho_0$ in $\mathscr{M}'(a',b')$.

The above theorem shows that equation $\mathscr{H}\rho = e$ has a unique solution if e satisfies certain conditions regarding to Mellin transform at $s = 1/2$, particularly $\mathscr{M}[e](1/2) = 0$. The existence result is followed by 4.4.2.

Theorem 4.3 *Let $e \in \mathscr{M}'(a,b)$ with $0 \le a < 1/2 < b \le 1$ and $\mathscr{M}[e](1/2) \neq 0$. Then $\mathscr{H}\rho = e$ has no solutions with $1/2 \in S_\rho$. Instead there are unique solutions $\rho_- \in \mathscr{M}'(a,1/2)$ and $\rho_+ \in \mathscr{M}'(1/2,b)$ and they satisfy*

$$\rho_+(t) - \rho_-(t) = \frac{4}{\pi}\mathscr{M}[e](1/2)\frac{1}{\sqrt{t}}. \tag{82}$$

Furthermore if $\rho' \in \mathscr{M}'(a',b')$ is another solution with $S(a',b')$ intersecting $S(a,1/2)$ or $S(1/2,b)$ then $\rho' = \rho_-$ or $\rho' = \rho_+$ in $\mathscr{M}'(a',b')$, respectively.

Proof 4 (Proof of Theorem 4.3) *The existence and non-existence follow from Lemma 4.4.2. Uniqueness is given by Lemma 4.4.3. All that's left to prove is the identity (82). The existence lemma gives us formulas for ρ_- and ρ_+ in the form of (75) and (76). These are just $(-td/dt)^m$ applied to the integrals in Section 4.4.4 with $f(s) = \mathscr{M}[e](s)$. Thus*

$$\rho_+(t) - \rho_-(t) = \frac{2^{m+2}}{\pi}\mathscr{M}[e](1/2)\left(-t\frac{d}{dt}\right)^m \frac{1}{\sqrt{t}}.$$

But $t^{-1/2}$ is an eigenfunction of $(-td/dt)$, since

$$(-td/dt)t^{-1/2} = -t\cdot(-1/2)t^{-1/2-1} = 2^{-1}t^{-1/2}.$$

Hence $(-td/dt)^m t^{-1/2} = 2^{-m}t^{-1/2}$ and the resul follows.

The Equation (82) shows that ρ_+ has a singularity of type $t^{-1/2}$ unless the Mellin transform of e vanishes at $s = 1/2$. This suggests that acoustically scattered waves from most cracks or screens will have a singularity at their ends. However, if

$$e(t) = \begin{cases} e^{i\sqrt{t}}, & 0 \le t \le (2\pi)^2, \\ 0, & t > (2\pi)^2, \end{cases}$$

it turns out that $\mathscr{M}[e](1/2) = 0$.

5 Conclusion

In this thesis, I considered the inverse scattering problem to determine the shape of a flat screen by using the Acoustic and EM waves and examining the singular behavior of scattered fields near the boundary of the scatterer. I explained the scattering analysis of Acoustic and EM waves in Chapter 3. Moreover, I described the curve shape scatterer's singularity containing equation $\mathscr{H}\rho = e$ by using the important integral transforms method in Chapter 4. The contents of the Chapters are based on the papers which are provided in the appendix.

In the introduction, I provided an overview of my research which consists of three articles. The study contributes to our understanding of passive sonars and their effect on sound patterns. The research focuses on the question that the shape and location of a passive sonar can be resolved by its sound reflection. This is quite a difficult problem, and I resolved that a single input-output pair of sound waves can specify the shape of both acoustic and EM flat screens. The motivation for studying the singularity of waves comes from analyzing the scattering of acoustic waves from a crack or screen in a two-dimensional domain.

In Chapter 2, I provided the definitions and discussed fundamental concepts essential for understanding the subsequent chapters. These concepts include addressing the direct scattering problem from screens and explaining the role of sound-soft screens in scattering phenomena. I defined some generalized functions that are helpful in the study of the scattered fields. I explored fundamental concepts such as Maxwell's equations, Silver Muller's radiation conditions, and the characterization of electric-magnetic far-field patterns in electromagnetic (EM) scattering problems. In addition, I introduced wave singularity concepts on the half-line and outlined the framework of the Mellin transform, establishing its connection with Hilbert and Fourier transforms. I provided a definition of test function spaces and corresponding dual spaces of distributions. These spaces play an important role in understanding how the convolution and Mellin transform work together.

Wave scattering with reduced measurement has recently become a topic of considerable interest. In Chapter 3, I explored the significance of a single far-field pattern in determining the characteristics of a screen using just one pair of input and output waves. The shape determining problem is known as Schiffer's problem, as Schiffer introduced this kind of inverse scattering problem and proved that sound-soft obstacles can be determined with many far-field patterns. My work in this Chapter shows that given the far-field caused by any single incident wave scattering off a smooth flat-screen. I considered the two-dimensional sound soft flat screen Ω in three-dimensional space. Acoustic waves scattered by Ω lead to the study of the Helmholtz equation. A direct scattering problem has a solution with a scattered field belonging to some function spaces. The fundamental solution for the Helmholtz equation and radiating boundary conditions has been discussed. The uniqueness of our passive inverse problem is proved as long as incident waves are not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$.

Furthermore, the unique determination of the unknown screen and supporting hyperplane corresponding to the single measurement of the far-field is another important result of the inverse EM scattering problem. The proof followed from the representation formula for the exterior solution of Maxwell's equations. Here, the main idea was to reduce the scattering problem to a single tangential integral equation on the screen, where the unknown is the jump of the tangential component of the scattered magnetic field. We showed that the far field uniquely determines the jump of the scattered field and also the screen, since the boundary of the field is the *support* of the jump.

Finally, in Chapter 4, I considered the singular behavior of solutions to the equation $\mathscr{H}\rho = e$ on a half-axis, where $\mathscr{H}$ represents the one-sided Hilbert transform, ρ is an unknown solution, and e is a known incident field. This investigation serves as a simplified model problem aimed at understanding wave field singularities caused by curve-shaped scatterers in a planar domain. I have demonstrated that the solution ρ exhibits a singularity of the form $\mathscr{M}[e](1/2)/\sqrt{t}$, where $\mathscr{M}$ denotes the Mellin transform. To obtain this, I used specially constructed function spaces $M'(a,b)$ introduced by Zemanian. These spaces allowed us to precisely explore the relationship between the Mellin and Hilbert transforms. Additionally, I discussed the Fourier transform, recognizing that the Mellin transform is essentially a Fourier transform on the locally compact Abelian multiplicative group of the half-line. This insight guided our investigation and provided a framework for our analysis.

In considering future directions, the following questions could arise:
Generalizing this work to compact real-analytic screens is a planned direction, however this will require more advanced techniques. As possible applications mentioned below:

- Suppose we have an inaccessible array of radars from which one can only obtain distant data. Such information could say whether the array uses classical dipole antennas or more advanced tripole antennas [29]. My result indicates that such information can, in principle, be obtained with a single measurement.
- In antenna theory, one can control better the far-field pattern of the antenna by taking into account the singularities of the electric-magnetic fields near the boundary of the reflector.

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Abstract

Inverse scattering of Acoustic and Electromagnetic waves from flat screens and properties of integral transforms on a half axis

In the field of mathematical physics, scattering theory has played a significant role in the theory of antennas. Recently, there has been rising interest in this area. In the context of this thesis, the problem of fixed frequency acoustic and electromagnetic scattering from a sound-soft flat screen and properties of integral transforms on a half axis have been considered. The screen is situated on a two–dimensional plane and interacts with incident waves, scattering them into three–dimensional space.

This study is relevant in applications such as reflecting sonars and antennas, where it's not safe to assume that the incident wave is a simple plane wave. The results of this work are significant. It demonstrates that it is possible to precisely determine the shape of the screen using the far-field pattern generated by a single arbitrary incident wave, given that it is not antisymmetric with respect to the plane. The method we are using is based on the idea of being partly motivated by a certain integral operator. The methodology we are following is in which we begin by demonstrating that the far-field pattern can be regarded as the restriction to a ball of radius k (representing the wavenumber) of the two-dimensional Fourier transform of a function that has support on the screen. To address the issue where the incident wave might vanish on parts of the screen, we establish a significant insight: the shape of the screen is precisely defined by the support of this function.

In addition, I studied that the far-field pattern of a scattered electromagnetic field, resulting from a single incoming plane wave, uniquely characterizes a bounded superconductive planar screen. This finding generalizes our acoustic results, highlighting the unique determination capabilities of electromagnetic scattering. The goal of this work is to discover the unique determination of supporting hyperplanes corresponding to the single measurement having non–vanishing far–fields. This work stands in contrast to some previous studies that primarily focused on simple scatterers like polyhedral shapes, balls, or discs, as well as smooth planar curves. These findings are remarkable because it holds true for screens of various shapes, even those with complex, smooth, and simply connected domains.

Furthermore, I examined the singular behavior at the origin of solutions to the equation $\mathscr{H}\rho = e$ on a half-axis, where H is the one-sided Hilbert transform, ρ an unknown solution and e a known function. More exactly, I studied the connection of the Mellin transform to the Hilbert and Fourier transforms in a half-axis $\mathbb{R}_+ = (0, \infty)$. This analysis is our first step into understanding the singular behavior of waves near the endpoint of cracks or screens in an acoustic medium. Our approach is to use the Mellin transform for generalized function and connection of this with Hilbert and Fourier transform.

To understand wave field singularities caused by curve-shaped scatterers in a planar domain, this is a simpler model problem. Our analysis determines that ρ exhibits a singularity pattern represented by $\mathscr{M}[e](1/2)\frac{1}{\sqrt{t}}$, where $\mathscr{M}$ denotes the Mellin transform. Our approach involves the utilization of specially designed function spaces These allow us to precisely investigate the relationship between the Mellin and Hilbert transforms. Additionally, Fourier analysis plays a vital role since the Mellin transform is essentially a form of the Fourier transform on the locally compact Abelian multiplicative group of the half-line. This familiar operator guides our investigation.

Kokkuvõte
Akustiliste ja elektromagnetlainete pöördhajumine lameekraanilt ja integraalteisenduste omadused poolteljel

Matemaatilise füüsika vallas, antenniteoorias on olulist rolli mänginud hajumise teooria. Viimasel ajal on huvi selle valdkonna vastu kasvanud. Käesolevas lõputöös on käsitletud fikseeritud sagedusega akustilise ja elektromagnetilise hajumise probleemi helipehmelt lameekraanilt ning integraalteisenduste omadusi poolteljel. Ekraan asub kahemõõtmelisel tasandil ja interakteerub langevate lainetega, hajutades need kolmemõõtmelisse ruumi.

Seda tüüpi uuring on eriti asjakohane niisuguste rakenduste puhul nagu peegeldavad sonarid ja antennid, mille puhul ei ole ohutu eeldada, et langev laine on lihtne tasapinnaline laine. Töö tulemused on märkimisväärsed. Need näitavad, et ekraani kuju on võimalik täpselt määrata, kasutades ühe suvalise langeva laine tekitatud kaugvälja mustrit eeldusel, et see ei ole tasandi suhtes antisümmeetriline. Meetod, mida me kasutame, põhineb ideel olla osaliselt motiveeritud teatud integraaloperaatori poolt. Meie järgitav metoodika on see, et me alustame sellega, et näitame, et kaugvälja mustrit võib vaadelda kui tugiekraanil kandjat omava funktsiooni kahemõõtmelise Fourier' teisenduse ahendit ringi raadiusega k (esindab lainearvu). Et käsitleda juhtu, kui langev laine võib ekraani osadel kaduda, jõuame olulisele tõdemusele: ekraani kuju on täpselt määratud selle funktsiooni toe põhjal.

Lisaks tegime kindlaks, et ühest sissetulevast tasapinnalisest lainest põhjustatud hajutatud elektromagnetvälja kaugvälja muster määrab üheselt tõkestatud ülijuhtiva tasapinnalise ekraani. See leid üldistab meie akustilisi tulemusi, tuues esile elektromagnetilise hajumise ainulaadsed määramisvõimalused. Selle töö eesmärk on teha kindlaks ühele mittehääbuvat kaugvälja omavale mõõtmisele vastava tugihüpertasapinna ühene määratus. See töö vastandub mõnele varasemale uuringule, mis keskendusid peamiselt lihtsatele objektidele, nagu hulktahukad, kerad või ringid, aga ka sujuvad tasapinnalised kõverad. Need leiud on tähelepanuväärsed, kuna need kehtivad erineva kujuga ekraanide puhul, isegi nende puhul, millel on keerukad, siledad ja ühelisidusad piirkonnad.

Peale selle me uurisime poolteljel antud võrrandi $\mathscr{H}\rho = e$ lahendite singulaarset käitumist nullpunktis, kus $\mathscr{H}$ on ühepoolne Hilberti teisendus, ρ tundmatu lahend ja e etteantud funktsioon. Täpsemalt uurime Mellini teisenduse seost Hilberti ja Fourier' teisendustega poolteljel $\mathbb{R}_+ = (0, \infty)$. See analüüs on meie esimene samm akustilises keskkonnas pragude või ekraanide lõpp-punkti lähedal olevate lainete singulaarse käitumise mõistmisel. Meie lähenemisviis seisneb Mellini teisenduse kasutamisel üldistatud funktsioonide jaoks ja selle ühendamisel Hilberti ja Fourier' teisendustega.

See on lihtne mudelprobleem, et mõista tasapinnalise piirkonna kõverakujuliste hajutajate põhjustatud lainevälja singulaarsusi. Meie analüüs teeb kindlaks, et ρ näitab singulaarsusmustrit, mida esindab $\mathscr{M}[e](1/2)\frac{1}{\sqrt{t}}$, kus $\mathscr{M}$ tähistab Mellini teisendust. Meie lähenemisviis hõlmab Zemaniani poolt kasutusele võetud spetsiaalselt loodud funktsiooniruume. Need võimaldavad meil täpselt uurida Mellini ja Hilberti teisenduste vahelisi seoseid. Lisaks on Fourier' analüüsil oluline roll, kuna Mellini teisendus on sisuliselt Fourier' teisenduse vorm poolsirge lokaalselt kompaktsel Abeli multiplikatiivsel rühmal. See tuntud operaator juhib meie uuringut.

Appendix 1

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mathematics

Article

Unique Determination of the Shape of a Scattering Screen from a Passive Measurement

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Abstract: We consider the problem of fixed frequency acoustic scattering from a sound-soft flat screen. More precisely, the obstacle is restricted to a two-dimensional plane and interacting with an arbitrary incident wave, it scatters acoustic waves to three-dimensional space. The model is particularly relevant in the study and design of reflecting sonars and antennas, cases where one cannot assume that the incident wave is a plane wave. Our main result is that given the plane where the screen is located, the far-field pattern produced by any single arbitrary incident wave determines the exact shape of the screen, as long as it is not antisymmetric with respect to the plane. This holds even for screens whose shape is an arbitrary simply connected smooth domain. This is in contrast to earlier work where the incident wave had to be a plane wave, or more recent work where only polygonal scatterers are determined.

Keywords: inverse scattering; screen; uniqueness; single measurement; passive measurement

MSC: 35R30; 35P25; 35A02

1. Introduction

1.1. Antennas

The motivation for the study of wave scattering from thin and large objects lies in the antenna theory. The starting point for this was when the Prussian Academy announced an open competition about who could be the first to show the existence or non-existence of electromagnetic (EM) waves in 1879. The existence of these waves were predicted 15 years earlier by the mathematical theory of James Clerk Maxwell [1]. The competition was won in 1882 by young Heinrich Hertz, in favor of Maxwell's theory. He did this by constructing a dipole antenna radiating EM waves which he could measure. It is needless to mention the importance which this experiment together with Maxwell's theory has had for modern society. Hertz's antenna consisted of two identical perfectly conducting planar bodies, in his case squares, which create radiating EM waves. Since, by reciprocity, radiating antennas are identical to receiving antennas, the theory of antennas is closely connected to EM scattering and inverse scattering theory.

A key question in antenna design for scientific radio arrays is how to choose the antenna topology so that its impedance and radiation patterns are frequency independent (FI) over a wide range of frequencies and, simultaneously, the radiation pattern supports beamforming. Well-known examples of FI antennas include log-periodic, log-spiral, and UHF fractal antennas on high-frequencies. While proven good for extremely wide band work, these are heavy and complicated structures and thus not cost-efficient for extremely large arrays.

Instead of relying on traditional antenna forms, we aim to derive general principles for designing antennas with frequency independent characteristics. A major step in such a design strategy is to solve the inverse scattering problem: given an input–output pair of waves, which antenna shape produces it? The input is a given incident wave, and the output is the far-field pattern produced by the antenna. The path to antenna design is a long one, so in this paper we study the technically easier acoustic scattering problem.

In acoustics, scattering surfaces or screens are not called antennas but sonars. Traditionally sonars are classified into active and passive sonars, depending on whether they act as a sound source or receiver. We consider acoustic scattering from screens, something which lies between these two extremes. It is more correct to call these screens passive sonars as they do not have an energy source, but they are active in the sense that their effect on the sound pattern is significant. In general the nomenclature "sonar" refers to probing using an active and passive sonar. Our research is rather in the domain of acoustic design. The mathematical question of finding a screen that scatters a given incident wave into a particular far-field has applications like the following, for example: how to reduce echo in an office space? How to direct acoustic vibrations or reduce them? Of course, it also answers the probing question: can we determine the shape and location of a passive sonar by how it reflects sound? These are complex questions, only one part of which we are going to solve, namely that a single input–output pair of sound waves uniquely determines the shape of a flat acoustic screen.

1.2. Mathematical Background

The problem of inverse scattering with reduced measurement data has gained a lot of interest lately. Traditionally determining a scatterer from far-field measurements requires sending all possible incident waves and recording the corresponding far-field patterns. The method of using complex geometrical optics solutions infinitely many far-field measurements in the fixed frequency setting was pioneered by Sylvester and Uhlmann in [2], and was the first method for uniquely determining an arbitrary smooth enough scattering potential by far-field measurements. The field has grown extremely fast since then, almost to the point of saturation, and we will only point the reader towards the surveys in [3] for references up to 2003, which gives a good picture of the situation except for scattering in two dimensions, which was solved by Bukhgeim [4] in 2007 and improved by several authors, e.g., [5–9].

In many applications the scatterer is impenetrable, or we are only interested in its shape or location. The shape determination problem is known as Schiffer's problem in the literature [10]. M. Schiffer showed that a sound-soft obstacle (with non-empty interior) can be uniquely determined by infinitely many far-field patterns. The proof appeared as a private communication in the monograph by Lax and Phillips [11]. Linear sampling [12] and factorization [13] methods were developed and they are very well suited for shape determination, also from the numerical point of view. These were applied in the context of curved screens in acoustic [14] and electromagnetic [15] scattering to determine the shape and location of the screen, also numerically. However, these methods require the full use of infinitely many far-field patterns, except for a case of interest in [14] to which we will return later in more detail.

There was still much to improve: counting dimensions shows that a single far-field (a mapping $\mathbb{S}^{n-1} \to \mathbb{C}$) should be enough to determine the shape (a manifold of dimension $n-1$). Colton and Sleeman reduced the requirements to finitely many far-field patterns [16]. It is widely conjectured that the uniqueness for Schiffer's problem follows from a single far-field pattern [10,17], and the situation for a general shape is wide open. This brings in the current results. Various authors proved at roughly the same time in the recent past that polyhedral sound-soft obstacles are uniquely determined by a single far-field pattern in various settings [18–24]. Part of the results above apply for screens as long as the screen is polygonal. A special case in [14] gives the unique determination of a flat screen by a single incident plane-wave measurement. Their proof requires that the incident wave has non-vanishing properties everywhere on the plane where the screen is located—an issue that we remedy completely. So far there is no proof for the unique determination of an obstacle's shape by one far-field pattern without restrictive a priori assumptions. The results in [25] come very close: the obstacle can be any

Lipschitz domain as long as its boundary is not an analytic manifold. It does not allow screens, which is our focus.

An alternative approach to unique determination which has gained interest recently, is to consider what can be determined with less data, e.g., one measurement, in the setting of penetrable scatterers which were usually treated with various methods based on the Sylvester—Uhlmann [2] or Bukhgeim [4] papers. Much of the recent work taking this point of view uses unique continuation results and precise analysis on the behavior of Fourier transforms of the characteristic functions of various shapes [26–32]. A very interesting point of view is determining the so-called convex scattering support [33,34] by one far-field measurement. Again, none of the above are applicable to screens per se.

Our work in this paper shows that given the far-field caused by any single given incident wave scattering off a smooth flat screen, the latter's shape is determined uniquely. Our methods are based on ideas which are partly motivated by the study of certain integral operators in [35,36]. As in [14], we first show that the far-field is the restriction to a ball of radius k (the wavenumber) of the two-dimensional Fourier transform of a function supported on the screen. Next, since the incident wave might vanish on part of the screen, we show that the shape of the screen is exactly the support of that function. This latter part involves a delicate analysis of the Taylor coefficients of the scattered wave at the screen, but it leads to our main theorem: that Schiffer's problem is uniquely solvable for flat screens on a plane in three dimensions, for any incident wave that causes scattering.

Let us discuss the significance of our result, with focus especially on our improvements over [14]: that any incident field is allowed. We will start with the mathematical challenges. Unlike for infinite measurements inverse problems such as [2,4], properties of the incident wave affect greatly the solvability of single measurement inverse problems. Complex plane waves make things technically simpler in many scattering problems because of their explicit form and non-vanishing everywhere. This often reduces the non-linear inverse scattering problem to the linear inverse source problem after a suitable interpretation, or avoids other challenges, as can be seen by comparing [26,27,31] to [28–30]. Furthermore, in situations involving scattering from multiple objects, the total incident field impinging on a given component is the sum of the original incident field and the fields scattered by the other components. This is relevant when one wishes to uniquely determine a screen where space contains other scatterers that are known. On the other hand, from the applied point of view, solving the inverse problems for any given incident field enables *passive measurements*. This means that even if we do not have control over the incident wave, or cannot afford to control it, the shape of the scatterer can be uniquely determined. This is both good and bad. It means that the flat screen design problem of finding its shape such that it scatters one given incident wave into a given far-field has no more than a unique solution. On the other hand it shows the impossibility of more complex input–output systems. One cannot require it to scatter two or more incident waves into their corresponding far-fields in general. The first incident wave and far-field pair already determines the shape.

Lastly, we remark that inverse scattering for screens has still many open problems. Current solutions require that the screen have at least a differentiable boundary, something which arises from the way that the direct scattering problem has been shown solvable in [37] and other sources. To bring forward the range characterization condition from [14] to the situations of let us say Herglotz incident waves, one would need to solve a deconvolution problem. A more difficult and certainly more interesting question mathematically and from the point of view of applications, is the unique determination of the shape of a curved screen from one measurement, passive or fully controlled. The problem is solved for infinitely many measurements in [14], but counting dimensions suggests that it should be solvable with one measurement.

1.3. Definitions and Theorems

Let us go forward to the mathematics. We start by defining what we mean by a screen and the scattering problem from screens. Then we state our three main theorems. They give representation formulas for the scattered wave, the far-field pattern, and the unique solvability of Schiffer's problem for determining the shape of a scattering screen using a single incident wave. In Section 2 we prove the representation formulas, and then in Section 3 we solve the inverse problem.

We consider the scattering of a two-dimensional sound-soft and flat obstacle Ω in three-dimensional space. We will assume that Ω is an open subset of $\mathbb{R}^2 \times \{0\}$.

Definition 1. *We call a set $\Omega \subset \mathbb{R}^3$ a screen, if $\Omega = \Omega_0 \times \{0\}$ for some simply connected bounded domain $\Omega_0 \subset \mathbb{R}^2$ whose boundary is smooth, and which we call its shape.*

The scattering of acoustic waves by Ω leads to the study of the Helmholtz equation $(\Delta + k^2)u = 0$ where the wave number k is given by the positive constant $k = \omega/c$ where c is the constant speed of sound in the background fluid (air, water, etc.) and ω is the angular frequency of the wave. The pressure of the total wave vanishes on the boundary of a sound-soft obstacle, and the total wave is a sum of the incident and scattered waves. This leads to the following set of partial differential equations.

Definition 2. *We define the direct scattering problem for a screen Ω as follows. Given an incident wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ and a screen Ω, the direct scattering problem has a solution if there is $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ that satisfies the following conditions*

$$(\Delta + k^2)u_s = 0, \qquad \mathbb{R}^3 \setminus \overline{\Omega}, \tag{1}$$

$$u_i(x) + u_s(x) = 0, \qquad x \in \Omega, \tag{2}$$

$$r\Big(\frac{\partial}{\partial r} - ik\Big)u_s = 0, \qquad r \to \infty, \tag{3}$$

where $r = |x|$ and the limit is uniform over all directions $\hat{x} = x/r \in \mathbb{S}^2$ as $r \to \infty$.

There are a few things above that we should clarify. By $H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ we mean the set of distributions ψ on $\mathbb{R}^3 \setminus \overline{\Omega}$ for which $\psi_{|U} \in H^1(U)$ for any bounded convex open set $U \subset \mathbb{R}^3 \setminus \overline{\Omega}$. Secondly, since strictly speaking u_s is not defined on Ω, by (2) we mean that the Sovolev trace of u_s both from above ($x_3 > 0$) and below ($x_3 < 0$) coincides, and is equal to $-u_i$ on Ω.

We shall start by showing a representation formula, the one in (4), for solutions u_s of the direct scattering problem for the screen. This is mainly done so that the reader would get a better intuition about this type of problems and to fix notation and function spaces clearly. This formula is well known, and it gives a unique solution to the direct problem [37]. After that we will show that the far-field, defined below, corresponding to a single given non-trivial incident wave uniquely determines the screen Ω. This type of theorem was shown in [14] on the condition that the incident wave does not vanish on the plane $\mathbb{R}^2 \times \{0\}$. To get rid of this assumption, we must show Lemma 7. We remark that the far-field pattern exists and is unique for each u_s satisfying the following assumptions. See [10] for reference.

Definition 3. *Let u_s satisfy the Sommerfeld radiation condition of (3) and the Helmholtz equation $(\Delta + k^2)u_s = 0$ outside a ball $B \subset \mathbb{R}^3$. We say that u_s^∞ is the far-field of u_s if*

$$u_s(x) = \frac{e^{ik|x|}}{|x|}\left(u_s^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right)$$

uniformly over $\hat{x}$ as $x \to \infty$.

We define some notation which will be useful throughout the whole text.

- $x, y, \ldots$ represent variables in $\mathbb{R}^3$, and we associate to them various projections described below.
- $x', y', \ldots$ mean variables in $\mathbb{R}^2$ or projections to $\mathbb{R}^2$. For example if $x = (1,2,3) \in \mathbb{R}^3$ then in that context $x' = (1,2) \in \mathbb{R}^2$, but we could have dy' in an integral over a subset of $\mathbb{R}^2$ without having to define the variable y separately.
- $x^0, y^0, \ldots$ denote lifts to $\mathbb{R}^3$, meaning $x^0 = (x', 0)$. For example if $x' = (-1,-2)$ then $x^0 = (-1,-2,0)$. This notation can also be used as a projection $\mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$. So, if $x = (1,2,3)$ then $x^0 = (1,2,0)$. Essentially $x'^0 = (x')^0 = x^0$ and $x^{0'} = (x^0)' = x'$ but we do not use this combined notation explicitly.
- Φ is reserved for the fundamental solution to $(\Delta + k^2)$, defined in Lemma 2.
- u^+, u^- mean the function u restricted to $\mathbb{R}^2 \times \mathbb{R}_+$ and $\mathbb{R}^2 \times \mathbb{R}_-$, respectively. If their variable is in $\mathbb{R}^2 \times \{0\}$ then they are the two-sided limits (traces) as $x_3 \to 0$. We often use $\partial_3 u^+$ and $\partial_3 u^-$. These are simply the derivatives in the x_3-direction of u^+ and u^-, respectively. Often this is evaluated on $\mathbb{R}^2 \times \{0\}$ where it then denotes the one-sided derivative, i.e., the trace of $\partial_3 u^\pm$.
- $\widetilde{H}^{-1/2}(\Omega_0)$: this is the set of $H^{-1/2}(\mathbb{R}^2)$ distributions whose support is contained in $\overline{\Omega_0}$, where we recall that Ω_0 signifies the shape of a screen Ω.

Let us discuss the direct scattering problem (1)–(3) first. In Section 2, Lemma 4, we will show the well-known representation formula

$$u_s(x) = \int_{\mathbb{R}^2} \Phi(x, y^0)\rho(y')dy' \tag{4}$$

for all $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, where

$$\rho(y') = \partial_3 u_s^+(y^0) - \partial_3 u_s^-(y^0) \tag{5}$$

is an element of $\widetilde{H}^{-1/2}(\Omega_0)$ and the integral in (4) is interpreted as a distribution pairing between ρ and the smooth test function Φ restricted to the screen. Taking the trace $x \to \Omega$ in (4) and recalling that $u_s = -u_i$ on Ω in the sense of traces, (2), we get

$$u_i(x) = -\int_{\mathbb{R}^2} \Phi(x, y^0)\rho(y')dy'. \tag{6}$$

Now, for any candidate solution $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$, it solves the direct problem (1)–(3) if and only if ρ, as defined above, is in $\widetilde{H}^{-1/2}(\Omega_0)$ and is the solution to (6). More precisely, given ρ solving the integral equation, we can define u_s by (4), and it would solve the direct scattering problem. This was shown in Theorem 2.5 in [37]. Theorem 2.7 in the same source proves that (6) has a unique solution $\rho \in \widetilde{H}^{-1/2}(\Omega_0)$ given any $u_i \in H^{1/2}(\Omega_0)$.

Our main contributions are the following. The first of which is the familiar far-field representation derived from (4) if ρ is a function. We generalize is to distributions in $H^{-1/2}(\mathbb{R}^2)$. This is required for consistency of the function spaces involved. This detail has not been stated explicitly in earlier work involving scattering from screens.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a screen and u_s satisfy the direct scattering problem for some incident field u_i and screen Ω. Then its far-field has the representation*

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\left\langle \left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0), e^{-ik\hat{x}\cdot y^0} \right\rangle_{y'} \tag{7}$$

for $\hat{x} \in \mathbb{S}^2$. If $\partial_3 u_s^+ - \partial_3 u_s^-$ is integrable on Ω, this formula is equivalent to

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y^0}\left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0)dy'.$$

Our main theorem shows that even with an unoptimal incident wave, the scattering caused by it from flat screens determines the shape uniquely.

Theorem 2. *Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^3$ be screens and $k \in \mathbb{R}_+$. Let u_i be an incident wave and $u_s, \tilde{u}_s$ be scattered waves that satisfy the direct scattering problem for screens $\Omega, \tilde{\Omega}$, respectively.*

If u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ and $u_s^\infty = \tilde{u}_s^\infty$, then $\Omega = \tilde{\Omega}$. If it is antisymmetric then $u_s^\infty = \tilde{u}_s^\infty = 0$ for any screens $\Omega, \tilde{\Omega}$.

2. Representation Theorems

In this section, we will prove that solutions to the direct scattering problem satisfy (4). In essence we present the well-known but very condensed argument of [37] in more detail for the convenience of the readers. We will start with representation formulas for smooth functions and then approximate the H^1-smooth u_s. At the end of the section we will prove Theorem 1.

Lemma 1. *Let $D \subset \mathbb{R}^3$ be a bounded domain whose boundary is piecewise of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed to the exterior of D. Then, for $u, v \in C^2(\overline{D})$ we have Green's second formula*

$$\int_D (v\Delta u - u\Delta v)dx = \int_{\partial D} \left(\frac{\partial u}{\partial \nu} v - u\frac{\partial v}{\partial \nu}\right) ds \tag{8}$$

where ds is the surface measure of ∂D.

Proof. Theorem 3 in Appendix C.2 of [38]. □

Lemma 2. *Let $D \subset \mathbb{R}^3$ be a bounded domain whose boundary is piecewise of class C^1 and $k \in \mathbb{R}_+$. Let*

$$\Phi(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

for $x, y \in \mathbb{R}^3$, $x \neq y$. Then for any $\varphi \in C^2(\overline{D})$ and $x \in \mathbb{R}^3 \setminus \partial D$ we have

$$\begin{aligned}\int_D \Phi(x,y)(\Delta + k^2)\varphi(y)dy = &\int_{\partial D} \big(\Phi(x,y)\partial_\nu \varphi(y) - \varphi(y)\partial_\nu \Phi(x,y)\big) ds(y) \\ &+ \begin{cases} 0, & x \in \mathbb{R}^3 \setminus \overline{D}, \\ -\varphi(x), & x \in D. \end{cases}\end{aligned} \tag{9}$$

Proof. We have $(\Delta + k^2)\varphi$ bounded and $y \mapsto \Phi(x, y)$ integrable for any x, so

$$\int_D \Phi(x,y)(\Delta + k^2)\varphi(y)dy = \lim_{r\to 0} \int_{D\setminus B(x,r)} \Phi(x,y)(\Delta + k^2)\varphi(y)dy.$$

Green's second formula, given in (8), applied to the integral on the right gives

$$\begin{aligned}\ldots = &\int_{D\setminus B(x,r)} (\Delta + k^2)\Phi(x,y)\varphi(y)dy \\ &+ \int_{S(x,r)\cap \overline{D}} \big(\Phi(x,y)\partial_\nu \varphi(y) - \varphi(y)\partial_\nu \Phi(x,y)\big) ds(y) \\ &+ \int_{\partial D\setminus \overline{B}(x,r)} \big(\Phi(x,y)\partial_\nu \varphi(y) - \varphi(y)\partial_\nu \Phi(x,y)\big) ds(y).\end{aligned}$$

The first integral here vanishes because $(\Delta_y + k^2)\Phi(x, y) = 0$ when $y \neq x$.

The integral over $\partial D \setminus \overline{B}(x,r)$ gives the second term in the claim when $r \to 0$ because $\Phi, \partial\Phi$ are integrable since $x \notin \partial D$. Let us estimate the first term in the first boundary integral. We have

$$\int_{S(x,r)\cap\overline{D}} \Phi(x,y)\partial_\nu \varphi(y)ds(y) = \int_{S(x,r)\cap\overline{D}} \frac{e^{ikr}}{4\pi r}\partial_\nu(y)ds(y)$$

and by the ML-inequality we have

$$\left|\int_{S(x,r)\cap\overline{D}} \Phi(x,y)\partial_\nu \varphi(y)ds(y)\right| \le \frac{1}{4\pi r} \sup_{y\in S(x,r)\cap\overline{D}} |\nabla\varphi(y)| 4\pi r^2 \to 0$$

as $r \to 0$ because $|\nabla\varphi|$ has a uniform bound in $\overline{D}$. In the last integral we have $\partial_n u\Phi(x,y) = -\partial_r\big(e^{ikr}/(4\pi r)\big) = -ike^{ikr}/(4\pi r) + e^{ikr}/(4\pi r^2)$. The integral involving $ike^{ikr}/(4\pi r)$ can be estimated as above to conclude that it vanishes when $r \to 0$. The remaining integral is

$$-\frac{e^{ikr}}{4\pi r^2}\int_{S(x,r)\cap\overline{D}} \varphi(y)ds(y) = -\frac{e^{ikr}}{4\pi r^2}\int_{S(x,r)\cap\overline{D}} \big(\varphi(y)-\varphi(x)\big)ds(y) - \frac{e^{ikr}}{4\pi r^2}\varphi(x)s\big(S(x,r)\cap\overline{D}\big).$$

We have $|\varphi(y)-\varphi(x)| \le \sup_{\xi\in\overline{D}}|\nabla\varphi(\xi)||x-y|$ so the absolute value of the first integral above can be estimated as

$$\ldots \le \frac{\sup|\nabla\varphi|}{4\pi r^2}\int_{S(x,r)\cap\overline{D}} |x-y|dy = \frac{\sup|\nabla\varphi|}{4\pi r^2} rs\big(S(x,r)\cap\overline{D}\big) \to 0$$

as $r \to 0$. The form of the remaining term implies the claim in each of the cases $x \in D$, $x \in \mathbb{R}^3 \setminus \overline{D}$. □

Lemma 3. *Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $k \in \mathbb{R}_+$. Let $u_s \in H^1(D)$ with $(\Delta + k^2)u_s \in L^2(D)$. Then*

$$u_s(x) = -\int_D \Phi(x,y)(\Delta+k^2)u_s(y)dy + \int_{\partial D}\big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y) \tag{10}$$

for $x \in D$ in the distribution sense. For $x \in \mathbb{R}^3 \setminus \overline{D}$ we have

$$0 = -\int_D \Phi(x,y)(\Delta+k^2)u_s(y)dy + \int_{\partial D}\big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y) \tag{11}$$

in the distribution sense. Here the boundary integrals involving $\partial_\nu u_s$ are to be interpreted as distribution pairings between a $H^{-1/2}(\partial D)$ function and a test function.

Proof. We will prove only the first case, namely $x \in D$. The second one follows similarly. Let $(\varphi_j)_{j=0}^\infty$ be a sequence of smooth functions defined on $\overline{D}$ such that

$$\|u_s - \varphi_j\|_{H^1(D)} + \left\|(\Delta+k^2)(u_s-\varphi_j)\right\|_{L^2(D)} \to 0$$

as $j \to \infty$. Such a sequence exists, for example by convolving u_s with a mollifier ψ_ε, as in $\varphi_j = (u_s * \psi_{1/j})_{|\overline{D}}$.

We have $\Phi(x,y) = \Psi(x-y)$ for $\Psi(z) = \exp(ik|z|)/(4\pi|z|)$ which is locally integrable in $\mathbb{R}^3$. Hence the first term in the right-hand side of (10), equal to $\Psi * (\Delta+k^2)u_s$, can be approximated by $\Psi * (\Delta+k^2)\varphi_j$ in the $L^2(D)$-sense.

For any $x \in D$ the second integral in (10) is well defined because $y \mapsto \Phi(x,y)$ and $y \mapsto \partial_\nu\Phi(x,y)$ are smooth on the smooth manifold ∂D. Moreover, the x-dependence is smooth, so the mapping

$$u_s \mapsto \int_{\partial D} u_s(y)\partial_\nu\Phi(x,y)ds(y)$$

is bounded $H^1(D) \to H^{1/2}(\partial D) \to C^0(D)$ and similarly

$$u_s \mapsto \int_{\partial D} \Phi(x,y)\partial_\nu u_s(y)\big)ds(y)$$

is bounded $H^1(D) \to H^{-1/2}(\partial D) \to C^0(D)$ when the integral is interpreted as a distribution pairing between a $H^{-1/2}(\partial D)$-function and a test function. The continuity does not necessarily hold up to the boundary. Because $\varphi_j \to u_s$ in $H^1(D)$ and the trace operators map $\mathrm{Tr} : H^1(D) \to H^{1/2}/D)$, $\partial_\nu : H^1(D) \to H^{-1/2}(\partial D)$, so the boundary integrals with u_s replaced by φ_j converge to the corresponding ones in $C^0(D)$, namely uniformly over compact subsets of D.

In conclusion, for a test function $\psi \in C_0^\infty(D)$ we have

$$\begin{aligned}
\langle u_s, \psi \rangle &= \lim_{j\to\infty} \langle \varphi_j, \psi \rangle \\
&= \lim_{j\to\infty} \Big\langle -\int_D \Phi(x,y)(\Delta + k^2)\varphi_j(y)dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu \varphi_j(y) - \varphi_j(y)\partial_\nu \Phi(x,y)\big)ds(y), \psi(x) \Big\rangle_x \\
&= \Big\langle -\int_D \Phi(x,y)(\Delta + k^2)u_s(y)dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu \Phi(x,y)\big)ds(y), \psi(x) \Big\rangle_x
\end{aligned}$$

so the equality holds in $\mathscr{D}'(D)$. □

Lemma 4. *Let $\Omega \subset \mathbb{R}^3$ be a screen, $k \in \mathbb{R}_+$ and Φ the fundamental solution from Lemma 2. Let $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$. If $(\Delta + k^2)u_s = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and it satisfies the Sommerfeld radiation condition, then*

$$u_s(x) = \int_{\mathbb{R}^2} \Phi(x,y^0)(\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)dy' \tag{12}$$

for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. Also $y' \mapsto (\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)$ is in $\widetilde{H}^{-1/2}(\Omega_0)$, and more precisely the integral above represents the distribution pairing of a $\widetilde{H}^{-1/2}(\Omega_0)$-function with the smooth test function Φ restricted to $\mathbb{R}^2 \times \{0\}$ on the y-variable.

Proof. Fix $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary for which $x \in D$ and $\Omega \subset \partial D$ and furthermore we want this set to be *on top* of Ω, namely that its boundary normal pointing to the interior at Ω is e_3 and not $-e_3$. Let $R > \sup_{z\in D} |x - z|$. We will use the formulas of Lemma 3 on D, which has Ω on its boundary, and $B(x,R) \setminus \overline{D}$.

First note that since $(\Delta + k^2)u_s = 0$ only the boundary integrals on the right-hand sides of (10) and (11) remain. We will see the first integral as is, namely

$$u_s(x) = \int_{\partial D} \big(\Phi(x,y)\partial_\nu^D u_s(y) - u_s(y)\partial_\nu^D \Phi(x,y)\big)ds(y), \tag{13}$$

where we denote by ∂_ν^D the internal boundary normal derivative of D, applied to functions on D. We will have the integrals in (11) to be over the set $B(x,R) \setminus \overline{D}$. The boundary of this set is $S(x,r) \cup \partial D$, and the boundary normal pointing to its interior is $-e_3$ on $\Omega \subset \partial(B(x,R) \setminus \overline{D})$. We will split the boundary integral accordingly, and in the integral over ∂D we denote by $\partial_\nu^{D_c}$ the *external* boundary normal derivative applied to function on $B(x,R) \setminus \overline{D}$. In conclusion (11) becomes

$$\begin{aligned}
0 = &\int_{S(x,R)} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu \Phi(x,y)\big)ds(y) \\
&+ \int_{\partial D} \big(\Phi(x,y)(-\partial_\nu^{D_c})u_s(y) - u_s(y)(-\partial_\nu^{D_c})\Phi(x,y)\big)ds(y).
\end{aligned} \tag{14}$$

Finally, by interior elliptic regularity we see that u_s is continuous (in fact real analytic) in some neighborhood of x. Also, because x is outside of ∂D and $S(x,R)$, the individual boundary integrals above are continuous. Hence the equality in the sense of distributions is in fact a pointwise equality for continuous functions. In other words, both of (13) and (14) hold as continuous functions. We still remind that the integrals involving $\partial_\nu u_s$ represent distribution pairings for an element of $H^{-1/2}(\partial D)$ with that of a smooth Φ.

Let us add (13) and (14). By smoothness, $\partial_\nu^D \Phi = \partial_\nu^{D_c}\Phi$. Please note that two-sided Sobolev traces of H^1-functions yield identical results, so the integrals of $u_s\partial_\nu^D\Phi$ and $u_s\partial_\nu^{D_c}\Phi$ in (13) and (14) cancel out. The sum then gives

$$u_s(x) = \int_{S(x,R)} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y) + \int_{\partial D} \Phi(x,y)\big(\partial_\nu^D u_s - \partial_\nu^{D_c} u_s\big)(y)ds(y). \quad (15)$$

Please note that as $R \to \infty$ the first integral in (15) vanishes because u_s satisfies the Sommerfeld radiation condition. Also, u_s is C^1 outside of $\overline{\Omega}$ by elliptic interior regularity, so the second integral's integrand is zero when $y \notin \overline{\Omega}$. Thus, letting $R \to \infty$ gives

$$u_s(x) = \int_\Omega \Phi(x,y)\big(\partial_\nu^D u_s - \partial_\nu^{D_c} u_s\big)(y)dy$$

which implies the claim as $\partial_\nu^D u_s = \partial_3 u_s^+$ and $\partial_\nu^{D_c} u_s = \partial_3 u_s^-$ on $\Omega \subset \mathbb{R}^2 \times \{0\}$. Furthermore, as above, since u_s is C^1 outside of $\overline{\Omega}$, we see that $\partial_3 u_s^+ - \partial_3 u_s^- = 0$ outside of $\overline{\Omega}$, so the integrand in the statement is in $\widetilde{H}^{-1/2}(\Omega_0)$, as claimed. □

With the proposition above, we are almost ready to prove the formula for the far-field of a wave scattered by a screen, Theorem 1. But first let us prove a lemma.

Lemma 5. *Let $k \in \mathbb{R}_+$ and $K \subset \mathbb{R}^3$ be a non-empty compact set. Then*

$$\lim_{r\to\infty} \sup_{|x|=r} \sup_{y\in K} |x| \left| \partial_y^\alpha \left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x}\cdot y} \right) \right| = 0$$

for any multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \le 1$. Recall that $\hat{x} = x/|x|$.

Proof. The case of $|\alpha| = 0$ is well known, see for example the proof of Theorem 2.5 in [10]. For $|\alpha| = 1$ we will instead show the equivalent statement with ∂_y^α replaced by ∇_y. Recall the following differentiation rules

- $\nabla_y |x-y|^s = -s\frac{x-y}{|x-y|}|x-y|^{s-1}$ for all $s \in \mathbb{R}$,
- $\nabla_y e^{ik|x-y|} = -ik\frac{x-y}{|x-y|}e^{ik|x-y|}$, and
- $\nabla_y e^{-ik\hat{x}\cdot y} = -ik\hat{x}e^{-ik\hat{x}\cdot y}$.

These imply that

$$\begin{aligned}\nabla_y \left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y} \right) &= -ik\frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|} + \frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2} + ik\hat{x}\frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y} \\ &= -ik\left(\frac{x-y}{|x-y|} - \hat{x}\right)\frac{e^{ik|x-y|}}{|x-y|} - ik\hat{x}\left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right) + \frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2}.\end{aligned}$$

Let us consider the three types of terms above. To prove the estimate, let us take the absolute value and multiply by $|x|$. The last one gives

$$|x|\left|\frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2}\right| = \frac{|x|}{|x-y|^2} \to 0$$

uniformly as $y \in K$, $|x| = r$ and $r \to \infty$. The first term gives

$$|x| \left| -ik \left(\frac{x-y}{|x-y|} - \hat{x} \right) \frac{e^{ik|x-y|}}{|x-y|} \right| = k \frac{|x|}{|x-y|} \left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right|$$

where can still estimate

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| = \left| \frac{x-y}{|x-y|} \frac{|x|-|x-y|}{|x|} - \frac{y}{|x|} \right| \leq \frac{||x|-|x-y||}{|x|} + \frac{|y|}{|x|} \leq 2\frac{|y|}{|x|}$$

because $||x| - |x-y|| \leq |y|$ by the triangle inequality. Thus, the first term also tends to zero uniformly as $r \to \infty$. Lastly, the second one is estimated as

$$|x| \left| -ik\hat{x} \left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x}\cdot y} \right) \right| = k|x| \left| \frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x}\cdot y} \right|$$

which tends to zero uniformly because this is the case $|\alpha| = 0$ proven at the beginning of this proof. □

Proof of Theorem 1. By the definition of the far-field there is a finite constant $C > 0$ independent of x such that

$$\left| u_\infty(\hat{x}) - |x| e^{-ik|x|} u_s(x) \right| \leq \frac{C}{|x|}$$

when $|x| \to \infty$. Let us denote $\rho(y') = (\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)$. Then (12) gives

$$u_s^\infty(\hat{x}) = \lim_{|x|\to\infty} |x| e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle_{y'}$$

should the limit exist. The distribution pairing is over $y' \in \mathbb{R}^2$. We can rewrite

$$|x| e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle = \left\langle \rho(y'), |x| e^{-ik|x|} \Phi(x, y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi} \right\rangle_{y'} + \frac{1}{4\pi} \langle \rho(y'), e^{-ik\hat{x}\cdot y^0} \rangle_{y'}.$$

We can write the C^1-test function in the first pairing on the right as

$$|x| e^{-ik|x|} \Phi(x, y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi} = \frac{e^{-ik|x|}|x|}{4\pi} \left(\frac{e^{ik|x-y^0|}}{|x-y^0|} - \frac{e^{ik|x|}}{|x|} e^{-ik\hat{x}\cdot y^0} \right)$$

which converges to zero in the C^1 topology over y', and a fortiori y^0, restricted to any compact set by Lemma 5. Please note that the C^1-seminorms are taken with respect to the y'-variable, and the absolute value makes the $e^{-ik|x|}$ that does not appear in the lemma disappear. Hence the application of the lemma is allowed. Elements of $\widetilde{H}^{-1/2}(\Omega_0)$ act well on C^1-functions, so the distribution pairing with ρ and the test function tends to zero. Thus

$$\lim_{|x|\to\infty} |x| e^{-ik|x|} \langle \rho(y'), \Phi(x, y^0) \rangle_{y'} = \frac{1}{4\pi} \left\langle \rho(y'), e^{-ik\hat{x}\cdot y^0} \right\rangle_{y'}$$

as claimed. □

3. Solving the Inverse Problem

We are ready to tackle the inverse problem in this section. In the following lemma, if ρ is integrable then $u_s^\infty(\hat{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y^0} \rho(y') dy'$.

Lemma 6. *Let $k \in \mathbb{R}_+$ and $\rho \in \mathscr{E}'(\mathbb{R}^2)$ be a distribution of compact support. Let*

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi} \left\langle \rho, e^{-ik\hat{x}\cdot y^0} \right\rangle \tag{16}$$

for $\hat{x} \in \mathbb{S}^2$ and where the distribution pairing is over the variable $y' = (y_1, y_2) \in \mathbb{R}^2$. Then ρ is uniquely determined by u_s^∞.

Proof. The operator mapping $\rho \mapsto u_s^\infty$ is bounded and linear $\mathcal{E}'(\mathbb{R}^2) \to C^0(\mathbb{S}^2)$. This is because $\hat{x} \mapsto \left(y' \mapsto \exp(-ik\hat{x}\cdot y^0)\right)$ is continuous $\mathbb{S}^2 \to \mathcal{E}(\mathbb{R}^2)$. So it is enough to show that $\rho = 0$ if $u_s^\infty = 0$. Let us assume the latter. For $\xi' \in \mathbb{R}^2$ we have

$$\hat{\rho}(\xi') = \frac{1}{2\pi} \left\langle \rho, e^{-i\xi'\cdot y'} \right\rangle$$

where the distribution pairing is over the variable $y' \in \mathbb{R}^2$. This looks similar to Formula (16) in the statement. We can rewrite

$$-ik\hat{x}\cdot y^0 = -ik(\hat{x}_1, \hat{x}_2, \hat{x}_3)\cdot(y_1, y_2, 0) = -i(k\hat{x}_1, k\hat{x}_2)\cdot(y_1, y_2).$$

Thus

$$u_s^\infty(\hat{x}) = \frac{1}{2}\hat{\rho}(k\hat{x}_1, k\hat{x}_2). \tag{17}$$

The left-hand side is zero for all $\hat{x} \in \mathbb{S}^2$. When $\hat{x}$ goes through the whole of $\mathbb{S}^2$, the sum including only two of the squares, $\hat{x}_1^2 + \hat{x}_2^2$, goes through the whole interval $(0, 1)$. Alternatively

$$\hat{\rho}(\xi') = 2u_s^\infty\left(\xi_1/k, \xi_2/k, \sqrt{k^2 - \xi_1^2 + \xi_2^2}/k\right) = 0$$

for all $|\xi'| \leq k$. Since ρ has compact support, $\hat{\rho}$ can be extended to an entire function on $\mathbb{C}^2$. Since it vanishes on an open subset of $\mathbb{R}^2$ it must be the zero function. Hence $u_s^\infty = 0$ implies $\rho = 0$. □

Lemma 7. *Let $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a screen and u_s satisfy the direct scattering problem of Definition 2. Denote*

$$\rho(x') = \partial_3 u_s^+(x^0) - \partial_3 u_s^-(x^0)$$

for $x' \in \mathbb{R}^2$ and its properties are given in Lemma 4. If $u_i(x', x_3) \neq -u_i(x', -x_3)$ for some $x \in \mathbb{R}^3$ then

$$\overline{\Omega_0} = \operatorname{supp} \rho \tag{18}$$

for the shape Ω_0 of the screen Ω.

Proof. The function ρ is a well-defined $H^{-1/2}(\Omega_0)$-function by Lemma 4 so in particular $\operatorname{supp} \rho \subset \overline{\Omega_0}$. It remains to prove that $\overline{\Omega_0} \subset \operatorname{supp} \rho$.

Assume the contrary that $\overline{\Omega_0}$ is not contained in the support of ρ. Then neither is Ω_0 because if $\Omega_0 \subset \operatorname{supp} \rho$ then $\overline{\Omega_0} \subset \overline{\operatorname{supp} \rho} = \operatorname{supp} \rho$. Because Ω_0 is an open set and $\operatorname{supp} \rho$ is closed there is $x_0' \in \Omega_0$ and $r > 0$ such that $B(x_0', r) \subset \Omega_0 \setminus \operatorname{supp} \rho$.

Let us study the behavior of u_s in the tube $B(x_0', r) \times \mathbb{R}$. We have $\rho = 0$ on $B(x_0', r)$. Recall Formula (4), which combined with the vanishing of ρ implies that $(\Delta + k^2)u_s = 0$ in the whole tube, and interior elliptic regularity implies that u_s is smooth there. In addition the formula implies that $u_s(x_1, x_2, x_3) = u_s(x_1, x_2, -x_3)$ for all x in the tube. The vanishing of ρ gives $\partial_3 u_s^+ = \partial_3 u_s^-$ on the base of the tube. These two imply that actually $\partial_3 u_s(x', 0) = 0$ for $x' \in B(x_0', r)$.

We have the following

$$u_s = -u_i, \tag{19}$$
$$\partial_3 u_s = 0 \tag{20}$$

on $B(x_0', r) \times \{0\}$. Let us calculate the higher order derivatives. Please note that ∂_3^j and $(\Delta + k^2)$ commute, and $(\Delta + k^2)u_s = 0$ in the tube. Thus

$$0 = \partial_3^j(\Delta + k^2)u_s = (\Delta + k^2)\partial_3^j u_s = (\Delta' + k^2)\partial_3^j u_s + \partial_3^{j+2} u_s$$

in the tube, and we denote $\Delta' = \partial_1^2 + \partial_2^2$. This gives $\partial_3^{j+2} u_s = -(\Delta' + k^2)\partial_3^j u_s$. Let us restrict ourselves to $B(x_0', r) \times \{0\}$ next. By induction and (19) and (20) we see that

$$\partial_3^j u_s = \begin{cases} (-1)^{j+1}(\Delta' + k^2)^j u_i, & j \in 2\mathbb{N}, \\ 0, & j \in 2\mathbb{N} + 1 \end{cases}$$

on $B(x_0', r) \times \{0\}$. This can still be simplified! Recall that u_i is an incident wave, so $(\Delta + k^2)u_i = 0$ everywhere. This means that $(\Delta' + k^2)u_i = -\partial_3^2 u_i$, and a fortiori $(\Delta' + k^2)^j u_j = (-\partial_3^2)^j u_i$ everywhere by the commuting of ∂_3^2 and $(\Delta' + k^2)$. This implies

$$\partial_3^j u_s = \begin{cases} -\partial_3^j u_i, & j \in 2\mathbb{N}, \\ 0, & j \in 2\mathbb{N} + 1. \end{cases} \tag{21}$$

The other derivatives, ∂_1 and ∂_2 commute with each other and ∂_3, so finally we have

$$\partial^\alpha u_s = \begin{cases} -\partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N} + 1 \end{cases} \tag{22}$$

on $B(x_0', r) \times \{0\}$ for all multi-indices $\alpha \in \mathbb{N}^3$.

Let us define

$$\tilde{u}_i(x) = \frac{1}{2}\big(u_i(x_1, x_2, x_3) + u_i(x_1, x_2, -x_3)\big)$$

for all $x \in \mathbb{R}^3$. This satisfies the Helmholtz equation everywhere, and is an incident wave because u_i is one. We see that

$$\partial^\alpha \tilde{u}_i(x) = \frac{1}{2}\big(\partial^\alpha u_i(x_1, x_2, x_3) + (-1)^{\alpha_3}\partial^\alpha u_i(x_1, x_2, -x_3)\big)$$

so

$$\partial^\alpha \tilde{u}_i = \begin{cases} \partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N} + 1 \end{cases} \tag{23}$$

on $B(x_0', r) \times \{0\}$. By (22) we see immediately that $\partial^\alpha u_s = -\partial^\alpha \tilde{u}_i$ on the base of the tube for all $\alpha \in \mathbb{N}^3$. Both functions u_s and $-\tilde{u}_i$ satisfy the Helmholtz equation not only in the tube but also in $\mathbb{R}^3 \setminus \overline{B}(0, R)$, where $R > 0$ is large enough that $\overline{\Omega} \subset B(0, R)$. Solutions of the Helmholtz equation are real analytic. Because their Taylor-expansions at $(x_0', 0)$ are equal, the functions are equal in the component of $\big(B(x_0', r) \times \mathbb{R}\big) \cup \big(\mathbb{R}^3 \setminus \overline{B}(0, R)\big)$ that contains $(x_0', 0)$, so in particular $u_s = -\tilde{u}_i$ in all of $\mathbb{R}^3 \setminus \overline{B}(0, R)$.

The function u_s satisfies the Sommerfeld radiation condition, so so does $\tilde{u}_i$. On the other hand $(\Delta + k^2)\tilde{u}_i = 0$ in all of $\mathbb{R}^3$, so $\tilde{u}_i$ is the zero function (Use e.g., (9) for a large ball whose radius grows to infinity. The boundary integral decreases to zero as was seen for the first integral in (15).), which means that u_i is antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$, a contradiction. Hence $\overline{\Omega_0} \subset \operatorname{supp} \rho$. □

The solution to the inverse problem of determining a screen Ω from the knowledge of a single incident wave u_i and the corresponding far-field u_s^∞ scattered from the screen comes from a combination of determining ρ from the far-field, and then Ω from ρ. There is a slight surprise, namely that the problem is only solvable for incident waves that are not too (anti)symmetric. However, one sees that antisymmetry is not the deciding factor: what matters is whether u_i is identically zero on the screen. By a similar argument as that at the end of the proof of Lemma 7, we see that if $u_i = 0$ on a non-empty open subset of $\mathbb{R}^2 \times \{0\}$ then $u_i(x', x_3) = -u_i(x', -x_3)$ for all $x \in \mathbb{R}^3$. It is interesting to see that partial invisibility is achieved inside thickened screens as long as the incident plane wave comes from a direction almost parallel to the screen's normal [39]. The direction of incident waves seems very important in scattering from objects that are thin in one direction.

Proof of Theorem 2. Theorem 1 and Lemma 6 imply that $\rho = \tilde{\rho}$ when $u_s^\infty = \tilde{u}_s^\infty$. If u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ then

$$\overline{\Omega_0} = \operatorname{supp} \rho = \operatorname{supp} \tilde{\rho} = \overline{\tilde{\Omega}_0}$$

by Lemma 7. Because Ω_0 is a smooth domain, we have $\Omega_0 = \operatorname{int} \overline{\Omega_0}$, and similarly for $\tilde{\Omega}_0$. Thus, the equation above implies $\Omega_0 = \tilde{\Omega}_0$ and by lifting, $\Omega = \tilde{\Omega}$.

If u_i is antisymmetric then $u_i = 0$ everywhere on $\mathbb{R}^2 \times \{0\}$ and $u_s = 0$ satisfies all conditions of the direct scattering problem. Since solutions to the direct scattering problem (2) are unique by ([37] [Thms 2.5–2.7]), this is the only solution. Thus, $u_s = \tilde{u}_s = 0$ and the same holds for their far-fields. This is irrespective of the shape of $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$. □

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Appendix 2

III
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Article

Unique Determination of a Planar Screen in Electromagnetic Inverse Scattering

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Abstract: The target of our research is the object being a highly conducting thin plate or a flat screen. We especially focus on the question of when a single measurement uniquely determines an object. By this, we mean that we have one fixed transmitted wave and the resulting scattered field is measured for all directions in the far field. Such measurements are called passive, since there is no need to move the transmitter after its position has been fixed. We show that the far field of a scattered electromagnetic field corresponding to a single incoming plane wave always uniquely determines a bounded super-conductive planar screen. This generalises a previous scalar result of Blåsten, Päivärinta and Sadique [3].

Keywords: Inverse conductivity problem; electrical impedance tomography; unknown boundary; Teichmüller mapping

MSC: 35R30; 35Q61; 78A46

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1. Introduction

The study of wave scattering from highly conductive objects is rooted in the field of antenna theory. This interest began with a competition initiated by the Prussian Academy in 1879 to demonstrate the existence or non-existence of electromagnetic waves. Maxwell's theory [27] predicted the existence of such waves 15 years earlier. In 1882, the competition was won by Heinrich Hertz, who constructed a dipole antenna that was able to radiate and measure EM waves, thereby confirming Maxwell's prediction.

Inverse scattering problems involve determining the properties of an object, such as its shape, size, and composition, from the produced scattered field when it is illuminated with, say, electromagnetic or acoustic waves. A well-studied problem is to determine the shape of an object by analysing the far field of the scattered wave. This is a problem that involves both mathematics and numerical techniques, and it has many practical applications. An overview of this topic can be found in the book [7] by Colton and Kress.

Recently, there has been an increasing number of publications about the inverse scattering problem with fewer measurements. Especially fascinating is the question of when just a single measurement determines an object uniquely. By this, we mean that we have one fixed transmitted wave and the resulting scattered field is measured for all directions in the far field. Such measurements are called passive, since you do not need to move the transmitter after its position has been fixed. This is exactly the target of the research here, the object being a highly conducting thin plate, i.e., a flat screen.

In [3], we considered the problem of fixed frequency acoustic scattering from a sound-soft flat screen. The main result of that paper is that the far field produced by any single incident wave determines the precise shape of the screen, given that it is not anti-symmetric with respect to the plane. Our current work is the generalisation of the result of [3] to

Maxwell's equations. This shape determination problem is known in the literature [7] as Schiffer's problem. The first uniqueness result for the case of the Dirichlet problem was presented by Schiffer in 1967 [31]. The Schiffer's uniqueness theorem for the inverse Dirichlet problem assumes a lot of information about the waves as it is using an infinite number of incident frequencies. After Schiffer's uniqueness result for sound-soft obstacles by countably many incident plane waves [7,31], extensive research in this direction has been conducted. Notable contributions include uniqueness results for general domain [11,21,24,32–34,38], polyhedral scatterers [2,8–10], for the ball or disc [6,18], and for smooth planar curves [22,23,29,35].

Important results on the inverse electromagnetic scattering problem in the TE polarisation case was conducted in [19]. They demonstrated that in the special case of a rectangular penetrable scatterer, it can be uniquely determined just by measuring the electric far-field pattern for a single incoming wave.

In [13], the authors studied uniqueness of an inverse acoustic obstacle scattering problem, demonstrating the unique determination of sound-hard and sound-soft polyhedral scatterers in $\mathbb{R}^n$. More precisely, they proved that N far-field measurements corresponding to N incident plane waves given by a fixed wave number and N linearly independent incident directions uniquely determine the obstacle. A few of the uniqueness results in inverse electromagnetic and acoustic obstacle scattering problems were obtained by Liu and Zou [24]. They emphasize recent developments in the unique determination of a general polyhedral scatterer using far-field data corresponding to one or several incident fields. For other recent results in time-harmonic inverse EM-scattering, see the short review by Rainer Kress [21].

A particular case in [1] gives the unique determination of a flat screen by a single incident plane-wave measurement with robin boundary condition. We also wish to mention the article [2], in which the authors demonstrate that the far-field pattern corresponding to one incident plane wave uniquely identifies a sound-soft polyhedral scatterer.

However, they consider the polyhedral sound-soft acoustic problem and not the electromagnetic problem.

In addition, [25] has established a reflection principle for the time-harmonic Maxwell equations. They derive a uniqueness result for the inverse electromagnetic scattering problem for a polyhedral scatterer. The scatterers considered can exhibit a wide variety; for instance, they might comprise a finite number of compact polyhedral shapes along with a finite number of portions from two-dimensional surfaces. Another important work is [17], where the authors consider an obstacle composed of finite solid polyhedra, and they prove that it can be uniquely characterised by the far-field pattern associated with a single incident electromagnetic plane wave. To our knowledge, there is no proof for the unique determination of a planar screen by one far-field pattern without restrictive a priori assumptions such as the assumptioned polyhedron shape. The research in [14] comes very close to ours. There, the obstacle can be any Lipschitz domain provided that its boundary is not an analytic manifold. But that work does not consider screens, which is our priority.

The main motivation for this study comes from antenna theory [20,26,36]. A typical (radar) antenna consists of a configuration of planar screens attached to a common stem, and understanding both the direct and inverse scattering of electromagnetic waves from such structures is a natural and important problem. Our study is the natural first step in analysing the inverse problem of recovering the shape of the antenna using exactly one incoming wave.

The goal of this work is to prove the unique determination of the unknown screen and supporting hyperplane corresponding to a single measurement of the far field. The proof follows from the representation formula for the exterior solution of Maxwell's equations. The main idea of our paper is to reduce the scattering problem to an integral equation on the screen. Here, the integral operator is the analogue of the double-curl layer potential on the screen, and has as its unknown the jump of the tangential component of the magnetic field. The inverse problem is then solved by showing that, first of all, the incoming plane wave uniquely determines the solution to this integral equation, and secondly, that when the tangential component of the incoming plane wave does not vanish on the screen, the support of the solution is full, i.e., the whole screen. More precisely, our main results state that a single far field corresponding to an incoming plane wave uniquely determines a planar screen, as follows:

Theorem 1. *Let S be a C^2–screen contained in a supporting hyperplane L and let*

$$E(\theta; p, q) = \mu^{1/2}(p \times \theta)e^{ik\langle\theta,x\rangle},\ H(\theta; p, q) = \varepsilon^{1/2}(q \times \theta)e^{ik\langle\theta,x\rangle}$$

describe the EM-plane wave with wavenumber $k = \omega\sqrt{\varepsilon\mu}$, propagation direction θ and polarizations p and q. Let $(e_{\rm sc}, h_{\rm sc})$ be the electromagnetic wave scattered by S and assume that it does not identically vanish. Then, the non-vanishing far-field pattern of $(e_{\rm sc}, h_{\rm sc})$ uniquely determines both the supporting hyperplane L and the screen S if neither p or q is parallel to θ.

Remark 1. *As will be clear from the proofs, the scattered field will vanish only if the electric polarization $p \times \theta$ is parallel to the screen.*

The plan of this paper is as follows: Analysis of mathematical proof and the concept for the direct scattering problem of EM waves are discussed in Section one. We start by giving a precise definition of a planar screen and discuss time-harmonic Maxwell's equations in the exterior of the screen. Representation theorem for the fundamental solution of vector Helmholtz equation is also analysed here. In Section 2, the solution of the inverse problem is presented, including also the unique determination of the supporting hyperplane.

2. Scattering from a Perfectly Conducting Screen

2.1. Formal Definitions

Definition 1. *A* planar C^k–screen, *$k = 1, \ldots, \infty$, in $\mathbb{R}^3$ is a compact, connected C^k–submanifold of an affine hyperplane $L \subset \mathbb{R}^3$. The affine hyperplane L is called* the supporting hyperplane *of S. In the sequel we also fix a globally defined unit normal vector field on S and denote it by ν. Also, the boundary of S as a submanifold of L is denoted by ∂S.*

Consider the time-harmonic Maxwell's equations in the exterior of a screen S:

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\varepsilon E \quad \text{in } \mathbb{R}^3 \setminus S. \tag{1}$$

Here, the *magnetic permeability* μ and the *dielectricity* ε are known positive constants and we assume also that the angular frequency $\omega > 0$ is known.

Given an incident filed (E_0, H_0), i.e., a solution of

$$\nabla \times E_0 = i\omega\mu H_0, \quad \nabla \times H_0 = -i\omega\varepsilon E_0 \quad \text{in } \mathbb{R}^3,$$

the corresponding *scattered field* (E_{sc}, H_{sc}) is (formally) defined by demanding that (E, H), where $E = E_0 + E_{sc}$ and $H = H_0 + H_{sc}$ satisfy (1), and the scattered field is outgoing in the sense that it satisfies the Silver–Müller radiation conditions,

$$\hat{r} \times E_{sc} + \sqrt{\frac{\varepsilon}{\mu}} H_{sc} = o(|x|^{-1}), \quad \hat{r} \times H_{sc} - \sqrt{\frac{\mu}{\varepsilon}} E_{sc} = o(|x|^{-1}), \quad \text{as } |x| \to \infty. \tag{2}$$

Here, $\hat{r} = x/|x|$. If we further assume that the screen is perfectly conducting, i.e., the total field vanishes on S, this leads to the *direct scattering problem for the perfectly conducting screen* S: for a given incident field, (E_0, H_0) show that there is a unique scattered field (E_{sc}, H_{sc}) s.t.

$$\nabla \times E_{sc} = i\omega\mu H_{sc}, \quad \nabla \times H_{sc} = -i\omega\varepsilon E_{sc} \quad \text{in } \mathbb{R}^3 \setminus S \tag{3}$$

satisfying (2) and such that

$$\nu \times (E_{sc} + E_0) = 0 \quad \text{on } S. \tag{4}$$

Note that we have not specified in what sense the boundary value (4) holds. This will depend on the spaces where we look for solutions and the availability of suitable trace theorems.

2.2. Representation Theorems

Assume for now that $S \subset \mathbb{R}^3$ is a C^2 screen. Denote by C_S^k the closed subspace of $C^k(\mathbb{R}^3 \setminus S)$ consisting of those $u \in C^k(\mathbb{R}^3 \setminus S)$ such that u and all its derivatives up to order k have normal limits on S, i.e., for all $|\alpha| \le k$, there are limits

$$\lim_{\delta \to +0} \partial_x^\alpha u(x \pm \delta \nu(x)) = u_\alpha^\pm(x), \quad x \in S,$$

where $u_\alpha^\pm \in C(S)$. Note that we do not assume that limits u_α^+ and u_α^- coincide on S.

Proposition 1. *Assume $(e, h) \in (C_S^1)^3 \times (C_S^1)^3$ solves*

$$\nabla \times e = i\omega\mu\, h, \quad \nabla \times h = -i\omega\varepsilon\, e \quad \text{in } \mathbb{R}^3 \setminus S,$$

and the Silver–Müller radiation condition

$$\hat{r} \times e + \sqrt{\frac{\varepsilon}{\mu}} h = o(|x|^{-1}), \quad \hat{r} \times h - \sqrt{\frac{\mu}{\varepsilon}} e = o(|x|^{-1}), \quad \text{as } |x| \to \infty, \quad \hat{r} = x/|x| > 0.$$

Then, for all $x \in \mathbb{R}^3$,

$$\begin{aligned} e(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y) \\ &- \frac{1}{i\omega\varepsilon}(\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \end{aligned}$$

and

$$\begin{aligned} h(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \\ &+ \frac{1}{i\omega\mu}(\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y). \end{aligned}$$

Proof. For $\delta > 0$, let ${}_\delta = \{x \pm t\nu(x);\, x \in S,\, 0 \le t < \delta\}$ be a collar neighbourhood of S. For sufficiently small δ, this is a bounded, piecewise analytic domain. The standard representation formulas (see for example [5]) give that for all $x \in \mathbb{R}^3 \setminus {}_\delta$, we have

$$e(x) = \nabla \times \int_{\partial_\delta} \Phi(x-y)(\nu_\delta(y) \times e(y))\, ds(y) - \frac{1}{i\omega\varepsilon}(\nabla\times)^2 \int_{\partial_\delta} \Phi(x-y)(\nu_\delta(y) \times h(y))\, ds(y)$$

and

$$h(x) = \nabla \times \int_{\partial_\delta} \Phi(x-y)(\nu_\delta(y) \times h(y))\, ds(y) + \frac{1}{i\omega\mu}(\nabla\times)^2 \int_{\partial_\delta} \Phi(x-y)(\nu_\delta(y) \times e(y))\, ds(y).$$

Here, Φ is the outgoing fundamental solution of the Helmholtz operator $\Delta + k^2$ and ν_δ is the exterior unit normal of ∂_δ. Then, as $\delta \to +0$,

$$\begin{aligned} e(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y) \\ &- \frac{1}{i\omega\varepsilon}(\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \end{aligned}$$

and

$$\begin{aligned} h(x) &= \nabla \times \int_S \Phi(x-y)(\nu \times \{h^+(y) - h^-(y)\})\, ds(y) \\ &+ \frac{1}{i\omega\mu}(\nabla\times)^2 \int_S \Phi(x-y)(\nu \times \{e^+(y) - e^-(y)\})\, ds(y), \end{aligned}$$

as claimed. □

In what follows, we will denote the jumps of a function (or a vector field) u across S by $[u]$, i.e.,

$$[u](y) = u^+(y) - u^-(y),\ y \in S.$$

2.3. EM-Plane Waves and Far-Field Patterns

Let $\theta,\ p \in S^2$ and denote $q = p \times \theta$. We call the field

$$E(\theta; p, q) = \mu^{1/2}(p \times \theta)e^{ik\langle\theta,x\rangle},\ H(\theta; p, q) = \varepsilon^{1/2}(q \times \theta)e^{ik\langle\theta,x\rangle}$$

the *EM-plane wave with wavenumber k, propagation direction θ and polarizations p and q.* It is easy to see that these fields satisfy the time-harmonic Maxwell's equations

$$\nabla \times E(\theta; p, q) = i\omega\mu\, H(\theta; p, q), \quad \nabla \times H(\theta; p, q) = -i\omega\varepsilon\, E(\theta; p, q).$$

when $k^2 = \varepsilon\mu$. Since the scalar components of the scattered electric and magnetic fields are solutions of the Helmholz equation $(\Delta + k^2)u = 0$ satisfying the Sommerfeld radiation condition, they have representations

$$E(x) = \frac{E^\infty(x)}{|x|} + o(|x|^{-1}), \quad H(x) = \frac{H^\infty(x)}{|x|} + o(|x|^{-1}$$

where E^∞ and H^∞ are the electric and magnetic far-field patterns. If the initial field is the EM-plane wave $(E(\theta; p, q), H(\theta; p, q))$, we denote the corresponding far-field patterns by $E^\infty(\theta; p, q)$ and $H^\infty(\theta; p, q)$.

2.4. Relevant Sobolev Spaces

Let (see [15,30]) $L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus S)$ be the space of measurable functions that are square integrable on compact subsets of $\mathbb{R}^3 \setminus S)$. This becomes a Fréchet space when equipped with semi-norms

$$\|f\|_R = \|f\|_{L^2(\mathbb{R}^3\setminus S)\cap B_R(0))}, \quad R > R_0,$$

where R_0 is so large that $S \subset B_{R_0}(0)$. Define also

$$L^2_{\mathrm{loc,\,curl}}(\mathbb{R}^3 \setminus S) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus S);\ \nabla \times u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus S)\},$$

$$L^2_{\mathrm{loc,\,div}}(\mathbb{R}^3 \setminus S) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus S);\ \nabla \cdot u \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus S)\},$$

and equip these space with semi-norms

$$\|f\|_{R,\mathrm{curl}} = (\|f\|_R^2 + \|\nabla \times f\|_R^2)^{1/2},\ \|f\|_{R,\mathrm{div}} = (\|f\|_R^2 + \|\nabla \cdot f\|_R^2)^{1/2}.$$

Also, let

$$TH^{-1/2}(S) = \{u \in H^{-1/2}(S)^3; \langle \nu, u\rangle = 0\},$$

i.e., the space of tangential $H^{-1/2}$ fields on S. We equip this with the norm induced from $H^{-1/2}(S)^3$. With Div denoting the surface divergence, we also define

$$TH^{-1/2}_{\mathrm{Div}}(S) = \{u \in TH^{-1/2}(S); \mathrm{Div}\,(u) \in H^{-1/2}(S)\}$$

and equip it with the Hilbert norm, defined by

$$\|u\|^2_{TH^{-1/2}_{\mathrm{Div}}(S)} = \|u\|^2_{TH^{-1/2}(S)} + \|\mathrm{Div}\,(u)\|^2_{H^{-1/2}(S)}.$$

Assume now that $U \in \mathbb{R}^3$ is a bounded C^2 domain with a connected complement, such that $S \subset \partial U$ is a compact C^2-submanifold and fix the unit normal ν of S so that it extends to a unit exterior normal $\tilde{\nu}$ of U. If $u,\ \varphi \in (C_0^\infty)^3$ then the vector Green's identities give

$$\int_{\partial U} \langle \tilde{\nu} \times u, \varphi \rangle \, ds = \int_U \langle \nabla \times u, \varphi \rangle - \langle u, \nabla \times \varphi \rangle \, dx,$$

and extending this by density to $\varphi \in H^1(\mathbb{R}^3)$ and $u \in L^2_{curl}(U)$ gives the existence of the tangential trace $\tilde{\nu} \times u|_{\partial U} \in TH^{-1/2}(\partial U)$. We can argue similarly for the exterior domain. Using this definition, we have well-defined tangential trace maps $t^\pm$ from the direction of $\pm\nu$,

$$t^\pm : L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S) \ni u \mapsto \nu \times u^\pm \in TH^{-1/2}(S)$$

Similarly, if $u \in (C_0^\infty)^3$ and $\psi \in C_0^\infty$, we get from the Divergence Theorem that

$$\int_U \langle \tilde{\nu}, u \rangle \, \psi \, ds = \int_U \langle u, \nabla \psi \rangle + \psi \nabla \cdot u \, dx,$$

and, using this, we have well-defined normal traces $n^\pm$,

$$n^\pm : L^2_{\mathrm{loc,\,div}}(\mathbb{R}^3 \setminus S) \ni u \mapsto \langle \nu, u^\pm \rangle \in H^{-1/2}(S).$$

Note also that if $u \in L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S)$, then $\nabla \times u \in L^2_{\mathrm{loc,\,div}}(\mathbb{R}^3 \setminus S)$ and

$$\mathrm{Div}\,(\nu \times u^\pm) = -\langle \nu, \nabla \times u^\pm \rangle \in H^{-1/2}(S),$$

i.e., the tangential traces of $L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S)$ are in $TH^{-1/2}_{\mathrm{Div}}(S)$. Note also that since extension by zero across a C^2 hypersurface is continuous in fractional Sobolev spaces, H^s when $s < 1/2$ the space $TC_0^\infty(S)$ is dense in $TH^s(S)$. However, this is not necessarily true for the Div spaces, and hence, the closure of $TC_0^\infty(S)$ in $TH^{-1/2}_{\mathrm{Div}}(S)$ is denoted by $T\dot{H}^{-1/2}_{\mathrm{Div}}(S)$.

2.5. Layer Potentials in Sobolev spaces

For $x \in \mathbb{R}^3 \setminus S)$ and $u \in C_0^\infty(S)^3$ define the (vector) single-layer potential of u by

$$V_{\mathbb{R}^3 \setminus S}(u)(x) = \int_S \Phi(x-y)\, u(y)\, ds(y)$$

and the electromagnetic layer operators by

$$K_{\mathbb{R}^3 \setminus S}(u)(x) = \nabla \times V_{\mathbb{R}^3 \setminus S}(u)(x),$$

and

$$N_{\mathbb{R}^3 \setminus S}(u)(x) = (\nabla \times)^2 V_{\mathbb{R}^3 \setminus S}(u)(x).$$

Proposition 2. *Assume that S can be extended to a boundary ∂U for some bounded C^2 domain U. Then, the single-layer potential has an extension to a bounded map*

$$V_{\mathbb{R}^3 \setminus S} : H^{-1/2}(S) \to H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus S)$$

and $V_{\mathbb{R}^3 \setminus S}(u)$ satisfies the Sommerfeld radiation condition. Also, the electromagnetic potentials have extensions into bounded maps

$$K_{\mathbb{R}^3 \setminus S},\ N_{\mathbb{R}^3 \setminus S} : T\dot{H}^{-1/2}_{\mathrm{Div}}(S) \to L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S)$$

and $K_{\mathbb{R}^3\setminus S}(u)$ *and* $N_{\mathbb{R}^3\setminus S}(u)$ *satisfy the Sommerfeld radiation conditions for any* $u \in T\dot{H}_{\mathrm{Div}}{}^{-1/2}(S)$.

Proof. By known continuity properties (see for example [28]), the single-layer potential defines a continuous map $H^{-1/2}(U) \to H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{U})$, and, in fact, $V_{\mathbb{R}^3\setminus\overline{U}}(\varphi)$ is continuous across U and the jump in the normal derivative is equal to $\varphi|_U$. Hence, the claim for V follows since $C_0^\infty(S)$ is dense in $H^{-1/2}(S)$. This also implies the claims for $K_{\mathbb{R}^3\setminus S}$ and $N_{\mathbb{R}^3\setminus S}$, since for $u \in TC_0^\infty(S)$, we have

$$(\nabla\times)^2 V_{\mathbb{R}^3\setminus S}(u) = -\nabla V_{\mathbb{R}^3\setminus S}(\mathrm{Div}\, u) + k^2 V_{\mathbb{R}^3\setminus S}(u)$$

and

$$(\nabla\times)^3 V_{\mathbb{R}^3\setminus S}(u) = k^2 \nabla \times V_{\mathbb{R}^3\setminus S}(u).$$

□

Using this, we can generalise the representation theorem 1 to weak solutions:

Proposition 3. *Let* $S \subset \mathbb{R}^3$ *be a* C^1 *screen. Assume* $(e,h) \in L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S)$ *solves*

$$\nabla \times e = i\omega\mu\, h, \quad \nabla \times h = -i\omega\varepsilon\, e \quad in \quad \mathbb{R}^3 \setminus S,$$

and the Silver–Müller radiation condition

$$\hat{r} \times e + \sqrt{\frac{\varepsilon}{\mu}} h = o(|x|^{-1}), \quad \hat{r} \times h - \sqrt{\frac{\mu}{\varepsilon}} e = o(|x|^{-1}), \quad as\ |x| \to \infty, \quad \hat{r} = x/|x| > 0.$$

If $\nu \times [e]$, $\nu \times [h] \in TH^{-1/2}_{\mathrm{Div}}(S)$, *then in* $\mathbb{R}^3 \setminus \overline{S}$,

$$e = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [e]) - \frac{1}{i\omega\varepsilon} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [h]),$$

and

$$h = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [h]) + \frac{1}{i\omega\mu} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [e]).$$

Here, ν *is the specified unit normal of* S.

2.6. Representation Formulas for the Scattered Field

Proposition 4. *Let* S *be a perfectly conducting* C^2 *screen, and let*

$$(E_{sc}, H_{sc}) \in L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S) \times L^2_{\mathrm{loc,curl}}(\mathbb{R}^3 \setminus S)$$

be the scattered field corresponding to an incoming field (E_0, H_0). *Then, in* $\mathbb{R}^3 \setminus \overline{S}$, *one has*

$$E_{\mathrm{sc}} = -\frac{1}{i\omega\varepsilon} N_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [H_{\mathrm{sc}}])$$

and

$$H_{\mathrm{sc}} = K_{\mathbb{R}^3\setminus\overline{S}}(\nu \times [H_{\mathrm{sc}}]).$$

These fields have the following asymptotic behaviour as $|x| \to \infty$:

$$E_{\mathrm{sc}}(x) = -\hat{x} \times \left(\hat{x} \times \frac{e^{ik|x|}}{4\pi i\omega\varepsilon|x|} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\mathrm{sc}}])(y)\, ds(y) \right) + O(|x|^{-2}),$$

$$H_{\mathrm{sc}}(x) = \hat{x} \times \frac{e^{ik|x|}}{4\pi i\omega\mu|x|} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\mathrm{sc}}])(y)\, ds(y) + O(|x|^{-2}),$$

where $\hat{x} = x/|x|$, $x \neq 0$.

Proof. Since on a perfectly conducting screen, $\nu \times [E_{\rm sc}] = -\nu \times [E_0] = 0$ and the representations of $E_{\rm sc}$ and $H_{\rm sc}$ follow from Proposition 1. The asymptotic behaviour is obvious since for $|x| \to \infty$, $y \in S$ and $a(y)$, a vector field on S,

$$\nabla_x \times \frac{e^{ik|x-y|}}{|x-y|} a(y) = \hat{x} \times \frac{e^{ik|x| - ik\langle \hat{x}, y\rangle}}{|x|} a(y) + O(|x|^{-2})$$

$$(\nabla_x \times)^2 \frac{e^{ik|x-y|}}{|x-y|} a(y) = \hat{x} \times (\hat{x} \times \frac{e^{ik|x| - ik\langle \hat{x}, y\rangle}}{|x|} a(y)) + O(|x|^{-2}).$$

□

In view of the above proposition, we can write

$$E_{\rm sc}(x) = \frac{e^{ik|x|}}{4\pi|x|} E^\infty(\hat{x}) + O(|x|^{-2}), \quad H_{\rm sc}(x) = \frac{e^{ik|x|}}{4\pi|x|} H^\infty(\hat{x}) + O(|x|^{-2})$$

where the far-field patterns E^∞ and H^∞ are given by

$$E^\infty(\hat{x}) = -\hat{x} \times \left(\hat{x} \times \frac{1}{i\omega\varepsilon} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\rm sc}])(y)\, ds(y) \right)$$

$$H^\infty(\hat{x}) = \hat{x} \times \frac{1}{i\omega\mu} \int_S e^{-ik\langle \hat{x}, y\rangle} (\nu \times [H_{\rm sc}])(y)\, ds(y).$$

Note also that $\varepsilon E^\infty(\hat{x}) = -\mu\, \hat{x} \times H^\infty(\hat{x})$, and that the uniqueness of the scattered field follows from the uniqueness of the Dirichlet and Neumann problems for the scalar Helmholtz equation [37].

2.7. Integral Equations for the Scattered Field

Assume now that $U \subset \mathbb{R}^3$ is a bounded C^2 domain with a connected complement such that $S \subset \partial U$ is a C^2 submanifold. Using the usual jump relations (see for example [5,30]), the tangential components of $N_{\mathbb{R}^3\setminus\overline{U}}(u)$ and $N_U(u)$ are continuous up to ∂U and they have equal traces for all $u \in TH^{-1/2}_{\rm Div}(\partial U)$. Furthermore, for $u \in TH^{-1/2}_{\rm Div}(\partial U)$, one has

$$\nu \times N_{\mathbb{R}^3\setminus\overline{U}}(u)|_{\partial U} = \nu \times N_U(u)|_{\partial U} = N(u),$$

where the surface integral operator N is given by

$$N(u) = \tilde{\nu} \times \nabla S(\operatorname{Div} u) + k^2 \tilde{\nu} \times S(u).$$

Here, $\tilde{\nu}$ is the exterior unit normal to U, which is assumed to agree with ν on S, and S is the direct boundary value of the single layer potential, i.e.,

$$S(f)(x) = \int_{\partial U} \Phi(x-y)\, f(y)\, ds(y), \quad x \in \mathbb{R}^3.$$

Note also that using the trace theorems given in Section 2.5, one has $N : TH^{-1/2}_{\rm Div}(\partial U) \to TH^{-1/2}_{\rm Div}(\partial U)$ continuously and for the restriction to S, one has $N : T\dot{H}^{-1/2}_{\rm Div}(S) \to TH^{-1/2}_{\rm Div}(S)$, again continuously.

Assuming now that (E_{sc}, H_{sc}) is the field scattered by the screen S and $\nu \times [H_{sc}] \in T\dot{H}^{-1/2}_{\rm Div}(S)$, we get from Proposition 4 and the continuity results of Section 2.5 that

$$\nu \times E_{sc} = iN(\nu \times [H_{sc}])/\omega\varepsilon,$$

and since the screen is perfectly conducting, one gets an integral equation for the jump of the tangential component of the magnetic field,

$$-\nu \times E_0 = iN(\nu \times [H_{sc}])/\omega\varepsilon, \quad \nu \times [H_{sc}] \in T\dot{H}^{-1/2}_{\mathrm{Div}}(S). \tag{5}$$

Solvability properties of this equation have been considered in [4]. More precisely, it is shown that (5) is uniquely solvable in $T\dot{H}^{-1/2}(S)$.

3. Solution of the Inverse Problem

3.1. Uniqueness When the Supporting Hyperplane Is Known.

The following lemma shows that for a planar screen, the tangential density of the far-field pattern is uniquely determined.

Lemma 1. *Assume that ρ is a compactly supported tangential distributional density on a hyperplane L. Let*

$$\rho^\infty(\hat{x}) = \hat{x} \times \langle \rho, \exp\{-ik\langle \hat{x}, \cdot\rangle\}\rangle, \quad \hat{x} \in \mathbb{S}^2.$$

Then the map $\rho \mapsto \rho^\infty$ is injective.

Proof. We may assume that coordinates have been chosen so that L is defined by $\{x;\, x_3 = 0\}$. Let $\rho = a d\sigma$ where $a = (a_1, a_2) \in \mathcal{E}'(\mathbb{R}^2)$ and $d\sigma$ is the surface measure on the hyperplane L. Then, $\rho^\infty = 0$ is equivalent to

$$\xi \times (\hat{a}_1(\xi'), \hat{a}_2(\xi'), 0) = 0,\, \xi = (\xi', (k^2 - |\xi'|^2)^{1/2}),\, |\xi'| < k,$$

and hence, $\hat{a}_1$ and $\hat{a}_2$ vanish in the unit ball of $\mathbb{R}^2$ and since they are entire functions, they are identically zero. □

This implies that the far field E^∞ (or H^∞ for that matter) uniquely determines the density $[\nu \times H_{\mathrm{sc}}]\, ds$ when the screen S is flat, i.e., included in a hyperplane.

Proposition 5. *Let S be a C^2 screen contained in a supporting hyperplane L and let (e_0, h_0) be an electromagnetic plane wave with wave number k with electric and magnetic polarisations p and q. Assume that $\rho \in T\dot{H}^{-1/2}(S)$ solves $-\nu \times e_0 = -iN(\nu \times \rho)/\omega\varepsilon$ on S. Then, if p or q are not parallel to θ and $\nu \times (p \times \theta) \neq 0$ the density ρ has full support, i.e.,* $\mathrm{supp}(\rho) = S$.

Proof. Assume coordinates chosen so that the $L = \{x \in \mathbb{R}^3;\, x_3 = 0\}$. Assume that there is a relatively open $U \subset S$ such that $\rho = 0$. Define

$$\tilde{h} = K_{\mathbb{R}^3 \setminus S}(\nu \times \rho), \quad \tilde{e} = \frac{i}{\omega\varepsilon} \nabla \times \tilde{h}.$$

Then, $\tilde{h}$, $\tilde{e} \in L^2_{\mathrm{loc,\,curl}}(\mathbb{R}^2 \setminus S)$ and they satisfy Maxwell's equations

$$\nabla \times \tilde{e} = i\omega\mu\, \tilde{h}, \quad \nabla \times \tilde{h} = -i\omega\varepsilon\, \tilde{e}.$$

The second equation follows from the definition and the first is an immediate consequence of the vector Green's formulas:

$$\begin{aligned}
\nabla \times \tilde{e} &= \frac{i}{\omega\varepsilon}(\nabla\times)^2\, \tilde{h} = \frac{i}{\omega\varepsilon}(\nabla\nabla\cdot - \Delta)(K_{\mathbb{R}^3 \setminus S}(\nu \times \rho)) = \frac{i}{\omega\varepsilon}(\nabla\nabla\cdot - \Delta)(\nabla \times S_{\mathbb{R}^3 \setminus S}(\nu \times \rho)) \\
&= \frac{i}{\omega\varepsilon} k^2\, \nabla \times S_{\mathbb{R}^3 \setminus S}(\nu \times \rho) = -i\omega\mu \nabla \times S_{\mathbb{R}^3 \setminus S}(\nu \times \rho) = -i\omega\mu\, \tilde{h}.
\end{aligned}$$

From the jump relations of the vector potentials, $[\nu \times \tilde{h}] = [\nu \times \tilde{\rho}] = 0$ on U and

$$-\tilde{e} = -\frac{i}{\omega\varepsilon}\nabla \times \tilde{h} = -\frac{i}{\omega\varepsilon}N_{\mathbb{R}^3\setminus S}(\nu \times \rho)$$

we have

$$\nu \times \tilde{e}|_S = iN(\nu \times \rho)/\omega\varepsilon|_S = \nu \times e_0.$$

Let $E = e_0 - \tilde{e}$ and $H = h_0 - \tilde{h}$. Then, from the above observations, (E, H) solves (1) and

$$\nu \times E|_S = 0, \quad [\nu \times H]|_U = 0.$$

Now, let $\hat{E}$ be an extension of E from the upper half-space $\{x_3 > 0\}$ to the lower half-space so that it is odd in the tangential component and even in the normal component, i.e.,

$$\hat{E}(x_1, x_2, -x_3) = (-E_1(x), -E_2(x), E_3(x)),\ x = (x_1, x_2, x_3),$$

and let $\hat{H}$ be an extension of H, which is even in the tangential component and odd in the normal component,

$$\hat{H}(x_1, x_2, -x_3) = (H_1(x), H_2(x), -H_3(x)),\ x = (x_1, x_2, x_3).$$

Then, a straightforward computation shows that

$$\nabla \times \hat{E} = i\omega\mu\,\hat{H}, \quad \nabla \times \hat{H} = -i\omega\varepsilon\,\hat{E}, \quad x_3 \neq 0.$$

Let $V = U \times \mathbb{R}$. Then, since the tangential components of E vanish on S and the tangential components of H are continuous across U, this holds also for the tangential components of $\hat{E}$ and $\hat{H}$ across U. Thus, $(\hat{E}, \hat{H})$ solves

$$\nabla \times \hat{E} = i\omega\mu\,\hat{H}, \quad \nabla \times \hat{H} = -i\omega\varepsilon\,\hat{E}, \quad x \in V,$$

and hence, by unique continuation, $E = \hat{E}$ and $H = \hat{H}$ in V and, thus, also in $\mathbb{R}^3 \setminus S$, since $\nu \times \hat{H} = \nu \times H$ on U. Hence, we can write

$$e_0 = \hat{E} + \tilde{e}, \quad h_0 = \hat{H} + \tilde{h}. \tag{6}$$

We say that a vector field has parity 1 if the tangential component is even and the normal component is odd with respect to $\{x_3 = 0\}$, and it has parity -1 if the tangential component is odd and the normal component is even. Notice, then, that since $\tilde{h}$ is the EM-double layer of a tangential density, it has parity -1. Hence, $\tilde{e} = -i\nabla \times \tilde{h}/\omega\varepsilon$ has parity 1. Also, the decomposition of a field as a sum of fields with parity $+1$ and -1 is unique. Since $\tilde{e}$ satisfies the Silver–Müller radiation condition, the incoming field e_0 must have parity -1 and similarly, h_0 must have parity 1. Recall that

$$e_0(x) = \mu^{1/2}(p \times \theta)e^{ik\langle\theta,x\rangle},\ h_0(x) = \varepsilon^{1/2}(q \times \theta)e^{ik\langle\theta,x\rangle}, \quad q = p \times \theta,$$

Hence, the parity 1 part of e_0 is given by

$$\mu^{1/2}(q_1 e_0^{(+)}, q_2 e_0^{(+)}, i q_3 e_0^{(-)}),$$

where $e_0^{(+)}(x) = \cos(k\langle\theta, x\rangle)$ and $e_0^{(-)}(x) = \sin(k\langle\theta, x\rangle)$. This vanishes identically if and only if $q = 0$, i.e., $p \times \theta = 0$. Similarly, the parity -1 part of h_0 is

$$\varepsilon^{1/2}(i(q \times \theta)_1 e_0^{(-)}, i(q \times \theta)_2 e_0^{(-)}, (q \times \theta)_3 e_0^{(+)}),$$

which vanishes identically if and only if $q \times \theta = 0$. Since p, q and θ are unit vectors and $q = p \times \theta$, this is not possible. $\square$

3.2. Unique Determination of a Planar Screen

We show that the supporting hyperplane uniquely determines the far field of a single scattering solution. This, combined with the unique determination results of the previous subsection, then proves Theorem 1.

Proposition 6. *Assume S_1 and S_2 are two planar screens contained in supporting hyperplanes π_1 and π_2, respectively. Assume $u_1 = (e_1, h_2)$ and $u_2 = (e_2, h_2)$ are scattering solutions for the screens S_1 and S_2 corresponding to the same initial field and having equal non-vanishing far fields. Then, $\pi_1 = \pi_2$.*

Proof. Let ρ_1 and ρ_2 be the jumps of $\nu_1 \times h_1$ and $\nu_2 \times h_2$ across S_1 and S_2, respectively. Here, ν_i is the specified unit normal to S_1. Since u_1 and u_2 have equal far fields and $\mathbb{R}^3 \setminus (S_1 \cup S_2)$ is connected, we must have $u_1 = u_2$ there. Hence, both fields must be smooth across $(S_1 \cup S_2) \setminus (S_1 \cap S_2)$, i.e., both densities ρ_1 and ρ_2 are supported in the intersection $S_1 \cap S_2$. If the planes π_1 and π_2 intersect transversally, the jumps are supported on a codimension 2 subspace, and since they belong to $\dot{\mathrm{TH}}^{-1/2}(S_1 \cup S_2)$, they must vanish (Note that a non-vanishing, compactly supported distribution density on a codimension 2 submanifold of $\mathbb{R}^3$ belongs to H^s if and only if $s < -1$. This follows, for example, from estimates at the end of Section 7.1 in [16] by applying these to a suitable dyadic decomposition.) if the intersection is transversal, so the far fields also vanish. □

4. Conclusions

In this article, we proved that a non-vanishing far-field pattern of a single plane wave uniquely determines a planar super-conducting screen. The proof was based on reduction of the scattering problem to a single tangential integral equation on the screen where the unknown is the jump of the tangential component of the scattered magnetic field. We showed that the far field uniquely determines the jump, and that screen is what supports the jump. We plan to generalise this to compact, real-analytic screens in a future work.

However, this will require more advanced techniques. As a possible application, we mention the following problem: Suppose we have an inaccessible array of radars from which we can only obtain distant data. Such information could be, say, whether the array uses classical dipole antennas or more advanced tripole antennas [20]. Our result indicates that such information can, in principle, be obtained with a single measurement.

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Appendix 3

II

Blåsten. E, Päivärinta. L, Sadique. S, "The Fourier, Hilbert, and Mellin Transforms on a Half-Line", *SIAM Journal on Mathematical Analysis*, vol. 55, p. 7529-7548, 2023, 10.1137/23M1560628

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THE FOURIER, HILBERT, AND MELLIN TRANSFORMS ON A HALF-LINE*

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Abstract. We are interested in the singular behavior at the origin of solutions to the equation $\mathscr{H}\rho = e$ on a half-axis, where $\mathscr{H}$ is the one-sided Hilbert transform, ρ an unknown solution, and e a known function. This is a simpler model problem on the path to understanding wave field singularities caused by curve-shaped scatterers in a planar domain. We prove that ρ has a singularity of the form $\mathscr{M}[e](1/2)/\sqrt{t}$, where $\mathscr{M}$ is the Mellin transform. To do this, we use specially built function spaces $\mathscr{M}'(a,b)$ by Zemanian, and these allow us to precisely investigate the relationship between the Mellin and Hilbert transforms. Fourier comes into play in the sense that the Mellin transform is simpy the Fourier transform on the locally compact Abelian multiplicative group of the half-line, and as a more familiar operator, it guides our investigation.

Key words. half-line, Mellin transform, singular behavior, vertical strip, unique solution

MSC codes. 46F12, 68R10, 68U05

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1. Introduction.

In the present article, we let R. H. Mellin meet J. B. J. Fourier and D. Hilbert. More exactly, we study the connection of the Mellin transform to the Hilbert and Fourier transforms in a half-axis $\mathbb{R}_+ = (0,\infty)$. Mellin defined his transform in 1886 [12] in connection with his studies on certain difference and differential equations. A bit more than a decade later, Hilbert presented a new singular integral transform [10] at the third International Congress of Mathematicians, 1904, where he gave a lecture about the Riemann–Hilbert problem. Fourier's work preceded these works of Mellin and Hilbert by more than 60 years [7].

The classical Hilbert transform on the real line is defined by the formula

$$\mathscr{H}f(x) = \text{Cauchy Principal value integral} \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy. \tag{1}$$

The connection to the Fourier transform $\mathscr{F}$ is the well-known formula

$$\mathscr{F}(\mathscr{H}f)(\xi) = i\,\mathrm{sgn}\,\xi\,\widehat{f}(\xi), \tag{2}$$

where $\widehat{f} = \mathscr{F}f$; see [11, 19, 20]. However, the Mellin transform is defined on a half-axis, and the connection to the Hilbert transform, and especially to the Fourier transform, is less widely known, despite being a quite old result [6, 9]. The secret to these connections is lying on the fact that the half-axis is a locally compact Abelian group with respect to multiplication. The Fourier transform is well defined in all such

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groups, and the convolution theorem holds [18]. Since the one-sided Hilbert transform [11] satisfies

$$\mathscr{H} f(t) = \text{p.v.} \int_0^\infty \frac{f(t/s)}{1-s} ds,$$

which is a convolution in the multiplicative group $(\mathbb{R}_+, \cdot)$, we have discovered the connection of the Hilbert and Fourier transforms[1] in $\mathbb{R}_+$. It remains to find out the Fourier transform in $\mathbb{R}_+$. After this lengthy introduction, it should be no big surprise that it is exactly the Mellin transform. All of this is explained with more detail in section 2 below.

In this article, we are interested in the so-called one-sided Hilbert transform

$$\mathscr{H} f(x) = \text{p.v.} \frac{1}{\pi} \int_0^{+\infty} \frac{f(y)}{x-y} dy. \tag{3}$$

Other terminology for this transform is the reduced Hilbert transform, the half-Hilbert transform, or the semi-infinite Hilbert transform [11, section 12.7]. Our interest is in understanding the existence, uniqueness, and behavior at the origin of solutions ρ to the inhomogenous equation

$$\mathscr{H} \rho = e \tag{4}$$

for a given e.

Equation (4) has previously been studied in a classical context, with ρ and e being classically smooth or Lebesgue integrable. See, for example, [5, 14, 17, 16]. These references have a practical point of view, with emphasis on computations or asymptotic expansions. Our motivation is to understand the singular behavior of the solution in cases where the righthand side might not be smooth or integrable in the classical sense. The motivation for this comes from studying scattering of quantum or acoustic waves from a crack or screen in a two-dimensional domain. The three-dimensional problem for a flat two-dimensional scattering screen was studied in [4]. In that paper, an incident probing wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ reacts with a screen S, and, as a consequence, a scattered wave u_s is emitted. These are tied together mathematically as follows:

$$(\Delta + k^2)u_s = 0, \qquad \mathbb{R}^3 \setminus \overline{S}, \tag{5}$$

$$u_i(x) + u_s(x) = 0, \qquad x \in S, \tag{6}$$

$$r\Big(\frac{\partial}{\partial r} - ik\Big)u_s = 0, \qquad r \to \infty, \tag{7}$$

where $r = |x|$ and the limit is uniform over all directions $\hat{x} = x/r$ as $r \to \infty$. The research question was whether the *far-field pattern* of u_s uniquely determines the shape S. Analyzing the problem leads to studying the support of a generalized function ρ, which satisfies an integral equation of the form

$$-\int_S \Phi(x-y)\rho(y)d\sigma(y) = u_i(x), \tag{8}$$

where Φ is the Green's function for $\Delta + k^2$ in three dimensions. Notice how it is analogous to (4). The methods in [4] apply to flat scatterers. For more general

[1]This is why we study the Hilbert transform on a half-axis and not on a finite interval as in section 4 of [23] or in [2].

objects, it is fruitful to study the singular behavior of solutions to inhomogenous integral equations as above; see [1, 8, 21] and the references therein related to the crack problem for the conductivity equation. The problem has yet to be solved in the acoustic setting.

This study is our first step into understanding the singular behavior of waves near the endpoint of cracks or screens in an acoustic medium. Simplifying the applied problem leads to the study of $\mathscr{H}\rho = e$ on the half-line in a class of generalized functions. Our approach is to use the Mellin transform

$$\mathscr{M}[f](s) = \int_0^\infty f(t)t^{s-1}dt \tag{9}$$

defined for generalized functions. We follow the approach of Zemanian [24]. See sections 3 and 4 for more details. We then see how the Hilbert transform applies to these generalized functions in section 5. In section 6, we prove the following theorems, but first, some explanation of the notation. An intuitive way of thinking of these spaces is that $u \in \mathscr{M}'(a,b)$ if informally

$$\begin{aligned} u(t) &= O(t^{-a}), \qquad t \to 0, \\ u(t) &= O(t^{-b}), \qquad t \to \infty. \end{aligned}$$

A more precise understanding is that $u \in \mathscr{M}'(a,b)$ if the Mellin transform $\mathscr{M}[u](s)$ is holomorphic in the vertical strip $s \in S(a,b)$ defined by $a < \Re(s) < b$ and has polynomial growth on vertical lines. This is enough to understand our theorems.

THEOREM 1.1. *Let $e \in \mathscr{M}'(a,b)$ with $0 \le a < b \le 1$. If $b \le 1/2$ or $1/2 \le a$ or $a < 1/2 < b$ and $\mathscr{M}[e](1/2) = 0$, then*

$$\mathscr{H}\rho = e$$

has a unique solution $\rho = \rho_0 \in \mathscr{M}'(a,b)$. Furthermore, if $\rho' \in \mathscr{M}'(a',b')$ is another solution with $S(a',b') \subset S(a,b)$, then $\rho' = \rho_0$ in $\mathscr{M}'(a',b')$.

THEOREM 1.2. *Let $e \in \mathscr{M}'(a,b)$ with $0 \le a < 1/2 < b \le 1$ and $\mathscr{M}[e](1/2) \ne 0$. Then, $\mathscr{H}\rho = e$ has no solutions ρ whose Mellin transform contains $s = 1/2$ in its strip of holomorphicity. Instead, there are unique solutions $\rho_- \in \mathscr{M}'(a,1/2)$ and $\rho_+ \in \mathscr{M}'(1/2,b)$, and they satisfy*

$$\rho_+(t) - \rho_-(t) = \frac{4}{\pi}\mathscr{M}[e](1/2)\frac{1}{\sqrt{t}}. \tag{10}$$

Furthermore, if $\rho' \in \mathscr{M}'(a',b')$ is another solution with $S(a',b')$ intersecting $S(a,1/2)$ or $S(1/2,b)$, then $\rho' = \rho_-$ or $\rho' = \rho_+$ in $\mathscr{M}'(a',b')$, respectively.

Equation (10) shows that ρ_+ has a singularity of type $t^{-1/2}$ unless the Mellin transform of e vanishes at $s = 1/2$. This suggests that acoustically scattered waves from most cracks or screens will have a singularity at their ends. However, if

$$e(t) = \begin{cases} e^{i\sqrt{t}}, & 0 \le t \le (2\pi)^2, \\ 0, & t > (2\pi)^2, \end{cases}$$

it turns out that $\mathscr{M}[e](1/2) = 0$. In this case, some incident plane wave might not have as strong a singularity at $t = 0$ for the curve $\Gamma(t) = (t, \sqrt{t})$ as for most other curves or incident waves. Further analysis is needed and will appear in forthcoming papers, but in this paper, we focus on the intrinsic properties of the one-sided Hilbert transform.

One might wonder what the role of the point $s = 1/2$ is in the theorems above. It arises as the only zero of the Mellin transform $\cot(\pi s)$ of the kernel of the Hilbert transform $\mathscr{H}$ that is in the strip $0 < \Re s < 1$. This strip comes from the technical proof showing that the kernel p. v. $1/(1-t)$ is Mellin transformable; see Lemma 5.1.

2. Hilbert and Mellin transforms for measurable functions.

In this section, we define the Hilbert transform and Mellin transform in $\mathbb{R}_+$ and establish their connection. Before that, we recall some known facts about Fourier transforms on locally compact Abelian groups. Then, we show that in the case of the multiplicative group $(\mathbb{R}_+, \cdot)$, we get exactly the Hilbert transform.

Definition of the LCA and Haar measure. Let $G = (X, \cdot)$ be any locally compact Abelian (LCA) group. Usually [18] the group operation is denoted by addition and identity element by 0. Since our main interest is the multiplicative group $G_+ = (\mathbb{R}_+, \cdot)$ we denote the group operation by a product xy, $x, y \in X$ and by 1 the identity element.

It is well known that there exists a measure m on X that is invariant in the group action; i.e.,

$$m(xE) = m(E) \tag{11}$$

for every $x \in X$ and every Borell set E. Such a measure is called the *Haar measure*, and it is unique up to a positive constant. If m and m' are two Haar measures on G, then $m' = \lambda m$ for some $\lambda > 0$. It is quite easy to see that, in $G_+ = (\mathbb{R}_+, \cdot)$, the Haar measure is dt/t, i.e., the measure m with

$$m(E) = \int_E \frac{dt}{t} \tag{12}$$

for any Borell set in $\mathbb{R}_+$. If m is a Haar measure on an LCA group G, we write $L^p(G)$ instead of $L^p(m)$. Note that

$$\|f\|_{L^p(G)} = \left(\int_X |f(x)|^p dm(x) \right)^{1/p} \tag{13}$$

is scaling invariant: If $f_x(y) = f(yx^{-1})$, then $\|f_x\|_{L^p(G)} = \|f\|_{L^p(G)}$. In particular, for G_+, we have $f_t(s) = f(s/t)$ and

$$\int_{\mathbb{R}_+} |f_t(s)|^p \frac{ds}{s} = \int_{\mathbb{R}_+} |f(s)|^p \frac{ds}{s}, \tag{14}$$

which can, of course, also be obtained directly by changing variables.

Fourier transforms in an LCA group. If $G = (X, \cdot)$ is an LCA group, we call a function $\gamma : X \to \mathbb{C}$ a *character*, if $|\gamma(x)| = 1$ for all $x \in X$ and

$$\gamma(x \cdot y) = \gamma(x)\gamma(y) \tag{15}$$

for every $x, y \in X$. So, a character on G is a homomorphism from G to T, where T is the group of rotations of the unit circle in the complex plane.

The set of all characters on a given LCA group is denoted by Γ. We equip it with multiplication

$$(\gamma_1\gamma_2)(x) = \gamma_1(x)\gamma_2(x) \tag{16}$$

for $x \in X$. This makes Γ a group. It is called the *dual group* of G.

We are ready to define the Fourier transform of $f \in L^1(G)$ by

$$\widehat{f}(\gamma) = \int_X f(x)\gamma(x^{-1})dm(x) \tag{17}$$

for $\gamma \in \Gamma$. We denote

$$\gamma(x) = (x, \gamma) \tag{18}$$

from now on.

Example 2.1.

1. If $G = (\mathbb{R}, +)$, we have, for $\xi \in \mathbb{R}$, that
$$\gamma_\xi(x) = e^{ix\xi}$$
is a character, and by denoting γ_ξ simply by ξ, the Fourier transform turns out to be
$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$
Hence, the dual group of $(\mathbb{R}, +)$ is $(\mathbb{R}, +)$ itself.
2. If $G = T$, the dual group is $(\mathbb{Z}, +)$, and
$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta.$$
3. By the Pontryagin duality theorem, the dual group of $\mathbb{Z}$ is T, and
$$\widehat{f}(e^{ix}) = \int_{-\infty}^{\infty} f(n)e^{-inx}dm_{\mathbb{Z}}(n) = \sum_{n=-\infty}^{\infty} f(n)e^{-inx}.$$

The *convolution* of $f \in L^1(G)$ and $g \in L^p(G)$, $1 \le p < \infty$ is defined as

$$f * g(x) = \int_X f(xy^{-1})g(y)dm(y), \tag{19}$$

and the *convolution theorem*

$$\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma) \tag{20}$$

holds in any LCA group [18].

To find out the Fourier transform in the group of our main interest, $G_+ = (\mathbb{R}_+, \cdot)$, we need to find its dual space Γ. But this is simple: For $z = ix$, $x \in \mathbb{R}$, define

$$\gamma_z(t) = t^z = t^{ix}, \qquad t \in \mathbb{R}_+. \tag{21}$$

Clearly, this is a character in G_+ since

$$\gamma_z(ts) = (ts)^{ix} = t^{ix}s^{ix}$$

for $s, t \in \mathbb{R}_+$.

It is not difficult to see [18, section 2.2] that there are no other characters. Hence, we can interpret that the dual group of G_+ is the additive imaginary axis of the complex plane, and the Fourier transform is given by

$$\widehat{f}(z) = \int_0^{\infty} t^z f(t)\frac{dt}{t} \tag{22}$$

for $f \in L^1(G_+)$ and $z \in i\mathbb{R}$. But this is exactly the definition of the Mellin transform [12, 22] whenever the righthand side is integrable. Thus, we have shown that the Mellin transform is nothing other than the Fourier transform in the multiplicative group on $\mathbb{R}_+$. Accordingly, all the results for the Fourier transforms in LCA groups, such as Plancherel's theorem, the inversion formula, and convolution theorem, follow now, as a matter of routine, from the general theory of Fourier analysis in LCA groups [18]. The connection to the Hilbert transform is in the formula

$$\mathscr{H} f(t) = \text{p.v.} \int_0^\infty \frac{1}{1-t/s} f(s) \frac{ds}{s} = h \vee f(t), \tag{23}$$

where $h = \text{p.v.} \frac{1}{1-t}$ and $\vee$ stands for the Mellin convolution $(\mathbb{R}_+, \cdot)$. The convolution theorem suggests that (23) implies that the Mellin transform of $\mathscr{H} f$ is

$$\mathscr{M}\mathscr{H} f(z) = \widehat{h}(z)\widehat{f}(z) = \cot(\pi z)\widehat{f}(z), \tag{24}$$

where $\widehat{\ }$ is the Fourier transform on the LCA group $(\mathbb{R}_+, \cdot)$, or, in other words, the Mellin transform. The second equality follows from Example 8.24.II in [15],

$$\text{p.v.} \int_0^\infty t^z \frac{1}{1-t} \frac{dt}{t} = \pi \cot(\pi z). \tag{25}$$

The problem is that h is not a function but a proper distribution. The theory of distributions does not exist for general LCA groups, and we must develop the theory for Mellin and Hilbert transforms specifically for the group $(\mathbb{R}_+, \cdot)$. This is done in the sections below.

Implications of LCA group theory. To end this introduction, we give an exercise on how to use this new connection of the Fourier transform in LCA groups and the Mellin transform to prove generally challenging results. For the reader's convenience, we also give its solution.

EXERCISE 2.1. *Assume that $f \in L^1(\mathbb{R}_+, dt/t)$ and that its Mellin transform $\mathscr{M} f \in L^1(i\mathbb{R})$. Then, f must be continuous, and*

$$\lim_{t \to 0+} f(t) = 0. \tag{26}$$

Before giving a solution, we make two remarks about the result. It is relatively easy to construct a function in $L^1(\mathbb{R}_+, dt/t)$ that is continuous, but the limit in (26) does not exist. We can even construct it so that it is positive and unbounded. However, if the limit exists, then it must be equal to zero.

SOLUTION. *We denote $G_+ = (\mathbb{R}_+, dt/t)$ and by Γ_+ its dual group $(i\mathbb{R}, +)$. For any LCA group G, the Fourier transform $\widehat{f}$ of a function belonging to $L^1(G, m)$, m being a Haar measure, is in the space $C_0(\Gamma)$, where Γ is its dual group and $C_0(\Gamma)$ is the closure of compactly supported continuous functions in $L^\infty(\Gamma)$* [18, section 1.2.3]. *Hence, in our case, $\widehat{f} \in L^1(i\mathbb{R}) \cap C_0(i\mathbb{R})$. We do not need this to solve the exercise but use instead* Pontryagin's duality theorem [18, section 1.5] *to get first $f(t) = \mathscr{F} g(-t)$, where g is the Fourier transform of f, namely, $g = \widehat{f}$. Next, we apply the above result in the context of the dual pair (Γ_+, G_+) instead of the original pair (G_+, Γ_+). We finally obtain that $f \in C_0(G_+)$, which means that f is continuous and $f(t) = 0$ when $t \to 0$.*

3. Space of Mellin transformable distributions. In this section, we define a class of distributions on the positive real axis. The Mellin transform of these distributions will be functions that are holomorphic on a vertical strip in the complex plane and also polynomially bounded as the imaginary part of the argument grows. This class of distributions will be denoted by $\mathscr{M}'(a_1, a_2)$, where $a_1, a_2 \in \mathbb{R}$ defines the strip of holomorphicity. The construction is analogous to how tempered distributions $\mathscr{S}'(\mathbb{R})$ are defined for extending the range of the Fourier transformation.

The strategy is loosely described in [3], which follows [13]. The general idea is to define spaces of ordinary smooth test functions on $\mathbb{R}_+$ that contain compactly supported smooth test functions $\mathscr{D}(\mathbb{R}_+)$ and also functions of the form t^{s-1} for some complex numbers s. One then defines the duals of these as the spaces of interest. We note that both [3] and [13] are scant on the precise details. In fact, the latter uses the notation $\mathscr{T}_{p,q}$ and implicitly $\mathscr{T}_{\alpha,\omega}$ to mean different things. This causes confusion when applied to real cases. For example, the function $g(t) = 1$ for $0 < t < 1$ and $g(t) = 0$ for $t \geq 1$ belongs to $\mathscr{T}_{0,1}$ when interpreted in the latter way but not in the former. A more reliable reference is [24]. Although the test function spaces are defined differently than in the former references, the final space of Mellin transformable distributions ends up being the same.

Section 11.3.3. in [13] compares their initial test function space $\mathscr{M}_{p,q}$ to spaces $\mathscr{M}(a, b)$ defined by Zemanian in [24] and concludes rightly that the function g above does not belong to $\mathscr{M}'_{0,\infty}$. However, these are not defined in Zemanian; instead, a larger space $\mathscr{M}'(0, \infty)$ is defined, and it does contain that function.

We start by describing a space of test functions that will be used to define the Mellin transform of a class of distributions. This summarizes section 4.2 of Zemanian [24].

DEFINITION 3.1. *Let $a_1 < a_2$ be real numbers. Then, $\mathscr{M}_{a_1,a_2}$ contains all smooth functions $\phi : \mathbb{R}_+ \to \mathbb{C}$ such that, for any $k \in \mathbb{N}$, we have $\|\phi\|_{a_1,a_2,k} < \infty$, where*

$$\|\phi\|_{a_1,a_2,k} = \sup_{0<t<\infty} \zeta_{a_1,a_2}(t) t^{k+1} \left| \frac{d^k}{dt^k} \phi(t) \right|, \tag{27}$$

$$\zeta_{a_1,a_2}(t) = \begin{cases} t^{-a_1}, & 0 < t \leq 1, \\ t^{-a_2}, & 1 < t < \infty. \end{cases} \tag{28}$$

A sequence $(\phi_j)_{j=1}^{\infty} \subset \mathscr{M}_{a_1,a_2}$ converges to $\phi \in \mathscr{M}_{a_1,a_2}$ if

$$\|\phi_j - \phi\|_{a_1,a_2,k} \to 0 \tag{29}$$

as $j \to \infty$ for each $k = 0, 1, 2, \ldots$.

For $a_1 < a_2$ real or $\pm\infty$, we define $\mathscr{M}(a_1, a_2)$ as follows. A function ϕ is an element of $\mathscr{M}(a_1, a_2)$ if $\phi \in \mathscr{M}_{a,b}$ for some $a_1 < a < b < a_2$. A sequence $(\phi_j)_{j=1}^{\infty} \subset \mathscr{M}(a_1, a_2)$ converges to it if a tail $(\phi_j)_{j=j_0}^{\infty}$, $j_0 \in \mathbb{N}$ converges to ϕ in some fixed space $\mathscr{M}_{a,b}$ with $a_1 < a < b < a_2$.

LEMMA 3.2. *Let a_1, a_2 be real numbers, and let $s \in \mathbb{C}$. Let $\phi(t) = t^{s-1}$ for $t > 0$. Then, $\phi \in \mathscr{M}_{a_1,a_2}$ if and only if $a_1 \leq \Re(s) \leq a_2$. As a consequence, $\phi \in \mathscr{M}(a_1, a_2)$ if and only if $a_1 < \Re(s) < a_2$.*

Proof. We have

$$t^{k+1-a_1} \left(\frac{d}{dt} \right)^k \phi(t) = (s-1)(s-2)\ldots(s-k) t^{s-a_1}, \tag{30}$$

and this is bounded in the interval $(0,1)$ if and only if $\Re(s) \geq a_1$. We see similarly that $t^{k+1-a_2}(d/dt)^k \phi(t)$ is bounded on $(1,\infty)$ if and only if $\Re(s) \leq a_2$, which proves the claim. □

The above and the following lemma show that the $\mathscr{M}(a_1,a_2)$, $a_1 < a_2$ are nontrivial. As a consequence of the following, we see that the linear functionals that we are building are in fact distributions $\mathscr{D}'(\mathbb{R}_+)$. We skip the proof. It is worth noting that they allow exponential growth and so cannot be interpreted as tempered distributions.

LEMMA 3.3. *Lets $\mathscr{D}(\mathbb{R}_+)$ be the space of compactly supported smooth test functions on $\mathbb{R}_+$ with the usual topology. Then, $\mathscr{D}(\mathbb{R}_+) \subset \mathscr{M}(a_1,a_2)$ continuously for any $a_1 < a_2$ real or infinite. The inclusion is dense.*

We will introduce the space of distributions, which will form a natural domain for the Mellin transform. For intuition, see section 4.3 in [24].

DEFINITION 3.4. *Let $a_1 < a_2$ be real or infinite. By $\mathscr{M}'(a_1,a_2)$, we mean the space of continuous linear functionals on $\mathscr{M}(a_1,a_2)$. In detail, $u \in \mathscr{M}'(a_1,a_2)$ if the following hold:*

1. *$\langle u,\phi\rangle$ is a complex number for each $\phi \in \mathscr{M}(a_1,a_2)$.*
2. *$\langle u, c_1\phi_1 + c_2\phi_2\rangle = c_1\langle u,\phi_1\rangle + c_2\langle u,\phi_2\rangle$ for all $c_1,c_2 \in \mathbb{C}$ and $\phi_1,\phi_2 \in \mathscr{M}(a_1,a_2)$.*
3. *$\langle u,\phi_j\rangle \to 0$ as $j\to\infty$ if $\phi_j \to 0$ in $\mathscr{M}(a_1,a_2)$*

Furthermore, we say that a sequence $u_j \to 0$ in $\mathscr{M}'(a_1,a_2)$ if $\langle u_j,\phi\rangle \to 0$ in $\mathbb{C}$ for all $\phi \in \mathscr{M}(a_1,a_2)$.

Example 3.1. Let

$$g(t) = \begin{cases} 1, & 0 < t < 1, \\ 0, & t \geq 1. \end{cases} \tag{31}$$

Then, $g \in \mathscr{M}'(a_1,a_2)$ if and only if $a_1 \geq 0$ and $a_2 > a_1$, where the latter is because we have not allowed $a_2 = a_1$ in the definitions. Let $a_2 > a_1 \geq 0$, $\phi \in \mathscr{M}(a_1,a_2)$, and $(\phi_j)_{j=1}^{\infty} \subset \mathscr{M}(a_1,a_2)$ converging to 0 in that space. Definition 3.1 implies that there is a,b such that $a_1 < a < b < a_2$ with $\phi,\phi_j \in \mathscr{M}_{a,b}$ and the latter converging to 0 in that same space. We have not defined it explicitly, but the interpretation of an ordinary function as a potential element of Mellin transformable distributions is by integrating the function multiplied by a test function. We see that

$$\langle g,\phi\rangle = \int_0^1 \phi(t)\,dt = \int_0^1 t^{a-1}t^{0+1-a}\phi(t)\,dt \leq \int_0^1 t^{a-1}\,dt \|\phi\|_{a,b,0} = \frac{1}{a}\|\phi\|_{a,b,0}. \tag{32}$$

The same implies that $\langle g,\phi_j\rangle \leq a^{-1}\|\phi_j\|_{a,b,0} \to 0$ as $j\to\infty$. Hence, $g \in \mathscr{M}'(a_1,a_2)$ when $a_2 > a_1 \geq 0$.

Next, assume that $a_2 > a_1 < 0$ and that $g \in \mathscr{M}'(a_1,a_2)$. Then, there is $p < 0$ such that $a_1 < p < a_2$. Let $\phi(t) = t^{p-1}$. By Lemma 3.2, we see that $\phi \in \mathscr{M}(a_1,a_2)$, but by (32), it is clear that $\langle g,\phi\rangle = \infty$. Hence, $g \notin \mathscr{M}(a_1,a_2)$ when $a_1 < 0$.

Remark 3.5. Lemma 3.3 implies that $\mathscr{M}'(a_1,a_2) \subset \mathscr{D}'(\mathbb{R}_+)$ for any $a_1 < a_2$ and that the inclusion is continuous. However, the converse does not hold because, for example, $t \to t^z$ is in $\mathscr{D}'(\mathbb{R}_+)\backslash\mathscr{M}'(a_1,a_2)$ for any $z \in \mathbb{C}$ and $a_1 < a_2$. Also, it looks like arbitrary elements of

$$L^{2,c}(\mathbb{R}_+) = \left\{ f : \mathbb{R}_+ \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |f(t)|^2 t^{2c-1} dt < \infty \right\} \tag{33}$$

do not belong to $\mathscr{M}'(a_1,a_2)$. However, it may happen that $f \in \mathscr{M}'(a_1,a_2)$ might satisfy $f \in L^{2,c}(\mathbb{R}_+)$, and then, a Plancherel-type theorem involving Mellin transform holds.

Our strategy for this section is the following. We will define the Mellin transform for elements of $\mathscr{M}'(a_1,a_2)$ and then study how the Hilbert transform on $\mathbb{R}_+$ acts on them. After this, we will prove estimates for elements in $L^{2,c}(\mathbb{R}_+)\cap\mathscr{M}'(a_1,a_2)$ (which are dense in $L^{2,c}(\mathbb{R}_+)$). Continuity will then imply the estimates for $L^{2,c}(\mathbb{R}_+)$. Note that $t^{z-1} \in \mathscr{M}(a_1,a_2)$ even though it is not in $\mathscr{M}'(a_1,a_2)$.

4. The Mellin transform for distributions. We are now ready to define the Mellin transform of $u \in \mathscr{M}'(a_1,a_2)$. Recall that if $u \in \mathscr{M}'(a_1,a_2)$ can be represented in the form

$$\langle u,\phi\rangle = \int_0^\infty f_u(t)\phi(t)\,dt, \qquad \phi \in \mathscr{M}(a_1,a_2) \tag{34}$$

for some measurable function $f_u : \mathbb{R}_+ \to \mathbb{C}$, then we identify u and f_u. Recall that the Mellin transform of a measurable function $f : \mathbb{R}_+ \to \mathbb{C}$ is given by

$$\mathscr{M}f(s) = \widetilde{f}(s) = \int_0^\infty f(t)t^{s-1}\,dt \tag{35}$$

for those $s \in \mathbb{C}$ for which the integral converges in the sense of Lebesgue. Inspired by these two observations, we define the following.

Definition 4.1. *Let $a_1,a_2 \in \{-\infty,+\infty\}\cup\mathbb{R}$ with $a_1 < a_2$, and let $u \in \mathscr{M}'(a_1,a_2)$. Then, the Mellin transform of u is*

$$\mathscr{M}u(s) = \widetilde{u}(s) = \langle u, t^{s-1}\rangle \tag{36}$$

for $s \in \mathbb{C}$, $a_1 < \Re(s) < a_2$.

Remark 4.2. Equation (36) is well defined because the test function $\phi(t) = t^{s-1}$ is in $\mathscr{M}(a_1,a_2)$ whenever $a_1 < \Re(s) < a_2$ by Lemma 3.2.

It turns out that the Mellin transform of a distribution in $\mathscr{M}'(a_1,a_2)$ has many nice properties. We summarize some of them. For proofs and details, see [24].

Lemma 4.3. *If $f \in \mathscr{M}'(a_1,a_2)$ with $a_1 < a_2$ real numbers or $-\infty,+\infty$, then $s \mapsto \mathscr{M}f(s)$ is holomorphic in $a_1 < \Re(s) < a_2$.*

Definition 4.4. *When we say $\mathscr{M}f$ has strip of holomorphicity S (or S_f), we mean that*

$$S = \{s \in \mathbb{C} \mid a_1 < \Re(s) < a_2\} \tag{37}$$

for some $a_1 < a_2$ and that $\mathscr{M}f$ is holomorphic on S. If $f \in \mathscr{M}'(a_1,a_2)$ with S as above, we write $f \in \mathscr{M}'_S$ or $f \in \mathscr{M}'_{S_f}$. Also, given $a_1,a_2 \in \mathbb{R}\cup\{-\infty,+\infty\}$, we denote

$$S(a_1,a_2) = \{s \in \mathbb{C} \mid a_1 < \Re(s) < a_2\}. \tag{38}$$

The Mellin transform for distributions has several properties.

Theorem 4.5. *In the following, we assume that $f \in \mathscr{M}'_{S_f}$ and $g \in \mathscr{M}'_{S_g}$. It holds that the following are true:*

1. *If* $n \in \mathbb{N}$, *then* $(-t\,d/dt)^n f \in \mathscr{M}'_{S_f}$ *and* $\mathscr{M}[(-t\,d/dt)^n f](s) = s^n \mathscr{M}[f](s)$.
2. *If* $S_f \cap S_g \neq \emptyset$ *and* $\mathscr{M} f = \mathscr{M} g$ *on* $S_f \cap S_g$, *then* $f = g$ *as distributions in* $\mathscr{M}'_{S_f \cap S_g}$ *and a fortiori in* $\mathscr{D}'(\mathbb{R}_+)$.
3. *A function* $F : S_f \to \mathbb{C}$ *is the Mellin transform of some* $f \in \mathscr{M}'_{S_f}$ *if and only if*
   (a) F *is holomorphic in* S_f *and*
   (b) *for any closed substrip of* S_f *of the form* $\alpha_1 \leq \Re(s) \leq \alpha_2$, *there is a polynomial* P *such that* $|F(s)| \leq P(|s|)$ *on that strip.*
4. *Let* $S_f \cap S_g = \{s \in \mathbb{C} \mid a_1 < \Re(s) < a_2\}$. *Then,*

$$\mathscr{M}[f \vee g](s) = \mathscr{M} f(s) \mathscr{M} g(s), \qquad a_1 < \Re(s) < a_2, \tag{39}$$

where

$$(f \vee g)(\tau) = \int_0^\infty f(t) g\left(\frac{\tau}{t}\right) \frac{dt}{t}, \qquad \tau > 0 \tag{40}$$

if f *and* g *are integrable functions, and otherwise,*

$$\langle f \vee g, \theta \rangle = \langle f, \psi \rangle, \qquad \psi(t) = \langle g, \theta_t \rangle \tag{41}$$

for $\theta \in \mathscr{M}(a_1, a_2)$, $t > 0$, *and* $\theta_t(\tau) = \theta(t\tau)$.

Recall from section 2 that $f \vee g$ in (40) is the convolution of the multiplicative group $(\mathbb{R}_+, \cdot)$, and dt/t is its Haar measure.

The following gives an inversion formula for the Mellin transform.

THEOREM 4.6. *If* $F : S(a_1, a_2) \to \mathbb{C}$ *is holomorphic and satisfies* $|F(s)| \leq K|s|^{-2}$ *for some finite constant* K, *and we set*

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) t^{-s}\, ds \tag{42}$$

for a fixed $\sigma \in (a_1, a_2)$, *then* $f : \mathbb{R}_+ \to \mathbb{C}$ *is continuous, does not depend on the choice of* σ, *and is in* $\mathscr{M}'(a_1, a_2)$. *Furthermore,* $\mathscr{M} f = F$ *on* $S(a_1, a_2)$.

The following corollary is Theorem 4.4.1 in [24]. In that reference, it is used to prove the result that corresponds to item 3 of Theorem 4.5 of our article,[2] and it gives another inversion formula for the cases where the theorem above cannot be applied, namely, if F has a singularity on the border of $S(a_1, a_2)$.

COROLLARY 4.7. *Let* $F : S(a_1, a_2) \to \mathbb{C}$ *be holomorphic, and let* $Q : \mathbb{C} \to \mathbb{C}$ *be a polynomial that has no zeroes in* $S(a_1, a_2)$ *such that*

$$\left| \frac{F(s)}{Q(s)} \right| \leq \frac{K}{|s|^2}, \qquad b_1 < \Re(s) < b_2 \tag{43}$$

for some $a_1 < b_1 < b_2 < a_2$ *and a finite constant* K. *Set*

$$g(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{F(s)}{Q(s)} t^{-s}\, ds \tag{44}$$

for some $b_1 < \sigma < b_2$. *Then,* $g : \mathbb{R}_+ \to \mathbb{C}$ *is continuous and belongs to* $\mathscr{M}'(b_1, b_2)$, *as does* $f(t) = Q(-t\,d/dt) g(t)$. *Furthermore,* $\mathscr{M} f = F$ *on* $S(b_1, b_2)$.

[2]Strictly speaking, this applies to the corresponding results for the Laplace transform. The results from the Mellin transform are only stated.

5. The Hilbert transform. We will need to know the Mellin transform of the distribution

$$\langle H,\phi\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left(\int_0^{1-\varepsilon}+\int_{1+\varepsilon}^{\infty}\right)\frac{\phi(t)}{1-t}dt, \tag{45}$$

namely, $H=\pi^{-1}/(1-t)$ in the principal value sense. It is almost the kernel of the Hilbert transform of a function vanishing on $\mathbb{R}_-$,

$$\mathscr{H}f(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_0^{\infty}\frac{f(y)}{x-y}\,dt. \tag{46}$$

In fact, formally

$$\mathscr{H}f(x) = -\frac{1}{\pi}\,\mathrm{p.v.}\int_0^{\infty}\frac{1}{1-t}f\left(\frac{x}{t}\right)\frac{dt}{t} = -(H\vee f)(x), \tag{47}$$

which can be deduced from (46) by change integration variables $y=x/t$ and $dy=-x\,dt/t^2$.

LEMMA 5.1. *The distribution* $1/(1-t)$ *in the principal value sense belongs to* $\mathscr{M}'(0,1)$. *Furthermore, it can be written as*

$$\left\langle\frac{1}{1-t},\phi\right\rangle = \left(\int_0^{1/2}+\int_{3/2}^{\infty}\right)\frac{\phi(t)}{1-t}dt - \int_{1/2}^{3/2}\frac{\phi(t)-\phi(1)}{t-1}dt, \tag{48}$$

where $1/(1-t)$ *is interpreted as a pointwise function on the righthand side. Lastly, there is a finite* C *such that* $|\langle 1/(1-t),\phi\rangle|\le C(\|\phi\|_{0,1,0}+\|\phi\|_{0,1,1})$.

Proof. Let us denote $u=1/(1-t)$, and recall that the distribution pairings are done with the principal value. We will first prove that $\langle u,\phi\rangle\in\mathbb{C}$ for $\phi\in\mathscr{M}(0,1)$. The latter means there are $0<a<b<1$ such that $\phi\in\mathscr{M}_{a,b}$. In particular, (27) implies that $\|\phi\|_{a,b,0}$ and $\|\phi\|_{a,b,1}$ are finite. Let $h(t)=1$ for $1/2<t<3/2$, and let $h(t)=0$ otherwise. Then,

$$\langle u,\phi\rangle = \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon}+\int_{1+\epsilon}^{\infty}\right)\left(\frac{\phi(t)-\phi(1)h(t)}{1-t}+\frac{\phi(1)h(t)}{1-t}\right)dt; \tag{49}$$

with $s=2-t$, we see that

$$\int_{1/2}^{1-\epsilon}\frac{\phi(1)h(t)}{1-t}dt = \phi(1)\int_{1/2}^{1-\epsilon}\frac{dt}{1-t} = \phi(1)\int_{3/2}^{1+\epsilon}\frac{-ds}{-1+s} = \phi(1)\int_{3/2}^{1+\epsilon}\frac{ds}{1-s}$$
$$= -\int_{1+\epsilon}^{3/2}\frac{\phi(1)h(s)}{1-s}ds,$$

and so, the last integral in (49) vanishes. For the first integral, recall that ϕ is smooth. Hence, the secant $(\phi(t)-\phi(1))/(t-1)$ is a continuous function of t. We see that

$$\left(\int_0^{1-\epsilon}+\int_{1+\epsilon}^{\infty}\right)\frac{\phi(t)-\phi(1)h(t)}{1-t}dt = \left(\int_0^{1/2}+\int_{3/2}^{\infty}\right)\frac{\phi(t)}{1-t}dt + \left(\int_{1/2}^{1-\epsilon}+\int_{1+\epsilon}^{3/2}\right)\frac{\phi(t)-\phi(1)}{1-t}dt. \tag{50}$$

This proves (48), as ϵ can be let equal to zero as the secant is continuous. The first integrand is continuous on $(0,1/2)\cup(3/2,\infty)$. It is also integrable since

$$\begin{aligned}\int_0^{1/2}\left|\frac{\phi(t)}{1-t}\right|dt &\le \int_0^{1/2} t^{a-1}t^{1-a}|\phi(t)|\cdot 2dt \le 2\Big/_0^{1/2}\frac{t^a}{a}\sup_{0<t<1/2} t^{1-a}|\phi(t)| \\ &\le \frac{2^{1-a}}{a}\|\phi\|_{a,b,0} < \infty. \qquad (51)\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{3/2}^{\infty}\left|\frac{\phi(t)}{1-t}\right|dt &\le \int_{3/2}^{\infty}\frac{t^{b-1}}{t-1}t^{1-b}|\phi(t)|dt \le \|\phi\|_{a,b,0}\int_{3/2}^{\infty}\frac{t^{b-1}}{t-1}dt \\ &\le \|\phi\|_{a,b,0}3\int_{3/2}^{\infty}\frac{t^{b-1}}{t}dt = \|\phi\|_{a,b,0}3\Big/_{3/2}^{\infty}\frac{t^{b-1}}{b-1} \\ &= \frac{3(3/2)^{b-1}}{1-b}\|\phi\|_{a,b,0} < \infty \qquad (52)\end{aligned}$$

because $1/(t-1)\le 3/t$ for $t\ge 3/2$.

For the second integrand in (50), note that

$$\left|\frac{\phi(t)-\phi(1)}{t-1}\right| = |\phi'(\xi)| \le \sup_{1/2<t<3/2}|\phi'(t)| \le C\|\phi\|_{a,b,1} < \infty \qquad (53)$$

for some finite constant C. Hence, we can take the limit and have

$$\lim_{\epsilon\to 0}\left(\int_{1/2}^{1-\epsilon}+\int_{1+\epsilon}^{3/2}\right)\frac{\phi(t)-\phi(1)}{1-t}dt = -\int_{1/2}^{3/2}\frac{\phi(t)-\phi(1)}{t-1}dt, \qquad (54)$$

which is bounded by

$$\int_{1/2}^{3/2} C\|\phi\|_{a,b,1}dt = C\|\phi\|_{a,b,1} < \infty. \qquad (55)$$

Hence, $\langle u,\phi\rangle\in\mathbb{C}$ for any $\phi\in\mathscr{M}_{a,b}$ with $0<a<b<1$. Similarly, by our calculation, so far we have $|\langle u,\phi\rangle|\le C\big(\|\phi\|_{a,b,0}+\|\phi\|_{a,b,1}\big)$ for a finite constant C whenever $\phi\in\mathscr{M}_{a,b}$. By (27), we can decrease a and increase b to get

$$|\langle u,\phi\rangle| \le C(\|\phi\|_{0,1,0}+\|\phi\|_{0,1,1})$$

for any $\phi\in\mathscr{M}_{a,b}$. Because this holds for arbitrary $0<a<b<1$, by Definition 3.1, the same estimate holds for all $\phi\in\mathscr{M}(0,1)$. So, the estimate in our claim is proven.

Now, let $(\phi_j)_{j=1}^{\infty}\to 0$ in $\mathscr{M}(0,1)$. This means that there is $0<a<b<1$ such that $(\phi_j)_{j=1}^{\infty}\subset\mathscr{M}_{a,b}$ and $\|\phi_j\|_{a,b,k}\to 0$ as $j\to\infty$ for each $k\in\mathbb{N}$. Thus,

$$|\langle u,\phi_j\rangle| \le C\left(\|\phi_j\|_{a,b,0}+\|\phi_j\|_{a,b,1}\right)\to 0,$$

and continuity is proven. The linearity property is trivial. Hence, $u\in\mathscr{M}'(0,1)$. ◻

LEMMA 5.2. *We have* $\mathscr{M}[1/(1-t)](s)=\pi\cot(\pi s)$ *in the principal value sense for* $0<\Re(s)<1$.

Proof. The distribution is in $\mathscr{M}(0,1)$. All we need to do is to calulate

$$\text{p.v.} \int_0^\infty \frac{t^{s-1}}{1-t} dt. \tag{56}$$

Refer to Example 8.24.II in [15], especially pages 219–220 for the calculations. ∎

DEFINITION 5.3. *For $f \in \mathscr{M}'(a,b)$ with $0 \le a < b \le 1$, define the Hilbert transform by*

$$\mathscr{H} f = -H \vee f, \tag{57}$$

where H is defined in (45) *and $\vee$ in* (41).

LEMMA 5.4. *The Hilbert transform is a well-defined element of $\mathscr{M}'(a,b)$, and if f is smooth and compactly supported in $\mathbb{R}_+$, we have* (46).

Proof. Lemma 5.1 implies that $H \in \mathscr{M}'(0,1)$, and so, Theorem 4.6.1 and the paragraph after it in [24] imply that $H \vee f \in \mathscr{M}'(a,b)$ when $f \in \mathscr{M}'(a,b)$.

Let f be smooth and compactly supported. We will use Theorem 4.6.2 by Zemanian [24]. In the sense of distributions on $\mathbb{R}_+$, we have $H \vee f$ equal to the following smooth function:

$$g(x) := \left\langle H, \frac{1}{t} f\left(\frac{x}{t}\right)\right\rangle_t = \lim_{\varepsilon\to 0} \frac{1}{\pi}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty\right) \frac{f(x/t)}{1-t}\frac{dt}{t}. \tag{58}$$

A change of integration variables $t = x/y$, $dt = -x dy/y^2$ gives

$$-g(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\left(\int_0^{x/(1+\varepsilon)} + \int_{x/(1-\varepsilon)}^\infty\right) \frac{f(y)}{x-y} dy, \tag{59}$$

which equals (46) by the following.

It remains to show that

$$\lim_{\varepsilon\to 0}\left(\int_0^{x/(1+\varepsilon)} + \int_{x/(1-\varepsilon)}^\infty\right) \frac{f(x/t)}{t-1}\frac{dt}{t} = \lim_{\varepsilon\to 0}\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right) \frac{1}{x-y} f(y) dy \tag{60}$$

for all x. We obtain

$$\begin{aligned}\left(\int_0^{x/(1+\varepsilon)} + \int_{x/(1-\varepsilon)}^\infty\right) \frac{f(x/t)}{t-1}\frac{dt}{t} &= \left(\int_0^{x-\varepsilon x} + \int_{x+\varepsilon x}^\infty\right) \frac{1}{x-y} f(y) dy \\ &\quad + \left(\int_{x-\varepsilon x}^{x/(1+\varepsilon)} + \int_{x/(1-\varepsilon)}^{x+\varepsilon x}\right) \frac{1}{x-y} f(y) dy.\end{aligned} \tag{61}$$

For any fixed $x \in (0,\infty)$, the first terms above clearly converges to the righthand side of (60). For $0 < \varepsilon < 1/2$, we have

$$0 < \frac{1}{1+\varepsilon} - (1-\varepsilon) \le \varepsilon^2, \tag{62}$$

$$0 < \frac{1}{1-\varepsilon} - (1+\varepsilon) \le 2\varepsilon^2, \tag{63}$$

$$0 < 1 - \frac{1}{1+\varepsilon} \le \frac{\varepsilon}{2}. \tag{64}$$

Hence, in the term

$$\int_{x-\varepsilon x}^{x/(1+\varepsilon)} \frac{1}{x-y} f(y)dy,$$

we have $|x-y| \geq \varepsilon x/2$ by (64). The length of the integration interval is less than $\varepsilon^2 x$ by (62). It follows that the absolute value of this term has the upper bound

$$\varepsilon^2 x \cdot \frac{2}{\varepsilon x} \max|f| \leq 2\varepsilon \max|f|,$$

and this tends to 0 as $\varepsilon \to 0$. Similarly, using (63) and (64), one can show that

$$\left| \int_{x/(1-\varepsilon)}^{1+\varepsilon x} \frac{1}{x-y} f(y)dy \right| \leq 2\varepsilon \max|f|.$$

We have thus shown that

$$\lim_{\varepsilon\to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty} \right) \frac{f(x/t)}{t-1} \frac{dt}{t} = \mathscr{H} f(x)$$

for every $x \in (0,\infty)$. ∎

The results of this section can be summarized as follows.

THEOREM 5.5. *The Hilbert transform $\mathscr{H}$ applied to test functions $f \in \mathscr{D}(\mathbb{R}_+)$ can be written as*

$$\mathscr{H} f(x) = \mathrm{p.v.}\, \frac{1}{\pi} \int_0^{\infty} \frac{f(y)}{x-y} dy = -\,\mathrm{p.v.}\, \frac{1}{\pi} \int_0^{\infty} \frac{f(x/t)}{1-t} \frac{dt}{t}. \tag{65}$$

Applied to a distribution $u \in \mathscr{M}'(a,b)$ with $0 \leq a < b \leq 1$, it is an element of $\mathscr{M}'(a,b)$ defined by $\mathscr{H} u = -H \vee u$ with

$$\langle H, \phi \rangle = \mathrm{p.v.}\, \frac{1}{\pi} \int_0^{\infty} \frac{\phi(t)}{1-t} dt, \tag{66}$$

$$\langle H \vee u, \theta \rangle = \langle H, \psi \rangle, \qquad \psi(t) = \langle u, \theta_t \rangle,\ \theta_t(s) = \theta(ts) \tag{67}$$

for $\theta \in \mathscr{M}(a,b)$. Lastly, if $u \in \mathscr{M}'(a,b)$ with $0 \leq a < b \leq 1$, then

$$\mathscr{M}[\mathscr{H} u](s) = -\cot(\pi s) \mathscr{M}[u](s) \tag{68}$$

for $a < \Re(s) < b$.

Proof. Equations (65), (66), and (67) are a restatement of Definition 5.3 and Lemma 5.4, the latter of which gives the mapping properties for $\mathscr{H}$ mentioned in the claim. Equation (68) follows from (39) in Theorem 4.5 and Lemma 5.2. ∎

6. Inhomogenous Hilbert transform on a half-line. In this section, we will prove that the solution ρ to

$$\mathscr{H} \rho = e, \qquad \mathbb{R}_+ \tag{69}$$

has a blow-up singularity at $x = 0$ when e is general but in a suitable function space.

LEMMA 6.1. *Let $0 \leq a \leq \alpha < \beta \leq b \leq 1$, and let $e \in \mathscr{M}'(a,b)$ and $\rho \in \mathscr{M}'(\alpha,\beta)$. Assume that* (69) *holds in $\mathscr{M}'(\alpha,\beta)$. If $1/2 \in (\alpha,\beta)$, then $\mathscr{M}[e](1/2) = 0$.*

Proof. Take the Mellin transform of (69). By Theorem 5.5, we have

$$-\cot(\pi s)\mathscr{M}[\rho](s) = \mathscr{M}[e](s)$$

for $\alpha < \Re(s) < \beta$. In particular, this holds at $s = 1/2$ if this point belongs to the interval (α, β). Since $\mathscr{M}'(a,b) \subset \mathscr{M}'(\alpha,\beta)$, we have $\rho, e \in \mathscr{M}'(\alpha,\beta)$. Then, by Lemma 4.3, both $\mathscr{M}[\rho]$ and $\mathscr{M}[e]$ are holomorphic in a complex neighborhood of $s = 1/2$; in particular, $\mathscr{M}[\rho](1/2)$ is a well-defined finite complex number. Since $\cot(\pi/2) = 0$, a value not changed by multiplying with a complex number, we have $\mathscr{M}[e](1/2) = 0$. ◻

LEMMA 6.2. *Let $x, y \in \mathbb{R}$. If x is at least $\varepsilon > 0$ distance from $1/2 + \mathbb{Z}$, then*

$$\left|\tan\left(\pi(x+iy)\right)\right|^2 \le \left(\cos\pi(1-2\varepsilon) + 1\right)^{-2}, \tag{70}$$

which is finite when such an x exists. Otherwise, if $|y| = M > 0$, we have

$$\left|\tan\left(\pi(x+iy)\right)\right|^2 \le \left(1 - (\cosh(2\pi M))^{-1}\right)^{-2}, \tag{71}$$

which is always finite and at most 4 when $M > 1/\pi$.

Proof. We start with the trigonometric identity

$$\tan\left(\pi(x+iy)\right) = \frac{\sin(2\pi x) + i\sinh(2\pi y)}{\cos(2\pi x) + \cosh(2\pi y)}. \tag{72}$$

Taking the square of the modulus and using $\sinh^2(2\pi y) = \cosh^2(2\pi y) - 1$, we get

$$\left|\tan\left(\pi(x+iy)\right)\right|^2 = \frac{\sin^2(2\pi x) + \cosh^2(2\pi y) - 1}{\left(\cos(2\pi x) + \cosh(2\pi y)\right)^2}. \tag{73}$$

If x is at least distance $\varepsilon > 0$ from $1/2 + \mathbb{Z}$, we must have $0 < \varepsilon \le 1/2$. Then, $\cos(2\pi x) \ge \cos(2\pi(1/2 - \varepsilon))$, and since $\cosh(2\pi y) \ge 1$ and $\sin^2(2\pi x) \le 1$, we get

$$\left|\tan\left(\pi(x+iy)\right)\right|^2 \le \frac{\cosh^2(2\pi y)}{\left(\cos\pi(1-2\varepsilon) + \cosh(2\pi y)\right)^2}.$$

This implies (70) after reducing the fraction by its numerator, noting that $-1 < \cos\pi(1-2\varepsilon) \le 0$ and using $\cosh(2\pi y) \ge 1$.

Now, if we just have $|y| = M > 0$, we can estimate $\cos(2\pi x) \ge -1$ and $\sin^2(2\pi x) \le 1$ in (73) to get

$$\left|\tan\left(\pi(x+iy)\right)\right|^2 \le \frac{\cosh^2(2\pi y)}{\left(-1 + \cosh(2\pi y)\right)^2}.$$

However, since $M > 0$ and the evenness of the hyperbolic cosine, we have $\cosh(2\pi y) = \cosh(2\pi M) > 1$, so the righthand side is a finite constant depending on M. The last claim follows since $M > 1/\pi$ implies that $\cosh(2\pi M) > 2$. ◻

LEMMA 6.3. *Let $e \in \mathscr{M}'(a,b)$ for some $0 \le a < b \le 1$. If*

$$a < b \le 1/2 \qquad \text{or} \qquad 1/2 \le a < b \qquad \text{or} \qquad \mathscr{M}[e](1/2) = 0,$$

then there is $\rho \in \mathscr{M}'(a,b)$ satisfying $\mathscr{H}\rho = e$. Furthermore, for any α, β, c with $a < \alpha < c < \beta < b$ for this ρ, it holds that

$$\rho(t) = \frac{-1}{2\pi i}\left(-t\,d/dt\right)^{m+2} \int_{c-i\infty}^{c+i\infty} s^{-m-2}\tan(\pi s)\mathscr{M}[e](s)t^{-s}ds \tag{74}$$

in $\mathscr{M}'(\alpha,\beta)$. Here, $m \in \mathbb{N}$ can be any number for which there is a polynomial P of degree m such that $|\mathscr{M}[e](s)| \le P(|x|)$ on $S(\alpha,\beta)$.

In the case where

$$a < 1/2 < b \qquad \text{and} \qquad \mathscr{M}[e](1/2) \neq 0,$$

there are no solutions in any $\mathscr{M}'(\alpha,\beta)$ with $\alpha < 1/2 < \beta$. Instead, there is $\rho_- \in \mathscr{M}'(a,1/2)$ and $\rho_+ \in \mathscr{M}'(1/2,b)$ such that $\mathscr{H}\rho_\pm = e$ in $\mathscr{M}'(a,1/2)$ and $\mathscr{M}'(1/2,b)$, respectively. They satisfy

$$\rho_-(t) = \frac{-1}{2\pi i}\,(-t\,d/dt)^{m+2}\int_{c_- - i\infty}^{c_- + i\infty} s^{-m-2}\tan(\pi s)\mathscr{M}[e](s)t^{-s}ds, \tag{75}$$

$$\rho_+(t) = \frac{-1}{2\pi i}\,(-t\,d/dt)^{m+2}\int_{c_+ - i\infty}^{c_+ + i\infty} s^{-m-2}\tan(\pi s)\mathscr{M}[e](s)t^{-s}ds \tag{76}$$

in $\mathscr{M}'(\alpha_-,\beta_-)$ and $\mathscr{M}'(\alpha_+,\beta_+)$, respectively, for any $a < \alpha_- < c_- < \beta_- < 1/2$ and $1/2 < \alpha_+ < c_+ < \beta_+ < b$. Here, $m \in \mathbb{N}$ can be any number for which there is a polynomial P of degree m such that $|\mathscr{M}[e](s)| \le P(|x|)$ on $S(\alpha_-,\beta_+)$.

Proof. Write $F(s) = -\tan(\pi s)\mathscr{M}[e](s)$. Then, $F : S(a,b) \to \mathbb{C}$ is holomorphic everywhere except at $s = 1/2$ if $\mathscr{M}[e](1/2) \neq 0$. We want to use the Mellin transform inversion formula. For that, we need to show an estimate for $|F(s)|$ that holds uniformly in a vertical strip of the complex plane.

Let us first consider the case "$a < b \le 1/2$, $1/2 \le a < b$, or $\mathscr{M}[e](1/2) = 0$." In that case, F is holomorphic on $S(a,b)$. We want to let ρ be the inverse Mellin transform of F, but for that, we need to prove some estimates first so that we can use item 3 of Theorem 4.5.

Consider an arbitrary closed substrip $\alpha_1 \le \Re(s) \le \alpha_2$ of $S(a,b)$. If it contains $s = 1/2$, then our assumptions imply that $\mathscr{M}[e](1/2) = 0$, in which case $|F(1/2)| < \infty$ so that there is $r > 0$ and $C < \infty$ such that $|F(s)| < C$ when $|s - 1/2| < r$. When $|s - 1/2| \ge r$, we have

$$|\tan(\pi s)| \le C_r$$

by Lemma 6.2. Furthermore, there is some polynomial P such that

$$|\mathscr{M}[e](s)| \le P(|s|) \tag{77}$$

on that closed vertical strip by item 3 of Theorem 4.5. In both cases, whether $\alpha_1 \le 1/2 \le \alpha_2$ or not, there is thus some finite constant K for which

$$|F(s)| \le K\big(1 + P(|s|)\big) \tag{78}$$

when $\alpha_1 \le \Re(s) \le \alpha_2$. Because this is an arbitrary vertical closed substrip of $S(a,b)$, then, by the same item of the same theorem, we see that there is $\rho \in \mathscr{M}'(a,b)$ such that $\mathscr{M}\rho = F$ on $S(a,b)$.

Next, by the Mellin transform formula for the Hilbert transform of Theorem 5.5, we have

$$\mathscr{M}[\mathscr{H}\rho](s) = -\cot(\pi s)(-\tan(\pi s))\mathscr{M}[e](s) = \mathscr{M}[e](s) \tag{79}$$

for $s \in S(a,b)$. So, by the uniqueness of the inverse Mellin transform (item 2 of Theorem 4.5), we have $\mathscr{H}\rho = e$ in $\mathscr{M}'(a,b)$. Next, let α,β,c be as in the assumptions.

Then, as in (77), we see that there is a polynomial P such that $|\mathscr{M}[e](s)| \le P(|s|)$ for $\alpha \le \Re(s) \le \beta$. Let $Q(x) = x^{2+m}$ and $m = \deg P$. By the estimate for $|F(s)|$ from (78), we have

$$\left|\frac{F(s)}{Q(s)}\right| \le \frac{K\big(1+P(|s|)\big)}{|s|^2|s|^m} \le \frac{C}{|s|^2}$$

when $\alpha \le \Re(s) \le \beta$. If we set

$$f(t) = \frac{1}{2\pi i}(-td/dt)^{m+2}\int_{c-i\infty}^{c+i\infty} s^{-m-2}F(s)t^{-s}ds,$$

then the integral gives a continuous function $\mathbb{R}_+ \to \mathbb{C}$ that is in $\mathscr{M}'(\alpha,\beta)$, and also, $f \in \mathscr{M}'(\alpha,\beta)$ satisfies $\mathscr{M}f = F$ in $S(\alpha,\beta)$ by Corollary 4.7. Because $\mathscr{M}\rho = F$ in $S(a,b)$, we have $f = \rho$ in $\mathscr{M}'(\alpha,\beta)$ by item 2 of Theorem 4.5. This concludes the proof of the first case. In the case where $a < 1/2 < b$ and $\mathscr{M}[e](1/2) \ne 0$, there are no solutions in $\mathscr{M}'(a,b)$ by Lemma 6.1. Note also that, in this case, F is holomorphic in $S(a,1/2)$ and $S(1/2,b)$ while having a singularity at $s = 1/2$. Consider the closed vertical strips $\alpha_1 \le \Re(s) \le \alpha_2$ and $\beta_1 \le \Re(s) \le \beta_2$ for arbitrary $a < \alpha_1 < \alpha_2 < 1/2$ and $1/2 < \beta_1 < \beta_2 < b$. As in the first part of the proof, we see that

$$|\tan(\pi s)| \le C_{\alpha_2,\beta_1}$$

by Lemma 6.2 when s belongs to either of these two closed strips because $\alpha_2 < 1/2$ and $1/2 < \beta_1$. As before, we have

$$|\mathscr{M}[e](s)| \le P(|s|)$$

on $\alpha_1 \le \Re(s) \le \beta_2$ by item 3 of Theorem 4.5. These two estimates give a polynomial upper bound for $|F(s)|$ on $\alpha_1 \le \Re(s) \le \alpha_2$ and on $\beta_1 \le \Re(s) \le \beta_2$ as in the first part of the proof. Since the closed substrips were arbitrary, these then imply the existence of $\rho_- \in \mathscr{M}'(a,1/2)$ and $\rho_+ \in \mathscr{M}'(1/2,b)$ satisfying $\mathscr{H}\rho_\pm = e$ in $\mathscr{M}'(a,1/2)$ and $\mathscr{M}'(1/2,b)$, respectively. With identical deductions as in the first part of the poof, we see the integral representation formulas for $\rho_\pm$ in $\mathscr{M}'(\alpha_\pm,\beta_\pm)$. ∎

LEMMA 6.4. *Let $0 \le a < b \le 1$, and let $\rho_1, \rho_2 \in \mathscr{M}'(a,b)$. If*

$$\mathscr{H}\rho_1 = \mathscr{H}\rho_2,$$

then $\rho_1 = \rho_2$ in $\mathscr{M}'(a,b)$.

Proof. By taking the Mellin transform of the equation and using the transformation properties of the Hilbert transform from Theorem 5.5, we see that

$$-\cot(\pi s)\mathscr{M}[\rho_1](s) = -\cot(\pi s)\mathscr{M}[\rho_2](s)$$

for $s \in S(a,b)$. When $s \ne 1/2$, we can divide by the cotangent and get

$$\mathscr{M}[\rho_1](s) = \mathscr{M}[\rho_2](s)$$

for $s \in S(a,b) \setminus \{1/2\}$. But $\rho_1 - \rho_2 \in \mathscr{M}'(a,b)$, so $\mathscr{M}[\rho_1] - \mathscr{M}[\rho_2]$ is holomorphic in $S(a,b)$. Hence, the equality holds in the whole $S(a,b)$. According to the properties of the Mellin transform in Theorem 4.5, we have $\rho_1 = \rho_2$ in $\mathscr{M}'(a,b)$. ∎

LEMMA 6.5. *The residue of* $\tan(\pi s)$ *at* $s = 1/2$ *is given by*

$$\operatorname{Res}\big(\tan(\pi s), 1/2\big) = -\frac{1}{\pi}.$$

Proof. We have $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, so the residue is given by the cosine. Then,

$$\lim_{s\to 1/2} \frac{s-1/2}{\cos(\pi s)} = \lim_{s\to 1/2} \frac{1}{\pi} \frac{\pi s - \pi/2}{\cos(\pi s) - \cos(\pi/2)} = \frac{1}{\pi} \lim_{\xi\to\pi/2} \frac{1}{\frac{\cos\xi - \cos(\pi/2)}{\xi - \pi/2}}$$
$$= \frac{1}{\pi} \frac{1}{\cos'(\pi/2)} = -\frac{1}{\pi} \frac{1}{\sin(\pi/2)} = -\frac{1}{\pi}.$$

Thus, $\operatorname{Res}(\tan(\pi s), 1/2) = \lim_{s\to 1/2}(s-1/2)\tan(\pi s) = -1/\pi$. ◻

LEMMA 6.6. *Let* $0 < \alpha < 1/2 < \beta < 1$, *and let* $f : S(\alpha,\beta) \to \mathbb{C}$ *be holomorphic with* $|f(s)| \le Cs^m$ *for some* $m \in \mathbb{N}$. *For* $\alpha < c_- < 1/2 < c_+ < \beta$, *define*

$$\bar\rho_-(t) = \frac{-1}{2\pi i} \int_{c_- - i\infty}^{c_- + i\infty} s^{-m-2} \tan(\pi s) f(s) t^{-s} ds,$$
$$\bar\rho_+(t) = \frac{-1}{2\pi i} \int_{c_+ - i\infty}^{c_+ + i\infty} s^{-m-2} \tan(\pi s) f(s) t^{-s} ds.$$

Then,

$$\bar\rho_+(t) = \frac{2^{m+2}}{\pi} f(\tfrac{1}{2}) t^{-1/2} + \bar\rho_-(t) \tag{80}$$

for all $t \in \mathbb{R}_+$.

Proof. The integrands in ρ_+, ρ_- are holomorphic in $S(a,b) \setminus \{1/2\}$ since f is holomorphic in $S(a,b)$. The estimates for the tangent function of Lemma 6.2 imply that

$$|\tan(\pi s)| \le C_{c_+}$$

when $\Re s = c_+$. This is because c_+ is fixed and away from half-integers. This and the estimate for f in the assumptions give

$$\big|s^{-m-2} \tan(\pi s) f(s)\big| \le K|s|^{-2} \tag{81}$$

for $\Re s = c_+$. Since $|s|^{-2}$ is integrable on $\{c_+ + it \mid t \in \mathbb{R}\}$, we get

$$\bar\rho_+(t) = \lim_{M\to\infty} \frac{-1}{2\pi i} \int_{c_+ - iM}^{c_+ + iM} s^{-m-2} \tan(\pi s) f(s) t^{-s} ds \tag{82}$$

for each $t \in \mathbb{R}_+$.

Define the following points and paths

$$\begin{cases} P_{+-} = c_+ - iM \\ P_{++} = c_+ + iM \\ P_{-+} = c_- + iM \\ P_{--} = c_- - iM \end{cases} \quad \begin{cases} \gamma_{+-}(r) = (1-r)P_{+-} + rP_{++}, \\ \gamma_{++}(r) = (1-r)P_{++} + rP_{-+}, \\ \gamma_{-+}(r) = (1-r)P_{-+} + rP_{--}, \\ \gamma_{--}(r) = (1-r)P_{--} + rP_{+-}, \end{cases} \tag{83}$$

which form a counterclockwise rectangle with the point $s=1/2$ in the interior of the loop. The integrand in (82) is holormorphic in a neighborhood of this rectangle as long as the neighborhood is small enough to not reach $s=1/2$. For any $t\in\mathbb{R}_+$, denote the integrand by

$$I_t(s)=s^{-m-2}\tan(\pi s)f(s)t^{-s},\qquad I_t:S(a,b)\setminus\{1/2\}\to\mathbb{C} \tag{84}$$

to save space. By Cauchy's residue theorem,

$$\frac{-1}{2\pi i}\left(\int_{\gamma_{+-}}+\int_{\gamma_{++}}+\int_{\gamma_{-+}}+\int_{\gamma_{--}}\right)I_t(s)ds=-\operatorname{Res}(I_t,1/2). \tag{85}$$

Let us calcuate the residue at $s = 1/2$. The factors of I_t are holomorphic around $s=1/2$ except for $\tan(\pi s)$, whose residue is given by Lemma 6.5. We have

$$\begin{aligned}\operatorname{Res}(I_t,\tfrac{1}{2})&=\left(\frac{1}{2}\right)^{-m-2}f(\tfrac{1}{2})t^{-1/2}\operatorname{Res}(\tan(\pi s),\tfrac{1}{2})\\&=-\frac{2^{m+2}}{\pi}f(\tfrac{1}{2})t^{-1/2}.\end{aligned} \tag{86}$$

Next, let us investigate what happens when we let $M\to\infty$ again. For the horizontal segments, recall the horizontal estimate for the tangent in Lemma 6.2. It implies that $|\tan(\pi s)|\le 2$ when $|\Im(s)|>1/\pi$. The estimate for f in the assumptions gives a uniform bound for $|f(s)/s^m|$. Furthermore, $|t^{-s}|=t^{-\Re(s)}\le t^{-\alpha}$ when $\Re(s)>\alpha$. This value is independent of M. Lastly, on γ_{++} and γ_{--}, we have $\left|s^{-2}\right|\le M^{-2}$, and the lengths of these paths are both c_+-c_-. Summarizing, on γ_{++} and γ_{--}, we have

$$|I_t(s)|\le Ct^{-a}M^{-2},$$

so the integrals over these horizontal paths vanish as $M\to+\infty$.

The integral over γ_{+-} multiplied by the constant in front of it in (85) equals $\bar{\rho}_+(t)$, as we saw above in (82) when we passed the integral limits to infinity. Lastly, just as at the beginning of this proof, we can let $M\to\infty$ in the integral over γ_{-+} and get $-\bar{\rho}_-(t)$. The claim follows. ∎

We have all the ingredients to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Existence is given by Lemma 6.3. Uniqueness follows from Lemma 6.4. ∎

Proof of Theorem 1.2. The existence and nonexistence follow from Lemma 6.3. Uniqueness is given by Lemma 6.4. All that is left to prove is the identity (10). The existence lemma gives us formulas for ρ_- and ρ_+ in the form of (75) and (76). These are just $(-td/dt)^m$ applied to the integrals in Lemma 6.6 with $f(s)=\mathscr{M}[e](s)$. Thus,

$$\rho_+(t)-\rho_-(t)=\frac{2^{m+2}}{\pi}\mathscr{M}[e](1/2)\left(-t\frac{d}{dt}\right)^m\frac{1}{\sqrt{t}}.$$

But $t^{-1/2}$ is an eigenfunction of $(-td/dt)$ since

$$(-td/dt)t^{-1/2}=-t\cdot(-1/2)t^{-1/2-1}=2^{-1}t^{-1/2}.$$

Hence, $(-td/dt)^m t^{-1/2}=2^{-m}t^{-1/2}$, and the result follows. ∎

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