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Transformation of Nonlinear State Equations into Observer Form

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for doctoral or equivalent academic degree.

/Vadim Kaparin/

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Contents

List of Publications

1. V. Kaparin and Ü. Kotta. Necessary conditions for transformation the nonlinear control system into the observer form via state and output coordinate changes. In *The 7th International Conference on Control and Automation*, pages 745–750, Christchurch, New Zealand, December 2009.

2. V. Kaparin and Ü. Kotta. Necessary and sufficient conditions in terms of differential-forms for linearization of the state equations up to input-output injections. In *UKACC International Conference on CONTROL*, pages 507–511, Coventry, UK, September 2010.

3. V. Kaparin and Ü. Kotta. Theorem on the differentiation of a composite function with a vector argument. *Proceedings of the Estonian Academy of Sciences*, 59(3):195–200, 2010.

4. V. Kaparin and Ü. Kotta. Extended observer form for discrete-time nonlinear control systems. In *The 9th International Conference on Control and Automation*, pages 507–512, Santiago, Chile, December 2011.

5. V. Kaparin, Ü. Kotta, and T. Mullari. Extended observer form: Simple existence conditions. *International Journal of Control*, 86(5):794–803, 2013.

6. V. Kaparin, Ü. Kotta, and M. Wyrwas. Observable space of nonlinear control system on homogeneous time scale. *Proceedings of the Estonian Academy of Sciences.* Accepted for publication.

Author's Contribution to the Publications

The results of all publications were obtained by the author of the thesis under the supervision of Dr. Ülle Kotta.

The fifth publication is a journal article based, in a sense, on the conference paper [79] with Tanel Mullari as coauthor. In [79] the solution was provided for the special case. Introducing the additional conditions, the author of the thesis extended the results of [79] to the general case. Moreover, he formulated the algorithm, which brings the system into the required extended observer form, whenever possible.

Regarding the sixth publication, the proofs of lemmas and propositions were performed by the author of the thesis as a result of joint discussion with Małgorzata Wyrwas, who acted as an expert in time scales.

Introduction

State of the Art

Observers

Various control tasks, based on the state-space representation of the system, require that the values of the states are available. Among such tasks are, for instance, computation of the state feedback and monitoring of the system behavior. However, in practice all state variables are rarely available for direct (on-line) measurement, or even if measurable, measurement can be expensive. In such situations the suitable estimates of the states can be used instead of their actual values. The tool which is intended to produce the required estimates is called observer. In brief, the observer is an auxiliary dynamical system which estimates the state from the knowledge of the inputs and outputs of the system under observation (generally, the model of the system is also required). Moreover, it is desirable that the observer is constructed in such a way that the error between the actual value of the state and its estimate vanishes as time increases. Though the estimation problem is also studied by the stochastic approaches (such as Kalman filter), this section focuses on the overview of the deterministic methods.

Since the first works of David G. Luenberger [71], [72], the problem of observer design for linear time-invariant systems is much studied and the respective theory is well established. Though for nonlinear systems there is no general solution, a great number of methods and approaches can be found in the literature. Among the most typical are the high-gain and sliding mode observers for the continuous-time systems, the Newton observer for the discrete-time systems and the Luenberger-type observer for the systems of both time domains. The methods for the construction of the high-gain observer were proposed, for instance, in [15], [30], [91]. The author of [91] proposed the linear and robust observer for the state and parameter estimation of nonlinear single-output autonomous systems. The main result of [30] claims that under certain technical assumptions (such as some functions are globally Lipschitz) an exponential observer can be built, which is in fact a kind of an extended Luenberger observer. The authors of [15] generalized the technique presented in [30] to the multi-input multi-output (MIMO) case. The design of the sliding mode observer was introduced in [87] and developed further in [74]. The development of the Newton observer design can be found in [13] and [77]. The authors of [77] proposed the application of the Newton's algorithm as the sampled-data observer. In [13] the modified Newton observer was introduced. The proposed observer has a hybrid structure, combining discrete-time iterations with the continuous-time filters. The methods for construction of the Luenberger-type observer were presented in [53] and [54]. In [53] the problem of observer design for continuous-time nonlinear systems was translated into the problem of solving a system of singular first-order linear partial differential equations. The similar approach was employed in [54] for the discrete-time systems in order to formulate the observer design problem via a system of first-order linear nonhomogeneous functional equations. In both cases rather general set of necessary and sufficient conditions was derived.

Observer Form Approach

A distinct from the above-mentioned approaches is the method of linearization by input-output injections. The main feature of the method is the intermediate step implying the transformation of the system into the special form, called the observer form. Roughly speaking, a system in the observer form is a linear observable system that is interconnected with an input-output-dependent nonlinearities, called input-output injections. Once the system is in the observer form, the construction of the nonlinear observer is relatively easy. The advantage of the approach is that the dynamics of the estimation error are linear. Furthermore, the method is, in a sense, systematic and can be applied to both continuous- and discrete-time systems. The pioneers of the research in this direction were the authors of two independent contributions [12], [62]. In both papers the existence of state transformation, bringing the nonlinear continuous-time autonomous system with single output into the observer form, was studied. The difference is that the authors of [12] investigated the problem for the time-varying systems, whereas in [62] the time-invariant systems were considered. Moreover, the paper [62] provided the necessary and sufficient solvability conditions using the tools of differential geometry. Afterwards, these results were extended to MIMO case in [63], [97]. The authors of [97] improved some results of [63] and proposed a different set of necessary and sufficient conditions as well as a procedure for the practical computation of the state transformation. The approach of [12] was developed further in [67], where the necessary and sufficient conditions for the existence of the state transformation were presented as rank conditions of certain matrices. The multi-output version of these conditions was given later in [96].

The problem in the context of discrete-time autonomous systems was addressed in the papers [23], [66]. In [23] the results of [62] were carried over to the discrete-time case with a view to investigating the effects of time-sampling on the solvability of the observer error linearization problem. The paper [66] originated as a parallel result to the continuous-time case [97]. The authors of [66] derived the necessary and sufficient solvability conditions for both single- and multi-output cases.

Extensions and Generalizations of the Observer Form

The observer form approach, relying on the state transformation only, has the disadvantage of imposing restrictive conditions for the existence of the observer form for nonlinear control system. This fact motivates various extensions and generalizations of the observer form as well as the generalization of the transformations to enlarge the class of systems for which the observer with linear error dynamics can be constructed. The majority of such generalizations, proposed in the literature, can be classified into three categories: (i) application of the output transformation in addition to the state transformation, (ii) generalization of the form of the matrix A (the coefficient matrix of the state vector in the matrix representation of the observer form), and (iii) generalization of the form of the input-output injections.

The generalization of category (i) has appeared already in the early paper [63], where the state transformation was supplemented by the output transformation, implying that the output in the observer form can be a function of the original output. The further development of this approach can be found in [83], [16], where certain algorithms for attaining the observer form are proposed. In [57] the output transformation is applied for the discrete-time systems in such a way that the usual addition operation in the observer form is replaced by a more general associative binary operation, yielding the associative observer form. Though this approach allows to reduce nonlinearities to the linearities in the representation of the observer form, its limitation is that the amount of the known associative binary operators is not very large.

The generalization of category (ii) can be found, for example, in [10], [36], [37], [88], [101]. The authors of [36] developed the necessary and sufficient conditions under which the continuous-time system is transformable into the observer form, where the matrix A is allowed to depend on the input. A different set of conditions and an alternative procedure for transformation were provided in [37]. The paper [10] is a kind of the discrete-time version of the results presented in [36]. In [101], the matrix A is allowed to depend on output, whereas in [88] on both input and output.

The observer forms with the generalizations of category (iii) are presented, for instance, in [70], [76], [84], [86], [92] for continuous-time systems and in [42], [68] for discrete-time systems. In [84] two types of generalized observer form were considered. In the first type the input-output injections are allowed to depend not only on the input and output but also on the derivatives of the input, whereas in the second type the derivatives of both the input and the output are allowed. The first type of the observer form was considered for MIMO systems, whereas the second for systems with single output and the extension to the MIMO case was provided in [70]. In [76] the another generalized observer form was proposed for MIMO systems. Though the input-output injections in the observer form can also depend on the derivatives of the input, this dependence is reduced. In both [86] and [92] the nonlinear term, playing the role of the input-output injection, is allowed to depend on certain state variables. In [92] such dependence leads to the triangular structure of the observer form. The extended observer form, proposed in [68] for the discrete-time multi-input single-output (MISO) systems, contains the input-output injections, which, besides the input and output, also depend on a finite number of their past values. In [42] less general case of the systems without inputs was considered. Though the extended observer form presented in [42] can also depend on the past values of the output, the input-output injections are structurally different from those considered in [68].

Furthermore, three categories of generalization may frequently occur in various combinations. The combination of categories (i) and (iii) can be found, for example, in [31], [40] for continuous- and discrete-time cases, respectively. The observer form presented in [31] is similar to the first type of the observer form considered in [84], but equipped with the output transformation. In [40] the output transformation was used to extend the results of [42]. The generalizations of categories (i) and (ii) were addressed in [9], [18], [19], [64], where the output transformation is employed and the matrix A is allowed to depend on input. In [9] the continuous-time MISO system was considered, whereas [18] and [19] addressed the cases of the discrete-time MISO and MIMO systems, respectively. The alternative approach to the discrete-time MISO systems is presented in [64], where, unlike [18], the solvability conditions are expressed in more simple form of certain partial derivatives and do not involve more complex elements of differential geometry.

It should be mentioned, that a number of methods, generalizing the observer form approach, are beyond the classification given above. For example, the method, proposed in [34], [85] for the continuous-time systems with single output, implies the application of output-dependent time-scaling transformation such that with respect to the new time-scale the system becomes transformable into the observer form. The approach of [34] is based on the language of differential forms and exterior calculus, whereas in [85] the tools of differential geometry are applied. The extension of the method to the multi-output case was presented in [93]. The attempt to develop a discrete-time counterpart of the time-scaling technique can be found in [69]. The other two related methods are the system immersion and the dynamic observer error linearization techniques (for the continuous-time case see [3], [4], [11], [46] and [5], [17], [81], [98], [99], respectively). The main feature of both methods is embedding the original system into the higher-dimensional one, being transformable into the observer form. The difference is that the immersion involves the application of the certain change of state variables, whereas in the case of dynamic observer error linearization the auxiliary dynamics and virtual outputs are introduced to augment the system. The discrete-time version of the dynamic observer error linearization and both the immersion and the dynamic observer error linearization can be found in [65] and [100], respectively.

Dynamical Systems on Time Scales

Roughly speaking, a time scale is a model of time. The most typical special cases instances of time scale are continuous and discrete time. However, there are a number of other time models, for example, the partly continuous and partly discrete time, q-models (the set of all integer powers of a number $q > 1$, including 0), sets of disjoint closed intervals, Cantor set, etc. The theory of dynamical equations on time scales is a new and popular research area, which was initiated in 1988 by Stefan Hilger in his PhD thesis [39] (supervised by Bernd Aulbach) with the intention of unifying continuous and discrete analysis. Afterwards, Martin Bohner and Allan Peterson published the first monograph on this topic [14]. From a modeling point of view, dynamical systems on time scales incorporate both the continuous- and discrete-time systems as the special cases, allowing that way to unify the study and consider the classical results as the special cases from the new theory. On the other hand, the study of dynamical systems on time scales helps to reveal and explain discrepancies, occasionally appearing between the results obtained for continuous-time systems and their discrete-time counterparts. However, it is important to note that the discrete-time model in the time scale formalism is given in terms of the difference operator, and not in terms of the more conventional shift operator as, for example, in [1], [2], [33]. The difference-based models, often referred to as delta-domain models, are not completely new for description of the discrete-time systems. They have been promoted during the last decades as the models closely linked to the continuous-time systems, being less sensitive to round-off errors at higher sampling rates [32], [73].

The properties of linear systems, defined on time scales, were studied, for instance, in [7] and [27]. In [6] the algebraic formalism in terms of differential one-forms has been developed for the study of nonlinear control systems defined on homogeneous time scales and used later to study different problems like realization, transfer equivalence, irreducibility and reduction of nonlinear input-output equations [8], [20], [56], [58].

NLControl package

The solutions of nonlinear control problems require a huge amount of symbolic computations. Unfortunately, the nonlinear control systems, unlike their linear counterparts, practically miss a support of professional software products. For this reason, the special software package NLControl was developed in the Control System Department of the Institute of Cybernetics at Tallinn University of Technology. The NLControl package is implemented within the computation system *Mathematica*, the commercial software developed by Wolfram Research company [95]. The purpose of the package is to provide the symbolic computational tools that assist the solution of different modeling, analysis, and synthesis problems for nonlinear control systems. The majority of provided tools rely on the algebraic approach of differential one-forms and the related methods based on the theory of the skew polynomial rings. The development of the NLControl package was originally initiated more than decade ago and at the first stages the functions were implemented mainly by Maris Tõnso (see, for example, [60], [61], [89]). At the moment the package consists of more than 100 functions/programs, assisting in research and teaching of nonlinear control theory.

Note, however, that the NLControl package is not stand-alone software and cannot be used outside of the *Mathematica* environment. In order to overcome this obstacle the NLControl website [43] was created. Employing web*Mathematica* technology (developed by Wolfram Research), the website makes available the most important functions from NLControl via the Internet, such that no other software except for a web browser needs to be installed on a computer [90].

Choice of Research Directions: Motivation

On the one hand, the transformation of the nonlinear system into the observer form is a recognized approach to the state estimation problem. Being, in a sense, *systematic* and *applicable to both the continuous- and discrete-time systems*, the method is attractive for researchers. Another advantage of the method is that it allows to construct the nonlinear observer in such a way that the dynamics of the estimation error are linear, which guaranties that the error converges asymptotically to zero. On the other hand, the research area related to the observer form approach is still active, which is confirmed by numerous recent publications (see [17], [64], [65], [69], [93], [98], [100]). There are still a number of unsolved problems and the techniques that can be improved. One of the possible directions for improvement is the development of the constructive algorithms, implementable in symbolic software. The main part of this thesis presents the results obtained by the author in this direction. Both the continuous- and discrete-time cases are investigated. The theoretical research was performed with the intention of implementing the results within the *Mathematica* based package NLControl. This motivated the author to develop more simple and direct formulas, as well as the algorithms which are constructive (at least in a sense of symbolic computations).

The Problems to Be Studied

In the continuous-time case the problem of transforming the single-input single-output (SISO) nonlinear control system into the observer form, using both the state and output transformations, is addressed. Regarding the classification from the previous section, the generalization belongs to the category (i). The starting point for our research served the contribution [31], where the approach based on differential forms was applied to the problem and the solution is found as a result of two-step procedure. The necessary condition, stated in [31] for the existence of the output transformation, is very mild and far from being sufficient. Obviously, its validity does not guarantee the solvability of the problem. To verify whether the problem is solvable, one has first to apply the output transformation and then check whether in the new output coordinates the system is transformable into the observer form by the state coordinate transformation only. For the latter the recursive algorithm was developed. Our aim was to find direct and simple solvability conditions, which would be both necessary and sufficient. The obtained conditions are formulated in terms of an unknown single-variable output dependent function and the differential one-forms, directly computable from the input-output equation, corresponding to the state equations. Note that the unknown function can be found solving certain differential equation. Once the function is obtained, the verification of the remaining conditions becomes straightforward.

In the discrete-time case the problem of transforming the SISO nonlinear state equations into the extended observer from was considered. The main feature of the extended observer form lies in the input-output injection terms, which, besides the input and output, depend also on a finite number of their past values. Though the past values of input and output require additional memory, the construction of the observer is as simple as in the case of traditional observer form, preserving the asymptotic error convergence to zero. Again, the output transformation in addition to the extended state coordinate change is employed. Thus, according to the classification from the previous section, the generalizations are of categories (i) and (iii). The problem, mentioned above, was addressed earlier in [40] for the systems without inputs. The approach of [40] relies on the sophisticated language of differential geometry and verification of the proposed conditions requires calculation of the exterior derivatives and wedge products of certain one-forms, associated with the system, as well as the Lie derivatives, Lie brackets and interior product, which leads to complicated computations. This motivated us to search for more simple conditions. Another goal of the research was extension of the theory to the input-dependent systems. As a result, two sets of necessary and sufficient conditions were obtained. The first set is formulated in terms of certain differential one-forms and has the advantage of being intrinsic. Though the second set of conditions is not intrinsic, it is formulated in terms of certain partial derivatives and, thence, is very simple to verify. Besides, certain algorithm for the transformation of the state equations into the extended observer form was proposed.

Another direction of the research presented in the thesis is study of nonlinear control systems, defined on homogenous time scales. Since the observability of the system is frequently assumed by the observer form approach, we decided to study this property for systems on time scales. Our intention was to unify the observability related results, obtained separately for continuous- and discrete-time systems, and to analyze the differences between two time domains. In particular, the observable space is always integrable in the continuous-time case, but is not necessarily so for the discrete-time systems.

Outline and Contributions of the Thesis

The main part of the thesis is divided into five chapters. Chapter 1 contains the introductory material. Chapters 2–4 present the theoretical results of the thesis, whereas Chapter 5 describes the implementation of the results from the previous chapters within the NLControl package.

Chapter 1

This chapter summarizes the main theoretical concepts and mathematical tools used throughout the thesis. The first section contains the introduction to the calculus on time scales. The next two sections define the object of the study, i.e. the nonlinear dynamical system, and recall the algebraic framework based on differential one-forms, respectively. The last section presents the theorem, stated and proved by the author of the thesis in [49]. The theorem is necessary for the proof of the main result of Chapter 2.

Chapter 2

The problem of transforming the continuous-time state equations into the observer form, using both the state and the output transformations, is addressed in the chapter. The main contributions of the chapter are the necessary and sufficient solvability conditions and the algorithm for finding the state and output transformation, bringing the system into the observer from. Moreover, the comparison of our results with those presented earlier in [31] is presented at the end of the chapter. The content of the chapter rests on the results published in [47] and [48].

Chapter 3

The chapter is devoted to the transformation of the discrete-time state equations into the extended observer form. In this form, the input-output injections depend not only on the input and the output but also on their past values, the number of which is determined by the integer, called the buffer. Furthermore, besides the extended coordinate change, the output transformation is allowed. Two sets of necessary and sufficient conditions are presented in the chapter. The first set of conditions is expressed in terms of certain differential one-forms. The second set of conditions, expressed in terms of certain partial derivatives, was initially proposed by Tanel Mullari et al. in [79] for the special case when the buffer equals 1. Moreover, the sufficient conditions presented in [79] are valid only for the buffer satisfying certain relation regarding the system order and/or the highest and the lowest shifts of input and output in system input-output equation. The necessary and sufficient conditions for the general case and an arbitrary buffer are presented in this chapter. The last contribution of the chapter is the algorithm for transforming the system into the extended observer form. The chapter is based on the results published in [50], [51].

Chapter 4

The chapter addresses the observability property of the MIMO nonlinear system, defined on homogeneous time scale. The observability necessary and sufficient condition is provided through the notion of the observable space. The related notions of the observability filtration and the observability indices are extended to the systems on homogeneous time scale. For the unobservable systems, whose observable space is completely integrable, the certain procedure of the decomposition into observable/unobservable subsystems is presented. The main contributions of the chapter are different lemmas and propositions proved in terms of the time scale formalism. Time scale analysis allows to consider the classical results, obtained separately for continuous- and discrete-time systems, as the special cases of the new theory. On the other hand, the time scale formalism includes the description of a discrete-time system based on the difference operator description (delta-domain approach), for which the results presented in this chapter are new, since the previous results have been obtained for discrete-time systems considered on the basis of the shift-operator formalism. The chapter rests on the results to be published in [52].

Chapter 5

The chapter presents several *Mathematica* functions, implementing the theoretical results from the previous chapters. The developed functions are the integral part of the NLControl package and can be also used online via the NLControl website [43]. Several functions were programmed to assist the transformation of the continuous- or discrete-time system into the respective observer forms. The other set of functions was developed for the system on homogeneous time scale to check the observability, find the observability filtration, observability indices, observable and unobservable spaces, as well as to decompose the system into the observable/unobservable subsystems, whenever possible.

Chapter 1

Preliminaries

The objective of this chapter is to introduce the basic notions and mathematical tools necessary in the main part of the thesis. The first section gives an overview of the basic concepts and computation rules of time scale analysis, necessary to understand the results of Chapter 4 of this thesis. The next two sections describe the nonlinear control system and the algebraic formalism of differential forms, applied in our studies. The theorem, presented in the last section, provides the formula used later for the proof of the main result in Chapter 2.

1.1 Time Scale Calculus

This section serves as a brief introduction to the time scale calculus. The basic notions and results, applied in the thesis, are recalled from [14].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise the continuous time case, $\mathbb{T} = \mathbb{R}$, the discrete time cases, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \tau\mathbb{Z} := \{\tau k \mid k \in \mathbb{Z}\}$ for $\tau > 0$, but also $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k \mid k \in \mathbb{Z}\} \cup \{0\}$ is a time scale. We assume that $\mathbb{T}$ is a topological space with the topology induced by $\mathbb{R}$. The so-called forward and backward jump operator are key notions in time scale calculus.

Definition 1.1. For $t \in \mathbb{T}$ the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} \mid s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup \{s \in \mathbb{T} \mid s < t\}.$$

In this definition we set in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$ and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. In the analysis on time scales the graininess function plays a central role.

Definition 1.2. For $t \in \mathbb{T}$ the *graininess functions* $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by

$$\mu(t) := \sigma(t) - t \quad \text{and} \quad \nu(t) := t - \rho(t),$$

respectively.

A time scale $\mathbb{T}$ is called *homogeneous* if $\mu = \nu \equiv \text{const}$. Let $\mathbb{T}^\kappa$ denote the truncated set consisting of $\mathbb{T}$ except for a possible maximal point $\max \mathbb{T}$ such that $\rho(\max \mathbb{T}) < \max \mathbb{T}$.

Definition 1.3. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then *delta derivative* of f at t, denoted by $f^\Delta(t)$, is the real number (provided it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Example 1.1.

- If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) \equiv 0$ and delta derivative $f^\Delta(t)$ is the ordinary time derivative, i.e. $f^\Delta(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t}$.
- If $\mathbb{T} = \tau\mathbb{Z}$ for $\tau > 0$, then $\sigma(t) = t + \tau$, $\mu(t) = \tau$ and
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+\tau) - f(t)}{\tau}$$
always exists.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for $q > 1$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$ and
$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}$$
for all $t \in \mathbb{T} \setminus \{0\}$.

It is easy to observe, the first two time scales are homogeneous, whereas the third is not, since in this case the graininess function depends on t.

For $f : \mathbb{T} \to \mathbb{R}$ define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) := f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e. $f^\sigma := f \circ \sigma$. The theorems below provide the general rules for application of delta derivative.

Theorem 1.1 ([14]). *Assume the functions $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. Then the delta derivative satisfies the following properties*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*
$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*
$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*
$$(fg)^\Delta(t) = f^\sigma(t)g^\Delta(t) + f^\Delta(t)g(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t).$$

(iv) *If $g(t)g^\sigma(t) \neq 0$, then $\frac{f}{g}$ is delta differentiable at t with*
$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

Theorem 1.2 (Chain Rule [14]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*
$$(f \circ g)^\Delta(t) = \left(\int_0^1 f'(g(t) + \lambda\mu(t)g^\Delta(t))\mathrm{d}\lambda\right) g^\Delta(t)$$
holds.

For a function $f : \mathbb{T} \to \mathbb{R}$ one can define the second delta derivative $f^{\langle 2\rangle} := \left(f^\Delta\right)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$ provided that the function f^Δ is delta differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$. In a similar manner one defines higher order delta derivatives $f^{\langle n\rangle} := \left(f^{\langle n-1\rangle}\right)^\Delta : \mathbb{T}^{\kappa^n} \to \mathbb{R}$.

1.2 Nonlinear Control Systems

In this section the nonlinear control systems are defined separately for different time domains. Both in continuous- and discrete-time cases the state space and the input-output representations of the system are given. For the system, defined on homogeneous time scale, only state space representation is used.

Note that throughout the thesis we use the abridged notations. First, in order to simplify the exposition we leave out the time argument t, so $x := x(t)$. Next, we apply Newton's notation for the first and second time derivatives, i.e. $\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}$, $\ddot{x} := \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, and a more general notation $x^{(k)} := \frac{\mathrm{d}^kx}{\mathrm{d}t^k}$ for the time derivative of an arbitrary order. Finally, in the discrete-time case we use symbols $^+$, $^-$ and $^{[k]}$ instead of the shifted time arguments, so $x^+ := x(t+1)$, $x^- := x(t-1)$ and $x^{[k]} := x(t+k)$.

1.2.1 Analytic and Meromorphic Functions

In this thesis we assume that the nonlinear control systems are described by real analytic functions. The choice of analytic functions is motivated by their nice properties, necessary for the construction of the algebraic framework, based on the differential one-forms. First, the analytic functions belong to class C^∞, i.e. are infinitely differentiable. However, unlike the ring of C^∞ functions, the ring of analytic functions is integral domain, meaning that it can be embedded into its quotient field whose elements are meromorphic functions. Moreover, the employment of analytic functions allows to study the *generic* properties of the systems, i.e. properties that hold in almost all situations except, so to say, the pathological ones. The notion of generic property does not make sense, in general, for systems defined by C^∞ functions, further details can be found in [25], [26].

1.2.2 Continuous-time Systems

Consider a single-input single-output (SISO) nonlinear continuous-time dynamical system, described either by the state equations

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1.1}$$

or by the higher order input-output differential equation

$$y^{(n)} = \phi\left(y, y^{(1)}, \ldots, y^{(n-1)}, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{1.2}$$

where $x(t) : \mathbb{R} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{R} \to \mathbb{U} \subset \mathbb{R}$ is an input and $y(t) : \mathbb{R} \to \mathbb{Y} \subset \mathbb{R}$ is an output. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $h : \mathbb{X} \to \mathbb{Y}$ and $\phi : \mathbb{Y}^n \times \mathbb{U}^n \to \mathbb{R}$ are assumed to be real analytic functions.

1.2.3 Discrete-time Systems

Consider a single-input single-output (SISO) nonlinear discrete-time dynamical system, described either by the state equations

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1.3}$$

or by the higher order input-output difference equation

$$y^{[n]} = \phi\left(y, y^{[1]}, \ldots, y^{[n-1]}, u, u^{[1]}, \ldots, u^{[n-1]}\right), \tag{1.4}$$

where $x(t) : \mathbb{Z} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{Z} \to \mathbb{U} \subset \mathbb{R}$ is an input and $y(t) : \mathbb{Z} \to \mathbb{Y} \subset \mathbb{R}$ is an output. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $h : \mathbb{X} \to \mathbb{Y}$ and $\phi : \mathbb{Y}^n \times \mathbb{U}^n \to \mathbb{R}$ are assumed to be real analytic functions.

The following assumption is necessary for the algebraic framework based on differential one-forms (see Section 1.3).

Assumption 1.1. *System* (1.3) *is generically submersive, i.e. almost everywhere, except on the set of measure zero, $f(x,u)$ satisfies the condition*

$$\operatorname{rank}\frac{\partial f(x,u)}{\partial(x,u)} = n.$$

Assumption 1.1 is not restrictive since it is a necessary condition for system accessibility [33]. Moreover, it is more general than the often assumed reversibility condition, which requires that

$$\operatorname{rank}\frac{\partial f(x,u)}{\partial x} = n$$

holds generically [29].

1.2.4 Systems Defined on Homogeneous Time Scales

Consider a multi-input multi-output (MIMO) nonlinear dynamical system, defined on homogeneous time scale[1] $\mathbb{T}$ and described by the state equations

$$\begin{aligned} x^{\Delta} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1.5}$$

where $x(t) : \mathbb{T} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{T} \to \mathbb{U} \subset \mathbb{R}^m$ is an m-dimensional input vector and $y(t) : \mathbb{T} \to \mathbb{Y} \subset \mathbb{R}^p$ is a p-dimensional output vector. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \to \mathbb{Y}$ are assumed to be real analytic functions.

Denote the state transition map of (1.5) by $\widetilde{f}(x,u) := x + \mu f(x,u)$. Like in the discrete-time case, the assumption below is necessary for the algebraic framework based on differential one-forms (see Section 1.3).

Assumption 1.2. *System* (1.5) *is generically submersive, i.e. almost everywhere, except on the set of measure zero, $\widetilde{f}(x,u)$ satisfies the condition*

$$\operatorname{rank}\frac{\partial \widetilde{f}(x,u)}{\partial(x,u)} = n.$$

Of course, for $\mu = 0$, that corresponds to the continuous-time case, the assumption above is always satisfied.

[1] Though the closed interval $[a,b]$ is also an example of homogeneous time scale, we restrict our consideration to infinite homogeneous time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \tau\mathbb{Z}$ for $\tau > 0$.

1.3 Algebraic Framework

This section is devoted to the description of the main algebraic structures employed in the thesis. First, we define the σ_f-differential and difference fields, associated with systems (1.5) and (1.3), respectively, and then we recall the basic concepts and facts from the theory of differential forms. Note that since the continuous-time system (1.1) is a special case of the system (1.5) for $\mathbb{T} = \mathbb{R}$, the differential field, associated with system (1.1), is a special case of the σ_f-differential field for $\sigma_f = \mathrm{id}$. By this reason we do not define the algebraic framework for continuous-time systems separately, referring the reader to [25] for details.

Henceforth, for notational convenience, denote $\xi^{\langle i \ldots n\rangle} := \left(\xi^{\langle i\rangle}, \ldots, \xi^{\langle n\rangle}\right)$ and $\xi^{[i \ldots n]} := \left(\xi^{[i]}, \ldots, \xi^{[n]}\right)$ for $0 \le i \le n$, implying $\xi^{\langle 0\rangle} = \xi^{[0]} := \xi$ and recalling that symbols ${}^{\langle i\rangle}$ and ${}^{[i]}$ stand for the ith delta derivative and ith time shift, respectively.

1.3.1 σ_f-Differential and Difference Fields

Consider the system described by the equations (1.5). Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of independent system variables from the infinite set

$$\mathcal{C} = \left\{x_i,\ i = 1, \ldots, n;\ u_v^{\langle k\rangle},\ v = 1, \ldots, m,\ k \ge 0\right\}.$$

For $F\left(x, u^{\langle 0 \ldots k\rangle}\right) \in \mathcal{K}$ the *forward-shift operator* $\sigma_f : \mathcal{K} \to \mathcal{K}$ is defined by

$$F^{\sigma_f}\left(x, u^{\langle 0 \ldots k+1\rangle}\right) := F\left(x + \mu f(x,u), u^{\langle 0 \ldots k\rangle} + \mu u^{\langle 1 \ldots k+1\rangle}\right),$$

where $f(x,u)$ is determined by (1.5). It is easy to note that in the continuous-time case ($\mu = 0$), the forward-shift operator is identity, i.e. $\sigma_f = \mathrm{id}$. Under Assumption 1.2, σ_f is injective endomorphism and so the operator σ_f is well defined on $\mathcal{K}$ [6]. Note that for the k-fold application of the forward-shift operator we use the notation $F^{\sigma_f^k} := \left(F^{\sigma_f^{k-1}}\right)^{\sigma_f}$. Furthermore, for $F\left(x, u^{\langle 0 \ldots k\rangle}\right) \in \mathcal{K}$ the *delta derivative operator* $\Delta_f : \mathcal{K} \to \mathcal{K}$ is defined by

$$F^{\Delta_f}\left(x, u^{\langle 0 \ldots k+1\rangle}\right) :=$$
$$= \begin{cases} \dfrac{F^{\sigma_f}\left(x, u^{\langle 0 \ldots k+1\rangle}\right) - F\left(x, u^{\langle 0 \ldots k\rangle}\right)}{\mu} & \text{if } \mu \ne 0, \\ \dfrac{\partial F}{\partial x}\left(x, u^{\langle 0 \ldots k\rangle}\right) f(x,u) + \displaystyle\sum_{k \ge 0} \dfrac{\partial F}{\partial u^{\langle 0 \ldots k\rangle}}\left(x, u^{\langle 0 \ldots k\rangle}\right) u^{\langle 1 \ldots k+1\rangle} & \text{if } \mu = 0. \end{cases}$$

Hereinafter, the k-fold application of the delta derivative operator is denoted by $F^{\langle k\rangle} := \left(F^{\langle k-1\rangle}\right)^{\Delta_f}$.

Proposition 1.1 ([6]). *For $F, G \in \mathcal{K}$ the delta derivative and forward-shift operators satisfy the following properties*

(i) $F^{\sigma_f} = F + \mu F^{\Delta_f}$,

(ii) $(\alpha F + \beta G)^{\Delta_f} = \alpha F^{\Delta_f} + \beta G^{\Delta_f}$, *for* $\alpha, \beta \in \mathbb{R}$,

(iii) $(FG)^{\Delta_f} = F^{\sigma_f} G^{\Delta_f} + F^{\Delta_f} G$ *(generalization of Leibniz rule)*,

(iv) *if* $GG^{\sigma_f} \neq 0$, *then* $(F/G)^{\Delta_f} = (F^{\Delta_f} G - FG^{\Delta_f})/(GG^{\sigma_f})$,

(v) *on homogeneous time scale operators* Δ_f *and* σ_f *commute, i.e*

$$\left(F^{\sigma_f}\right)^{\Delta_f} = \left(F^{\Delta_f}\right)^{\sigma_f}.$$

An operator satisfying the generalized Leibniz rule is called a σ_f*-derivation* and a commutative field endowed with a σ_f-derivation is called a σ_f*-differential field* [24]. Therefore, under Assumption 1.2, $\mathcal{K}$ endowed with the delta derivative operator Δ_f is a σ_f-differential field. For $\mu = 0$, $\sigma_f = \sigma_f^{-1} = \mathrm{id}$ and $\mathcal{K}$ is inversive. Though $\mathcal{K}$ is not inversive in general, it is always possible to embed $\mathcal{K}$ into an inversive σ_f-differential overfield $\mathcal{K}^*$, called the *inversive closure* of $\mathcal{K}$ [24]. Since σ_f is injective endomorphism, it can be extended to $\mathcal{K}^*$ so that $\sigma_f : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism. A practical procedure for construction of $\mathcal{K}^*$ for $(\mu \neq 0)$ is given in [6].

In order to define the difference field, associated with system (1.1), denote by $\overline{\mathcal{K}}$ the field of meromorphic functions in a finite number of independent system variables from the infinite set

$$\overline{\mathcal{C}} = \left\{x_i,\ i = 1, \ldots, n;\ u^{[k]},\ k \geq 0\right\}.$$

For $F\left(x, u^{[0\ldots k]}\right) \in \overline{\mathcal{K}}$ the *forward-shift operator* $\delta : \overline{\mathcal{K}} \to \overline{\mathcal{K}}$ is defined by

$$\delta\left(F\left(x, u^{[0\ldots k]}\right)\right) := F\left(f(x, u), u^{[1\ldots k+1]}\right),$$

where $f(x, u)$ is determined by (1.3). Note that throughout the thesis the alternative notation $F^+ := \delta(F)$ is employed and the k-fold application of the operator δ is denoted by $F^{[k]} := \delta\left(F^{[k-1]}\right)$. Under Assumption 1.1, δ is injective and $\overline{\mathcal{K}}$ endowed with the forward-shift operator δ is a *difference field*. Like the σ_f-differential field, the difference field $\overline{\mathcal{K}}$ is not inversive in general, necessitating to embed $\overline{\mathcal{K}}$ into its inversive closure $\overline{\mathcal{K}^*}$, which is always possible. A construction of $\overline{\mathcal{K}^*}$ for practical computations is given in [2].

1.3.2 Differential Forms

Consider the infinite set of symbols

$$\mathrm{d}\mathcal{C} = \left\{ \mathrm{d}x_i,\ i = 1, \ldots, n;\ \mathrm{d}u_\upsilon^{\langle k \rangle},\ \upsilon = 1, \ldots, m,\ k \geq 0 \right\}$$

and denote by $\mathcal{E}$ the vector space spanned over the field $\mathcal{K}^*$ by the elements of $\mathrm{d}\mathcal{C}$, namely $\mathcal{E} := \mathrm{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}$. Any element of $\mathcal{E}$ has the form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k \geq 0} \sum_{\upsilon=1}^{m} B_{\upsilon,k} \mathrm{d}u_\upsilon^{\langle k \rangle},$$

where only a finite number of coefficients $B_{\upsilon,k}$ are nonzero elements of $\mathcal{K}^*$. The elements of $\mathcal{E}$ are called the *differential one-forms*.

For $F\left(x, u^{\langle 0 \ldots k \rangle}\right) \in \mathcal{K}^*$ define the operator $\mathrm{d} : \mathcal{K}^* \to \mathcal{E}$ as follows

$$\mathrm{d}F := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \mathrm{d}x_i + \sum_{l=0}^{k} \sum_{\upsilon=1}^{m} \frac{\partial F}{\partial u_\upsilon^{\langle l \rangle}} \mathrm{d}u_\upsilon^{\langle l \rangle}. \tag{1.6}$$

One says that $\omega \in \mathcal{E}$ is an *exact one-form* if $\omega = \mathrm{d}F$ for some $F \in \mathcal{K}^*$. We will refer to $\mathrm{d}F$ as to the *total differential* of F.

For one-form $\omega = \sum_i A_i \mathrm{d}\zeta_i$, where $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}$, one can define the operators $\Delta_f : \mathcal{E} \to \mathcal{E}$ and $\sigma_f : \mathcal{E} \to \mathcal{E}$ by

$$\omega^{\Delta_f} := \sum_i \left(A_i^{\Delta_f} \mathrm{d}\zeta_i + A_i^{\sigma_f} \mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right) \tag{1.7}$$

and

$$\omega^{\sigma_f} := \sum_i A_i^{\sigma_f} \mathrm{d}\left(\zeta_i^{\sigma_f}\right).$$

Since $A_i^{\sigma_f} = A_i + \mu A_i^{\Delta_f}$,

$$\omega^{\Delta_f} = \sum_i \left(A_i^{\Delta_f} \mathrm{d}\zeta_i + \left(A_i + \mu A_i^{\Delta_f}\right) \mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right).$$

Adapting Leibniz rule for an arbitrary-order derivatives of product to the computation of an arbitrary-order delta derivative of one-form, we obtain

$$\omega^{\langle r \rangle} = \sum_{q=0}^{r} C_r^q \sum_i \left(A_i^{\langle r-q \rangle}\right)^{\sigma_f^q} \mathrm{d}\zeta_i^{\langle q \rangle}, \tag{1.8}$$

where C_r^q is a binomial coefficient and σ_f^0 implies id.

In the same manner, over the field $\overline{\mathcal{K}^*}$ one can define the difference vector space $\overline{\mathcal{E}} := \mathrm{span}_{\overline{\mathcal{K}^*}} \mathrm{d}\overline{\mathcal{C}}$ of differential one-forms, where

$$\mathrm{d}\overline{\mathcal{C}} = \left\{ \mathrm{d}x_i,\ i = 1, \ldots, n;\ \mathrm{d}u^{[k]},\ k \geq 0 \right\}.$$

The vector space $\overline{\mathcal{E}}$ may be also endowed with the forward-shift operator $\delta : \overline{\mathcal{E}} \to \overline{\mathcal{E}}$, defined by

$$\delta\left(\sum_i A_i \mathrm{d}\zeta_i\right) := \sum_i \delta\left(A_i\right) \mathrm{d}\delta\left(\zeta_i\right),$$

where $A_i \in \overline{\mathcal{K}^*}$ and $\zeta_i \in \overline{\mathcal{C}}$.

Note that throughout the thesis the symbol $\mathrm{d}\omega$ means the exterior derivative of the differential form ω and $\wedge$ means the exterior or wedge product (for details see [22]). A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed.* It is well known that exact forms are closed, while closed forms are only locally exact. Integrability of the subspace of one-forms may be checked by the Frobenius theorem below.

Theorem 1.3 (Frobenius theorem [22])**.** *Let $\mathcal{V} = \mathrm{span}_{\mathcal{K}^*}\{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathcal{E}$. $\mathcal{V}$ is integrable if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$$

for any $i = 1, \ldots, r$.

1.4 Theorem on the Differentiation of a Composite Function with a Vector Argument

The theorem below was stated and proved by the author of the thesis with the intention of proving the main result of Section 2. The theorem shows how the partial derivative of the total derivative of the composite function with a vector argument can be expressed through the total derivative of the partial derivative of it.

Theorem 1.4. *Assume that $\Phi(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))$ is a composite function for which derivatives up to order $a + b$ are defined; then*

$$\frac{\partial\left(\Phi(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(a+b)}}{\partial\left(\xi_l^{(a)}(t)\right)} = C_{a+b}^{b}\left(\frac{\partial\Phi(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))}{\partial\xi_l(t)}\right)^{(b)},$$

where $l = 1, 2, \ldots, r$, C_{a+b}^{b} is the binomial coefficient and a, b are nonnegative integers.

The proof of the theorem is given in the Appendix. Some useful corollaries of the theorem are given below.

Corollary 1.1. *Under the assumptions of Theorem 1.4*

$$\frac{\partial\left(\Phi(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(a+b)}}{\partial\xi_l(t)} = \left(\frac{\partial\left(\Phi(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(a)}}{\partial\xi_l(t)}\right)^{(b)},$$

where a and b are nonnegative integers.

Corollary 1.2. *Under the assumptions of Theorem 1.4*

$$\frac{\partial\left(\Phi(\xi_1(t),\xi_2(t),\ldots,\xi_r(t))\right)^{(a)}}{\partial\xi_l(t)} = \left(\frac{\partial\Phi(\xi_1(t),\xi_2(t),\ldots,\xi_r(t))}{\partial\xi_l(t)}\right)^{(a)},$$

where a is nonnegative integer.

The following example illustrates the statement of Theorem 1.4.

Example 1.2. Consider the composite function $\Phi(u(t),y(t))$ and assume that we need to take the partial derivative with respect to $\ddot{y}(t)$ of the 3rd-order total derivative of the function. Direct computations yield

$$\frac{\partial\left(\Phi(u(t),y(t))\right)^{(3)}}{\partial\ddot{y}(t)} = 3\frac{\partial^2\Phi(u(t),y(t))}{\partial y(t)^2}\dot{y}(t) + 3\frac{\partial^2\Phi(u(t),y(t))}{\partial u(t)\partial y(t)}\dot{u}(t).$$

On the other hand, taking the partial derivative of $\Phi(u(t),y(t))$ with respect to $y(t)$ and the total derivative of the obtained result, one gets

$$\left(\frac{\partial\Phi(u(t),y(t))}{\partial y(t)}\right)^{(1)} = \frac{\partial^2\Phi(u(t),y(t))}{\partial y(t)^2}\dot{y}(t) + \frac{\partial^2\Phi(u(t),y(t))}{\partial u(t)\partial y(t)}\dot{u}(t).$$

Multiplying both sides of the equality above by C_3^1, we have

$$C_3^1\left(\frac{\partial\Phi(u(t),y(t))}{\partial y(t)}\right)^{(1)} = 3\frac{\partial^2\Phi(u(t),y(t))}{\partial y(t)^2}\dot{y}(t) + 3\frac{\partial^2\Phi(u(t),y(t))}{\partial u(t)\partial y(t)}\dot{u}(t).$$

It is not difficult to check that

$$\frac{\partial\left(\Phi(u(t),y(t))\right)^{(3)}}{\partial\ddot{y}(t)} = C_3^1\left(\frac{\partial\Phi(u(t),y(t))}{\partial y(t)}\right)^{(1)}.$$

Chapter 2

Generalized Observer Form for Continuous-Time Systems

This chapter is devoted to the problem of transforming the continuous-time nonlinear state equations into the observer form, using both the state and output transformations. The brief introduction into the issue is followed by the solvability conditions and the algorithm for finding the transformations, whenever they exist. Section 2.4 shows the advantage of the presented conditions over those suggested earlier in [31]. The example in the last section illustrates the application of the theory.

2.1 Problem Statement

Recall from Subsection 1.2.2 a SISO system, described by the state equations

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{2.1}$$

and the higher order input-output (i/o) differential equation

$$y^{(n)} = \phi\left(y, y^{(1)}, \ldots, y^{(n-1)}, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{2.2}$$

where $x(t) : \mathbb{R} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{R} \to \mathbb{U} \subset \mathbb{R}$ is an input and $y(t) : \mathbb{R} \to \mathbb{Y} \subset \mathbb{R}$ is an output. The functions $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $h : \mathbb{X} \to \mathbb{Y}$ and $\phi : \mathbb{Y}^n \times \mathbb{U}^n \to \mathbb{R}$ are real analytic. From now on, we consider system (2.1) to be observable in a sense of rank condition [25].

The purpose is to find the conditions under which there exist a local state transformation (i.e. diffeomorphism) $\psi : \mathbb{X} \to \mathbb{X}$ defined by

$$z = \psi(x) \tag{2.3}$$

and a real analytic output transformation $\Psi : \mathbb{Y} \to \mathbb{Y}$, defined by

$$Y = \Psi(y) \tag{2.4}$$

such that in the new coordinates the state equations (2.1) are in the observer form

$$\begin{aligned}
\dot{z}_1 &= z_2 + \varphi_1(Y, u) \\
&\vdots \\
\dot{z}_{n-1} &= z_n + \varphi_{n-1}(Y, u) \\
\dot{z}_n &= \varphi_n(Y, u) \\
Y &= z_1.
\end{aligned} \tag{2.5}$$

Note that the state equations (2.1) can be transformed into the observer form (2.5) with the state transformation (2.3) and output transformation (2.4), if the i/o equation (2.2), corresponding to (2.1), can be rewritten in the form

$$Y^{(n)} = (\varphi_1(Y, u))^{(n-1)} + \cdots + (\varphi_2(Y, u))^{(1)} + \varphi_n(Y, u). \tag{2.6}$$

Remark 2.1. Note that under observability assumption, one may always find the i/o representation (2.2), at least locally, using the state elimination algorithm [25], [28]. However, the global state elimination problem is a difficult task that results generally in an implicit i/o equation accompanied with a number of inequations [28].

If (2.6) holds, one can define the new state variables as

$$\begin{aligned}
z_1 &= Y, \\
z_2 &= \dot{Y} - \varphi_1(Y, u), \\
z_3 &= \ddot{Y} - (\varphi_1(Y, u))^{(1)} - \varphi_2(Y, u), \\
&\vdots \\
z_n &= Y^{(n-1)} - (\varphi_1(Y, u))^{(n-2)} - \cdots - (\varphi_2(Y, u))^{(1)} - \varphi_{n-1}(Y, u),
\end{aligned} \tag{2.7}$$

yielding the state equations in the observer form (2.5). Though (2.7) is expressed via the elements of the i/o equation (2.6), it implicitly determine the state transformation (2.3) (see Step 6 of Example 2.1).

2.2 Necessary and Sufficient Conditions

The necessary and sufficient solvability conditions are formulated in terms of certain one-forms ω_i which can be derived stepwise from the i/o equation (2.2) by means of the algorithm given below.

Consider first the one-forms

$$P_i := \sum_{q=0}^{n-1} A_i^q \mathrm{d}y^{(q)} + \sum_{q=0}^{n-1} B_i^q \mathrm{d}u^{(q)}, \quad i = 1, \dots, n,$$

whose coefficients $A_i^q, B_i^q \in \mathcal{K}^*$ can be found by setting $P_1 = \mathrm{d}\phi$ and then computing recursively, for $i = 1, \dots, n-1$

$$P_{i+1} := P_i - \omega_i^{(n-i)}, \tag{2.8a}$$

where $\omega_i^{(n-i)}$ denotes the $(n-i)$th time derivative of the one-form ω_i, defined by

$$\omega_i := A_i^{n-i} \mathrm{d}y + B_i^{n-i} \mathrm{d}u, \quad i = 1, \dots, n. \tag{2.8b}$$

Note that the algorithm above is a modification of the one given in [25]. The main difference is in skipping the step where the integration of the one-forms is required. Unlike [25], in (2.8a) instead of the functions we use the one-forms, which are not required to be integrable.

The proposition below provides the direct formula for computation of the one-forms[1] ω_i, $i = 1, \dots, n$. This formula simplifies the computations and will be used in the sequel to prove the main result, that is Theorem 2.1.

Proposition 2.1. *The one-forms ω_i, $i = 1, \dots, n$ in* (2.8) *can be computed directly by the formula*

$$\omega_i = \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \left[\left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right], \tag{2.9}$$

where C_{n-i+j}^j denotes the binomial coefficient.

The proof of Proposition 2.1 is given in the Appendix. Moreover, in order to prove Theorem 2.1, we need the following technical lemma.

Lemma 2.1.

(i) $\displaystyle\sum_{j=1}^{\varsigma} (-1)^{j-1} C_{\varsigma}^{j-1} = (-1)^{\varsigma-1}$ *for* $\varsigma \geq 1$,

[1] Alternatively the one-forms ω_i can be computed using the approach based on the notion of adjoint polynomial [35].

(ii) $\sum_{j=1}^{\varsigma-s+1}(-1)^{j-1}C_{\varsigma-s}^{j-1}=0$ *for* $s=1,\ldots,\varsigma-1$ *and* $\varsigma\geq 2$.

The proof of Lemma 2.1 is also given in the Appendix.

Furthermore, denote the composite function of $\varphi_s(Y,u)$ and $\Psi(y)$, for $s=1,\ldots,n$, as

$$\bar{\varphi}_s(y,u):=\varphi_s(\Psi(y),u)\,. \tag{2.10}$$

Now we are ready to prove the main result of this chapter.

Theorem 2.1. *The system* (2.1) *can be transformed by the state transformation* (2.3) *and the output transformation* (2.4) *into the observer form* (2.5) *iff there exists a function* $\lambda(y)$*, such that for* $\varsigma=1,\ldots,n$ *the one-forms*

$$(-1)^{\varsigma-1}C_n^{\varsigma}\lambda^{(\varsigma)}\mathrm{d}y+\sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}\lambda^{(\varsigma-i)}\omega_i, \tag{2.11}$$

where ω_i*'s are defined by* (2.9)*, are closed.*

Proof. Necessity. Assume that system (2.1) is transformable into the form (2.5). Consequently, the i/o equation (2.2) can be rewritten in the form (2.6). Complete the following steps:

- Take the partial derivatives of both sides of the i/o equation (2.6) with respect to $y^{(n-\varsigma+j-1)}$ for $j=1,\ldots,\varsigma$.

- Next, take the $(j-1)$th time-derivative of each expression, obtained in the previous step.

- Denote
$$\alpha_j:=(-1)^{j-1}C_{n-\varsigma+j-1}^{j-1} \tag{2.12}$$
and multiply by α_j both sides of the equalities obtained in the previous step.

- Sum the obtained equalities over $j=1,\ldots,\varsigma$.

Repeat the same steps with respect to the control variable u. As a result, one obtains the equalities

$$LY=RY \text{ and } LU=RU, \tag{2.13}$$

where

$$LY := \sum_{j=1}^{\varsigma} \alpha_j \left(\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)},$$

$$RY := \sum_{j=1}^{\varsigma} \sum_{s=1}^{n} \alpha_j \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)},$$

$$LU := \sum_{j=1}^{\varsigma} \alpha_j \left(\frac{\partial \Psi^{(n)}}{\partial u^{(n-\varsigma+j-1)}} \right)^{(j-1)},$$

$$RU := \sum_{j=1}^{\varsigma} \sum_{s=1}^{n} \alpha_j \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial u^{(n-\varsigma+j-1)}} \right)^{(j-1)}.$$

Note that $\Psi^{(n)}$ in LY and LU depends, besides other arguments, on $y^{(n)}$ which, according to (2.2), must be replaced by the function ϕ. In order to take this replacement into account, consider the explicit formula of the nth derivative of the output transformation $\Psi(y)$, which, according to Faà di Bruno's Formula [45], reads

$$\Psi^{(n)} = \sum \frac{n!}{k_1! \cdots k_n!} \Psi^{\underline{\overline{K}}} \prod_{\iota=1}^{n} \left(\frac{y^{(\iota)}}{\iota!} \right)^{k_\iota}, \tag{2.14}$$

where $\underline{\overline{K}} = k_1 + \cdots + k_n$ denotes the order of derivative with respect to y and the sum is taken over all possible different sets of nonnegative integers $k_1, \ldots, k_n$ being the solutions of the Diophantine equation $k_1 + 2k_2 + \cdots + nk_n = n$. It is easy to observe that $y^{(n)}$ appears in (2.14) only in the term defined by $\iota = n$ and $k_n = 1$. In this case $k_1 = \cdots = k_{n-1} = 0$ and the corresponding addend of the sum is $\Psi' y^{(n)}$, where the prime means the derivative with respect to y. In order to take the replacement (2.2) into account and avoid the complication in the further transformations of $\Psi^{(n)}$ we add to LY a formal zero term, such that LY now reads as

$$LY = \sum_{j=1}^{\varsigma} \alpha_j \left(\left(\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} + \left(\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} \right),$$

where in $\Psi^{(n)}$ we consider $y^{(n)}$ as a symbol which we do not have to replace. This trick simplifies the proof below by allowing to use Theorem 1.4.

By Theorem 1.4 for $r = 1$, $a = n - \varsigma + j - 1$ and $b = \varsigma - j + 1$,

$$\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} = C_n^{\varsigma-j+1} (\Psi')^{(\varsigma-j+1)},$$

yielding

$$LY = \sum_{j=1}^{\varsigma} \alpha_j \left(C_n^{\varsigma-j+1} (\Psi')^{(\varsigma)} + \left(\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} \right).$$

Using product rule for finding the derivative one can write

$$\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} = \Psi' \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} - \frac{\partial y^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right) + \\ + \left(\phi - y^{(n)} \right) \frac{\partial \Psi'}{\partial y^{(n-\varsigma+j-1)}}.$$

Since $n-\varsigma+j-1 < n$ for $\varsigma = 1, \ldots, n$ and $j = 1, \ldots, \varsigma$, then $\frac{\partial y^{(n)}}{\partial y^{(n-\varsigma+j-1)}} = 0$. Also taking into account that, according to (2.2), $y^{(n)} = \phi$, one obtains

$$\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} = \Psi' \frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}},$$

which, using the Leibniz Formula for the higher order derivative of the product, yields

$$\left(\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} = \sum_{i=0}^{j-1} C_{j-1}^{i} (\Psi')^{(j-1-i)} \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} \right)^{(i)}.$$

Thus, LY can be rewritten as follows

$$LY = \sum_{j=1}^{\varsigma} \alpha_j C_n^{\varsigma-j+1} (\Psi')^{(\varsigma)} + \sum_{j=1}^{\varsigma} \sum_{i=0}^{j-1} \alpha_j C_{j-1}^{i} (\Psi')^{(j-1-i)} \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} \right)^{(i)}.$$

Changing the summation order $\sum_{j=1}^{\varsigma} \sum_{i=0}^{j-1} a_{j,i} = \sum_{i=1}^{\varsigma} \sum_{j=0}^{i-1} a_{\varsigma-i+j+1,j}$ one obtains

$$LY = \sum_{j=1}^{\varsigma} \alpha_j C_n^{\varsigma-j+1} (\Psi')^{(\varsigma)} + \\ + \sum_{i=1}^{\varsigma} \sum_{j=0}^{i-1} \alpha_{\varsigma-i+j+1} C_{\varsigma-i+j}^{j} (\Psi')^{(\varsigma-i)} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)}.$$

Using (2.12) and taking into account that $(-1)^{\varsigma-i+j} = (-1)^{\varsigma-i}(-1)^j$ and that by direct computations

$$C_{n-\varsigma+j-1}^{j-1} C_n^{\varsigma-j+1} = C_n^{\varsigma} C_{\varsigma}^{j-1} \text{ and } C_{n-i+j}^{\varsigma-i+j} C_{\varsigma-i+j}^{j} = C_{n-i}^{\varsigma-i} C_{n-i+j}^{j},$$

we obtain

$$LY = C_n^{\varsigma}(\Psi')^{(\varsigma)}\sum_{j=1}^{\varsigma}(-1)^{j-1}C_{\varsigma}^{j-1}+$$
$$+\sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}\left(\Psi'\right)^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^{j}C_{n-i+j}^{j}\left(\frac{\partial\phi}{\partial y^{(n-i+j)}}\right)^{(j)}.$$

Applying (i) of Lemma 2.1 we obtain

$$LY = (-1)^{\varsigma-1}C_n^{\varsigma}(\Psi')^{(\varsigma)}+$$
$$+\sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}\left(\Psi'\right)^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^{j}C_{n-i+j}^{j}\left(\frac{\partial\phi}{\partial y^{(n-i+j)}}\right)^{(j)}.$$

Since LY and LU have a similar structure, the transformations made with LY can be made also with LU, yielding

$$LU = (-1)^{\varsigma-1}C_n^{\varsigma}\left(\frac{\partial\Psi}{\partial u}\right)^{(\varsigma)}+$$
$$+\sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}(\Psi')^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^{j}C_{n-i+j}^{j}\left(\frac{\partial\phi}{\partial u^{(n-i+j)}}\right)^{(j)},$$

which, taking into account that $\frac{\partial\Psi}{\partial u}=0$, yields

$$LU = \sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}(\Psi')^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^{j}C_{n-i+j}^{j}\left(\frac{\partial\phi}{\partial u^{(n-i+j)}}\right)^{(j)}.$$

Next consider RY. Note that if $s>\varsigma-j+1$ then $n-s<n-\varsigma+j-1$ and so $\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}}=0$. Therefore, instead of taking $s=1,\dots,n$ we can take $s=1,\dots,\varsigma-j+1$. Moreover, by Theorem 1.4 for $r=2$, $a=n-\varsigma+j-1$ and $b=\varsigma-s-j+1$

$$\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}} = C_{n-s}^{\varsigma-s-j+1}\left(\frac{\partial\bar{\varphi}_s}{\partial y}\right)^{(\varsigma-s-j+1)}.$$

Thus, one can write

$$RY = \sum_{j=1}^{\varsigma}\sum_{s=1}^{\varsigma-j+1}\alpha_j C_{n-s}^{\varsigma-s-j+1}\left(\frac{\partial\bar{\varphi}_s}{\partial y}\right)^{(\varsigma-s)}.$$

Changing the summation order $\sum_{j=1}^{\varsigma}\sum_{s=1}^{\varsigma-j+1}a_{j,s}=\sum_{s=1}^{\varsigma}\sum_{j=1}^{\varsigma-s+1}a_{j,s}$, applying (2.12) and taking into account that by direct computations

$$C_{n-\varsigma+j-1}^{j-1}C_{n-s}^{\varsigma-s-j+1}=C_{n-s}^{\varsigma-s}C_{\varsigma-s}^{j-1},$$

one obtains

$$RY = \sum_{s=1}^{\varsigma} C_{n-s}^{\varsigma-s} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s)} \sum_{j=1}^{\varsigma-s+1} (-1)^{j-1} C_{\varsigma-s}^{j-1}.$$

Note that for $\varsigma = 1$ $RY = \frac{\partial \bar{\varphi}_1}{\partial y}$. In case $\varsigma \geq 2$, one can separate the last addend of the sum RY, yielding

$$RY = \frac{\partial \bar{\varphi}_\varsigma}{\partial y} + \sum_{s=1}^{\varsigma-1} C_{n-s}^{\varsigma-s} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s)} \sum_{j=1}^{\varsigma-s+1} (-1)^{j-1} C_{\varsigma-s}^{j-1}.$$

By (ii) of Lemma 2.1, $RY = \frac{\partial \bar{\varphi}_\varsigma}{\partial y}$. In the same manner we get $RU = \frac{\partial \bar{\varphi}_\varsigma}{\partial u}$, for $\varsigma = 1, \ldots, n$.

As a result, (2.13) can be rewritten as

$$\begin{aligned}
\frac{\partial \bar{\varphi}_\varsigma}{\partial y} &= (-1)^{\varsigma-1} C_n^\varsigma (\Psi')^{(\varsigma)} + \\
&\quad + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)}, \\
\frac{\partial \bar{\varphi}_\varsigma}{\partial u} &= \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)}.
\end{aligned}$$

Adding together the equalities above and taking into account (2.9) and the notation

$$\lambda := \Psi', \tag{2.15}$$

we finally obtain the closed differential one-forms

$$\mathrm{d}\bar{\varphi}_\varsigma = (-1)^{\varsigma-1} C_n^\varsigma \lambda^{(\varsigma)} \mathrm{d}y + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} \lambda^{(\varsigma-i)} \omega_i. \tag{2.16}$$

Obviously the right-hand side of equality (2.16) equals (2.11).

Sufficiency. If there exists a function $\lambda(y)$, such that the one-forms (2.11) where ω_i's are defined by (2.9), are closed, then the function $\Psi(y)$ for the output transformation (2.4) can be calculated as an integral

$$\Psi(y) = \int \lambda(y) \mathrm{d}y. \tag{2.17}$$

Integrating the closed one-forms (2.16) one can find functions $\bar{\varphi}_\varsigma$ for $\varsigma = 1, \ldots, n$. By means of functions $\Psi(y)$ and $\bar{\varphi}_\varsigma$ the state equations in the observer form (2.5) can be easily constructed. $\square$

2.3 Algorithm

In this section we represent the algorithm for transformation of the state equations (2.1) into the observer form (2.5), whenever possible. Note that the algorithm is applied to the i/o representation (2.2) of the system (2.1) (see Remark 2.1).

Algorithm 2.1.

Step 1. Using (2.9), compute the one-forms ω_i for $i = 1, \ldots, n$.

Step 2. Keeping in mind that $\dot{\lambda} = \lambda' \dot{y}$, where prime means the derivative with respect to y, take the exterior derivative of the one-form (2.11) for $\varsigma = 1$. For this one-form to be closed its exterior derivative has to equal zero, which yields the differential two-form[2]

$$n\lambda' \mathrm{d}\dot{y} \wedge \mathrm{d}y + \lambda' \mathrm{d}y \wedge \omega_1 + \lambda \mathrm{d}\omega_1 = 0,$$

which, using (2.9), can be rewritten as

$$\begin{aligned} n\lambda' \mathrm{d}\dot{y} \wedge \mathrm{d}y + \left[\lambda' \frac{\partial \phi}{\partial u^{(n-1)}} + \lambda \left(\frac{\partial^2 \phi}{\partial y \partial u^{(n-1)}} - \frac{\partial^2 \phi}{\partial u \partial y^{(n-1)}}\right)\right] \mathrm{d}y \wedge \mathrm{d}u + \\ + \lambda \sum_{i=1}^{n-1} \left(\frac{\partial^2 \phi}{\partial y^{(i)} \partial y^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}y + \frac{\partial^2 \phi}{\partial u^{(i)} \partial y^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}y + \right. \\ \left. + \frac{\partial^2 \phi}{\partial y^{(i)} \partial u^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}u + \frac{\partial^2 \phi}{\partial u^{(i)} \partial u^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}u \right) = 0. \quad (2.18) \end{aligned}$$

To satisfy the equality, all the components of the two-form on the left-hand side of (2.18) must be zero. Summation of these components leads to the differential equation

$$\begin{aligned} \lambda' \left(n + \frac{\partial \phi}{\partial u^{(n-1)}}\right) + \lambda \left(\frac{\partial^2 \phi}{\partial y \partial u^{(n-1)}} - \frac{\partial^2 \phi}{\partial u \partial y^{(n-1)}} + \right. \\ + \sum_{i=1}^{n-1} \left(\frac{\partial^2 \phi}{\partial y^{(i)} \partial y^{(n-1)}} + \frac{\partial^2 \phi}{\partial u^{(i)} \partial y^{(n-1)}} + \right. \\ \left. \left. + \frac{\partial^2 \phi}{\partial y^{(i)} \partial u^{(n-1)}} + \frac{\partial^2 \phi}{\partial u^{(i)} \partial u^{(n-1)}} \right) \right) = 0, \quad (2.19) \end{aligned}$$

that has to be solved with respect to $\lambda(y)$. If the solution does not exist, the problem is not solvable; stop.

Step 3. Using ω_i's and $\lambda(y)$ compute the one-forms (2.11), for $\varsigma = 1, \ldots, n$.

[2]For the detailed definition of differential k-from see [25]

Step 4. Check whether the one-forms (2.11) are closed or not. If at least one of them is not closed, the problem is not solvable; stop.

Step 5. Rewrite the (closed) one-forms (2.11) as $\mathrm{d}\bar{\varphi}_\varsigma$ (see (2.16)) and integrate them, yielding $\bar{\varphi}_\varsigma$ for $\varsigma = 1, \ldots, n$. Using $\lambda(y)$ and (2.17) one can find the output transformation $\Psi(y)$ and the functions φ_ς in terms of which the system in the observer form (2.5) can be easily constructed.

Step 6. Using the functions $\varphi_1, \ldots, \varphi_n$ and the output transformation (2.4), find the system equations in the observer form (2.5).

One can observe that the most difficult part of the algorithm is Step 2, which requires the solution of partial differential equation.

2.4 Comparison with the Earlier Results

The alternative solvability condition was given earlier in [31]. We recall this condition in Theorem 2.2.

Theorem 2.2. *If the system* (2.1) *can be transformed by the state transformation* (2.3) *and the output transformation* (2.4) *into the observer form* (2.5), *then*

$$\mathrm{d}\left(\frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}}\right) \wedge \mathrm{d}y = 0. \tag{2.20}$$

Moreover, if (2.20) *is satisfied, then the possible output transformation* $\Psi(y)$ *is a solution of*

$$\Psi' \frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}} + n\Psi'' = 0, \tag{2.21}$$

where the prime and the double prime mean, respectively, the first and the second derivatives with respect to y.

Using (2.15), equality (2.18) can be rewritten as

$$\begin{aligned}
&\left(\Psi' \frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}} + n\Psi''\right) \mathrm{d}\dot{y} \wedge \mathrm{d}y + \\
&\quad + \left[\Psi'' \frac{\partial \phi}{\partial u^{(n-1)}} + \Psi' \left(\frac{\partial^2 \phi}{\partial y \partial u^{(n-1)}} - \frac{\partial^2 \phi}{\partial u \partial y^{(n-1)}}\right)\right] \mathrm{d}y \wedge \mathrm{d}u + \\
&\quad + \Psi' \sum_{i=2}^{n-1} \frac{\partial^2 \phi}{\partial y^{(i)} \partial y^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}y + \Psi' \sum_{i=1}^{n-1} \left(\frac{\partial^2 \phi}{\partial u^{(i)} \partial y^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}y + \right. \\
&\quad \left. + \frac{\partial^2 \phi}{\partial y^{(i)} \partial u^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}u + \frac{\partial^2 \phi}{\partial u^{(i)} \partial u^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}u \right) = 0.
\end{aligned} \tag{2.22}$$

Recall that the equality implies that all the components of the two-form on the left-hand side of (2.22) are zero. Note that the coefficient of $\mathrm{d}\dot{y} \wedge \mathrm{d}y$ is exactly the left-hand side of condition (2.21). Thus, (2.18) obviously implies (2.21). However, the converse does not hold, since the condition (2.21) does not guarantee that the other components of the two-form (2.18) equal to zero. It should be mentioned that (2.20) is just the exterior derivative of (2.21) multiplied by $\mathrm{d}y$. To summarize, it can be stated that conditions from Theorem 2.2 are very mild and far from being sufficient. The output transformation obtained from (2.21) does not guarantee that the i/o equation (2.2) can be represented in the form (2.6). To verify whether the problem is solvable, one has to apply the output transformation to the i/o equation (2.2) and check whether the obtained i/o equation is transformable into the observer form by the state coordinate transformation only [31].

2.5 Example

Example 2.1. Examine the model of a direct current (DC) motor, described by the equations (see [25])

$$\begin{aligned}
\dot{x}_1 &= -K_m x_1 x_2 - \frac{R_a + R_f}{K} x_1 + u \\
\dot{x}_2 &= -\frac{B}{J} x_2 - x_3 + \frac{K_m}{J} K x_1^2 \\
\dot{x}_3 &= 0 \\
y &= x_1,
\end{aligned} \tag{2.23}$$

where x_1 denotes the magnetic flux and verifies $x_1 > 0$; x_2 denotes the rotor speed; x_3 denotes the constant load torque; R_a and R_f denote the stator and the inductor resistances, respectively; B is the viscous friction coefficient, and K_m is the constant motor torque. The i/o equation, corresponding to (2.23), is

$$\begin{aligned}
y^{(3)} = \frac{B}{J}\left(\dot{u} - \ddot{y} + \frac{\dot{y}\,(\dot{y} - u)}{y}\right) - \frac{2KK_m^2 y^2 \dot{y}}{J} + \\
+ \ddot{u} - \frac{2\dot{u}\dot{y} + \ddot{y}\,(u - 3\dot{y})}{y} + \frac{2\dot{y}^2\,(u - \dot{y})}{y^2}.
\end{aligned}$$

We will follow Algorithm 2.1.

Step 1. Compute, according to (2.9),

$$\begin{aligned}
\omega_1 &= \left(-\frac{B}{J} - \frac{u-3\dot{y}}{y}\right)\mathrm{d}y + \mathrm{d}u,\\
\omega_2 &= \left(\frac{B\,(2\dot{y}-u)}{Jy} - \frac{3\ddot{y}}{y} + \frac{2u\dot{y}}{y^2} - \frac{2KK_m^2y^2}{J}\right)\mathrm{d}y+\\
&\quad+\left(\frac{B}{J} - \frac{2\dot{y}}{y}\right)\mathrm{d}u,\\
\omega_3 &= \left(\frac{(3\dot{y}-2u)\,\ddot{y}-2\dot{u}\dot{y}}{y^2} + \frac{B}{J}\left(\frac{\dot{u}-2\ddot{y}}{y} + \frac{\dot{y}^2}{y^2}\right)+\right.\\
&\quad\left.+\frac{2\,(u-\dot{y})\,\dot{y}^2}{y^3} + \frac{\ddot{u}}{y}\right)\mathrm{d}y + \left(\frac{\ddot{y}}{y} - \frac{B\dot{y}}{Jy}\right)\mathrm{d}u.
\end{aligned} \tag{2.24}$$

Step 2. The differential equation (2.19) reads as

$$4\left(\frac{\lambda}{y} + \lambda'\right) = 0,$$

solving which with respect to $\lambda(y)$, one obtains

$$\lambda(y) = \frac{1}{y}. \tag{2.25}$$

Step 3. For the case $n = 3$ the one-forms (2.11) read as

$$\begin{aligned}
3\dot{\lambda}\mathrm{d}y + \lambda\omega_1,\\
-3\ddot{\lambda}\mathrm{d}y - 2\dot{\lambda}\omega_1 + \lambda\omega_2,\\
\lambda^{(3)}\mathrm{d}y + \ddot{\lambda}\omega_1 - \dot{\lambda}\omega_2 + \lambda\omega_3,
\end{aligned}$$

which, according to (2.24) and (2.25), yield

$$\begin{aligned}
-\frac{Ju+By}{Jy^2}\mathrm{d}y + \frac{1}{y}\mathrm{d}u,\\
-\frac{Bu+2KK_m^2y^3}{Jy^2}\mathrm{d}y + \frac{B}{Jy}\mathrm{d}u,\\
0.
\end{aligned}$$

Step 4. It is not hard to verify that all three one-forms, given above, are closed, meaning that the conditions for transformation of the system (2.23) into the observer form (2.5) are satisfied.

Step 5. Now one can define

$$\begin{aligned}
\mathrm{d}\bar{\varphi}_1 &:= -\frac{Ju+By}{Jy^2}\mathrm{d}y + \frac{1}{y}\mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_2 &:= -\frac{Bu+2KK_m^2y^3}{Jy^2}\mathrm{d}y + \frac{B}{Jy}\mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_3 &:= 0,
\end{aligned}$$

integration of which yields

$$\begin{aligned}
\bar{\varphi}_1 &= -\frac{B\ln y}{J} + \frac{u}{y},\\
\bar{\varphi}_2 &= \frac{Bu}{Jy} - \frac{KK_m^2y^2}{J},\\
\bar{\varphi}_3 &= 0.
\end{aligned}$$

Taking into account (2.4), (2.17) and (2.25), one finds the output transformation

$$Y = \Psi(y) = \ln y, \tag{2.26}$$

which, according to (2.10), leads to

$$\begin{aligned}
\varphi_1 &= -\frac{BY}{J} + \frac{u}{\mathrm{e}^Y},\\
\varphi_2 &= \frac{Bu}{J\mathrm{e}^Y} - \frac{KK_m^2\mathrm{e}^{2Y}}{J},\\
\varphi_3 &= 0.
\end{aligned}$$

Step 6. By (2.7) for $n=3$, one can define the new state variables as

$$\begin{aligned}
z_1 &= Y,\\
z_2 &= \dot{Y} + \frac{BY}{J} - \frac{u}{\mathrm{e}^Y},\\
z_3 &= \ddot{Y} + \frac{B\dot{Y}}{J} - \frac{\dot{u}-u\dot{Y}}{\mathrm{e}^Y} - \frac{Bu}{J\mathrm{e}^Y} + \frac{KK_m^2\mathrm{e}^{2Y}}{J},
\end{aligned}$$

which, due to the output transformation (2.26) and state equations (2.23), can be rewritten as

$$\begin{aligned}
z_1 &= \ln x_1,\\
z_2 &= -\frac{R_a+R_f}{K} + \frac{B\ln x_1}{J} - K_m x_2,\\
z_3 &= -\frac{B\left(R_a+R_f\right)}{JK} + K_m x_3,
\end{aligned} \tag{2.27}$$

that leads to the new state equations in the observer form:

$$
\begin{aligned}
\dot{z}_1 &= z_2 - \frac{Bz_1}{J} + \frac{u}{\mathrm{e}^{z_1}} \\
\dot{z}_2 &= z_3 + \frac{Bu}{J\mathrm{e}^{z_1}} - \frac{KK_m^2\mathrm{e}^{2z_1}}{J} \\
\dot{z}_3 &= 0 \\
Y &= z_1.
\end{aligned}
\tag{2.28}
$$

Remark 2.2. Note that in [25] the output of the system (2.23) was already chosen as $y = \ln x_1$. Such farsighted choice allowed to transform the system into the observer form only by the state transformation and to avoid the necessity in the output transformation, which was not considered in [25]. Our task was to show how the output transformation $Y = \ln x_1$ can be computed. Therefore, we used the output $y = x_1$, which is more natural for the model of DC motor [21].

Chapter 3

Extended Observer Form for Discrete-Time Systems

This chapter addresses the problem of transforming the discrete-time nonlinear state equations into the extended observer form, which, besides the input and output, also depends on a finite number of their past values. The first section introduces the reader into the issue. Sections 3.2 and 3.3 provide two sets of the necessary and sufficient solvability conditions. The first set of conditions is formulated in terms of the differential one-forms and have the advantage of being intrinsic. The conditions of the second set are expressed in terms of the certain partial derivatives, related to the input-output equation of the system. Due to their matrix representation the validity of the conditions can be checked almost by direct inspection. Furthermore, Section 3.4 presents the algorithm for transforming the state equations into the extended observer form. The objective of the last section is to illustrate the application of the theory by means of examples.

3.1 Problem Statement

Recall from Subsection 1.2.3 a SISO system, described by the state equations

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x), \end{aligned} \tag{3.1}$$

and the higher order input-output (i/o) difference equation

$$y^{[n]} = \phi\left(y, y^{[1]}, \dots, y^{[n-1]}, u, u^{[1]}, \dots, u^{[n-1]}\right), \tag{3.2}$$

where $x(t) : \mathbb{Z} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{Z} \to \mathbb{U} \subset \mathbb{R}$ is an input and $y(t) : \mathbb{Z} \to \mathbb{Y} \subset \mathbb{R}$ is an output. The functions $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, $h : \mathbb{X} \to \mathbb{Y}$ and $\phi : \mathbb{Y}^n \times \mathbb{U}^n \to \mathbb{R}$ are real analytic.

From now on, we consider system (3.1) to be observable in a sense of rank condition [55].

The purpose is to find the conditions under which there exist an extended coordinate change[1] $\psi(\cdot, \xi_1, \ldots, \xi_{2N}) : \mathbb{X} \to \mathbb{X}$, parameterized by $(\xi_1, \ldots, \xi_{2N})$ and defined by

$$z = \psi\left(x, y^{[-1]}, \ldots, y^{[-N]}, u^{[-1]}, \ldots, u^{[-N]}\right), \tag{3.3}$$

as well as an output transformation (i.e. diffeomorphism) $\Psi : \mathbb{Y} \to \mathbb{Y}$, defined by

$$Y = \Psi(y), \tag{3.4}$$

such that in the new coordinates the state equations (3.1) are in the following extended observer form with buffer $N \in \{1, \ldots, n-2\}$

$$\begin{aligned}
z_1^+ &= z_2 + \varphi_1\left(Y, Y^{[-1]}, \ldots, Y^{[-N]}, u, u^{[-1]}, \ldots, u^{[-N]}\right) \\
&\vdots \\
z_{n-N}^+ &= z_{n-N+1} + \varphi_{n-N}\left(Y, Y^{[-1]}, \ldots, Y^{[-N]}, u, u^{[-1]}, \ldots, u^{[-N]}\right) \\
z_{n-N+1}^+ &= z_{n-N+2} \\
&\vdots \\
z_{n-1}^+ &= z_n \\
z_n^+ &= 0 \\
Y &= z_1,
\end{aligned} \tag{3.5}$$

where the forward shift of the coordinates z depends besides the input u and the output y also on their past values $u^{[-1]}, \ldots, u^{[-N]}$, and $y^{[-1]}, \ldots, y^{[-N]}$. This form without inputs was considered earlier in [40], [41]. We do not address the case when the buffer $N = n-1$ since, as shown in by Huijberts et al. in [41], the system can be always transformed into such form (even without the output transformation), whenever the system under consideration is strongly observable. The proof carries over to systems depending on control too. Therefore, it is obvious that the results of this chapter address only the case $n \geq 3$.

Note that the state equations (3.1) can be transformed into the extended observer form (3.5) by means of the extended coordinate change (3.3) and the output transformation (3.4), if the i/o equation (3.2) corresponding to (3.1), can be rewritten in the form

$$\Psi \circ \phi = \sum_{l=1}^{n-N} \varphi_l\left(Y^{[n-l]}, \ldots, Y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]}\right). \tag{3.6}$$

[1]The details about the properties of the extended coordinate change can be found in [40] for the case of autonomous systems.

Remark 3.1. Note that under observability assumption, one may always find the i/o representation (3.2), at least locally, using the state elimination algorithm. However, the global state elimination problem is a difficult task that results generally in an implicit i/o equation accompanied with a number of inequations [28].

If (3.6) holds, one can define the new state variables as

$$\begin{aligned} z_1 &= Y, \\ z_i &= Y^{[i-1]} - \sum_{l=1}^{\min(i-1,n-N)} \varphi_l\Big(Y^{[i-1-l]}, \dots, Y^{[i-1-l-N]}, \\ &\qquad u^{[i-1-l]}, \dots, u^{[i-1-l-N]}\Big), \quad i = 2, \dots, n, \end{aligned} \tag{3.7}$$

that leads to the state equations in the extended observer form (3.5).

3.2 Intrinsic Necessary and Sufficient Conditions

In this section the conditions will be formulated in terms of differential one-form, associated with the i/o equation (3.2), corresponding to the state equations (3.1). Define for $i = 0, \dots, n-1$ the differential one-forms

$$\omega_i := \frac{\partial \phi}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \phi}{\partial u^{[i]}} \mathrm{d}u^{[i]} \tag{3.8}$$

and the codistributions

$$\begin{aligned} \Omega_i := \operatorname{span}_{\overline{\mathcal{K}^*}} \Big\{ \omega_k, \mathrm{d}u^{[k]} \mid k \neq i,\ k = \max(0, i-N), \dots, \\ \min(i+N, n-1) \Big\}. \end{aligned} \tag{3.9}$$

For example, if $N = 1$ and $n = 5$, then

$$\begin{aligned} \Omega_0 &= \operatorname{span}_{\overline{\mathcal{K}^*}} \left\{ \omega_1, \mathrm{d}u^+ \right\}, \\ \Omega_1 &= \operatorname{span}_{\overline{\mathcal{K}^*}} \left\{ \omega_0, \mathrm{d}u, \omega_2, \mathrm{d}u^{++} \right\}, \\ \Omega_2 &= \operatorname{span}_{\overline{\mathcal{K}^*}} \left\{ \omega_1, \mathrm{d}u^+, \omega_3, \mathrm{d}u^{[3]} \right\}, \\ \Omega_3 &= \operatorname{span}_{\overline{\mathcal{K}^*}} \left\{ \omega_2, \mathrm{d}u^{++}, \omega_4, \mathrm{d}u^{[4]} \right\}, \\ \Omega_4 &= \operatorname{span}_{\overline{\mathcal{K}^*}} \left\{ \omega_3, \mathrm{d}u^{[3]} \right\}. \end{aligned}$$

The minimal number of independent generators of a codistribution is called its dimension. For an arbitrary one-form ω and an r-dimensional codistribution $\Omega = \operatorname{span}_{\overline{\mathcal{K}^*}} \{\upsilon_1, \dots, \upsilon_r\}$, we will say that

$$\mathrm{d}\omega \equiv 0 \mod \Omega$$

if and only if

$$\mathrm{d}\omega \wedge \upsilon_1 \wedge \cdots \wedge \upsilon_r = 0.$$

Moreover, define the composite functions of φ_l and Ψ as

$$\bar{\varphi}_l\left(y, y^{[-1]}, \ldots, y^{[-N]}, u, u^{[-1]}, \ldots, u^{[-N]}\right) := \\ \varphi_l\left(Y, Y^{[-1]}, \ldots, Y^{[-N]}, u, u^{[-1]}, \ldots, u^{[-N]}\right)$$

and the vector argument

$$\nu_l := \left[y^{[n-l]}, \ldots, y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]}\right] \tag{3.10}$$

for $l = 1, \ldots, n-N$. The latter will be used in the sequel to simplify the exposition. In order to prove the main result of this section, that is Theorem 3.1 below, we need the following lemma, the proof of which is given in the Appendix.

Lemma 3.1. *For functions $\bar{\varphi}_1(\nu_1), \ldots, \bar{\varphi}_{n-N}(\nu_{n-N})$ the following holds*

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{i=0}^{n-1} \Upsilon_i, \tag{3.11}$$

where

$$\Upsilon_i := \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \right. \\ \left. + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}} \mathrm{d}u^{[i]} \right). \tag{3.12}$$

Theorem 3.1. *The system* (3.1) *can be transformed by the extended coordinate change* (3.3) *and the output transformation* (3.4) *into the extended observer form* (3.5) *with buffer $N \in \{1, \ldots, n-2\}$ if and only if for all $0 \leq i, j \leq n-1$*

$$\mathrm{d}\omega_i \wedge \omega_j + \mathrm{d}\omega_j \wedge \omega_i \equiv 0 \mod \left((\Omega_i + \Omega_j) \setminus \operatorname{span}_{\overline{\mathcal{K}^*}} \{\omega_i, \omega_j\}\right). \tag{3.13}$$

Remark 3.2. Note that in (3.13) the buffer N is hidden inside the definition of the codistributions Ω_i and Ω_j.

Now we are ready to prove the main result of this section.

Proof. Necessity. Assume that system (3.1) is transformable into the extended observer form (3.5). Consequently, the i/o equation (3.2), corresponding to (3.1), can be rewritten in the form (3.6), the total differential of which reads as

$$\left(\Psi' \circ \phi\right) \mathrm{d}\phi = \left(\Psi' \circ \phi\right) \sum_{i=0}^{n-1} \omega_i = \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l\left(\nu_l\right),$$

where $\Psi' \circ \phi$ means the derivative of the function Ψ evaluated at ϕ. According to Lemma 3.1,

$$\left(\Psi' \circ \phi\right) \sum_{i=0}^{n-1} \omega_i = \sum_{i=0}^{n-1} \Upsilon_i. \tag{3.14}$$

From (3.14) we have for $i = 0, \ldots, n-1$

$$\left(\Psi' \circ \phi\right) \omega_i = \Upsilon_i. \tag{3.15}$$

Consider the functions $\bar{\varphi}_{n-N-l}\left(\nu_{n-N-l}\right)$ for $l = \max(0, i-N), \ldots, \min(i, n-1-N)$. Taking into account (3.10) for new index $n-N-l$, one can write

$$\mathrm{d}\bar{\varphi}_{n-N-l}\left(\nu_{n-N-l}\right) = \sum_{s=0}^{N}\Bigg(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}}\mathrm{d}y^{[l+s]} + \\ + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}}\mathrm{d}u^{[l+s]}\Bigg). \tag{3.16}$$

Note that the codistribution Ω_i, defined by (3.9), can be rewritten as

$$\Omega_i = \operatorname{span}_{\overline{\mathcal{K}^*}}\Big\{\mathrm{d}y^{[l+s]}, \mathrm{d}u^{[l+s]} \mid l+s \neq i, s = 0, \ldots, N, \\ l = \max(0, i-N), \ldots, \min(i, n-1-N)\Big\}. \tag{3.17}$$

As a consequence, from (3.16) one obtains

$$\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) \equiv \Bigg(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}}\mathrm{d}y^{[i]} + \\ + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}}\mathrm{d}u^{[i]}\Bigg) \mod \Omega_i, \tag{3.18}$$

which, by (3.15) and (3.12), leads to

$$\left(\Psi' \circ \phi\right) \omega_i \equiv \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) \mod \Omega_i.$$

Application of the exterior derivative to the equality above yields

$$\mathrm{d}\left(\Psi' \circ \phi\right) \wedge \omega_i + \left(\Psi' \circ \phi\right) \mathrm{d}\omega_i \equiv 0 \quad \mathrm{mod}\ \Omega_i.$$

From the relationship above we obtain

$$\mathrm{d}\omega_i \equiv -\mathrm{d}\ln\left|\Psi' \circ \phi\right| \wedge \omega_i \quad \mathrm{mod}\ \Omega_i$$

for $i = 0, \ldots, n-1$. Obviously,

$$\mathrm{d}\omega_i \wedge \omega_j \equiv -\mathrm{d}\ln\left|\Psi' \circ \phi\right| \wedge \omega_i \wedge \omega_j \quad \mathrm{mod}\ \left((\Omega_i + \Omega_j) \setminus \mathrm{span}_{\overline{\mathcal{K}^*}}\left\{\omega_i, \omega_j\right\}\right),$$

using which, one gets for $i, j = 0, \ldots, n-1$

$$\begin{aligned}\mathrm{d}\omega_i \wedge \omega_j + \mathrm{d}\omega_j \wedge \omega_i \equiv -\mathrm{d}\ln\left|\Psi' \circ \phi\right| \wedge (\omega_i \wedge \omega_j +& \\ + \omega_j \wedge \omega_i) \quad \mathrm{mod}\ \left((\Omega_i + \Omega_j) \setminus \mathrm{span}_{\overline{\mathcal{K}^*}}\left\{\omega_i, \omega_j\right\}\right).& \quad (3.19)\end{aligned}$$

Since the wedge product is anticommutative, the expression in the parentheses on the right-hand side of (3.19) is always zero, which yields (3.13).

Sufficiency. The proof consists of three steps. On the first step (i) we will show that under the conditions (3.13) there exist functions $\chi_l(\nu_l)$ for $l = 1, \ldots, n-N$, such that

$$\omega_i \equiv \lambda_i \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\chi_{n-N-l}(\nu_{n-N-l}) \quad \mathrm{mod}\ \Omega_i \tag{3.20}$$

for $i = 0, \ldots, n-1$. On the second step (ii) we will prove that for all ω_i there exists the common integrating factor λ, and finally, on the last step (iii) we will show that from steps (i) and (ii) follows the existence of output transformation Ψ such that its composition with ϕ yields (3.6).

(i) Note that in case $i = j$ (3.13) yields

$$\mathrm{d}\omega_i \wedge \omega_i \equiv 0 \quad \mathrm{mod}\ \Omega_i, \tag{3.21}$$

from which follows the existence of the integrating factor $\lambda_i(y,\, y^{[-1]},\, \ldots,\, y^{[n-1]},\, u,\, u^{[-1]},\, \ldots,\, u^{[n-1]})$ such that

$$\omega_i \equiv \lambda_i \mathrm{d}\bar{\chi}_i(\bar{\nu}_i) \quad \mathrm{mod}\ \Omega_i \tag{3.22}$$

for some functions[2] $\bar{\chi}_i(\bar{\nu}_i)$, where $\bar{\nu}_i$ is the vector argument which consists of the elements of the set $\left\{y^{[k]}, u^{[k]} \mid k = \max(0, i-N), \ldots, \min(i+N, n-1)\right\}$. Note that, taking into account (3.8) and (3.9), according to (3.22),

$$\frac{1}{\lambda_i}\omega_i = \frac{\partial \bar{\chi}_i(\bar{\nu}_i)}{\partial y^{[i]}}\mathrm{d}y^{[i]} + \frac{\partial \bar{\chi}_i(\bar{\nu}_i)}{\partial u^{[i]}}\mathrm{d}u^{[i]}. \tag{3.23}$$

[2]The functions $\bar{\chi}_i(\bar{\nu}_i)$ should not be confused with the functions $\chi_l(\nu_l)$ in (3.20). The number of functions $\bar{\chi}_i(\bar{\nu}_i)$ is n, but the number of functions $\chi_l(\nu_l)$ is $n-N$. Moreover, the vector arguments $\bar{\nu}_i$ and ν_l have different number of elements.

Choose the function $\zeta\left(y, y^{[1]}, \dots, y^{[n-1]}, u, u^{[1]}, \dots, u^{[n-1]}\right)$ such that

$$\frac{\partial\zeta}{\partial y^{[i]}}\mathrm{d}y^{[i]} + \frac{\partial\zeta}{\partial u^{[i]}}\mathrm{d}u^{[i]} = \frac{\partial\bar{\chi}_i(\bar{\nu}_i)}{\partial y^{[i]}}\mathrm{d}y^{[i]} + \frac{\partial\bar{\chi}_i(\bar{\nu}_i)}{\partial u^{[i]}}\mathrm{d}u^{[i]} \tag{3.24}$$

for $i = 1, \dots, n-1$, and consequently

$$\sum_{i=0}^{n-1} \frac{1}{\lambda_i}\omega_i = \mathrm{d}\zeta.$$

As we will show in the sequel, the function ζ really exists and can be represented in the form

$$\zeta = \sum_{l=1}^{n-N} \chi_l(\nu_l) \tag{3.25}$$

for some functions $\chi_1(\nu_1), \dots, \chi_{n-N}(\nu_{n-N})$. Note that (3.25) holds, if the following second order partial derivatives of ζ equal zero,

$$\begin{aligned}
&\frac{\partial^2\zeta}{\partial y^{[i]}\partial y^{[j]}} = 0, \quad \frac{\partial^2\zeta}{\partial u^{[i]}\partial u^{[j]}} = 0,\\
&\frac{\partial^2\zeta}{\partial u^{[i]}\partial y^{[j]}} = 0, \quad \frac{\partial^2\zeta}{\partial y^{[i]}\partial u^{[j]}} = 0
\end{aligned} \tag{3.26}$$

for $i, j = 0, \dots, n-1$, $j \neq i-N, \dots, i+N$. Our next purpose is to prove that (3.26) holds. Taking into account (3.23), (3.24) and (3.8), one can rewrite (3.26) as follows

$$\begin{aligned}
\frac{\partial\lambda_i^{-1}}{\partial y^{[j]}}\frac{\partial\phi}{\partial y^{[i]}} + \lambda_i^{-1}\frac{\partial^2\phi}{\partial y^{[i]}\partial y^{[j]}} &= 0,\\
\frac{\partial\lambda_i^{-1}}{\partial u^{[j]}}\frac{\partial\phi}{\partial u^{[i]}} + \lambda_i^{-1}\frac{\partial^2\phi}{\partial u^{[i]}\partial u^{[j]}} &= 0,\\
\frac{\partial\lambda_i^{-1}}{\partial y^{[j]}}\frac{\partial\phi}{\partial u^{[i]}} + \lambda_i^{-1}\frac{\partial^2\phi}{\partial u^{[i]}\partial y^{[j]}} &= 0,\\
\frac{\partial\lambda_i^{-1}}{\partial u^{[j]}}\frac{\partial\phi}{\partial y^{[i]}} + \lambda_i^{-1}\frac{\partial^2\phi}{\partial y^{[i]}\partial u^{[j]}} &= 0.
\end{aligned} \tag{3.27}$$

Expressing $\partial\lambda_i^{-1}/\partial y^{[j]}$ from the first equality of (3.27) and substituting it into the third equality, and also expressing $\partial\lambda_i^{-1}/\partial u^{[j]}$ from the second equality and substituting it into the fourth equality, one obtains

$$\begin{aligned}
\frac{\partial\phi}{\partial y^{[i]}}\frac{\partial^2\phi}{\partial u^{[i]}\partial y^{[j]}} - \frac{\partial\phi}{\partial u^{[i]}}\frac{\partial^2\phi}{\partial y^{[i]}\partial y^{[j]}} &= 0,\\
\frac{\partial\phi}{\partial u^{[i]}}\frac{\partial^2\phi}{\partial y^{[i]}\partial u^{[j]}} - \frac{\partial\phi}{\partial y^{[i]}}\frac{\partial^2\phi}{\partial u^{[i]}\partial u^{[j]}} &= 0.
\end{aligned}$$

It is easy to verify that under the conditions (3.21) the equalities above are satisfied and, as a consequence, the function ζ really exists, satisfying (3.25), which yields

$$\sum_{i=0}^{n-1} \frac{1}{\lambda_i}\omega_i = \sum_{l=1}^{n-N} \mathrm{d}\chi_l(\nu_l), \tag{3.28}$$

from which, using Lemma 3.1 and (3.18) for functions $\chi_l(\nu_l)$, one obtains (3.20).

(ii) Take the exterior derivative of (3.20) and then apply (3.20) as follows

$$\sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\chi_{n-N-l}(\nu_{n-N-l}) \equiv \frac{1}{\lambda_i}\omega_i \mod \Omega_i.$$

This yields

$$\mathrm{d}\omega_i \equiv \mathrm{d}\lambda_i \wedge \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\chi_{n-N-l}(\nu_{n-N-l}) \equiv \mathrm{d}\ln|\lambda_i| \wedge \omega_i \mod \Omega_i.$$

By the conditions (3.13)

$$(\mathrm{d}\ln|\lambda_i| - \mathrm{d}\ln|\lambda_j|) \wedge \omega_i \wedge \omega_j \equiv 0 \mod \left((\Omega_i + \Omega_j) \setminus \mathrm{span}_{\overline{\mathcal{K}^*}}\{\omega_i, \omega_j\}\right),$$

from which follows $\lambda_i = \lambda_j = \lambda$ for $i, j = 0, \ldots, n-1$.

(iii) From (i) and (ii) follows that one can find functions $\bar{\varphi}_l(\nu_l)$, $l = 1, \ldots, n-N$ for which there exists the common integrating factor λ such that (3.28) can be rewritten as

$$\sum_{i=0}^{n-1} \omega_i = \lambda \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l). \tag{3.29}$$

Since $\mathrm{d}\phi$ is a total differential, its exterior derivative

$$\mathrm{d}^2\phi = \sum_{i=0}^{n-1} \mathrm{d}\omega_i = \mathrm{d}\lambda \wedge \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \mathrm{d}\ln|\lambda| \wedge \sum_{i=0}^{n-1} \omega_i = \mathrm{d}\ln|\lambda| \wedge \mathrm{d}\phi$$

equals zero and by Cartan's Lemma $\mathrm{d}\ln|\lambda| \in \mathrm{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}\phi\}$. Therefore, λ can be represented as a composite function of ϕ and some other function. We will show below that the choice $\lambda = 1/(\Psi' \circ \phi)$ guarantees, that the composite function $\Psi \circ \phi$ has the form (3.6). First, we prove that $\Psi' \circ \phi$ is the common integrating factor for all one-forms ω_i, that is

$$\left(\Psi' \circ \phi\right)\omega_i \equiv \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) \mod \Omega_i.$$

Taking the exterior derivative of $(\Psi' \circ \phi)\,\omega_i$, one obtains

$$\begin{aligned}\mathrm{d}\left[\left(\Psi' \circ \phi\right)\omega_i\right] &\equiv \left(\Psi'' \circ \phi\right)\mathrm{d}\phi \wedge \omega_i + \left(\Psi' \circ \phi\right)\mathrm{d}\omega_i \equiv \\ &\equiv \left(\Psi'' \circ \phi\right)\mathrm{d}\phi \wedge \omega_i + \left(\Psi' \circ \phi\right)\mathrm{d}\ln|\lambda| \wedge \omega_i \equiv \\ &\equiv \left(\Psi'' \circ \phi\right)\mathrm{d}\phi \wedge \omega_i - \mathrm{d}\left(\ln\left|\Psi' \circ \phi\right|\right)\left(\Psi' \circ \phi\right) \wedge \omega_i \equiv 0 \mod \Omega_i,\end{aligned}$$

meaning the functions $\bar{\varphi}_l(\nu_l)$ really exist. Finally, multiplying $\mathrm{d}\phi$ by $\Psi' \circ \phi$ and taking into account (3.29), one obtains

$$\mathrm{d}\left(\Psi \circ \phi\right) = \left(\Psi' \circ \phi\right)\mathrm{d}\phi = \left(\Psi' \circ \phi\right)\sum_{i=0}^{n-1}\omega_i = \sum_{l=1}^{n-N}\mathrm{d}\bar{\varphi}_l(\nu_l),$$

yielding (3.6). □

3.3 Simple Necessary and Sufficient Conditions

This section presents the conditions expressed in terms of partial derivatives, related to the i/o equation (3.2), corresponding to the state equations (3.1). In order to present the theorem and the proof in a more compact form, denote by α the variable, which can be either u or y. Then by β is denoted u, if α is y and y if α is u. Moreover, denote by $\overline{j_\alpha}$ and $\underline{j_\alpha}$, respectively, the highest and the lowest shifts of α the function ϕ depends on. For example, if $\phi\left(y, y^{[2]}, y^{[3]}, u^{[1]}, u^{[2]}, u^{[4]}\right)$, then $\underline{j_y} = 0$, $\underline{j_u} = 1$, $\overline{j_y} = 3$ and $\overline{j_u} = 4$.

Theorem 3.2. *The system* (3.1) *can be transformed by the extended coordinate change* (3.3) *and the output transformation* (3.4) *into the extended observer form* (3.5) *with buffer* $N \in \{1, \ldots, n-2\}$ *if and only if there exists a function* $S\left(y, \ldots, y^{[n-1]}, u, \ldots, u^{[n-1]}\right)$ *such that for* $i, j = 0, \ldots, n-1$, $j \neq i - N, \ldots, i + N$

$$\frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial \phi}{\partial \alpha^{[i]}}\right|\right) = \frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial \phi}{\partial \beta^{[i]}}\right|\right) =: \frac{\partial S}{\partial \alpha^{[j]}}, \tag{3.30a}$$

and in case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ *the function* S *satisfies for* $r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$ *and an arbitrary* $j \neq r$ *the following additional conditions*

$$\frac{\partial S}{\partial \alpha^{[j]}}\frac{\partial \phi}{\partial \alpha^{[r]}}\left(\frac{\partial \phi}{\partial \alpha^{[j]}}\right)^{-1} = \frac{\partial S}{\partial \beta^{[j]}}\frac{\partial \phi}{\partial \alpha^{[r]}}\left(\frac{\partial \phi}{\partial \beta^{[j]}}\right)^{-1} =: \frac{\partial S}{\partial \alpha^{[r]}}. \tag{3.30b}$$

Remark 3.3. Note that in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ the conditions (3.30a) are both necessary and sufficient, but in the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ they are only necessary, and for sufficiency[3] one needs the additional conditions (3.30b), which in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ hold by (3.30a).

Remark 3.4. Suppose that for some $q = 0, \ldots, n-1$ the function ϕ (and S, as a consequence) does not depend on the variable $y^{[q]}$ (or $u^{[q]}$). In this case either the left-hand side or the middle part in the corresponding condition of (3.30a) should be omitted, depending on whether the $\alpha^{[i]}$ or $\beta^{[i]}$ stands for $y^{[q]}$ (or $u^{[q]}$). Thus, for instance, in the case of system without input one obtains the conditions (3.30a) where $\alpha = y$ and middle part is omitted.

Remark 3.5. If the conditions (3.30a) are satisfied it is enough to check the conditions (3.30b) only for one $j \neq r$. However, one has to choose (if possible) j such that the function ϕ (and S, as a consequence) depends on both $y^{[j]}$ and $u^{[j]}$. If such a choice is not possible, then either the left-hand side or the middle part of (3.30b) should be omitted, depending on whether $\alpha^{[j]}$ or $\beta^{[j]}$ stands for the variable, the function ϕ does not depend on. Thus, for instance, in the case of system without input one obtains the conditions (3.30b) where $\alpha = y$ and the middle part is omitted.

Remark 3.6. Taking $N = 0$, the conditions (3.30a) (and (3.30b) for the special case $\overline{j_\alpha} = \underline{j_\alpha}$) can be used to check whether the system is transformable into the observer form without the buffer (see the different results in [40] for systems without input and [78] for input dependent systems).

In order to prove Theorem 3.2, we need the following lemma, the proof of which is given in the Appendix.

Lemma 3.2. *From conditions* (3.30a) (*and in the case* $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ (3.30b)) *follows*

$$\mathrm{d}S \wedge \mathrm{d}\phi = 0. \tag{3.31}$$

Now we are ready to prove the main result of this section.

Proof. Necessity. Assume that system (3.1) is transformable into the extended observer form (3.5). Consequently, the i/o equation (3.2), corresponding to (3.1), can be rewritten in the form (3.6), yielding that the following second-order partial derivatives of the composition $\Psi \circ \phi$ equal to

[3] Without going into details, one can say that in order to prove sufficiency we need $\frac{\partial S}{\partial \alpha^{[j]}}$ for all $j = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. However, we should take into account that in conditions (3.30a) index j depends on index i and buffer N. This dependency implies that $j = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1, \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}$, which in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ yields that j runs from $\underline{j_\alpha}$ to $\overline{j_\alpha}$ without interruption, whereas in the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ there is a gap between $\overline{j_\alpha} - N - 1$ and $\underline{j_\alpha} + N + 1$. To compensate this gap we use (3.30b) in addition to (3.30b) (in other words, index r complements j).

zero for $i,j=0,\dots,n-1$, $j\neq i-N,\dots,i+N$:

$$\begin{aligned}\frac{\partial^2\left(\Psi\circ\phi\right)}{\partial\alpha^{[i]}\partial\alpha^{[j]}}&=\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}\frac{\partial\phi}{\partial\alpha^{[i]}}+\left(\Psi'\circ\phi\right)\frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\alpha^{[j]}}=0,\\ \frac{\partial^2\left(\Psi\circ\phi\right)}{\partial\beta^{[i]}\partial\alpha^{[j]}}&=\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}\frac{\partial\phi}{\partial\beta^{[i]}}+\left(\Psi'\circ\phi\right)\frac{\partial^2\phi}{\partial\beta^{[i]}\partial\alpha^{[j]}}=0,\end{aligned}\tag{3.32}$$

where $\Psi'\circ\phi$ means the derivative of the function Ψ evaluated at ϕ. Dividing the first equation of (3.32) by $(\Psi'\circ\phi)\left(\partial\phi/\partial\alpha^{[i]}\right)$ and the second equation by $(\Psi'\circ\phi)\left(\partial\phi/\partial\beta^{[i]}\right)$ yields

$$\begin{aligned}&\frac{1}{\Psi'\circ\phi}\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}+\frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\alpha^{[j]}}\left(\frac{\partial\phi}{\partial\alpha^{[i]}}\right)^{-1}=\\ &\qquad=\frac{\partial\ln\left|\Psi'\circ\phi\right|}{\partial\alpha^{[j]}}+\frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\alpha^{[i]}}\right|\right)=0,\\ &\frac{1}{\Psi'\circ\phi}\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}+\frac{\partial^2\phi}{\partial\beta^{[i]}\partial\alpha^{[j]}}\left(\frac{\partial\phi}{\partial\beta^{[i]}}\right)^{-1}=\\ &\qquad=\frac{\partial\ln\left|\Psi'\circ\phi\right|}{\partial\alpha^{[j]}}+\frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\beta^{[i]}}\right|\right)=0.\end{aligned}$$

The equalities above suggest that the function

$$S=-\ln\left|\Psi'\circ\phi\right|\tag{3.33}$$

will make the conditions (3.30a) and (3.30b) to hold.

Sufficiency. Suppose the conditions (3.30a) and (3.30b) are satisfied. Then, according to Lemma 3.2, $\mathrm{d}S\wedge\mathrm{d}\phi=0$, which by Cartan's Lemma yields $\mathrm{d}S\in\operatorname{span}_{\overline{\mathcal{K}^*}}\{\mathrm{d}\phi\}$. Therefore, the function S can be represented as a composition of some function $\widehat{\Psi}$ with ϕ, i.e. $S=\widehat{\Psi}\circ\phi$. We will show below that the choice $S=-\ln|\Psi'\circ\phi|$ guarantees that the equalities (3.32) are satisfied, meaning that the composition $\Psi\circ\phi$ has the form (3.6). Replacing the function S in (3.30a) by the expression $-\ln|\Psi'\circ\phi|$, one obtains

$$\begin{aligned}\frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\alpha^{[i]}}\right|\right)&=-\frac{\partial\ln|\Psi'\circ\phi|}{\partial\alpha^{[j]}},\\ \frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\beta^{[i]}}\right|\right)&=-\frac{\partial\ln|\Psi'\circ\phi|}{\partial\alpha^{[j]}}.\end{aligned}$$

By the derivative of the logarithmic function, one can rewrite the equalities, given above, as

$$\begin{aligned}\left(\frac{\partial\phi}{\partial\alpha^{[i]}}\right)^{-1}\frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\alpha^{[j]}}+\frac{1}{\Psi'\circ\phi}\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}&=0,\\ \left(\frac{\partial\phi}{\partial\beta^{[i]}}\right)^{-1}\frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\beta^{[j]}}+\frac{1}{\Psi'\circ\phi}\frac{\partial\left(\Psi'\circ\phi\right)}{\partial\alpha^{[j]}}&=0.\end{aligned}$$

Multiplying the first equality by $(\Psi' \circ \phi)\left(\partial\phi/\partial\alpha^{[i]}\right)$ and the second by $(\Psi' \circ \phi)\left(\partial\phi/\partial\beta^{[i]}\right)$ yields (3.32). This completes the proof. $\square$

3.3.1 Matrix Representation of the Conditions

In this subsection we represent the conditions (3.30a) and (3.30b) in the matrix form, which makes them easier to check by direct inspection.

Denote by $A^{\alpha,\alpha}$ and $A^{\alpha,\beta}$ the $n \times n$ matrices, whose elements are defined by ($i = 0, \dots, n-1$ pointing to the row and $j = 0, \dots, n-1$ to the column)

$$a_{i,j}^{\alpha,\alpha} := \begin{cases} 0, & \begin{array}{l} j = i - N, \dots, i + N, \text{ or} \\ \phi \text{ does not depend on } \alpha^{[i]}, \end{array} \\ \dfrac{\partial}{\partial \alpha^{[j]}} \left(\ln \left| \dfrac{\partial \phi}{\partial \alpha^{[i]}} \right| \right), & \text{otherwise,} \end{cases}$$

and

$$a_{i,j}^{\alpha,\beta} := \begin{cases} 0, & \begin{array}{l} j = i - N, \dots, i + N, \text{ or} \\ \phi \text{ does not depend on } \beta^{[i]}, \end{array} \\ \dfrac{\partial}{\partial \alpha^{[j]}} \left(\ln \left| \dfrac{\partial \phi}{\partial \beta^{[i]}} \right| \right), & \text{otherwise,} \end{cases}$$

respectively. Thus, the matrices contain zeros on the main diagonal and N diagonals above and below it. Moreover, if the function ϕ does not depend on the variable $y^{[i]}$ or $u^{[i]}$ for some $i = 0, \dots, n-1$, then the corresponding elements of the matrices are zeros too. Also denote the $2n \times 2n$ matrix as

$$A := \begin{bmatrix} A^{y,y} & A^{u,y} \\ A^{y,u} & A^{u,u} \end{bmatrix}. \tag{3.34}$$

Proposition 3.1. *If the conditions* (3.30a) *hold, then in every column of the matrix A all nonzero elements are equal.*

Remark 3.7. Note that if the function ϕ depends on the variables $y^{[q]}$ or $u^{[q]}$ for all $q = 0, \dots, n-1$, then $A^{\alpha,\alpha} = A^{\alpha,\beta}$.

If in every column of the matrix A all nonzero elements are equal, one needs to check whether there exists a function S such that for $j = 0, \dots, n-1$ the nonzero elements of the $(j+1)$th and $(j+1+n)$th columns are equal to $\partial S/\partial y^{[j]}$ and $\partial S/\partial u^{[j]}$, respectively. In the case $2N \geq \overline{j} - \underline{j}$ (where $\overline{j} := \max(\overline{j_y}, \overline{j_u})$ and $\underline{j} := \min(\underline{j_y}, \underline{j_u})$), the matrix A does not contain nonzero elements in the $(\overline{j} - N + 1)$th up to $(\underline{j} + N + 1)$th and $(\overline{j} - N + n + 1)$th up to $(\underline{j} + N + n + 1)$th columns. As a consequence, the conditions for corresponding partial derivatives of S are absent. The additional conditions (3.30b) compensate this aspect. In order to represent the conditions (3.30b) in the matrix form, denote by $B^{\alpha,\alpha}$ and $B^{\alpha,\beta}$ the $(2N + \underline{j_\alpha} - \overline{j_\alpha} + 1) \times 1$ vectors whose elements are defined by

$$b_r^{\alpha,\alpha} := \frac{\partial S}{\partial \alpha^{[j]}} \frac{\partial \phi}{\partial \alpha^{[r]}} \left(\frac{\partial \phi}{\partial \alpha^{[j]}} \right)^{-1}$$

and

$$b_r^{\alpha,\beta} := \frac{\partial S}{\partial \beta^{[j]}} \frac{\partial \phi}{\partial \alpha^{[r]}} \left(\frac{\partial \phi}{\partial \beta^{[j]}} \right)^{-1},$$

respectively, where $r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$. The value of index j should be chosen according to Remark 3.5 and $\partial S/\partial \alpha^{[j]}$, $\partial S/\partial \beta^{[j]}$ can be calculated from (3.30a).

Proposition 3.2. *If the conditions* (3.30b) *hold, then* $B^{\alpha,\alpha} = B^{\alpha,\beta}$.

If $B^{\alpha,\alpha} = B^{\alpha,\beta}$ and the function S satisfying the conditions (3.30a) exists, additionally one needs to check whether S is such that $\partial S/\partial \alpha^{[r]}$ is equal to $b_r^{\alpha,\alpha}$ (and $b_r^{\alpha,\beta}$, as a consequence).

3.4 Algorithm

In this section we represent the algorithm for transformation of the system (3.1) into the observer form (3.5), whenever possible. First, taking into account (3.8) and (3.12), compare the coefficients of $\mathrm{d}y^{[i]}$ and $\mathrm{d}u^{[i]}$ at both sides of equality (3.15), to obtain

$$\left(\Psi' \circ \phi\right) \frac{\partial \phi}{\partial \alpha^{[i]}} = \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \frac{\partial \bar{\varphi}_{n-N-l} \left(\nu_{n-N-l}\right)}{\partial \alpha^{[i]}} \tag{3.35}$$

for $i = 0, \ldots, n-1$.

The algorithm is applied to the i/o representation (3.2) of the system (3.1) (see Remark 3.1).

Algorithm 3.1.

Step 1, option 1. Check the validity of conditions (3.13). If they are not satisfied, the problem is not solvable; stop.

Step 1, option 2. Check for every column of the matrix A whether the nonzero elements are equal. For $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ check also whether $B^{\alpha,\alpha} = B^{\alpha,\beta}$ (if both matrices can be constructed, see Remark 3.5). If the above-mentioned conditions are not satisfied, the problem is not solvable; stop.

Step 2. Differentiate both sides of (3.33) with respect to $\alpha^{[j]}$ and compare the obtained equality with (3.30a). This yields for $i, j = 0, \ldots, n-1$, $j \neq i-N, \ldots, i+N$

$$\begin{aligned}
\left(\ln\left|\Psi' \circ \phi\right|\right)' &= \\
&= -\left(\frac{\partial\phi}{\partial\alpha^{[j]}}\right)^{-1} \frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\alpha^{[i]}}\right|\right) = \\
&= -\left(\frac{\partial\phi}{\partial\alpha^{[j]}}\right)^{-1} \frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial\phi}{\partial\beta^{[i]}}\right|\right). \quad (3.36)
\end{aligned}$$

Note that in order to obtain $(\ln|\Psi' \circ \phi|)'$ either the middle part or the right-hand side of the equality above can be used, whereas the indices j and i should be chosen such that the function ϕ depends on both $\alpha^{[j]}$, $\alpha^{[i]}$ (or $\alpha^{[j]}$, $\beta^{[i]}$). Next, solving the i/o equation (3.2) with respect to an arbitrary variable $\alpha^{[i]}$, find the replacement rule $\alpha^{[i]} = F(\cdot)$. The application of the replacement rule to $(\ln|\Psi' \circ \phi|)'$ yields $\left(\ln\left|\Psi' \circ y^{[n]}\right|\right)'$, which can be shifted backward n times to obtain $(\ln|\Psi' \circ y|)'$, where now prime means the derivative with respect to y. Thus, the output transformation can be computed as

$$Y = \Psi \circ y = \int \mathrm{e}^{\int (\ln|\Psi' \circ y|)' \mathrm{d}y} \mathrm{d}y.$$

Step 3. Solve, if possible, the system of partial differential equations (3.35) to find the functions $\bar{\varphi}_1, \ldots, \bar{\varphi}_{n-N}$, from which the functions $\varphi_1, \ldots, \varphi_{n-N}$ can be obtained applying the output transformation.

Step 4. Using the functions $\varphi_1, \ldots, \varphi_{n-N}$ and the output transformation (3.4), construct the system in the extended observer form (3.5).

One can note that, requiring the integration and solution of differential equations, Steps 2 and 3 represent the most difficult part of Algorithm 3.1. The additional difficulties were related to the implementation of the algorithm in NLControl package (see Chapter 5). First, the computation system *Mathematica* has no built-in function, which allows to solve the system of partial differential equations (3.35). Therefore, the step by step procedure was developed to find the functions $\bar{\varphi}_1, \ldots, \bar{\varphi}_{n-N}$ from (3.35). Moreover, the relation (3.36) implies the necessity to program the algorithm for choosing the value of the indices i and j, satisfying the requirements described in Step 2.

3.5 Examples

Example 3.1. Examine the following state equations

$$\begin{aligned}
x_1^+ &= x_2 u \\
x_2^+ &= x_3 \\
x_3^+ &= (x_1 + x_2 u + u)(x_2 u + x_3)(x_1 u + x_4) \\
x_4^+ &= x_1 + u \\
y &= x_1.
\end{aligned} \tag{3.37}$$

The i/o equation, corresponding to (3.37), is

$$y^{[4]} = \left(y + u + y^+ u^+\right)\left(y^+ + u^+ + y^{++}\right) u^{[3]} \left(y^{++} + \frac{y^{[3]}}{u^{++}}\right). \tag{3.38}$$

Note that once the system is transformable into the extended observer form with some arbitrary buffer N it is also transformable into the extended observer forms with the buffers that are greater than N. Therefore, our goal is to find the least buffer N, for which the system (3.37) is transformable into the extended observer form (3.5). Consequently, it is reasonable to initiate Algorithm 3.1 with $N = 1$.

Step 1, option 1. Compute, according to (3.8),

$$\begin{aligned}
\omega_0 &= \left(y^+ + u^+ + y^{++}\right) u^{[3]} \left(y^{++} + \frac{y^{[3]}}{u^{++}}\right)(\mathrm{d}y + \mathrm{d}u), \\
\omega_1 &= u^{[3]} \left(y^{++} + \frac{y^{[3]}}{u^{++}}\right) \left(\left(y + u + u^+\left(u^+ + 2y^+ + y^{++}\right)\right)\mathrm{d}y^+ + \right. \\
&\quad \left. + \left(y + u + y^+\left(y^+ + 2u^+ + y^{++}\right)\right)\mathrm{d}u^+\right), \\
\omega_2 &= u^{[3]}\left(y + u + u^+ y^+\right)\left(\left(y^+ + u^+ + 2y^{++} + \frac{y^{[3]}}{u^{++}}\right)\mathrm{d}y^{++} - \right. \\
&\quad \left. - \left(\left(y^+ + u^+ + y^{++}\right)\frac{y^{[3]}}{(u^{++})^2}\right)\mathrm{d}u^{++}\right), \\
\omega_3 &= \frac{\left(y + u + y^+ u^+\right)\left(u^+ + y^+ + y^{++}\right)}{u^{++}}\left(u^{[3]}\mathrm{d}y^{[3]} + \right. \\
&\quad \left. + \left(y^{++}u^{++} + y^{[3]}\right)\mathrm{d}u^{[3]}\right)
\end{aligned}$$

and, according to (3.9),

$$\begin{aligned}
\Omega_0 &= \operatorname{span}_{\overline{\mathcal{K}^*}}\{\omega_1, u^+\},\\
\Omega_1 &= \operatorname{span}_{\overline{\mathcal{K}^*}}\{\omega_0, u, \omega_2, u^{++}\},\\
\Omega_2 &= \operatorname{span}_{\overline{\mathcal{K}^*}}\{\omega_1, u^+, \omega_3, u^{[3]}\},\\
\Omega_3 &= \operatorname{span}_{\overline{\mathcal{K}^*}}\{\omega_2, u^{++}\}.
\end{aligned}$$

For the case $n = 4$ and $N = 1$ conditions (3.13) are the following

$$\begin{aligned}
\mathrm{d}\omega_0 \wedge \omega_0 &\equiv 0 \bmod \Omega_0,\\
\mathrm{d}\omega_1 \wedge \omega_1 &\equiv 0 \bmod \Omega_1,\\
\mathrm{d}\omega_2 \wedge \omega_2 &\equiv 0 \bmod \Omega_2,\\
\mathrm{d}\omega_3 \wedge \omega_3 &\equiv 0 \bmod \Omega_3,\\
\mathrm{d}\omega_0 \wedge \omega_1 + \mathrm{d}\omega_1 \wedge \omega_0 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\mathrm{d}u, \mathrm{d}u^+, \omega_2, \mathrm{d}u^{++}\right\},\\
\mathrm{d}\omega_0 \wedge \omega_2 + \mathrm{d}\omega_2 \wedge \omega_0 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\omega_1, \mathrm{d}u^+, \omega_3, \mathrm{d}u^{[3]}\right\},\\
\mathrm{d}\omega_0 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_0 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\omega_1, \mathrm{d}u^+, \omega_2, \mathrm{d}u^{++}\right\},\\
\mathrm{d}\omega_1 \wedge \omega_2 + \mathrm{d}\omega_2 \wedge \omega_1 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\omega_0, \mathrm{d}u, \mathrm{d}u^+, \mathrm{d}u^{++}, \omega_3, \mathrm{d}u^{[3]}\right\},\\
\mathrm{d}\omega_1 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_1 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\omega_0, \mathrm{d}u, \omega_2, \mathrm{d}u^{++}\right\},\\
\mathrm{d}\omega_2 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_2 &\equiv 0 \bmod \operatorname{span}_{\overline{\mathcal{K}^*}}\left\{\omega_1, \mathrm{d}u^+, \mathrm{d}u^{++}, \mathrm{d}u^{[3]}\right\}.
\end{aligned}$$

By direct computations one can confirm that all conditions above are satisfied, which means that system (3.37) is transformable via the extended coordinate change and output transformation into the extended observer form with buffer $N = 1$.

Step 2. According to (3.36)

$$\left(\ln\left|\Psi' \circ \phi\right|\right)' = \\ = \frac{-u^{++}}{u^{[3]}\left(u + y + u^+ y^+\right)\left(u^+ + y^+ + y^{++}\right)\left(u^{++} y^{++} + y^{[3]}\right)}.$$

The easiest way is to solve the i/o equation (3.38) with respect to $u^{[3]}$. This yields the following replacement rule

$$u^{[3]} = \frac{u^{++} y^{[4]}}{\left(u + y + u^+ y^+\right)\left(u^+ + y^+ + y^{++}\right)\left(u^{++} y^{++} + y^{[3]}\right)},$$

applying which to $(\ln|\Psi' \circ \phi|)'$, one obtains

$$\left(\ln\left|\Psi' \circ y^{[4]}\right|\right)' = -\frac{1}{y^{[4]}}. \tag{3.39}$$

The equality (3.39) shifted backward 4 times leads to $(\ln|\Psi' \circ y|)' = -1/y$ yielding the output transformation

$$Y = \Psi \circ y = \int \mathrm{e}^{\int -\frac{1}{y}\mathrm{d}y}\mathrm{d}y = \ln y. \tag{3.40}$$

Step 3. The system of partial differential equations (3.35) for $n = 4$, $N = 1$ and α being both y and u reads as

$$\begin{aligned}
\frac{1}{u+y+u^+y^+} &= \frac{\partial\bar{\varphi}_3}{\partial y} \\
\frac{u^+}{u+y+u^+y^+} + \frac{1}{u^+ + y^+ + y^{++}} &= \frac{\partial\bar{\varphi}_3}{\partial y^+} + \frac{\partial\bar{\varphi}_2}{\partial y^+} \\
\frac{1}{u^+ + y^+ + y^{++}} + \frac{u^{++}}{u^{++}y^{++} + y^{[3]}} &= \frac{\partial\bar{\varphi}_2}{\partial y^{++}} + \frac{\partial\bar{\varphi}_1}{\partial y^{++}} \\
\frac{1}{u^{++}y^{++} + y^{[3]}} &= \frac{\partial\bar{\varphi}_1}{\partial y^{[3]}} \\
\frac{1}{u+y+u^+y^+} &= \frac{\partial\bar{\varphi}_3}{\partial u} \\
\frac{y^+}{u+y+u^+y^+} + \frac{1}{u^+ + y^+ + y^{++}} &= \frac{\partial\bar{\varphi}_3}{\partial u^+} + \frac{\partial\bar{\varphi}_2}{\partial u^+} \\
\frac{y^{[3]}}{(u^{++})^2 y^{++} + u^{++}y^{[3]}} &= \frac{\partial\bar{\varphi}_2}{\partial u^{++}} + \frac{\partial\bar{\varphi}_1}{\partial u^{++}} \\
\frac{1}{u^{[3]}} &= \frac{\partial\bar{\varphi}_1}{\partial u^{[3]}},
\end{aligned}$$

leading to

$$\begin{aligned}
\bar{\varphi}_1 &= \ln\left|u^{[3]}\right| + \ln\left|y^{++} + \frac{y^{[3]}}{u^{++}}\right|, \\
\bar{\varphi}_2 &= \ln\left|y^+ + u^+ + y^{++}\right|, \\
\bar{\varphi}_3 &= \ln\left|y + u + y^+u^+\right|,
\end{aligned}$$

which, due to the output transformation (3.40), yields

$$\begin{aligned}
\varphi_1 &= \ln\left|u^{[3]}\right| + \ln\left|e^{Y^{++}} + \frac{e^{Y^{[3]}}}{u^{++}}\right|, \\
\varphi_2 &= \ln\left|e^{Y^+} + u^+ + e^{Y^{++}}\right|, \\
\varphi_3 &= \ln\left|e^{Y} + u + e^{Y^+}u^+\right|.
\end{aligned}$$

Step 4. Using (3.7), one can define the new state variables

$$\begin{aligned}
z_1 &= Y, \\
z_2 &= Y^{+} - \ln|u| - \ln\left|e^{Y^{-}} + \frac{e^{Y}}{u^{-}}\right|, \\
z_3 &= Y^{++} - \ln\left|u^{+}\right| - \ln\left|e^{Y} + \frac{e^{Y^{+}}}{u}\right| - \ln\left|e^{Y^{-}} + u^{-} + e^{Y}\right|, \\
z_4 &= Y^{[3]} - \ln\left|u^{++}\right| - \ln\left|e^{Y^{+}} + \frac{e^{Y^{++}}}{u^{+}}\right| - \ln\left|e^{Y} + u + e^{Y^{+}}\right| - \\
&\quad - \ln\left|e^{Y^{-}} + u^{-} + e^{Y}u\right|,
\end{aligned}$$

which, due to the output transformation (3.40) and state equations (3.37), can be rewritten as

$$\begin{aligned}
z_1 &= \ln|x_1|, \\
z_2 &= \ln|x_2| - \ln\left|x_1^{-} + \frac{x_1}{u^{-}}\right|, \\
z_3 &= \ln|x_3| - \ln|x_1 + x_2| - \ln\left|x_1^{-} + u^{-} + x_1\right|, \\
z_4 &= \ln|x_1 u + x_4| - \ln\left|x_1^{-} + u^{-} + x_1 u\right|,
\end{aligned} \tag{3.41}$$

that leads to the state equations in the extended observer form

$$\begin{aligned}
z_1^{+} &= z_2 + \ln|u| + \ln\left|e^{z_1^{-}} + \frac{e^{z_1}}{u^{-}}\right| \\
z_2^{+} &= z_3 + \ln\left|e^{z_1^{-}} + u^{-} + e^{z_1}\right| \\
z_3^{+} &= z_4 + \ln\left|e^{z_1^{-}} + u^{-} + e^{z_1}u\right| \\
z_4^{+} &= 0 \\
Y &= z_1.
\end{aligned}$$

Example 3.2. Examine the following state equations

$$\begin{aligned}
x_1^{+} &= x_1 + x_2 - x_3 \\
x_2^{+} &= -x_1 - x_2 \\
x_3^{+} &= -\frac{x_1 x_2}{u x_3 + x_1 x_2 x_4} \\
x_4^{+} &= -\frac{u\left(x_2\right)^2}{\left(x_1 + x_2\right)\left(x_1 + x_2 - x_3\right)} - \frac{x_5}{u} \\
x_5^{+} &= \frac{x_2 - u\left(x_1 + x_2\right)}{x_3} \\
y &= x_2.
\end{aligned} \tag{3.42}$$

The i/o equation, corresponding to (3.42), is

$$y^{[5]} = \frac{u^+ y^{++}\left(y^{++} + y^{[3]}\right)}{\lambda}, \tag{3.43}$$

where in order to simplify the exposition we denoted $\lambda := \left(u^+\right)^2 \left(y^+\right)^2 + \left(y + uy^+\right)\left(y^{++} + y^{[3]}\right) + u^+ u^{++} y^{[4]}$. Using the conditions of Theorems 3.1 or 3.2, one can verify that the system (3.42) is not transformable into the extended observer form with buffer $N = 1$. Next, take $N = 2$ and follow the Algorithm (3.1).

Step 1, option 2. Using (3.34), one obtains

$$A = \begin{bmatrix} 0 & 0 & 0 & sy_3 & sy_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sy_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ sy_0 & 0 & 0 & 0 & 0 & su_0 & 0 & 0 & 0 & 0 \\ sy_0 & sy_1 & 0 & 0 & 0 & su_0 & su_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & sy_3 & sy_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sy_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where we use the notations

$$\begin{aligned}
sy_0 &:= -\frac{2\left(y^{++} + y^{[3]}\right)}{\lambda}, \\
sy_1 &:= -\frac{2\left(2\left(u^+\right)^2 y^+ + u\left(y^{++} + y^{[3]}\right)\right)}{\lambda}, \\
sy_3 &:= \frac{2u^+\left(u^+\left(y^+\right)^2 + u^{++} y^{[4]}\right)}{\left(y^{++} + y^{[3]}\right)\lambda}, \\
sy_4 &:= -\frac{2u^+ u^{++}}{\lambda}, \\
su_0 &:= -\frac{2y^+\left(y^{++} + y^{[3]}\right)}{\lambda}, \\
su_1 &:= -\frac{2\left(\left(y + uy^+\right)\left(y^{++} + y^{[3]}\right) - \left(u^+\right)^2\left(y^+\right)^2\right)}{u^+\lambda}.
\end{aligned}$$

Since $\underline{j_y} = 0$, $\overline{j_y} = 4$, $\underline{j_u} = 0$, $\overline{j_u} = 2$ and $N = 2$, both inequalities $2N \geq \overline{j_y} - \underline{j_y}$ and $2N \geq \overline{j_u} - \underline{j_u}$ are satisfied and, as a consequence, one has to check the additional conditions. Choosing j according to

Remark 3.5, one obtains the following matrices

$$B^{y,y} = B^{y,u} = \begin{bmatrix} sy_2 \end{bmatrix}, \quad B^{u,u} = B^{u,y} = \begin{bmatrix} su_0 \\ su_1 \\ su_2 \end{bmatrix}, \tag{3.44}$$

where

$$sy_2 := 2\left(\frac{1}{y^{++} + y^{[3]}} + \frac{1}{y^{++}} - \frac{y + uy^{+}}{\lambda}\right), \quad su_2 := -\frac{2u^{+}y^{[4]}}{\lambda}.$$

Taking into consideration (3.44) and the fact that all the nonzero elements of every column of the matrix A are equal, one may conclude that the necessary conditions for transformation of the system (3.42) into the extended observer form with buffer $N = 2$ are satisfied.

Step 2. According to (3.36)

$$\begin{aligned}\left(\ln\left|\Psi' \circ \phi\right|\right)' &= \\ &= -\frac{2\left(\left(u^{+}\right)^2\left(y^{+}\right)^2 + \left(y + uy^{+}\right)\left(y^{++} + y^{[3]}\right) + u^{+}u^{++}y^{[4]}\right)}{u^{+}y^{++}\left(y^{++} + y^{[3]}\right)}.\end{aligned}$$

Solving the i/o equation (3.43) with respect to y, the following replacement rule can be obtained

$$y = \frac{u^{+}y^{++}}{y^{[5]}} - \frac{\left(u^{+}\right)^2\left(y^{+}\right)^2 + uy^{+}\left(y^{++} + y^{[3]}\right) + u^{+}u^{++}y^{[4]}}{y^{++} + y^{[3]}},$$

applying which to $(\ln|\Psi' \circ \phi|)'$, one obtains

$$\left(\ln\left|\Psi' \circ y^{[5]}\right|\right)' = -\frac{2}{y^{[5]}}. \tag{3.45}$$

The equality (3.45) shifted backward 5 times leads to $(\ln|\Psi' \circ y|)' = -2/y$ yielding the output transformation

$$Y = \Psi \circ y = \int \mathrm{e}^{\int -\frac{2}{y}\mathrm{d}y}\mathrm{d}y = -\frac{1}{y}. \tag{3.46}$$

Step 3. The system of partial differential equations (3.35) for $n = 5$, $N = 2$

and α being both y and u reads as

$$\begin{aligned}
-\frac{1}{u^{+}y^{++}} &= \frac{\partial\bar{\varphi}_3}{\partial y}\\
-\frac{2\left(u^{+}\right)^2 y^{+} + u\left(y^{++}+y^{[3]}\right)}{u^{+}y^{++}\left(y^{++}+y^{[3]}\right)} &= \frac{\partial\bar{\varphi}_3}{\partial y^{+}} + \frac{\partial\bar{\varphi}_2}{\partial y^{+}}\\
\frac{\left(2y^{++}+y^{[3]}\right)\left(u^{+}\left(y^{+}\right)^2 + u^{++}y^{[4]}\right)}{\left(y^{++}\right)^2\left(y^{++}+y^{[3]}\right)^2}+ &\\
+\frac{y+uy^{+}}{u^{+}\left(y^{++}\right)^2} &= \frac{\partial\bar{\varphi}_3}{\partial y^{++}} + \frac{\partial\bar{\varphi}_2}{\partial y^{++}} + \frac{\partial\bar{\varphi}_1}{\partial y^{++}}\\
\frac{u^{+}\left(y^{+}\right)^2 + u^{++}y^{[4]}}{y^{++}\left(y^{++}+y^{[3]}\right)^2} &= \frac{\partial\bar{\varphi}_2}{\partial y^{[3]}} + \frac{\partial\bar{\varphi}_1}{\partial y^{[3]}}\\
-\frac{u^{++}}{y^{++}\left(y^{++}+y^{[3]}\right)} &= \frac{\partial\bar{\varphi}_1}{\partial y^{[4]}}\\
-\frac{y^{+}}{u^{+}y^{++}} &= \frac{\partial\bar{\varphi}_3}{\partial u}\\
\frac{y+uy^{+}}{\left(u^{+}\right)^2 y^{++}} - \frac{\left(y^{+}\right)^2}{y^{++}\left(y^{++}+y^{[3]}\right)} &= \frac{\partial\bar{\varphi}_3}{\partial u^{+}} + \frac{\partial\bar{\varphi}_2}{\partial u^{+}}\\
-\frac{y^{[4]}}{y^{++}\left(y^{++}+y^{[3]}\right)} &= \frac{\partial\bar{\varphi}_3}{\partial u^{++}} + \frac{\partial\bar{\varphi}_2}{\partial u^{++}} + \frac{\partial\bar{\varphi}_1}{\partial u^{++}},
\end{aligned}$$

leading to

$$\begin{aligned}
\bar{\varphi}_1 &= -\frac{y^{[4]}u^{++}}{\left(y^{[3]}+y^{++}\right)y^{++}},\\
\bar{\varphi}_2 &= -\frac{\left(y^{+}\right)^2 u^{+}}{\left(y^{[3]}+y^{++}\right)y^{++}},\\
\bar{\varphi}_3 &= -\frac{y+y^{+}u}{y^{++}u^{+}},
\end{aligned}$$

which, due to the output transformation (3.46) yields

$$\begin{aligned}
\varphi_1 &= \frac{\left(Y^{++}\right)^2 Y^{[3]}u^{++}}{\left(Y^{++}+Y^{[3]}\right)Y^{[4]}},\\
\varphi_2 &= -\frac{\left(Y^{++}\right)^2 Y^{[3]}u^{+}}{\left(Y^{+}\right)^2\left(Y^{++}+Y^{[3]}\right)},\\
\varphi_3 &= -\frac{\left(Yu+Y^{+}\right)Y^{++}}{YY^{+}u^{+}}.
\end{aligned}$$

Step 4. Using (3.7), one can define the new state variables

$$
\begin{aligned}
z_1 &= Y,\\
z_2 &= Y^{+} - \frac{\left(Y^{--}\right)^2 Y^{-} u^{--}}{\left(Y^{--} + Y^{-}\right) Y},\\
z_3 &= Y^{++} - \frac{\left(Y^{-}\right)^2 Y u^{-}}{\left(Y^{-} + Y\right) Y^{+}} + \frac{\left(Y^{-}\right)^2 Y u^{--}}{\left(Y^{--}\right)^2 \left(Y^{-} + Y\right)},\\
z_4 &= Y^{[3]} - \frac{\left(Y\right)^2 Y^{+} u}{\left(Y + Y^{+}\right) Y^{++}} + \frac{\left(Y\right)^2 Y^{+} u^{-}}{\left(Y^{-}\right)^2 \left(Y + Y^{+}\right)} +\\
&\quad + \frac{\left(Y^{--} u^{--} + Y^{-}\right) Y}{Y^{--} Y^{-} u^{-}},\\
z_5 &= Y^{[4]} - \frac{\left(Y^{+}\right)^2 Y^{++} u^{+}}{\left(Y^{+} + Y^{++}\right) Y^{[3]}} + \frac{\left(Y^{+}\right)^2 Y^{++} u}{\left(Y\right)^2 \left(Y^{+} + Y^{++}\right)} +\\
&\quad + \frac{\left(Y^{-} u^{-} + Y\right) Y^{+}}{Y^{-} Y u},
\end{aligned}
$$

which, due to the output transformation (3.46) and state equations (3.42), can be rewritten as

$$
\begin{aligned}
z_1 &= -\frac{1}{x_2},\\
z_2 &= \frac{x_2 u^{--}}{\left(x_2^{--}\right)^2 + x_2^{--} x_2^{-}} + \frac{1}{x_1 + x_2},\\
z_3 &= -\frac{1}{x_3} + \frac{\left(x_2^{--}\right)^2 u^{--} - u^{-}\left(x_1 + x_2\right)}{x_2^{-}\left(x_2^{-} + x_2\right)},\\
z_4 &= x_4 - \frac{\left(x_2^{-}\right)^2 u^{-}}{x_1 x_2} + \frac{x_2^{--} + x_2^{-} u^{--}}{x_2 u^{-}},\\
z_5 &= -\frac{x_2^{-} - x_2 u^{-} - x_1 x_5 - x_2 x_5}{u\left(x_1 + x_2\right)},
\end{aligned}
\tag{3.47}
$$

that leads to the state equations in the extended observer form

$$
\begin{aligned}
z_1^{+} &= z_2 + \frac{\left(z_1^{--}\right)^2 z_1^{-} u^{--}}{\left(z_1^{--} + z_1^{-}\right) z_1}\\
z_2^{+} &= z_3 - \frac{\left(z_1^{-}\right)^2 z_1 u^{--}}{\left(z_1^{--}\right)^2 \left(z_1^{-} + z_1\right)}\\
z_3^{+} &= z_4 - \frac{z_1\left(z_1^{-} + z_1^{--} u^{--}\right)}{z_1^{-} z_1^{--} u^{-}}\\
z_4^{+} &= z_5\\
z_5^{+} &= 0\\
Y &= z_1.
\end{aligned}
$$

Chapter 4

Observable Space of the System on Homogeneous Time Scale

The observability property of the system, defined on homogeneous time scale, is studied in this chapter. The definition of the observability is given through the observability rank condition, commonly used both in continuous- and discrete-time cases. This, however, differs from the standard definition, where the concept of (in)distinguishable states is employed. The observability condition is presented in terms of the observable space. Moreover, the notions of the observability filtration and observability indices are extended to the systems on homogeneous time scales and the decomposition of the system into the observable/unobservable subsystems is addressed. The examples throughout the chapter are intended to illustrate different aspects of the theory.

4.1 Observability and Observable Space

Recall form Subsection 1.2.4 the MIMO system, defined on homogeneous time scale $\mathbb{T}$, that is,

$$\begin{aligned} x^{\Delta} &= f(x, u) \\ y &= h(x), \end{aligned} \tag{4.1}$$

where $x(t) : \mathbb{T} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{T} \to \mathbb{U} \subset \mathbb{R}^m$ is an m-dimensional input vector $y(t) : \mathbb{T} \to \mathbb{Y} \subset \mathbb{R}^p$ is a p-dimensional output vector. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \to \mathbb{Y}$ are assumed to be real analytic functions.

Frequently the observability rank condition is used to check whether the continuous-time nonlinear system is locally weakly observable [25], [38].

This condition is sufficient for arbitrary initial state and necessary for almost all initial states. Thus, we introduce the definition of (generic) observability for nonlinear systems, defined on homogeneous time scales, through the observability rank condition.

Definition 4.1. System (4.1) is called *generically (single-experiment) observable* if the rank of the observability matrix is generically equal to n, i.e. if

$$\operatorname{rank}_{\mathcal{K}^*}\left[\frac{\partial\left(h_1, h_1^{\langle 1\rangle}, \ldots, h_1^{\langle n-1\rangle}, \ldots, h_p, h_p^{\langle 1\rangle}, \ldots, h_p^{\langle n-1\rangle}\right)}{\partial x}\right] = n. \tag{4.2}$$

Recall that the superscript $^{\langle i\rangle}$ denotes the ith delta derivative. Take into account, that for $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$ the higher order delta derivative can be computed explicitly as

$$h_\nu^{\langle i\rangle} = \frac{1}{\tau^i}\sum_{k=0}^{i}(-1)^k C_i^k h_\nu^{\sigma_f^{i-k}}, \tag{4.3}$$

where C_i^k is the binomial coefficient, i.e. $C_i^k = \frac{i!}{(i-k)!k!}$.

Proposition 4.1. *For $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, the following holds*

$$\operatorname{rank}_{\mathcal{K}^*}\left[\frac{\partial\left(h_1, h_1^{\langle 1\rangle}, \ldots, h_1^{\langle n-1\rangle}, \ldots, h_p, h_p^{\langle 1\rangle}, \ldots, h_p^{\langle n-1\rangle}\right)^{\mathrm{T}}}{\partial x}\right] =$$

$$= \operatorname{rank}_{\mathcal{K}^*}\left[\frac{\partial\left(h_1, h_1^{\sigma_f}, \ldots, h_1^{\sigma_f^{n-1}}, \ldots, h_p, h_p^{\sigma_f}, \ldots, h_p^{\sigma_f^{n-1}}\right)^{\mathrm{T}}}{\partial x}\right]. \tag{4.4}$$

Proof. Separate the first addend of the sum in the right-hand side of (4.3) and then apply the relation (A.2), given in Appendix, to obtain

$$h_\nu^{\langle i\rangle} = \frac{1}{\tau^i}\left(h_\nu^{\sigma_f^i} - \sum_{k=1}^{i} C_i^k h_\nu^{\sigma_f^{i-k}} \sum_{l=0}^{k-1} C_k^l \,(-1)^l\right). \tag{4.5}$$

Changing the summation order

$$\sum_{k=1}^{i}\sum_{l=0}^{k-1} a_{k,l} = \sum_{k=1}^{i}\sum_{l=0}^{i-k} a_{k+l,l}$$

and then using the relations $C_i^{k+l}C_{k+l}^l = C_i^k C_{i-k}^l$ and $\frac{1}{\tau^i} = \frac{1}{\tau^k}\frac{1}{\tau^{i-k}}$, one can rewrite (4.5) as

$$h_\nu^{\langle i \rangle} = \frac{1}{\tau^i} h_\nu^{\sigma_f^i} - \sum_{k=1}^{i} \frac{1}{\tau^k} C_i^k \frac{1}{\tau^{i-k}} \sum_{l=0}^{i-k} (-1)^l \, C_{i-k}^l h_\nu^{\sigma_f^{i-k-l}},$$

which, according to (4.3), yields

$$h_\nu^{\langle i \rangle} = \frac{1}{\tau^i} h_\nu^{\sigma_f^i} - \sum_{k=1}^{i} \frac{1}{\tau^k} C_i^k h_\nu^{\langle i-k \rangle}.$$

Using the relation above, the arbitrary row of the left-hand matrix in (4.4) may be rewritten as

$$\frac{\partial h_\nu^{\langle i \rangle}}{\partial x} = \frac{1}{\tau^i} \frac{\partial h_\nu^{\sigma_f^i}}{\partial x} - \sum_{k=1}^{i} \frac{1}{\tau^k} C_i^k \frac{\partial h_\nu^{\langle i-k \rangle}}{\partial x}$$

for $\nu = 1, \ldots, p$ and $i = 1, \ldots, n-1$. It is easy to observe, that the sum in the relation above represents the linear combination of the previous rows of the matrix and, therefore, can be removed without changing the rank of the matrix. Since $\partial h_\nu^{\sigma_f^i} / \partial x$ is the row of the right-hand matrix of (4.4) for $i = 1, \ldots, n-1$, the statement of the proposition holds. $\square$

Remark 4.1. Since for $\mathbb{T} = \mathbb{R}$ the delta derivative coincides with the classical time derivative, the condition (4.2) is equivalent to observability rank condition given in [25] for continuous-time systems. By Proposition 4.1 in the case $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, the condition (4.2) is equivalent to the observability rank condition given in [55] for discrete-time systems, described in terms of shift operator.

Though Definition 4.1 may be applied to check observability, it is easier to be done using the concept of observable space like in the continuous-time case [25]. Moreover, the observable space, whenever integrable, allows to decompose the system into the observable/unobservable subsystems. In the rest of this section we extend the concept of observable space to MIMO systems, defined on homogeneous time scales, and, using the notion of observable space, provide the necessary and sufficient observability condition.

Given the system (4.1), denote by $\mathcal{X}$, $\mathcal{Y}^k$, $\mathcal{Y}$ and $\mathcal{U}$ the following subspaces of the differential one-forms

$$\begin{aligned} \mathcal{X} &:= \operatorname{span}_{\mathcal{K}^*} \{\mathrm{d}x\}, \\ \mathcal{Y}^k &:= \operatorname{span}_{\mathcal{K}^*} \left\{\mathrm{d}h_\nu^{\langle j \rangle},\ \nu = 0, \ldots, p,\ 0 \le j \le k\right\}, \\ \mathcal{Y} &:= \operatorname{span}_{\mathcal{K}^*} \left\{\mathrm{d}h_\nu^{\langle j \rangle},\ \nu = 0, \ldots, p,\ j \ge 0\right\}, \\ \mathcal{U} &:= \operatorname{span}_{\mathcal{K}^*} \left\{\mathrm{d}u_\upsilon^{\langle l \rangle},\ \upsilon = 1, \ldots, m,\ l \ge 0\right\}. \end{aligned} \tag{4.6}$$

By analogy with [25], the chain of subspaces

$$0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \cdots \subset \mathcal{O}_k \subset \cdots \subset \mathcal{O}_{k^*-1} = \mathcal{O}_{k^*} =: \mathcal{O}_\infty, \tag{4.7}$$

where

$$\mathcal{O}_k := \mathcal{X} \cap \left(\mathcal{Y}^k + \mathcal{U}\right) \tag{4.8}$$

is called the *observability filtration.* Denote by $\mathcal{O}_\infty$ the limit of the observability filtration. It is easy to see that

$$\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$$

and analogously with [25] we call the subspace $\mathcal{O}_\infty$ of $\mathcal{X}$ the *observable space*[1] of the system (4.1). The *unobservable space* of system (4.1), denoted by $\mathcal{X}_{\bar{O}}$, is defined as a subspace of $\mathcal{X}$, which satisfies

$$\mathcal{X}_{\bar{O}} \cong \mathcal{X}/\mathcal{O}_\infty, \quad \mathcal{X}_{\bar{O}} \oplus \mathcal{O}_\infty = \mathcal{X},$$

where $\mathcal{X}/\mathcal{O}_\infty$ denotes the factor-space.

From (4.6), taking into account (1.6) and using the linear transformations, one obtains

$$\mathcal{Y}^k + \mathcal{U} =$$
$$= \operatorname{span}_{\mathcal{K}^*}\left\{\frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x,\ \nu = 1,\dots,p,\ 0 \le j \le k;\ \mathrm{d}u_\upsilon^{\langle l\rangle},\ \upsilon = 1,\dots,m,\ l \ge 0\right\}.$$

Consequently, according to (4.8)

$$\mathcal{O}_k = \operatorname{span}_{\mathcal{K}^*}\left\{\frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x,\ \nu = 1,\dots,p,\ 0 \le j \le k\right\}, \tag{4.9}$$

yielding

$$\mathcal{O}_\infty = \operatorname{span}_{\mathcal{K}^*}\left\{\frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x,\ \nu = 1,\dots,p,\ j \ge 0\right\}.$$

Before studying the properties of the observable space we provide Lemma 4.1. Denote the one-forms which generate the observable space $\mathcal{O}_\infty$ as $\omega_{\nu,j} := \frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x$ for $\nu = 1,\dots,p, j \ge 0$ and arrange them in the form of the following matrix:

$$\Omega := \begin{bmatrix} \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \\ \omega_{p,0} & \omega_{p,1} & \omega_{p,2} & \cdots \end{bmatrix}.$$

Denote the arbitrary row of the matrix Ω by Ω_ν.

[1]Note that $\mathcal{O}_\infty$ is in general not the observation space (as in [94]), associated with the concept of the multi-experiment observability.

Lemma 4.1. *If Ω_ν contains the one-form $\omega_{\nu,i}$, being a linear combination of the former one-forms $\omega_{\nu,0},\ldots,\omega_{\nu,i-1}$ from Ω_ν, then the next one-forms $\omega_{\nu,j}$'s for $j>i$ can also be represented as a linear combination of the one-forms $\omega_{\nu,0},\ldots,\omega_{\nu,i-1}$.*

The proof of Lemma 4.1 is given in the Appendix.

The proposition below describes the property of the subspace $\mathcal{O}_\infty$.

Proposition 4.2.

$$\dim_{\mathcal{K}^*}\mathcal{O}_\infty=\operatorname{rank}_{\mathcal{K}^*}\left[\frac{\partial\left(h_1,h_1^{\langle 1\rangle},\ldots,h_1^{\langle n-1\rangle},\ldots,h_p,h_p^{\langle 1\rangle},\ldots,h_p^{\langle n-1\rangle}\right)}{\partial x}\right].$$

Proof. Represent the observable space as

$$\mathcal{O}_\infty=\mathcal{O}_\infty^1+\mathcal{O}_\infty^2+\cdots+\mathcal{O}_\infty^p,$$

where $\mathcal{O}_\infty^\nu$ is generated by the elements of Ω_ν. Since $\mathcal{O}_\infty^\nu\subseteq\mathcal{O}_\infty\subseteq\mathcal{X}$ and, as a consequence, $\dim\mathcal{O}_\infty^\nu\le\dim\mathcal{O}_\infty\le\dim\mathcal{X}=n$, it is enough to use n independent differential one-forms $\omega_{\nu,j}$ to generate $\mathcal{O}_\infty^\nu$. Lemma 4.1 guarantees that the first n one-forms $\omega_{\nu,j}$, $0\le j\le n-1$, span the subspace $\mathcal{O}_\infty^\nu$. Consequently,

$$\begin{aligned}\operatorname{span}_{\mathcal{K}^*}&\left\{\frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x,\ \nu=1,\ldots,p,\ j\ge 0\right\}=\\&=\operatorname{span}_{\mathcal{K}^*}\left\{\frac{\partial h_\nu^{\langle j\rangle}}{\partial x}\mathrm{d}x,\ \nu=1,\ldots,p,\ 0\le j\le n-1\right\}.\end{aligned}$$

Thus, the rows of the observability matrix

$$\left[\frac{\partial\left(h_1,h_1^{\langle 1\rangle},\ldots,h_1^{\langle n-1\rangle},\ldots,h_p,h_p^{\langle 1\rangle},\ldots,h_p^{\langle n-1\rangle}\right)}{\partial x}\right]\tag{4.10}$$

with n columns can be regarded as the representation of the elements of the codistribution $\mathcal{O}_\infty$. Therefore, the number of linearly independent vectors of $\mathcal{O}_\infty$, i.e. $\dim_{\mathcal{K}^*}\mathcal{O}_\infty$, can be found as the rank of the matrix (4.10). □

The following theorem is the direct consequence of Definition 4.1 and Proposition 4.2 and provides the characterization of the observability of the system.

Theorem 4.1. *A system* (4.1) *is (single-experiment) observable if and only if $\mathcal{O}_\infty=\mathcal{X}$.*

The following example illustrates how the observability of the system can be checked using the observable space.

Example 4.1. Consider the continuous-time model of unicycle [25] and its discrete-time approximation, based on Euler sampling scheme, as a single model defined on the homogeneous time scale $\mathbb{T}$

$$\begin{aligned} x_1^{\Delta} &= u_1 \cos x_3 \\ x_2^{\Delta} &= u_1 \sin x_3 \\ x_3^{\Delta} &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned} \tag{4.11}$$

Using (4.9), the observability filtration (4.7) of the system (4.11) may be computed as follows

$$\begin{aligned} \mathcal{O}_0 &= \operatorname{span}_{\mathcal{K}^*} \{\mathrm{d}x_1, \mathrm{d}x_2\}, \\ \mathcal{O}_\infty = \mathcal{O}_1 &= \operatorname{span}_{\mathcal{K}^*} \{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}. \end{aligned}$$

Since the observable space $\mathcal{O}_\infty = \mathcal{X}$, the system is observable. Alternatively, one may check that direct application of Definition 4.1 yields the same result though requires more computations:

$$\operatorname{rank}_{\mathcal{K}^*} \left[\frac{\partial \left(h_1, h_1^{\Delta_f}, h_1^{\langle 2 \rangle}, h_2, h_2^{\Delta_f}, h_2^{\langle 2 \rangle} \right)}{\partial x} \right] = $$

$$= \operatorname{rank}_{\mathcal{K}^*} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -u_1 \sin x_3 \\ 0 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & u_1 \cos x_3 \\ 0 & 0 & b \end{bmatrix} = 3,$$

where

$$a := \begin{cases} \dfrac{u_1 \sin x_3 - \left(u_1 + \tau u_1^{\Delta}\right) \sin\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ -u_1 u_2 \cos x_3 - \dot{u}_1 \sin x_3 & \text{if } \mathbb{T} = \mathbb{R}, \end{cases}$$

$$b := \begin{cases} \dfrac{-u_1 \cos x_3 + \left(u_1 + \tau u_1^{\Delta}\right) \cos\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ -u_1 u_2 \sin x_3 + \dot{u}_1 \cos x_3 & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

Given a system of the form (4.1), its observability filtration (4.7), like in the continuous-time case [25], defines a set of structural indices σ_j for $j = 1, \ldots, k^*$ by

$$\begin{aligned}\sigma_1 &:= \dim_{\mathcal{K}^*} \mathcal{O}_0, \\ \sigma_j &:= \dim_{\mathcal{K}^*} \left(\mathcal{O}_{j-1}/\mathcal{O}_{j-2}\right), \quad j = 2, \ldots, k^*.\end{aligned} \tag{4.12}$$

Another set of indices s_i for $i = 1, \ldots, p$, being dual to the set $\{\sigma_j, j = 1, \ldots, k^*\}$, is defined by

$$s_i := \operatorname{card}\{\sigma_j \mid \sigma_j \geq i\} \tag{4.13}$$

and called the set of *observability indices* of system (4.1). The integer σ_j represents the number of observability indices s_i which are greater than or equal to j, and duality implies that $\sigma_j = \operatorname{card}\{s_i \mid s_i \geq j\}$.

Observability indices determine how many delta derivatives of the respective output components one needs to use for computation of the initial state x on the basis of the inputs and outputs and their delta derivatives. The following proposition describes the key property of the observability indices.

Proposition 4.3. *Given a system of the form* (4.1)*, one has*

$$\dim_{\mathcal{K}^*} \mathcal{O}_\infty = s_1 + \cdots + s_p.$$

Proof. Note that $\dim_{\mathcal{K}^*}(\mathcal{O}_{j-1}/\mathcal{O}_{j-2}) = \dim_{\mathcal{K}^*} \mathcal{O}_{j-1} - \dim_{\mathcal{K}^*} \mathcal{O}_{j-2}$. Using (4.12) one can write

$$\sum_{j=1}^{k^*} \sigma_j = \sum_{j=1}^{k^*} \dim_{\mathcal{K}^*} \mathcal{O}_{j-1} - \sum_{j=2}^{k^*} \dim_{\mathcal{K}^*} \mathcal{O}_{j-2}. \tag{4.14}$$

Separating the last addend of the first sum in the right-hand side of (4.14), replacing in this sum index j by $j-1$ and taking into account that $\mathcal{O}_{k^*-1} = \mathcal{O}_\infty$, we obtain

$$\sum_{j=1}^{k^*} \sigma_j = \dim_{\mathcal{K}} \mathcal{O}_\infty + \sum_{j=2}^{k^*} \dim_{\mathcal{K}} \mathcal{O}_{j-2} - \sum_{j=2}^{k^*} \dim_{\mathcal{K}} \mathcal{O}_{j-2} = \dim_{\mathcal{K}} \mathcal{O}_\infty. \tag{4.15}$$

The relation between indices σ_j and s_i can be expressed by means of a $k^* \times p$ table, whose (j, i)th element is defined by ($j = 1, \ldots, k^*$ pointing to the row and $i = 1, \ldots, p$ to the column)

$$a_{j,i} := \begin{cases} 1, & 1 \leq i \leq \sigma_j, \\ 0, & (\sigma_j + 1) \leq i \leq p, \end{cases} = \begin{cases} 1, & 1 \leq j \leq s_i, \\ 0, & (s_i + 1) \leq j \leq k^*. \end{cases}$$

Thus, the indices σ_j and s_i are the sums of elements in the jth row and ith column, respectively, i.e.

$$\sigma_j = \sum_{i=1}^{p} a_{j,i}, \quad s_i = \sum_{j=1}^{k^*} a_{j,i}. \tag{4.16}$$

Taking into account (4.15) and (4.16), one obtains

$$\sum_{i=1}^{p} s_i = \sum_{i=1}^{p}\sum_{j=1}^{k^*} a_{j,i} = \sum_{j=1}^{k^*} \sigma_j = \dim_{\mathcal{K}} \mathcal{O}_\infty,$$

which completes the proof. □

Example 4.2. (Continuation of Example 4.1). One has $\sigma_1 = 2$, $\sigma_2 = 1$ and so, the observability indices are $s_1 = 2$, $s_2 = 1$. Taking delta derivatives of y_1 and y_2 up to the orders $s_1 - 1$ and $s_2 - 1$, respectively, we obtain $y_1 = x_1$, $y_1^\Delta = u_1 \cos x_3$, $y_2 = x_2$, yielding

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ x_3 &= \arccos \frac{y_1^\Delta}{u_2}. \end{aligned}$$

4.2 Decomposition of the System into Observable and Unobservable Subsystems

For certain applications it will be useful to have system representations in which the observable and unobservable state variables can be explicitly distinguished. For a continuous-time nonlinear control system the decomposition into observable/unobservable subsystems has been carried out both via differential geometric [44], [80] and linear algebraic methods [25] and is proved to be always doable. For example, in [25] the decomposition was first carried out for linearized system defined in terms of one-forms, and then, it was proved that the observable subspace of differential one-forms is always (generically) integrable. Therefore, the observable subspace of one-forms can be (at least locally) spanned by exact one-forms whose integrals define the observable state coordinates. As demonstrated in [55], for the discrete-time nonlinear control systems described in terms of the shift operator σ_f the decomposition at the level of equations (state variables) is not always possible since the observable space of one-forms is not necessarily completely integrable. Moreover, the paper [59] provides a general subclass of systems with non-integrable observable subspace.

The purpose of this section is to study the possibility to decompose the nonlinear control system defined on the homogeneous time scale into the observable and unobservable subsystems. Since the delta-domain model obtained via sampling [32] behaves similarly to the continuous-time system and at the limit, when the sampling frequency tends to infinity, approaches the continuous-time system, it was our working hypotheses that the delta-domain models are, in general, decomposable into observable/unobservable parts.

The latter would mean that the respective observable space $\mathcal{O}_\infty$, as a space of differential one-forms, is completely integrable. In the case $\mu \equiv 0$ ($\mathbb{T} = \mathbb{R}$), the observable space $\mathcal{O}_\infty$ is proved to be integrable [25]. Unfortunately, unlike the case $\mathbb{T} = \mathbb{R}$ for the case $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, $\mathcal{O}_\infty$ is not necessarily integrable. We give a number of counterexamples.

Example 4.3. Consider the control system, defined on homogeneous time scale

$$\begin{aligned} x_1^\Delta &= x_3 + ux_3 - x_1 \\ x_2^\Delta &= u - x_2 \\ x_3^\Delta &= ux_1 - x_3 - x_2 \\ y &= x_3. \end{aligned} \tag{4.17}$$

By (4.7), for this system,

$$\mathcal{O}_\infty = \mathcal{O}_2 = \operatorname{span}_{\mathcal{K}^*}\left\{\mathrm{d}x_3, 2\mathrm{d}x_2 + \left(u^\Delta - \mu u^\Delta - 2u\right)\mathrm{d}x_1, \mathrm{d}x_2 - u\mathrm{d}x_1\right\}.$$

If $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and obviously[2], $\mathcal{O}_\infty = \mathcal{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathcal{O}_\infty = \mathcal{X}$, except for the case $\mu = \tau = 1$ when

$$\mathcal{O}_\infty = \operatorname{span}_{\mathcal{K}^*}\{\mathrm{d}x_3, \mathrm{d}x_2 - u\mathrm{d}x_1\},$$

being non-integrable subspace by Theorem 1.3, since $\mathrm{d}(\mathrm{d}x_2 - u\mathrm{d}x_1) \wedge \mathrm{d}x_3 \wedge (\mathrm{d}x_2 - u\mathrm{d}x_1) = \mathrm{d}u \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \neq 0$.

Next example demonstrates that the loss of integrability does not necessarily occur only at $\mu = 1$.

Example 4.4. Consider the system

$$\begin{aligned} x_1^\Delta &= x_2 - \frac{x_1}{3} \\ x_2^\Delta &= ux_1 + x_3 - x_2 \\ x_3^\Delta &= \mathrm{e}^{u^2x_1 + ux_3} - \frac{x_3}{3} \\ y &= x_2 \end{aligned} \tag{4.18}$$

[2] Of course, for $\mu \equiv 0$ the result also follows from continuous-time theory [25].

the observable space of which is

$$\mathcal{O}_\infty = \mathcal{O}_2 = \operatorname{span}_{\mathcal{K}}\left\{ \mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3, \left(u^\Delta - \frac{\mu u^\Delta}{3} \right) \mathrm{d}x_1 \right\}.$$

Like in the previous example, if $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and $\mathcal{O}_\infty = \mathcal{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathcal{O}_\infty = \mathcal{X}$, except for the case $\mu = \tau = 3$ when $\mathcal{O}_\infty = \operatorname{span}_{\mathcal{K}^*} \{\mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3\}$, again non-integrable by the Frobenius theorem.

Finally, we provide an example of the system for which the observable space $\mathcal{O}_\infty$ is integrable for every choice of the value of μ.

Example 4.5. Consider the system

$$\begin{aligned}
x_1^\Delta &= \tan(x_1 - x_2)u_1 \\
x_2^\Delta &= u_1 \tan(x_1 - x_2) - u_2 \cos^2(x_1 - x_2) \\
x_3^\Delta &= u_1 \\
y_1 &= x_3 \\
y_2 &= x_1 - x_2.
\end{aligned} \tag{4.19}$$

The observable space $\mathcal{O}_\infty = \mathcal{O}_0 = \operatorname{span}_{\mathcal{K}} \{\mathrm{d}x_1 - \mathrm{d}x_2, \mathrm{d}x_3\}$ is obviously integrable by direct inspection.

To conclude, we conjecture that the observable space $\mathcal{O}_\infty$ is in general integrable except for a few possible μ values where these values correspond to the sampling frequencies at which the state transition map of the sampled system is not reversible. The following example illustrates this conjecture.

Example 4.6. (Continuation of Examples 4.3 – 4.5). The state transition map of system (4.17) is

$$\begin{aligned}
x_1^\sigma &= \mu\,(x_3 + ux_3 - x_1) + x_1 \\
x_2^\sigma &= \mu\,(u - x_2) + x_2 \\
x_3^\sigma &= \mu\,(ux_1 - x_3 - x_2) + x_3.
\end{aligned} \tag{4.20}$$

In order to check the reversibility of the system, one needs to verify whether the Jacobian matrix $\partial \widetilde{f}(x,u)/\partial x$ is nonsingular. The Jacobian matrix of system (4.20) is

$$\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1-\mu & 0 & \mu(1+u) \\ 0 & 1-\mu & 0 \\ \mu u & -\mu & 1-\mu \end{bmatrix}.$$

Note that the matrix above is singular for $\mu = 1$, implying that the state transition map (4.20) is not reversible at the sampling frequency equal 1.

Next, consider the state transition map of system (4.18), which reads as

$$\begin{aligned} x_1^\sigma &= \mu\left(x_2 - \frac{x_1}{3}\right) + x_1 \\ x_2^\sigma &= \mu\left(ux_1 + x_3 - x_2\right) + x_2 \\ x_3^\sigma &= \mu\left(\mathrm{e}^{u^2 x_1 + ux_3} - \frac{x_3}{3}\right) + x_3. \end{aligned} \tag{4.21}$$

The Jacobian matrix of system (4.21), i.e.

$$\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1 - \dfrac{\mu}{3} & \mu & 0 \\ \mu u & 1-\mu & \mu \\ \mathrm{e}^{u(ux_1+x_3)}\mu u^2 & 0 & 1 - \dfrac{\mu}{3} + \mathrm{e}^{u(ux_1+x_3)}\mu u \end{bmatrix},$$

is singular for $\mu = 3$. Consequently, the state transition map (4.21) is not reversible at the sampling frequency equal 3. Finally, the state transition map of system (4.19) is

$$\begin{aligned} x_1^\sigma &= \mu \tan(x_1 - x_2)u_1 + x_1 \\ x_2^\sigma &= \mu\left(u_1 \tan(x_1 - x_2) - u_2 \cos^2(x_1 - x_2)\right) + x_2 \\ x_3^\sigma &= \mu u_1 + x_3 \end{aligned} \tag{4.22}$$

and its Jacobian matrix reads as

$$\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1 + \dfrac{\mu u_1}{\cos^2(x_1 - x_2)} & \dfrac{-\mu u_1}{\cos^2(x_1 - x_2)} & 0 \\ a & 1-a & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$a := \mu\left(\frac{u_1}{\cos(x_1 - x_2)^2} + u_2 \sin\left(2\left(x_1 - x_2\right)\right)\right).$$

One can verify that the matrix above is nonsingular for any $\mu \equiv \text{const}$, meaning that the state transition map (4.22) is reversible at any sampling frequency. To conclude, comparing the result above with those presented in Examples 4.3 – 4.5, one can observe the consistency of the sampling frequencies at which the state transition maps are not reversible and the values of μ for which the observable spaces $\mathcal{O}_\infty$ are not integrable. These examples support our conjecture.

If the observable space $\mathcal{O}_\infty$ is integrable, and therefore, has locally an exact basis $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$, one can complete the set $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$ to a basis $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r, \mathrm{d}\zeta_{r+1}, \ldots, \mathrm{d}\zeta_n\}$ of $\mathcal{X}$. Then, in the coordinates $\{\zeta_1, \ldots, \zeta_n\}$, the system reads as

$$\begin{aligned}
\zeta_1^\Delta &= f_1(\zeta_1, \ldots, \zeta_r, u), \\
&\vdots \\
\zeta_r^\Delta &= f_r(\zeta_1, \ldots, \zeta_r, u), \\
\zeta_{r+1}^\Delta &= f_{r+1}(\zeta, u), \\
&\vdots \\
\zeta_n^\Delta &= f_n(\zeta, u), \\
y &= h(\zeta_1, \ldots, \zeta_r),
\end{aligned}$$

being the decomposition of the system into the observable (first r equations) and unobservable (equations from $(r+1)$th to nth) subsystems.

Example 4.7. (Continuation of Example 4.5). Integrating the observable space $\mathcal{O}_\infty$ of the system, we get the set of the observable state variables $\zeta_1 = x_1 - x_2$ and $\zeta_2 = x_3$. Next we complete $\mathcal{O}_\infty$ to a basis of $\mathcal{X}$, taking, for example, $\zeta_3 = x_1$. In these coordinates the system equations read as

$$\begin{aligned}
\zeta_1^\Delta &= u_1 \\
\zeta_2^\Delta &= u_2 \cos^2 \zeta_2 \\
\zeta_3^\Delta &= u_1 \tan \zeta_2 \\
y_1 &= \zeta_1 \\
y_2 &= \zeta_2,
\end{aligned}$$

where the first two equations (together with the output equations) define the observable subsystem. The state ζ_3 is unobservable.

Chapter 5

Implementation of the Results in the NLControl Package

The purpose of this chapter is to present several *Mathematica* functions, implementing the theoretical results of the thesis. The functions are developed within the package NLControl, which provides the symbolic computational tools that assist the solution of different modeling, analysis, and synthesis problems for nonlinear control systems. Therefore, the first section provides the brief introduction into the basics of the NLControl package. The next section describes the functions developed to facilitate the transformation of the continuous- or discrete-time system equations into the (extended) observer form. The observability related functions are presented in the last section. The description of the functions is accompanied by illustrative examples.

5.1 Outline of the NLControl Package

In this section the essential information about the NLControl package is recalled and the description of its basic functions, necessary in the sequel, is provided.

To take advantage of the NLControl package, first it should be properly installed within *Mathematica* environment and then loaded by means of the following command

```
In[1]:= << NLControl`Master`
```

In the further examples of this chapter we assume that the command above is evaluated and the NLControl package is loaded.

In order to operate with the system described by the state equations (1.1), (1.3) or (1.5), it should be entered in the following form

```
StateSpace[f, Xt, Ut, t, h, Yt, Type],
```

where `f` is a list containing the components of the state function; `Xt`, `Ut` and `Yt` define the lists of the state, input and output variables, respectively; `t` is a time argument and `h` defines the output function. The argument `Type` may have one of the following values: `TimeDerivative` and `Shift` stand for continuous- and discrete-time cases, respectively, whereas `TimeScale` indicates the system defined on time scale.

Another special object in the NLControl package is

```
SpanK[{{a11, ..., a1i}, ..., {ak1, ..., aki}}, {x1, ..., xi}, -1, t]
```

which represents the subspace spanned over $\mathcal{K}^*$ by the differential one-forms $a_{j1} \mathrm{d} x_1 + \cdots + a_{ji} \mathrm{d} x_i$ for $j = 1, \ldots, k$. Argument `t` means that all symbols depending on `t` are considered as variables.

Note that the form of some objects, determined by *Mathematica* and the NLControl package, differs from the traditional and familiar form. The function `BookForm` is intended to display such objects in a more accustomed form. For instance, it allows to represent the system, specified by the function `StateSpace`, in a form of equations. One of the optional arguments of the function `BookForm` is `TimeArgument -> False`, which allows to leave out the time argument `t` and display the result in the abridged notation. Note that throughout this chapter we will use this option to make the exposition more compact.

5.2 Transformation of the System into Observer Form

The functions, presented in this section, were developed within the NLControl package to facilitate the transformation of the nonlinear control system into the observer form. The function

```
ObserverFormTransformability[contSys]
```

checks the validity of the conditions (2.11) in order to verify whether the continuous-time system, determined by the argument `contSys`, is transformable into the observer form (2.5) or not. Besides, the function

```
ObserverFormTransformability[discrSys, bN, opts]
```

checks whether the discrete-time system, given by the argument `discrSys`, is transformable into the extended observer form (3.5) with the buffer, determined by the argument *bN*. The optional argument *opts* specifies the type of the conditions to check. If the value of *opts* is *Method -> OneForms* the function uses the conditions (3.13). The default value of the argument *opts* is *Method -> PartialDerivatives*, yielding the verification of the conditions (3.30). In both continuous- and discrete-time cases the function `ObserverFormTransformability` returns `True` if the transformability conditions are satisfied. Otherwise, the output of the function is `False`.

For the discrete-time systems was programmed the function

`MinBuffer[`*discrSys*`]`,

which recursively applies the function `ObserverFormTransformability` to find the minimal buffer, allowing to transform the system into the extended observer form (3.5).

The function

`ObserverForm[`*contSys*, *newXt*, *newYt*`]`

applies Algorithm 2.1 to transform the continuous-time system *contSys* into the observer form (2.5), whenever possible. Morover, the same function called as

`ObserverForm[`*discrSys*, *bN*, *newXt*, *newYt*`]`

applies Algorithm 3.1 to transform the discrete-time system *discrSys* into the extended observer form (3.5) with buffer *bN*, whenever possible. The argument *newXt* defines the new state variables, in terms of which the transformed system will be represented. The argument can be entered as a list of variables or as a pure function $z_{\#}$`[t]&`, which implicitly generates the list `{`z_1`[t], ..., `z_n`[t]}` of the demanded length. The last argument *newYt* denotes the new output variable, necessary to represent the transformed output. If all steps of the corresponding algorithm can be completed, the function `ObserverForm` returns the system in the required observer form together with the change of coordinates ((2.7) or (3.7), respectively) and the output transformation ((2.4) or (3.4), respectively). In the case when the transformability conditions are not satisfied, the output of the function is an empty list `{}`.

Furthermore, the functions, described above, were implemented online at the NLContol website [43]. To take advantage of this online tool, one has to select the option Observer Form either from the section Continuous or Discrete in the main menu of the site and, after filling the corresponding text fields, push the button Evaluate.

The following examples illustrate the application of the functions, described above.

Example 5.1. Consider the model of a DC motor from Example 2.1. The state equations (2.23) can be entered as follows

```
In[2]:= f = {-Km x1[t] x2[t] - (Ra + Rf) / k x1[t] + u[t],
           -B / J x2[t] - x3[t] + Km / J k x1[t]^2, 0};
        Xt = {x1[t], x2[t], x3[t]};
        Ut = {u[t]};
        Yt = {y[t]};
        h = {x1[t]};
        DCMotor = StateSpace[f, Xt, Ut, t, h, Yt,
           TimeDerivative];
        BookForm[DCMotor, TimeArgument -> False]
```

yielding

Out[8]=
$$
\begin{aligned}
x_1' &= \frac{k\,u - x_1\,(Ra + Rf + k\,Km\,x_2)}{k}\\
x_2' &= \frac{k\,Km\,x_1^2 - B\,x_2 - J\,x_3}{J}\\
x_3' &= 0\\
y &= x_1
\end{aligned}
$$

Applying the function `ObserverFormTransformability`, one obtains

```
In[9]:= ObserverFormTransformability[DCMotor]

Out[9]= True
```

meaning that the state equations can be transformed into the observer form. To perform the transformation we apply the function `ObserverForm` as follows

```
In[10]:= BookForm[ObserverForm[DCMotor, z#[t]&, Y[t]],
          TimeArgument -> False]
```

Out[10]=
$$
\begin{aligned}
z_1' &= e^{-z_1}\,u - \frac{B\,z_1}{J} + z_2\\
z_2' &= -\frac{e^{2\,z_1}\,k\,Km^2}{J} + \frac{B\,e^{-z_1}\,u}{J} + z_3\\
z_3' &= 0\\
Y &= z_1
\end{aligned}
$$

$$
\begin{aligned}
z_1 &= \mathrm{Log}[x_1]\\
z_2 &= \frac{-J\,(Ra + Rf) + B\,k\,\mathrm{Log}[x_1] - J\,k\,Km\,x_2}{J\,k}\\
z_3 &= -\frac{B\,(Ra + Rf)}{J\,k} + Km\,x_3\\
Y &= \mathrm{Log}[y]
\end{aligned}
$$

The first four rows of the result represent the system in the observer form (2.28), whereas the rest contains the change of coordinates (2.27) and the output transformation (2.26).

Example 5.2. Recall the discrete-time system (3.37) from Example 3.1. One can enter the system as follows

```
In[11]:= f = {u[t] x2[t], x3[t],
             (x1[t] + x2[t] u[t] + u[t]) (u[t] x2[t] + x3[t])
              (u[t] x1[t] + x4[t]), u[t] + x1[t]};
         Xt = {x1[t], x2[t], x3[t], x4[t]};
         Ut = {u[t]};
         Yt = {y[t]};
         h = {x1[t]};
         system1 = StateSpace[f, Xt, Ut, t, h, Yt, Shift];
         BookForm[system1, TimeArgument -> False]
```

Out[17]=

$$
\begin{aligned}
x_1^+ &= u\,x_2\\
x_2^+ &= x_3\\
x_3^+ &= (x_1 + u\,(1 + x_2))\,(u\,x_2 + x_3)\,(u\,x_1 + x_4)\\
x_4^+ &= u + x_1\\
y &= x_1
\end{aligned}
$$

In order to check whether the system above satisfies the conditions (3.30) for buffer $N = 1$, we run the command

```
In[18]:= ObserverFormTransformability[system1, 1]

Out[18]= True
```

The validity of the conditions (3.13) for $N = 1$ can be checked by means of the command

```
In[19]:= ObserverFormTransformability[system1, 1,
          Method -> OneForms]

Out[19]= True
```

Both results show that the system can be transformed into the extended observer form with buffer $N = 1$. Note that entering 0 as the value of the buffer, one can check whether the system is transformable into the observer form without buffer.

```
In[20]:= ObserverFormTransformability[system1, 0]

Out[20]= False
```

The consequence of the result above is that $N = 1$ is the minimal buffer for which the system can be transformed into the observer form. The function `MinBuffer` confirms this fact.

```
In[21]:= MinBuffer[system1]

Out[21]= 1
```

To transform the system into the extended observer form with buffer $N = 1$, one can call the function `ObserverForm` as follows

```
In[22]:= BookForm[ObserverForm[system1, 1, z#[t]&, Y[t]],
           TimeArgument -> False]
```

$$
\begin{aligned}
z_1^+ &= -\mathrm{Log}[u^-] + \mathrm{Log}\left[e^{z_1} + e^{z_1^-} u^-\right] + \mathrm{Log}[u] + z_2 \\
z_2^+ &= \mathrm{Log}\left[e^{z_1^-} + e^{z_1} + u^-\right] + z_3 \\
z_3^+ &= \mathrm{Log}\left[e^{z_1^-} + u^- + e^{z_1} u\right] + z_4 \\
z_4^+ &= 0 \\
Y &= z_1
\end{aligned}
$$

Out[22]=

$$
\begin{aligned}
z_1 &= \mathrm{Log}[x_1] \\
z_2 &= \mathrm{Log}[u^-] - \mathrm{Log}[u^- x_1^- + x_1] + \mathrm{Log}[x_2] \\
z_3 &= \mathrm{Log}[u] - \mathrm{Log}[u^- + x_1^- + x_1] - \mathrm{Log}[u\,(x_1 + x_2)] + \mathrm{Log}[x_3] \\
z_4 &= \mathrm{Log}[u^+] - \mathrm{Log}[u^- + x_1^- + u\,x_1] + \mathrm{Log}[u\,x_2 + x_3] - \\
&\quad \mathrm{Log}[u^+\,(u\,x_2 + x3)] + \mathrm{Log}[u\,x_1 + x_4] \\
Y &= \mathrm{Log}[y]
\end{aligned}
$$

The upper block of equations above is the system in the extended observer form with buffer $N = 1$. The lower block represents the change of coordinates (3.41) and the output transformation (3.40), respectively.

Example 5.3. Examine the discrete-time system (3.42) from Example 3.2. The system can be entered as follows

```
In[23]:= f = {x1[t] + x2[t] - x3[t], -x1[t] - x2[t],
           -(x1[t] x2[t]) / (u[t] x3[t] + x1[t] x2[t] x4[t]),
           -(u[t]^2 x2[t]^2 + (x1[t] + x2[t])
                  (x1[t] + x2[t] - x3[t]) x5[t]) /
              (u[t] (x1[t] + x2[t]) (x1[t] + x2[t] - x3[t])),
           (x2[t] - u[t] (x1[t] + x2[t])) / x3[t]};
         Xt = {x1[t], x2[t], x3[t], x4[t], x5[t]};
         Ut = {u[t]};
         Yt = {y[t]};
         h = {x2[t]};
         system2 = StateSpace[f, Xt, Ut, t, h, Yt, Shift];
         BookForm[system2, TimeArgument -> False]
```

Out[29]=

$$
\begin{aligned}
x_1^+ &= x_1 + x_2 - x_3 \\
x_2^+ &= -x_1 - x_2 \\
x_3^+ &= -\frac{x_1\, x_2}{u\, x_3 + x_1\, x_2\, x_4} \\
x_4^+ &= -\frac{u^2\, x_2{}^2 + (x_1 + x_2)\,(x_1 + x_2 - x_3)\, x_5}{u\,(x_1 + x_2)\,(x_1 + x_2 - x_3)} \\
x_5^+ &= \frac{x_2 - u\,(x_1 + x_2)}{x_3} \\
y &= x_2
\end{aligned}
$$

First, find the minimal buffer, allowing to transform the system into the extended observer form.

In[30]:= **bN = MinBuffer[system2]**

Out[30]= 2

Next, transform the system into the extended observer form with obtained buffer.

In[31]:= **BookForm[ObserverForm[system2, bN, z$_{\#}$[t]&, Y[t]], TimeArgument -> False]**

$$
\begin{aligned}
z_1^+ &= \frac{u^{--}\, z_1^{--2}\, z_1^-}{(z_1^{--} + z_1^-)\, z_1} + z_2 \\
z_2^+ &= -\frac{u^{--}\, z_1^{-2}\, z_1}{z_1^{--2}\,(z_1^- + z_1)} + z_3 \\
z_3^+ &= \frac{\left(-\frac{1}{z_1^{--}} - \frac{u^{--}}{z_1^-}\right) z_1}{u^-} + z_4 \\
z_4^+ &= z_5 \\
z_5^+ &= 0 \\
Y &= z_1
\end{aligned}
$$

Out[31]=

$$
\begin{aligned}
z_1 &= -\frac{1}{x_2} \\
z_2 &= \frac{u^{--}\, x_2}{x_2^{--2} + x_2^{--}\, x_2^-} + \frac{1}{x_1 + x_2} \\
z_3 &= -\frac{x_2^{-2} + x_2^-\, x_2 + \left(-u^{--}\, x_2^{--2} + u^-\,(x_1 + x_2)\right) x_3}{x_2^-\,(x_2^- + x_2)\, x_3} \\
z_4 &= \frac{-u^{-2}\, x_2^{-2} + x_1\,(x_2^{--} + u^{--}\, x_2^- + u^-\, x_2\, x_4)}{u^-\, x_1\, x_2} \\
z_5 &= -\frac{x_2^- + u^-\, x_2 + (x_1 + x_2)\, x_5}{u\,(x_1 + x_2)} \\
Y &= -\frac{1}{y}
\end{aligned}
$$

The upper block of equations above is the system in the extended observer form with buffer $N = 2$. The lower block represents the change of coordinates (3.47) and the output transformation (3.46), respectively.

5.3 Observability Related Functions

The following set of functions was developed within the NLControl package in order to assist in the verification of the observability condition, construction of the observability filtration and the observable space, computation of the observability indices and decomposition of the system into the observable and unobservable subsystems. Though in the context of the thesis the functions are applied to the systems, defined on the homogeneous time scale, they are also applicable to the continuous- and discrete-time systems (see for details [82]).

The functions and their assignments are listed below.

- `Observability[`*`Sys`*`]` uses the observability rank condition (4.2) to check whether the system is observable or not.
- `ObservabilityFiltration[`*`Sys`*`]` constructs the observability filtration (4.7) of the system.
- `ObservableSpace[`*`Sys`*`]` constructs the observable subspace $\mathcal{O}_\infty$ of the system, i.e. $\mathcal{O}_\infty \subseteq \mathcal{X}$.
- `UnObservableSpace[`*`Sys`*`]` constructs the unobservable subspace $\mathcal{X}_{\bar{O}}$ of the system, i.e. $\mathcal{X}_{\bar{O}} \cong \mathcal{X}/\mathcal{O}_\infty$.
- `ObservabilityIndices[`*`Sys`*`]` computes the sets of the indices σ_j and the observability indices s_i, defined by (4.12) and (4.13), respectively.
- `ObservabilityDecomposition[`*`Sys, newXt`*`]` decomposes the system into observable and unobservable subsystems, whenever possible.

In the functions, described above, the arguments *`Sys`* and *`newXt`* define the system under consideration and a list of new state variables, respectively. Moreover, the function `ObservabilityDecomposition` optionally may have the argument *`PrintInfo -> True`*, which provides the additional information.

Furthermore, the online implementation of the above-mentioned functions is available at the NLContol website [43]. To take advantage of this online tool, reveal the content of the section Time Scales in the main menu of the site, then choose the option Observability and, after filling the corresponding text fields, push the button Evaluate.

The application of the functions is illustrated by means of the following examples.

Example 5.4. Consider the model of unicycle from Example 4.1. One can enter the equation (4.11) as follows

```
In[32]:= f = {u1[t] Cos[x3[t]], u1[t] Sin[x3[t]], u2[t]};
         Xt = {x1[t], x2[t], x3[t]};
         Ut = {u1[t], u2[t]};
         Yt = {y1[t], y2[t]};
         h = {x1[t], x2[t]};
         unicycle = StateSpace[f, Xt, Ut, t, h, Yt, TimeScale];
         BookForm[unicycle]
```

yielding

$$\text{Out[38]=}\quad \begin{aligned} x_1^{[1]}[t] &= \mathrm{Cos}[x_3[t]]\,u_1[t]\\ x_2^{[1]}[t] &= \mathrm{Sin}[x_3[t]]\,u_1[t]\\ x_3^{[1]}[t] &= u_2[t]\\ y_1[t] &= x_1[t]\\ y_2[t] &= x_2[t] \end{aligned}$$

Note that, unlike the notations in the thesis, in the NLControl package the superscript $^{[1]}$ stands for the delta derivative. Applying the function `Observability` one obtains

```
In[39]:= Observability[unicycle]

Out[39]= True
```

meaning that the system is observable. The observability filtration of the system can be found as follows

```
In[40]:= BookForm[ObservabilityFiltration[unicycle]
          TimeArgument -> False]

Out[40]= {SpanK[dx1, dx2],
          SpanK[dx1, dx2, dx3], SpanK[dx1, dx2, dx3]}
```

The last element of the result is the observable space of the system. To compute the sets of indices σ_j and s_i, the function `ObservabilityIndices` can be used

```
In[41]:= ObservabilityIndices[unicycle]

Out[41]= {{2, 1}, {2, 1}}
```

In above, the first set of integers represents the indices σ_j, defined by (4.12), whereas the second set of integers stands for the observability indices s_i, defined by (4.13).

Example 5.5. Consider the system (4.19) from Example 4.5. The equations can be entered by means of the following commands

```
In[42]:= f = {Tan[x1[t] - x2[t]] u1[t],
             u1[t] Tan[x1[t] - x2[t]] - Cos[x1[t] - x2[t]]^2 u2[t],
             u1[t]};
         Xt = {x1[t], x2[t], x3[t]};
         Ut = {u1[t], u2[t]};
         Yt = {y1[t], y2[t]};
         h = {x3[t], x1[t] - x2[t]};
         system3 = StateSpace[f, Xt, Ut, t, h, Yt, TimeScale];
         BookForm[system3]
```

Out[48]=

$$\begin{aligned} x_1^{[1]} &= \mathrm{Tan}[x_1 - x_2]\, u_1 \\ x_2^{[1]} &= \mathrm{Tan}[x_1 - x_2]\, u_1 - \mathrm{Cos}[x_1 - x_2]^2\, u_2 \\ x_3^{[1]} &= u_1 \\ y_1 &= x_3 \\ y_2 &= x_1 - x_2 \end{aligned}$$

Running the command

```
In[49]:= ObservabilityIndices[system3]

Out[49]= {{2}, {1, 1}}
```

we obtain the sets of indices σ_j and s_i, respectively. Next, compute the observable and unobservable spaces of the system.

```
In[50]:= BookForm[ObservableSpace[system3],
          TimeArgument -> False]
```

Out[50]= $\mathrm{SpanK}[dx_1 - dx_2,\ dx_3]$

```
In[51]:= BookForm[UnObservableSpace[system3]],
          TimeArgument -> False]
```

Out[51]= $\mathrm{SpanK}[dx_1]$

Since the observable space is not equal to $\mathcal{X}$ (or alternatively, the unobservable space is not $\{0\}$), the system is not observable, and one can employ the function `ObservabilityDecomposition` to decompose the system into the observable and unobservable subsystems.

In[52]:= **BookForm[ObservabilityDecomposition[system3, z# [t]&, PrintInfo -> True], TimeArgument -> False]**

```
{z1[t], z2[t]} are observable variables.
{z3[t]} is unobservable variable.
```

Out[52]=

$$
\begin{aligned}
z_1^{[1]} &= u_1 \\
z_2^{[1]} &= \mathrm{Cos}[z_2]^2\, u_2 \\
z_3^{[1]} &= \mathrm{Tan}[z_2]\, u_1 \\
y_1 &= z_1 \\
y_2 &= z_2 \\
\\
z_1 &= x_3 \\
z_2 &= x_1 - x_2 \\
z_3 &= x_1
\end{aligned}
$$

The first two rows of the result provide an additional information about the new coordinates, the next five rows represent the decomposed system and the remaining rows show the coordinate transformation, which yielded the decomposition.

Conclusions

Concluding Remarks

In such practical control tasks as computation of the state feedback and monitoring of the system behavior the observer plays an important role, providing the state estimate, whenever the state itself is not directly measurable or its measurement is very expensive. The thesis is mainly devoted to the problem of transforming the nonlinear state equations into the (extended) observer form, for which the observer can be easily constructed. Both continuous- and discrete-time systems are considered. Our approach is based on the analysis of the structure of the input-output equation, corresponding to the state equations. Under observability assumption, one may always find the input-output equation, at least locally, using the state elimination algorithm (see, for example, [25]). The observer form approach, relying on the state transformation only, imposes restrictive conditions for transformability of nonlinear control system into the observer form. Therefore, the aim is to relax the conditions by employing the output transformation in addition to the state transformation and considering the extended observer form with generalized input-output injections. The results of the thesis can be divided into four parts.

- First, the necessary and sufficient conditions are given for the existence of the state and output coordinate transformations, that bring the continuous-time state equations into the observer form. The conditions require that certain differential one-forms, associated with the input-output equation of the system, are closed. Once the input-output equation is obtained by the state elimination, the conditions to be checked can be easily constructed due to the direct formula for computation of the necessary one-forms. However, note that the conditions depend on an unknown single-variable output-dependent function. As a consequence, the verification of the conditions requires to solve certain differential equation, which, sometimes, can be difficult task. The algorithm is also given for transformation of the state equations into the observer form. The presented results can be considered as an improvement of those given in [31]. Unlike [31], where the two-step procedure was proposed, the conditions presented in the thesis are more straightforward for verification.

- Second, the thesis presents two alternative (complementary) sets of necessary and sufficient conditions for the existence of the extended coordinate change and the output transformation that allow to transform the discrete-time state equations into the extended observer form with buffer (i.e. the nonnegative integer determining the number of past values of the input and output, necessary for transformation). The first set of conditions is expressed in terms of the exterior derivatives and the exterior products of certain one-forms, associated with the input-output equation, corresponding to the state equations. These conditions have the advantage of being intrinsic. The other set of conditions is formulated in terms of certain partial derivatives, related to the i/o equation of the system, and due to the matrix representation can be checked almost by direct inspection. Moreover, the matrix representation simplifies the determination of the minimal value of the buffer allowing the transformation. Besides the conditions, we proposed the algorithm for transformation of the state equations into the extended observer form. The presented results generalize those given in [40], [78], [79]. In [78] the authors provided the necessary and sufficient conditions in terms of one-forms for the special case of the observer form without the buffer, whereas in [79] the special case of the buffer being equal to 1 was considered and the solvability conditions were given in terms of partial derivatives. Though in [40] the arbitrary buffer was considered, the conditions, relying on the sophisticated language of differential geometry, were given only for systems without inputs.

- The third part of the thesis presents the results on observability property of the nonlinear system, defined on homogeneous time scale. Time scale analysis allows to unify continuous- and discrete-time theories, presenting both of them simultaneously under the same language. The related notions such as observability, observability filtration, observable space and observability indices were extended to the systems, defined on time scale. Moreover, the possibility to decompose the system into observable/unobservable subsystems is discussed.

- Finally, the fourth part describes implementation of the theoretical results of the thesis in the form of *Mathematica* functions within the framework of NLControl package. In total nine functions were presented. Three of them facilitate the transformation of the continuous-or discrete-time state equation into the corresponding (extended) observer form, whereas another six functions assist in the verification of the observability condition, construction of the observability filtration and the observable (unobservable) space, computation of the observability indices and decomposition of the system into the observable/unobservable subsystems, whenever possible.

Future Research

The results of the thesis may be extended in several ways. One of the open topics for future research is the extension of our results for the continuous-time systems to the case, when the input-output injections in the observer form are allowed to depend also on the derivatives of the input (as in [31]). Unlike [31], where the solution was given as a two-step procedure, we are intended to derive more direct conditions. Moreover, we have an intention to compare our results with those presented in [63], [85], which rely on the tools from differential geometry.

In the discrete-time case the future goal is to compare our results with the dynamic observer error linearization technique, presented in [100], where in order to transform the system into the generalized observer form, it is suggested to augment the system by means of the so called dynamic auxiliary system of the specific linear form. It is our conjecture that the two approaches are closely related, since, in principle, the past values of input and output may be possibly expressed in terms of system extensions.

Regarding the systems, defined on homogeneous time scales, one of the future goals is to define the observability property using the concept of (in)distinguishable states. Moreover, we intend to carry over the observer form approach to the systems on homogeneous time scale.

Appendix

For the proof of Lemmas A.1 and 2.1 we will use the binomial theorem $(a+b)^k = \sum_{l=0}^{k} C_k^l a^l b^{k-l}$, which for $a=-1$, $b=1$ and $k \geq 1$ gives

$$\sum_{l=0}^{k} C_k^l (-1)^l = 0. \tag{A.1}$$

Separating the last addend of the sum above and placing it into the right-hand side of the equality, yields

$$\sum_{l=0}^{k-1} C_k^l (-1)^l = -(-1)^k. \tag{A.2}$$

Proof of Theorem 1.4

Proof. In the proof we omit the variable t, i.e. use instead of $\xi_i(t)$ a shorter notation ξ_i, which allows to write the bulky formulas in a more compact form. According to Mishkov's formula [75], the $(a+b)$th derivative of the composite function Φ can be computed by the formula

$$\begin{aligned}(\Phi(\xi_1, \xi_2, \ldots, \xi_r))^{(a+b)} = \sum_0 \sum_1 \sum_2 \cdots \sum_{a+b} \frac{(a+b)!}{\prod_{i=1}^{a+b} (i!)^{k_i} \prod_{i=1}^{a+b} \prod_{j=1}^{r} q_{i,j}!} \cdot \\ \cdot \frac{\partial^k \Phi}{\partial \xi_1^{p_1} \partial \xi_2^{p_2} \cdots \partial \xi_r^{p_r}} \prod_{i=1}^{a+b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}},\end{aligned} \tag{A.3}$$

where the respective sums are taken over all nonnegative integer solutions of the Diophantine equations, as follows

$$\sum_0 \to k_1 + 2k_2 + \cdots + (a+b)k_{a+b} = a+b, \tag{A.4a}$$

$$\sum_i \to q_{i,1} + q_{i,2} + \cdots + q_{i,r} = k_i, \tag{A.4b}$$

for $i = 1, \dots, a+b$, and p_j and k on the right-hand side of (A.3) satisfy the relations

$$\begin{aligned} p_j &= q_{1,j} + q_{2,j} + \dots + q_{a+b,j}, \quad j = 1, 2, \dots, r, \\ k &= p_1 + p_2 + \dots + p_r = k_1 + k_2 + \dots + k_{a+b}. \end{aligned} \tag{A.5}$$

In derivation of sum (A.3) with respect to $\xi_l^{(a)}$, only addends of sum (A.3) with $q_{a,l} \neq 0$ will matter. Denote by $H(\cdot)$ and $G(\cdot)$ the parts of sum (A.3) corresponding to $q_{a,l} \neq 0$ and $q_{a,l} = 0$, respectively; then

$$(\Phi(\xi_1, \xi_2, \dots, \xi_r))^{(a+b)} = H(\cdot) + G(\cdot). \tag{A.6}$$

Note that it is possible to state that $H(\cdot)$ equals to expression in the right-hand side of (A.3) where in addition to the restrictions expressed by (A.4) and (A.5), the condition $q_{a,l} \neq 0$ has to be satisfied. Note also that if $q_{a,l} \neq 0$, then $k_a \neq 0$. We prove the formula separately for the cases $a > b$ and $a \leq b$.

First, consider the case when $a > b$. Since $k_a \neq 0$ and $q_{a,l} \neq 0$, in order to satisfy (A.4) the following must hold

$$\begin{aligned} k_a &= 1, \quad k_i = 0, \quad b < i \leq a+b,\ i \neq a, \\ q_{a,l} &= 1, \quad q_{a,j} = 0, \quad j = 1, 2, \dots, r,\ j \neq l, \\ q_{i,j} &= 0, \quad b < i \leq a+b,\ i \neq a,\ j = 1, 2, \dots, r. \end{aligned} \tag{A.7}$$

As a result, under the condition $q_{a,l} \neq 0$, one can rewrite (A.4a) as follows

$$\sum_0 \to k_1 + 2k_2 + \dots + bk_b = b, \tag{A.8}$$

and in (A.4b), now $i = 1, \dots, b$.

Using (A.7) and changing the notations, taking $\bar{p}_j = p_j$ for $j = 1, 2, \dots, r$, $j \neq l$, $\bar{p}_l = p_l - 1$ and $\bar{k} = k - 1$, equations (A.5) may be rewritten as

$$\begin{aligned} \bar{p}_j &= q_{1,j} + q_{2,j} + \dots + q_{b,j}, \quad j = 1, 2, \dots, r, \\ \bar{k} &= \bar{p}_1 + \bar{p}_2 + \dots + \bar{p}_r = k_1 + k_2 + \dots + k_b. \end{aligned} \tag{A.9}$$

Note also that under conditions (A.7)

$$\prod_{i=1}^{a+b} (i!)^{k_i} = a! \prod_{i=1}^{b} (i!)^{k_i}, \qquad \prod_{i=1}^{a+b} \prod_{j=1}^{r} q_{i,j}! = \prod_{i=1}^{b} \prod_{j=1}^{r} q_{i,j}!,$$

and

$$\prod_{i=1}^{a+b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}} = \xi_l^{(a)} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}}.$$

Taking into account the equalities above and the fact that the partial derivative of $G(\cdot)$ in (A.6) with respect to $\xi_l^{(a)}$ is 0, the variables $\bar{p}_j$ and $\bar{k}$ yield

$$\frac{\partial\left(\Phi(\xi_1,\xi_2,\ldots,\xi_r)\right)^{(a+b)}}{\partial\xi_l^{(a)}} = \sum_0\sum_1\sum_2\cdots\sum_b \frac{(a+b)!}{a!\prod_{i=1}^{b}(i!)^{k_i}\prod_{i=1}^{b}\prod_{j=1}^{r}q_{i,j}!}\cdot$$
$$\cdot\frac{\partial^{\bar{k}+1}\Phi}{\partial\xi_1^{\bar{p}_1}\cdots\partial\xi_{l-1}^{\bar{p}_{l-1}}\partial\xi_l^{\bar{p}_l+1}\partial\xi_{l+1}^{\bar{p}_{l+1}}\cdots\partial\xi_r^{\bar{p}_r}}\prod_{i=1}^{b}\prod_{j=1}^{r}\left(\xi_j^{(i)}\right)^{q_{i,j}}. \quad \text{(A.10)}$$

Note that in (A.10) all the partial derivatives with respect to ξ_j are of order $\bar{p}_j$, except ξ_l where the order of the partial derivative is $\bar{p}_l+1$. For the unification of the orders denote $\bar{\Phi} := \frac{\partial\Phi}{\partial\xi_l}$. Also we multiply the right-hand side of equation (A.10) by $b!/b!$ to obtain

$$\frac{\partial\left(\Phi(\xi_1,\xi_2,\ldots,\xi_r)\right)^{(a+b)}}{\partial\xi_l^{(a)}} = C_{a+b}^{b}\sum_0\sum_1\sum_2\cdots\sum_b \frac{b!}{\prod_{i=1}^{b}(i!)^{k_i}\prod_{i=1}^{b}\prod_{j=1}^{r}q_{i,j}!}\cdot$$
$$\cdot\frac{\partial^{\bar{k}}\bar{\Phi}}{\partial\xi_1^{\bar{p}_1}\partial\xi_2^{\bar{p}_2}\cdots\partial\xi_r^{\bar{p}_r}}\prod_{i=1}^{b}\prod_{j=1}^{r}\left(\xi_j^{(i)}\right)^{q_{i,j}}. \quad \text{(A.11)}$$

It is easy to observe now that, according to Mishkov's formula, the sum on the right-hand side of (A.11) together with the conditions (A.4b) for $i=1,\ldots,b$, (A.8) and (A.9) is the bth order total derivative of the function $\bar{\Phi}$. Consequently,

$$\frac{\partial\left(\Phi(\xi_1,\xi_2,\ldots,\xi_r)\right)^{(a+b)}}{\partial\xi_l^{(a)}} = C_{a+b}^{b}\bar{\Phi}^{(b)} =$$
$$= C_{a+b}^{b}\left(\frac{\partial\Phi(\xi_1,\xi_2,\ldots,\xi_r)}{\partial\xi_l}\right)^{(b)}. \quad \text{(A.12)}$$

Second, consider the case $a \le b$. Since $k_a \neq 0$, in order to satisfy (A.4), the following must hold

$$\begin{aligned} k_i &= 0, \quad b < i \le a+b, \\ q_{i,j} &= 0, \quad b < i \le a+b,\ j=1,2,\ldots,r. \end{aligned} \quad \text{(A.13)}$$

Therefore, it is possible to rewrite condition (A.4a) as

$$\sum_0 \to k_1+\cdots+(a-1)k_{a-1}+a(k_a-1)+(a+1)k_{a+1}+\cdots+bk_b = b, \quad \text{(A.14)}$$

and in (A.4b), now $i=1,\ldots,b$.

Again, in order to unify the notation in (A.14), one can take $\bar{k}_i = k_i$ for $i = 1, 2, \ldots, b$, $i \neq a$ and $\bar{k}_a = k_a - 1$. This allows to rewrite (A.14) as follows

$$\sum_0 \to \bar{k}_1 + 2\bar{k}_2 + \cdots + b\bar{k}_b = b, \tag{A.15}$$

and (A.4b) as

$$\begin{aligned} &\sum_i \to q_{i,1} + q_{i,2} + \cdots + q_{i,r} = \bar{k}_i, \quad i = 1, \ldots, b, \ i \neq a, \\ &\sum_a \to q_{a,1} + q_{a,2} + \cdots + q_{a,r} = \bar{k}_a + 1. \end{aligned} \tag{A.16}$$

Since $q_{a,l} \geq 1$ we can denote $\bar{q}_{a,l} := q_{a,l} - 1$ and the remaining q's as $\bar{q}_{i,j} := q_{i,j}$. Thereby (A.16) can by rewritten in the unified notation as

$$\sum_i \to \bar{q}_{i,1} + \bar{q}_{i,2} + \cdots + \bar{q}_{i,r} = \bar{k}_i, \tag{A.17}$$

for $i = 1, \ldots, b$. Changing notations, taking $\bar{p}_j = p_j$ for $j = 1, 2, \ldots, r$, $j \neq l$, $\bar{p}_l = p_l - 1$ and $\bar{k} = k - 1$, equations (A.5) may be rewritten as

$$\begin{aligned} \bar{p}_j &= \bar{q}_{1,j} + \bar{q}_{2,j} + \cdots + \bar{q}_{b,j}, \quad j = 1, 2, \ldots, r, \\ \bar{k} &= \bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_r = \bar{k}_1 + \bar{k}_2 + \cdots + \bar{k}_b. \end{aligned} \tag{A.18}$$

Taking into account (A.13) and using variables $\bar{k}_i$ and $\bar{q}_{i,j}$ we have

$$\prod_{i=1}^{a+b} (i!)^{k_i} = a! \prod_{i=1}^{b} (i!)^{\bar{k}_i}, \qquad \prod_{i=1}^{a+b} \prod_{j=1}^{r} q_{i,j}! = (\bar{q}_{a,l} + 1) \prod_{i=1}^{b} \prod_{j=1}^{r} \bar{q}_{i,j}!,$$

and

$$\begin{aligned} \prod_{i=1}^{a+b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}} &= \left(\xi_l^{(1)}\right)^{\bar{q}_{1,l}} \cdots \left(\xi_l^{(a-1)}\right)^{\bar{q}_{a-1,l}} \left(\xi_l^{(a)}\right)^{\bar{q}_{a,l}+1} \cdot \\ &\cdot \left(\xi_l^{(a+1)}\right)^{\bar{q}_{a+1,l}} \cdots \left(\xi_l^{(b)}\right)^{\bar{q}_{b,l}} \prod_{i=1}^{b} \prod_{\substack{j=1 \\ j \neq l}}^{r} \left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}. \end{aligned}$$

Furthermore, based on the equalities above and the fact that the partial derivative of $G(\cdot)$ in (A.6) with respect to $\xi_l^{(a)}$ equals 0, we obtain, in new variables $\bar{p}_j$ and $\bar{k}$

$$\begin{aligned} \frac{\partial \left(\Phi(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} &= \sum_0 \sum_1 \sum_2 \cdots \sum_b \frac{(a+b)!}{a! \prod_{i=1}^{b} (i!)^{\bar{k}_i} \prod_{i=1}^{b} \prod_{j=1}^{r} \bar{q}_{i,j}!} \cdot \\ &\cdot \frac{\partial^{\bar{k}+1} \Phi}{\partial \xi_1^{\bar{p}_1} \cdots \partial \xi_{l-1}^{\bar{p}_{l-1}} \partial \xi_l^{\bar{p}_l+1} \partial \xi_{l+1}^{\bar{p}_{l+1}} \cdots \partial \xi_r^{\bar{p}_r}} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}. \end{aligned}$$

Like in case $a > b$ we denote $\bar{\Phi} = \frac{\partial \Phi}{\partial \xi_l}$ and multiply the right-hand side of the equality above by $\frac{b!}{b!}$ to obtain

$$\frac{\partial \left(\Phi(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} = C_{a+b}^{b} \sum_{0} \sum_{1} \sum_{2} \cdots \sum_{b} \frac{b!}{\prod_{i=1}^{b} (i!)^{\bar{k}_i} \prod_{i=1}^{b} \prod_{j=1}^{r} \bar{q}_{i,j}!} \cdot$$
$$\cdot \frac{\partial^{\bar{k}} \bar{\Phi}}{\partial \xi_1^{\bar{p}_1}, \partial \xi_2^{\bar{p}_2} \cdots \partial \xi_r^{\bar{p}_r}} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}. \quad \text{(A.19)}$$

Again it is not difficult to observe that according to Mishkov's formula the sum on the right-hand side of the equation (A.19) together with the conditions (A.15), (A.17) and (A.18) is the bth order total derivative of the function $\bar{\Phi}$. Consequently, (A.12) holds again, and this completes the proof. □

Proof of Proposition 2.1

In order to prove Proposition 2.1, we need Lemma A.1 below

Lemma A.1. *For $j \geq 1$ and $r \geq j+1$ the following holds*

$$\sum_{i=1}^{j} (-1)^i C_{r-1}^{i-1} C_{r-i}^{r-j-1} = (-1)^j C_{r-1}^{j}. \quad \text{(A.20)}$$

Proof. Take (A.2) for $k = j$ and $l = i - 1$ and multiply both sides of the equality by $-C_{r-1}^{j}$, where $r \geq j+1$, to obtain

$$\sum_{i=1}^{j} (-1)^i C_j^{i-1} C_{r-1}^{j} = (-1)^j C_{r-1}^{j}.$$

Taking into account the definition of binomial coefficient, i.e. $C_n^k = \frac{n!}{(n-k)!k!}$, one can easily verify that $C_j^{i-1} C_{r-1}^{j} = C_{r-1}^{i-1} C_{r-i}^{r-j-1}$, which implies the validity of (A.20). □

Moreover, rewriting (1.8) for the continuous-time case ($\sigma_f = \mathrm{id}$), we recall that the rth time derivative of an arbitrary one-form $\omega = \sum_i A_i \mathrm{d}\zeta_i$ may be computed as

$$\omega^{(r)} = \sum_{q=0}^{r} C_r^q \sum_i A_i^{(r-q)} \mathrm{d}\zeta_i^{(q)}. \quad \text{(A.21)}$$

Now we are ready to prove Proposition 2.1.

Proof. The proof is by mathematical induction on i. One can easily verify that (2.8b) and (2.9) coincide for $i = 1$. Next we assume that the statement of Proposition holds for $i \le k$ (k is an arbitrary integer from 1 to $n-1$) and show that it is true for $i = k+1$.

From (2.8a) one obtains

$$P_{k+1} = \mathrm{d}\phi - \sum_{i=1}^{k} \omega_i^{(n-i)}.$$

From the assumption that (2.9) holds for $i \le k$ we have

$$P_{k+1} = \mathrm{d}\phi - $$
$$- \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left[\left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right]^{(n-i)}.$$

Using the rule (A.21), the latter yields

$$P_{k+1} = \mathrm{d}\phi - \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \sum_{q=0}^{n-i} C_{n-i}^{q} \cdot$$
$$\cdot \left[\left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j+n-i-q)} \mathrm{d}y^{(q)} + \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j+n-i-q)} \mathrm{d}u^{(q)} \right].$$

From (2.8b) follows that in order to find ω_{k+1}, only A_{k+1}^{n-k-1} and B_{k+1}^{n-k-1} are necessary. In other words, we are interested only in such elements of P_{k+1} where the order of differentiation of $\mathrm{d}y$ and $\mathrm{d}u$ is $q = n-k-1$. Thus, we have

$$A_{k+1}^{n-k-1} = \frac{\partial \phi}{\partial y^{(n-k-1)}} - $$
$$- \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} C_{n-i}^{n-k-1} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j+k-i+1)} \tag{A.22a}$$

and

$$B_{k+1}^{n-k-1} = \frac{\partial \phi}{\partial u^{(n-k-1)}} - $$
$$- \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} C_{n-i}^{n-k-1} \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j+k-i+1)}. \tag{A.22b}$$

Note that (A.22a) and (A.22b) have a similar structure. Thus, all the transformations made with one expression will be similar for the other.

Taking into account that $-(-1)^{i-1} = (-1)^i$ and changing the summation order $\sum_{i=1}^{k}\sum_{j=0}^{i-1} a_{i,j} = \sum_{j=1}^{k}\sum_{i=1}^{j} a_{k-j+i,i-1}$, rewrite (A.22a) as follows

$$\begin{aligned} A_{k+1}^{n-k-1} &= \frac{\partial\phi}{\partial y^{(n-k-1)}} + \sum_{j=1}^{k}\sum_{i=1}^{j}(-1)^i C_{n-k+j-1}^{i-1}\cdot \\ &\quad \cdot C_{n-k+j-i}^{n-k-1}\left(\frac{\partial\phi}{\partial y^{(n-k+j-1)}}\right)^{(j)} = \frac{\partial\phi}{\partial y^{(n-k-1)}} + \\ &\quad + \sum_{j=1}^{k}\left[\left(\frac{\partial\phi}{\partial y^{(n-k+j-1)}}\right)^{(j)}\sum_{i=1}^{j}(-1)^i C_{n-k+j-1}^{i-1} C_{n-k+j-i}^{n-k-1}\right]. \end{aligned} \tag{A.23}$$

Taking into account that $\frac{\partial\phi}{\partial y^{(n-k-1)}} = (-1)^j C_{n-k-1+j}^{j}\left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}}\right)^{(j)}$ for $j=0$ and using Lemma A.1 for $r = n-k+j$, one can rewrite (A.23) as

$$A_{k+1}^{n-k-1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j}\left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}}\right)^{(j)}. \tag{A.24a}$$

Analogously, from (A.22b) we get

$$B_{k+1}^{n-k-1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j}\left(\frac{\partial\phi}{\partial u^{(n-k-1+j)}}\right)^{(j)}. \tag{A.24b}$$

Using (2.8b) and (A.24)

$$\omega_{k+1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j}\left[\left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}}\right)^{(j)}\mathrm{d}y + \left(\frac{\partial\phi}{\partial u^{(n-k-1+j)}}\right)^{(j)}\mathrm{d}u\right]$$

being (2.9) for $i = k+1$. □

Proof of Lemma 2.1

Proof. (i) Taking into account that $-(-1)^{\varsigma} = (-1)^{\varsigma-1}$, the equality (A.2) taken for $k=\varsigma$ and $l = j-1$ confirms statement (i).

(ii) Since $s = 1,\ldots,\varsigma-1$ and $\varsigma \geq 2$, then $\varsigma - s \geq 1$ and it is eligible to take (A.1) for $k = \varsigma - s$, which after replacing the summation index l by $j-1$, leads to (ii). □

Proof of Lemma 3.1

Proof. It is easy to observe that

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{l=1}^{n-N}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_l(\nu_l)}{\partial y^{[n-l-s]}}\mathrm{d}y^{[n-l-s]} + \frac{\partial\bar{\varphi}_l(\nu_l)}{\partial u^{[n-l-s]}}\mathrm{d}u^{[n-l-s]}\right).$$

Replace on the right-hand side of the relationship, given above, the summation index l by $l+1$. In this case $l = 0, \dots, n-N-1$ and one can change the summation order

$$\sum_{l=1}^{n-N}\sum_{s=0}^{N} a_{l,s} = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{n-N-l,N-s},$$

which yields

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}}\mathrm{d}y^{[l+s]} + \right.$$
$$\left. + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}}\mathrm{d}u^{[l+s]}\right).$$

Change the summation indices l and s for $i = l+s$ and l. It is easy to see, that in this case i changes from 0 to $n-1$ and $l = i-s$. Since $s = 0, \dots, N$, the minimal and maximal values of $i-s$ are $i-N$ and i, respectively. On the other hand, l changes from 0 to $n-N-1$. Thus, we take $l = \max(0, i-N), \dots, \min(i, n-1-N)$. As a result, one can use the following relation

$$\sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{l,l+s} = \sum_{i=0}^{n-1}\sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} a_{l,i},$$

which leads to (3.11). □

Proof of Lemma 3.2

Proof. The proof of lemma, though in principle not very difficult, is technically rather demanding. Figures A.1, A.2 and A.3 below help to follow the separate steps of the proof. First, let us mention that the cases $2N < \overline{j_\alpha} - \underline{j_\alpha}$ and $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ are treated separately. The reason is that conditions (3.30b) are unnecessary for the first case.

Second, in the proof we will use the matrices (tables) with elements $\Theta_{k,l}$ (k pointing to the row and l to the column), where k and l may take values from different sets of non-negative integers at different steps of the proof. However, unlike the typical case when the matrix element is a number or expression, here its content is two relations (equalities). We do not manipulate with those relations, the role of the matrix is just to keep the track of the steps in the proof.

Evaluating the total differentials $\mathrm{d}S$ and $\mathrm{d}\phi$ as well as their wedge product $\mathrm{d}S \wedge \mathrm{d}\phi$ in (3.31), it is easy to observe by direct inspection, after tedious calculations, that the condition (3.31) is equivalent to the equalities (A.25) below

$$\begin{aligned} \frac{\partial S}{\partial \alpha^{[l]}} \frac{\partial \phi}{\partial \alpha^{[k]}} &= \frac{\partial S}{\partial \alpha^{[k]}} \frac{\partial \phi}{\partial \alpha^{[l]}}, \\ \frac{\partial S}{\partial \alpha^{[l]}} \frac{\partial \phi}{\partial \beta^{[k]}} &= \frac{\partial S}{\partial \beta^{[k]}} \frac{\partial \phi}{\partial \alpha^{[l]}}, \end{aligned} \tag{A.25}$$

where $k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. Recall that here, like in the assumptions (3.30a) and (3.30b) of the lemma, the variable α denotes either input u or output y, and by β is denoted the other variable; i.e. if $\alpha = y$, then $\beta = u$ and vice versa, if $\alpha = u$, then $\beta = y$. These notations help to make the presentation more compact[1]. Now, the contents of $\Theta_{k,l}$ are the equalities (A.25).

Before turning to separate steps of the proof we rewrite the assumptions (3.30a) and (3.30b) into the form, suitable for the proof. Namely, the conditions (3.30a) may be given as (A.26) below by evaluation of the derivative of the logarithmic function and rewriting the conditions separately for α alone as well as for α and β.

$$\frac{\partial S}{\partial \alpha^{[j]}} = \left(\frac{\partial \phi}{\partial \alpha^{[i]}} \right)^{-1} \frac{\partial^2 \phi}{\partial \alpha^{[i]} \partial \alpha^{[j]}}, \tag{A.26a}$$

$$\frac{\partial S}{\partial \alpha^{[j]}} = \left(\frac{\partial \phi}{\partial \beta^{[i]}} \right)^{-1} \frac{\partial^2 \phi}{\partial \beta^{[i]} \partial \alpha^{[j]}}, \tag{A.26b}$$

where $i, j = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$, $j \neq i - N, \ldots, i + N$. Taking into account that α and β can be mutually interchanged, rewrite the conditions (3.30b) as

$$\frac{\partial S}{\partial \alpha^{[j]}} = \frac{\partial S}{\partial \alpha^{[r]}} \frac{\partial \phi}{\partial \alpha^{[j]}} \left(\frac{\partial \phi}{\partial \alpha^{[r]}} \right)^{-1}, \tag{A.27a}$$

$$\frac{\partial S}{\partial \alpha^{[j]}} = \frac{\partial S}{\partial \beta^{[r]}} \frac{\partial \phi}{\partial \alpha^{[j]}} \left(\frac{\partial \phi}{\partial \beta^{[r]}} \right)^{-1}, \tag{A.27b}$$

where $r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$.

[1] Note that equalities (A.25) are redundant. Namely, since the first relation is symmetric it gives the identical sets of equalities for $k = \underline{j_\alpha}, \ldots, \overline{j_\alpha}, l = \underline{j_\alpha}, \ldots, k-1$ and $k = \underline{j_\alpha}, \ldots, \overline{j_\alpha}, l = k+1, \ldots, \overline{j_\alpha}$. Moreover, the first relation is trivial for $k = l$. The second relation gives the identical sets of equalities for $\alpha = u, \beta = y$ and $\alpha = y, \beta = u$. Nevertheless, for the compactness of the presentation we omit these details.

Now we turn to the separate steps of the proof. The separate steps (i)–(ix) prove the relations in $\Theta_{k,l}$ for different sets of k and l values so that jointly the steps cover all necessary k, l values in (A.25). On the steps (i)–(iv) we will focus on the conditions (3.30a) and will prove that in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ they yield $\Theta_{k,l}$ for $k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$ (see Figure A.1 and the top of Figure A.3). On the steps (v)–(ix) we will prove that using additionally the conditions (3.30b), the outcome of the previous four steps can be complemented to obtain the same result for the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ (see Figure A.2 and the middle part of the Figure A.3).

(i) Consider first (A.26) for $j = \overline{j_\alpha}$. Since $j \neq i - N, \ldots, i + N$, now $i \neq \overline{j_\alpha} - N, \ldots, \overline{j_\alpha} + N$ and consequently $i \leq \overline{j_\alpha} - N - 1$. In (A.26a) denote index i by index l and compare successively the obtained equality first, with (A.26a) and second with (A.26b), where in both equalities index i is replaced by index k. This yields

$$\left(\frac{\partial\phi}{\partial\alpha^{[l]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[l]}\partial\alpha^{[\overline{j_\alpha}]}} = \left(\frac{\partial\phi}{\partial\alpha^{[k]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[k]}\partial\alpha^{[\overline{j_\alpha}]}},$$
$$\left(\frac{\partial\phi}{\partial\alpha^{[l]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[l]}\partial\alpha^{[\overline{j_\alpha}]}} = \left(\frac{\partial\phi}{\partial\beta^{[k]}}\right)^{-1} \frac{\partial^2\phi}{\partial\beta^{[k]}\partial\alpha^{[\overline{j_\alpha}]}}.$$

Divide both sides of both obtained equalities by $\left(\partial\phi/\partial\alpha^{[\overline{j_\alpha}]}\right)$ to get

$$\left(\frac{\partial\phi}{\partial\alpha^{[\overline{j_\alpha}]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[l]}\partial\alpha^{[\overline{j_\alpha}]}} \frac{\partial\phi}{\partial\alpha^{[k]}} = \left(\frac{\partial\phi}{\partial\alpha^{[\overline{j_\alpha}]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[k]}\partial\alpha^{[\overline{j_\alpha}]}} \frac{\partial\phi}{\partial\alpha^{[l]}},$$
$$\left(\frac{\partial\phi}{\partial\alpha^{[\overline{j_\alpha}]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[l]}\partial\alpha^{[\overline{j_\alpha}]}} \frac{\partial\phi}{\partial\beta^{[k]}} = \left(\frac{\partial\phi}{\partial\alpha^{[\overline{j_\alpha}]}}\right)^{-1} \frac{\partial^2\phi}{\partial\beta^{[k]}\partial\alpha^{[\overline{j_\alpha}]}} \frac{\partial\phi}{\partial\alpha^{[l]}}.$$

Take the conditions (A.26) for $i = \overline{j_\alpha}$ and in (A.26b) interchange mutually variables α and β, which is eligible by the definition of α and β. In this case $j \leq \overline{j_\alpha} - N - 1$. Since j changes in the same range as indices k and l, one can apply (A.26a) for $j := l$ to the left-hand sides of the equalities above, as well as (A.26a) and (A.26b) for $j := k$ to the right-hand sides of the first and second equalities above, respectively. This yields

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1. \tag{A.28}$$

(ii) Using (A.26a), rewrite the elements of (A.28) for $l = \underline{j_\alpha}$ as follows

$$\left(\frac{\partial\phi}{\partial\alpha^{[i]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\alpha^{[\underline{j_\alpha}]}} \frac{\partial\phi}{\partial\alpha^{[k]}} = \frac{\partial S}{\partial\alpha^{[k]}} \left(\frac{\partial S}{\partial\alpha^{[j]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[\underline{j_\alpha}]}\partial\alpha^{[j]}},$$
$$\left(\frac{\partial\phi}{\partial\alpha^{[i]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[i]}\partial\alpha^{[\underline{j_\alpha}]}} \frac{\partial\phi}{\partial\beta^{[k]}} = \frac{\partial S}{\partial\beta^{[k]}} \left(\frac{\partial S}{\partial\alpha^{[j]}}\right)^{-1} \frac{\partial^2\phi}{\partial\alpha^{[\underline{j_\alpha}]}\partial\alpha^{[j]}},$$

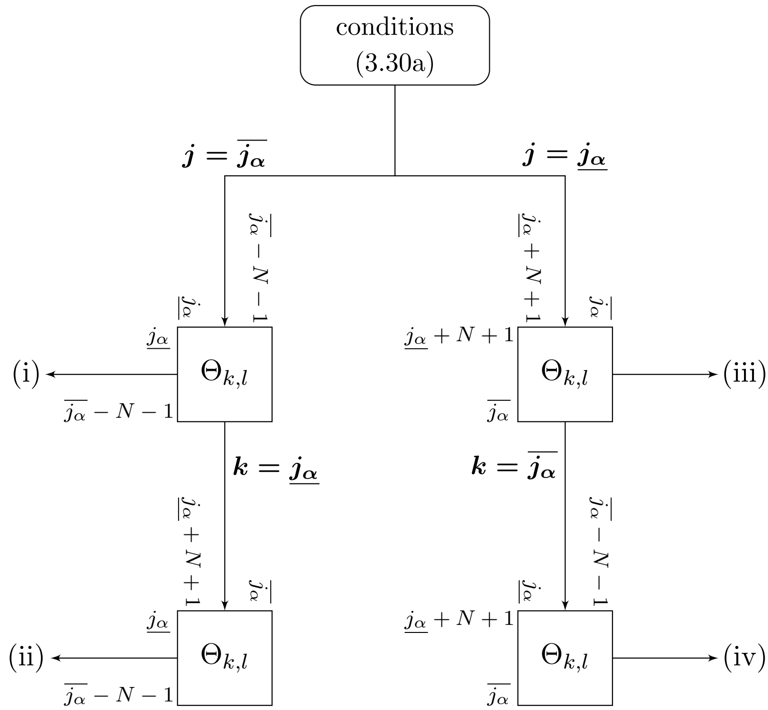

Figure A.1: Steps (i)–(iv) of the proof of Lemma 3.2.

where $i, j = \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}$ and $k = \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1$. Denoting $l := i = j$, after simplification we obtain

$$\Theta_{k,l}, \qquad \begin{aligned} k &= \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1, \\ l &= \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}. \end{aligned} \tag{A.29}$$

(iii) Next, consider (A.26) for $j = \underline{j_\alpha}$. Since $j \neq i - N, \dots, i + N$, now $i \neq \underline{j_\alpha} - N, \dots, \underline{j_\alpha} + N$ and consequently $i \geq \underline{j_\alpha} + N + 1$. Performing the analogical steps as in (i) we obtain

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}. \tag{A.30}$$

(iv) Using (A.26a), rewrite the elements of (A.30) for $l = \overline{j_\alpha}$ as follows

$$\begin{aligned}
\left(\frac{\partial \phi}{\partial \alpha^{[i]}}\right)^{-1} \frac{\partial^2 \phi}{\partial \alpha^{[i]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial \phi}{\partial \alpha^{[k]}} &= \frac{\partial S}{\partial \alpha^{[k]}} \left(\frac{\partial S}{\partial \alpha^{[j]}}\right)^{-1} \frac{\partial^2 \phi}{\partial \alpha^{[\overline{j_\alpha}]} \partial \alpha^{[j]}}, \\
\left(\frac{\partial \phi}{\partial \alpha^{[i]}}\right)^{-1} \frac{\partial^2 \phi}{\partial \alpha^{[i]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial \phi}{\partial \beta^{[k]}} &= \frac{\partial S}{\partial \beta^{[k]}} \left(\frac{\partial S}{\partial \alpha^{[j]}}\right)^{-1} \frac{\partial^2 \phi}{\partial \alpha^{[\overline{j_\alpha}]} \partial \alpha^{[j]}},
\end{aligned}$$

where $i, j = \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1$ and $k = \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}$. Denotation

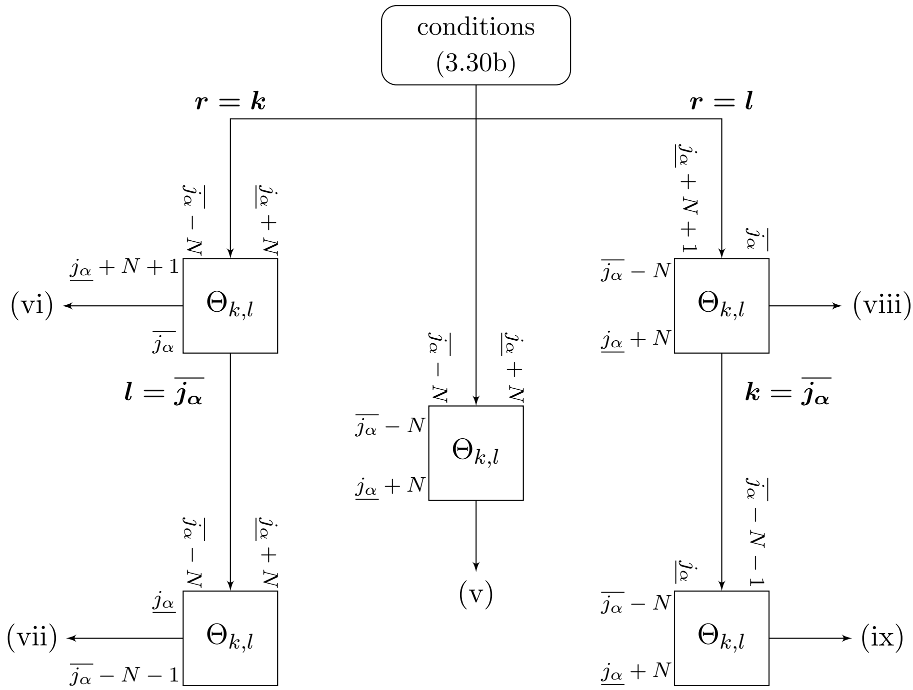

Figure A.2: Steps (v)–(ix) of the proof of Lemma 3.2.

$l := i = j$ and simplification yields

$$\Theta_{k,l}, \qquad \begin{aligned} k &= \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}, \\ l &= \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1. \end{aligned} \tag{A.31}$$

It is not hard to verify (see Figure A.3) that joining together tables (A.28), (A.29), (A.30) and (A.31) yields

$$\Theta_{k,l} \quad \begin{cases} k, l = \underline{j_\alpha}, \dots, \overline{j_\alpha}, & \text{if } 2N < \overline{j_\alpha} - \underline{j_\alpha}, \\ k, l = \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1, & \text{if } 2N \geq \overline{j_\alpha} - \underline{j_\alpha}. \\ \qquad\quad \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}, & \end{cases} \tag{A.32}$$

(v) Consider the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$. In (A.27a) replace index r by index k and compare successively the obtained equality first, with (A.27a) and second with (A.27b), where in both equalities index r is replaced by index l. After simplification we obtain

$$\Theta_{k,l}, \quad k, l = \overline{j_\alpha} - N, \dots, \underline{j_\alpha} + N. \tag{A.33}$$

(vi) Next take (A.27) for $r = l$, $j = \underline{j_\alpha}$ and perform the similar operations as in step (ii) to get

$$\Theta_{k,l}, \qquad \begin{aligned} k &= \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}, \\ l &= \overline{j_\alpha} - N, \dots, \underline{j_\alpha} + N. \end{aligned} \tag{A.34}$$

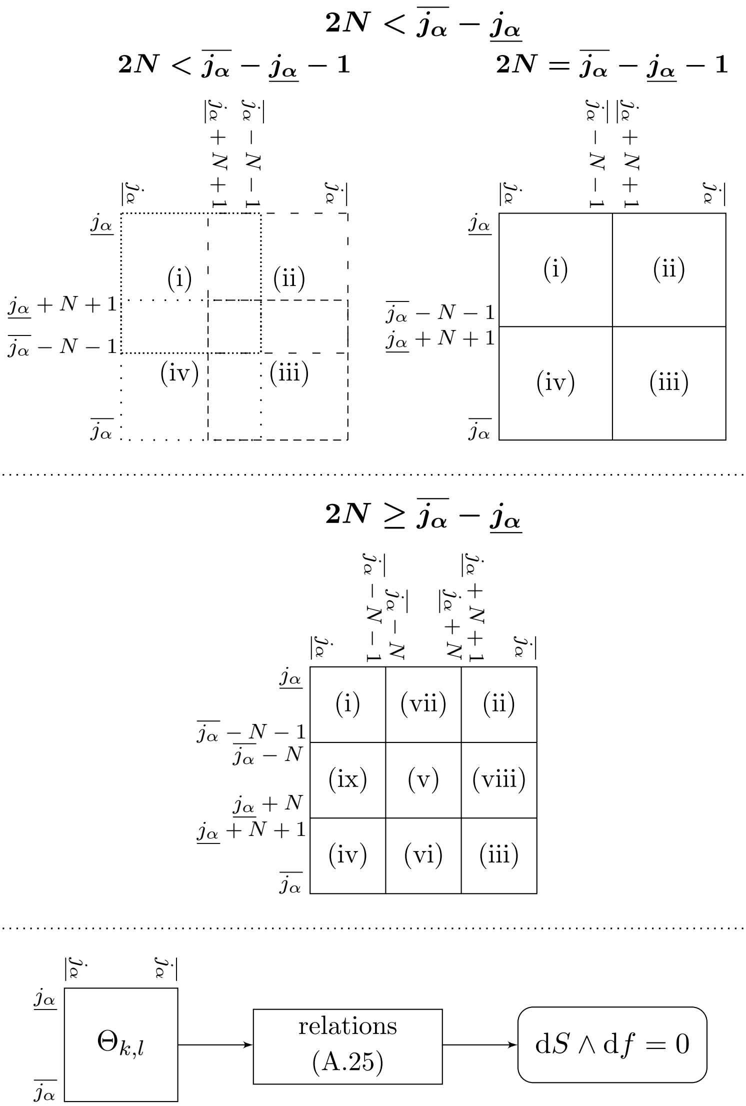

Figure A.3: The main steps of the proof of Lemma 3.2.

(vii) Taking the elements of (A.34) for $k = \overline{j_\alpha}$ by analogy with step (iv) one obtains

$$\Theta_{k,l}, \qquad \begin{aligned} k &= \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1, \\ l &= \overline{j_\alpha} - N, \dots, \underline{j_\alpha} + N. \end{aligned} \tag{A.35}$$

(viii) Next, take (A.27) for $r = k$, $j = \underline{j_\alpha}$ and perform the similar operations as in step (ii) to get

$$\Theta_{k,l}, \qquad \begin{aligned} k &= \overline{j_\alpha} - N, \dots, \underline{j_\alpha} + N, \\ l &= \underline{j_\alpha} + N + 1, \dots, \overline{j_\alpha}. \end{aligned} \tag{A.36}$$

(ix) Taking the elements of (A.36) for $l = \overline{j_\alpha}$ by analogy with step (iv) one obtains

$$\Theta_{k,l}, \qquad \begin{array}{c} k = \overline{j_\alpha} - N, \dots, \underline{j_\alpha} + N, \\ l = \underline{j_\alpha}, \dots, \overline{j_\alpha} - N - 1. \end{array} \tag{A.37}$$

As a result, complementary tables (A.33), (A.34), (A.35), (A.36) and (A.37) allow to rewrite (A.32) as

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha}, \dots, \overline{j_\alpha}$$

for arbitrary N (see Figure A.3), which means that under conditions (3.30a) and (3.30b) the equalities (A.25) are satisfied for $k, l = \underline{j_\alpha}, \dots, \overline{j_\alpha}$. This completes the proof. $\square$

Proof of Lemma 4.1

In order to prove Lemma 4.1, we need Lemma A.2 below.

Lemma A.2. *For the homogeneous time scale* $\mathbb{T}$ *one has*

$$\frac{\partial h_\nu^{\langle i+1\rangle}}{\partial x} = \frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\frac{\partial f(x,u)}{\partial x} + \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right),$$
$$\nu = 1, \dots p, \quad i = 0, 1, \dots, \tag{A.38}$$

where I_n *is* $n \times n$ *identity matrix.*

Proof. By commutativity of operators d and Δ_f [6],

$$\mathrm{d}\left(h_\nu^{\langle i+1\rangle}\right) = \left(\mathrm{d}h_\nu^{\langle i\rangle}\right)^{\Delta_f}. \tag{A.39}$$

In what follows, we omit in (A.39) the parts involving the terms $\mathrm{d}u_v^{\langle l\rangle}$ in the expressions of total differentials, therefore we have

$$\frac{\partial h_\nu^{\langle i+1\rangle}}{\partial x}\mathrm{d}x + \cdots = \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\mathrm{d}x\right)^{\Delta_f} + \cdots. \tag{A.40}$$

We compute the delta derivative of the one-form at the right-hand side of (A.40), using (1.7). Since $(\mathrm{d}x)^{\Delta_f} = \mathrm{d}f(x,u)$, and again, omitting the parts involving the terms $\mathrm{d}u_v$, we get

$$\left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\mathrm{d}x\right)^{\Delta_f} = \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\Delta_f}\mathrm{d}x + \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\sigma_f}\frac{\partial f(x,u)}{\partial x}\mathrm{d}x + \cdots.$$

Since the vectors $\mathrm{d}x$, $\mathrm{d}u_v, \ldots,$ $\mathrm{d}u_v^{\langle i-1\rangle}$ are independent over the field $\mathcal{K}^*$, comparing the coefficients of $\mathrm{d}x$ at both sides of equality (A.40), we get

$$\frac{\partial h_\nu^{\langle i+1\rangle}}{\partial x} = \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\Delta_f} + \left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\sigma_f} \frac{\partial f(x,u)}{\partial x}.$$

Finally, applying (i) of Proposition 1.1 to $\left(\frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\right)^{\sigma_f}$ we obtain (A.38). □

Now we are ready to prove Lemma 4.1.

Proof. According to the condition of the lemma

$$\omega_{\nu,i} := \frac{\partial h_\nu^{\langle i\rangle}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1} \alpha_k \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\mathrm{d}x. \tag{A.41}$$

We first prove that the statement of the lemma holds for $j = i+1$, i.e.

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \beta_k \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1} \beta_k \omega_{\nu,k} \tag{A.42}$$

for some β_k's. By Lemma A.2 and (A.41)

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left[\alpha_k \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\frac{\partial f(x,u)}{\partial x} + \left(\alpha_k \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right)\right]\mathrm{d}x.$$

Using (iii) of Proposition 1.1 for $\left(\alpha_k \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\right)^{\Delta_f}$ and then (i) of Proposition 1.1 for α_k, we get

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left[\frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\left(\frac{\partial f(x,u)}{\partial x}\alpha_k^{\sigma_f} + \alpha_k^{\Delta_f}\right) + \right.$$
$$\left. + \alpha_k^{\sigma_f}\left(\frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right)\right]\mathrm{d}x.$$

By Lemma A.2

$$\left(\frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right) = \frac{\partial h_\nu^{\langle k+1\rangle}}{\partial x} - \frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\frac{\partial f(x,u)}{\partial x},$$

yielding

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \alpha_k^{\Delta_f}\frac{\partial h_\nu^{\langle k\rangle}}{\partial x}\mathrm{d}x + \sum_{k=0}^{i-1} \alpha_k^{\sigma_f}\frac{\partial h_\nu^{\langle k+1\rangle}}{\partial x}\mathrm{d}x.$$

Changing the summation index of the second sum for $s = k+1$, separating the last addend of the second sum, and applying (A.41) to it, we obtain

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_k\right) \frac{\partial h_\nu^{\langle k\rangle}}{\partial x} \mathrm{d}x + \sum_{s=1}^{i-1} \alpha_{s-1}^{\sigma_f} \frac{\partial h_\nu^{\langle s\rangle}}{\partial x} \mathrm{d}x.$$

Separating the first addend of the first sum yields

$$\omega_{\nu,i+1} = \sum_{k=1}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_k + \alpha_{k-1}^{\sigma_f}\right) \frac{\partial h_\nu^{\langle k\rangle}}{\partial x} \mathrm{d}x + \left(\alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_0\right) \frac{\partial h_\nu}{\partial x} \mathrm{d}x.$$

Denoting $\beta_0 := \alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_0$ and $\beta_k := \alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_k + \alpha_{k-1}^{\sigma_f}$ we get (A.42). The similar arguments can be applied for the cases $j > i+1$. □

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Kokkuvõte

Mittelineaarsete olekuvõrrandite olekutaastaja kujule teisendamine

Väitekirja põhiliseks uurimisvaldkonnaks on mittelineaarsete ühe sisendi ja ühe väljundiga juhtimissüstemide mudelite olekutaastaja kujule teisendamine, kasutades teisendusi olekute ja väljundite ruumis. Probleemi uurimiseks rakendatakse valdavalt diferentsiaalvormidel põhinevat algebralist formalismi. Uuritakse nii pideva ajaga kui ka diskreetaja süsteeme. Pidevate süsteemide juhul on leitud tarvilikud ja piisavad tingimused süsteemi olekutaastaja kujule teisendamiseks, kasutades lisaks olekuteisendusele ka väljundteisendust. Diskreetsel juhul vaadeldakse süsteemi nn. laiendatud olekutaastaja kujule teisendamist, mille võrrandid sõltuvad mitte ainult sisendist ja väljundist, vaid ka nende teatud väärtustest minevikus. Ka siin lubatakse kasutada lisaks üldistatud olekuteisendusele väljundteisendust. Leitud on kaks tarvilike ja piisavate tingimuste alternatiivset komplekti. Esimese komplekti eelis võrreldes pideva juhuga seisneb selles, et tingimused ei sõltu mingist otsitavast funktsioonist. Teise komplekti tingimused on eriti lihtsad ja esitatud nii, et neid võib kontrollida (peale teatud osatuletiste leidmist) praktiliselt pealevaatamise teel.

Väitekirja täiendav osa on pühendatud homogeensel ajaskaalal defineeritud mittelineaarse mitme sisendi ja mitme väljundiga juhtimissüsteemi vaadeldavuse uurimisele. Vaadeldavuse tarvilik ja piisav tingimus on esitatud vaadeldava ruumi mõiste abil. Juhul kui süsteem ei ole vaadeldav, aga vaadeldav ruum, mille elementideks on diferentsiaalsed üks-vormid, on täielikult integreeruv, on süsteem dekomponeeritav vaadeldavaks ja mittevaadeldavaks alamsüsteemiks.

Teoreetilised tulemused on viidud konkreetsete algoritmide kujule ja programmeeritud *Mathematica* funktsioonidena. Funktsioonid on integreeritud Küberneetika Instituudis loodud tarkvarapaketti NLControl ja on kasutatavad paketi NLControl veebisaidil interneti vahendusel ilma *Mathematica* tarkvara lokaalsesse arvutisse installeerimata.

Abstract

Transformation of Nonlinear State Equations into Observer Form

The main subject of the thesis is transformation of the nonlinear single-input single-output state equations into the observer form via change of coordinates in both state- and output-spaces. An algebraic framework based on differential forms is used as a main tool of the research. Both the continuous- and discrete-time cases are considered. In the continuous-time case necessary and sufficient conditions are given for the existence of the state and output coordinate transformations, bringing the system into the observer form. In the discrete-time case we present two different sets of necessary and sufficient conditions for the existence of the extended coordinate change and the output transformation that allow to transform the system into the extended observer form, which, besides the input and output, depends also on a finite number of their past values. The first set of conditions has an advantage of being intrinsic, implying that they do not depend on some unknown function, whereas the conditions of the second set are very simple and due to the matrix representation can be checked almost by direct inspection.

The supplementary part of the thesis is devoted to studying the observability property of the nonlinear multi-input multi-output system, defined on homogeneous time scale. The necessary and sufficient observability condition is given in terms of the observable space. If the system is not observable and its observable space, whose elements are differential one-forms, is completely integrable, then the system can be decomposed into the observable/unobservable subsystems.

Theoretical results of the thesis are implemented as *Mathematica* functions, and integrated into the software package NLControl (developed in the Institute of Cybernetics). Moreover, the functions can be used online at NLControl website without the necessity to install *Mathematica* on a local computer.

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6. Teadustegevus

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7. Kaitstud lõputööd

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   Mittelineaarsete juhtimissüsteemide vaadeldavuse kontroll ja olekutaastaja konstrueerimine paketi *Mathematica* abil, MSc, Tallinna Tehnikaülikool, 2008.

8. Teadustöö põhisuunad

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Curriculum Vitae

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Estonian	fluent
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Period	Organization	Position
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6. Scientific work

1. V. Kaparin and Ü. Kotta. Necessary conditions for transformation the nonlinear control system into the observer form via state and output coordinate changes. In *The 7th International Conference on Control and Automation*, pages 745–750, Christchurch, New Zealand, December 2009.

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7. Defended theses

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   Nonlinear Control Systems with Computer Algebra System *Mathematica*: Observability and Observer Forms, MSc, Tallinn University of Technology, 2008.

8. Main areas of scientific work/Current research topics

   Nonlinear control systems.

Publications

Publication 1

V. Kaparin and Ü. Kotta. Necessary conditions for transformation the non-linear control system into the observer form via state and output coordinate changes. In *The 7th International Conference on Control and Automation*, pages 745–750, Christchurch, New Zealand, December 2009.

Necessary Conditions for Transformation the Nonlinear Control System into the Observer Form via State and Output Coordinate Changes

Vadim Kaparin and Ülle Kotta

***Abstract*— The paper gives more direct and simple necessary conditions for the existence of state and output coordinate transformations, allowing to transform the nonlinear single-input single-output control system into the observer form. Both the old and new conditions require that the certain n differential one-forms, associated with the nth order differential input-output equation (corresponding to the state equations), and depending on a unknown single-variable output function, are the total differentials of certain functions.**

***Index Terms*— nonlinear control system, state and output transformations, observer form, differential one-form.**

I. INTRODUCTION

Conditions for the existence of observer form for nonlinear control system using the state coordinate transformation are known to be very restrictive (see [2], [9]), motivating various extensions to enlarge the class of systems for which observers with linear error dynamics can be designed. Either the class of transformations was enlarged as in [10], were in addition to state transformation also output transformation is allowed or different generalized observer forms were introduced as in [6] or both as in [1] and [4]. Moreover, system immersion into higher dimensional system [8], [13] or output-dependent time scale transformation [5] were also applied to reach the desired goal.

The paper addresses the problem of transforming the single-input single-output continuous-time nonlinear control system into the observer form using both the state and output transformations. This problem has been addressed earlier using the approach based on differential forms in [4] and [11]. In [4] the necessary solvability condition has been formulated in terms of differential forms, yielding to a partial differential equation, the solution of which provides a candidate output transformation function. However, the necessary condition in [4] is very mild and far from being sufficient as shown in [11]. Its existence does not guarantee the solution of the problems. To see, if the problem is solvable, one has to apply the output transformation and check whether in the new output coordinates the system is transformable into the observer form by the state coordinate transformation. The paper [11] strengthened the necessary solvability condition, given in [4]. If the conditions in [11] are satisfied, so are the conditions in [4] but the opposite does not hold. Moreover, the conditions in [11] are expressed in terms of original system equations and do not require to apply the output transformation to check the problem solvability. It is conjectured in [11] that the conditions are also sufficient, but the proof was left for the future studies.

Note that the conditions in [11] depend on certain n differential one-forms, associated with the nth order input-output equation of the control system. In this paper, in order to simplify the necessary conditions from [11], a different set of one-forms, associated with a control system, is suggested to use. Our results, like those of [4] and [11], are neither local nor global but generic, i.e. they hold in almost all situations except the pathological cases, see [3].

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a single-input single-output nonlinear continuous-time system, described by the state equations

$$\begin{array}{rcl} \dot{x} & = & f(x,u) \\ y & = & h(x), \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. Our purpose is to find the conditions under which system (1) can be transformed into the observer form

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + \varphi_1(Y,u) \\ & \vdots & \\ \dot{z}_{n-1} & = & z_n + \varphi_{n-1}(Y,u) \\ \dot{z}_n & = & \varphi_n(Y,u) \\ Y & = & z_1, \end{array} \tag{2}$$

using the state transformation

$$z = \psi(x) \tag{3}$$

and the output transformation

$$Y = F(y). \tag{4}$$

Consider the input-output equation

$$y^{(n)} = P(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-1)}), \tag{5}$$

corresponding to (1)[1], and define the differential one-forms θ_i for $i = 1, \ldots, n$ as follows

$$\theta_i = \frac{\partial P}{\partial y^{(n-i)}}\mathrm{d}y + \frac{\partial P}{\partial u^{(n-i)}}\mathrm{d}u. \tag{6}$$

Moreover, define the composite functions $\bar{\varphi}_i(y,u) = \varphi_i(F(y),u)$.

In [11] the following theorem was formulated.

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[1] In case when it is difficult to obtain the input-output equation (5) from the state equations (1), one can compute $\mathrm{d}P$ from the tangent linearized equations $\mathrm{d}\dot{x} = \mathrm{d}f(x,u), \mathrm{d}y = \mathrm{d}h(x)$ like in [14]

Theorem 1: One can transform the system (1) into the observer form (2) by the state transformation (3) and the output transformation (4) if there exists a function $\lambda(y)$, such that the one-forms

$$\mathrm{d}\bar{\varphi}_i := C_n^i \lambda^{(i)} \mathrm{d}y + \lambda\theta_i - \\ - \sum_{s=1}^{i-1} \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-i)}} \mathrm{d}y + \frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial u^{(n-i)}} \mathrm{d}u \right) \quad (7)$$

for $i = 1, \ldots, n$ are the total differentials. The function F for the output transformation (4) can be calculated as an integral

$$F(y) = \int \lambda(y) \mathrm{d}y. \quad (8)$$

Denote $P_1 = P$ and compute

$$\mathrm{d}P_{i+1} = \mathrm{d}P_i - \omega_i^{(n-i)}, \qquad i = 1, \ldots, n-1, \quad (9)$$

where by $\omega_i^{(n-i)}$ is denoted the $(n-i)$th order derivative of the one-form ω_i, defined by

$$\omega_i = \frac{\partial P_i}{\partial y^{(n-i)}} \mathrm{d}y + \frac{\partial P_i}{\partial u^{(n-i)}} \mathrm{d}u. \quad (10)$$

Note that for an one-form $\omega = \alpha_1 \mathrm{d}y + \alpha_2 \mathrm{d}u$, the rth derivative can be computed as follows:

$$\omega^{(r)} = \sum_{m=0}^{r} C_r^m \left(\alpha_1^{(r-m)} \mathrm{d}y^{(m)} + \alpha_2^{(r-m)} \mathrm{d}u^{(m)} \right). \quad (11)$$

The purpose of this paper is to show that using the one-forms ω_i instead of θ_i one can simplify the expression (7).

III. RELATIONSHIP BETWEEN ω_i AND θ_i

First we will find the relationship between the one-forms ω_i, defined by (10), and θ_i, defined by (6). In order to prove Proposition 1, presenting this relationship, we need the following lemma, the proof of which is given in the Appendix.

Lemma 1: For $j = 1, \ldots, k$ and $r = n - k + j$, the following holds

$$(-1)^j C_{r-1}^j = \sum_{i=1}^{j} (-1)^i C_{r-1}^{i-1} C_{r-i}^{r-j-1}. \quad (12)$$

Proposition 1: Given a control system of the form (5), the relationship between the one-forms ω_i and θ_i, for $i = 1, \ldots, n$, defined by (10) and (6) respectively, is as follows

$$\omega_i = \theta_i + \sum_{j=1}^{i-1} (-1)^j C_{n-i+j}^j \cdot \\ \cdot \left[\left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right]. \quad (13)$$

Proof: To simplify the proof, we rewrite (13) in a more compact form as

$$\omega_i = \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \left[\left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \\ + \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right], \quad (14)$$

where the first element of the sum is θ_i. We will use the mathematical induction to prove that equation (14) holds for $i = 1, \ldots, n$. It is easy to check by (6) and (10) that the statement holds when $i = 1$. Assume that the statement (14) holds for $i = k$ and prove that it is true for $i = k + 1$. According to (9) and (10) one has

$$\mathrm{d}P_{k+1} = \mathrm{d}P - \sum_{i=1}^{k} \omega_i^{(n-i)}.$$

From the assumption that the statement (14) holds for $i = k$ we have:

$$\mathrm{d}P_{k+1} = \mathrm{d}P - \sum_{i=1}^{k} \left(\sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \cdot \\ \cdot \left[\left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right] \right)^{(n-i)}.$$

Using the relationship (11), the latter yields

$$\mathrm{d}P_{k+1} = \mathrm{d}P - \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \sum_{m=0}^{n-i} C_{n-i}^m \cdot \\ \cdot \left[\left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j+n-i-m)} \mathrm{d}y^{(m)} + \\ + \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j+n-i-m)} \mathrm{d}u^{(m)} \right].$$

From (10) it follows that in order to find ω_{k+1}, only $\frac{\partial P_{k+1}}{\partial y^{(n-k-1)}}$ and $\frac{\partial P_{k+1}}{\partial u^{(n-k-1)}}$ are necessary. Therefore, we are interested only in such elements of $\mathrm{d}P_{k+1}$ where the order of differentiation of $\mathrm{d}y$ and $\mathrm{d}u$ is $m := n - k - 1$. Thus, we have:

$$\frac{\partial P_{k+1}}{\partial y^{(n-k-1)}} = \frac{\partial P}{\partial y^{(n-k-1)}} - \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \cdot \\ \cdot C_{n-i}^{n-k-1} \left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j+k-i+1)} \quad (15)$$

and

$$\frac{\partial P_{k+1}}{\partial u^{(n-k-1)}} = \frac{\partial P}{\partial u^{(n-k-1)}} - \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \cdot \\ \cdot C_{n-i}^{n-k-1} \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j+k-i+1)}. \quad (16)$$

Note that (15) and (16) have a similar structure. Thus, all the transformations made with one expression will be similar for the other.

Changing the summation order $\sum_{i=1}^{k} \sum_{j=0}^{i-1} a_{i,j} = \sum_{j=1}^{k} \sum_{i=1}^{j} a_{k-j+i,i-1}$, and taking into account that

$-(-1)^{i-1}=(-1)^i$, rewrite (15) as follows:

$$\frac{\partial P_{k+1}}{\partial y^{(n-k-1)}}=\frac{\partial P}{\partial y^{(n-k-1)}}+\sum_{j=1}^{k}\sum_{i=1}^{j}(-1)^i\cdot$$
$$\cdot C_{n-k+j-1}^{i-1}C_{n-k+j-i}^{n-k-1}\left(\frac{\partial P}{\partial y^{(n-k+j-1)}}\right)^{(j)}=$$
$$=\frac{\partial P}{\partial y^{(n-k-1)}}+\sum_{j=1}^{k}\left[\left(\frac{\partial P}{\partial y^{(n-k+j-1)}}\right)^{(j)}\cdot\right.$$
$$\left.\cdot\sum_{i=1}^{j}(-1)^iC_{n-k+j-1}^{i-1}C_{n-k+j-i}^{n-k-1}\right].\quad(17)$$

By Lemma 1 one can rewrite (17) as

$$\frac{\partial P_{k+1}}{\partial y^{(n-k-1)}}=\sum_{j=0}^{k}(-1)^jC_{n-k-1+j}^{j}\cdot$$
$$\cdot\left(\frac{\partial P}{\partial y^{(n-k-1+j)}}\right)^{(j)},\quad(18)$$

since $\frac{\partial P}{\partial y^{(n-k-1)}}=(-1)^jC_{n-k-1+j}^{j}\left(\frac{\partial P}{\partial y^{(n-k-1+j)}}\right)^{(j)}$ for $j=0$. Analogously, we get

$$\frac{\partial P_{k+1}}{\partial u^{(n-k-1)}}=\sum_{j=0}^{k}(-1)^jC_{n-k-1+j}^{j}\cdot$$
$$\cdot\left(\frac{\partial P}{\partial u^{(n-k-1+j)}}\right)^{(j)}.\quad(19)$$

Using the definition of ω_{k+1}, (18) and (19)

$$\omega_{k+1}=\sum_{j=0}^{k}(-1)^jC_{n-k-1+j}^{j}\cdot$$
$$\cdot\left[\left(\frac{\partial P}{\partial y^{(n-k-1+j)}}\right)^{(j)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-k-1+j)}}\right)^{(j)}\mathrm{d}u\right]$$

being (14) for $i=k+1$. ■

IV. NEW NECESSARY CONDITIONS

Theorem 2, given below, provides an alternative but equivalent formulation of Theorem 1 in terms of one-forms ω_i. In order to prove Theorem 2, we need the following lemma, the proof of which is given in the Appendix.

Lemma 2: For $k=1,\dots,n$ the following holds

$$\text{(i)}\quad\sum_{s=1}^{k}\sum_{j=1}^{s}(-1)^{s-j}C_{n-j}^{s-j}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}C_{n-s}^{n-k}\cdot$$
$$\cdot\sum_{m=0}^{k-s}C_{k-s}^{m}\lambda^{(k-j-m)}\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}y+\right.$$
$$\left.+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}u\right)=\lambda\theta_k,$$

$$\text{(ii)}\quad C_n^k-\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}=(-1)^{k-1}C_n^k.$$

Theorem 2: The system (1) can be transformed by the state transformation (3) and the output transformation (4) into the observer form (2) if there exists a function $\lambda(y)$, such that the one-forms, denoted by $\mathrm{d}\bar{\varphi}_i$

$$\mathrm{d}\bar{\varphi}_i:=(-1)^{i-1}C_n^i\lambda^{(i)}\mathrm{d}y+\sum_{j=1}^{i}(-1)^{i-j}C_{n-j}^{i-j}\lambda^{(i-j)}\omega_j\quad(20)$$

for $i=1,\dots,n$ are the total differentials.

Proof: The main idea of the proof is to show by the mathematical induction that the right hand side of (7) and the right hand side of (20) are equal. It is easy to check that the statement holds when $i=1$. Assume that statement (20) holds for $i=k-1$ and prove that it is true for $i=k$. From (7) we have:

$$\mathrm{d}\bar{\varphi}_k=C_n^k\lambda^{(k)}\mathrm{d}y+\lambda\theta_k-$$
$$-\sum_{s=1}^{k-1}\left(\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial y^{(n-k)}}\mathrm{d}y+\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial u^{(n-k)}}\mathrm{d}u\right).\quad(21)$$

To simplify the proof let us denote

$$A:=\sum_{s=1}^{k-1}\left(\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial y^{(n-k)}}\mathrm{d}y+\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial u^{(n-k)}}\mathrm{d}u\right).$$

From the assumption that statement (20) holds for $i=k-1$ one can write:

$$\sum_{s=1}^{k-1}\mathrm{d}\bar{\varphi}_s^{(n-s)}=\sum_{s=1}^{k-1}\left((-1)^{s-1}C_n^s\lambda^{(s)}\mathrm{d}y+\right.$$
$$\left.+\sum_{j=1}^{s}(-1)^{s-1}C_{n-j}^{s-j}\lambda^{(s-j)}\omega_j\right)^{(n-s)}.$$

Using the compact description of ω_i, given by (14), and the formula (11) for the $(n-s)$th order derivative of the one-forms $\lambda^{(s)}\mathrm{d}y$ and $\lambda^{(s-j)}\omega_j$, the latter yields:

$$\sum_{s=1}^{k-1}\mathrm{d}\bar{\varphi}_s^{(n-s)}=\sum_{s=1}^{k-1}(-1)^{s-1}C_n^s\sum_{m=0}^{n-s}C_{n-s}^m\lambda^{(n-m)}\mathrm{d}y^{(m)}+$$
$$+\sum_{s=1}^{k-1}\sum_{j=1}^{s}(-1)^{s-j}C_{n-j}^{s-j}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}^{p}\left[\sum_{m=0}^{n-s}C_{n-s}^m\cdot\right.$$
$$\cdot\left(\lambda^{(s-j)}\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p)}\right)^{(n-s-m)}\mathrm{d}y^{(m)}+\sum_{m=0}^{n-s}C_{n-s}^m\cdot$$
$$\left.\cdot\left(\lambda^{(s-j)}\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p)}\right)^{(n-s-m)}\mathrm{d}u^{(m)}\right].$$

In order to find A, one needs the elements of $\sum_{s=1}^{k-1}\mathrm{d}\bar{\varphi}_s^{(n-s)}$ with the order of differentiation of $\mathrm{d}y$ and

$\mathrm{d}u$ being $m := n-k$:

$$A = \lambda^{(k)}\mathrm{d}y\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}+ + \sum_{s=1}^{k-1}\sum_{j=1}^{s}(-1)^{s-j}C_{n-j}^{s-j}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}C_{n-s}^{n-k}\cdot \cdot\left[\left(\lambda^{(s-j)}\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p)}\right)^{(k-s)}\mathrm{d}y+ + \left(\lambda^{(s-j)}\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p)}\right)^{(k-s)}\mathrm{d}u\right] = = \lambda^{(k)}\mathrm{d}y\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}+\sum_{s=1}^{k-1}\sum_{j=1}^{s}(-1)^{s-j}\cdot \cdot C_{n-j}^{s-j}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}C_{n-s}^{n-k}\sum_{m=0}^{k-s}C_{k-s}^m\lambda^{(k-j-m)}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}u\right).$$

If we add and subtract the addend of the second sum with $s=k$, we get:

$$A = \lambda^{(k)}\mathrm{d}y\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}+\sum_{s=1}^{k}\sum_{j=1}^{s}(-1)^{s-j}C_{n-j}^{s-j}\cdot \cdot\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}C_{n-s}^{n-k}\sum_{m=0}^{k-s}C_{k-s}^m\lambda^{(k-j-m)}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+m)}\mathrm{d}u\right)- -\sum_{j=1}^{k}(-1)^{k-j}C_{n-j}^{k-j}\lambda^{(k-j)}\omega_j,$$

which by (i) is

$$A = \lambda^{(k)}\mathrm{d}y\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}+\lambda\theta_k- -\sum_{j=1}^{k}(-1)^{k-j}C_{n-j}^{k-j}\lambda^{(k-j)}\omega_j.$$

Note that from (21)

$$\mathrm{d}\bar\varphi_k = C_n^k\lambda^{(k)}\mathrm{d}y+\lambda\theta_k - A = \lambda^{(k)}\mathrm{d}y\bigg(C_n^k- -\sum_{s=1}^{k-1}(-1)^{s-1}C_n^sC_{n-s}^{n-k}\bigg)+\sum_{j=1}^{k}(-1)^{k-j}C_{n-j}^{k-j}\lambda^{(k-j)}\omega_j.$$

Finally, applying (ii) we obtain:

$$\mathrm{d}\bar\varphi_k = (-1)^{k-1}C_n^k\lambda^{(k)}\mathrm{d}y+\sum_{j=1}^{k}(-1)^{k-j}C_{n-j}^{k-j}\lambda^{(k-j)}\omega_j.$$

■

V. EXAMPLE

Examine the following example, where we suppose $x_1 > 0$:

$$\begin{array}{rcl} \dot x_1 &=& x_2+ux_1 \\ \dot x_2 &=& -x_1+\dfrac{x_2^2}{x_1}+x_3+u(x_1+x_2+x_1\ln x_1) \\ \dot x_3 &=& -x_1+x_3+\dfrac{x_2x_3}{x_1}+x_1\ln x_1+ \\ && +u(x_1+x_3+x_1\ln x_1) \\ y &=& x_1. \end{array} \tag{22}$$

The input-output equation, corresponding to (22), is

$$y^{(3)} = \ddot y+3\frac{\ddot y\dot y}{y}-2\frac{\dot y^3}{y^2}-\frac{\dot y^2}{y}+\dot yu+y\ddot u+y\dot u\ln y+y\ln y.$$

To check, whether it is possible to transform system (22) via the state and output coordinate transformations into the observer form (2), one has to check the validity of conditions (20), which for the case $n=3$ require that the one-forms at the right hand side of

$$\begin{array}{rcl} \mathrm{d}\bar\varphi_1 &:=& 3\dot\lambda\mathrm{d}y+\lambda\omega_1, \\ \mathrm{d}\bar\varphi_2 &:=& -3\ddot\lambda\mathrm{d}y-2\dot\lambda\omega_1+\lambda\omega_2, \\ \mathrm{d}\bar\varphi_3 &:=& \dddot\lambda\mathrm{d}y+\ddot\lambda\omega_1-\dot\lambda\omega_2+\lambda\omega_3 \end{array} \tag{23}$$

are, for some function $\lambda(y)$, the total differentials.

Step 1. Compute, according to (13),

$$\begin{array}{rcl} \omega_1 &=& \left(1+3\dfrac{\dot y}{y}\right)\mathrm{d}y+y\mathrm{d}u, \\ \omega_2 &=& \left(-3\dfrac{\ddot y}{y}-2\dfrac{\dot y}{y}+u\right)\mathrm{d}y+(y\ln y-2\dot y)\mathrm{d}u, \\ \omega_3 &=& \bigg(3\dfrac{\dot y\ddot y}{y^2}+2\dfrac{\ddot y}{y}-2\dfrac{\dot y^3}{y^3}-\dfrac{\dot y^2}{y^2}+\ddot u+\ln y+ \\ && +\dot u\ln y+1\bigg)\mathrm{d}y+(\ddot y-\dot y\ln y)\,\mathrm{d}u. \end{array} \tag{24}$$

Step 2. To find whether there exists $\lambda(y)$, consider the first equation in (23) and take the exterior derivative of both side. Taking into account that $\dot\lambda = \lambda'\dot y$ where the prime means the derivative with respect to y, this yields the differential equation $3\lambda'\mathrm{d}\dot y\wedge\mathrm{d}y+\lambda'\mathrm{d}y\wedge\omega_1+\lambda\mathrm{d}\omega_1 = 0$ that we have to solve with respect to $\lambda(y)$. Doing this we get

$$\lambda(y) = 1/y, \tag{25}$$

yielding by (8)

$$F(y) = \ln y. \tag{26}$$

Step 3. Using (24) and (25), compute according to (23)

$$\begin{array}{rcl} \mathrm{d}\bar\varphi_1 &:=& \dfrac{\mathrm{d}y}{y}+\mathrm{d}u, \\ \mathrm{d}\bar\varphi_2 &:=& \dfrac{u}{y}\mathrm{d}y+\ln y\mathrm{d}u, \\ \mathrm{d}\bar\varphi_3 &:=& \dfrac{\mathrm{d}y}{y}. \end{array} \tag{27}$$

All three expressions are total differentials, which means that the necessary conditions for transformation of the system

(22) into the observer form (2) are satisfied. Integration of (27) yields $\bar{\varphi}_1 = \ln y + u$, $\bar{\varphi}_2 = u \ln y$, $\bar{\varphi}_3 = \ln y$. Due to the output transformation (26), $Y = F(y) = \ln y$, and therefore, $\varphi_1 = Y + u$, $\varphi_2 = uY$, $\varphi_3 = Y$. From (2) one can define the new state variables as follows:

$$\begin{array}{rclcl} z_1 & = & Y & = & \ln x_1, \\ z_2 & = & \dot{Y} - \varphi_1 & = & \dfrac{x_2}{x_1} - \ln x_1, \\ z_3 & = & \ddot{Y} - \dot{\varphi}_1 - \varphi_2 & = & \dfrac{x_3}{x_1} - \dfrac{x_2}{x_1} - 1, \end{array}$$

that leads to the new state equations in the observer form:

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 + z_1 + u \\ \dot{z}_2 & = & z_3 + z_1 u \\ \dot{z}_3 & = & z_1 \\ Y & = & z_1. \end{array}$$

VI. CONCLUSIONS

Alternative necessary conditions were suggested for transformability of a single-input single-output state equations into the observer form using both the state and the output transformations. Both conditions, the old ones and those given in this paper, require that certain n differential one-forms, associated with the nth order differential input-output equation (corresponding to the state equations), and depending on an unknown single-variable output dependent function, are the total differentials of some functions. Although the new conditions are more direct and more simple, they still require to check whether a certain partial differential equation is solvable or not, that might be difficult to test. On the other hand, in the discrete-time case simple necessary and sufficient conditions exist that are directly computable from the input-output equation and do not depend on the unknown output function. These conditions [12] are expressed in terms of exterior derivatives and the exterior products of the one-forms, similar to those in the continuous-time case. Our further goal is to find out whether it is possible to extend these conditions to the continuous-time case. Note that the output transformation and shift operator commute, whereas this does not hold for the output transformation and derivative operator, and therefore, the answer is not immediate. In case the extension turns out to be impossible, our goal is to reformulate the discrete-time conditions in such a way that using the mathematical machinery of the pseudo-linear algebra, the discrete- and continuous-time cases can be unified into one single condition, from which both results follow as the special cases. Note that this has been done for the case when only a state transformation is allowed to transform the system equations into the observer form [7].

APPENDIX

For the proof of Lemma 1 and Lemma 2 we will use the Newton's binomial theorem $(a+b)^k = \sum_{s=0}^{k} C_k^s a^s b^{k-s}$, which for $a = -1$, $b = 1$ and $k > 0$ gives

$$\sum_{s=0}^{k} C_k^s (-1)^s = 0. \tag{28}$$

A. Proof of Lemma 1

Proof: Since $C_n^k = \frac{n!}{(n-k)!k!}$, proving (12) is equivalent to prove that

$$(-1)^j = \sum_{i=1}^{j} (-1)^i C_j^{i-1}. \tag{29}$$

Note that from (28), taking $0 = \sum_{p=0}^{j} C_j^p (-1)^p$ and changing the summation index, i.e. taking $p = i - 1$, we have $0 = \sum_{i=1}^{j+1} C_j^{i-1} (-1)^{i-1}$. Finally, separating the last addend of the sum, putting it into the left side of the equation and multiplying both sides by -1 yields $(-1)^j = \sum_{i=1}^{j} (-1)^i C_j^{i-1}$. Consequently, (29) is true and therefore (12) is true, too. ■

B. Proof of Lemma 2

Proof: (i) To prove the first part of Lemma 2, we denote the left hand side of (i) by B and rewrite it in the following way:

$$\begin{aligned} B := \sum_{s=1}^{k} \sum_{j=1}^{s} \sum_{m=0}^{k-s} (-1)^{s-j} C_{n-j}^{s-j} \sum_{p=0}^{j-1} (-1)^p \cdot \\ \cdot C_{n-j+p} C_{n-s}^{n-k} C_{k-s}^{m} \lambda^{(k-j-m)} \cdot \\ \cdot \left(\left(\frac{\partial P}{\partial y^{(n-j+p)}} \right)^{(p+m)} \mathrm{d}y + \left(\frac{\partial P}{\partial u^{(n-j+p)}} \right)^{(p+m)} \mathrm{d}u \right). \end{aligned}$$

After changing the summation order

$$\sum_{s=1}^{k} \sum_{j=1}^{s} \sum_{m=0}^{k-s} a_{s,j,m} = \sum_{s=1}^{k} \sum_{j=1}^{s} \sum_{m=0}^{k-s} a_{m+j,j,s-j},$$

we obtain:

$$\begin{aligned} B = \sum_{s=1}^{k} \sum_{j=1}^{s} \sum_{m=0}^{k-s} (-1)^m C_{n-j}^{m} \sum_{p=0}^{j-1} (-1)^p C_{n-j+p} C_{n-m-j}^{n-k} \cdot \\ \cdot C_{k-m-j}^{s-j} \lambda^{(k-s)} \left(\left(\frac{\partial P}{\partial y^{(n-j+p)}} \right)^{(p+s-j)} \mathrm{d}y + \right. \\ \left. + \left(\frac{\partial P}{\partial u^{(n-j+p)}} \right)^{(p+s-j)} \mathrm{d}u \right) = \sum_{s=1}^{k} \lambda^{(k-s)} \cdot \\ \cdot \sum_{j=1}^{s} \sum_{p=0}^{j-1} (-1)^p C_{n-j+p} \left(\left(\frac{\partial P}{\partial y^{(n-j+p)}} \right)^{(p+s-j)} \mathrm{d}y + \right. \\ \left. + \left(\frac{\partial P}{\partial u^{(n-j+p)}} \right)^{(p+s-j)} \mathrm{d}u \right) \cdot \\ \cdot \sum_{m=0}^{k-s} (-1)^m C_{n-j}^{m} C_{n-m-j}^{n-k} C_{k-m-j}^{s-j}. \end{aligned}$$

Separating from B the last addend,

$$B=\sum_{s=1}^{k-1}\lambda^{(k-s)}\sum_{j=1}^{s}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+s-j)}\mathrm{d}y + +\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+s-j)}\mathrm{d}u\right)\sum_{m=0}^{k-s}(-1)^mC^m_{n-j}\cdot \cdot C^{n-k}_{n-m-j}C^{s-j}_{k-m-j}+\lambda\sum_{j=1}^{k}\sum_{p=0}^{j-1}(-1)^pC^p_{n-j+p}C^{n-k}_{n-j}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}u\right).$$

It is easy to check by direct computations that $C^m_{n-j}C^{n-k}_{n-m-j}C^{s-j}_{k-m-j}=C^{n-s}_{n-j}C^{k-s}_{n-s}C^m_{k-s}$, which yields:

$$B=\sum_{s=1}^{k-1}\lambda^{(k-s)}\sum_{j=1}^{s}\sum_{p=0}^{j-1}(-1)^pC_{n-j+p}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+s-j)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+s-j)}\mathrm{d}u\right)\cdot \cdot C^{n-s}_{n-j}C^{k-s}_{n-s}\sum_{m=0}^{k-s}C^m_{k-s}(-1)^m+\lambda\sum_{j=1}^{k}\sum_{p=0}^{j-1}(-1)^p\cdot \cdot C^p_{n-j+p}C^{n-k}_{n-j}\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}y + +\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}u\right),$$

and taking into account (28)

$$B=\lambda\sum_{j=1}^{k}\sum_{p=0}^{j-1}(-1)^pC^p_{n-j+p}C^{n-k}_{n-j}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-j+p)}}\right)^{(p+k-j)}\mathrm{d}u\right).$$

Changing the summation order

$$\sum_{j=1}^{k}\sum_{p=0}^{j-1}a_{j,p}=\sum_{p=0}^{k-1}\sum_{j=0}^{p}a_{k-p+j,j},$$

we have:

$$B=\lambda\sum_{p=0}^{k-1}\sum_{j=0}^{p}(-1)^jC^j_{n-k+p}C^{n-k}_{n-k+p-j}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-k+p)}}\right)^{(p)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-k+p)}}\right)^{(p)}\mathrm{d}u\right).$$

By definition (6), one can easily verify that the first addend of B is $\lambda\theta_k$, thus we can rewrite B as follows:

$$B=\lambda\theta_k+\lambda\sum_{p=1}^{k-1}\sum_{j=0}^{p}(-1)^jC^j_{n-k+p}C^{n-k}_{n-k+p-j}\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-k+p)}}\right)^{(p)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-k+p)}}\right)^{(p)}\mathrm{d}u\right).$$

It is easy to check by direct computations that $C^j_{n-k+p}C^{n-k}_{n-k+p-j}=C^j_pC^p_{n-k+p}$, which allows us to rewrite B as follows:

$$B=\lambda\theta_k+\lambda\sum_{p=1}^{k-1}C^p_{n-k+p}\sum_{j=0}^{p}C^j_p(-1)^j\cdot \cdot\left(\left(\frac{\partial P}{\partial y^{(n-k+p)}}\right)^{(p)}\mathrm{d}y+\left(\frac{\partial P}{\partial u^{(n-k+p)}}\right)^{(p)}\mathrm{d}u\right).$$

By (28), the second addend in B is zero, therefore $B=\lambda\theta_k$.

(ii) To prove the second part of Lemma 2, we multiply both sides of (28) by $C^k_n\frac{(n-s)!}{(n-s)!}$ which yields $0=\sum_{s=0}^{k}(-1)^sC^s_nC^{n-k}_{n-s}$. If we separate the last and the first addends of the sum and put the last addend into the left hand side of the equation, we get $-(-1)^kC^k_n=C^{n-k}_n+\sum_{s=1}^{k-1}(-1)^sC^s_nC^{n-k}_{n-s}$. Finally, using $C^{n-k}_n=C^k_n$ and $-(-1)^k=(-1)^{k-1}$, we prove (ii). ∎

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Necessary and Sufficient Conditions in Terms of Differential-Forms for Linearization of the State Equations up to Input-Output Injections ⋆

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Abstract: The simple necessary and sufficient conditions are derived for transformability of the single-input single-output state equations into the observer form, using both the state and output transformations. The conditions require that certain differential one-forms, associated with the input-output equation of the control system, and depending on a unknown single variable output function, are closed.

Keywords: nonlinear control system, state and output transformations, observer form, differential one-form.

1. INTRODUCTION

The paper addresses the problem of transforming the continuous-time single-input single-output nonlinear control system into the nonlinear observer form using both the state and output transformations. Unlike the traditional methods as in Conte et al. [2007] or Isidori [1985], where only the state transformation is used, the approach, where in addition the output transformation is applied, enlarges the class of systems for which the observer with linear error dynamic can be constructed. This problem has been addressed before in papers by Glumineau et al. [1996], Mullari et al. [2008] and Kaparin et al. [2009]. In Glumineau et al. [1996] the necessary solvability condition is formulated in terms of the differential forms. This condition yields to a partial differential equation, the solution of which provides a candidate output transformation function. The necessary condition is very mild and far from being sufficient. Obviously, its existence does not guarantee the solvability of the problem. To check whether the problem is solvable, one has to apply the output transformation and check if in the new output coordinates the system is transformable into the observer form by the state coordinate transformation only. In Mullari et al. [2008] the new necessary solvability conditions have been found for the systems up to the order 4. The latter conditions were expressed directly in terms of the original system equations and do not require to apply the intermediate output transformation to check the problem solvability. In Kaparin et al. [2009], in order to simplify the necessary conditions from Mullari et al. [2008], a different set of one-forms, associated with a control system, is suggested to use. Moreover, the closed formulas that show the relationship between the two sets of one-forms, is given. However, since the results of Kaparin et al. [2009] rely on the relationship between the two sets of one-forms, the results are also valid for systems up to the order 4.

The purpose of this paper is to present the necessary and sufficient conditions in terms of the one-form, introduced in Kaparin et al. [2009] for the systems of arbitrary order. Of course, our approach, as well as those in Glumineau et al. [1996], Mullari et al. [2008] and Kaparin et al. [2009] assumes the knowledge of the input-output equation corresponding to the state space description. Under the observability assumption, using the state elimination algorithm, one can, at least locally, always find for the state equation a corresponding input-output equation, see for example Conte et al. [2007]. However, globally elimination of the state is a difficult problem, that results, in general, an implicit input-output equation accompanied to a number of inequations, see for example Diop [1991]. Alternative result, formulated in terms of vector fields is suggested in Boutat et al. [2009] for systems without inputs. Extension for systems depending on inputs was also discussed, but then the conditions are only sufficient and the assumed observer form is more general. Namely, the coefficients of the linear part also depend on control. The different kind of generalized observer form was also assumed in Besançon [1999].

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2. PROBLEM STATEMENT AND PRELIMINARIES

Consider a single-input single-output nonlinear continuous-time system, described by the state equations

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. Our purpose is to find the conditions under which system (1) can be transformed into the observer form

$$\begin{aligned} \dot{z}_1 &= z_2 + \varphi_1(Y,u) \\ &\vdots \\ \dot{z}_{n-1} &= z_n + \varphi_{n-1}(Y,u) \\ \dot{z}_n &= \varphi_n(Y,u) \\ Y &= z_1, \end{aligned} \tag{2}$$

using the state transformation

$$z = \psi(x) \tag{3}$$

and the output transformation

$$Y = F(y). \tag{4}$$

Control system (1) can be transformed into the observer form (2) if the input-output (i/o) equation

$$y^{(n)} = P(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-1)}), \tag{5}$$

corresponding to (1), can be written in the form

$$Y^{(n)} = \varphi_1^{(n-1)} + \varphi_2^{(n-2)} + \ldots + \varphi_n, \tag{6}$$

where all functions φ_i depend only on the new output Y and the input u:

$$\varphi_i : (Y, u) \to \mathbb{R} \quad \forall i = 1, \ldots, n.$$

If this holds, one can define the new state variables as follows:

$$\begin{aligned} z_1 &= Y, \\ z_2 &= \dot{Y} - \varphi_1, \\ z_3 &= \ddot{Y} - \dot{\varphi}_1 - \varphi_2, \\ &\vdots \\ z_n &= Y^{(n-1)} - \varphi_1^{(n-2)} - \varphi_2^{(n-3)} - \ldots - \varphi_{n-1}, \end{aligned} \tag{7}$$

leading to system (2).

For the further search of the necessary and sufficient conditions we define the differential one-forms by means of the following algorithm which is the extension of the algorithm described by Conte et al. [2007]:

$$P_1 := P, \quad \mathrm{d}P_{i+1} := \mathrm{d}P_i - \omega_i^{(n-i)}, \qquad i = 1, \ldots, n-1,$$

where by $\omega_i^{(n-i)}$ is denoted the $(n-i)$th order derivative of the one-form ω_i, defined by

$$\omega_i = \frac{\partial P_i}{\partial y^{(n-i)}}\mathrm{d}y + \frac{\partial P_i}{\partial u^{(n-i)}}\mathrm{d}u.$$

According to Kaparin et al. [2009], the closed form of the one-forms ω_i for $i = 1, \ldots, n$ is the following:

$$\omega_i = \sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^j \left[\left(\frac{\partial P}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial P}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right]. \tag{8}$$

Moreover, define the composite functions $\bar{\varphi}_i(y, u) := \varphi_i(F(y), u)$.

3. NECESSARY AND SUFFICIENT CONDITIONS

The theorem proved in this section provides the necessary and sufficient conditions, which allow us to transform the i/o equation (5) into the form (6) and, as a consequence, the state equations (1) into the observer form (2). In order to prove Theorem 1, we need the following proposition and lemma.

Proposition 1. (Kaparin et al. [2010]). Assume that $f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))$ is a composite function for which derivatives up to order $a + b$ are defined; then

$$\frac{\partial \left(f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(a+b)}}{\partial \xi_l^{(a)}(t)} = C_{a+b}^b \left(\frac{\partial f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))}{\partial \xi_l(t)} \right)^{(b)},$$

where $l = 1, 2, \ldots, r$, C_{a+b}^b is the binomial coefficient and a, b are nonnegative integers.

Lemma 1.

(i) $\displaystyle\sum_{j=1}^{i}(-1)^{j-1}C_i^{j-1} = (-1)^{i-1}.$

(ii) $\displaystyle\sum_{j=1}^{i-s+1}(-1)^{j-1}C_{i-s}^{j-1} = 0,$ for $s = 1, \ldots, i-1$ and $i > 1$.

Proof. For the proof we use the binomial formula $(a+b)^k = \sum_{m=0}^{k} C_k^m a^m b^{k-m}$, which for $a = -1$, $b = 1$ and $k > 0$ gives

$$\sum_{m=0}^{k} C_k^m (-1)^m = 0. \tag{9}$$

(i) Using (9) for $k = i$ we get

$$\sum_{m=0}^{i} C_i^m (-1)^m = 0.$$

Separating the last addend of the sum and putting it into the right-hand side yields:

$$\sum_{m=0}^{i-1} C_i^m (-1)^m = -(-1)^i.$$

Finally, if we change the summation index, taking $m = j - 1$, and take into account that $-(-1)^i = (-1)^{i-1}$, we obtain (i).

(ii) Since $s = 1, \ldots, i-1$ and $i > 1$, then $i - s > 0$ and one can apply (9) for $k = i - s$:

$$\sum_{m=0}^{i-s} C_{i-s}^m (-1)^m = 0.$$

Changing the summation index, taking $m = j - 1$, we obtain (ii).

$\square$

Now we are ready to prove our main result.

Theorem 1. The system (1) can be transformed by the state transformation (3) and the output transformation (4) into the observer form (2) if and only if there exists a function $\lambda(y)$, such that the one-forms

$$(-1)^{i-1}C_n^i \lambda^{(i)}\mathrm{d}y + \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}\lambda^{(i-p)}\omega_p, \qquad i = 1, \ldots, n, \tag{10}$$

where ω_p's are defined by (8), are exact.

Proof. *Necessity:* Assume that system (1) is transformable into the observer form (2). Consequently, the i/o equation (5) can be rewritten in the form (6). Complete the following steps:

- Take the partial derivatives of both sides of the i/o equation (6) with respect to $y^{(n-i+j-1)}$, for $j = 1, \ldots, i$.
- Next, take the $(j-1)$-th order time-derivative of each expression, obtained in the previous step.
- Denote
$$\alpha_j := (-1)^{j-1}C_{n-i+j-1}^{j-1} \tag{11}$$
and multiply both sides of the equation by α_j.
- Add the obtained equations for $j = 1, \ldots, i$.

As a result, the following equations are obtained:

$$\sum_{j=1}^{i} \alpha_j \left(\frac{\partial F^{(n)}}{\partial y^{(n-i+j-1)}} \right)^{(j-1)} = \sum_{j=1}^{i}\sum_{s=1}^{n} \alpha_j \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-i+j-1)}} \right)^{(j-1)}. \tag{12}$$

Repeating the same with respect to control variable u, one obtains

$$\sum_{j=1}^{i}\alpha_j\left(\frac{\partial F^{(n)}}{\partial u^{(n-i+j-1)}}\right)^{(j-1)} = \\ = \sum_{j=1}^{i}\sum_{s=1}^{n}\alpha_j\left(\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial u^{(n-i+j-1)}}\right)^{(j-1)}. \quad (13)$$

Consider separately each side of equations (12) and (13). Denote

$$LY := \sum_{j=1}^{i}\alpha_j\left(\frac{\partial F^{(n)}}{\partial y^{(n-i+j-1)}}\right)^{(j-1)},$$

$$LU := \sum_{j=1}^{i}\alpha_j\left(\frac{\partial F^{(n)}}{\partial u^{(n-i+j-1)}}\right)^{(j-1)},$$

$$RY := \sum_{j=1}^{i}\sum_{s=1}^{n}\alpha_j\left(\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial y^{(n-i+j-1)}}\right)^{(j-1)},$$

$$RU := \sum_{j=1}^{i}\sum_{s=1}^{n}\alpha_j\left(\frac{\partial\bar{\varphi}_s^{(n-s)}}{\partial u^{(n-i+j-1)}}\right)^{(j-1)}.$$

According to Faà di Bruno's Formula (Johnson [2002]), the n-th order time derivative of output transformation F reads as follows:

$$F^{(n)} = \sum\frac{n!}{k_1!\dots k_n!}F^{\overline{K}}\prod_{p=1}^{n}\left(\frac{y^{(p)}}{p!}\right)^{k_p},$$

where $\underline{K}$ is the order of derivative with respect to y and is defined as $\underline{K} = k_1 + \ldots + k_n$ where the sum is taken over all possible partitions of n, i.e., over the values of $k_1, ..., k_n$ such that $k_1 + 2k_2 + \ldots + nk_n = n$.

Obviously, the addend of $F^{(n)}$ corresponding to $k_n = 1$ and $k_1, \ldots, k_{n-1} = 0$, equals $F'y^{(n)}$. According to (5), $y^{(n)}$ must be replaced by the function P. To take this replacement into account and avoid the complication in the further transformations of $F^{(n)}$ we add to LY a zero term, such that LY now reads as

$$LY = \sum_{j=1}^{i}(\alpha_j\left(\left(\frac{\partial F^{(n)}}{\partial y^{(n-i+j-1)}}\right)^{(j-1)} + \right. \\ \left. + \left(\frac{\partial\left(F'P - F'y^{(n)}\right)}{\partial y^{(n-i+j-1)}}\right)^{(j-1)}\right),$$

where in $F^{(n)}$ we consider $y^{(n)}$ as symbol which we do not need to replace. This trick simplifies the proof below by allowing to use Proposition 1.

By Proposition 1 for $r = 1$, $a = n-i+j-1$ and $b = i-j+1$,

$$\frac{\partial F^{(n)}}{\partial y^{(n-i+j-1)}} = C_n^{i-j+1}(F')^{(i-j+1)},$$

yielding

$$LY = \sum_{j=1}^{i}\alpha_j\left(C_n^{i-j+1}(F')^{(i)} + \right. \\ \left. + \left(\frac{\partial\left(F'P - F'y^{(n)}\right)}{\partial y^{(n-i+j-1)}}\right)^{(j-1)}\right).$$

Using product rule for finding the derivative one can write:

$$\frac{\partial\left(F'P - F'y^{(n)}\right)}{\partial y^{(n-i+j-1)}} = F'\frac{\partial P}{\partial y^{(n-i+j-1)}} + P\frac{\partial F'}{\partial y^{(n-i+j-1)}} - \\ - y^{(n)}\frac{\partial F'}{\partial y^{(n-i+j-1)}} - F'\frac{\partial y^{(n)}}{\partial y^{(n-i+j-1)}}.$$

Since $n-i+j-1 < n$ for $i = 1, \ldots, n$ and $j = 1, \ldots, i$ then $\partial y^{(n)}/\partial y^{(n-i+j-1)} = 0$. Also taking into account that $y^{(n)} = P$ one obtains

$$\frac{\partial\left(F'P - F'y^{(n)}\right)}{\partial y^{(n-i+j-1)}} = F'\frac{\partial P}{\partial y^{(n-i+j-1)}}.$$

Thus LY can be rewritten as follows:

$$LY = \sum_{j=1}^{i}\alpha_j\left(C_n^{i-j+1}(F')^{(i)} + \left(F'\frac{\partial P}{\partial y^{(n-i+j-1)}}\right)^{(j-1)}\right).$$

By direct computation $C_{n-i+j-1}^{j-1}C_n^{i-j+1} = C_n^iC_i^{j-1}$ and taking into account (11), we obtain:

$$LY = C_n^i(F')^{(i)}\sum_{j=1}^{i}(-1)^{j-1}C_i^{j-1} + \\ + \sum_{j=1}^{i}\alpha_j\left(F'\frac{\partial P}{\partial y^{(n-i+j-1)}}\right)^{(j-1)}.$$

By (i) of Lemma 1 we obtain:

$$LY = (-1)^{i-1}C_n^i(F')^{(i)} + \sum_{j=1}^{i}\alpha_j\left(F'\frac{\partial P}{\partial y^{(n-i+j-1)}}\right)^{(j-1)}.$$

To rewrite the obtained expression in a more compact form denote

$$\beta := (-1)^{i-1}C_n^i(F')^{(i)}. \quad (14)$$

After the application of the Leibnitz Formula for the higher order derivative of the product, we have:

$$LY = \beta + \sum_{j=1}^{i}\alpha_j\sum_{p=0}^{j-1}C_{j-1}^p(F')^{(j-1-p)}\cdot \\ \cdot\left(\frac{\partial P}{\partial y^{(n-i+j-1)}}\right)^{(p)} = \beta + \sum_{j=1}^{i}\sum_{p=0}^{j-1}\alpha_j\cdot \\ \cdot C_{j-1}^p(F')^{(j-1-p)}\left(\frac{\partial P}{\partial y^{(n-i+j-1)}}\right)^{(p)}.$$

Usind (11) and changing the summation order $\sum_{j=1}^{i}\sum_{p=0}^{j-1}a_{j,p} = \sum_{p=1}^{i}\sum_{j=0}^{p-1}a_{i-p+j+1,j}$, rewrite LY as follows:

$$LY = \beta + \sum_{p=1}^{i}\sum_{j=0}^{p-1}(-1)^{i-p+j}C_{n-p+j}^{i-p+j}\cdot \\ \cdot C_{i-p+j}^j(F')^{(i-p)}\left(\frac{\partial P}{\partial y^{(n-p+j)}}\right)^{(j)}.$$

By direct computation $C_{n-p+j}^{i-p+j}C_{i-p+j}^j = C_{n-p}^{i-p}C_{n-p+j}^j$, and taking also into account that $(-1)^{i-p+j} = (-1)^{i-p}(-1)^j$, we finally obtain:

$$LY = \beta + \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}(F')^{(i-p)} \cdot$$

$$\cdot \sum_{j=0}^{p-1}(-1)^{j}C_{n-p+j}^{j}\left(\frac{\partial P}{\partial y^{(n-p+j)}}\right)^{(j)}.$$

Since LY and LU have a similar structure, the transformations made with LY can be made also with LU, yielding

$$LU = (-1)^{i-1}C_n^i\left(\frac{\partial F}{\partial u}\right)^{(i)} + \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}(F')^{(i-p)} \cdot$$

$$\cdot \sum_{j=0}^{p-1}(-1)^{j}C_{n-p+j}^{j}\left(\frac{\partial P}{\partial u^{(n-p+j)}}\right)^{(j)}.$$

Since $\partial F/\partial u = 0$,

$$LU = \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}(F')^{(i-p)} \cdot$$

$$\cdot \sum_{j=0}^{p-1}(-1)^{j}C_{n-p+j}^{j}\left(\frac{\partial P}{\partial u^{(n-p+j)}}\right)^{(j)}.$$

Next consider RY. Note, that if $s > i-j+1$ then $n-s < n-i+j-1$ and so $[\partial\bar\varphi_s^{(n-s)}/\partial y^{(n-i+j-1)}] = 0$. Therefore, instead of taking $s = 1,\ldots,n$ we can take $s = 1,\ldots,i-j+1$ and rewrite RY as follows:

$$RY := \sum_{j=1}^{i}\sum_{s=1}^{i-j+1}\alpha_j\left(\frac{\partial\bar\varphi_s^{(n-s)}}{\partial y^{(n-i+j-1)}}\right)^{(j-1)},$$

By Proposition 1 for $r = 2$, $a = n-i+j-1$ and $b = i-s-j+1$

$$\frac{\partial\bar\varphi_s^{(n-s)}}{\partial y^{(n-i+j-1)}} = C_{n-s}^{i-s-j+1}\left(\frac{\partial\bar\varphi_s}{\partial y}\right)^{(i-s-j+1)},$$

and therefore,

$$RY := \sum_{j=1}^{i}\sum_{s=1}^{i-j+1}\alpha_j C_{n-s}^{i-s-j+1}\left(\frac{\partial\bar\varphi_s}{\partial y}\right)^{(i-s)},$$

Changing the summation order $\sum_{j=1}^{i}\sum_{s=1}^{i-j+1}a_{j,s} = \sum_{s=1}^{i}\sum_{j=1}^{i-s+1}a_{j,s}$, we obtain:

$$RY := \sum_{s=1}^{i}\sum_{j=1}^{i-s+1}\alpha_j C_{n-s}^{i-s-j+1}\left(\frac{\partial\bar\varphi_s}{\partial y}\right)^{(i-s)}.$$

By direct computation $C_{n-i+j-1}^{j-1}C_{n-s}^{i-s-j+1} = C_{n-s}^{i-s}C_{i-s}^{j-1}$ and taking into account (11), we have

$$RY := \sum_{s=1}^{i}C_{n-s}^{i-s}\left(\frac{\partial\bar\varphi_s}{\partial y}\right)^{(i-s)}\sum_{j=1}^{i-s+1}(-1)^{j-1}C_{i-s}^{j-1}.$$

Note that for $i = 1$ $RY := [\partial\bar\varphi_1/\partial y]$. In case $i > 1$, one can separate the last addend of the sum RY, yielding

$$RY := \frac{\partial\bar\varphi_i}{\partial y} + \sum_{s=1}^{i-1}C_{n-s}^{i-s}\left(\frac{\partial\bar\varphi_s}{\partial y}\right)^{(i-s)}\sum_{j=1}^{i-s+1}(-1)^{j-1}C_{i-s}^{j-1}.$$

By (ii) of Lemma 1, $RY := [\partial\bar\varphi_i/\partial y]$. Analogously we get $RU := [\partial\bar\varphi_i/\partial u]$, for $i = 1,\ldots,n$. Thus, taking into account (14), one can rewrite (12) and (13) as follows

$$\frac{\partial\bar\varphi_i}{\partial y} = (-1)^{i-1}C_n^i(F')^{(i)} + \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}(F')^{i-p} \cdot$$

$$\cdot \sum_{j=0}^{p-1}(-1)^{j}C_{n-p+j}^{j}\left(\frac{\partial P}{\partial y^{(n-p+j)}}\right)^{(j)},$$

$$\frac{\partial\bar\varphi_i}{\partial u} = \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}(F')^{i-p} \cdot$$

$$\cdot \sum_{j=0}^{p-1}(-1)^{j}C_{n-p+j}^{j}\left(\frac{\partial P}{\partial u^{(n-p+j)}}\right)^{(j)}.$$

If we add together the above equations, taking into account (8) and notation $\lambda := F'$, we finally obtain the exact differential one-forms

$$\mathrm{d}\bar\varphi_i := (-1)^{i-1}C_n^i\lambda^{(i)}\mathrm{d}y + \sum_{p=1}^{i}(-1)^{i-p}C_{n-p}^{i-p}\lambda^{(i-p)}\omega_p. \quad (15)$$

Obviously the right-hand side of equations (15) equals (10).

Sufficiency: If there exists a function $\lambda(y)$, such that the one-forms (10), where ω_p's are defined by (8), are exact, then the function $F(y)$ for the output transformation (4) can be calculated as an integral

$$F(y) = \int \lambda(y)\mathrm{d}y. \quad (16)$$

Integrating the exact one-forms (15) one can find functions $\bar\varphi_i$ for $i = 1,\ldots,n$. By means of functions $F(y)$ and $\bar\varphi_i$ the system equations in the observer form (2) can be easily constructed.

□

4. ALGORITHM

In this section we represent the algorithm for transformation the system (1) into the observer form (2). First, one has to find the i/o representation (5) of the system (1) and then perform the following steps:

Step 1. Using (8), compute the one-forms ω_i for $i = 1,\ldots,n$.

Step 2. Take the exterior derivative of the one-form (10) for $i = 1$. For this one-form to be exact the exterior derivative has to equal zero. Taking into account that $\dot\lambda = \lambda'\dot y$ where the prime means the derivative with respect to y, this yields the differential equation $C_n^1\lambda'\mathrm{d}\dot y \wedge \mathrm{d}y + \lambda'\mathrm{d}y \wedge \omega_1 + \lambda\mathrm{d}\omega_1 = 0$ which we have to solve with respect to $\lambda(y)$. If the solution does not exist, the problem is not solvable; stop.

Step 3. Using ω_i's and $\lambda(y)$ compute the one-forms (10), for $i = 2,\ldots,n$.

Step 4. Check whether the one-forms (10) are exact or not. If at least one of them is not exact, the problem is not solvable; stop.

Step 5. Rewrite the (exact) one-forms (10) as $\mathrm{d}\bar\varphi_i$ (see (15)). Integrate the one-forms $\mathrm{d}\bar\varphi_i$, yielding $\bar\varphi_i$ for $i = 1,\ldots,n$. Using $\lambda(y)$ and (16) one can find φ_i and $F(y)$ in terms of which the system in the observer form (2) can be easily constructed.

5. EXAMPLE

Examine the following example, where we suppose $x_1 > 0$ and $x_2 \neq 0$:

$$
\begin{aligned}
\dot{x}_1 &= x_1(u^2 + x_2)\\
\dot{x}_2 &= \frac{u}{x_1} - u^2 x_2 + x_2 x_3\\
\dot{x}_3 &= 1 - \frac{2u}{x_1} + \frac{u^2}{x_2} + \frac{u \ln x_1}{x_1 x_2} + u^2 x_2 - x_2^2 -\\
&\quad - \frac{u x_3}{x_1 x_2} - 3x_2 x_3 - x_3^2\\
y &= \ln x_1.
\end{aligned} \tag{17}
$$

The input-output equation, corresponding to (17), is

$$
\begin{aligned}
y^{(3)} = \dot{u}\left(e^{-y} + 2\dot{u}\right) + u\left(e^{-y} y + 4\dot{u}\dot{y} + 2\ddot{u}\right) +\\
+ u^2\left(\dot{y}^2 + \ddot{y}\right) - \dot{y}\left(\dot{y}^2 + 3\ddot{y} - 1\right).
\end{aligned}
$$

To check, whether it is possible to transform system (17) via the state and output coordinate transformations into the observer form (2), one has to check the validity of conditions from Theorem 1, which for the case $n = 3$ require that the one-forms

$$
\begin{aligned}
&3\dot{\lambda}\mathrm{d}y + \lambda\omega_1,\\
&-3\ddot{\lambda}\mathrm{d}y - 2\dot{\lambda}\omega_1 + \lambda\omega_2,\\
&\dddot{\lambda}\mathrm{d}y + \ddot{\lambda}\omega_1 - \dot{\lambda}\omega_2 + \lambda\omega_3
\end{aligned} \tag{18}
$$

are exact, for some function $\lambda(y)$. We will follow the algorithm, described in the previous section:

Step 1. Compute, according to (8),

$$
\begin{aligned}
\omega_1 &= \left(u^2 - 3\dot{y}\right)\mathrm{d}y + 2u\mathrm{d}u,\\
\omega_2 &= \left(1 + 2u^2\dot{y} - 3\dot{y}^2 + 2\ddot{y}\right)\mathrm{d}y + \left(e^{-y} + 4u\dot{y}\right)\mathrm{d}u,\\
\omega_3 &= \big(e^{-y}\left(u - uy - \dot{u}\right) - 2u\left(2\dot{u}\dot{y} + \ddot{u} + u\ddot{y}\right) +\\
&\quad -2\dot{u}^2 + 6\dot{y}\ddot{y}\big)\mathrm{d}y + \big(e^{-y}\left(y + \dot{y}\right) +\\
&\quad +2u\left(\dot{y}^2 - \ddot{y}\right)\big)\mathrm{d}u.
\end{aligned} \tag{19}
$$

Step 2. To find whether there exists $\lambda(y)$, consider the differential equation $3\lambda'\mathrm{d}\dot{y} \wedge \mathrm{d}y + \lambda'\mathrm{d}y \wedge \omega_1 + \lambda\mathrm{d}\omega_1 = 0$ which we have to solve with respect to $\lambda(y)$. Doing this we get

$$
\lambda(y) = e^y, \tag{20}
$$

yielding by (16)

$$
F(y) = e^y. \tag{21}
$$

Step 3. Using (19) and (20), compute according to (18)

$$
\begin{aligned}
&e^y u^2 \mathrm{d}y + 2e^y u\mathrm{d}u,\\
&e^y \mathrm{d}y + \mathrm{d}u,\\
&u\mathrm{d}y + y\mathrm{d}u.
\end{aligned} \tag{22}
$$

Step 4. All three expressions are total differentials, which means that the necessary conditions for transformation of the system (17) into the observer form (2) are satisfied.

Step 5. Since the necessary and sufficient conditions are satisfied one can define

$$
\begin{aligned}
\mathrm{d}\bar{\varphi}_1 &:= e^y u^2 \mathrm{d}y + 2e^y u\mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_2 &:= e^y \mathrm{d}y + \mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_3 &:= u\mathrm{d}y + y\mathrm{d}u.
\end{aligned} \tag{23}
$$

Integration of (23) yields

$$
\begin{aligned}
\bar{\varphi}_1 &:= e^y u^2,\\
\bar{\varphi}_2 &:= e^y + u,\\
\bar{\varphi}_3 &:= uy.
\end{aligned}
$$

Due to the output transformation (21), $Y = F(y) = e^y$, and therefore,

$$
\begin{aligned}
\varphi_1 &:= Y u^2,\\
\varphi_2 &:= Y + u,\\
\varphi_3 &:= u \ln Y.
\end{aligned}
$$

From (7) one can define the new state variables as follows:

$$
\begin{aligned}
z_1 &= x_1,\\
z_2 &= x_1 x_2,\\
z_3 &= x_1\left(x_2\left(x_2 + x_3\right) - 1\right),
\end{aligned}
$$

that leads to the new state equations in the observer form:

$$
\begin{aligned}
\dot{z}_1 &= z_2 + z_1 u^2\\
\dot{z}_2 &= z_3 + z_1 + u\\
\dot{z}_3 &= u \ln z_1\\
Y &= z_1.
\end{aligned}
$$

6. CONCLUSIONS

The simple necessary and sufficient conditions were derived for the existence of the state and output transformations that allow to transform the state equations into the observer form. Since the conditions are formulated in terms of the one-forms, computed from the i/o equation of the system, the conditions are applicable in case when the i/o equation can be (easily) found from the state equations. The conditions are simple and require to check whether certain one-forms, associated with the i/o equation are closed or not. However, note that these one-forms depend on an unknown single-variable output function, and the most difficult part of the transformation procedure is to solve a certain differential equation with respect to this unknown function.

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MATHEMATICS

Theorem on the differentiation of a composite function with a vector argument

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Abstract. The paper provides a theorem on the differentiation of a composite function with a vector argument. The theorem shows how the partial derivative of the total derivative of the composite function can be expressed through the total derivative of the partial derivative of the composite function. The proof of the theorem is based on Mishkov's formula, which is the generalization of the well-known Faà di Bruno's formula for a composite function with a vector argument.

Key words: differential calculus, partial derivative, total derivative, composite function.

1. INTRODUCTION

The theorem proved in this paper was required as an intermediate result in solving the problem of the transformation of the nonlinear control system, described by state equations, into the observer form and finding the necessary conditions for the possibility of such transformation [2]. The deduction of the necessary conditions involves frequent application of the differentiation of the composite functions with respect to time argument and taking the partial derivatives of the differentiated composite function with respect to one of the variables or its derivatives. The goal of this paper is to present and prove a formula (commutation rule) which allows changing the order of taking the total higher-order derivatives of the composite function and their partial derivatives with respect to one of the variables or its derivative. Since this result may be useful in the solution of other nonlinear control problems, we propose it as a separate contribution. For example, probably the main result, provided in the paper, can be applied for observer design in [4].

The main tool for proving the theorem (commutation rule) is Mishkov's theorem [3] which provides the explicit formula for the nth derivative of a composite function with a vector argument. Mishkov's formula is a straightforward generalization of the well-known Faà di Bruno's formula [1] which gives an explicit equation for the nth-order derivative of the composite function with a scalar argument.

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2. MAIN RESULT

The following theorem shows how the partial derivative of the total derivative of the composite function can be expressed through the total derivative of the partial derivative of this function. The composite function with the vector argument with an arbitrary number of components is considered.

Theorem 1. *Assume that $f(\xi_1(t),\xi_2(t),\ldots,\xi_r(t))$ is a composite function for which derivatives up to order $a+b$ are defined; then*

$$\frac{\partial\left(f(\xi_1(t),\xi_2(t),\ldots,\xi_r(t))\right)^{(a+b)}}{\partial\xi_l^{(a)}(t)}=C_{a+b}^{b}\left(\frac{\partial f(\xi_1(t),\xi_2(t),\ldots,\xi_r(t))}{\partial\xi_l(t)}\right)^{(b)},$$

where $l=1,2,\ldots,r$, C_{a+b}^{b} is the binomial coefficient and a,b are nonnegative integers.

Proof. In the proof we omit the variable t of $\xi_i(t)$, i.e. use instead of $\xi_i(t)$ a shorter notation ξ_i, which allows the bulky formulas to be written in a more compact form. According to Mishkov's formula [3], the $(a+b)$th derivative of the composite function with a vector argument can be computed by the formula

$$(f(\xi_1,\xi_2,\ldots,\xi_r))^{(a+b)}=\sum_0\sum_1\sum_2\cdots\sum_{a+b}\frac{(a+b)!}{\prod_{i=1}^{a+b}(i!)^{k_i}\prod_{i=1}^{a+b}\prod_{j=1}^{r}q_{i,j}!}\frac{\partial^k f}{\partial\xi_1^{p_1}\partial\xi_2^{p_2}\cdots\partial\xi_r^{p_r}}\prod_{i=1}^{a+b}\prod_{j=1}^{r}\left(\xi_j^{(i)}\right)^{q_{i,j}},\tag{1}$$

where the respective sums are taken over all nonnegative integer solutions of the Diophantine equations as follows:

$$\sum_0\to k_1+2k_2+\cdots+(a+b)k_{a+b}=a+b,\tag{2}$$

$$\sum_i\to q_{i,1}+q_{i,2}+\cdots+q_{i,r}=k_i,\tag{3}$$

for $i=1,\ldots,a+b$, and p_j and k on the right-hand side of (1) satisfy the relations

$$\begin{aligned}&p_j=q_{1,j}+q_{2,j}+\cdots+q_{a+b,j},\quad j=1,2,\ldots,r,\\&k=p_1+p_2+\cdots+p_r=k_1+k_2+\cdots+k_{a+b}.\end{aligned}\tag{4}$$

In taking the partial derivative of sum (1) with respect to $\xi_l^{(a)}$, only addends of sum (1) with $q_{a,l}\neq 0$ will matter. Denote by $h(\cdot)$ and $g(\cdot)$ the parts of sum (1) corresponding to $q_{a,l}\neq 0$ and $q_{a,l}=0$, respectively; then

$$(f(\xi_1,\xi_2,\ldots,\xi_r))^{(a+b)}=h(\cdot)+g(\cdot).\tag{5}$$

Note that it is possible to state that $h(\cdot)$ equals the expression in the right-hand side of (1) where, in addition to the restrictions expressed by (2), (3), and (4), the condition $q_{a,l}\neq 0$ has to be satisfied. Note also that if $q_{a,l}\neq 0$, then $k_a\neq 0$. We prove the formula separately for the cases $a>b$ and $a\leq b$.

First, consider the case when $a>b$. Since $k_a\neq 0$ and $q_{a,l}\neq 0$, in order to satisfy (2) and (3), the following must hold

$$\begin{aligned}&k_a=1,\quad k_i=0,\quad b<i\leq a+b,\ i\neq a,\\&q_{a,l}=1,\quad q_{a,j}=0,\quad j=1,2,\ldots,r,\ j\neq l,\\&q_{i,j}=0,\quad b<i\leq a+b,\ i\neq a,\ j=1,2,\ldots,r.\end{aligned}\tag{6}$$

As a result, under the condition $q_{a,l}\neq 0$, one can rewrite (2) as follows:

$$\sum_0\ \to\ k_1+2k_2+\cdots+bk_b=b,\tag{7}$$

and in (3), now $i=1,\ldots,b$.

Using (6) and changing the notations, taking $\bar{p}_j = p_j$ for $j = 1, 2, \ldots, r$, $j \neq l$, $\bar{p}_l = p_l - 1$ and $\bar{k} = k - 1$, equations (4) may be rewritten as

$$\begin{aligned} &\bar{p}_j = q_{1,j} + q_{2,j} + \cdots + q_{b,j}, \quad j = 1, 2, \ldots, r, \\ &\bar{k} = \bar{p}_1 + \bar{p}_2 + \cdots + \bar{p}_r = k_1 + k_2 + \cdots + k_b. \end{aligned} \tag{8}$$

Note also that under conditions (6)

$$\prod_{i=1}^{a+b} (i!)^{k_i} = a! \prod_{i=1}^{b} (i!)^{k_i}, \qquad \prod_{i=1}^{a+b} \prod_{j=1}^{r} q_{i,j}! = \prod_{i=1}^{b} \prod_{j=1}^{r} q_{i,j}!,$$

and

$$\prod_{i=1}^{a+b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}} = \xi_l^{(a)} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}}.$$

Taking into account the above equations and the fact that the partial derivative of $g(\cdot)$ in (5) with respect to $\xi_l^{(a)}$ equals 0, we obtain, in new variables $\bar{p}_j$ and $\bar{k}$:

$$\begin{aligned} \frac{\partial \left(f(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} &= \sum_0 \sum_1 \sum_2 \cdots \sum_b \frac{(a+b)!}{a! \prod_{i=1}^{b} (i!)^{k_i} \prod_{i=1}^{b} \prod_{j=1}^{r} q_{i,j}!} \\ &\quad \times \frac{\partial^{\bar{k}+1} f}{\partial \xi_1^{\bar{p}_1} \cdots \partial \xi_{l-1}^{\bar{p}_{l-1}} \partial \xi_l^{\bar{p}_l + 1} \partial \xi_{l+1}^{\bar{p}_{l+1}} \cdots \partial \xi_r^{\bar{p}_r}} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}}. \end{aligned} \tag{9}$$

Note that in (9) all the partial derivatives with respect to ξ_j are of order $\bar{p}_j$ except with respect to ξ_l when the order of the partial derivative is $\bar{p}_l + 1$. In order to unify the orders, denote $\bar{f} := \frac{\partial f}{\partial \xi_l}$. We also multiply the right-hand side of equation (9) by $\frac{b!}{b!}$ to obtain

$$\begin{aligned} \frac{\partial \left(f(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} &= C_{a+b}^{b} \sum_0 \sum_1 \sum_2 \cdots \sum_b \frac{b!}{\prod_{i=1}^{b} (i!)^{k_i} \prod_{i=1}^{b} \prod_{j=1}^{r} q_{i,j}!} \\ &\quad \times \frac{\partial^{\bar{k}} \bar{f}}{\partial \xi_1^{\bar{p}_1} \partial \xi_2^{\bar{p}_2} \cdots \partial \xi_r^{\bar{p}_r}} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{q_{i,j}}. \end{aligned} \tag{10}$$

It is easy to observe now that, according to Mishkov's formula, the sum on the right-hand side of (10) together with the conditions (3) for $i = 1, \ldots, b$, (7) and (8), is the bth-order total derivative of the function $\bar{f}$. Consequently,

$$\frac{\partial \left(f(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} = C_{a+b}^{b} \bar{f}^{(b)} = C_{a+b}^{b} \left(\frac{\partial f(\xi_1, \xi_2, \ldots, \xi_r)}{\partial \xi_l}\right)^{(b)}. \tag{11}$$

Second, consider the case $a \leq b$. Since $k_a \neq 0$, in order to satisfy (2) and (3), the following must hold:

$$\begin{aligned} &k_i = 0, \quad b < i \leq a + b, \\ &q_{i,j} = 0, \quad b < i \leq a + b, \ j = 1, 2, \ldots, r. \end{aligned} \tag{12}$$

Therefore, it is possible to rewrite condition (2) as

$$\sum_0 \;\rightarrow\; k_1+\cdots+(a-1)k_{a-1}+a(k_a-1)+(a+1)k_{a+1}+\cdots+bk_b=b, \tag{13}$$

and in (3), now $i=1,\ldots,b$.

Again, in order to unify the notation in (13), one can take $\bar{k}_i=k_i$ for $i=1,2,\ldots,b$, $i\neq a$ and $\bar{k}_a=k_a-1$. This allows (13) to be rewritten as follows:

$$\sum_0 \;\rightarrow\; \bar{k}_1+2\bar{k}_2+\cdots+b\bar{k}_b=b, \tag{14}$$

and (3) as

$$\begin{aligned} &\sum_i \;\rightarrow\; q_{i,1}+q_{i,2}+\cdots+q_{i,r}=\bar{k}_i, \quad i=1,\ldots,b,\ i\neq a,\\ &\sum_a \;\rightarrow\; q_{a,1}+q_{a,2}+\cdots+q_{a,r}=\bar{k}_a+1. \end{aligned} \tag{15}$$

Since $q_{a,l}\geq 1$, we can denote $\bar{q}_{a,l}:=q_{a,l}-1$ and the remaining q's as $\bar{q}_{i,j}:=q_{i,j}$. Thereby (15) can be rewritten in unified notation as

$$\sum_i \;\rightarrow\; \bar{q}_{i,1}+\bar{q}_{i,2}+\cdots+\bar{q}_{i,r}=\bar{k}_i, \tag{16}$$

for $i=1,\ldots,b$. Changing notations, taking $\bar{p}_j=p_j$ for $j=1,2,\ldots,r$, $j\neq l$, $\bar{p}_l=p_l-1$ and $\bar{k}=k-1$, equations (4) may be rewritten as

$$\begin{aligned} &\bar{p}_j=\bar{q}_{1,j}+\bar{q}_{2,j}+\cdots+\bar{q}_{b,j}, \quad j=1,2,\ldots,r,\\ &\bar{k}=\bar{p}_1+\bar{p}_2+\cdots+\bar{p}_r=\bar{k}_1+\bar{k}_2+\cdots+\bar{k}_b. \end{aligned} \tag{17}$$

Taking (12) into account and using variables $\bar{k}_i$ and $\bar{q}_{i,j}$, we have

$$\begin{aligned} &\prod_{i=1}^{a+b}(i!)^{k_i}=a!\prod_{i=1}^{b}(i!)^{\bar{k}_i}, \qquad \prod_{i=1}^{a+b}\prod_{j=1}^{r}q_{i,j}!=(\bar{q}_{a,l}+1)\prod_{i=1}^{b}\prod_{j=1}^{r}\bar{q}_{i,j}!,\\ &\prod_{i=1}^{a+b}\prod_{j=1}^{r}\left(\xi_j^{(i)}\right)^{q_{i,j}}\\ &=\left(\xi_l^{(1)}\right)^{\bar{q}_{1,l}}\cdots\left(\xi_l^{(a-1)}\right)^{\bar{q}_{a-1,l}}\left(\xi_l^{(a)}\right)^{\bar{q}_{a,l}+1}\left(\xi_l^{(a+1)}\right)^{\bar{q}_{a+1,l}}\cdots\left(\xi_l^{(b)}\right)^{\bar{q}_{b,l}}\prod_{i=1}^{b}\prod_{\substack{j=1\\ j\neq l}}^{r}\left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}. \end{aligned} \tag{18}$$

Furthermore, on the basis of (18) and the fact that the partial derivative of $g(\cdot)$ in (5) with respect to $\xi_l^{(a)}$ equals 0, we obtain, in new variables $\bar{p}_j$ and $\bar{k}$

$$\frac{\partial\left(f(\xi_1,\xi_2,\ldots,\xi_r)\right)^{(a+b)}}{\partial\xi_l^{(a)}}=\sum_0\sum_1\sum_2\cdots\sum_b\frac{(a+b)!}{a!\prod_{i=1}^{b}(i!)^{\bar{k}_i}\prod_{i=1}^{b}\prod_{j=1}^{r}\bar{q}_{i,j}!}$$
$$\times\frac{\partial^{\bar{k}+1}f}{\partial\xi_1^{\bar{p}_1}\cdots\partial\xi_{l-1}^{\bar{p}_{l-1}}\partial\xi_l^{\bar{p}_l+1}\partial\xi_{l+1}^{\bar{p}_{l+1}}\cdots\partial\xi_r^{\bar{p}_r}}\prod_{i=1}^{b}\prod_{j=1}^{r}\left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}.$$

Like in case $a > b$ we denote $\bar{f} = \frac{\partial f}{\partial \xi_l}$ and multiply the right-hand side of the equality given above by $\frac{b!}{b!}$ to obtain

$$\frac{\partial \left(f(\xi_1, \xi_2, \ldots, \xi_r)\right)^{(a+b)}}{\partial \xi_l^{(a)}} = C_{a+b}^{b} \sum_0 \sum_1 \sum_2 \cdots \sum_b \frac{b!}{\prod_{i=1}^{b} (i!)^{\bar{k}_i} \prod_{i=1}^{b} \prod_{j=1}^{r} \bar{q}_{i,j}!}$$

$$\times \frac{\partial^{\bar{k}} \bar{f}}{\partial \xi_1^{\bar{p}_1}, \partial \xi_2^{\bar{p}_2} \cdots \partial \xi_r^{\bar{p}_r}} \prod_{i=1}^{b} \prod_{j=1}^{r} \left(\xi_j^{(i)}\right)^{\bar{q}_{i,j}}. \quad (19)$$

Again it is not difficult to observe that according to Mishkov's formula, the sum on the right-hand side of equation (19), together with the conditions (14), (16), and (17), is the bth-order total derivative of the function $\bar{f}$. Consequently, (11) holds again, and this completes the proof. □

Some useful corollaries of the theorem are given below.

Corollary 1. *Under the assumptions of Theorem* 1

$$\frac{\partial \left(f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(m+n)}}{\partial \xi_l(t)} = \left(\frac{\partial \left(f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(m)}}{\partial \xi_l(t)}\right)^{(n)},$$

where m and n are nonnegative integers.

Corollary 2. *Under the assumptions of Theorem* 1

$$\frac{\partial \left(f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))\right)^{(n)}}{\partial \xi_l(t)} = \left(\frac{\partial f(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))}{\partial \xi_l(t)}\right)^{(n)},$$

where n is a nonnegative integer.

3. EXAMPLE

The example in this section illustrates the statement of Theorem 1. Consider the composite function $f(x(t), y(t))$ and assume that we need to take the partial derivative with respect to $\ddot{y}(t)$ of the 3rd-order total derivative of the function. Direct computations yield

$$\frac{\partial \left(f(x(t), y(t))\right)^{(3)}}{\partial \ddot{y}(t)} = 3 \frac{\partial^2 f(x(t), y(t))}{\partial y(t)^2} \dot{y}(t) + 3 \frac{\partial^2 f(x(t), y(t))}{\partial x(t) \partial y(t)} \dot{x}(t).$$

On the other hand, taking the partial derivative of $f(x(t), y(t))$ with respect to $y(t)$ and the total derivative of the obtained result, one gets

$$\left(\frac{\partial f(x(t), y(t))}{\partial y(t)}\right)^{(1)} = \frac{\partial^2 f(x(t), y(t))}{\partial y(t)^2} \dot{y}(t) + \frac{\partial^2 f(x(t), y(t))}{\partial x(t) \partial y(t)} \dot{x}(t).$$

Multiplying both sides of the above equality by C_3^1, we have

$$C_3^1 \left(\frac{\partial f(x(t), y(t))}{\partial y(t)}\right)^{(1)} = 3 \frac{\partial^2 f(x(t), y(t))}{\partial y(t)^2} \dot{y}(t) + 3 \frac{\partial^2 f(x(t), y(t))}{\partial x(t) \partial y(t)} \dot{x}(t).$$

It is not difficult to check that

$$\frac{\partial \left(f(x(t), y(t))\right)^{(3)}}{\partial \ddot{y}(t)} = C_3^1 \left(\frac{\partial f(x(t), y(t))}{\partial y(t)}\right)^{(1)}.$$

4. CONCLUSIONS

The paper shows how to commute the operations of taking higher-order total and partial derivatives of composite functions with vector arguments. The formula, provided in the paper, may be applicable not only in differential calculus. As already mentioned in the introduction, the theorem was a useful tool in deriving solvability conditions of a certain problem in nonlinear control theory. With high probability it may be useful in dealing with other nonlinear control problems where the derivatives of the composite functions with a vector argument often show up.

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Teoreem vektorargumendiga liitfunktsiooni diferentseerimisest

Vadim Kaparin ja Ülle Kotta

On tõestatud teoreem vektorargumendiga liitfunktsiooni diferentseerimise kohta. Teoreemis esitatud valem näitab, kuidas liitfunktsiooni täistuletise osatuletist saab väljendada tema osatuletise täistuletise kaudu. Teoreemi tõestus põhineb Mishkovi valemil, mis omakorda kujutab endast tuntud Faà di Bruno valemi üldistust vektorargumendiga liitfunktsiooni jaoks. Näide illustreerib teoreetilist tulemust.

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Extended Observer Form for Discrete-Time Nonlinear Control Systems

Vadim Kaparin and Ülle Kotta

***Abstract*— The paper addresses the problem of transforming the discrete-time single-input single-output nonlinear state equations into the extended observer form, which, besides the input and output, also depends on a finite number of their past values. The simple necessary and sufficient conditions for the existence of the extended coordinate change and the output transformation, allowing to solve the problem, are formulated in terms of differential one-forms, associated with the input-output equation, corresponding to the state equations.**

I. INTRODUCTION

The design of nonlinear observer with linearizable error dynamics is relatively easy if the state equations are in the observer form. Conditions for the existence of the observer form for nonlinear control system using the state coordinate transformation are known to be quite restrictive, motivating various extensions to enlarge the class of systems for which observers with linear error dynamics can be constructed [1], [2], [3], [4], [5]. In [1], [4], [5], for instance, the matrix A in the observer form is allowed to depend on control variable u. In [4] both the state and the output transformations are allowed and [5] extends the results of [4] into the multi-input multi-output case. In [2], [3] and [6] the past measurements of the system output are used in the extended observer form. In [3] and [6] the problem of transforming the discrete-time system without inputs into the extended observer form was studied, which consists of an observable linear system interconnected with a nonlinearity, which, besides the output of the system, also depends on a finite number of its past values. The paper [3] provides the conditions under which a given single-output discrete-time system may be transformed into the extended observer form by means of an extended coordinate change (i.e. a coordinate transformation that depends on the state of the system and a finite number of past output values) and an output transformation. A corollary of the results considered in [7] is that when the number of past output values equals $n-1$ (where n is the dimension of the state space of the system under consideration), the system can be always transformed into the extended observer form, provided the system under consideration is strongly observable. The necessary and sufficient conditions for transformation of the input dependent system into the extended observer form were presented in [8], where certain partial derivatives related to the input-output equation have been computed, and the function, necessary for the output transformation, is easy to find from the conditions.

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The purpose of this paper is to present the simple intrinsic necessary and sufficient conditions for the existence of the extended coordinate change and output transformation, allowing to transform the discrete-time single-input single-output nonlinear state equations into the extended observer form. The conditions are formulated in terms of differential one-forms, associated with the input-output equation, corresponding to the state equations. The results generalize and simplify those stated in [3].

The advantages of the new conditions are the following. First, unlike those in [3], our conditions do not require the calculation of the Lie derivatives of the dual vector fields, corresponding to certain one-forms, as well as the interior products of the one-forms and the vector fields. Moreover, in order to simplify the result, we suggest to use the different set of one-forms, which contain less terms than those in [3]. As a consequence, our conditions are easier to check and implement. Second, the conditions given in [3] are applicable only for the systems without inputs, but the conditions given in the present paper work also in the case of input dependent system. Although the input dependent system was considered in [2], the extended observer form presented in [2] differs from the one, addressed in this paper. In contrast with [2] our approach does not require, in general, the maximal buffer of past measurements of the system output. Finally, likewise [3], we investigate the problem through the sophisticated language of differential geometry, and therefore, in comparison with [8], our results are intrinsic, not equation dependent.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a single-input single-output nonlinear discrete-time system, described by the state equations

$$\begin{array}{rcl} x^+ & = & F(x,u) \\ y & = & h(x), \end{array} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$ is the state, $u \in U \subset \mathbb{R}$ is the input, $y \in Y \subset \mathbb{R}$ is the output, $F : X \times U \to X$ and $h : X \to Y$ are assumed to be real meromorphic functions. Notice that in this paper we use symbols $^+$, $^-$ and $^{[i]}$ instead of the arguments $t+1$, $t-1$ and $t+i$, respectively, to simplify the exposition, so $x^+ := x(t+1)$, $x^- := x(t-1)$, $x := x(t)$ and $x^{[i]} = x(t+i)$. Our purpose is to find the conditions under which there exist the extended coordinate change $\Phi(\cdot, \xi_1, \ldots, \xi_{2N+1}) : X \to X$, parameterized by $(\xi_1, \ldots, \xi_{2N+1})$ and defined by

$$z = \Phi(x, y^-, \ldots, y^{[-N]}, u, u^-, \ldots, u^{[-N]}) \tag{2}$$

and the output transformation $p : Y \to Y$, defined by

$$Y = p(y) \tag{3}$$

such that in the new state and output coordinates the state equations (1) are in the following extended observer form[1] with buffer $N \in \{1, \dots, n-2\}$

$$\begin{array}{rcl} z_1^+ & = & z_2 + \varphi_1(Y, \dots, Y^{[-N]}, u, \dots, u^{[-N]}) \\ & \vdots & \\ z_{n-N}^+ & = & z_{n-N+1} + \\ & & +\varphi_{n-N}(Y, \dots, Y^{[-N]}, u, \dots, u^{[-N]}) \\ z_{n-N+1}^+ & = & z_{n-N+2} \\ & \vdots & \\ z_{n-1}^+ & = & z_n \\ z_n^+ & = & 0 \\ Y & = & z_1, \end{array} \quad (4)$$

where the forward shift of the coordinates z depends besides the input u and the output y also on their past values $u^-, \dots, u^{[-N]}$, and $y^-, \dots, y^{[-N]}$.

Note that the state equations (1) can be transformed into the extended observer form (4) with the extended coordinate change (2) and output transformation (3), if the input-output (i/o) equation

$$y^{[n]} = f(y, \dots, y^{[n-1]}, u, \dots, u^{[n-1]}), \quad (5)$$

corresponding to (1), can be written in the form [8]

$$p \circ f = \sum_{l=1}^{n-N} \varphi_l(Y^{[n-l]}, \dots, Y^{[n-l-N]}, u^{[n-l]}, \dots, u^{[n-l-N]}). \quad (6)$$

If (6) holds, one can define the new state variables as follows:

$$z_1 = Y,$$
$$z_i = Y^{[i-1]} - \sum_{j=1}^{k} \varphi_j(Y^{[i-1-j]}, \dots, Y^{[i-1-j-N]}, u^{[i-1-j]}, \dots, u^{[i-1-j-N]}), \quad i = 2, \dots, n, \quad (7)$$

where

$$k = \begin{cases} i-1, & \text{for } i = 2, \dots, n-N+1, \\ n-N, & \text{for } i = n-N+2, \dots, n, \end{cases}$$

that leads to the new state equations in the extended observer form (4).

III. NECESSARY AND SUFFICIENT CONDITIONS

Define for $i = 0, \dots, n-1$ the differential one-forms

$$\omega_i = \frac{\partial f}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial f}{\partial u^{[i]}} \mathrm{d}u^{[i]} \quad (8)$$

and codistributions

$$\Omega_i = \mathrm{span}\big\{\omega_k, \mathrm{d}u^{[k]} \mid k \neq i, \; k = \max(0, i-N), \dots, \min(i+N, n-1)\big\}. \quad (9)$$

[1]The extended observer form without inputs was considered earlier in [3].

For example, if $N = 1$ and $n = 5$, then

$$\begin{array}{rcl} \Omega_0 & = & \mathrm{span}\{\omega_1, \mathrm{d}u^+\}, \\ \Omega_1 & = & \mathrm{span}\{\omega_0, \mathrm{d}u, \omega_2, \mathrm{d}u^{++}\}, \\ \Omega_2 & = & \mathrm{span}\{\omega_1, \mathrm{d}u^+, \omega_3, \mathrm{d}u^{[3]}\}, \\ \Omega_3 & = & \mathrm{span}\{\omega_2, \mathrm{d}u^{++}, \omega_4, \mathrm{d}u^{[4]}\}, \\ \Omega_4 & = & \mathrm{span}\{\omega_3, \mathrm{d}u^{[3]}\}. \end{array}$$

The minimal number of independent generators of a codistribution is called its dimension. For a one form ω and a r-dimensional codistribution $\Omega = \mathrm{span}\{\upsilon_1, \dots, \upsilon_r\}$, we will say that

$$\mathrm{d}\omega \equiv 0 \mod \Omega$$

if and only if

$$\mathrm{d}\omega \wedge \upsilon_1 \wedge \dots \wedge \upsilon_r = 0.$$

Moreover, define the composite functions of φ_l and p

$$\bar{\varphi}_l(y, \dots, y^{[-N]}, u, \dots, u^{[-N]}) := \varphi_l(Y, \dots, Y^{[-N]}, u, \dots, u^{[-N]})$$

and the vector argument

$$\nu_l := \left[y^{[n-l]}, \dots, y^{[n-l-N]}, u^{[n-l]}, \dots, u^{[n-l-N]}\right] \quad (10)$$

for $l = 1, \dots, n-N$. In order to prove our main result, that is Theorem 1 below, we need the following lemma, the proof of which is given in the Appendix.

Lemma 1: For functions $\bar{\varphi}_1(\nu_1), \dots, \bar{\varphi}_{n-N}(\nu_{n-N})$ the following holds

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{i=0}^{n-1} A_i\left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right), \quad (11)$$

where

$$A_i\left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right) = \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}} \mathrm{d}u^{[i]}\right). \quad (12)$$

Now we are ready to prove our main result.

Theorem 1: The system (1) can be transformed by the extended coordinate change (2) and the output transformation (3) into the extended observer form (4) with buffer $N \in \{1, \dots, n-2\}$ if and only if for all $0 \le i, j \le n-1$

$$\mathrm{d}\omega_i \wedge \omega_j + \mathrm{d}\omega_j \wedge \omega_i \equiv \equiv 0 \mod \left((\Omega_i + \Omega_j) \setminus \mathrm{span}\{\omega_i, \omega_j\}\right). \quad (13)$$

Proof: Necessity. Assume that system (1) is transformable into the extended observer form (4). Consequently, the i/o equation (5), corresponding to (1), can be rewritten in the form (6), the total differential of which reads as

$$(p' \circ f)\mathrm{d}f = (p' \circ f) \sum_{i=0}^{n-1} \omega_i = \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l),$$

where $p' \circ f$ means the derivative of the function p evaluated at f. According to Lemma 1,

$$(p' \circ f) \sum_{i=0}^{n-1} \omega_i = \sum_{i=0}^{n-1} A_i \left(\varphi_{n-N-l}(\nu_{n-N-l})\right). \qquad (14)$$

From (14) we have for $i = 0, \ldots, n-1$

$$(p' \circ f)\omega_i = A_i \left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right). \qquad (15)$$

Consider the functions $\bar{\varphi}_{n-N-l}(\nu_{n-N-l})$ for $l = \max(0, i-N), \ldots, \min(i, n-1-N)$. Taking into account (10) for new index $n-N-l$, one can write

$$\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) = \sum_{s=0}^{N} \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}} \mathrm{d}y^{[l+s]} + \right. \\ \left. + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}} \mathrm{d}u^{[l+s]} \right). \qquad (16)$$

Note, that the codistribution Ω_i, defined by (9), can be rewritten as

$$\Omega_i = \mathrm{span}\{\mathrm{d}y^{[k+s]}, \mathrm{d}u^{[k+s]} \mid \\ k = \max(0, i-N), \ldots, \min(i, n-1-N), \\ s = 0, \ldots, N\} \setminus \mathrm{span}\{\mathrm{d}y^{[i]}, \mathrm{d}u^{[i]}\}. \qquad (17)$$

As a consequence, taking into account that the indices l in (16) and k in (17) have the same ranges, from (16) one obtains

$$\mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) \equiv \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \right. \\ \left. + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}} \mathrm{d}u^{[i]} \right) \mod \Omega_i, \qquad (18)$$

which, by (15) and (12), leads to

$$(p' \circ f)\omega_i \equiv \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\bar{\varphi}_{n-N-l}(\nu_{n-N-l}) \mod \Omega_i.$$

Applying the exterior derivative to the above equality yields

$$\mathrm{d}(p' \circ f) \wedge \omega_i + (p' \circ f)\mathrm{d}\omega_i \equiv 0 \mod \Omega_i.$$

From the above relationship we obtain

$$\mathrm{d}\omega_i \equiv -\mathrm{d}\ln|p' \circ f| \wedge \omega_i \mod \Omega_i$$

for $i = 0, \ldots, n-1$. Obviously,

$$\mathrm{d}\omega_i \wedge \omega_j \equiv -\mathrm{d}\ln|p' \circ f| \wedge \\ \wedge \omega_i \wedge \omega_j \mod \left((\Omega_i + \Omega_j) \setminus \mathrm{span}\{\omega_i, \omega_j\}\right),$$

using which, one gets for $i, j = 0, \ldots, n-1$

$$\mathrm{d}\omega_i \wedge \omega_j + \mathrm{d}\omega_j \wedge \omega_i \equiv -\mathrm{d}\ln|p' \circ f| \wedge (\omega_i \wedge \omega_j + \\ + \omega_j \wedge \omega_i) \mod \left((\Omega_i + \Omega_j) \setminus \mathrm{span}\{\omega_i, \omega_j\}\right). \qquad (19)$$

Since the wedge product is anticommutative, the expression in the parentheses on the right-hand side of (19) is always zero, which yields (13).

Sufficiency. The proof consists of three steps. On the first step (i) we will show that under the conditions (13) there exist functions $\psi_l(\nu_l)$ for $l = 1, \ldots, n-N$, such that

$$\omega_i \equiv \lambda_i \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\psi_{n-N-l}(\nu_{n-N-l}) \mod \Omega_i \qquad (20)$$

for $i = 0, \ldots, n-1$. On the second step (ii) we will prove that for all ω_i there exists the common integrating factor λ, and finally, on the last step (iii) we will show that from steps (i) and (ii) follows the existence of output transformation p such that its composition with f yields (6).

(i) Note that in case $i = j$ (13) yields

$$\mathrm{d}\omega_i \wedge \omega_i \equiv 0 \mod \Omega_i, \qquad (21)$$

from which follows the existence of the integrating factor $\lambda_i(y, \ldots, y^{[n-1]}, u, \ldots, u^{[n-1]})$ such that

$$\omega_i \equiv \lambda_i \mathrm{d}\bar{\psi}_i(\bar{\nu}_i) \mod \Omega_i \qquad (22)$$

for some functions[2] $\bar{\psi}_i(\bar{\nu}_i)$, where $\bar{\nu}_i$ is the vector argument which consists of the elements of the set $\{y^{[k]}, u^{[k]} \mid k = \max(0, i-N), \ldots, \min(i+N, n-1)\}$. Note that, taking into account (8) and (9), according to (22),

$$\frac{1}{\lambda_i}\omega_i = \left(\frac{\partial \bar{\psi}_i(\bar{\nu}_i)}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \bar{\psi}_i(\bar{\nu}_i)}{\partial u^{[i]}} \mathrm{d}u^{[i]} \right). \qquad (23)$$

Choose the function $\zeta(y, \ldots, y^{[n-1]}, u, \ldots, u^{[n-1]})$ such that

$$\frac{\partial \zeta}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \zeta}{\partial u^{[i]}} \mathrm{d}u^{[i]} = \frac{\partial \bar{\psi}_i(\bar{\nu}_i)}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \bar{\psi}_i(\bar{\nu}_i)}{\partial u^{[i]}} \mathrm{d}u^{[i]} \qquad (24)$$

for $i = 1, \ldots, n-1$, and consequently

$$\sum_{i=0}^{n-1} \frac{1}{\lambda_i}\omega_i = \mathrm{d}\zeta.$$

As we will show in the sequel, the function ζ really exist and can be represented in the form

$$\zeta = \sum_{l=1}^{n-N} \psi_l(\nu_l) \qquad (25)$$

for some functions $\psi_1(\nu_1), \ldots, \psi_{n-N}(\nu_{n-N})$. Note that (25) holds, if the following second order partial derivatives of ζ equal zero,

$$\begin{aligned} \frac{\partial^2 \zeta}{\partial y^{[i]} \partial y^{[j]}} &= 0, \\ \frac{\partial^2 \zeta}{\partial u^{[i]} \partial u^{[j]}} &= 0, \\ \frac{\partial^2 \zeta}{\partial u^{[i]} \partial y^{[j]}} &= 0, \\ \frac{\partial^2 \zeta}{\partial y^{[i]} \partial u^{[j]}} &= 0 \end{aligned} \qquad (26)$$

for $i, j = 0, \ldots, n-1$, except for $j \neq \max(0, i-N), \ldots, \min(i+N, n-1)$. Our next purpose is to prove

[2]The functions $\bar{\psi}_i(\bar{\nu}_i)$ should not be confused with the functions $\psi_l(\nu_l)$ in (20). The number of functions $\bar{\psi}_i(\bar{\nu}_i)$ is n, but the number of functions $\psi_l(\nu_l)$ is $n-N$. Moreover, the vector arguments $\bar{\nu}_i$ and ν_l have different number of elements.

that (26) holds. Denoting $\alpha_i = 1/\lambda_i$ and taking into account (23), (24) and (8), one can rewrite (26) as follows

$$\begin{aligned} \frac{\partial \alpha_i}{\partial y^{[j]}}\frac{\partial f}{\partial y^{[i]}} + \alpha_i \frac{\partial^2 f}{\partial y^{[i]}\partial y^{[j]}} &= 0,\\ \frac{\partial \alpha_i}{\partial u^{[j]}}\frac{\partial f}{\partial u^{[i]}} + \alpha_i \frac{\partial^2 f}{\partial u^{[i]}\partial u^{[j]}} &= 0,\\ \frac{\partial \alpha_i}{\partial y^{[j]}}\frac{\partial f}{\partial u^{[i]}} + \alpha_i \frac{\partial^2 f}{\partial u^{[i]}\partial y^{[j]}} &= 0,\\ \frac{\partial \alpha_i}{\partial u^{[j]}}\frac{\partial f}{\partial y^{[i]}} + \alpha_i \frac{\partial^2 f}{\partial y^{[i]}\partial u^{[j]}} &= 0. \end{aligned} \tag{27}$$

Expressing $\partial\alpha_i/\partial y^{[j]}$ from the first equation of (27) and substituting it into the third equation, and also expressing $\partial\alpha_i/\partial u^{[j]}$ from the second equation and substituting it into the fourth equation, one obtains

$$\begin{aligned} \frac{\partial f}{\partial y^{[i]}}\frac{\partial^2 f}{\partial u^{[i]}\partial y^{[j]}} - \frac{\partial f}{\partial u^{[i]}}\frac{\partial^2 f}{\partial y^{[i]}\partial y^{[j]}} &= 0,\\ \frac{\partial f}{\partial u^{[i]}}\frac{\partial^2 f}{\partial y^{[i]}\partial u^{[j]}} - \frac{\partial f}{\partial y^{[i]}}\frac{\partial^2 f}{\partial u^{[i]}\partial u^{[j]}} &= 0. \end{aligned}$$

It is easy to verify that under the conditions (21) the above equalities are satisfied and, as a consequence, the function ζ really exists, satisfying (25), which yields

$$\sum_{i=0}^{n-1}\frac{1}{\lambda_i}\omega_i = \sum_{l=1}^{n-N}\mathrm{d}\psi_l(\nu_l), \tag{28}$$

from which, using Lemma 1 and (18) for functions $\psi_l(\nu_l)$, one obtains (20).

(ii) Take the exterior derivative of (20) and then apply (20) as follows:

$$\sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\psi_{n-N-l}(\nu_{n-N-l}) \equiv \frac{1}{\lambda_i}\omega_i \mod \Omega_i.$$

This yields

$$\begin{aligned} \mathrm{d}\omega_i \equiv \mathrm{d}\lambda_i \wedge \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\psi_{n-N-l}(\nu_{n-N-l}) \equiv \\ \equiv \mathrm{d}\ln|\lambda_i| \wedge \omega_i \mod \Omega_i. \end{aligned}$$

By the conditions (13):

$$\begin{aligned} (\mathrm{d}\ln|\lambda_i| - \mathrm{d}\ln|\lambda_j|)\wedge\omega_i\wedge\omega_j \equiv \\ \equiv 0 \mod ((\Omega_i+\Omega_j)\setminus \mathrm{span}\{\omega_i,\omega_j\}), \end{aligned}$$

from which follows

$$\lambda_i = \lambda_j = \lambda$$

for $i, j = 0, \ldots, n-1$.

(iii) From (i) and (ii) follows that one can find functions $\bar\varphi_1(\nu_1), \ldots, \bar\varphi_{n-N}(\nu_{n-N})$, for which there exists the common integrating factor λ such that (28) can be rewritten as

$$\sum_{i=0}^{n-1}\omega_i = \lambda\sum_{l=1}^{n-N}\mathrm{d}\bar\varphi_l(\nu_l). \tag{29}$$

Since $\mathrm{d}f$ is a total differential, its exterior derivative

$$\begin{aligned} \mathrm{d}^2 f = \sum_{i=0}^{n-1}\mathrm{d}\omega_i = \mathrm{d}\lambda \wedge \sum_{l=1}^{n-N}\mathrm{d}\bar\varphi_l(\nu_l) = \mathrm{d}\ln|\lambda|\sum_{i=0}^{n-1}\omega_i = \\ = \mathrm{d}\ln|\lambda|\wedge\mathrm{d}f \end{aligned}$$

equals zero and by Cartan's Lemma $\mathrm{d}\ln|\lambda| \in \mathrm{span}\{\mathrm{d}f\}$. Therefore, λ can be represented as a composite function of f and some other function. We will show below that the choice $\lambda = 1/(p'\circ f)$ guarantees, that the composite function $p\circ f$ has the form (6). First, we prove that $p'\circ f$ is the common integrating factor for all one-forms ω_i, that is

$$(p'\circ f)\omega_i \equiv \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \mathrm{d}\bar\varphi_{n-N-l}(\nu_{n-N-l}) \mod \Omega_i.$$

Taking the exterior derivative of $(p'\circ f)\omega_i$, one obtains

$$\begin{aligned} \mathrm{d}\left[(p'\circ f)\omega_i\right] \equiv (p''\circ f)\mathrm{d}f\wedge\omega_i + \\ + (p'\circ f)\mathrm{d}\omega_i \equiv (p''\circ f)\mathrm{d}f\wedge\omega_i + \\ + (p'\circ f)\mathrm{d}\ln|\lambda|\wedge\omega_i \equiv (p''\circ f)\mathrm{d}f\wedge\omega_i - \\ - \mathrm{d}\left(\ln|p'\circ f|\right)(p'\circ f)\wedge\omega_i \equiv 0 \mod \Omega_i, \end{aligned}$$

meaning the functions $\bar\varphi_l(\nu_l)$ really exist. Finally, multiplying $\mathrm{d}f$ by $p'\circ f$ and taking into account (29), one obtains

$$\mathrm{d}(p\circ f) = (p'\circ f)\mathrm{d}f = (p'\circ f)\sum_{i=0}^{n-1}\omega_i = \sum_{l=1}^{n-N}\mathrm{d}\bar\varphi_l(\nu_l),$$

yielding (6). ■

IV. EXAMPLE

Examine the following state equations:

$$\begin{aligned} x_1^+ &= x_2 u\\ x_2^+ &= x_3\\ x_3^+ &= (x_1 + x_2 u + u)(x_2 u + x_3)(x_1 u + x_4)\\ x_4^+ &= x_1 + u\\ y &= x_1. \end{aligned} \tag{30}$$

The i/o equation, corresponding to (30), is

$$\begin{aligned} y^{[4]} = \left(y + u + y^+ u^+\right)\left(y^+ + u^+ + y^{++}\right)u^{[3]}\cdot \\ \cdot\left(y^{++} + \frac{y^{[3]}}{u^{++}}\right). \end{aligned} \tag{31}$$

Note, that once the system is transformable into the extended observer form with some arbitrary buffer N it is also transformable into the extended observer forms with the buffers that are greater than N. Therefore, our goal is to find the least buffer N, for which the system (30) is transformable into the extended observer form (4). Consequently, it is reasonable to start with $N = 1$. Compute, according to (8),

$$
\begin{aligned}
\omega_0 &= \left(y^+ + u^+ + y^{++}\right) u^{[3]} \left(y^{++} + \frac{y^{[3]}}{u^{++}}\right) \cdot \\
&\quad \cdot \left(\mathrm{d}y + \mathrm{d}u\right), \\
\omega_1 &= u^{[3]} \left(y^{++} + \frac{y^{[3]}}{u^{++}}\right) \cdot \\
&\quad \cdot \left(\left(y + u + u^+ \left(u^+ + 2y^+ + y^{++}\right)\right) \mathrm{d}y^+ + \right. \\
&\quad \left. + \left(y + u + y^+ \left(y^+ + 2u^+ + y^{++}\right)\right) \mathrm{d}u^+\right), \\
\omega_2 &= u^{[3]} \left(y + u + u^+ y^+\right) \cdot \\
&\quad \cdot \left(\left(y^+ + u^+ + 2y^{++} + \frac{y^{[3]}}{u^{++}}\right) \mathrm{d}y^{++} - \right. \\
&\quad \left. - \left(\left(y^+ + u^+ + y^{++}\right) \frac{y^{[3]}}{\left(u^{++}\right)^2}\right) \mathrm{d}u^{++}\right), \\
\omega_3 &= \frac{\left(y + u + y^+ u^+\right)\left(u^+ + y^+ + y^{++}\right)}{u^{++}} \cdot \\
&\quad \cdot \left(u^{[3]} \mathrm{d}y^{[3]} + \left(y^{++} u^{++} + y^{[3]}\right) \mathrm{d}u^{[3]}\right)
\end{aligned}
$$

and, according to (9),

$$
\begin{aligned}
\Omega_0 &= \mathrm{span}\{\omega_1, u^+\}, \\
\Omega_1 &= \mathrm{span}\{\omega_0, u, \omega_2, u^{++}\}, \\
\Omega_2 &= \mathrm{span}\{\omega_1, u^+, \omega_3, u^{[3]}\}, \\
\Omega_3 &= \mathrm{span}\{\omega_2, u^{++}\}.
\end{aligned}
$$

To verify whether the system (30) is transformable into the extended observer form (4), one has to check the validity of conditions (13), which in case $n = 4$ and $N = 1$ are the following:

$$
\begin{aligned}
&\mathrm{d}\omega_0 \wedge \omega_0 \equiv 0 \mod \Omega_0, \\
&\mathrm{d}\omega_1 \wedge \omega_1 \equiv 0 \mod \Omega_1, \\
&\mathrm{d}\omega_2 \wedge \omega_2 \equiv 0 \mod \Omega_2, \\
&\mathrm{d}\omega_3 \wedge \omega_3 \equiv 0 \mod \Omega_3, \\
&\mathrm{d}\omega_0 \wedge \omega_1 + \mathrm{d}\omega_1 \wedge \omega_0 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\mathrm{d}u, \mathrm{d}u^+, \omega_2, \mathrm{d}u^{++}\right\}, \\
&\mathrm{d}\omega_0 \wedge \omega_2 + \mathrm{d}\omega_2 \wedge \omega_0 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\omega_1, \mathrm{d}u^+, \omega_3, \mathrm{d}u^{[3]}\right\}, \\
&\mathrm{d}\omega_0 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_0 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\omega_1, \mathrm{d}u^+, \omega_2, \mathrm{d}u^{++}\right\}, \\
&\mathrm{d}\omega_1 \wedge \omega_2 + \mathrm{d}\omega_2 \wedge \omega_1 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\omega_0, \mathrm{d}u, \mathrm{d}u^+, \mathrm{d}u^{++}, \omega_3, \mathrm{d}u^{[3]}\right\}, \\
&\mathrm{d}\omega_1 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_1 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\omega_0, \mathrm{d}u, \omega_2, \mathrm{d}u^{++}\right\}, \\
&\mathrm{d}\omega_2 \wedge \omega_3 + \mathrm{d}\omega_3 \wedge \omega_2 \equiv \\
&\qquad \equiv 0 \mod \mathrm{span}\left\{\omega_1, \mathrm{d}u^+, \mathrm{d}u^{++}, \mathrm{d}u^{[3]}\right\}.
\end{aligned}
$$

By direct computations one can confirm that all above conditions are satisfied, which means that system (30) is transformable via the extended coordinate change and output transformation into the extended observer form with buffer $N = 1$. In this paper we do not provide the precise algorithm for computation of the output transformation $p(y)$ and the functions $\varphi_1, \ldots, \varphi_{n-N}$. However, in [9] such procedure was given for the case $N = 0$ and we conjecture that it can be extended for the case $N > 0$. Due to simplicity of this academic example one can intuitively choose the output transformation $Y = p(y) = \ln|y|$, applying which to (31), one obtains the i/o equation in the form (6):

$$
\begin{aligned}
Y^{[4]} = \ln\left|e^Y + u + e^{Y^+} u^+\right| + \ln\left|e^{Y^+} + u^+ + e^{Y^{++}}\right| + \\
+ \ln\left|u^{[3]}\right| + \ln\left|e^{Y^{++}} + \frac{e^{Y^{[3]}}}{u^{++}}\right|,
\end{aligned}
$$

from which we have by simple inspection

$$
\begin{aligned}
\varphi_1(\nu_1) &= \ln\left|u^{[3]}\right| + \ln\left|e^{Y^{++}} + \frac{e^{Y^{[3]}}}{u^{++}}\right|, \\
\varphi_2(\nu_2) &= \ln\left|e^{Y^+} + u^+ + e^{Y^{++}}\right|, \\
\varphi_3(\nu_3) &= \ln\left|e^Y + u + e^{Y^+} u^+\right|.
\end{aligned}
$$

Using (7) one can define the new state variables as follows:

$$
\begin{aligned}
z_1 &= Y, \\
z_2 &= Y^+ - \ln|u| - \ln\left|e^{Y^-} + \frac{e^Y}{u^-}\right|, \\
z_3 &= Y^{++} - \ln\left|u^+\right| - \ln\left|e^Y + \frac{e^{Y^+}}{u}\right| - \\
&\quad - \ln\left|e^{Y^-} + u^- + e^Y\right|, \\
z_4 &= Y^{[3]} - \ln\left|u^{++}\right| - \ln\left|e^{Y^+} + \frac{e^{Y^{++}}}{u^+}\right| - \\
&\quad - \ln\left|e^Y + u + e^{Y^+}\right| - \ln\left|e^{Y^-} + u^- + e^Y u\right|,
\end{aligned}
$$

which, due to the output transformation $Y = \ln|y|$ and state equations (30), can be rewritten as

$$
\begin{aligned}
z_1 &= \ln|x_1|, \\
z_2 &= \ln|x_2| - \ln\left|x_1^- + \frac{x_1}{u^-}\right|, \\
z_3 &= \ln|x_3| - \ln|x_1 + x_2| - \ln\left|x_1^- + u^- + x_1\right|, \\
z_4 &= \ln|x_1 u + x_4| - \ln\left|x_1^- + u^- + x_1 u\right|,
\end{aligned}
$$

that leads to the new state equations in the extended observer form

$$
\begin{aligned}
z_1^+ &= z_2 + \ln|u| + \ln\left|e^{z_1^-} + \frac{e^{z_1}}{u^-}\right| \\
z_2^+ &= z_3 + \ln\left|e^{z_1^-} + u^- + e^{z_1}\right| \\
z_3^+ &= z_4 + \ln\left|e^{z_1^-} + u^- + e^{z_1} u\right| \\
z_4^+ &= 0 \\
Y &= z_1.
\end{aligned}
$$

V. CONCLUSIONS

A. Conclusions

Alternative necessary and sufficient conditions were derived for the existence of the extended coordinate change and output transformation that allow to transform the discrete-time single-input single-output nonlinear state equations into the extended observer form. The conditions are expressed in terms of the exterior derivatives and the exterior products of the one-forms, associated with the input-output equation, corresponding to the state equations, and, consequently, are directly computable whenever the input-output equation is easily computable from the state equations. Moreover, in case when the buffer $N = 0$, our conditions coincide with those in [9]. Thus, our result can be considered as the direct generalization of the conditions derived in [9], not depending on the past values of input and output.

B. Future Works

Although the paper suggests the solvability conditions, that do not depend on the existence of unknown functions, and are therefore easily checkable, likewise in [3], in this paper no procedure is given to compute the extended coordinate change and the output transformation. The development of the constructive algorithm for transformation the system into the extended observer form remains a topic for future research. Another future goal is to compare our results with the method developed in [10], where in order to transform the system into the generalized observer form, it is suggested to extend the system by means of the so-called dynamic auxiliary system of the specific linear form. It is our conjecture that the two approaches are closely related, since, in principle, the past values of input and output may be possibly expressed in terms of system extensions. Moreover, we intend to find also the relationship between the results of our approach and the conditions developed in [11] for transformation of the system into the output-scaled observer form. Both apply certain output functions, though in different purposes. Whereas we use the output function to transform the original output into a new output, the paper [11] multiplies the right-hand side of equations of the observer form by this output function. However, note that both [10] and [11] address systems without control variable.

APPENDIX

A. Proof of Lemma 1

Proof: It is easy to observe that

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{l=1}^{n-N}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_l(\nu_l)}{\partial y^{[n-l-s]}}\mathrm{d}y^{[n-l-s]}+ + \frac{\partial\bar{\varphi}_l(\nu_l)}{\partial u^{[n-l-s]}}\mathrm{d}u^{[n-l-s]}\right).$$

Replace on the right-hand side of the above relationship the summation index l by $l+1$. In this case $l = 0,\ldots,n-N-1$ and one can change the summation order

$$\sum_{l=1}^{n-N}\sum_{s=0}^{N} a_{l,s} = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{n-N-l,N-s},$$

which yields

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = = \sum_{l=0}^{n-N-1}\sum_{s=0}^{N}\left(\frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[l+s]}}\mathrm{d}y^{[l+s]}+ + \frac{\partial\bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[l+s]}}\mathrm{d}u^{[l+s]}\right).$$

Change the summation indices l and s for $i = l+s$ and l. It is easy to see, that in this case i changes from 0 to $n-1$ and $l = i-s$. Since $s = 0,\ldots,N$, the minimal and maximal values of $i-s$ are $i-N$ and i, respectively. On the other hand, l changes from 0 to $n-N-1$. Thus, we take $l = \max(0, i-N),\ldots,\min(i, n-1-N)$. As a result, one can use the following relation:

$$\sum_{l=0}^{n-N-1}\sum_{s=0}^{N} a_{l,l+s} = \sum_{i=0}^{n-1}\sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} a_{l,i},$$

which leads to (11). ■

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Extended observer form: simple existence conditions

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The paper focuses on the problem of transforming the discrete-time single-input single-output nonlinear state equations into the extended observer form, which, besides the input and output, also depends on a finite number of their past values. The simple necessary and sufficient conditions for the existence of the extended coordinate change and the output transformation, allowing to solve the problem, are formulated in terms of certain partial derivatives, related to the input–output equation, corresponding to the state equations. Moreover, a certain algorithm for transforming the state equations into the observer form is proposed.

Keywords: nonlinear control system; discrete-time system; extended coordinate change; output transformation; extended observer form

1. Introduction

Conditions for transformability of the discrete-time nonlinear state equations by state coordinate transformation into the observer (input–output injection) form are known to be very restrictive, see Lee and Nam (1991). This has motivated to generalise the observer form in such a manner that one may still construct the observers with linear error dynamics (Besançon and Bornard, 1995; Califano, Monaco, and Normand-Cyrot, 2003, 2009; Lee and Hong, 2011; Lin and Wei, 2009; Zhang, Feng, and Xu, 2010) and/or to allow additionally output transformation. For example, in Besançon and Bornard (1995), Califano et al. (2003), Califano et al. (2009) and Lee and Hong (2011), the matrix A in the observer form is allowed to depend on control variable u. In Besançon and Bornard (1995), only state transformation was allowed whereas Califano et al. (2003) and Lee and Hong (2011) allow the output transformation also. The paper by Califano et al. (2009) extends the results to multi-input multi-output case. The paper by Lin and Wei (2009) relies on the time-scaling technique, which means that the right-hand side of the standard observer form is multiplied by a single-variable function, depending on the output variable, and the paper only allows the state transformation. A method called dynamic observer error linearisation was provided in Zhang, Feng, and Xu (2010), where in order to transform the system into the generalised observer form, it is suggested to augment the system by means of the so-called dynamic auxiliary system of the specific linear form. In some other papers, the output injection term is allowed to depend, besides the current output value, also on a finite number of its past values, reducing that way the restrictions on possibility to construct such transformations (Huijberts, 1999; Huijberts, Lilge, and Nijmeijer, 1999). A corollary of the results, obtained in Huijberts et al. (1999), is that when the number of past output values equals $n - 1$ (where n is denoted the state dimension), the system can always be transformed into the extended observer form, provided the system under consideration is strongly observable.

We build up from this result and are looking for necessary and sufficient conditions for transformability the state equations into the extended observer form with buffer N less than $n - 1$. In the problem addressed, we allow both the state and output transformations. The paper may be understood as an extension of the paper by Huijberts (1999) where the systems without control were considered. However, our conditions do not mimic those in Huijberts (1999), relying on the sophisticated language of differential geometry. Checking the conditions of Huijberts (1999) requires calculation of the exterior derivatives and wedge products of certain one forms, associated with the system, as well as the Lie derivatives of the corresponding dual vector fields, which leads to complicated computations. In comparison, our conditions are extremely easy to check. Once certain partial derivatives, related to the input–output (i/o) equation are found, the conditions may be checked practically by direct inspection. That way, our conditions are similar in spirit to those in Lee and Hong (2011) except that they consider the buffer-free case and instead of relying on the input–output equation of system, transform the state equations into the specific canonical form. Nevertheless, the computation of the state transformation, leading to the abovementioned canonical form, is not easier than finding the

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corresponding i/o equation. Furthermore, the output transformation may also be found from our conditions, unlike in the paper by Kaparin and Kotta (2011), where the simple intrinsic necessary and sufficient solvability conditions were formulated in terms of differential one-forms, associated with the i/o equation, corresponding to the state equations. Though very simple, the results of this paper have the disadvantages of not being intrinsic. Another point to mention is that our results assume (but so do those in Huijberts (1999) and Huijberts et al. (1999)) that the i/o equation, corresponding to the state equations, can be easily found from the state equations. Under observability assumption, one may always find the i/o equations, at least locally, using the state elimination algorithm. For example, The Nonlinear Control Webpage (a webMathematica-based application developed in the Institute of Cybernetics at Tallinn University of Technology) provides the tool which, using the state elimination algorithm, finds the i/o equations starting from the state equations. The site is available at www.nlcontrol.ioc.ee and does not require Mathematica software to be installed in a computer. However, the global state elimination problem is a difficult task that results, in general, an implicit i/o equation accompanied with the number of inequations, see Diop (1991).

Preliminary results of the paper were published in conference article Mullari and Kotta (2011), where the conditions were proved only for the special case $N = 1$. Moreover, the sufficient conditions presented in Mullari and Kotta (2011) are valid only for N satisfying certain relation regarding the system order and/or the highest and the lowest shifts of input and output in system i/o equation. In this paper, by means of the additional conditions, the necessary and sufficient conditions are given and proved for an arbitrary buffer N. Finally, this paper presents the algorithm for computation of the extended coordinate change and the output transformation, necessary for transformation of the state equations into the extended observer form.

2. Preliminaries and problem statement

Consider a single-input single-output nonlinear discrete-time system, described by the state equations

$$\begin{aligned} x^{[1]} &= F(x, u) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subset \mathbb{R}$ is the input, $y \in \mathcal{Y} \subset \mathbb{R}$ is the output, $F : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{Y}$ are assumed to be real meromorphic functions. To simplify the exposition of the paper, we use symbol $^{[i]}$ instead of the argument $t + i$ for $i \in \mathbb{Z}$, except for $i = 0$ when we omit symbol $^{[0]}$, so $x := x(t)$ and $x^{[i]} = x(t + i)$. Our purpose is to find the conditions under which there exists the extended coordinate change $\Phi(\cdot, \xi_1, \dots, \xi_{2N+1}) : \mathcal{X} \to \mathcal{X}$, parameterised by $(\xi_1, \dots, \xi_{2N+1})$ and defined by

$$z = \Phi\left(x, y^{[-1]}, \dots, y^{[-N]}, u, u^{[-1]}, \dots, u^{[-N]}\right) \tag{2}$$

and the output transformation $p : \mathcal{Y} \to \mathcal{Y}$, defined by

$$Y = p(y), \tag{3}$$

such that in the new state and output coordinates the state equations (1) are in the following extended observer form with buffer $N \in \{1, \dots, n-2\}$:

$$\begin{aligned} z_1^{[1]} &= z_2 + \varphi_1\left(Y, \dots, Y^{[-N]}, u, \dots, u^{[-N]}\right) \\ &\vdots \\ z_{n-N}^{[1]} &= z_{n-N+1} + \varphi_{n-N}\left(Y, \dots, Y^{[-N]}, u, \dots, u^{[-N]}\right) \\ z_{n-N+1}^{[1]} &= z_{n-N+2} \\ &\vdots \\ z_{n-1}^{[1]} &= z_n \\ z_n^{[1]} &= 0 \\ Y &= z_1, \end{aligned} \tag{4}$$

where the forward shift of the coordinates z depends besides the input u and the output y also on their past values $u^{[-1]}, \dots, u^{[-N]}$, and $y^{[-1]}, \dots, y^{[-N]}$. This form without inputs was considered earlier in Huijberts (1999). We do not address the case when the buffer $N = n - 1$ since, as shown by Huijberts, the system can always be transformed into such form (even without the output transformation), whenever the system under consideration is strongly observable. The proof of Huijberts carries over to systems depending on control too. Therefore, it is obvious that the results of this paper address only the case $n \geq 3$.

Note that the state equations (1) can be transformed into the extended observer form (4) with the extended coordinate change (2) and the output transformation (3), if the input–output (i/o) equation,

$$y^{[n]} = f\left(y, \dots, y^{[n-1]}, u, \dots, u^{[n-1]}\right), \tag{5}$$

corresponding to Equation (1), can be written in the form,

$$\begin{aligned} p \circ f = \sum_{l=1}^{n-N} \varphi_l \big(& Y^{[n-l]}, \dots, Y^{[n-l-N]}, \\ & u^{[n-l]}, \dots, u^{[n-l-N]}\big). \end{aligned} \tag{6}$$

If Equation (6) holds, one can define the new state variables as follows:

$$\begin{aligned} z_1 &= Y, \\ z_i &= Y^{[i-1]} - \sum_{j=1}^{\min(i-1, n-N)} \varphi_j\big(Y^{[i-1-j]}, \dots, Y^{[i-1-j-N]}, \\ &\qquad u^{[i-1-j]}, \dots, u^{[i-1-j-N]}\big), \quad i = 2, \dots, n, \end{aligned} \tag{7}$$

that leads to the new state equations in the extended observer form (4).

3. Necessary and sufficient conditions

To present the theorem and the proof in a more compact form, denote by α the variable, which can be either u or y. Then by β is denoted u, if α is y and y if α is u. Moreover, denote by $\overline{j_\alpha}$ and $\underline{j_\alpha}$, respectively, the highest and the lowest shifts of α the function f depends on. For example, if $f\left(y, y^{[2]}, y^{[3]}, u^{[1]}, u^{[2]}, u^{[4]}\right)$, then $\underline{j_y} = 0$, $\underline{j_u} = 1$, $\overline{j_y} = 3$ and $\overline{j_u} = 4$.

Theorem 3.1: *The system (1) can be transformed by the extended coordinate change (2) and the output transformation (3) into the extended observer form (4) with buffer $N \in \{1, \ldots, n-2\}$ if and only if there exists a function $S\left(y, \ldots, y^{[n-1]}, u, \ldots, u^{[n-1]}\right)$ such that for $i, j = 0, \ldots, n-1$, $j \neq i - N, \ldots, i + N$,*

$$\frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial \alpha^{[i]}}\right|\right) = \frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial \beta^{[i]}}\right|\right) =: \frac{\partial S}{\partial \alpha^{[j]}}, \quad (8)$$

and in case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ the function S satisfies for $r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$ and an arbitrary $j \neq r$ the following additional conditions

$$\begin{aligned}\frac{\partial S}{\partial \alpha^{[j]}}\frac{\partial f}{\partial \alpha^{[r]}}\left(\frac{\partial f}{\partial \alpha^{[j]}}\right)^{-1} &= \frac{\partial S}{\partial \beta^{[j]}}\frac{\partial f}{\partial \alpha^{[r]}}\left(\frac{\partial f}{\partial \beta^{[j]}}\right)^{-1} \\ &=: \frac{\partial S}{\partial \alpha^{[r]}}. \end{aligned} \quad (9)$$

Remark 1: Note that in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ the conditions (8) are both necessary and sufficient, but in the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ they are only necessary, and for sufficiency[1] one needs the additional conditions (9) (which in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ hold by (8)).

Remark 2: Suppose that for some $q = 0, \ldots, n-1$ the function f (and S, as a consequence) does not depend on the variable $y^{[q]}$ (or $u^{[q]}$). In this case either the left-hand side or the middle part in the corresponding condition of Equation (8) should be omitted, depending on whether the $\alpha^{[i]}$ or $\beta^{[i]}$ stands for $y^{[q]}$ (or $u^{[q]}$). Thus, for instance, in the case of system without input one obtains the conditions (8) where $\alpha = y$ and middle part is excluded.

Remark 3: If the conditions (8) are satisfied it is enough to check the conditions (9) only for one $j \neq r$. However, one has to choose (if possible) j such that the function f (and S, as a consequence) depends on both $y^{[j]}$ and $u^{[j]}$. If such a choice is not possible, then either the left-hand side or the middle part of Equation (9) should be omitted, depending on whether $\alpha^{[j]}$ or $\beta^{[j]}$ stands for the variable, the function f does not depend on. Thus, for instance, in the case of system without input, one obtains the conditions (9) where $\alpha = y$ and the middle part is excluded.

Remark 4: Taking $N = 0$, the conditions (8) (and Equation (9) for the special case $\overline{j_\alpha} = \underline{j_\alpha}$) can be used to check whether the system is transformable into the observer form without the buffer (see the different results in Huijberts (1999) for systems without input and Mullari and Kotta (2009) for input dependent systems).

To prove Theorem 3.1, we need the following lemma, the proof of which is given in Appendix.

Lemma 3.2: *From conditions (8) (and in the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ (9)) follows*

$$\mathrm{d}S \wedge \mathrm{d}f = 0. \quad (10)$$

Now we are ready to prove the main result.

Proof:

Necessity. Assume that system (1) is transformable into the extended observer form (4). Consequently, the i/o Equation (5), corresponding to Equation (1), can be rewritten in the form (6), yielding that the following second-order partial derivatives of the composition $p \circ f$ equal to zero for $i, j = 0, \ldots, n-1$, $j \neq i - N, \ldots, i + N$,

$$\begin{aligned}\frac{\partial^2 (p \circ f)}{\partial \alpha^{[i]} \partial \alpha^{[j]}} &= \frac{\partial (p' \circ f)}{\partial \alpha^{[j]}}\frac{\partial f}{\partial \alpha^{[i]}} + \left(p' \circ f\right)\frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[j]}} = 0, \\ \frac{\partial^2 (p \circ f)}{\partial \beta^{[i]} \partial \alpha^{[j]}} &= \frac{\partial (p' \circ f)}{\partial \alpha^{[j]}}\frac{\partial f}{\partial \beta^{[i]}} \\ &\quad + \left(p' \circ f\right)\frac{\partial^2 f}{\partial \beta^{[i]} \partial \alpha^{[j]}} = 0, \end{aligned} \quad (11)$$

where $p' \circ f$ means the derivative of the function p evaluated at f. Dividing the first equation of Equation (11) by $(p' \circ f)\left(\partial f / \partial \alpha^{[i]}\right)$ and the second equation by $(p' \circ f)\left(\partial f / \partial \beta^{[i]}\right)$ yields

$$\begin{aligned}&\frac{1}{p' \circ f}\frac{\partial (p' \circ f)}{\partial \alpha^{[j]}} + \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[j]}}\left(\frac{\partial f}{\partial \alpha^{[i]}}\right)^{-1} \\ &\quad = \frac{\partial \ln\left|p' \circ f\right|}{\partial \alpha^{[j]}} + \frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial \alpha^{[i]}}\right|\right) = 0, \\ &\frac{1}{p' \circ f}\frac{\partial (p' \circ f)}{\partial \alpha^{[j]}} + \frac{\partial^2 f}{\partial \beta^{[i]} \partial \alpha^{[j]}}\left(\frac{\partial f}{\partial \beta^{[i]}}\right)^{-1} \\ &\quad = \frac{\partial \ln\left|p' \circ f\right|}{\partial \alpha^{[j]}} + \frac{\partial}{\partial \alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial \beta^{[i]}}\right|\right) = 0. \end{aligned}$$

The above equalities suggest that the function

$$S = -\ln\left|p' \circ f\right| \quad (12)$$

will make the conditions (8) and (9) to hold.

Sufficiency. Suppose the conditions (8) and (9) are satisfied. Then, according to Lemma 3.2, $\mathrm{d}S \wedge \mathrm{d}f = 0$, which by Cartan's Lemma yields $\mathrm{d}S \in \operatorname{span}\{\mathrm{d}f\}$. Therefore, the function S can be represented as a composition of some function Ψ with f, i.e. $S = \Psi \circ f$. We will show below that the choice $S = -\ln|p' \circ f|$ guarantees that the equalities (11) are satisfied, meaning that the composition $p \circ f$ has the form (6). Replacing the function S in Equation (8) by the expression $-\ln\left|p' \circ f\right|$, one obtains

$$\frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial\alpha^{[i]}}\right|\right) = -\frac{\partial \ln\left|p' \circ f\right|}{\partial\alpha^{[j]}},$$
$$\frac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\frac{\partial f}{\partial\beta^{[i]}}\right|\right) = -\frac{\partial \ln\left|p' \circ f\right|}{\partial\alpha^{[j]}}.$$

By the derivative of the logarithmic function, one can rewrite the above equalities as

$$\left(\frac{\partial f}{\partial\alpha^{[i]}}\right)^{-1}\frac{\partial^2 f}{\partial\alpha^{[i]}\partial\alpha^{[j]}} + \frac{1}{p' \circ f}\frac{\partial\left(p' \circ f\right)}{\partial\alpha^{[j]}} = 0,$$
$$\left(\frac{\partial f}{\partial\beta^{[i]}}\right)^{-1}\frac{\partial^2 f}{\partial\alpha^{[i]}\partial\beta^{[j]}} + \frac{1}{p' \circ f}\frac{\partial\left(p' \circ f\right)}{\partial\alpha^{[j]}} = 0.$$

Multiplying the first equality by $(p' \circ f)\left(\partial f/\partial\alpha^{[i]}\right)$ and the second by $(p' \circ f)\left(\partial f/\partial\beta^{[i]}\right)$ yields (11). This completes the proof. □

4. Matrix representation of the conditions

In this section, we represent the conditions (8) and (9) in the matrix form, which makes them easier to check by direct inspection.

Denote by $A^{\alpha,\alpha}$ and $A^{\alpha,\beta}$ the $n \times n$ matrices, whose elements are defined by ($i = 0, \ldots, n-1$ pointing to the row and $j = 0, \ldots, n-1$ to the column)

$$a^{\alpha,\alpha}_{i,j} = \begin{cases} 0, & j = i-N, \ldots, i+N, \text{ or } f \text{ does not depend on } \alpha^{[i]}, \\ \dfrac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\dfrac{\partial f}{\partial\alpha^{[i]}}\right|\right), & \text{otherwise,} \end{cases}$$

and

$$a^{\alpha,\beta}_{i,j} = \begin{cases} 0, & j = i-N, \ldots, i+N, \text{ or } f \text{ does not depend on } \beta^{[i]}, \\ \dfrac{\partial}{\partial\alpha^{[j]}}\left(\ln\left|\dfrac{\partial f}{\partial\beta^{[i]}}\right|\right), & \text{otherwise,} \end{cases}$$

respectively. Thus, the matrices contain zeros on the main diagonal and N diagonals above and below it. Moreover, if for some $i = 0, \ldots, n-1$ the function f does not depend on the variable $y^{[i]}$ or $u^{[i]}$, then the corresponding elements of the matrices are zeros too. Also denote the $2n \times 2n$ matrix as

$$A = \begin{pmatrix} A^{y,y} & A^{u,y} \\ A^{y,u} & A^{u,u} \end{pmatrix}. \tag{13}$$

Proposition 4.1: *If the conditions (8) hold, then in every column of the matrix A all non-zero elements are equal.*

Remark 5: Note that if the function f depends on the variables $y^{[q]}$ or $u^{[q]}$ for all $q = 0, \ldots, n-1$, then $A^{\alpha,\alpha} = A^{\alpha,\beta}$.

If in every column of the matrix A all non-zero elements are equal, one needs to check whether there exists a function S such that for $j = 0, \ldots, n-1$ the non-zero elements of the $(j+1)$th and $(j+1+n)$th columns are equal to $\partial S/\partial y^{[j]}$ and $\partial S/\partial u^{[j]}$, respectively. In the case $2N \geq \overline{j} - \underline{j}$ (where $\overline{j} := \max(\overline{j_y}, \overline{j_u})$ and $\underline{j} := \min(\underline{j_y}, \underline{j_u})$), the matrix A does not contain non-zero elements in the $(\overline{j} - N + 1)$th up to $(\underline{j} + N + 1)$th and $(\overline{j} - N + n + 1)$th up to $(\underline{j} + N + n + 1)$th columns. As a consequence, the conditions for corresponding partial derivatives of S are absent. The additional conditions (9) compensate this aspect. To represent the conditions (9) in the matrix form, denote by $B^{\alpha,\alpha}$ and $B^{\alpha,\beta}$ the $(2N + \underline{j_\alpha} - \overline{j_\alpha} + 1) \times 1$ vectors whose elements are defined by

$$b^{\alpha,\alpha}_r = \frac{\partial S}{\partial\alpha^{[j]}}\frac{\partial f}{\partial\alpha^{[r]}}\left(\frac{\partial f}{\partial\alpha^{[j]}}\right)^{-1}, \quad r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N,$$
$$b^{\alpha,\beta}_r = \frac{\partial S}{\partial\beta^{[j]}}\frac{\partial f}{\partial\alpha^{[r]}}\left(\frac{\partial f}{\partial\beta^{[j]}}\right)^{-1}, \quad r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N,$$

respectively, where j should be chosen according to Remark 3 and $\partial S/\partial\alpha^{[j]}$, $\partial S/\partial\beta^{[j]}$ can be calculated from Equation (8).

Proposition 4.2: *If the conditions (9) hold, then $B^{\alpha,\alpha} = B^{\alpha,\beta}$.*

If $B^{\alpha,\alpha} = B^{\alpha,\beta}$ and the function S satisfying the conditions (8) exists, additionally one needs to check whether S is such that $\partial S/\partial\alpha^{[r]}$ is equal to $b^{\alpha,\alpha}_r$ (and $b^{\alpha,\beta}_r$).

5. Algorithm

Denote the composite functions of $\varphi_l(Y, \ldots, Y^{[-N]}, u, \ldots, u^{[-N]})$ and p as

$$\bar{\varphi}_l\left(y, \ldots, y^{[-N]}, u, \ldots, u^{[-N]}\right) := \varphi_l\left(p(y), \ldots, (p(y))^{[-N]}, u, \ldots, u^{[-N]}\right). \tag{14}$$

To present the algorithm for transformation the system (1) into the extended observer form (4), we need the following proposition, the proof of which is given in Appendix.

Proposition 5.1: *If the i/o Equation (5) can be rewritten in the form (6), then*

$$\left(p' \circ f\right)\frac{\partial f}{\partial\alpha^{[i]}} = \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \frac{\partial\bar{\varphi}_{n-N-l}\left(y^{[l+N]}, \ldots, y^{[l]}, u^{[l+N]}, \ldots, u^{[l]}\right)}{\partial\alpha^{[i]}} \tag{15}$$

for $i = 0, \ldots, n-1$.

The algorithm is applied to the i/o representation (5) of the system (1).

Algorithm 1.

Step 1: Check for every column of the matrix A whether the non-zero elements are equal. For $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ also check whether $B^{\alpha,\alpha} = B^{\alpha,\beta}$ (if both matrices can be constructed, see Remark 3). If the above conditions are not satisfied, the problem is not solvable; stop.

Step 2: From Equations (8) and (9) construct the partial differential equation and solve it with respect to S. If the solution does not exist, the problem is not solvable; stop.

Step 3: Using Equation (12), find $p' \circ f = \mathrm{e}^{-S}$. Then, to find the replacement rule $y^{[-1]} = \phi(\cdot)$, shift backwards both sides of the i/o equation (5) a sufficient number of times until the variable $y^{[-1]}$ will appear and solve the obtained equation with respect to $y^{[-1]}$ (If the function f in Equation (5) does not depend on y and its time shifts, then the replacement rule should be found for $u^{[-1]}$ in the similar manner). Next, shift backwards $p' \circ f$ and apply the replacement rule to the obtained expression. Iterating this procedure $n-1$ times, we obtain $p' \circ y$, from which the output transformation can be computed as $Y = p \circ y = \int (p' \circ y)\,\mathrm{d}y$.

Step 4: Solve, if possible, the system of partial differential equations (15) to find the functions $\bar{\varphi}_1, \ldots, \bar{\varphi}_{n-N}$, from which the functions $\varphi_1, \ldots, \varphi_{n-N}$ can be obtained applying the output transformation.

Step 5: Using the output transformation (3) and the functions $\varphi_1, \ldots, \varphi_{n-N}$, construct the system in the extended observer form (4).

6. Example

To illustrate the above theory, examine the following example:

$$\begin{aligned}
x_1^{[1]} &= x_1 + x_2 - x_3 \\
x_2^{[1]} &= -x_1 - x_2 \\
x_3^{[1]} &= -\frac{x_1 x_2}{u x_3 + x_1 x_2 x_4} \\
x_4^{[1]} &= -\frac{u\,(x_2)^2}{(x_1+x_2)(x_1+x_2-x_3)} - \frac{x_5}{u} \\
x_5^{[1]} &= \frac{x_2 - u\,(x_1+x_2)}{x_3} \\
y &= x_2 .
\end{aligned} \tag{16}$$

The i/o equation, corresponding to Equation (16), is

$$y^{[5]} = \frac{u^{[1]} y^{[2]} \left(y^{[2]} + y^{[3]}\right)}{\lambda}, \tag{17}$$

where to simplify the exposition we denoted $\lambda := (u^{[1]})^2 (y^{[1]})^2 + (y + u y^{[1]})(y^{[2]} + y^{[3]}) + u^{[1]} u^{[2]} y^{[4]}$. Note that we are interested in the least buffer N, for which the system (16) is transformable into the extended observer form (4). Constructing the matrix A for $N = 0$ and $N = 1$, one can verify that in both cases the non-zero elements of every column of the matrix are not equal. First, this means that the system (16) is not transformable into the standard observer form (this fact can also be checked by means of the conditions presented in Mullari and Kotta (2009)). Second, the system is not transformable into the extended observer form with buffer $N = 1$. Next, take $N = 2$ and follow algorithm.

Step 1. Using Equation (13), one obtains

$$A = \begin{pmatrix}
0 & 0 & 0 & sy_3 & sy_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & sy_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
sy_0 & 0 & 0 & 0 & 0 & su_0 & 0 & 0 & 0 & 0 \\
sy_0 & sy_1 & 0 & 0 & 0 & su_0 & su_1 & 0 & 0 & 0 \\
0 & 0 & 0 & sy_3 & sy_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & sy_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

where we use the notations

$$\begin{aligned}
sy_0 &:= -\frac{2(y^{[2]} + y^{[3]})}{\lambda}, \\
sy_1 &:= -\frac{2\left(2\left(u^{[1]}\right)^2 y^{[1]} + u\left(y^{[2]} + y^{[3]}\right)\right)}{\lambda}, \\
sy_3 &:= \frac{2u^{[1]}\left(u^{[1]}\left(y^{[1]}\right)^2 + u^{[2]} y^{[4]}\right)}{\left(y^{[2]} + y^{[3]}\right)\lambda}, \\
sy_4 &:= -\frac{2u^{[1]} u^{[2]}}{\lambda}, \\
su_0 &:= -\frac{2y^{[1]}\left(y^{[2]} + y^{[3]}\right)}{\lambda}, \\
su_1 &:= -\frac{2\left(\left(y + u y^{[1]}\right)\left(y^{[2]} + y^{[3]}\right) - \left(u^{[1]}\right)^2 \left(y^{[1]}\right)^2\right)}{u^{[1]}\lambda}.
\end{aligned}$$

Since $\underline{j_y} = 0$, $\overline{j_y} = 4$, $\underline{j_u} = 0$, $\overline{j_u} = 2$ and $N = 2$, both inequalities $2N \geq \overline{j_y} - \underline{j_y}$ and $2N \geq \overline{j_u} - \underline{j_u}$ are satisfied and, as a consequence, one has to check the additional conditions. Choosing j according to Remark 3, one obtains the following matrices:

$$B^{y,y} = B^{y,u} = (sy_2), \quad B^{u,u} = B^{u,y} = \begin{pmatrix} su_0 \\ su_1 \\ su_2 \end{pmatrix}, \quad (18)$$

where

$$sy_2 := 2\left(\frac{1}{y^{[2]} + y^{[3]}} + \frac{1}{y^{[2]}} - \frac{y + uy^{[1]}}{\lambda}\right),$$
$$su_2 := -\frac{2u^{[1]}y^{[4]}}{\lambda}.$$

Taking into consideration Equation (18) and the fact that all the non-zero elements of every column of the matrix A are equal, one may conclude that the necessary conditions for transformation of the system (16) into the extended observer form with buffer $N = 2$ are satisfied.

Step 2. The differential equation

$$\sum_{j=0}^{4} \frac{\partial S}{\partial y^{[j]}} + \sum_{j=0}^{2} \frac{\partial S}{\partial u^{[j]}} = \sum_{j=0}^{4} sy_j + \sum_{j=0}^{2} su_j$$

yields

$$S = 2\left(\ln u^{[1]} + \ln y^{[2]} + \ln\left(y^{[2]} + y^{[3]}\right) - \ln\lambda\right). \quad (19)$$

Step 3. Using Equation (19), compute

$$p' \circ f = \mathrm{e}^{-S} = \frac{\lambda^2}{(u^{[1]})^2(y^{[2]})^2(y^{[2]} + y^{[3]})^2}.$$

Shifting both sides of the i/o equation (17) backwards the following replacement rule can be obtained for $y^{[-1]}$:

$$y^{[-1]} = \frac{uy^{[1]}}{y^{[4]}} - \frac{u^2y^2 + uu^{[1]}y^{[3]}}{y^{[1]} + y^{[2]}} - u^{[-1]}y,$$

applying which to $(p' \circ f)^{[-1]}$ one obtains

$$(p' \circ f)^{[-1]} = \frac{1}{(y^{[4]})^2}. \quad (20)$$

Shifting the equality (20) backwards four times leads to $p' \circ y = 1/y^2$ yielding the output transformation,

$$Y = p \circ y = \int (p' \circ y)\mathrm{d}y = -\frac{1}{y}. \quad (21)$$

Step 4. The system of partial differential equations (15) for $n = 5$, $N = 2$ and α being both y and u reads as

$$-\frac{1}{u^{[1]}y^{[2]}} = \frac{\partial\bar{\varphi}_3}{\partial y}$$

$$-\frac{2\left(u^{[1]}\right)^2 y^{[1]} + u\left(y^{[2]} + y^{[3]}\right)}{u^{[1]}y^{[2]}\left(y^{[2]} + y^{[3]}\right)} = \frac{\partial\bar{\varphi}_3}{\partial y^{[1]}} + \frac{\partial\bar{\varphi}_2}{\partial y^{[1]}}$$

$$\frac{y + uy^{[1]}}{u^{[1]}\left(y^{[2]}\right)^2} + \frac{\left(2y^{[2]} + y^{[3]}\right)\left(u^{[1]}\left(y^{[1]}\right)^2 + u^{[2]}y^{[4]}\right)}{\left(y^{[2]}\right)^2\left(y^{[2]} + y^{[3]}\right)^2} = \frac{\partial\bar{\varphi}_3}{\partial y^{[2]}} + \frac{\partial\bar{\varphi}_2}{\partial y^{[2]}} + \frac{\partial\bar{\varphi}_1}{\partial y^{[2]}}$$

$$\frac{u^{[1]}\left(y^{[1]}\right)^2 + u^{[2]}y^{[4]}}{y^{[2]}\left(y^{[2]} + y^{[3]}\right)^2} = \frac{\partial\bar{\varphi}_2}{\partial y^{[3]}} + \frac{\partial\bar{\varphi}_1}{\partial y^{[3]}}$$

$$-\frac{u^{[2]}}{y^{[2]}\left(y^{[2]} + y^{[3]}\right)} = \frac{\partial\bar{\varphi}_1}{\partial y^{[4]}}$$

$$-\frac{y^{[1]}}{u^{[1]}y^{[2]}} = \frac{\partial\bar{\varphi}_3}{\partial u}$$

$$\frac{y + uy^{[1]}}{\left(u^{[1]}\right)^2 y^{[2]}} - \frac{\left(y^{[1]}\right)^2}{y^{[2]}\left(y^{[2]} + y^{[3]}\right)} = \frac{\partial\bar{\varphi}_3}{\partial u^{[1]}} + \frac{\partial\bar{\varphi}_2}{\partial u^{[1]}}$$

$$-\frac{y^{[4]}}{y^{[2]}\left(y^{[2]} + y^{[3]}\right)} = \frac{\partial\bar{\varphi}_3}{\partial u^{[2]}} + \frac{\partial\bar{\varphi}_2}{\partial u^{[2]}} + \frac{\partial\bar{\varphi}_1}{\partial u^{[2]}},$$

leading to

$$\bar{\varphi}_1 = -\frac{y^{[4]}u^{[2]}}{\left(y^{[3]} + y^{[2]}\right)y^{[2]}}, \quad \bar{\varphi}_2 = -\frac{\left(y^{[1]}\right)^2 u^{[1]}}{\left(y^{[3]} + y^{[2]}\right)y^{[2]}},$$
$$\bar{\varphi}_3 = -\frac{y + y^{[1]}u}{y^{[2]}u^{[1]}},$$

which, due to the output transformation (21), yields

$$\varphi_1 = \frac{\left(Y^{[2]}\right)^2 Y^{[3]}u^{[2]}}{\left(Y^{[2]} + Y^{[3]}\right)Y^{[4]}}, \quad \varphi_2 = -\frac{\left(Y^{[2]}\right)^2 Y^{[3]}u^{[1]}}{\left(Y^{[1]}\right)^2\left(Y^{[2]} + Y^{[3]}\right)},$$
$$\varphi_3 = -\frac{\left(Yu + Y^{[1]}\right)Y^{[2]}}{YY^{[1]}u^{[1]}}.$$

Step 5. Using Equation (7), one can define the new state variables

$$z_1 = Y,$$
$$z_2 = Y^{[1]} - \frac{\left(Y^{[-2]}\right)^2 Y^{[-1]}u^{[-2]}}{\left(Y^{[-2]} + Y^{[-1]}\right)Y},$$
$$z_3 = Y^{[2]} - \frac{\left(Y^{[-1]}\right)^2 Yu^{[-1]}}{\left(Y^{[-1]} + Y\right)Y^{[1]}} + \frac{\left(Y^{[-1]}\right)^2 Yu^{[-2]}}{\left(Y^{[-2]}\right)^2\left(Y^{[-1]} + Y\right)},$$
$$z_4 = Y^{[3]} - \frac{(Y)^2 Y^{[1]}u}{\left(Y + Y^{[1]}\right)Y^{[2]}} + \frac{(Y)^2 Y^{[1]}u^{[-1]}}{\left(Y^{[-1]}\right)^2\left(Y + Y^{[1]}\right)}$$
$$+\frac{\left(Y^{[-2]}u^{[-2]} + Y^{[-1]}\right)Y}{Y^{[-2]}Y^{[-1]}u^{[-1]}},$$

$$z_5 = Y^{[4]} - \frac{\left(Y^{[1]}\right)^2 Y^{[2]} u^{[1]}}{\left(Y^{[1]} + Y^{[2]}\right) Y^{[3]}} + \frac{\left(Y^{[1]}\right)^2 Y^{[2]} u}{(Y)^2 \left(Y^{[1]} + Y^{[2]}\right)} + \frac{\left(Y^{[-1]} u^{[-1]} + Y\right) Y^{[1]}}{Y^{[-1]} Y u},$$

which, due to the output transformation (21) and state equations (16), can be rewritten as

$$\begin{aligned}
z_1 &= -\frac{1}{x_2},\\
z_2 &= \frac{x_2 u^{[-2]}}{\left(x_2^{[-2]}\right)^2 + x_2^{[-2]} x_2^{[-1]}} + \frac{1}{x_1 + x_2},\\
z_3 &= -\frac{1}{x_3} + \frac{\left(x_2^{[-2]}\right)^2 u^{[-2]} - u^{[-1]}(x_1 + x_2)}{x_2^{[-1]}\left(x_2^{[-1]} + x_2\right)},\\
z_4 &= x_4 - \frac{\left(x_2^{[-1]}\right)^2 u^{[-1]}}{x_1 x_2} + \frac{x_2^{[-2]} + x_2^{[-1]} u^{[-2]}}{x_2 u^{[-1]}},\\
z_5 &= -\frac{x_2^{[-1]} - x_2 u^{[-1]} - x_1 x_5 - x_2 x_5}{u(x_1 + x_2)},
\end{aligned}$$

that leads to the state equations in the extended observer form

$$\begin{aligned}
z_1^{[1]} &= z_2 + \frac{\left(z_1^{[-2]}\right)^2 z_1^{[-1]} u^{[-2]}}{\left(z_1^{[-2]} + z_1^{[-1]}\right) z_1}\\
z_2^{[1]} &= z_3 - \frac{\left(z_1^{[-1]}\right)^2 z_1 u^{[-2]}}{\left(z_1^{[-2]}\right)^2 \left(z_1^{[-1]} + z_1\right)}\\
z_3^{[1]} &= z_4 - \frac{z_1\left(z_1^{[-1]} + z_1^{[-2]} u^{[-2]}\right)}{z_1^{[-1]} z_1^{[-2]} u^{[-1]}}\\
z_4^{[1]} &= z_5\\
z_5^{[1]} &= 0\\
Y &= z_1.
\end{aligned}$$

7. Conclusions

The paper provides simple necessary and sufficient conditions for the existence of the extended coordinate change and the output transformation that allow to transform the discrete-time single-input single-output nonlinear state equations into the extended observer form with buffer. The conditions are expressed in terms of certain partial derivatives and due to the matrix representation can be checked almost by direct inspection. Moreover, the matrix representation simplifies the determination of the minimal value of the buffer (i.e. the minimal number of past values of the input and output), necessary for transformation. The algorithm is also given for transformation of the state equations into the extended observer form that requires the solution of certain partial differential equations for the computation of the output transformation and the extended coordinate change. Although the proposed approach is beneficial for discrete-time systems, it cannot be applied in the case of continuous-time systems, due to the structural differences between the shift operator and the time derivative.

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Note

1. Without going into details, one can say that in order to prove sufficiency we need $\frac{\partial S}{\partial \alpha^{[j]}}$ for all $j = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. However, we should take into account that in conditions (8) index j depends on index i and buffer N. This dependency implies that $j = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1, \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}$, which in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ yields that j runs from $\underline{j_\alpha}$ to $\overline{j_\alpha}$ without interruption, whereas in the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ there is a gap between $\overline{j_\alpha} - N - 1$ and $\underline{j_\alpha} + N + 1$. To compensate this gap we use Equation (9) in addition to Equation (8) (in other words, index r complements j).

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Appendix 1: Proof of Lemma 3.2

The proof of lemma, though in principle not very difficult, is technically rather demanding. Figure A1 helps to follow the separate steps of the proof. First, let us mention that the cases $2N < \overline{j_\alpha} - \underline{j_\alpha}$ and $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ are treated separately. The reason is that conditions (9) are unnecessary for the first case.

Second, in the proof we will use the matrices (tables) with elements $\Theta_{k,l}$ (k pointing to the column and l to the row), where k and l may take values from different sets of non-negative integer numbers at different steps of the proof. However, unlike the typical case when the matrix element is a number or expression, here its content is two relations (equalities). We do not manipulate with those relations, the role of the matrix is just to keep the track of the steps in the proof.

Evaluating the total differentials $\mathrm{d}S$ and $\mathrm{d}f$ as well as their wedge product $\mathrm{d}S \wedge \mathrm{d}f$ in Equation (10), it is easy to observe by direct inspection, after tedious calculations, that the condition (10) is equivalent to the equalities (A1),

$$\begin{aligned} \frac{\partial S}{\partial \alpha^{[k]}} \frac{\partial f}{\partial \alpha^{[l]}} &= \frac{\partial S}{\partial \alpha^{[l]}} \frac{\partial f}{\partial \alpha^{[k]}}, \\ \frac{\partial S}{\partial \alpha^{[k]}} \frac{\partial f}{\partial \beta^{[l]}} &= \frac{\partial S}{\partial \beta^{[l]}} \frac{\partial f}{\partial \alpha^{[k]}}, \end{aligned} \tag{A1}$$

where $k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. Recall that here, like in the assumptions (8) and (9) of the lemma, the variable α denotes either input u or output y, and by β is denoted the other variable; i.e. if $\alpha = y$, then $\beta = u$ and vice versa, if $\alpha = u$, then $\beta = y$. These notations help to make the presentation more compact[2]. Now, the content of $\Theta_{k,l}$ is two respective equalities in Equation (A1).

Before turning to separate steps of the proof, we rewrite the assumptions (8) and (9) into the form, suitable for proof. Namely, the conditions (8) may be given as Equation (A2) by evaluation of the derivative of the logarithmic function and rewriting the conditions separately for α alone as well as for α and β,

$$\frac{\partial S}{\partial \alpha^{[j]}} = \left(\frac{\partial f}{\partial \alpha^{[i]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[j]}}, \tag{A2a}$$

$$\frac{\partial S}{\partial \alpha^{[j]}} = \left(\frac{\partial f}{\partial \beta^{[i]}} \right)^{-1} \frac{\partial^2 f}{\partial \beta^{[i]} \partial \alpha^{[j]}}, \tag{A2b}$$

where $i, j = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$, $j \neq i - N, \ldots, i + N$. Taking into account that α and β can be mutually interchanged, rewrite the conditions (9) as

$$\frac{\partial S}{\partial \alpha^{[j]}} = \frac{\partial S}{\partial \alpha^{[r]}} \frac{\partial f}{\partial \alpha^{[j]}} \left(\frac{\partial f}{\partial \alpha^{[r]}} \right)^{-1}, \tag{A3a}$$

$$\frac{\partial S}{\partial \alpha^{[j]}} = \frac{\partial S}{\partial \beta^{[r]}} \frac{\partial f}{\partial \alpha^{[j]}} \left(\frac{\partial f}{\partial \beta^{[r]}} \right)^{-1}, \tag{A3b}$$

where $r = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$.

Now we turn to the separate steps of the proof. The separate steps (i)–(ix) prove the relations in $\Theta_{k,l}$ for different sets of k and l values so that jointly the steps cover all necessary k, l values in Equation (A1). In steps (i)–(iv), we will focus on the conditions (8) and prove that in the case $2N < \overline{j_\alpha} - \underline{j_\alpha}$ they yield $\Theta_{k,l}$ for $k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. In steps (v)–(ix), we will prove that using the conditions (9), the outcome of the previous four steps can be complemented to obtain the same result for the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$ (see the boxes on the middle part of the Figure A1).

(i) Consider first (A2) for $j = \overline{j_\alpha}$. Since $j \neq i - N, \ldots, i + N$, now $i \neq \overline{j_\alpha} - N, \ldots, \overline{j_\alpha} + N$ and consequently $i \leq \overline{j_\alpha} - N - 1$. In Equation (A2a), denote index i by index k and compare successively the obtained equality first, with Equation (A2a) and second with Equation (A2b), where in both equalities index i is replaced by index l. This yields

$$\begin{aligned} \left(\frac{\partial f}{\partial \alpha^{[k]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[k]} \partial \alpha^{[\overline{j_\alpha}]}} &= \left(\frac{\partial f}{\partial \alpha^{[l]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[l]} \partial \alpha^{[\overline{j_\alpha}]}}, \\ \left(\frac{\partial f}{\partial \alpha^{[k]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[k]} \partial \alpha^{[\overline{j_\alpha}]}} &= \left(\frac{\partial f}{\partial \beta^{[l]}} \right)^{-1} \frac{\partial^2 f}{\partial \beta^{[l]} \partial \alpha^{[\overline{j_\alpha}]}}. \end{aligned}$$

Divide both sides of both obtained equalities by $(\partial f / \partial \alpha^{[\overline{j_\alpha}]})$ to get

$$\begin{aligned} \left(\frac{\partial f}{\partial \alpha^{[\overline{j_\alpha}]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[k]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \alpha^{[l]}} &= \left(\frac{\partial f}{\partial \alpha^{[\overline{j_\alpha}]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[l]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \alpha^{[k]}}, \\ \left(\frac{\partial f}{\partial \alpha^{[\overline{j_\alpha}]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[k]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \beta^{[l]}} &= \left(\frac{\partial f}{\partial \alpha^{[\overline{j_\alpha}]}} \right)^{-1} \frac{\partial^2 f}{\partial \beta^{[l]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \alpha^{[k]}}. \end{aligned}$$

Take the conditions (A2) for $i = \overline{j_\alpha}$ and in Equation (A2b) interchange variables α and β mutually, which is eligible by the definition of α and β. In this case, $j \leq \overline{j_\alpha} - N - 1$. Since j changes in the same range as indices k and l, one can apply Equation (A2a) for $j := k$ to the left-hand sides of the above equalities, as well as Equations (A2a) and (A2b) for $j := l$ to the right-hand sides of the first and second above equalities, respectively. This yields

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1. \tag{A4}$$

(ii) Using Equation (A2a), rewrite the elements of Equation (A4) for $k = \underline{j_\alpha}$ as follows:

$$\begin{aligned} \left(\frac{\partial f}{\partial \alpha^{[i]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[\underline{j_\alpha}]}} \frac{\partial f}{\partial \alpha^{[l]}} &= \frac{\partial S}{\partial \alpha^{[l]}} \left(\frac{\partial S}{\partial \alpha^{[j]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[\underline{j_\alpha}]} \partial \alpha^{[j]}}, \\ \left(\frac{\partial f}{\partial \alpha^{[i]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[\underline{j_\alpha}]}} \frac{\partial f}{\partial \beta^{[l]}} &= \frac{\partial S}{\partial \beta^{[l]}} \left(\frac{\partial S}{\partial \alpha^{[j]}} \right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[\underline{j_\alpha}]} \partial \alpha^{[j]}}, \end{aligned}$$

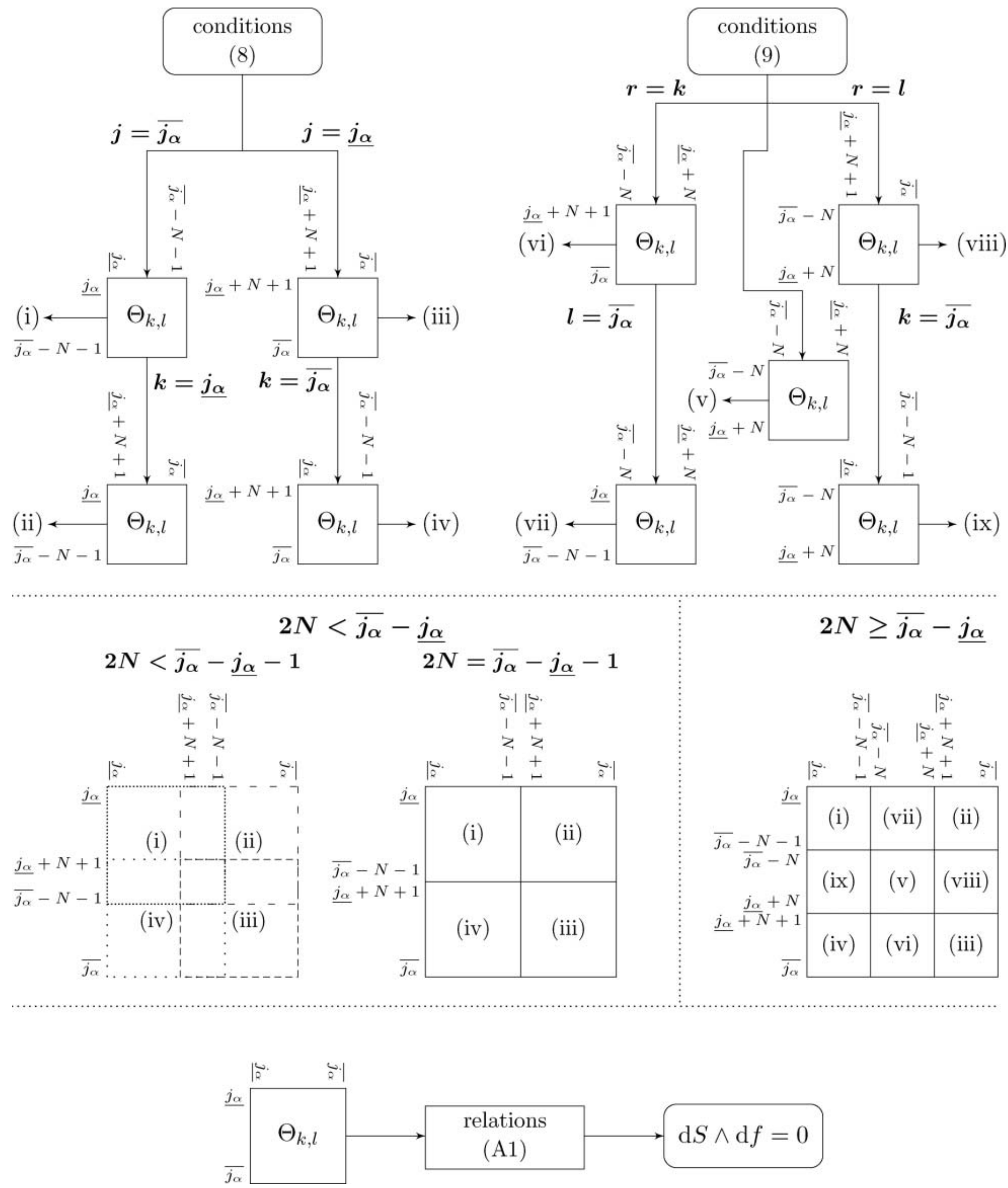

Figure A1. The main steps of the proof of Lemma 3.2.

where $i, j = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}$ and $l = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1$. Denoting $k := i = j$, after simplification, we obtain

$$\Theta_{k,l}, \quad \begin{aligned} k &= \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}, \\ l &= \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1. \end{aligned} \tag{A5}$$

(iii) Next, consider Equation (A2) for $j = \underline{j_\alpha}$. Since $j \neq i - N, \ldots, i + N$, now $i \neq \underline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N$ and consequently $i \geq \underline{j_\alpha} + N + 1$. Performing the analogical steps as in step (i), we obtain

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}. \tag{A6}$$

(iv) Using Equation (A2a), rewrite the elements of Equation (A6) for $k = \overline{j_\alpha}$ as follows:

$$\left(\frac{\partial f}{\partial \alpha^{[i]}}\right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \alpha^{[l]}} = \frac{\partial S}{\partial \alpha^{[l]}} \left(\frac{\partial S}{\partial \alpha^{[j]}}\right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[\overline{j_\alpha}]} \partial \alpha^{[j]}},$$

$$\left(\frac{\partial f}{\partial \alpha^{[i]}}\right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[i]} \partial \alpha^{[\overline{j_\alpha}]}} \frac{\partial f}{\partial \beta^{[l]}} = \frac{\partial S}{\partial \beta^{[l]}} \left(\frac{\partial S}{\partial \alpha^{[j]}}\right)^{-1} \frac{\partial^2 f}{\partial \alpha^{[\overline{j_\alpha}]} \partial \alpha^{[j]}},$$

where $i, j = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1$ and $l = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}$. Denotation $k := i = j$ and simplification yield

$$\Theta_{k,l}, \quad \begin{array}{l} k = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1, \\ l = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}. \end{array} \tag{A7}$$

It is not hard to verify (see Figure A1) that joining together tables (A4)–(A7) yields

$$\Theta_{k,l}, \quad \begin{cases} k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}, & \text{if } 2N < \overline{j_\alpha} - \underline{j_\alpha}, \\ k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1, & \\ \quad \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}, & \text{if } 2N \geq \overline{j_\alpha} - \underline{j_\alpha}. \end{cases} \tag{A8}$$

(v) Consider the case $2N \geq \overline{j_\alpha} - \underline{j_\alpha}$. In Equation (A3a) replace index r by index k and compare successively the obtained equality first, with Equation (A3a) and second with Equation (A3b), where in both equalities index r is replaced by index l. After simplification, we obtain

$$\Theta_{k,l}, \quad k, l = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N. \tag{A9}$$

(vi) Next take Equation (A3) for $r = k$, $j = \underline{j_\alpha}$ and perform the similar operations as in step (ii) to get

$$\Theta_{k,l}, \quad \begin{array}{l} k = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N, \\ l = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}. \end{array} \tag{A10}$$

(vii) Taking the elements of (A10) for $l = \overline{j_\alpha}$ by analogy with step (iv), one obtains

$$\Theta_{k,l}, \quad \begin{array}{l} k = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N, \\ l = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1. \end{array} \tag{A11}$$

(viii) Next, take Equation (A3) for $r = l$ and perform the similar operations as in step (ii) to get

$$\Theta_{k,l}, \quad \begin{array}{l} k = \underline{j_\alpha} + N + 1, \ldots, \overline{j_\alpha}, \\ l = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N. \end{array} \tag{A12}$$

(ix) Taking the elements of Equation (A12) for $k = \overline{j_\alpha}$ by analogy with step (iv), one obtains

$$\Theta_{k,l}, \quad \begin{array}{l} k = \underline{j_\alpha}, \ldots, \overline{j_\alpha} - N - 1, \\ l = \overline{j_\alpha} - N, \ldots, \underline{j_\alpha} + N. \end{array} \tag{A13}$$

As a result, complementary tables (A9)–(A13) allow to rewrite Equation (A8) as

$$\Theta_{k,l}, \quad k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$$

for arbitrary N, which means that under conditions (8) and (9) the equalities (A1) are satisfied for $k, l = \underline{j_\alpha}, \ldots, \overline{j_\alpha}$. This completes the proof. □

Appendix 2: Proof of Proposition 5.1

To make the exposition of the proof more compact, define for $i = 0, \ldots, n - 1$ the differential one-forms

$$\omega_i = \frac{\partial f}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial f}{\partial u^{[i]}} \mathrm{d}u^{[i]}, \tag{B1}$$

and the vector argument

$$\nu_l := \left[y^{[n-l]}, \ldots, y^{[n-l-N]}, u^{[n-l]}, \ldots, u^{[n-l-N]}\right] \tag{B2}$$

for $l = 1, \ldots, n - N$. The proof of Proposition 5.1 relies on the following lemma.

Lemma A.1 (Kaparin and Kotta, 2011): *For functions $\bar{\varphi}_1(\nu_1), \ldots, \bar{\varphi}_{n-N}(\nu_{n-N})$ the following holds*

$$\sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l) = \sum_{i=0}^{n-1} \Upsilon_i \left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right),$$

where

$$\Upsilon_i \left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right) = \sum_{l=\max(0,i-N)}^{\min(i,n-1-N)} \left(\frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \frac{\partial \bar{\varphi}_{n-N-l}(\nu_{n-N-l})}{\partial u^{[i]}} \mathrm{d}u^{[i]} \right). \tag{A16}$$

Now we are ready to prove Proposition 5.1.

Proof: The total differential of Equation (6) reads as

$$(p' \circ f)\mathrm{d}f = (p' \circ f) \sum_{i=0}^{n-1} \omega_i = \sum_{l=1}^{n-N} \mathrm{d}\bar{\varphi}_l(\nu_l),$$

which, according to Lemma B.1, can be rewritten as

$$(p' \circ f) \sum_{i=0}^{n-1} \omega_i = \sum_{i=0}^{n-1} \Upsilon_i \left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right). \tag{A17}$$

From Equation (A17) we have for $i = 0, \ldots, n - 1$,

$$(p' \circ f)\omega_i = \Upsilon_i \left(\bar{\varphi}_{n-N-l}(\nu_{n-N-l})\right). \tag{A18}$$

Taking into account Equations (A14)–(A16), compare the coefficients of $\mathrm{d}y^{[i]}$ and $\mathrm{d}u^{[i]}$ at both sides of equality (A18), to obtain Equation (A15). This completes the proof. □

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Observable Space of Nonlinear Control System on Homogeneous Time Scale

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Abstract

The observability property of the nonlinear system, defined on homogeneous time scale, is studied in the paper. The observability condition is provided through the notion of the observable space. Moreover, the observability filtration and observability indices are defined and the decomposition of the system into the observable/unobservable subsystems is considered.
Keywords: nonlinear control system, time scale, observability, observable space

1 INTRODUCTION

The theory of dynamical systems on time scales is a new and popular research area. From a modeling point of view, dynamical systems on time scales incorporate both the continuous- and discrete-time systems as the special cases, allowing that way to unify the study and consider the classical results as the special cases from the new theory. However, it is important to note that the discrete-time model in the time scale formalism is given in terms of the difference operator, and not in terms of the more conventional shift operator as, for example, in [1], [2], [3], [13]. The difference-based models, often referred to as delta-domain models, are not completely new for description of the discrete-time systems. They have been promoted during the last 20 years as the models closely linked to the continuous-time systems, being less sensitive to round-off errors at higher sampling rates [12], [20].

The properties (including observability) of linear systems, defined on time scales, were studied, for instance, in [5] and [11]. In [4] the algebraic formalism in terms of differential one-forms has been developed for the study of nonlinear control systems defined on homogeneous time scales and used later to study different problems like transfer equivalence, irreducibility, reduction and realization of nonlinear input-output equations [7], [17], [18]. The formalism constructs the vector space of differential one-forms, defined over the differential field of meromorphic functions, associated with the control system.

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In this paper we apply this formalism to define and construct the observable space for nonlinear control system on homogeneous time scale and define the observability indices of the system. Moreover, we provide the necessary and sufficient condition to check the single-experiment observability[1] of the system using the notion of the observable space. Finally, we discuss the possibility to decompose the system into observable/unobservable subsystems.

The paper is organized as follows. The preliminary information about the time scale calculus and algebraic framework is given in Section 2. The notions of observability, observability filtration, observable space and observability indices are provided in Section 3. In Section 4 the decomposition of the system into the observable/unobservable subsystems is studied. Section 5 provides the brief conclusions.

2 PRELIMINARIES

2.1 Time Scale Calculus

For a general introduction to the time scale calculus, see [6]. Here we recall only those notions and facts that we need in this paper, in particular, the concept of delta derivative for real function defined on homogeneous time scale.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise the continuous time case, $\mathbb{T} = \mathbb{R}$, and the discrete time cases, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \tau\mathbb{Z}$ for $\tau > 0$. We assume that $\mathbb{T}$ is a topological space with the topology induced by $\mathbb{R}$. In the definition of the delta derivative, the so-called forward jump operator plays an important role. For $t \in \mathbb{T}$ the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} \mid \ s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup \{s \in \mathbb{T} \mid \ s < t\}.$$

In this definition we set in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$. Obviously $\sigma(t)$ is in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. The *graininess functions* $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, respectively. A time scale $\mathbb{T}$ is called *homogeneous*[2] if $\mu = \nu \equiv \mathrm{const}$. Let $\mathbb{T}^\kappa$ denote truncated set consisting of $\mathbb{T}$ except for a possible maximal point such that $\rho(\max \mathbb{T}) < \max \mathbb{T}$.

Definition 2.1. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. *Delta derivative* of f at t, denoted by $f^\Delta(t)$, is the real number (provided it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

[1]The multi-experiment observability of nonlinear control systems, defined on time scales, was studied in [22].

[2]Though the closed interval $[a, b]$ is also an example of homogeneous time scale, we restrict our consideration to infinite homogeneous time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \tau\mathbb{Z}$ for $\tau > 0$.

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Example 2.2.

- If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ and delta derivative is the ordinary time derivative.
- If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mu(t) = \tau$ and $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)} = \frac{f(t+\tau)-f(t)}{\tau}$ is the difference operator.

For a function $f : \mathbb{T} \to \mathbb{R}$ one can define the 2nd delta derivative $f^{[2]} := \left(f^\Delta\right)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$ provided that f^Δ is delta differentiable on $\mathbb{T}^{\kappa^2} := \left(\mathbb{T}^\kappa\right)^\kappa$. In a similar manner one defines higher order delta derivatives $f^{[n]} := \left(f^{[n-1]}\right)^\Delta : \mathbb{T}^{\kappa^n} \to \mathbb{R}$, where $\mathbb{T}^{\kappa^n} = \left(\mathbb{T}^{\kappa^{n-1}}\right)^\kappa$, $n \geq 1$. For notational convenience, denote $f^{[i\dots n]} := \left(f^{[i]}, \dots, f^{[n]}\right)$, for $0 \leq i \leq n$ and $f^{[0]} := f$.

2.2 Algebraic Framework

In this subsection we recall some notions and facts from [4], necessary for our study.

Consider a multi-input multi-output (MIMO) nonlinear control system, defined on homogeneous time scale $\mathbb{T}$, and described by the state equations

$$\begin{aligned} x^\Delta &= f(x,u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x(t) : \mathbb{T} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u(t) : \mathbb{T} \to \mathbb{U} \subset \mathbb{R}^m$ is an m-dimensional input vector and $y(t) : \mathbb{T} \to \mathbb{Y} \subset \mathbb{R}^p$ is a p-dimensional output vector. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \to \mathbb{Y}$ are assumed to be real analytic functions.

Remark 2.3. Note that we are focusing neither on local nor global, but on the generic properties of the system, i.e. the properties that hold almost everywhere, except on a set of measure zero. Though the notion of generic property does not make sense, in general, for systems defined by C^∞ functions, the choice of analytic functions allows to employ the generic approach. Moreover, unlike the ring of C^∞ functions, the ring of analytic functions is integral domain, meaning that it can be embedded into its quotient field whose elements are meromorphic functions.

Assume that the map $(x,u) \mapsto \widetilde{f}(x,u) := x + \mu f(x,u)$ generically defines a submersion, that is generically

$$\operatorname{rank} \frac{\partial \widetilde{f}(x,u)}{\partial (x,u)} = n \tag{2}$$

holds. Assumption (2) is not restrictive since it is a necessary condition for system accessibility [13] and always satisfied in case $\mu \equiv 0$. Consider the infinite set of (independent) real indeterminates $\mathscr{C} := \left\{x_i,\, i = 1, \dots, n;\, u_\upsilon^{[k]},\, \upsilon = 1, \dots, m,\, k \geq 0\right\}$. Let $\mathscr{K}$ denote the field of meromorphic functions in a finite number of variables from the set $\mathscr{C}$. Thus for each $F \in \mathscr{K}$ there is $k \geq 0$ such that F depends on x and $u^{[0\ldots k]}$. Let $\sigma_f : \mathscr{K} \to \mathscr{K}$ be the *forward shift operator* defined by

$$F^{\sigma_f}\left(x, u^{[0\ldots k+1]}\right) := F\left(x + \mu f(x,u), u^{[0\ldots k]} + \mu u^{[1\ldots k+1]}\right).$$

Under the submersivity assumption, σ_f is injective endomorphism and so the operator σ_f is well defined on $\mathscr{K}$ (see [4]). Furthermore, define the operator $\Delta_f : \mathscr{K} \to \mathscr{K}$ by

$$\begin{aligned} F^{\Delta_f}&\left(x, u^{[0\ldots k+1]}\right) := \\ &= \begin{cases} \dfrac{F^{\sigma_f}\left(x, u^{[0\ldots k+1]}\right) - F\left(x, u^{[0\ldots k]}\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ \dfrac{\partial F}{\partial x}\left(x, u^{[0\ldots k]}\right) f(x,u) + \displaystyle\sum_{k \geq 0} \dfrac{\partial F}{\partial u^{[0\ldots k]}}\left(x, u^{[0\ldots k]}\right) u^{[1\ldots k+1]} & \text{if } \mathbb{T} = \mathbb{R}. \end{cases} \end{aligned}$$

Proposition 2.4. *Let $F : \mathscr{K} \to \mathscr{K}$, $G : \mathscr{K} \to \mathscr{K}$. The delta derivative satisfies the following properties*

(i) $F^{\sigma_f} = F + \mu F^{\Delta_f}$,

(ii) $(\alpha F + \beta G)^{\Delta_f} = \alpha F^{\Delta_f} + \beta G^{\Delta_f}$, *for* $\alpha, \beta \in \mathbb{R}$,

(iii) $(FG)^{\Delta_f} = F^{\sigma_f} G^{\Delta_f} + F^{\Delta_f} G$ (*generalization of Leibniz rule*),

(iv) *On homogeneous time scale operators Δ_f and σ_f commute, i.e*

$$\left(F^{\sigma_f}\right)^{\Delta_f} = \left(F^{\Delta_f}\right)^{\sigma_f}.$$

An operator Δ_f satisfying the rule (iii) of Proposition 2.4 is called a *"σ_f-derivation"* [9]. A commutative field endowed with a σ_f-derivation is called a *differential field*. The field $\mathscr{K}$ is endowed with a σ_f-differential structure, determined by system (1), and there exists the differential overfield $\mathscr{K}^*$, called the *inversive closure* of $\mathscr{K}$. In [4] the construction of the inversive closure $\mathscr{K}^*$ for system (1) is given. The extension of σ_f to $\mathscr{K}^*$ is an automorphism [9].

Consider the infinite set of symbols $\mathrm{d}\mathscr{C} = \left\{\mathrm{d}x_i,\ i = 1, \ldots, n;\ \mathrm{d}u_\upsilon^{[k]},\ \upsilon = 1, \ldots, m,\ k \geq 0\right\}$ and denote by $\mathscr{E}$ the vector space over the field $\mathscr{K}^*$ spanned by the elements of $\mathrm{d}\mathscr{C}$, namely

$$\mathscr{E} = \operatorname{span}_{\mathscr{K}^*} \mathrm{d}\mathscr{C}.$$

Any element of $\mathscr{E}$ has the form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k \geq 0} \sum_{\upsilon=1}^{m} B_{\upsilon k} \mathrm{d}u_\upsilon^{[k]},$$

where only a finite number of coefficients $B_{\upsilon k}$ are nonzero elements of $\mathscr{K}^*$. The elements of $\mathscr{E}$ will be called the *differential one-forms*.

Let us define the operator $\mathrm{d} : \mathscr{K}^* \to \mathscr{E}$ as follows

$$\mathrm{d}F\left(x, u^{[0\ldots k]}\right) := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\left(x, u^{[0\ldots k]}\right) \mathrm{d}x_i + \sum_{l=0}^{k} \sum_{\upsilon=1}^{m} \frac{\partial F}{\partial u_\upsilon^{[l]}}\left(x, u^{[0\ldots k]}\right) \mathrm{d}u_\upsilon^{[l]}. \tag{3}$$

Let $\omega = \sum_i A_i \mathrm{d}\zeta_i$ be a one-form, where $A_i \in \mathscr{K}^*$ and $\zeta_i \in \mathscr{C}$. We define the operators $\Delta_f : \mathscr{E} \to \mathscr{E}$ and $\sigma_f : \mathscr{E} \to \mathscr{E}$ by

$$\omega^{\Delta_f} := \sum_i \left(A_i^{\Delta_f} \mathrm{d}\zeta_i + A_i^{\sigma_f} \mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right) \tag{4}$$

and

$$\omega^{\sigma_f} := \sum_i A_i^{\sigma_f} \mathrm{d}\left(\zeta_i^{\sigma_f}\right).$$

Since $A_i^{\sigma_f} = A_i + \mu A_i^{\Delta_f}$,

$$\omega^{\Delta_f} = \sum_i \left(A_i^{\Delta_f} \mathrm{d}\zeta_i + \left(A_i + \mu A_i^{\Delta_f}\right) \mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right).$$

One says that $\omega \in \mathscr{E}$ is an *exact* one-form if $\omega = \mathrm{d}F$ for some $F \in \mathscr{K}^*$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. It is well known that exact forms are closed, while closed forms are only locally exact. Integrability of the subspace of one-forms may be checked by the Frobenius theorem below, where the symbol $\mathrm{d}\omega_i$ means the exterior derivative of one-form ω_i and $\wedge$ means the exterior or wedge product (for details see [8]).

Theorem 2.5 ([8]). *Let $\mathscr{V} = \mathrm{span}_{\mathscr{K}^*}\{\omega_1, \ldots, \omega_r\}$ be a subspace of $\mathscr{E}$. $\mathscr{V}$ is integrable if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$$

for any $i = 1, \ldots, r$.

3 OBSERVABILITY AND OBSERVABLE SPACE

Frequently the observability rank condition is used to check whether the continuous-time nonlinear system is locally weakly observable [10], [14]. This condition is sufficient for arbitrary initial state and necessary for almost all initial states. Thus, we introduce the definition of observability for nonlinear systems, defined on homogeneous time scales, through the observability rank condition.

Definition 3.1. System (1) is called *generically (single-experiment) observable* if the rank of the observability matrix is generically equal to n, i.e. if

$$\mathrm{rank}_{\mathscr{K}^*} \left[\frac{\partial \left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]} \right)}{\partial x} \right] = n. \tag{5}$$

Observe that $h_\nu^{\sigma_f^k} := \left(h_\nu^{\sigma_f^{k-1}} \right)^{\sigma_f}$ for $k \geq 2$ and take into account, that for $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$ the higher order delta derivative can be computed explicitly as

$$h_\nu^{[i]} = \frac{1}{\tau^i} \sum_{k=0}^{i} (-1)^k C_i^k h_\nu^{\sigma_f^{i-k}}, \tag{6}$$

where C_i^k is the binomial coefficient, i.e. $C_i^k = \frac{i!}{(i-k)!k!}$.

Proposition 3.2. *For $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, the following holds*

$$\operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]}\right)^{\mathrm{T}}}{\partial x}\right] =$$
$$= \operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\sigma_f}, \ldots, h_1^{\sigma_f^{n-1}}, \ldots, h_p, h_p^{\sigma_f}, \ldots, h_p^{\sigma_f^{n-1}}\right)^{\mathrm{T}}}{\partial x}\right]. \tag{7}$$

Proof. Using (6), the arbitrary row of the left-hand side matrix in (7) may be rewritten as

$$\frac{\partial h_\nu^{[i]}}{\partial x} = \frac{1}{\tau^i}\sum_{k=0}^{i}(-1)^k C_i^k \cdot \frac{\partial h_\nu^{\sigma_f^{i-k}}}{\partial x}$$

for $\nu = 1, \ldots, p$ and $i = 1, \ldots, n-1$. Separating the first addend of the above sum yields

$$\frac{\partial h_\nu^{[i]}}{\partial x} = \frac{1}{\tau^i}\left(\frac{\partial h_\nu^{\sigma_f^i}}{\partial x} + \sum_{k=1}^{i}(-1)^k C_i^k \cdot \frac{\partial h_\nu^{\sigma_f^{i-k}}}{\partial x}\right).$$

Now the sum $\sum_{k=1}^{i}$ in the above equality is the linear combination of the previous rows of the matrix and therefore, can be removed without changing the rank of the matrix. Since $\partial h_\nu^{\sigma_f^i}/\partial x$ is the row of the right-hand side matrix of (7) for $i = 1, \ldots, n-1$, the statement of the proposition holds. $\square$

Remark 3.3. Since for $\mathbb{T} = \mathbb{R}$ the delta derivative coincides with the classical time derivative, the condition (5) is equivalent to observability rank condition in [10]. By Proposition 3.2 in the discrete-time case the condition (5) is equivalent to the observability rank condition given in [16].

Though Definition 3.1 may be applied to check observability, it is easier to be done using a concept of observable space like in the continuous-time case [10]. Moreover, the observable space, if integrable, allows to decompose the system into the observable/unobservable subsystems. In the remaining part of this section we extend the concept of observable space to the case of (MIMO) systems, defined on homogeneous time scales, and, using the notion of observable space, provide the necessary and sufficient observability condition.

Given system (1), denote by $\mathscr{X}$, $\mathscr{Y}^k$, $\mathscr{Y}$ and $\mathscr{U}$ the following subspaces of the differential one-forms:

$$\begin{aligned}
\mathscr{X} &:= \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x\},\\
\mathscr{Y}^k &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}h_\nu^{[j]}, \nu = 0, \ldots, p, 0 \le j \le k\right\},\\
\mathscr{Y} &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}h_\nu^{[j]}, \nu = 0, \ldots, p, j \ge 0\right\},\\
\mathscr{U} &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}u_\upsilon^{[l]}, \upsilon = 1, \ldots, m, l \ge 0\right\}.
\end{aligned} \tag{8}$$

By analogy with [10], the finite chain of subspaces

$$0 \subset \mathscr{O}_0 \subset \mathscr{O}_1 \subset \cdots \subset \mathscr{O}_k \subset \cdots \subset \mathscr{O}_{k^*-1} = \mathscr{O}_{k^*} =: \mathscr{O}_\infty, \tag{9}$$

where

$$\mathscr{O}_k := \mathscr{X} \cap \left(\mathscr{Y}^k + \mathscr{U}\right) \tag{10}$$

is called the *observability filtration*. Denote by $\mathscr{O}_\infty$ the limit of the observability filtration; it is easy to see that

$$\mathscr{O}_\infty = \mathscr{X} \cap (\mathscr{Y} + \mathscr{U})$$

and analogously with [10] we call the subspace $\mathscr{O}_\infty$ of $\mathscr{X}$ the *observable space*[3] of the system (1). The unobservable space of system (1), denoted by $\mathscr{X}_{\bar{O}}$, is defined as a subspace of $\mathscr{X}$, which satisfies

$$\mathscr{X}_{\bar{O}} \cong \mathscr{X}/\mathscr{O}_\infty, \quad \mathscr{X}_{\bar{O}} \oplus \mathscr{O}_\infty = \mathscr{X},$$

where $\mathscr{X}/\mathscr{O}_\infty$ denotes the factor-space.

From (8), taking into account (3) and using the linear transformations, one obtains

$$\mathscr{Y}^k + \mathscr{U} = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1, \ldots, p, 0 \leq j \leq k; \mathrm{d}u_\upsilon^{[l]}, \upsilon = 1, \ldots, m, l \geq 0\right\}.$$

Consequently, according to (10)

$$\mathscr{O}_k = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1, \ldots, p, 0 \leq j \leq k\right\}, \tag{11}$$

yielding

$$\mathscr{O}_\infty = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1, \ldots, p, j \geq 0\right\}.$$

Before studying the properties of the observable space we provide Lemma 3.4. Denote the one-forms which generate the observable space $\mathscr{O}_\infty$ as $\omega_{\nu,j} := \frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x$ for $\nu = 1, \ldots, p, j \geq 0$ and arrange them in the form of the following matrix:

$$\Omega = \begin{bmatrix} \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \\ \omega_{p,0} & \omega_{p,1} & \omega_{p,2} & \cdots \end{bmatrix}.$$

Also denote the arbitrary row of the above matrix by Ω_ν.

Lemma 3.4. *If Ω_ν contains the one-form $\omega_{\nu,i}$, being a linear combination of the former one-forms $\omega_{\nu,0}, \ldots, \omega_{\nu,i-1}$ from Ω_ν, then the next one-forms $\omega_{\nu,j}$'s for $j > i$ can also be represented as a linear combination of the one-forms $\omega_{\nu,0}, \ldots, \omega_{\nu,i-1}$.*

The proof of Lemma 3.4 is given in the Appendix.

The proposition below describes the property of the subspace $\mathscr{O}_\infty$.

[3]Note that $\mathscr{O}_\infty$ is in general not the observation space as in [23], associated with the concept of the multi-experiment observability

Proposition 3.5.

$$\dim_{\mathscr{K}^*} \mathscr{O}_\infty = \operatorname{rank}_{\mathscr{K}^*} \left[\frac{\partial \left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]} \right)}{\partial x} \right].$$

Proof. Represent the observable space as

$$\mathscr{O}_\infty = \mathscr{O}_\infty^1 + \mathscr{O}_\infty^2 + \cdots + \mathscr{O}_\infty^p,$$

where $\mathscr{O}_\infty^\nu$ is generated by the elements of Ω_ν. Since $\mathscr{O}_\infty^\nu \subseteq \mathscr{O}_\infty \subseteq \mathscr{X}$ and, as a consequence, $\dim \mathscr{O}_\infty^\nu \leq \dim \mathscr{O}_\infty \leq \dim \mathscr{X} = n$, it is enough to use n independent differential one-forms $\omega_{\nu,j}$ to generate $\mathscr{O}_\infty^\nu$. Lemma 3.4 guarantees that the first n one-forms $\omega_{\nu,j}$, $0 \leq j \leq n-1$, span the subspace $\mathscr{O}_\infty^\nu$. Consequently,

$$\begin{aligned} &\operatorname{span}_{\mathscr{K}^*} \left\{ \frac{\partial h_\nu^{[j]}}{\partial x} \mathrm{d}x, \nu = 1, \ldots, p, j \geq 0 \right\} = \\ &\qquad = \operatorname{span}_{\mathscr{K}^*} \left\{ \frac{\partial h_\nu^{[j]}}{\partial x} \mathrm{d}x, \nu = 1, \ldots, p, 0 \leq j \leq n-1 \right\}. \end{aligned}$$

Thus, the rows of the observability matrix

$$\left[\frac{\partial \left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]} \right)}{\partial x} \right] \tag{12}$$

with n columns can be regarded as the representation of the elements of the codistribution $\mathscr{O}_\infty$. Therefore, the number of linearly independent vectors of $\mathscr{O}_\infty$, i.e. $\dim_{\mathscr{K}^*} \mathscr{O}_\infty$, can be found as the rank of the matrix (12). □

The following theorem is the direct consequence of Definition 3.1 and Proposition 3.5 and provides the characterization of the observability of the system.

Theorem 3.6. *A system* (1) *is (single-experiment) observable if and only if* $\mathscr{O}_\infty = \mathscr{X}$.

Example 3.7. Consider the continuous-time model of unicycle [10] and its discrete-time approximation, based on Euler sampling scheme, as a single model defined on the homogeneous time scale $\mathbb{T}$

$$\begin{aligned} x_1^\Delta &= u_1 \cos x_3 \\ x_2^\Delta &= u_1 \sin x_3 \\ x_3^\Delta &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned} \tag{13}$$

Using (11), the observability filtration (9) of the system (13) may be computed as follows

$$\begin{aligned} \mathscr{O}_0 &= \operatorname{span}_{\mathscr{K}^*} \{\mathrm{d}x_1, \mathrm{d}x_2\}, \\ \mathscr{O}_\infty = \mathscr{O}_1 &= \operatorname{span}_{\mathscr{K}^*} \{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}. \end{aligned}$$

Since the observable space $\mathscr{O}_\infty = \mathscr{X}$, the system is observable. Alternatively, one may check that direct application of Definition 3.1 yields the same result though requires more computations:

$$\operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, h_1^{[2]}, h_2, h_2^{\Delta_f}, h_2^{[2]}\right)}{\partial x}\right] = \operatorname{rank}_{\mathscr{K}^*}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -u_1\sin x_3\\ 0 & 0 & a\\ 0 & 1 & 0\\ 0 & 0 & u_1\cos x_3\\ 0 & 0 & b\end{bmatrix} = 3,$$

where

$$a := \begin{cases}\dfrac{u_1\sin x_3 - \left(u_1 + \tau u_1^{\Delta}\right)\sin\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0,\\ -u_1u_2\cos x_3 - \dot{u}_1\sin x_3 & \text{if } \mathbb{T} = \mathbb{R},\end{cases}$$

$$b := \begin{cases}\dfrac{-u_1\cos x_3 + \left(u_1 + \tau u_1^{\Delta}\right)\cos\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0,\\ -u_1u_2\sin x_3 + \dot{u}_1\cos x_3 & \text{if } \mathbb{T} = \mathbb{R}.\end{cases}$$

Given a system of the form (1), its observability filtration (9), like in the continuous-time case [10], defines a set of structural indices σ_j for $j = 1, \ldots, k^*$ by

$$\begin{aligned}\sigma_1 &= \dim_{\mathscr{K}^*}\mathscr{O}_0,\\ \sigma_j &= \dim_{\mathscr{K}^*}\left(\mathscr{O}_{j-1}/\mathscr{O}_{j-2}\right), \quad j = 2, \ldots, k^*.\end{aligned} \tag{14}$$

Another set of indices s_i, for $i = 1, \ldots, p$, being dual to the set $\{\sigma_j, j = 1, \ldots, k^*\}$, is defined by

$$s_i = \operatorname{card}\{\sigma_j \mid \sigma_j \geq i\}$$

and called the set of *observability indices* of system (1). The integer σ_j represents the number of observability indices s_i which are greater than or equal to j, and duality implies that $\sigma_j = \operatorname{card}\{s_i \mid s_i \geq j\}$.

Observability indices determine how many delta derivatives of the respective output components one needs to use for computation of the initial state x on the basis of the inputs and outputs and their delta derivatives. The following proposition describes the key property of the observability indices.

Proposition 3.8. *Given a system of the form* (1)*, one has*

$$\dim_{\mathscr{K}^*}\mathscr{O}_\infty = s_1 + \cdots + s_p.$$

Proof. Note, that $\dim_{\mathscr{K}^*}(\mathscr{O}_{j-1}/\mathscr{O}_{j-2}) = \dim_{\mathscr{K}^*}\mathscr{O}_{j-1} - \dim_{\mathscr{K}^*}\mathscr{O}_{j-2}$. Using (14) one can write

$$\sum_{j=1}^{k^*}\sigma_j = \sum_{j=1}^{k^*}\dim_{\mathscr{K}^*}\mathscr{O}_{j-1} - \sum_{j=2}^{k^*}\dim_{\mathscr{K}^*}\mathscr{O}_{j-2}. \tag{15}$$

Separating the last addend of the first sum in the right-hand side of (15), replacing in this sum index j by $j-1$ and taking into account that $\mathscr{O}_{k^*-1} = \mathscr{O}_\infty$, we obtain

$$\sum_{j=1}^{k^*}\sigma_j = \dim_{\mathscr{K}^*}\mathscr{O}_\infty + \sum_{j=2}^{k^*}\dim_{\mathscr{K}^*}\mathscr{O}_{j-2} - \sum_{j=2}^{k^*}\dim_{\mathscr{K}^*}\mathscr{O}_{j-2} = \dim_{\mathscr{K}^*}\mathscr{O}_\infty. \tag{16}$$

The relation between indices σ_j and s_i can be expressed by means of a $k^* \times p$ table, whose (j,i)th element is defined by ($j = 1, \dots, k^*$ pointing to the row and $i = 1, \dots, p$ to the column)

$$a_{j,i} = \begin{cases} 1, & 1 \le i \le \sigma_j, \\ 0, & (\sigma_j + 1) \le i \le p, \end{cases} = \begin{cases} 1, & 1 \le j \le s_i, \\ 0, & (s_i + 1) \le j \le k^*. \end{cases}$$

Thus, the indices σ_j and s_i are the sums of elements in the jth row and ith column, respectively, i.e.

$$\sigma_j = \sum_{i=1}^{p} a_{j,i}, \quad s_i = \sum_{j=1}^{k^*} a_{j,i}. \tag{17}$$

Taking into account (16) and (17), one obtains

$$\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} \sum_{j=1}^{k^*} a_{j,i} = \sum_{j=1}^{k^*} \sigma_j = \dim_{\mathscr{K}^*} \mathscr{O}_\infty,$$

which completes the proof. □

Example 3.9. (Continuation of Example 3.7). One has $\sigma_1 = 2$, $\sigma_2 = 1$ and so, the observability indices are $s_1 = 2$, $s_2 = 1$. Taking delta derivatives of y_1 and y_2 up to the orders $s_1 - 1$ and $s_2 - 1$, respectively, we obtain $y_1 = x_1$, $y_1^\Delta = u_1 \cos x_3$, $y_2 = x_2$, yielding

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ x_3 &= \arccos \frac{y_1^\Delta}{u_2}. \end{aligned}$$

4 DECOMPOSITION

For certain applications it will be useful to have system representations in which the observable and unobservable state variables can be explicitly distinguished. For a continuous-time nonlinear control system the decomposition into observable/unobservable subsystems has been carried out both via differential geometric [15], [21] and linear algebraic methods [10] and is proved to be always doable. For example, in [10] the decomposition was first carried out for linearized system defined in terms of one-forms, and then, it was proved that the observable subspace of differential one-forms is always (generically) integrable. Therefore, the observable subspace of one-forms can be (at least locally) spanned by exact one-forms whose integrals define the observable state coordinates. As demonstrated in [16], for the discrete-time nonlinear control systems described in terms of the shift operator σ_f the decomposition at the level of equations (state variables) is not always possible since the observable space of one-forms is not necessarily completely integrable. Moreover, the paper [19] provides a general subclass of systems with non-integrable observable subspace.

The purpose of this section is to study the possibility to decompose the nonlinear control system defined on the homogeneous time scale into the observable and unobservable subsystems. Since the delta-domain model obtained via sampling [12] behaves similarly to the continuous-time system and at the limit, when the sampling frequency increases infinitely, approaches the continuous-time system, it was our working hypotheses that the delta-domain models are, in general, decomposable into observable/unobservable parts.

The latter would mean that the respective observable space $\mathscr{O}_\infty$, as a space of differential one-forms, is completely integrable. In the case $\mu \equiv 0$ ($\mathbb{T} = \mathbb{R}$), the observable space $\mathscr{O}_\infty$ is proved to be integrable in [10]. Unfortunately, unlike the case $\mathbb{T} = \mathbb{R}$ for the case $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, $\mathscr{O}_\infty$ is not necessarily integrable. We give a number of counterexamples.

Example 4.1. Consider the control system, defined on homogeneous time scale

$$
\begin{aligned}
x_1^\Delta &= x_3 + ux_3 - x_1 \\
x_2^\Delta &= u - x_2 \\
x_3^\Delta &= ux_1 - x_3 - x_2 \\
y &= x_3.
\end{aligned}
\tag{18}
$$

By (9), for this system, $\mathscr{O}_\infty = \mathscr{O}_2 = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_3, 2\mathrm{d}x_2 + \left(u^\Delta - \mu u^\Delta - 2u\right)\mathrm{d}x_1, \mathrm{d}x_2 - -u\mathrm{d}x_1\}$. If $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and obviously[4], $\mathscr{O}_\infty = \mathscr{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathscr{O}_\infty = \mathscr{X}$, except for the case $\mu = \tau = 1$ when $\mathscr{O}_\infty = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_3, \mathrm{d}x_2 - u\mathrm{d}x_1\}$, being non-integrable subspace by Theorem 2.5, since $\mathrm{d}(\mathrm{d}x_2 - u\mathrm{d}x_1) \wedge \mathrm{d}x_3 \wedge (\mathrm{d}x_2 - u\mathrm{d}x_1) = \mathrm{d}u \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \neq 0$.

Next example demonstrates that the loss of integrability does not necessarily occur only at $\mu = 1$.

Example 4.2. Consider the system

$$
\begin{aligned}
x_1^\Delta &= x_2 - \frac{x_1}{3} \\
x_2^\Delta &= ux_1 + x_3 - x_2 \\
x_3^\Delta &= \mathrm{e}^{u^2x_1 + ux_3} - \frac{x_3}{3} \\
y &= x_2.
\end{aligned}
\tag{19}
$$

The observable space of the system

$$
\mathscr{O}_\infty = \mathscr{O}_2 = \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3, \left(u^\Delta - \frac{\mu u^\Delta}{3}\right)\mathrm{d}x_1\right\}.
$$

Like in the previous example, if $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and $\mathscr{O}_\infty = \mathscr{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathscr{O}_\infty = \mathscr{X}$, except for the case $\mu = \tau = 3$ when $\mathscr{O}_\infty = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3\}$, again non-integrable by the Frobenius theorem.

Finally, we provide an example of the system for which the observable space $\mathscr{O}_\infty$ is integrable for every choice of the value of μ.

[4] Of course, for $\mu \equiv 0$ the result also follows from continuous-time theory [10]

Example 4.3. Consider the system

$$
\begin{aligned}
x_1^\Delta &= \tan(x_1 - x_2)u_1 \\
x_2^\Delta &= u_1 \tan(x_1 - x_2) - u_2 \cos^2(x_1 - x_2) \\
x_3^\Delta &= u_1 \\
y_1 &= x_3 \\
y_2 &= x_1 - x_2.
\end{aligned} \tag{20}
$$

The observable space $\mathscr{O}_\infty = \mathscr{O}_0 = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_1 - \mathrm{d}x_2, \mathrm{d}x_3\}$ is obviously integrable by direct inspection.

To conclude, we conjecture that the observable space $\mathscr{O}_\infty$ is in general integrable except for a few possible μ values where these values correspond to the sampling frequencies at which the state transition map of the sampled system is not reversible. The following example illustrates this conjecture.

Example 4.4. (Continuation of Examples 4.1 – 4.3). The state transition map of system (18) is

$$
\begin{aligned}
x_1^+ &= \mu\,(x_3 + ux_3 - x_1) + x_1 \\
x_2^+ &= \mu\,(u - x_2) + x_2 \\
x_3^+ &= \mu\,(ux_1 - x_3 - x_2) + x_3,
\end{aligned} \tag{21}
$$

where we use the notation $x^+ := x(t+\mu)$. In order to check the reversibility of the system, one needs to verify whether the Jacobian matrix $\partial\widetilde{f}(x,u)/\partial x$ is nonsingular. The Jacobian matrix of system (21) is

$$
\frac{\partial\widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1-\mu & 0 & \mu(1+u) \\ 0 & 1-\mu & 0 \\ \mu u & -\mu & 1-\mu \end{bmatrix}.
$$

One can verify that the above matrix is singular for $\mu = 1$, implying that the state transition map (21) is not reversible at the sampling frequency equal 1. Next, consider the state transition map of system (19), which reads as

$$
\begin{aligned}
x_1^+ &= \mu\left(x_2 - \frac{x_1}{3}\right) + x_1 \\
x_2^+ &= \mu\,(ux_1 + x_3 - x_2) + x_2 \\
x_3^+ &= \mu\left(\mathrm{e}^{u^2x_1 + ux_3} - \frac{x_3}{3}\right) + x_3.
\end{aligned} \tag{22}
$$

The Jacobian matrix of system (22), i.e.

$$
\frac{\partial\widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1-\frac{\mu}{3} & \mu & 0 \\ \mu u & 1-\mu & \mu \\ \mathrm{e}^{u(ux_1+x_3)}\mu u^2 & 0 & 1-\frac{\mu}{3}+\mathrm{e}^{u(ux_1+x_3)}\mu u \end{bmatrix}
$$

is singular for $\mu = 3$. Consequently, the state transition map (22) is not reversible at the sampling frequency equal 3. Finally, the state transition map of system (20) is

$$
\begin{aligned}
x_1^+ &= \mu\tan(x_1 - x_2)u_1 + x_1 \\
x_2^+ &= \mu\left(u_1\tan(x_1 - x_2) - u_2\cos^2(x_1 - x_2)\right) + x_2 \\
x_3^+ &= \mu u_1 + x_3
\end{aligned} \tag{23}
$$

and its Jacobian matrix reads as

$$\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1+\frac{\mu u_1}{\cos^2(x_1-x_2)} & \frac{-\mu u_1}{\cos^2(x_1-x_2)} & 0 \\ a & 1-a & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a := \mu\left(\frac{u_1}{\cos^2(x_1-x_2)} + u_2 \sin\left(2\left(x_1-x_2\right)\right)\right)$. One can verify that the above matrix is nonsingular for any $\mu \equiv \mathrm{const}$, meaning that the state transition map (23) is reversible at any sampling frequency. To conclude, comparing the above result with those presented in Examples 4.1 – 4.3, one can observe the consistency of the sampling frequencies at which the state transition maps are not reversible and the values of μ for which the observable spaces $\mathscr{O}_\infty$ are not integrable. These examples support our conjecture.

If $\mathscr{O}_\infty$ is integrable, and therefore, has locally an exact basis $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$, one can complete the set $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$ to a basis $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r, \mathrm{d}\zeta_{r+1}, \ldots, \mathrm{d}\zeta_n\}$ of $\mathscr{X}$. Then, in the coordinates $\{\zeta_1, \ldots, \zeta_n\}$, the system can be decomposed into an observable and unobservable subsystems

$$\begin{aligned} \zeta_1^\Delta &= f_1\left(\zeta_1, \ldots, \zeta_r, u\right), \\ &\vdots \\ \zeta_r^\Delta &= f_r\left(\zeta_1, \ldots, \zeta_r, u\right), \\ y &= h\left(\zeta_1, \ldots, \zeta_r\right) \end{aligned}$$

and

$$\begin{aligned} \zeta_{r+1}^\Delta &= f_{r+1}\left(\zeta, u\right), \\ &\vdots \\ \zeta_n^\Delta &= f_n\left(\zeta, u\right), \end{aligned}$$

respectively.

Example 4.5. (Continuation of Example 4.3). Integrating the observable space $\mathscr{O}_\infty$ of the system, we get the set of the observable state variables $\zeta_1 = x_1 - x_2$ and $\zeta_2 = x_3$. Next we complete this set to a basis $\{\zeta_1, \zeta_2, \zeta_3\}$ of $\mathbb{R}^3$, taking, for example, $\zeta_3 = x_1$. In these coordinates the system equations read as

$$\begin{aligned} \zeta_1^\Delta &= u_1 \\ \zeta_2^\Delta &= u_2 \cos^2 \zeta_2 \\ \zeta_3^\Delta &= u_1 \tan \zeta_2 \\ y_1 &= \zeta_1 \\ y_2 &= \zeta_2, \end{aligned}$$

where the first two equations (together with the output equations) define the observable subsystem. The state ζ_3 is unobservable.

5 CONCLUSIONS

Though the theory of continuous- and discrete-time dynamical systems as presented in the literature is different, the analysis on time scales is nowadays recognized as the right tool to unify the seemingly separate fields of discrete dynamical systems (i.e. difference equations) and continuous dynamical systems (i.e. differential equations). In the paper we studied the observability of multi-input multi-output control systems on homogeneous time scale, which allows us to unify continuous- and discrete-time theories, presenting both of them simultaneously under the same language. The presented approach covers the continuous- and discrete-time cases in such a manner that those are the special cases of the formalism. Since delta derivative (used in our paper to describe the dynamical systems) coincides with the time derivative for the continuous-time case, the results available in the literature can be obtained from our results as a special case, namely the case in which the time scale is the set of real numbers. On the other hand, our formalism includes the description of a discrete-time system based on the difference operator description (delta-domain approach), for which the results shown in the paper are new, since previous results have been obtained for discrete-time systems considered on the basis of the shift-operator formalism. Therefore, in our paper the discrete-time systems are described in terms of the difference operator unlike in the majority of papers where the system is described via the shift-operator. To conclude, though the computation of the delta-derivative is different in the continuous- and discrete-time cases, the results obtained by means of it are the same for both time domains.

In the paper the notion of the observable space was used to provide the observability condition that can be easily checked. However, note that the definition of the observability was introduced through the observability rank condition, commonly used both in continuous- and discrete-time cases. One of the future goals is to define the observability of the nonlinear system on homogeneous time scale using the concept of (in)distinguishable states. Another goal is to find the conditions under which the nonlinear system defined on homogeneous time scale is transformable into the observer form, which allows to construct an observer with linearizable error dynamics.

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APPENDIX. PROOF OF LEMMA 3.4

In order to prove Lemma 3.4, we need Lemma 5.1 below.

Lemma 5.1. *For the homogeneous time scale* $\mathbb{T}$ *one has*

$$\frac{\partial h_\nu^{[i+1]}}{\partial x} = \frac{\partial h_\nu^{[i]}}{\partial x}\frac{\partial f(x,u)}{\partial x} + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right),$$
$$\nu = 1,\dots p, \quad i = 0,1,\dots, \tag{24}$$

where I_n *is* $n \times n$ *identity matrix.*

Proof. By commutativity of operators d and Δ_f [4],

$$\mathrm{d}\left(h_\nu^{[i+1]}\right) = \left(\mathrm{d}h_\nu^{[i]}\right)^{\Delta_f}. \tag{25}$$

In what follows, we omit in (25) the parts involving the terms $\mathrm{d}u_\upsilon^{[l]}$ in the expressions of total differentials, therefore we have

$$\frac{\partial h_\nu^{[i+1]}}{\partial x}\mathrm{d}x + \cdots = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x\right)^{\Delta_f} + \cdots. \tag{26}$$

We compute the delta derivative of the one-form at the right-hand side of (26), using (4). Since $(\mathrm{d}x)^{\Delta_f} = \mathrm{d}f(x,u)$, and again, omitting the parts involving the terms $\mathrm{d}u_\upsilon$, we get

$$\left(\frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x\right)^{\Delta_f} = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f}\mathrm{d}x + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}\frac{\partial f(x,u)}{\partial x}\mathrm{d}x + \cdots.$$

Since the vectors $\mathrm{d}x$, $\mathrm{d}u_\upsilon$,..., $\mathrm{d}u_\upsilon^{[i-1]}$ are independent over the field $\mathscr{K}^*$, by comparing the coefficients of $\mathrm{d}x$ at both sides of equality (26) we get

$$\frac{\partial h_\nu^{[i+1]}}{\partial x} = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f} + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}\frac{\partial f(x,u)}{\partial x}.$$

Finally, applying (i) of Proposition 2.4 to $\left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}$ we obtain (24). □

Now we are ready to prove Lemma 3.4.

Proof. According to the condition of the lemma

$$\omega_{\nu,i} := \frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1}\alpha_k\frac{\partial h_\nu^{[k]}}{\partial x}\mathrm{d}x. \tag{27}$$

We first prove that the statement of the lemma holds for $j = i+1$, i.e.

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1}\beta_k\frac{\partial h_\nu^{[k]}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1}\beta_k\omega_{\nu,k} \tag{28}$$

for some β_k's. By Lemma 5.1 and (27)

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1}\left[\alpha_k\frac{\partial h_\nu^{[k]}}{\partial x}\frac{\partial f(x,u)}{\partial x} + \left(\alpha_k\frac{\partial h_\nu^{[k]}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right)\right]\mathrm{d}x.$$

Using (iii) of Proposition 2.4 for $\left(\alpha_k \frac{\partial h_\nu^{[k]}}{\partial x}\right)^{\Delta_f}$ and then (i) of Proposition 2.4 for α_k, we get

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left[\frac{\partial h_\nu^{[k]}}{\partial x} \left(\frac{\partial f(x,u)}{\partial x} \alpha_k^{\sigma_f} + \alpha_k^{\Delta_f} \right) + \right.$$
$$\left. + \alpha_k^{\sigma_f} \left(\frac{\partial h_\nu^{[k]}}{\partial x} \right)^{\Delta_f} \left(I_n + \mu \frac{\partial f(x,u)}{\partial x} \right) \right] \mathrm{d}x.$$

By Lemma 5.1

$$\left(\frac{\partial h_\nu^{[k]}}{\partial x} \right)^{\Delta_f} \left(I_n + \mu \frac{\partial f(x,u)}{\partial x} \right) = \frac{\partial h_\nu^{[k+1]}}{\partial x} - \frac{\partial h_\nu^{[k]}}{\partial x} \frac{\partial f(x,u)}{\partial x},$$

yielding

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \alpha_k^{\Delta_f} \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \sum_{k=0}^{i-1} \alpha_k^{\sigma_f} \frac{\partial h_\nu^{[k+1]}}{\partial x} \mathrm{d}x.$$

Changing the summation index of the second sum for $s = k+1$, separating the last addend of the second sum, and applying (27) to it, we obtain

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_k \right) \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \sum_{s=1}^{i-1} \alpha_{s-1}^{\sigma_f} \frac{\partial h_\nu^{[s]}}{\partial x} \mathrm{d}x.$$

Separating the first addend of the first sum yields

$$\omega_{\nu,i+1} = \sum_{k=1}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_k + \alpha_{k-1}^{\sigma_f} \right) \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \left(\alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_0 \right) \frac{\partial h_\nu}{\partial x} \mathrm{d}x.$$

Denoting $\beta_0 := \alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_0$ and $\beta_k := \alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f}\alpha_k + \alpha_{k-1}^{\sigma_f}$ we get (28). The similar arguments can be applied for the case $j > i+1$. $\square$

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Homogeensel ajaskaalal defineeritud mittelineaarse juhtimissüsteemi vaadeldav ruum

Vadim Kaparin, Ülle Kotta ja Małgorzata Wyrwas

Uuriti homogeensel ajaskaalal defineeritud mittelineaarse juhtimissüsteemi vaadeldavust. Vaadeldavus tähendab võimalust määrata (leida) süsteemi mittemõõdetav algolek mõõdetavate juhttoimete ja väljundite abil. Vaadeldavuse tingimus on esitatud vaadeldava ruumi mõiste kaudu. Juhul kui süsteem ei ole vaadeldav, aga vaadeldav ruum, mille elementideks on diferentsiaalsed üks-vormid, on täielikult integreeruv, on süsteem dekomponeeritav vaadeldavaks ja mittevaadeldavaks alamsüsteemiks.

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