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Advanced Design of Nonlinear Discrete-time and Delayed Systems

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology and Ecole Centrale de Nantes has not been submitted for doctoral or equivalent academic degree.

/Arvo Kaldmäe/

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ARVO KALDMÄE

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List of Publications

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2. A. Kaldmäe and Ü. Kotta. Input-output decoupling of discrete-time nonlinear systems by measurement feedback. In *The 21st International Symposium on Mathematical Theory of Networks and Systems*, pages 311-316, Groningen, The Netherlands, 2014.
3. A. Kaldmäe and Ü. Kotta. Disturbance decoupling by measurement feedback. In *The 19th IFAC World Congress*, pages 7735-7740, Cape Town, South Africa, 2014.
4. A. Kaldmäe and C. H. Moog. Disturbance decoupling of time delay systems. *Asian Journal of Control*, DOI: 10.1002/asjc.1169.
5. A. Kaldmäe, C. Califano and C. H. Moog. Integrability for nonlinear time-delay systems. *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2015.2482003.

Author's Contribution to the Publications

In all the publications the author of the thesis is the first author and main contributor.

The results of first three publications were obtained under the supervision of Dr. Ülle Kotta. All the main results were achieved and writing the papers was done by the author of the thesis. The last two articles are the results of joint work with C. Califano and Dr. C.H. Moog. The writing of the papers as well as the proofs were performed by the author of the thesis.

Introduction

State of the Art

In this thesis two classes of nonlinear control systems are studied: the discrete-time delay-free systems and the continuous-time systems with delays. The mathematical approach used in the thesis fits well to both system classes, since the delays can be handled similarly as the forward-shifts in the discrete-time case. Nevertheless, the study of time-delay systems is more complex, since besides the delay operator there is additionally the time derivative operator acting on system variables.

Discrete-time Systems

The majority of dynamical systems are modeled by a set of differential equations rather than by difference equations. In the classical control theory it is usually assumed that the model of a continuous-time control system is given by a set of first order differential equations. This is the so-called state space representation. The majority of research in control theory is done for this kind of system descriptions. Since most of the control algorithms are implemented digitally, one needs to construct digital controllers. There are two main approaches for the design of digital controllers. The first (and most often used) is to construct the controller for the continuous-time plant and then discretize it. The second approach is to discretize the continuous-time plant and construct the controller for the discrete-time model. See [71] for an overview of nonlinear digital control.

There are several methods to find a discrete-time model of a continuous-time system. The most known is the Euler forward discretization, which gives the 1st order approximation of the continuous-time system. The drawback of the Euler discretization is that there is no advantage compared to the case, when the controller of a continuous-time plant is discretized. Though the exact discrete-time models are, in general, impossible to obtain for nonlinear systems, one may construct higher order approximations to define 'good enough' sampled-data models. See [98] for an overview on the effects of sampling on system properties.

Another reason for addressing discrete-time control systems is that in some cases it is more natural to describe a system by a set of difference equations. This is the case when, for example, some system variables have discrete values. Such examples can be found in many fields, like computer science [41], biology [74], economics [28] etc. Moreover, the models developed via identification are in majority of cases also discrete-time models, see [65].

A lot of research has been done to study the properties or the control methods for nonlinear discrete-time systems. Nevertheless, there are still some missing pieces, that needs to be filled. When looking for solutions to different synthesis problems, the majority of the contributions use the state feedback. This choice is problematic when all the states are not measured. Unfortunately, there are not a lot of contributions for discrete-time systems when a measurement feedback is applied [52, 78, 56, 73]. Also, the study of the flatness property (or the dynamic feedback linearization problem) of discrete-time systems does not have a full computable solution.

Time-delay Systems

In some cases the ordinary differential equations do not describe all the physical effects of dynamical systems in the best way. For example, transporting information over long distances takes always some time and often more precise models are necessary, that take such effects into account. This gives rise to the so-called time-delay control systems, described by a set of functional differential equations. In some other cases the delays can be introduced by actuators and sensors. The time-delay systems are used in many application areas, like telecommunications, control over networks, medicine and biological systems (see [84] and the references therein). When working with time-delay systems, the typical assumption made is that the different delays are commensurable, i.e. multiples of some fixed minimal delay.

Just like in the delay-free case, one can discretize continuous time-delay systems by the Euler discretization scheme [26], or use more advanced methods [99]. However, exact discretization is much more difficult to perform and if the states are delayed, not even always possible [90]. Under the assumption that *the delay is a multiple of the discretization step*, the approximate discretization yields a discrete time-delay system, described by the equations of the form

$$x(k+1) = F(x(k), x(k-1), \dots, x(k-p), u(k), u(k-1), \dots, u(k-q)). \quad (1)$$

Observe that the above assumption results in a discrete-time model that depends only on a single invertible shift operator (delays can be viewed as a result of applying backward-shift operator), whereas the continuous time-delay system depends on two operators, acting on the system variables: the time-derivative operator and the delay operator. Due to the form (1), one can always eliminate the delay from the state variables. For that, one just has to extend the state by defining the new state variables as $z(k) = (x(k), x(k-1), \dots, x(k-p))$ which yields a system of the form

$$z(k+1) = F_*(z(k), u(k), u(k-1), \dots, u(k-q)),$$

where there is no delay in the state variables. Moreover, often[1] one can eliminate all the delays from (1) and work instead with a higher dimensional discrete-time system. For that reason, in this thesis the more challenging case of continuous time-delay systems is studied instead of discrete time-delay systems.

The study of time-delay systems has received much attention over the past decades. However, most of the results are only valid for linear systems and majority of work is devoted to stability issues. The structural control problems, like accessibility, observability, feedback linearization, are not much studied, which makes the study of nonlinear time-delay systems practically unexplored area.

Methodology and Background

In nonlinear control theory there are two successful mathematical approaches to study the structural properties or address the structural design problems for nonlinear systems. The first is based on differential geometry (vector fields, their Lie derivatives and Lie brackets), see [89, 17], and the second, used in this thesis, is based on differential/difference algebra [24] and differential forms [23]. Both approaches have been applied to continuous- [45, 76, 25] and discrete-time [6, 39, 49] systems to solve various control problems. While the geometric approach assumes that the system is described by the state equations, the algebraic method can handle also the systems, defined by their input-output equations.

During the past 15 years, attempts have been made to generalize the well-known geometric and algebraic methods to a more general class of systems - nonlinear time-delay systems. The problem one faces here is infinite dimensionality of such systems, which does not suit well to neither of the approaches mentioned above. In [18] the extended Lie bracket was introduced and shown to be useful for analysis of nonlinear time-delay systems. A different approach, based on modules over non-commutative polynomial rings, is discussed in [94]. This method generalizes the differential algebraic approach to time-delay systems by looking the differential 1-forms

[1]When the equations (1) are realizable in the state space form.

as elements of a module. The module is defined over a non-commutative polynomial ring, and allows to describe nonlinear time-delay systems by a finite number of equations. The main gap in application of the algebraic methods for time-delay systems is integrability of a set of 1-forms.

The notion of involutive distribution or integrable codistribution is of major importance both in geometric and algebraic methods. In the delay-free case the property of integrability is checked by the Frobenius theorem, which unfortunately is no more valid for the time-delay case. More precisely, the Frobenius theorem can still be used in the study of time-delay systems, but often it does not yield meaningful results, because the effect of delays is not taken into account. Therefore, a more general notion of integrability must be defined. The problem has been studied in [18] using the extended Lie bracket and in [72, 69] using the modules over non-commutative polynomial rings. The latter approach is more natural, when studying integrability of the 1-forms, since they are defined as elements of modules for which an exact basis is looked for. The papers [72] and [69] gave necessary and sufficient integrability conditions for a single 1-form. Note that the definitions of integrability in these papers are different. Namely, in [69] a 1-form is said to be integrable if the module it generates has an exact basis (i.e. the basis that is a differential of a vector function), but in [72] an exact basis of a closure of the generated module is searched for. The case where multiple 1-forms are considered seems to be much more difficult and so far only sufficient conditions are given in [69].

In the difference algebraic approach, used in this thesis, one works with global linearized system descriptions. It means that solutions to various problems are found in terms of 1-forms and the integrability property allows to transform the solutions back to the level of functions. This is also true regarding the feedback linearization problem. Historically, linear control systems have been much more studied than nonlinear systems. This is obvious, because linearity is just a special case of nonlinearity. Moreover, control methods for linear systems are more developed than those for nonlinear systems. Unfortunately, in many applications, nonlinearities are common and unavoidable. The feedback linearization problem studies possibilities to transform the nonlinear system equations into linear by applying a feedback and a state coordinate transformation. A regular feedback defines, in some sense, an invertible transformation, which allows to control the original system by finding the control law for the transformed system. Therefore, if a nonlinear system can be linearized by a regular feedback and a state transformation, then further one can control, instead of the given nonlinear system, the corresponding linearized system, and later transform the solutions back to the original nonlinear system.

Most often, a state feedback is used to linearize nonlinear systems. The problem has received a lot of attention [37, 48, 47, 31, 8, 6, 32, 33, 77]. The complete solution exists when a static state feedback is looked for [37, 48, 6], but the dynamic feedback case is still not completely solved. It is a well-known fact that the dynamic state feedback linearization problem is closely linked to the system property, called flatness, introduced in [31, 32]. In simple words, flatness means that all the system variables can be written in terms of m functions (called flat outputs) and their derivatives/forward-shifts, where m is the number of system inputs. Despite a great number of publications on flatness (see for instance, [33, 88, 63, 64, 22] and the references therein), a complete, computable solution is still missing. Recently an algorithmic procedure was suggested in [64, 63] and implemented in [3] to find flat outputs of continuous-time system and generalized for the discrete-time case in [55, 53]. However, the procedure has two major flaws: it is not finite, i.e. it stops only when a flat output is found, and it requires searching for certain polynomial matrix, which may be a difficult task, because the polynomial matrix one looks for must be unimodular (i.e. invertible in the ring of given square matrices). Compared to the huge number of publications on dynamic feedback linearization that address continuous-time systems, there exist only a few which work with discrete-time systems [6, 55, 77, 85, 19]. The paper [77] focused on the dynamic equivalence of discrete-time systems and as an application, the results on feedback linearization were obtained. In [6] a necessary and sufficient condition was given for dynamic feedback linearization by an endogenous feedback, but it depends on the existence of certain unimodular polynomial matrix and thus is not constructive. Later, in [55], it was proved that the condition of [6] is equivalent to flatness property. In [19] a partial feedback linearization problem is studied. Finally, note that in the continuous-time case flatness is equivalent to dynamic feedback linearizability, which is not true for discrete-time case as shown in [7], where exogenous feedback solutions were found.

The state feedback solutions are common in nonlinear control theory, but this requires the knowledge of the states. If the states are not measurable, then one can either construct an observer or use an output/measurement feedback. Since an observer construction is in general not an easy task [96], then the study of output feedback solutions is useful. An advantage of an output feedback is also the fact that one does not need state equations of a system to apply it. Since a realization of an input-output model does not always exist, see for example [61], then in such a case, output feedback solutions are the only possibility. When searching an output feedback to linearize nonlinear system equations, there is no point of considering systems given by state equations, because one does not use the state variables. Thus, one usually linearizes the input-output (i/o) equations instead of state equations. Because the output feedback linearization problem is more restrictive (one uses only partial information on states to obtain the

	static feedback	dynamic feedback
SISO systems	[50, 81, 83]	[95, 52, 81]
MIMO systems	[51]	[2, 13]

Table 1: References to the DDP by measurement feedback.

feedback) than state feedback linearization, only few contributions can be found for the i/o linearization problem by output feedback. The papers [67, 82] solve the problem for continuous-time multi-input multi-output (MIMO) systems and the paper [78] for discrete-time single-input single-output (SISO) systems; both only look for static output feedback. The dynamic feedback solution was also given in [25] for continuous-time SISO case and in [52, 78] for discrete-time SISO systems, where sufficient conditions were derived. As for MIMO case, there are no results on dynamic output feedback linearization.

The i/o linearization is also useful in solving different decoupling problems by dynamic output or measurement feedback, as demonstrated in [52, 95, 80]. In [95, 52] the solution of the i/o linearization problem was applied to solve the disturbance decoupling problem (DDP) via dynamic measurement feedback. The goal of the DDP is to find a feedback such that in the closed-loop system, the system outputs do not depend (explicitly) on the disturbances anymore. The derived conditions in [95, 52] were only sufficient, since the solution of the i/o linearization problem, it relied on, was sufficient. Due to the fact that disturbances are very common in real applications, the DDP has received much attention, but most contributions look for the state feedback solutions [45, 76, 25, 5, 4, 70, 38, 30]. The first paper that applied measurement feedback to solve the DDP was [46], where sufficient solvability conditions were given for continuous-time systems. In [56], similar results as in [46] were given for the discrete-time case. The feedback used in [46, 56] is slightly different (less general) than the one in [95, 52], where the state of the compensator is not a function of the state of the system, but can be chosen independently of it. Different approaches (i.e. those not based on the i/o linearization) have been applied in [2] and [13] to study the solvability of the DDP by dynamic output/measurement feedback. A necessary condition is given in [2] and a sufficient condition in [13] for solvability of the DDP. An overview of papers considering a measurement feedback to solve the DDP and where the problem statement is as in this thesis, is given by Table 1.

One may also apply the results of i/o linearization to solve the i/o decoupling problem by dynamic output or measurement feedback, see [80].

	Continuous-time systems	Discrete-time systems
Static feedback	[12, 43, 79, 14, 44]	[73]
Dynamic feedback	[12, 80]	

Table 2: Output/measurement feedback references for i/o decoupling problem.

The problem is to find a feedback, such that every output of a closed-loop system depends exactly on a single distinct input of the closed-loop system. In this case one can control every system output separately. The i/o decoupling has many possible applications, see [66, 35, 20, 9]. In [66, 35] the i/o decoupling was used to control induction motors. In [20] the decoupling approach was used to control the model of proton exchange membrane fuel cells and in [9] to control the heating, ventilation and air-conditioning (HVAC) system. Most of the contributions that address the i/o decoupling problem use the state feedback (static or dynamic) to solve the problem, see for example [75, 76, 45, 25, 57]. The output or measurement feedback solutions, being more complicated than the state feedback solutions, are not that much studied, see Table 2. The static output and measurement feedback solutions are given in [12, 43, 79, 14, 44] for continuous-time case and in [73] for discrete-time case, respectively. The more complicated dynamic output or measurement feedback cases have only been studied in [12, 80] for continuous-time systems and provide only sufficient solvability conditions. Because majority of papers solve the i/o decoupling problem by state feedback, almost all of them, including the ones that consider output or measurement feedback, assume that the system is described by state equations. Only the papers [59, 73] address the case when the system is given by the set of higher-order input-output difference or differential equations. The structure of the feedback in [59] is very different from those, that consider output feedback. In particular, it depends on the values of past inputs and outputs and as such is more close to state feedback. In [73] necessary and sufficient solvability conditions by static output feedback were developed.

For time-delay nonlinear systems, the decoupling problems are also not much studied. Although the delay allows to use a more general feedback (see [72]), it also adds complexity. Typically one prefers a causal feedback, i.e. the feedback not depending on the future values of the system variables. If one follows the standard delay-free state feedback disturbance decoupling procedure, then it is possible to end up with non-causal solution. To avoid this to happen, feedback is allowed to depend only on certain variables. This makes the solution to the DDP of time-delay system by state feedback similar to that of the measurement feedback solution for the delay-free case, where the feedback is also allowed to depend only on some functions of the states (measurements). For nonlinear time-delay systems, the disturbance decoupling problem has been studied in [93, 72, 36, 68, 97]. In [93] the SISO case was considered and sufficient solvability condition via static feedback was derived. The results were extended for MIMO case in [36] when the number of inputs equals the number of outputs. The full solution for nonlinear SISO systems by bicausal static feedback was given in [72] and extended for MIMO systems in [68], where the concept of controlled invariant submodule was used to give (non-computable) necessary and sufficient solvability conditions of the DDP by a causal compatible compensator. In this thesis the conditions from [68] have been shown to be only sufficient. Except some sufficient conditions in [72] for SISO systems, there are no results for solvability of the dynamic DDP.

Contributions and Outline of the Thesis

The thesis focuses on feedback linearization of nonlinear discrete-time control systems and some related problems. Three main problems to be solved are:

(i) i/o linearization by dynamic output feedback

(ii) linearization of state equations by dynamic endogenous state feedback (flatness problem)

(iii) integrability conditions for differential 1-forms in the case of continuous time-delay systems.

Moreover, the solution of the i/o linearization problem is used to solve the decoupling problems by output or measurement feedback, which are, in turn, very similar to state feedback solutions in case of time-delay systems.

At this point it needs to be explained how the time-delay integrability problem is linked to the solution of the feedback linearization problem of discrete-time system. As said above, in the algebraic setting used in this thesis, in case of time-delay systems the modules of 1-forms are defined over certain polynomial ring, which allows to handle such systems as finite dimensional. More precisely, in such module the effect of delays in system variables is taken into account, which is not the case when the delayed variables are looked as elements of a vector space over the field of meromorphic functions (as in the delay-free case). In nonlinear discrete-time setting, a similar polynomial ring is often used to solve various control problems, see for example [6, 15]. The difference is that in the time-delay case the polynomial variable acts as a delay operator on an 1-form, but in the discrete-time case as a forward-shift operator. Now, it is proved in [6] that a nonlinear discrete-time system is dynamic endogenous feedback linearizable if and only if there exists an unimodular polynomial matrix, which transforms certain vector of 1-forms into a vector of exact 1-forms. As it turns out, the time-delay integrability problem of a set of 1-forms reduces to a similar mathematical problem, i.e. one has to find a polynomial matrix, which transforms a given vector of 1-forms into a vector of exact 1-forms. In that sense, the problem of integrability of the 1-forms in the time-delay case is closely related to the works of [64, 63, 53], where similar polynomial matrices were searched.

Next, the contributions of the thesis are highlighted. The work is divided into chapters based on the three main problems considered, which are listed above as: (i), (ii) and (iii), whereas the first chapter is devoted to introduction of the methodology.

Chapter 1

The chapter is divided into three sections, which describe the objects of the study, i.e. the classes of systems considered in the thesis, as well as the mathematical tools. The first section introduces the systems and the second describes the main tools for discrete-time systems, given by state equations. For the other system descriptions, the methodology is similar and only the main differences are commented. The difference field is constructed from the system description and the vector spaces of differential forms over the difference field are introduced. In the third section another algebraic method is described for the study of nonlinear discrete-time control systems. It is based on lattice theory and allows to consider non-smooth functions. The method will be used in Chapter 3 to study the possibilities to linearize non-smooth systems by static state feedback and a state transformation.

Chapter 2

In this chapter discrete-time systems are considered. The chapter is devoted to the study of the i/o linearization problem via dynamic output feedback. A complete solution is given, which generalizes the previous results. The necessary and sufficient conditions for the existence of a linearizing feedback are given in terms of certain functions, which are computed from the system equations. The conditions guarantee the solvability of certain system of algebraic equations, the solution of which gives the required feedback. In the second and third sections, the i/o linearization problem is shown to be useful in solving the i/o decoupling and disturbance decoupling problems by dynamic measurement feedback, respectfully. For the i/o decoupling problem necessary and sufficient solvability conditions are found for systems, described either by state or i/o equations. The sufficient condition for solvability of the disturbance decoupling problem generalizes the results of [52] to the MIMO case.

Chapter 3

In Chapter 3 one works with nonlinear discrete-time systems described by state equations. A state feedback and a state transformation are used to linearize the system. First, the algebraic method, based on the lattice theory, is used to study linearization of possibly non-smooth systems by the state transformation and the static state feedback. Necessary and sufficient solvability conditions are given and compared to the well-known results from [6], when the system under consideration is analytic.

The second section is devoted for finding flat outputs. A different approach compared to [6, 64, 63, 55, 53] is used, which has some similarities with the papers [86, 87], addressing the continuous-time case. Following this approach the original system equations are transformed into certain form by the state transformation and the static state feedback, which allows to eliminate some of the system equations. Now, one can continue with a lower dimensional system and repeat the process. It is proved that a discrete-time system is flat if and only if one is able to eliminate step-by-step all the system equations. Compared to the previous results, the computations needed here to verify flatness are much easier.

Chapter 4

The last chapter is devoted to nonlinear time-delay systems. The integrability definition of a set of 1-forms is generalized to time-delay case. Note that in the delay-free case, if a set of 1-forms is integrable, there exists an invertible matrix, which is defined over *the field of functions*, such that the matrix transforms the vector of exact 1-forms into the given vector of 1-forms. Since in the time-delay case, the 1-forms are looked as elements of a module, now that matrix is defined over *a ring of polynomials*. Thus, there are two possibilities to generalize the integrability notion: either to require that the matrix has a full rank or to require that it is invertible in the ring of square polynomial matrices. The first case leads to the weak integrability and the second to the strong integrability, respectivly. For both cases the necessary and sufficient conditions are found to check the property. The new concepts of integrability are shown to be useful in studying the accessibility property of time-delay systems. It has turned out that in majority of cases weak integrability is enough to solve various problems. The strong integrability is mostly used in verifying the weak integrability property.

Moreover, in the second and third sections the disturbance and i/o decoupling problems are considered, respectively, where the weak integrability property is applied. A pure shift dynamic feedback is used to solve the disturbance decoupling problem as well as the i/o decoupling problem. A more general dynamic feedback is also used to decouple the disturbances from the SISO time-delay system.

Chapter 1

Preliminaries

The goal of this chapter is to introduce the objects of study as well as the tools used to solve different problems in the following chapters. The first section gives a short overview of system classes, considered in this thesis: discrete-time systems described by state or input-output (i/o) equations and time-delay systems. In the second section, the main algebraic setting, used in the thesis, is described. In this approach, analysis of systems and construction of feedback are based on the global linearized system description, which is expressed via differential 1-forms. First, a difference field, defined by the system equations, is introduced. Second, the vector spaces of differential forms over the field of meromorphic functions are defined and the main concepts about integrability of differential 1-forms and system invertibility are discussed.

A different algebraic setting, based on the lattice theory, is developed in the last section. It will be used in Section 3.1 to study the static state feedback linearization problem of non-smooth discrete-time systems.

In most part of this thesis, instead of smooth functions, a more restrictive class of analytic functions, which define the system equations, is considered. The reason is that the ring of smooth functions is not an integral domain, i.e. it contains zero divisors, and thus, cannot be embedded into a field of fractions, which is necessary in order to construct the difference field. Therefore, the ring of analytic functions is considered instead, which is an integral domain. The elements of the field of fractions of analytic functions, are called meromorphic functions. The use of meromorphic functions is essential for carrying out divisions in the computations. Additionally, the use of analytic and meromorphic functions allows to study the generic properties of the systems. The latter means that the properties hold on some open and dense subsets of suitable domains if they hold at some point of this domain, see [25]. That is, generic properties hold in almost all situations. The study of such properties allows to express the solutions in a more compact way, since there is no need to specify the working point and its neighborhood.

Note that throughout the thesis all the functions and transformations are assumed to be meromorphic, if not stated otherwise.

1.1 System Descriptions

In this section systems, which will be studied in this thesis, are introduced. The discrete-time systems can be described either by state equations or by i/o equations. These systems are assumed to have certain properties, which allow to build the algebraic approach to study them. Also, time-delay systems are introduced, where the delays are commensurable.

1.1.1 Nonlinear Discrete-time Systems

The nonlinear discrete-time system is typically described by a set of first-order difference equations, called the state space representation,

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1.1}$$

where $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the input, $y(t) \in Y \subset \mathbb{R}^p$ is the output of a system. It is assumed, that the system (1.1) is submersive, that is it satisfies generically, i.e. on an open and dense subset of $X \times U$, condition

$$\operatorname{rank}\Big[\frac{\partial f(\cdot)}{\partial(x(t), u(t))}\Big] = n. \tag{1.2}$$

The condition (1.2) is specific for discrete-time systems and needed to construct an inversive difference field $(\mathcal{K}, \delta)$ below. In the case of continuous-time systems, such condition (1.2) is not necessary, since the differential field $(\mathcal{K}, \frac{\mathrm{d}}{\mathrm{d}t})$ is already inversive.

An important notion in nonlinear control is the relative degree of the output component y_i, which is defined for system (1.1) as follows.

Definition 1.1. The relative degree r_i of an output component $y_i(t)$ with respect to the control input $u(t)$ is defined as

$$r_i := \min\{k \in \mathbb{N} \mid \frac{\partial y_i(t+k)}{\partial u_j(t)} \not\equiv 0 \ \text{ for some } \ j \in \{1, \ldots, m\}\}.$$

Alternatively, a nonlinear discrete-time system can be described by the set of higher order difference equations, called input-output (i/o) equations, that relate the system inputs, outputs and their forward-shifts:

$$y_i(t+n_i) = \Phi_i(y_\tau(t), \ldots, y_\tau(t+n_{i\tau}), u_j(t), \ldots, u_j(t+q_i)) \tag{1.3}$$

for $i, \tau = 1, \ldots, p$, $j = 1, \ldots, m$. The indices in (1.3) are supposed to satisfy the conditions

$$\begin{array}{lll} n_1 \le n_2 \le \cdots \le n_p, & n_{i\tau} < n_\tau & q_i < n_i \\ n_{i\tau} < n_i, & \tau \le i & \\ n_{i\tau} \le n_i, & \tau > i. & \end{array} \tag{1.4}$$

The restrictions (1.4) mean that the equations (1.3) are assumed to be in the Popov form [11]. This guarantees that the indices n_i are unique up to permutation [21]. Note that under mild conditions, one can always transform an arbitrary set of i/o equations, at least locally, into the form (1.3), see [60, 10]. The systems (1.3) are often obtained as the results of identification process and thus are important objects of study.

Like in the case of state equations (1.1), the equations (1.3) are also assumed to satisfy the submersivity condition, i.e. the map $\Phi = (\Phi_1, \ldots, \Phi_p)^T$ satisfies generically the condition

$$\mathrm{rank}\left[\frac{\partial \Phi(\cdot)}{\partial(y(t), u(t))}\right] = p,$$

where $y(t) = (y_1(t), \ldots, y_p(t))$ and $u(t) = (u_1(t), \ldots, u_m(t))$.

1.1.2 Time-delay Systems

In this thesis the nonlinear continuous-time systems with commensurable time-delays (i.e. all the delays are multiples of a fixed minimal delay) are considered, described by the equations

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t), x(t-1), \ldots, x(t-D), u(t), u(t-1), \ldots, u(t-D)) \\ y(t) & = & h(x(t), \ldots, x(t-D)), \end{array} \tag{1.5}$$

where $D > 0$, $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the control input, $y(t) \in Y \subset \mathbb{R}^p$ is the output of the system.

The relative degrees of system outputs y_i, $i = 1, \ldots, p$, are defined similarly as for the discrete-time systems.

Definition 1.2. [72] The relative degree r_i of output $y_i(t)$ with respect to the control input $u(t)$ is defined as

$$r_i = \min\{k \in \mathbb{N} \mid \exists \tau \in \mathbb{N} \text{ and } j \in \{1, \ldots, m\} \text{ s.t. } \frac{\partial y_i^{(k)}(t)}{\partial u_j(t-\tau)} \not\equiv 0\}.$$

It is also useful to characterize the minimal τ, that appears in the last definition.

Definition 1.3. [72] The relative shift μ_i of $y_i(t)$ is defined as

$$\mu_i = \min\{\tau \in \mathbb{N} \mid \exists j \in \{1, \ldots, m\} \text{ s.t. } \frac{\partial y_i^{(r_i)}(t)}{\partial u_j(t-\tau)} \not\equiv 0\}.$$

1.2 Algebraic Setting

In this section the algebraic methods and different tools, that will be used in the thesis, are briefly described. In what follows, the notations x, $x^{[k]}$, $k \in \mathbb{Z}$ are used, instead of $x(t)$ and $x(t+k)$. Similar notations are used for the other variables. The element $x^{[k]}$ must be understood as a variable of the field $\mathcal{K}$ below, and not as a function of time t.

Introduce, for system (1.1), the set of *independent* (in the sense that they are not functionally dependent) system variables $\mathcal{C}_1 = \{x, u^{[k]}; k \geq 0\}$. For system (1.3), this set is $\mathcal{C}_2 = \{y_i^{[n_i-1]}, \ldots, y_i, u_j^{[k]}; i = 1, \ldots, p; j = 1, \ldots, m; k \geq 0\}$ and for the time-delay system (1.5), $\mathcal{C}_3 = \{x^{[-d]}, (u^{(k)})^{[-d]}; d, k \geq 0\}$. To continue, one has to choose the appropriate set $\mathcal{C}_i$, $i = 1, 2, 3$, depending on the given system description. In this section, the set $\mathcal{C}_1$ is chosen, i.e. the algebraic setting is described for system (1.1). The other two cases are similar and only the most important differences will be commented below.

A more detailed description of the approach for system (1.1) can be found, for instance, from [6].

1.2.1 Difference Field

In this subsection the difference field will be constructed, defined by the equations (1.1). Consider the field $\mathcal{K}$ of meromorphic functions in the variables from the set $\mathcal{C}_1$, i.e. the field of fractions of the ring of analytic functions, depending on variables from $\mathcal{C}_1$. In the field $\mathcal{K}$, the forward shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined for the elements of $\mathcal{C}_1$ by the relations $\delta x = x^{[1]} = f(x, u)$, $\delta u^{[k]} = u^{[k+1]}$, for $k \geq 0$. Applying the forward shift to a function means shifting all the arguments of the function, i.e.

$$\delta[\varphi(x, u, \ldots, u^{[k]})] = \varphi(f(x, u), u^{[1]}, \ldots, u^{[k+1]}).$$

Under the submersivity assumption (1.2) the operator δ is an injective endomorphism and the pair $(\mathcal{K}, \delta)$ a difference field [24]. In general, the difference field $(\mathcal{K}, \delta)$ is not inversive, i.e. the operator δ is not an automorphism. Nevertheless, one can always find an overfield $\mathcal{K}^*$ of $\mathcal{K}$, such that if δ is extended to $\mathcal{K}^*$, it becomes an automorphism [24]. The extension is made by adding variables $z = \chi(x, u)$ to $\mathcal{K}$, such that the map $\bar{f} = (f, \chi)^T$ becomes generically invertible. The difference field $(\mathcal{K}^*, \delta)$ is called an inversive difference field. In this thesis, the inverse of δ is denoted by δ^{-1}, and defined analogously to δ. For more information on difference algebra, see [24], and on construction of the field $\mathcal{K}^*$, see [6]. With slight abuse of notation, in the thesis, the inversive difference field $(\mathcal{K}^*, \delta)$ is often denoted simply by $\mathcal{K}$.

Remark 1.1. In the case of time-delay systems (1.5), the operator δ, called delay operator, does not shift a function forward, but backward, and is defined as $\delta\xi^{[-k]} = \xi^{[-k-1]}$ for any $\xi \in \mathcal{C}_3$ and $k \geq 0$. Moreover, a differential field $(\mathcal{K}, \frac{\mathrm{d}}{\mathrm{d}t}, \delta)$ is constructed, which depends on two operators - the time-derivative operator and the delay operator. For simplicity, the difference field $(\mathcal{K}, \frac{\mathrm{d}}{\mathrm{d}t}, \delta)$ is denoted simply by $\mathcal{K}$.

The difference field $\mathcal{K}$ and the operator δ induce a non-commutative polynomial ring, denoted by $\mathcal{K}[\vartheta]$. A element of $\mathcal{K}[\vartheta]$ is a polynomial $p(\vartheta)$ of the form

$$p(\vartheta) = \sum_{i=0}^{\gamma} a_i \vartheta^i,$$

where $a_i \in \mathcal{K}$, $i = 0, \ldots, \gamma$, and by ϑ is denoted the polynomial indeterminate. The multiplication in $\mathcal{K}[\vartheta]$ is defined by the rule:

$$\vartheta\varphi = \delta(\varphi)\vartheta$$

for a function $\varphi \in \mathcal{K}$. In $\mathcal{K}[\vartheta]$ addition is defined in a usual way. Let us recall some important properties of the ring $\mathcal{K}[\vartheta]$:

- $\mathcal{K}[\vartheta]$ is an integral domain;
- $\mathcal{K}[\vartheta]$ satisfies the left Ore condition (see [27] for discrete-time case and [94] for time-delay case).

The set of $s \times q$ matrices over $\mathcal{K}[\vartheta]$ is denoted by $\mathcal{K}[\vartheta]^{s\times q}$. A special subset of $\mathcal{K}[\vartheta]^{q\times q}$ is the set of unimodular matrices, denoted by $\mathcal{U}_q[\vartheta]$. A unimodular matrix is defined as follows.

Definition 1.4. A matrix $U \in \mathcal{K}[\vartheta]^{q\times q}$ is called unimodular if there exists a matrix $U^{-1} \in \mathcal{K}[\vartheta]^{q\times q}$ such that $UU^{-1} = U^{-1}U = I_q$.

A useful property for polynomial matrices in $\mathcal{K}[\vartheta]^{s\times q}$ is the Jacobson decomposition, see [24].

Theorem 1.1. *[24] For every $M(\vartheta) \in \mathcal{K}[\vartheta]^{s\times q}$, there exist matrices $V(\vartheta) \in \mathcal{U}_s[\vartheta]$ and $U(\vartheta) \in \mathcal{U}_q[\vartheta]$ such that*

$$V(\vartheta)M(\vartheta)U(\vartheta) = \begin{cases} (\Delta_s, 0_{s,q-s}), & \text{if } s \leq q; \\ \begin{pmatrix} \Delta_q \\ 0_{s-q,q} \end{pmatrix}, & \text{if } s \geq q, \end{cases} \tag{1.6}$$

where $0_{s,q-s}$ and $0_{s-q,q}$ are the matrices with zero entries, Δ_s and Δ_q are square diagonal matrices with elements $(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ such that $\sigma_i \in \mathcal{K}[\vartheta]$, for $i = 1, \ldots, k$, and σ_i is a divisor of σ_{i+1} for all $i = 1, \ldots, k-1$, i.e. $\sigma_{i+1} = \alpha\sigma_i$ for some $\alpha \in \mathcal{K}[\vartheta]$.

Note that the matrices $U(\vartheta)$ and $V(\vartheta)$ in Theorem 1.1 are not unique whereas Δ_s (respectively Δ_q) is. The matrix $(\Delta_s, 0_{s,q-s})$ (respectively $(\Delta_q, 0_{s-q,q})^T$) is called the Jacobson form of the matrix $M(\vartheta)$.

1.2.2 Differential Forms

In this subsection a brief overview of differential forms is given. For a more complete description, see [23].

Consider the set of symbols

$$\mathrm{d}\mathcal{C}_1 = \{\mathrm{d}x, \mathrm{d}u^{[k]}; k \geq 0\}.$$

A differential q-form α is an object of the form

$$\alpha = \sum_{\xi_i \in \mathrm{d}\mathcal{C}} a_{\xi_1,\ldots,\xi_q} \xi_1 \wedge \cdots \wedge \xi_q,$$

where a finite number of functions $a_{\xi_1,\ldots,\xi_q} \in \mathcal{K}$ are non-zero. Let $\mathcal{E}_q$ denote the set of q-forms, which has the structure of a vector field. Also, denote the field (of functions) $\mathcal{K}$ as the set of 0-forms $\mathcal{E}_0$. Next, two operations, the exterior product and exterior differential, are defined for the differential forms.

The exterior (or wedge) product $\wedge : \mathcal{E}_q \times \mathcal{E}_s \to \mathcal{E}_{q+s}$ is a bilinear and associative map, which has the properties

- $\alpha \wedge \beta = (-1)^{qs} \beta \wedge \alpha$, where $\alpha \in \mathcal{E}_q$ and $\beta \in \mathcal{E}_s$;
- $\alpha \wedge \alpha = 0$, if $q = s$ is odd number.

The exterior differential $\mathrm{d} : \mathcal{E}_q \to \mathcal{E}_{q+1}$ is an operator, satisfying the following properties:

- $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^s \alpha \wedge \mathrm{d}\beta$, where $\alpha \in \mathcal{E}_s$;
- $\mathrm{d}\alpha$ coincides with the ordinary differential if α is a 0-form, i.e. $\alpha \in \mathcal{K}$;
- $\mathrm{d}^2 = 0$.

The vector space of 1-forms can be now defined as $\mathcal{E}_1 = \mathrm{span}_{\mathcal{K}} \mathrm{d}\mathcal{C}_1$ and a 1-form has the form

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i + \sum_{k \geq 0} \sum_{j=1}^{m} b_{j,k} \mathrm{d}u_j^{[k]}, \tag{1.7}$$

where only a finite number of coefficients $b_{j,k}$ are non-zero. Often, the vector space of 1-forms is simply denoted by $\mathcal{E}$.

The shift operator δ can be extended to $\mathcal{E}$ in the following way. For a 1-form ω, given by (1.7), its forward shift is defined by the rule

$$\delta\omega = \sum_{i=1}^{n} \delta(a_i)\mathrm{d}\big(\delta(x_i)\big) + \sum_{k\geq 0}\sum_{j=1}^{m} \delta(b_{j,k})\mathrm{d}\big(\delta(u_j^{[k]})\big).$$

The backward shift δ^{-1} may be extended to $\mathcal{E}$ in a similar manner.

The set $\mathcal{E}$ of 1-forms is a vector space, but it can also be given a structure of a module. Roughly speaking, a module is a vector space, defined over a ring, not a field. Unlike a vector space, not every module has a basis. The modules, that do have a basis, are called free modules. Since $\mathcal{K}[\vartheta]$ satisfies the left Ore condition, any two basis of a free module have the same cardinality, which is called the rank of the free module. See [1] for overview of module theory.

Now, the set of 1-forms has also the structure of a module, since every element of $\mathcal{E}$ is also an element of

$$\mathcal{M} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u\}$$

and vice-versa, every element of $\mathcal{M}$ is an element of a vector space $\mathcal{E}$.

Definition 1.5. [94] The closure of a free submodule $\mathcal{A}$ of $\mathcal{M}$, denoted by $cl_{\mathcal{K}[\vartheta]}(\mathcal{A})$, is defined as

$$cl_{\mathcal{K}[\vartheta]}(\mathcal{A}) = \{\omega \in \mathcal{M} \mid \exists p(\vartheta) \in \mathcal{K}[\vartheta], \text{ s.t. } p(\vartheta)\omega \in \mathcal{A}\}.$$

By definition, the closure of the free submodule $\mathcal{A}$ is the largest free submodule, containing $\mathcal{A}$, and having the same rank as $\mathcal{A}$.

1.2.3 Integrability of 1-forms

In the thesis one usually works with the generically (globally) linearized system description, i.e. with the 1-forms, and not with the system equations themselves. In the final step of solution (regarding any problem) one should go back to the level of functions. For that, one has to integrate the 1-forms. Unfortunately, since the differential operator $\mathrm{d} : \mathcal{K} \to \mathcal{E}$ is not one-to-one, one can not just inverse d to go from $\mathcal{E}$ to $\mathcal{K}$. The elements ω of $\mathcal{E}$ for which the inverse exists (at least locally) are called exact and can be written as $\omega = \mathrm{d}\varphi$ for some function $\varphi \in \mathcal{K}$. Note that the result is not unique, since $\omega = \mathrm{d}\varphi = \mathrm{d}(\varphi + c)$, where $c \in \mathbb{R}$. An integrable 1-form is exact up to multiplication by a function $\lambda \in \mathcal{K}$, called the integrating coefficient. Therefore, integrable 1-form ω can be written as $\omega = \lambda\mathrm{d}\varphi$ for some functions $\lambda, \varphi \in \mathcal{K}$. A vector space $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k\}$ is said to be integrable if it has a basis consisting of exact 1-forms. A condition to check whether a vector space $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k\}$ is integrable is given by the Frobenius theorem.

Theorem 1.2. *[25] A subspace* $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k\}$*, where* ω_i*,* $i = 1, \ldots, k$ *are independent, is integrable if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$$

for $i = 1, \ldots, k$*.*

Note that by this theorem, a 1-form ω is integrable if and only if $\mathrm{d}\omega \wedge \omega = 0$. Sometimes the following notation is also used: $\mathrm{d}\omega = 0 \bmod \mathrm{span}_{\mathcal{K}}\{\bar{\omega}_1, \ldots, \bar{\omega}_k\}$, which means that $\mathrm{d}\omega \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k = 0$.

A concept which generalizes the notion of integrability, is the rank of a 1-form.

Definition 1.6. The rank of a 1-form ω, denoted by $\mathrm{rank}\,\omega$, is the minimal number $\gamma \in \mathbb{N}$ such that

$$\omega \wedge (\mathrm{d}\omega)^{\gamma} = 0$$

where $(\mathrm{d}\omega)^{\gamma}$ denotes the γ-fold wedge product of $\mathrm{d}\omega$ with itself.

According to this definition, a 1-form is integrable if and only if its rank is 1. If $\gamma \in \mathbb{N}$ is the rank of a 1-form ω, then there exist functions φ_i, $i = 1, \ldots, \gamma$ such that

$$\omega = \sum_{i=1}^{\gamma} a_i \mathrm{d}\varphi_i$$

for some functions $a_i \in \mathcal{K}$. Moreover, γ is equal to the minimal number of exact 1-forms whose linear combination gives ω. Thus, the rank of a 1-form is equal to the dimension of a minimal integrable subspace that contains ω. Based on this meaning, the next lemma is given.

Lemma 1.1. *A minimal integrable subspace that contains the vector space* $\mathcal{A} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k\}$ *has the dimension*

$$\gamma := \max_{\alpha_i \in \mathcal{K}} [\mathrm{rank}(\alpha_1\omega_1 + \cdots + \alpha_k\omega_k)]. \tag{1.8}$$

Proof. The integrable subspace $\bar{\mathcal{A}}$ that contains the vector space $\mathcal{A}$ has to contain every element of $\mathcal{A}$, i.e. $\alpha_1\omega_1 + \cdots + \alpha_k\omega_k$. Thus, the dimension γ of $\bar{\mathcal{A}}$ satisfies

$$\gamma \geq \max_{\alpha_i \in \mathcal{K}} [\mathrm{rank}(\alpha_1\omega_1 + \cdots + \alpha_k\omega_k)].$$

Let $\alpha_1\omega_1 + \cdots + \alpha_k\omega_k$ be a 1-form that has rank γ. Then, there exist γ functions φ_i, $i = 1, \ldots, \gamma$, such that

$$\alpha_1\omega_1 + \cdots + \alpha_k\omega_k \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_{\gamma}\}.$$

Note that $\alpha_i \not\equiv 0$, $i = 1, \ldots, k$, since otherwise the 1-forms ω_i, $i = 1, \ldots, k$, are not independent, and thus for $i = 1, \ldots, k$,

$$\omega_i \in \text{span}_{\mathcal{K}}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_\gamma\}. \tag{1.9}$$

Really, if for example, (1.9) is not true for ω_1, then the 1-form

$$(1 + \alpha_1)\omega_1 + \alpha_2\omega_2 + \cdots + \alpha_k\omega_k$$

should have a bigger rank than γ. This is not possible and therefore, (1.9) is true and $\bar{\mathcal{A}} = \text{span}_{\mathcal{K}}\{\text{d}\varphi_1, \ldots, \text{d}\varphi_\gamma\}$. □

1.2.4 Inversion Algorithm and Invertibility

The inversion algorithm for system (1.1), where the output $y = h(x, u)$ is also allowed to depend on the input u and $\dim y = \dim u = m$, is recalled in this subsection. Based on this algorithm the conditions for invertiblity of the system (1.1) are given. In the case of systems of the form (1.3), the right-invertibility is defined, since in this case one does not require that $m = p$. For more about inversion algorithm and invertibility, see [57] for the case when the system is described by the state equations and [58] when it is given by the i/o equations.

Inversion Algorithm:

Step 0. Let $\xi_0 := \text{rank}_{\mathcal{K}} \frac{\partial y}{\partial u}$. Decompose the output y into two parts $y = (\tilde{y}_0, \bar{y}_0)$ such that $\dim \tilde{y}_0 = \xi_0$ and

$$\text{rank}_{\mathcal{K}} \frac{\partial \tilde{y}_0}{\partial u} = \xi_0.$$

Step i. Let $\xi_i := \text{rank}_{\mathcal{K}} \frac{\partial(\tilde{y}_0, \ldots, \tilde{y}_{i-1}^{[i-1]}, \bar{y}_{i-1}^{[i]})}{\partial u}$. Decompose $\bar{y}_{i-1}^{[i]}$ into two parts $\bar{y}_{i-1}^{[i]} = (\tilde{y}_i^{[i]}, \bar{y}_i^{[i]})$ such that $\dim \tilde{y}_i^{[i]} = \xi_i - \xi_{i-1}$ and

$$\text{rank}_{\mathcal{K}} \frac{\partial(\tilde{y}_0, \ldots, \tilde{y}_i^{[i]})}{\partial u} = \xi_i.$$

The algorithm stops when

$$\text{rank}_{\mathcal{K}} \frac{\partial(\bar{y}_0, \ldots, \bar{y}_\rho^{[\rho]})}{\partial x} = \text{rank}_{\mathcal{K}} \frac{\partial(\bar{y}_0, \ldots, \bar{y}_{\rho-1}^{[\rho-1]})}{\partial x}$$

for some $\rho \geq 0$.

Next, the definition of invertibility of system (1.1) with the (extended) output $y = h(x, u, \ldots, u^{[l]})$, $y \in Y \subset \mathbb{R}^m$, is given. Such invertibility has an important role in the study of flatness property in Section 3.2. Namely, a system (1.1) with a flat output $y = h(x, u, \ldots, u^{[l]})$, $y \in Y \subset \mathbb{R}^m$ is invertible.

Definition 1.7. The system (1.1) with an output $y = h(x, u, \ldots, u^{[l]})$ is said to be invertible if one can write the system input u as a function of the state variable x, the output variable y and a finite number of its shifts $y^{[i]}$, $i = 1, \ldots, s$.

Lemma 1.2. *The system (1.1) with the output $y = h(x, u)$ is invertible if and only if*

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\tilde{y}_0, \ldots, \tilde{y}_\rho^{[\rho]})}{\partial u} = m,$$

where $\tilde{y}_i^{[i]}$, $i = 0, \ldots, \rho$, are defined by the inversion algorithm.

For systems of the form (1.3) the right-invertibility is defined as follows. Note that here one does not require that $m = p$.

Definition 1.8. The system (1.3) is said to be right-invertible if the polynomial matrix

$$\sum_{j=0}^{\max\{q_i\}} \frac{\partial(\Phi_1, \ldots, \Phi_p)}{\partial u^{[j]}} \vartheta^j$$

has rank p. Recall that q_i is defined by equations (1.3) as the highest shift of input u the function Φ_i depends on. When $m = p$, then one says that system (1.3) is invertible.

1.3 Functions' Algebra

In this section another algebraic approach, called functions' algebra [62, 100], is described, which is developed in analogy of the algebra of partitions [40]. The advantage of this method over difference algebraic and differential geometric methods is that it allows to handle also certain types of non-smooth functions. It will be used later, in Section 3.1, to develop conditions for static state feedback linearization of possibly non-smooth nonlinear discrete-time systems, described by the state equations. In this setting, one does not work with 1-forms, but directly with functions.

Consider the discrete-time system

$$x(t+1) = f(x(t), u(t)), \tag{1.10}$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$ and f is possibly non-smooth.

Denote by $S_{X \times U}$ the set of vector functions with the domain $X \times U$. The elements of $S_{X \times U}$ are vectors with finite dimension, whose elements are (possibly non-smooth) functions depending on the variables x and u. Note that in some cases the knowledge of one vector function yields the knowledge of another vector function. For example, if one knows the value of the vector function $\alpha = [x_1, x_2 x_3]^T =: [\alpha_1, \alpha_2]^T$ one knows also the value of the vector function $\beta = x_1x_2x_3$, since $\beta = \alpha_1\alpha_2$. Based on this, the relation of preorder $\leq$ is defined on the set $S_{X\times U}$.

Definition 1.9. Given $\alpha, \beta \in S_{X\times U}$, one says that $\alpha \leq \beta$ if for all $x \in X$, $u \in U$ there exists a vector function γ such that $\beta(x, u) = \gamma(\alpha(x, u))$.

Note that there exist non-equal vector functions $\alpha, \beta \in S_{X\times U}$ such that $\alpha \leq \beta$ and $\beta \leq \alpha$, which means that the relation $\leq$ is not a partial order. For example, the vector functions $\alpha = [x_1, x_2]^T$ and $\beta = [x_1 + x_2, x_2]^T$ are not equal, but they satisfy the conditions $\alpha \leq \beta$ and $\beta \leq \alpha$. To be able to build an algebraic structure for the study of nonlinear systems (1.10), such functions are defined to be equivalent.

Definition 1.10. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called equivalent, denoted by $\alpha \cong \beta$.

The relation $\cong$ is reflexive ($\alpha \cong \alpha$ for all $\alpha \in S_{X\times U}$), symmetric ($\alpha \cong \beta \Rightarrow \beta \cong \alpha$) and transitive ($\alpha \cong \beta$ and $\beta \cong \theta$ yield $\alpha \cong \theta$) and thus an equivalence relation. The equivalence relation divides the set $S_{X\times U}$ into the equivalence classes, containing the equivalent functions. If $S_{X\times U}/\cong$ is the set of all equivalence classes, then the relation $\leq$ becomes a partial order on this set. In this algebraic setting, one works with the set of equivalence classes $S_{X\times U}/\cong$ (or rather with their simplest representatives). This also means that in this setting the symbol "=" should be understood as "$\cong$".

There exist two special equivalence classes. The equivalence class $\mathbf{1}$ contains all the constant functions and satisfies $\alpha \leq \mathbf{1}$ for all $\alpha \in S_{X\times U}$. On the other hand, the equivalence class $\mathbf{0} := [x, u]^T$ satisfies $\mathbf{0} \leq \alpha$ for all $\alpha \in S_{X\times U}$. Therefore, for any two equivalence classes $\alpha, \beta \in S_{X\times U}/\cong$, there exist a minimal equivalence class γ, satisfying $\alpha \leq \gamma$, $\beta \leq \gamma$ and a maximal equivalence class ζ, satisfying $\zeta \leq \alpha$, $\zeta \leq \beta$. Thus, $(S_{X\times U}/\cong, \leq)$ has a structure of a lattice[1].

A lattice can be viewed as an algebraic structure with two binary operations $\times$ and $\oplus$, such that for every two elements α, β both operations are commutative and associative and moreover, $\alpha\times(\alpha\oplus\beta) = \alpha$, $\alpha\oplus(\alpha\times\beta) = \alpha$. The binary operations $\times$ and $\oplus$ are defined as

$$\begin{aligned} \alpha \times \beta &= \inf(\alpha, \beta) \\ \alpha \oplus \beta &= \sup(\alpha, \beta), \end{aligned} \tag{1.11}$$

where the ordering is with respect to relation $\leq$. Therefore, the triple $(S_{X\times U}/\cong, \times, \oplus)$ can also be viewed as a lattice. With a slight abuse of notation, the notation $S_{X\times U}$ is used in this thesis instead of $S_{X\times U}/\cong$.

[1]Recall that a lattice is a set with a partial order where every two elements α and β have a unique supremum (least upper bound) $\sup(\alpha, \beta)$ and a unique infimum (greatest lower bound) $\inf(\alpha, \beta)$.

To compute $\oplus$, in the simple cases, the definition (1.11) may be used. Computation of $\times$ is much more simple: $\alpha \times \beta = [\alpha, \beta]^T$. However, the product may contain functionally dependent components that have to be found and removed, which just means finding the simplest representative in the equivalence class for $\alpha \times \beta$.

Next, the lattice $(S_{X\times U}/\cong, \times, \oplus)$ is connected with the system dynamics (1.10) through the following definition of binary relation Δ. Since (1.10) defines only the forward shift of x, but not that of u, in the following definitions the vector functions must belong to S_X.

Definition 1.11. Given $\alpha, \beta \in S_X$, one says that (α, β) satisfy binary relation Δ, denoted as $\alpha\Delta\beta$, if for all $x \in X$ and $u \in U$, there exists a function f_* such that

$$\beta(f(x, u)) = f_*(\alpha(x), u).$$

The binary relation Δ is mostly used for definition of the operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 1.12. (i) $\mathbf{m}(\alpha)$ is a minimal vector function $\beta \in S_X$ that satisfies $\alpha\Delta\beta$;

(ii) $\mathbf{M}(\beta)$ is a maximal vector function $\alpha \in S_X$ that satisfies $\alpha\Delta\beta$.

Computation of the operator $\mathbf{m}$. Note that by the definition of Δ, the condition

$$\mathbf{m}(\alpha)(f) \geq \alpha \times u$$

must be satisfied for the vector function $\alpha(x)$. Obviously, $\mathbf{m}(\alpha)(f) \geq f$. Therefore, by the definition of operator $\oplus$

$$\mathbf{m}(\alpha)(f) = (\alpha \times u) \oplus f.$$

Finally, observe that $\mathbf{m}(\alpha)(x)$ can be computed by shifting the function $(\alpha \times u) \oplus f$ back once:

$$\mathbf{m}(\alpha)(x) = [(\alpha \times u) \oplus f]^{[-1]}. \tag{1.12}$$

Computation of the operator $\mathbf{M}$. To compute $\mathbf{M}$, there is no general formula. In the special case when $\beta(f(x, u))$ can be represented in the form

$$\beta(f(x, u)) = \sum_{i=1}^{d} a_i(x)b_i(u)$$

where a_1, a_2, ..., a_d are arbitrary functions and all the non-constant functions b_1, b_2, ..., b_d are functionally independent, then $\mathbf{M}(\beta) := a_1 \times a_2 \times \cdots \times a_d$.

Example 1.1. Consider the system

$$\begin{aligned} x_1^{[1]} &= x_2 u \\ x_2^{[1]} &= x_1 + x_3 \\ x_3^{[1]} &= x_3 + u \end{aligned}$$

and the vector function $\alpha = [x_1, x_2]^T$. First, compute $\mathbf{m}(\alpha)$ by (1.12):

$$\begin{aligned} \alpha \times u &= [x_1, x_2, u]^T \\ (\alpha \times u) \oplus f &= [x_1, x_2, u]^T \oplus [x_2 u, x_1 + x_3, x_3 + u]^T \\ &= [x_2 u, x_1 - u]^T = [x_1^{[1]}, x_2^{[1]} - x_3^{[1]}]^T \\ \mathbf{m}(\alpha) &= [(\alpha \times u) \oplus f]^{[-1]} = [x_1, x_2 - x_3]^T. \end{aligned}$$

Now, compute $\mathbf{M}(\alpha)$ using the discussion above. Since $\alpha(f(x,u)) = [x_2 u, x_1 + x_3]^T$, then $a_1(x) = x_2$, $b_1(u) = u$, $a_2(x) = x_1 + x_3$, $b_2(u) = 1$ and thus

$$\mathbf{M}(\alpha) = a_1 \times a_2 = x_2 \times (x_1 + x_3) = [x_2, x_1 + x_3]^T.$$

1.4 Conclusions

In this chapter different classes of systems were described and overview of algebraic approaches, used in this thesis, was given. In most parts of the thesis the difference algebraic approach, described in Section 1.2, will be used. Only in Section 3.1 the functions' algebra (see Section 1.3) will be applied. The difference algebraic approach was characterized in detail for discrete-time system, given by the state representation. For systems described by the i/o difference equations and for time-delay systems all the algebraic objects can be built similarly and thus these descriptions were omitted.

An important concept in the given algebraic approach is the rank of an 1-form. It can be used to compute the dimension of the minimal integrable vector space, that contains a given vector space of 1-forms, see Lemma 1.1. The latter plays an essential role in the solutions of the flatness problem (see Section 3.2) and the i/o linearization problem (see Section 2.1).

Chapter 2

Input-Output Linearization

Given the system description in the form (1.3), the main goal of this chapter is to find necessary and sufficient conditions, under which there exists a dynamic output feedback, which linearizes the equations (1.3). This problem is called input-output (i/o) linearization problem.

In the first section, the i/o linearization problem is solved. The necessary and sufficient conditions are expressed in terms of certain functions, computed from the system equations (1.3). The remaining two sections apply the results on i/o linearization to solve two decoupling problems. In Section 2.2, the i/o decoupling problem is considered. The goal is to transform a system into a form, where every system output depends on exactly one different system input, using dynamic measurement feedback. The problem is solved, i.e. necessary and sufficient solvability conditions are found and feedback constructed, for systems, described either by state equations or by i/o equations. In Section 2.3, sufficient conditions are given to eliminate the effects of disturbances from the system outputs by dynamic measurement feedback.

2.1 Input-Output Linearization by Dynamic Output Feedback

In this section, the necessary and sufficient conditions are found for solvability of the i/o linearization problem via dynamic output feedback for multi-input multi-output (MIMO) discrete-time systems of the form (1.3). The results generalize those from [52] to MIMO case and additionally, compared to [52], necessity of the conditions is proved.

To simplify the presentation, the following notation is used in this section: $\mathcal{E}^k := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{[k-1]}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{[k-1]}; i = 1, \ldots, p; j = 1, \ldots, m\}$ for any $k \in \mathbb{N}$.

2.1.1 Necessary and Sufficient Condition

Problem statement. Given a discrete-time system of the form (1.3), the goal is to find a dynamic output feedback of the form

$$\begin{aligned}\eta(t+1) &= F(\eta(t), y(t), v(t)) \\ u(t) &= H(\eta(t), y(t), v(t)),\end{aligned} \tag{2.1}$$

where $\eta(t) \in \Delta \subset \mathbb{R}^\rho$ and $v(t) \in V \subset \mathbb{R}^m$ are the state and the input of the compensator (2.1) respectively, such that the equations of the closed-loop system (1.3),(2.1) are linear. More precisely, one requires that for $i, \tau = 1, \ldots, p$ and $j = 1, \ldots, m$

$$\begin{aligned}\mathrm{d}y_i^{[n_i]} &\in \operatorname{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v\} \\ \mathrm{d}y_i^{[n_i]} &\notin \operatorname{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau\}.\end{aligned} \tag{2.2}$$

If such feedback exists, then one says that system (1.3) is i/o linearizable by dynamic output feedback. Additionally, it is required that the compensator (2.1) is regular, i.e. it is invertible.

The Theorem 2.1 below, that gives the necessary and sufficient solvability conditions, is expressed in terms of certain functions. These functions can be computed from the functions Φ_i, $i = 1, \ldots, p$, that define the system (1.3).

Note that there may be some terms on the right-hand side of (1.3), that depend already linearly on outputs and their forward shifts. Since one does not need to do anything with these terms, the first task is to eliminate such terms by defining the 1-forms $\tilde{\omega}_i$, $i = 1, \ldots, p$, as

$$\tilde{\omega}_i := \mathrm{d}\Phi_i \mod \operatorname{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau; \tau = 1, \ldots, p\}.$$

For solvability of the i/o linearization problem, it is necessary that[1] for $i = 1, \ldots, p$

$$\tilde{\omega}_i \in \mathcal{E}^{n_i - r_i + 1}, \tag{2.3}$$

where r_i is defined as $r_i := n_i - q_i$, since otherwise nonlinearities in (1.3) appear before the input u starts to affect the output y. The goal is to find a feedback of the form (2.1), such that in the closed-loop system $\operatorname{span}_{\mathbb{R}}\{\tilde{\omega}_i\} \subseteq \operatorname{span}_{\mathbb{R}}\{\mathrm{d}v\}$. To continue, only p_1 independent (over $\mathbb{R}$) 1-forms $\tilde{\omega}_i$, $i = 1, \ldots, p$, are kept. Therefore, let ω_i, $i = 1, \ldots, p_1$, be the basis elements[2] of $\operatorname{span}_{\mathbb{R}}\{\tilde{\omega}_i\}$. To simplify the presentation, in the rest of this subsection assume that $i, \tau = 1, \ldots, p_1$ and $j = 1, \ldots, m$.

Let σ_i be such that

$$\omega_i \in \mathcal{E}^{\sigma_i}.$$

[1] Observe that for $r_i = 1$, the condition (2.3) is always satisfied.

[2] These basis elements are exact, since 1-forms $\tilde{\omega}_i$ are exact.

Next, define the 1-forms

$$\bar{\omega}_{i,l} \in \text{span}_{\mathcal{K}}\{\mathrm{d}y^{[\sigma_i - l]}, \ldots, \mathrm{d}y^{[\sigma_i - 1]}, \mathrm{d}u^{[\sigma_i - l]}, \ldots, \mathrm{d}u^{[\sigma_i - 1]}\},$$

where $l = 1, \ldots, \sigma_i - 1$, such that

$$\omega_i - \bar{\omega}_{i,l} \in \mathcal{E}^{\sigma_i - l} \tag{2.4}$$

and

$$\bar{\omega}_{i,\sigma_i} := \omega_i. \tag{2.5}$$

Let $\gamma_{i,l}$ be the rank of a 1-form $\bar{\omega}_{i,l}$ for $l = 1, \ldots, \sigma_i$. Then there exist $\gamma_{i,l}$ functions $\tilde{\phi}^k_{i,l}(y^{[\sigma_i - l]}, \ldots, y^{[\sigma_i - 1]}, u^{[\sigma_i - l]}, \ldots, u^{[\sigma_i - 1]})$ such that

$$\bar{\omega}_{i,l} \in \text{span}_{\mathcal{K}}\{\mathrm{d}\tilde{\phi}^1_{i,l}, \ldots, \mathrm{d}\tilde{\phi}^{\gamma_{i,l}}_{i,l}\}.$$

Finally, define the function $\phi^k_{i,l}$ as the $(\sigma_i - l)$ step backward shift of the function $\tilde{\phi}^k_{i,l}$, i.e.

$$\phi^k_{i,l} := (\delta^{-1})^{\sigma_i - l}\tilde{\phi}^k_{i,l} = \delta^{l - \sigma_i}\tilde{\phi}^k_{i,l}$$

for $l = 1, \ldots, \sigma_i$ and $k = 1, \ldots, \gamma_{i,l}$.

Theorem 2.1. *Under the assumption (2.3) the system (1.3) is i/o linearizable by dynamic output feedback of the form (2.1) if and only if*

$$\dim(\text{span}_{\mathcal{K}}\{\mathrm{d}\phi^k_{i,l}\}) = \text{rank}_{\mathcal{K}} \frac{\partial \phi^k_{i,l}}{\partial(u, \delta\phi^k_{i,l^*})}, \tag{2.6}$$

for $l = 1, \ldots, \sigma_i$, $l^* = 1, \ldots, \sigma_i - 1$, $k = 1, \ldots, \gamma_{i,l}$ *and functions* ϕ^1_{i,σ_i} *are independent from* ϕ^k_{i,l^*}.

Proof. Sufficiency. Construct the feedback that solves the input-output linearization problem in the following way. Take all the independent functions ϕ^k_{i,l^*}, as the states of the compensator (2.1), i.e.

$$\eta_{i,l,k} := \phi^k_{i,l^*}. \tag{2.7}$$

Also, let

$$v_i := \phi^1_{i,\sigma_i}. \tag{2.8}$$

By (2.6) the system of equations (2.7), (2.8) is solvable with respect to the variables $\{u, \eta^{[1]}_{i,l,k}\}$. Note that if $p_1 < m$, the number of equations is less than that of variables, and so $m - p_1$ variables are free. Take these free variables equal to the new input v_π, $\pi = p_1 + 1, \ldots, m$. Solution of the equations (2.7), (2.8), with respect to the variables $\{u, \eta^{[1]}_{i,l,k}\}$, results in a feedback of the form (2.1). This feedback yields, because of (2.5) and

(2.8), $\omega_i = \mathrm{d}v_i$. From the definition of the 1-forms ω_i and $\tilde{\omega}_i$, one concludes $\mathrm{d}y_i^{[n_i]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v\}$, i.e. the system (1.3) is input-output linearized.

Necessity. To prove the necessity of condition (2.6), the following 1-forms are used: $\psi_{i,l} := \delta^{l-\sigma_i}\bar{\omega}_{i,l}$, $l = 1, \ldots, \sigma_i$. These 1-forms can be recursively computed as

$$
\begin{array}{rcl}
\psi_{i,1} & = & \bar{\psi}_{i,1} \\
\psi_{i,2} & = & \delta\psi_{i,1} + \bar{\psi}_{i,2} \\
& \vdots & \\
\psi_{i,\sigma_i-1} & = & \delta\psi_{i,\sigma_i-2} + \bar{\psi}_{i,\sigma_i-1} \\
\psi_{i,\sigma_i} & = & \delta\psi_{i,\sigma_i-1} + \bar{\psi}_{i,\sigma_i},
\end{array}
\tag{2.9}
$$

where $\bar{\psi}_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y\}$, $l = 1, \ldots, \sigma_i$. Also, it is obvious from the definition of 1-forms $\psi_{i,l}$ that $\psi_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,l}^k\}$, where $l = 1, \ldots, \sigma_i$ and $k = 1, \ldots, \gamma_{i,l}$.

Because of (2.2), in the closed-loop system, one has $\omega_i = \mathrm{d}v_i$. Since $\omega_i = \bar{\omega}_{i,\sigma_i} = \psi_{i,\sigma_i}$, one gets that $\psi_{i,\sigma_i} = \mathrm{d}v_i$. Thus, to find a feedback, that guarantees $\omega_i = \mathrm{d}v_i$, one has to take $\psi_{i,\sigma_i} = \mathrm{d}v_i$ in (2.9) and solve the set of equations in $\mathrm{d}u$ and $\delta\psi_{i,l}$, $l = 1, \ldots, \sigma_i - 1$. Now, use the concept of rank of a 1-form. Choose the state coordinates η of a feedback as the integrals of the basis elements of a 1-forms $\psi_{i,l}$, i.e. $\psi_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\eta\}$ like in (2.7). Since the given system is feedback linearizable, the system of equations (2.7)-(2.8) must be solvable with respect to the variables $\{u, \eta_{i,l,l}^{[1]}\}$. This means that (2.6) must be satisfied. Finally, the regularity of the feedback (2.1) guarantees that the functions ϕ_{i,σ_i}^1 are independent from ϕ_{i,l^*}^k, $l^* = 1, \ldots, \sigma_i - 1$, $k = 1, \ldots, \gamma_{i,l}$. □

Example 2.1. Consider the system

$$
\begin{array}{rcl}
y_1^{[4]} & = & y_1^{[3]} + u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1 \\
y_2^{[2]} & = & y_1^{[1]} u_1^{[1]} + u_3 y_2.
\end{array}
\tag{2.10}
$$

Observe that $r_1 = 2$ and $r_2 = 1$. Check the condition (2.6) for system (2.10). First, compute the functions $\phi_{i,l}^k$. For that, define the 1-forms $\tilde{\omega}_1 = \mathrm{d}(u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1)$ and $\tilde{\omega}_2 = \mathrm{d}y_2^{[2]}$. It is easy to see that the condition (2.3) is satisfied in both cases. Note that for this example $\omega_i = \tilde{\omega}_i$, $i = 1, 2$, and $\sigma_1 = 3$, $\sigma_2 = 2$. Next compute the 1-forms $\bar{\omega}_{i,l}$,

$i=1,2,\ l=1,\ldots,\sigma_i$:

$$\begin{array}{rcl} \bar{\omega}_{1,1} & = & u_1^{[1]}\mathrm{d}(y_1^{[2]}u_1^{[2]}) \\ \bar{\omega}_{1,2} & = & \mathrm{d}(u_1^{[1]}y_1^{[2]}u_1^{[2]}) + y_2\mathrm{d}u_2^{[1]} \\ \bar{\omega}_{1,3} & = & \mathrm{d}(u_1^{[1]}y_1^{[2]}u_1^{[2]} + y_2u_2^{[1]} + y_2u_1) \\ \bar{\omega}_{2,1} & = & \mathrm{d}(y_1^{[1]}u_1^{[1]}) \\ \bar{\omega}_{2,2} & = & \mathrm{d}(y_1^{[1]}u_1^{[1]} + u_3y_2). \end{array} \tag{2.11}$$

From (2.11) it is easy to see that $\gamma_{1,2}=2$ and $\gamma_{1,1}=\gamma_{1,3}=\gamma_{2,1}=\gamma_{2,2}=1$. Finally, one can define the functions $\phi_{i,l}^k$, $i=1,2,\ l=1,\ldots,\sigma_i,\ k=1,\ldots,\gamma_{i,l}$ as follows:

$$\begin{array}{rcll} \phi_{1,1}^1 & = & y_1u_1 & \\ \phi_{1,2}^1 & = & u_1y_1^{[1]}u_1^{[1]} & \qquad \phi_{1,2}^2 = u_2 \\ \phi_{1,3}^1 & = & u_1^{[1]}y_1^{[2]}u_1^{[2]} + y_2u_2^{[1]} + y_2u_1 & \\ \phi_{2,1}^1 & = & y_1u_1 & \\ \phi_{2,2}^1 & = & y_1^{[1]}u_1^{[1]} + u_3y_2. & \end{array}$$

Now, since $\phi_{1,1}^1=\phi_{2,1}^1$ and all the other functions depend on some different independent variables,

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{1,1}^1, \mathrm{d}\phi_{1,2}^1, \mathrm{d}\phi_{1,2}^2, \mathrm{d}\phi_{1,3}^3, \mathrm{d}\phi_{2,1}^1, \mathrm{d}\phi_{2,2}^1\}) = 5.$$

Also,

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\phi_{1,1}^1, \phi_{2,1}^1, \phi_{1,2}^1, \phi_{1,2}^2, \phi_{2,2}^1, \phi_{1,3}^1)^T}{\partial(u, \delta\phi_{1,1}^1, \delta\phi_{2,1}^1, \delta\phi_{1,2}^1, \delta\phi_{1,2}^2)}$$

$$= \mathrm{rank}_{\mathcal{K}} \begin{pmatrix} y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta\phi_{1,1}^1 & 0 & 0 & u_1 & u_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 1 & 1 & 0 & 0 \\ y_2 & 0 & 0 & 0 & 0 & 1 & y_2 \end{pmatrix} = 5.$$

Thus, the condition (2.6) is satisfied. The feedback of the form (2.1) can be found by taking $\eta_{i,l,k}$ and new input v as follows

$$\begin{array}{rcl} \eta_{1,1,1} & = & \phi_{1,1}^1 = y_1u_1 \\ \eta_{1,2,1} & = & \phi_{1,2}^1 = u_1y_1^{[1]}u_1^{[1]} \\ \eta_{1,2,2} & = & \phi_{1,2}^2 = u_2 \\ v_1 & = & \phi_{1,3}^1 = u_1^{[1]}y_1^{[2]}u_1^{[2]} + y_2u_2^{[1]} + y_2u_1 \\ v_2 & = & \phi_{2,2}^1 = y_1^{[1]}u_1^{[1]} + u_3y_2. \end{array}$$

This set of equations has to be solved with respect to variables $\{\eta^{[1]}_{1,1,1}, \eta^{[1]}_{1,2,1}, \eta^{[1]}_{1,2,2}, u_1, u_2, u_3\}$. Since there are five equations, but six unknowns, then one unknown, for example $\eta^{[1]}_{1,2,2}$, will remain free. This variable will be taken equal to the new input v_3. To conclude, the feedback

$$\begin{aligned}
\eta^{[1]}_{1,1,1} &= \frac{y_1\eta_{1,2,1}}{\eta_{1,1,1}}\\
\eta^{[1]}_{1,2,1} &= v_1 - y_2v_3 - \frac{y_2\eta_{1,1,1}}{y_1}\\
\eta^{[1]}_{1,2,2} &= v_3\\
u_1 &= \frac{\eta_{1,1,1}}{y_1}\\
u_2 &= \eta_{1,2,2}\\
u_3 &= \frac{v_2\eta_{1,1,1} - y_1\eta_{1,2,1}}{y_2\eta_{1,1,1}}
\end{aligned}$$

solves the input-output linearization problem for system (2.10).

Example 2.2. Consider the model of the liquid level system of interconnected tanks [16], defined by the i/o equation

$$\begin{aligned}
y^{[3]} &= 0.43y^{[2]} + 0.681y^{[1]} - 0.149y + 0.396u^{[2]} + 0.014u^{[1]} - 0.071u\\
&- 0.351y^{[2]}u^{[2]} - 0.03(y^{[1]})^2 - 0.135y^{[1]}u^{[1]} - 0.027(y^{[1]})^3\\
&- 0.108(y^{[1]})^2u^{[1]} - 0.099(u^{[1]})^3.
\end{aligned} \tag{2.12}$$

Since the system (2.12) is a SISO system, the indices, that are not needed, are omitted. It is straightforward to define

$$\begin{aligned}
\omega &= \tilde{\omega} = \mathrm{d}\big(0.396u^{[2]} + 0.014u^{[1]} - 0.071u - 0.351y^{[2]}u^{[2]} - 0.03(y^{[1]})^2\\
&- 0.135y^{[1]}u^{[1]} - 0.027(y^{[1]})^3 - 0.108(y^{[1]})^2u^{[1]} - 0.099(u^{[1]})^3\big)
\end{aligned}$$

and since $\omega \in \mathcal{E}^3$

$$\begin{aligned}
\bar{\omega}_1 &= \mathrm{d}\big(0.396u^{[2]} - 0.351y^{[2]}u^{[2]}\big)\\
\bar{\omega}_2 &= \mathrm{d}\big(0.396u^{[2]} + 0.014u^{[1]} - 0.351y^{[2]}u^{[2]} - 0.03(y^{[1]})^2 - 0.135y^{[1]}u^{[1]}\\
&- 0.027(y^{[1]})^3 - 0.108(y^{[1]})^2u^{[1]} - 0.099(u^{[1]})^3\big).
\end{aligned}$$

Now,

$$\begin{aligned}
\phi_1 &= 0.396u - 0.351yu\\
\phi_2 &= 0.396\phi_1^{[1]} + 0.014u - 0.03y^2 - 0.135yu - 0.027y^3\\
&- 0.108y^2u - 0.099u^3\\
\phi_3 &= \phi_2^{[1]} - 0.071u.
\end{aligned}$$

Clearly, the condition (2.6) is satisfied and the feedback

$$\begin{aligned}
\eta_1^{[1]} &= \eta_2 + 0.03y^2 + 0.027y^3 \\
&- (0.014 - 0.135y - 0.108y^2)\frac{\eta_1}{0.396 - 0.351y} + \Big(\frac{0.463\eta_1}{0.183 - 0.163y}\Big)^3 \\
\eta_2^{[1]} &= v + \frac{0.071\eta_1}{0.028 - 0.025y} \\
u &= \frac{\eta_1}{0.396 - 0.351y}
\end{aligned}$$

linearizes the system (2.12), yielding the closed-loop system equation $y^{[3]} = 0.43y^{[2]} + 0.681y^{[1]} - 0.149y + v$.

2.1.2 Generalized Problem Statement

Next, the problem statement from the previous subsection, is generalized. It is shown that under the assumption that system (1.3) is right-invertible the conditions of Theorem 2.1 are also necessary and sufficient for solvability of the generalized problem.

In the generalized problem statement the conditions (2.2) are replaced by weaker conditions

$$\begin{aligned}
\mathrm{d}y_i^{[n_i]} &\in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v_j^{[n_i-1]}, \ldots, \mathrm{d}v_j\} \\
\mathrm{d}y_i^{[n_i]} &\notin \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau\},
\end{aligned} \tag{2.13}$$

where $i, \tau = 1, \ldots, p$ and $j = 1, \ldots, m$. Unlike the relations (2.2), $y_i^{[n_i]}$ in the closed-loop system is now allowed to depend also on the forward-shifts of the new control v.

Lemma 2.1. *Assume that system (1.3) is right-invertible. Then there exists a feedback of the form (2.1), such that (2.13) is satisfied for the closed-loop system if and only if the conditions of Theorem 2.1 are satisfied.*

Proof. Necessity. Assume that there exists a regular feedback such that (2.13) is satisfied for the closed-loop system. It is shown that then there exists another regular feedback, such that (2.2) is satisfied for the closed-loop system. The latter means that the conditions of Theorem 2.1 are satisfied. Clearly, since a regular feedback is applied and system (1.3) is right-invertible, the closed-loop system is right-invertible. Next, it is shown that every right-invertible system satisfying (2.13) satisfies the conditions of Theorem 2.1. Since the closed-loop system is linear,

$$\begin{aligned}
\phi_{i,1} &= \psi_{i,1}(u) \\
\phi_{i,l} &= \delta\phi_{i,l-1} + \psi_{i,l}(u)
\end{aligned}$$

for $l = 2, \dots, \sigma_i$, $i = 1, \dots, p_1$ and some functions $\psi_{i,l}(\cdot)$[3]. Therefore

$$\dim(\text{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,l}\}) = \text{rank}_{\mathcal{K}} \frac{\partial \phi_{i,l}}{\partial(u, \delta\phi_{i,l^*})}$$

for $i = 1, \dots, p_1$, $l = 1, \dots, \sigma_i$ and $l^* = 1, \dots, \sigma_i - 1$. The right-invertibility guarantees that the functions ϕ_{i,σ_i} are independent from all the other functions $\phi_{i,l}$, i.e. one can define the system of equations (2.7), (2.8). Thus, the conditions of Theorem 2.1 are satisfied.

Sufficiency. This is obvious. □

In the following sections, the notion of i/o linearizability of a set of functions is used in the solutions of the decoupling problems. The functions $\varphi_i(y, \dots, y^{[s-1]}, u, \dots, u^{[s-1]})$, $i = 1, \dots, p$, are said to be linearizable if there exists a feedback of the form (2.1), such that the system

$$y_i^{[s]} = \varphi_i(y, \dots, y^{[s-1]}, u, \dots, u^{[s-1]}),$$

$i = 1, \dots, p$, is i/o linearizable. The functions φ_i, $i = 1, \dots, p$, are said to be *strictly* linearizable, if the application of the linearizing feedback yields $\mathrm{d}\varphi_i \in \text{span}_{\mathbb{R}}\{\mathrm{d}v\}$, for $i = 1, \dots, p$.

2.2 Input-Output Decoupling

In this section, the results on i/o linearization are applied to give a solution to the i/o decoupling problem by dynamic measurement feedback, i.e. by a feedback that depends on outputs that are measured. Compared to the majority of contributions, that address the i/o decoupling problem by state feedback, here a measurement feedback is used. It allows to apply the i/o decoupling techniques when all the states are not measurable and construction of the observer is too complicated. Of course, the solvability conditions are more restrictive.

The techniques similar to those in [44, 79, 73, 80] are applied. Two cases are considered: when the system is described by 1) the state equations and 2) the i/o equations. Necessary and sufficient solvability conditions are derived. The results of this section generalize the results of [73] from output feedback (the case when exactly the controlled outputs are measurable) to measurement feedback case and from the static feedback to the dynamic feedback case.

[3]Note that here we write $\phi_{i,l}$ instead of $\phi_{i,l}^k$ since k is equal to 1 for all the functions.

2.2.1 Systems Described by State Equations

First, the case when the nonlinear discrete-time system is given by state equations, is considered. Compared to (1.1), here one has two types of outputs: the measured output $y_*(t) \in Y_* \subset \mathbb{R}^p$ and the output-to-be-controlled $y(t) \in Y \subset \mathbb{R}^m$. Thus, the system is described by the equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y_*(t) &= h_*(x(t)) \\ y(t) &= h(x(t)). \end{aligned} \tag{2.14}$$

Assume that system (2.14) is invertible, i.e.

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial (h_1(x^{[j_1]}), \ldots, h_m(x^{[j_m]}))^T}{\partial u} = m,$$

for some $j_i \in \mathbb{N}$, $i = 1, \ldots, m$. Also, let $j_{max} := \max\{j_1, \ldots, j_m\}$ and assume that the relative degree r_i of output y_i is finite for every $i = 1, \ldots, m$.

Next, as in [80], for each output component y_i a subspace Ω_i of $\mathcal{X} := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$ is defined in the following way:

$$\begin{aligned} \Omega_i = \; & \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \\ & \delta^k \omega \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+k-1]}\}\}. \end{aligned} \tag{2.15}$$

The subspaces Ω_i are essential to solve the i/o decoupling problem, since the forward-shifts of the elements of Ω_i do not depend on the input u explicitly. Suppose $\Omega_i = \operatorname{span}_{\mathcal{K}}\{\theta_1, \ldots, \theta_l\}$. Define the forward-shift of subspace Ω_i element-wise by $\Omega_i^{[1]} = \operatorname{span}_{\mathcal{K}}\{\delta\theta_1, \ldots, \delta\theta_l\}$. Let $\Omega_i^{[0]} := \Omega_i$, and $\Omega_i^{[k]} := (\Omega_i^{[k-1]})^{[1]}$. The following lemma gives a procedure for computation of the subspaces Ω_i.

Lemma 2.2. *The subspace Ω_i may be computed as the limit of the following algorithm:*

$$\begin{aligned} \Omega_i^0 &= \mathcal{X} \\ \Omega_i^{k+1} &= \{\omega \in \Omega_i^k \mid \delta\omega \in \Omega_i^k + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\}\}, \quad k \geq 0. \end{aligned} \tag{2.16}$$

Proof. Clearly, the sequence Ω_i^k converges, since it is decreasing sequence of vector spaces. Let the limit be $\Omega_i^{k_*}$. Now, if $\omega \in \Omega_i^{k_*}$, then

$$\delta^k \omega \in \Omega_i^{k_*} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+k-1]}\}$$

for $\forall k$. Since $\Omega_i^{k_*} \subset \mathcal{X}$, one has that $\omega \in \Omega_i$ and $\Omega_i^{k_*} \subset \Omega_i$.

Now, let $\omega \in \Omega_i$, then

$$\delta\omega = \omega_0 + \omega_y,$$

where $\omega_0 \in \mathcal{X}$ and $\omega_y \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\}$. Since $\omega \in \Omega_i$, then ω_0 must also belong to Ω_i. Thus, $\Omega_i^{[1]} \subset \Omega_i + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\}$. Since, $\Omega_i^{k_*}$, as the limit of (2.16), is maximal subspace of $\mathcal{X}$ satisfying $(\Omega_i^{k_*})^{[1]} \subset \Omega_i^{k_*} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\}$, then $\Omega_i \subset \Omega_i^{k_*}$. Therefore, $\Omega_i = \Omega_i^{k_*}$. □

Problem statement. The goal is to find a regular dynamic measurement feedback of the form

$$\begin{aligned} \eta(t+1) &= F(\eta(t), y_*(t), v(t)) \\ u(t) &= H(\eta(t), y_*(t), v(t)), \end{aligned} \tag{2.17}$$

where $v(t) \subset V \in \mathbb{R}^m$ is the new input, $\eta(t) \subset \Lambda \in \mathbb{R}^\rho$ is the state of the feedback and V, Λ are open and dense, such that the in closed-loop system different outputs y_i are affected by different inputs v_i for every time instant, i.e.

$$\mathrm{d}y_i^{[k]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v_i, \ldots, \mathrm{d}v_i^{[k-\bar{r}_i]}\}$$

for $k \geq \bar{r}_i$, where $\bar{r}_i$ is the relative degree of output y_i of the closed-loop system.

To check whether the system (2.14) is already i/o decoupled, the following Lemma can be used.

Lemma 2.3. *Under the assumption $r_i < \infty$, for $i = 1, \ldots, m$, the system (2.14) is i/o decoupled if and only if for $i = 1, \ldots, m$*

$$\mathrm{d}y_i^{[r_i]} \in \Omega_i + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u_i\}. \tag{2.18}$$

Proof. Necessity. Let throughout this proof $i = 1, \ldots, m$. If the system (2.14) is i/o decoupled, then

$$\mathrm{d}y_i^{[r_i]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i\}.$$

Thus, there exists $\omega_i \in \mathcal{X}$ and $\lambda_i \in \mathcal{K}$, such that $\mathrm{d}y_i^{[r_i]} = \omega_i + \lambda_i \mathrm{d}u_i$. It will be shown that $\omega_i \in \Omega_i$. Note that for every $\sigma \in \mathbb{N}$,

$$\delta^\sigma \omega_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\}. \tag{2.19}$$

Since $\mathrm{d}y_i^{[k]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-r_i]}\}$ for $k \geq 0$, then

$$\begin{aligned} &\operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\} \\ &\quad = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+\sigma-1]}\}. \end{aligned} \tag{2.20}$$

Thus, (2.19) and (2.20) give

$$\delta^\sigma \omega_i \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+\sigma-1]}\}$$

for every $\sigma \in \mathbb{N}$, which, by definition of Ω_i, means that $\omega_i \in \Omega_i$.

Sufficiency. By Lemma 2.2 and (2.18), one gets

$$\Omega_i^{[1]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i^{[r_i]}\} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i\}.$$

Thus, $\Omega_i^{[k]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-1]}\}$ and therefore,

$$\begin{aligned} \mathrm{d}y_i^{[r_i+k]} &\in \Omega_i^{[k]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i^{[k]}\} \\ &\subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\} \\ &\subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\}, \end{aligned}$$

which means, that the system (2.14) is i/o decoupled. □

In Theorem 2.2 below, the necessary and sufficient conditions for solvability of the i/o decoupling problem by dynamic measurement feedback are given.

Theorem 2.2. *The invertible system (2.14) can be i/o decoupled by the dynamic measurement feedback (2.17) if and only if the following conditions are satisfied for $i = 1, \ldots, m$:*

(i) there exists $s \geq j_{max} - r_i + 1$ such that[4]

$$\begin{aligned} \mathrm{d}y_i^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} \\ + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}; \end{aligned}$$

(ii) there exist integrable 1-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ such that

$$\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]};$$

(iii) for $\omega_i = \lambda_i \mathrm{d}\varphi_i$, the functions $\varphi_i(y_*, \ldots, y_*^{[s-1]}, u, \ldots, u^{[s-1]})$ are independent and strictly linearizable by dynamic feedback (2.17).

Proof. Necessity. Let $s \geq 1$ be such that in the closed-loop system the relative degree $\bar{r}_i$ of output y_i is $\bar{r}_i = r_i + s - 1$. By Lemma 2.3 and the fact that the closed-loop system is i/o decoupled,

$$\mathrm{d}y_i^{[\bar{r}_i]} \in \bar{\Omega}_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}, \tag{2.21}$$

[4]Note that, one can, in principle, search, instead of the joint index s, a separate s_i that satisfies $s_i \geq j_{max} - r_i + 1$. Then s can be taken as $s = \max_i\{s_i\}$.

where by $\bar{\Omega}_i$ is denoted the subspace Ω_i for the closed-loop system. Next, it is shown that $\bar{\Omega}_i = \Omega_i + \cdots + \Omega_i^{[s-1]}$. From the definition (2.15) of Ω_i,

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+s-2]}\}.$$

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}. \tag{2.22}$$

Thus,

$$\begin{aligned}\Omega_i + \cdots + \Omega_i^{[s-1]} &= \{\bar{\omega} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ \bar{\omega}^{[k]} &\in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_i^{[r_i+s-1]}, \ldots, \mathrm{d}y_i^{[r_i+s-k-2]}\}\} = \bar{\Omega}_i.\end{aligned}$$

The last equality comes from the definition (2.15) of the subspace $\bar{\Omega}_i$. Therefore, (2.21) becomes

$$\mathrm{d}y_i^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v_i\}.$$

Then one can define the 1-forms $\omega_i = \lambda_i \mathrm{d}v_i$ such that $\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]}$. Now, the conditions (i) and (ii) must be satisfied, since otherwise the feedback would not be measurement feedback. Since the conditions (i) and (ii) are satisfied, $\omega_i = \lambda_i \mathrm{d}\varphi_i(u, \ldots, u^{[s-1]}, y_*, \ldots, y_*^{[s-1]})$ for some functions φ_i. Note that under the feedback $\omega_i = \lambda_i \mathrm{d}v_i$, i.e. the functions φ_i are strictly linearizable.

Sufficiency. It will be shown that the feedback that linearizes strictly the functions φ_i in (iii), solves the i/o decoupling problem.

Because the functions φ_i, $i = 1, \ldots, m$, are independent and strictly linearizable, for the closed-loop system one has $\mathrm{d}\varphi_i = \mathrm{d}v_i$, and the relative degree of output y_i is $r_i + s - 1$. Thus

$$\mathrm{d}y_i^{[r_i+j]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \tag{2.23}$$

for $j = 0, \ldots, s-2$. From the definition (2.15) of the subspace Ω_i one concludes $\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}$.

Now, like in the proof of necessity part, one can show that $\Omega_i + \cdots + \Omega_i^{[s-1]} = \bar{\Omega}_i$, where $\bar{\Omega}_i$ is the subspace Ω_i for the closed-loop system. Therefore, by (i), (ii) and (iii), $\mathrm{d}y_i^{[r_i+s-1]} \in \bar{\Omega}_i + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}v_i\}$. Finally, by Lemma 2.3, the system (2.14) is i/o decoupled. □

Remark 2.1. The assumption of invertibility of system (2.14) in Theorem 2.2 is necessary for linearizability of independent functions φ_i, $i = 1, \ldots, m$, defined in (iii) of Theorem 2.2.

Remark 2.2. If $s = 1$ in (*ii*) and (*iii*) of Theorem 2.2, one gets the conditions for solvability of the i/o decoupling problem by static measurement feedback, that in the special case $y = y_*$ recover the result obtained in [73] for output feedback case.

The drawback of the conditions in Theorem 2.2 is that in general the 1-forms ω_i and the functions φ_i, $i = 1, \dots, m$, are *not unique*, and as such it is not completely constructive to check the conditions of Theorem 2.2. Below a constructive way to check whether the system (2.14) can be i/o decoupled by the feedback (2.17), is suggested.

First, find the minimal integer s such that $s \geq j_{max} - r_i + 1$ and

$$\begin{aligned} \mathrm{d}y_i^{[r_i+s-1]} &\in \Omega_i + \cdots + \Omega_i^{[s-1]} \\ &+ \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \dots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \dots, \mathrm{d}u^{[s-1]}\}. \end{aligned}$$

for $i = 1, \dots, m$.

In the following an assumption is made, that guarantees the uniqueness of the 1-forms ω_i, defined in (*ii*) of Theorem 2.2.

Assumption 1. The subspaces $\Omega_i + \cdots + \Omega_i^{[s-1]}$ are integrable for $i = 1, \dots, m$.

From Assumption 1, $\Omega_i + \cdots + \Omega_i^{[s-1]} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\beta_{i1}, \dots, \mathrm{d}\beta_{il}\}$ for some functions $\beta_{ij} \in \mathcal{K}$, $j = 1, \dots, l$. Then, define the 1-forms ω_i, $i = 1, \dots, m$, as

$$\omega_i := \mathrm{d}y_i^{[r_i+s-1]} - \sum_{j=1}^{l} \frac{\partial y_i^{[r_i+s-1]}}{\partial \beta_{ij}} \mathrm{d}\beta_{ij}.$$

Now, the condition (*ii*) of Theorem 2.2 is satisfied if and only if for $i = 1, \dots, m$

$$\omega_i = a_i \mathrm{d}\tilde{\varphi}_i + \sum_{j=1}^{l_i} b_{ij} \bar{\omega}_{ij},$$

where $a_i, b_{ij} \in \mathcal{K}$ and

$$\begin{aligned} &(\Omega_i + \cdots + \Omega_i^{[s-1]}) \cap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \dots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \dots, \mathrm{d}u^{[s-1]}\} \\ &= \mathrm{span}_{\mathcal{K}}\{\bar{\omega}_{i1}, \dots, \bar{\omega}_{il_i}\}. \end{aligned}$$

So, under the Assumption 1 Theorem 2.2 can be rewritten in the following form.

Corollary 2.1. *An invertible system (2.14) can be i/o decoupled by the dynamic measurement feedback (2.17) if and only if the functions* $\tilde{\varphi}_i$, $i = 1, \dots, m$ *are strictly linearizable.*

Proof. Necessity. The beginning of the proof follows that of Theorem 2.2. Now, since one has

$$\mathrm{d}y_i^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\},$$

then one gets $\tilde{\varphi}_i = v_i$. This means that the feedback that solves the i/o decoupling problem, linearizes strictly the functions $\tilde{\varphi}_i$.

Sufficiency. By construction, the 1-forms

$$\omega_i - \sum_{j=1}^{l_i} b_{ij}\bar{\omega}_{ij} = a_i \mathrm{d}\tilde{\varphi}_i$$

and the functions $\tilde{\varphi}_i$ satisfy the conditions (*ii*) and (*iii*) of Theorem 2.2. Therefore, the i/o decoupling problem has a solution. □

Remark 2.3. When one sets $y_* = y$ in Theorem 2.2, then one gets the conditions under which the i/o decoupling problem is solvable by dynamic output feedback. Moreover, when $y_* = x$, then the state feedback solution is obtained. In this case the conditions of Theorem 2.2 are always satisfied under the assumption of invertibility.

Example 2.3. Consider the system described by the difference equations

$$\begin{array}{rcl} x_1^{[1]} & = & (x_3+x_4)u_1 - x_2 \\ x_2^{[1]} & = & \frac{u_1 x_5}{x_4} + x_1 \\ x_3^{[1]} & = & x_1 x_3 \\ x_4^{[1]} & = & (x_3+x_4)u_1 x_5 \\ x_5^{[1]} & = & \frac{u_2 x_5}{x_4} \\ y_1 & = & x_1, \quad y_2 = x_4 \\ y_{*1} & = & x_3 + x_4 \quad y_{*2} = \frac{x_5}{x_4}. \end{array} \tag{2.24}$$

Check whether the conditions of Theorem 2.2 are satisfied for system (2.24). First, note that the relative degrees of outputs y_1 and y_2 are $r_1 = r_2 = 1$. Since

$$\begin{array}{rcl} y_1^{[1]} & = & (x_3+x_4)u_1 - x_2 \\ y_2^{[2]} & = & \left(y_1^{[2]} + x_1 + \dfrac{u_1 x_5}{x_4}\right)\dfrac{u_2 x_5}{x_4}, \end{array}$$

one gets $\mathrm{rank}_{\mathcal{K}} \frac{\partial (y_1^{[1]}, y_2^{[2]})^T}{\partial u} = 2$. Therefore, the system (2.24) is invertible and $j_1 = 1$, $j_2 = 2$. The subspaces Ω_i are, according to Lemma 2.2, $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_3\}$ and $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_4\}$.

Now, check whether the condition (i) of Theorem 2.2 is satisfied. Since s has to satisfy the inequalities $s \geq j_{max} - r_i + 1$ for $i = 1, 2$, the first choice for s is $s = 2$. Compute

$$\begin{aligned}
\mathrm{d}y_1^{[2]} &= u_1^{[1]}\mathrm{d}y_{*1}^{[1]} + y_{*1}^{[1]}\mathrm{d}u_1^{[1]} - y_{*2}\mathrm{d}u_1 - u_1\mathrm{d}y_{*2} - \mathrm{d}x_1 \\
&\in \Omega_1 + \Omega_1^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y_*, \mathrm{d}u^{[1]}, \mathrm{d}y_*^{[1]}\} \\
\mathrm{d}y_2^{[2]} &= u_2y_{*2}y_{*1}^{[1]}\mathrm{d}u_1^{[1]} + u_2y_{*2}u_1^{[1]}\mathrm{d}y_{*1}^{[1]} + y_{*2}u_1^{[1]}y_{*1}^{[1]}\mathrm{d}u_2 + u_2u_1^{[1]}y_{*1}^{[1]}\mathrm{d}y_{*2} \\
&\in \Omega_2 + \Omega_2^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y_*, \mathrm{d}u^{[1]}, \mathrm{d}y_*^{[1]}\}
\end{aligned}$$

and thus, condition (i) of Theorem 2.2 is satisfied. Next, the integrable 1-forms ω_i, $i = 1, 2$, satisfying the condition (ii) of Theorem 2.2, are found:

$$\begin{aligned}
\omega_1 &= u_1^{[1]}\mathrm{d}y_{*1}^{[1]} + y_{*1}^{[1]}\mathrm{d}u_1^{[1]} - y_{*2}\mathrm{d}u_1 - u_1\mathrm{d}y_{*2} \\
&= \mathrm{d}(y_{*1}^{[1]}u_1^{[1]} - y_{*2}u_1) \\
\omega_2 &= u_2y_{*2}y_{*1}^{[1]}\mathrm{d}u_1^{[1]} + u_2y_{*2}u_1^{[1]}\mathrm{d}y_{*1}^{[1]} + y_{*2}u_1^{[1]}y_{*1}^{[1]}\mathrm{d}u_2 \\
&\quad + u_2u_1^{[1]}y_{*1}^{[1]}\mathrm{d}y_{*2} = \mathrm{d}(u_1^{[1]}y_{*1}^{[1]}u_2y_{*2}).
\end{aligned}$$

It remains to be checked whether the functions $\varphi_1 = y_{*1}^{[1]}u_1^{[1]} - y_{*2}u_1$ and $\varphi_2 = u_1^{[1]}y_{*1}^{[1]}u_2y_{*2}$ are strictly linearizable. In fact, the feedback

$$\begin{aligned}
\eta_{1,1}^{[1]} &= v_1 + \tfrac{\eta_{1,1}y_{*2}}{y_{*1}} \\
u_1 &= \tfrac{\eta_{1,1}}{y_{*1}} \\
u_2 &= \tfrac{v_2y_{*1}}{y_{*2}(y_{*1}v_1 + \eta_{1,1}y_{*2})}
\end{aligned} \tag{2.25}$$

linearizes the functions φ_1, φ_2 and solves the i/o decoupling problem for system (2.24). For the closed-loop system one gets $\bar{r}_1 = \bar{r}_2 = r_i + s - 1 = 2$ and

$$\begin{aligned}
\mathrm{d}y_1^{[2]} &= \mathrm{d}v_1 - \mathrm{d}x_1 \in \bar{\Omega}_1 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_1\} \\
\mathrm{d}y_2^{[2]} &= \mathrm{d}v_2 \in \bar{\Omega}_2 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_2\},
\end{aligned}$$

which means that by Lemma 2.3, the closed-loop system is i/o decoupled.

2.2.2 Systems Described by Input-Output Equations

In this subsection, systems of the form (1.3), where $p = m$, i.e.

$$y_i(t + n_i) = \Phi_i(y_j(t), \ldots, y_j(t + n_{ij}), u_j(t), \ldots, u_j(t + q_i)) \tag{2.26}$$

for $i, j = 1, \ldots, m$, are considered.

The system (2.26) is said to be i/o decoupled if, after possibly reordering the inputs, one has

$$\mathrm{d}\Phi_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{[n_i-1]}, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[q_i]}\}$$

or equivalently,

$$\mathrm{d}\Phi_i \in \Delta_i + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u_i\}, \tag{2.27}$$

where

$$\Delta_i := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{[n_i-1]}\} \tag{2.28}$$

for $i = 1, \ldots, m$.

Problem statement. A regular dynamic output feedback of the form

$$\begin{aligned} \eta(t+1) &= F(\eta(t), y(t), v(t)) \\ u(t) &= G(\eta(t), y(t), v(t)), \end{aligned} \tag{2.29}$$

where $v(t) \subset V \in \mathbb{R}^m$ and $\eta(t) \subset \Lambda \in \mathbb{R}^\rho$, is searched, such that the closed-loop system (2.26), (2.29) is i/o decoupled.

Theorem 2.3. *The invertible system (2.26) can be i/o decoupled by the dynamic output feedback of the form (2.29) if and only if there exist strictly linearizable functions $\varphi_i \in \mathcal{K}$, such that*

$$\mathrm{d}\Phi_i \in \Delta_i + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_i\} \tag{2.30}$$

for $i = 1, \ldots, m$.

Proof. Necessity. If the invertible system (2.26) is i/o decoupled, then the condition (2.27) must be satisfied. Take for instance $\varphi_i = u_i$ and the condition (2.30) is satisfied.

Sufficiency. By assumption, the functions φ_i, $i = 1, \ldots, m$ are strictly linearizable via the dynamic output feedback. Thus, under this feedback $\varphi_i = v_i$ for some new inputs v_i. Therefore, (2.30) becomes (2.27) under the feedback that linearizes the functions φ_i. □

Typically there exist multiple possibilities to choose the functions φ_i that satisfy the condition (2.30). If the system can be i/o decoupled, then at least one set of the functions φ_i, $i = 1, \ldots, m$, is strictly linearizable. The problem one faces is: how to choose the functions φ_i in the best manner? Obviously, one can always take $\varphi_i = \Phi_i$, $i = 1, \ldots, m$ in Theorem 2.3, but this yields only a sufficient solvability condition.

Corollary 2.2. *The invertible system (2.26) can be i/o decoupled by the dynamic output feedback if the functions Φ_i, $i = 1, \ldots, m$, are strictly linearizable.*

Next, certain unique 1-forms ω_i, $i = 1, \ldots, m$, will be defined. From these 1-forms, one can calculate, by integration, the functions φ_i, $i = 1, \ldots, m$, that satisfy the condition (2.30).

Consider the 1-forms

$$\omega_i := \mathrm{d}\Phi_i - \sum_{j=0}^{n_i-1} \frac{\partial \Phi_i}{\partial y_i^{[j]}} \mathrm{d}y_i^{[j]} \tag{2.31}$$

for $i = 1, \ldots, m$. Note that the ring $\mathcal{K}[\vartheta]$ is defined such that one can interpret a shift of variable as multiplication by the polynomial indeterminate ϑ from left. Therefore, one can construct m modules $\mathcal{A}_i := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_i\}$. Let for each $i = 1, \ldots, m$, the 1-form $\bar{\omega}_i$ be the basis element of the closure of $\mathcal{A}_i$.

Next, let $\bar{\Delta}_i$ be the smallest integrable subspace of Δ_i such that

$$\mathrm{d}\bar{\omega}_i \wedge \bar{\omega}_i = 0 \mod \bar{\Delta}_i. \tag{2.32}$$

The subspace $\bar{\Delta}_i$ is unique and always exists. In extreme cases $\bar{\Delta}_i = \Delta_i$ (see (2.28)) or $\bar{\Delta}_i = 0$. If the condition (2.32) is satisfied, then there exist functions $\bar{\varphi}_i \in \mathcal{K}$ such that

$$\bar{\omega}_i = a_i \mathrm{d}\bar{\varphi}_i + \sum_{j=1}^{l_i} b_{ij} \mathrm{d}\alpha_{ij}$$

for some $a_i, b_{ij} \in \mathcal{K}$, where $\bar{\Delta}_i = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\alpha_{i1}, \ldots, \mathrm{d}\alpha_{il_i}\}$.

Corollary 2.3. *The invertible system (2.26) can be i/o decoupled by the dynamic output feedback (2.29) if and only if the functions $\bar{\varphi}_i$, $i = 1, \ldots, m$, are strictly linearizable.*

Proof. Necessity. If the system (2.26) is i/o decoupled, then the condition (2.27) is satisfied. Thus, one gets $\bar{\omega}_i = \mathrm{d}u_i$ and $\bar{\Delta}_i = \emptyset$. Therefore, $\bar{\varphi}_i = u_i$ and these functions are clearly strictly linear.

Sufficiency. By construction

$$\begin{aligned}
\mathrm{d}\Phi_i &= \sum_{j=0}^{n_i-1} \frac{\partial \Phi_i}{\partial y_i^{[j]}} \mathrm{d}y_i^{[j]} + p_i(\vartheta)\bar{\omega}_i \\
&= \sum_{j=0}^{n_i-1} \frac{\partial \Phi_i}{\partial y_i^{[j]}} \mathrm{d}y_i^{[j]} + p_i(\vartheta) \sum_{j=1}^{l_i} b_{ij} \mathrm{d}\alpha_{ij} + p_i(\vartheta) a_i \mathrm{d}\bar{\varphi}_i
\end{aligned}$$

for $i = 1, \ldots, m$ and some $p_i(\vartheta) \in \mathcal{K}[\vartheta]$. Therefore, since

$$\sum_{j=0}^{n_i-1} \frac{\partial \Phi_i}{\partial y_i^{[j]}} \mathrm{d}y_i^{[j]} + p_i(\vartheta) \sum_{j=1}^{l_i} b_{ij} \mathrm{d}\alpha_{ij} \in \Delta_i,$$

the functions $\bar{\varphi}_i$ satisfy the condition (2.30). Since these functions are strictly linearizable, the conditions of Theorem 2.3 are satisfied and thus the system (2.26) can be i/o decoupled. □

Example 2.4. Consider an invertible system

$$
\begin{aligned}
y_1^{[3]} &= y_1^{[2]} + y_1^{[1]} y_2^{[1]} u_1^{[1]} + u_2 \\
y_2^{[2]} &= y_1^{[1]} u_2^{[1]} + y_2 .
\end{aligned}
\tag{2.33}
$$

Compute, by (2.31)

$$
\begin{aligned}
\omega_1 &= y_1^{[1]} u_1^{[1]} \mathrm{d}y_2^{[1]} + y_1^{[1]} y_2^{[1]} \mathrm{d}u_1^{[1]} + \mathrm{d}u_2 \\
\omega_2 &= y_1^{[1]} \mathrm{d}u_2^{[1]} + u_2^{[1]} \mathrm{d}y_1^{[1]}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\omega}_1 &= \omega_1 \\
\bar{\omega}_2 &= \vartheta \omega_2 .
\end{aligned}
$$

Clearly, $\mathrm{d}\bar{\omega}_1 \wedge \bar{\omega}_1 \wedge \mathrm{d}y_1^{[1]} = 0$ and $\mathrm{d}\bar{\omega}_2 \wedge \bar{\omega}_2 = 0$, i.e. $\bar{\Delta}_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1^{[1]}\}$ and $\bar{\Delta}_2 = 0$. Thus,

$$
\begin{aligned}
\bar{\omega}_1 &= \mathrm{d}\bar{\varphi}_1 - u_1^{[1]} y_2^{[1]} \mathrm{d}y_1^{[1]} \\
\bar{\omega}_2 &= \mathrm{d}\bar{\varphi}_2 ,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\varphi}_1 &= y_1^{[1]} y_2^{[1]} u_1^{[1]} + u_2 \\
\bar{\varphi}_2 &= y_1 u_2 .
\end{aligned}
$$

These functions can be strictly linearized by a dynamic feedback

$$
\begin{aligned}
\eta^{[1]} &= v_1 - \frac{v_2}{y_1} \\
u_1 &= \frac{\eta}{y_1 y_2} \\
u_2 &= \frac{v_2}{y_1} ,
\end{aligned}
$$

which, by Corollary 2.3 also solves the i/o decoupling problem. After applying the feedback, the equations (2.33) become

$$
\begin{aligned}
y_1^{[3]} &= y_1^{[2]} + v_1 \\
y_2^{[2]} &= v_2^{[1]} + y_2 .
\end{aligned}
$$

2.3 Disturbance Decoupling

A problem similar to the i/o decoupling is the disturbance decoupling problem (DDP). Just like in the previous section, here the i/o linearization is used to solve the DDP by measurement feedback. The results, presented here, are direct extension of those from [52]. It is shown that a feedback, that strictly linearizes certain functions, also solves the disturbance decoupling problem. In this section, the systems, described by the state equations, are studied. Compared to the description (1.1), here the system has two types of inputs - the control input $u(t) \in U \subset \mathbb{R}^m$ and the disturbance input $w(t) \in W \subset \mathbb{R}^\iota$ - and two kinds of outputs - the controlled output $y(t) \subset Y \in \mathbb{R}^p$ and the measured output $y_*(t) \subset Y_* \in \mathbb{R}^q$. Thus, the systems of the form

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)) \\ y(t) &= h(x(t)) \\ y_*(t) &= h_*(x(t)) \end{aligned} \tag{2.34}$$

are studied. Let r_i be the relative degree of output y_i, for $i = 1, \dots, p$, with respect to the control input u.

Next, two subspaces Ω and Ω_u of $\mathcal{X} := \mathrm{span}_\mathcal{K}\{\mathrm{d}x\}$ are defined, which play an important role in the solution of the problem:

$$\begin{aligned} \Omega = \; & \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \\ & \delta^k\omega \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \dots, \mathrm{d}y_i^{[r_i+k-1]}; i = 1, \dots, p\}\} \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \Omega_u = \; & \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \delta^k\omega \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}u, \\ & \dots, \mathrm{d}u^{[k-1]}, \mathrm{d}y_i^{[r_i]}, \dots, \mathrm{d}y_i^{[r_i+k-1]}; i = 1, \dots, p\}\}. \end{aligned} \tag{2.36}$$

Obviously, by definitions, $\Omega \subseteq \Omega_u$. Note that for SISO systems $\Omega = \Omega_u$, since $\mathrm{d}u$ can be written as a linear combination of $\mathrm{d}x$ and $\mathrm{d}y^{[r]}$. The subspaces Ω and Ω_u can be computed using the following lemmas.

Lemma 2.4. *The subspace Ω may be computed as the limit of the algorithm:*

$$\begin{aligned} \Omega^0 &= \mathcal{X} \\ \Omega^{k+1} &= \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_\mathcal{K}\{\mathrm{d}y_i^{[r_i]}; i = 1, \dots, p\}\}. \end{aligned} \tag{2.37}$$

Lemma 2.5. *The subspace Ω_u may be computed as the limit of the algorithm:*

$$\begin{aligned} \Omega^0 &= \mathcal{X} \\ \Omega^{k+1} &= \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_\mathcal{K}\{\mathrm{d}u, \mathrm{d}y_i^{[r_i]}; i = 1, \dots, p\}\}. \end{aligned} \tag{2.38}$$

The proofs of Lemmas 2.4 and 2.5 are similar to that of Lemma 2.2, except the vector spaces one works with are different. The forward-shift of Ω (or Ω_u) is defined element-wise: $\Omega^{[k]} = \mathrm{span}_{\mathcal{K}}\{\delta^k\theta_1, \ldots, \delta^k\theta_s\}$ for $k \geq 1$, where $\Omega = \mathrm{span}_{\mathcal{K}}\{\theta_1, \ldots, \theta_s\}$.

Problem statement. The goal of this section is to find a measurement feedback of the form

$$\begin{aligned} \eta(t+1) &= F(\eta(t), y_*(t), v(t)) \\ u(t) &= H(\eta(t), y_*(t), v(t)), \end{aligned} \tag{2.39}$$

where $\eta(t) \in \mathbb{R}^\rho$ and $v(t) \in \mathbb{R}^m$, such that the components y_i $(i = 1, \ldots, p)$ of the output-to-be-controlled y of the closed-loop system do not depend on the disturbance w at any time instant, i.e.

$$\begin{aligned} \mathrm{d}y_i^{[k]} &\in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \quad k < \tilde{r}_i \\ \mathrm{d}y_i^{[k]} &\in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v, \ldots, \mathrm{d}v^{[k-\tilde{r}_i]}\} \quad k \geq \tilde{r}_i, \end{aligned}$$

where $\tilde{r}_i$ is the relative degree of y_i of the closed loop system with respect to v.

Assumption 2. The relative degree r_i of y_i with respect to the input u is finite, for $i = 1, \ldots, p$.

Given a system of the form (2.34), one can check whether it is disturbance decoupled or not by the lemma below.

Lemma 2.6. *Under the Assumption 2, system (2.34) is disturbance decoupled if and only if for* $i = 1, \ldots, p$

$$\mathrm{d}y_i^{[r_i]} \in \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}. \tag{2.40}$$

Proof. Necessity. Let throughout the proof $i = 1, \ldots, p$. By the definition of relative degree r_i

$$\mathrm{d}y_i^{[r_i]} = \omega_{i0} + \sum_{j=1}^{m} b_{i,j}\mathrm{d}u_j,$$

where $b_{i,j} \in \mathcal{K}$ and $\omega_{i0} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. Next, it is shown that $\omega_{i0} \in \Omega_u$. Assume, by contradiction, that $\omega_{i0} \notin \Omega_u$ for some i. Then there exists $k \in \mathbb{N}$ such that

$$\delta^k\omega_{i0} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{[k-1]}\}.$$

This means that the 1-form ω_{i0} is not disturbance decoupled and thus y_i is not disturbance decoupled either. This is a contradiction and thus, $\omega_{i0} \in \Omega_u$.

Sufficiency. If (2.40) is true, then by Lemma 2.5, $\Omega_u^{[1]} \subseteq \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$. Thus, Ω_u is invariant with respect to the system dynamics and since $\mathrm{d}y_i \in \Omega_u$, the system (2.34) is disturbance decoupled. □

2.3.1 Sufficient Solvability Conditions

The theorem below gives sufficient conditions for the existence of the feedback in the form (2.39), that solves the disturbance decoupling problem by dynamic measurement feedback.

Theorem 2.4. *Under Assumption 2, the DDP by dynamic measurement feedback is solvable for system (2.34), if*

(i) *there exist 1-forms* $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$, *such that*

$$\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega + \cdots + \Omega^{[s-1]}$$

for $i = 1, \ldots, p$ *and some* $s \geq 1$;

(ii) *if* $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_l\}$ *is the minimal integrable subspace containing* $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_p\}$, *the functions* $\alpha_j(y_*, \ldots, y_*^{[s-1]}, u, \ldots, u^{[s-1]})$, $j = 1, \ldots, l$, *are strictly linearizable by dynamic measurement feedback (2.39).*

Proof. It will be shown that the feedback that linearizes strictly the functions α_j, in *(ii)*, solves the DDP. In the rest of the proof $i = 1, \ldots, p$ and $j = 1, \ldots, l$.

From the linearization process the relative degree of y_i with respect to v is $\bar{r}_i = r_i + s - 1$. Since by *(ii)* the functions α_j are strictly linearizable, in the closed-loop system $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$. Now, from *(i)*

$$\mathrm{d}y_i^{[\bar{r}_i]} \in \Omega + \cdots + \Omega^{[s-1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}.$$

Next, it is shown that $\bar{\Omega} = \Omega + \cdots + \Omega^{[s-1]}$, where by $\bar{\Omega}$ is denoted the subspace Ω for the closed-loop system. From the definition of Ω (see (2.35)),

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \ldots, \mathrm{d}y_i^{[r_i+s-2]}\}.$$

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}.$$

Thus,

$$\begin{aligned} \Omega + \cdots + \Omega^{[s-1]} = \{&\bar{\omega} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ &\delta^k \bar{\omega} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_i^{[r_i+s-1]}, \ldots, \mathrm{d}y_i^{[r_i+s-k-2]}\}\} \\ &= \bar{\Omega}. \end{aligned}$$

The last equality comes from the definition (2.35) of the subspace $\bar{\Omega}$.

Since $\bar{\Omega} \subseteq \bar{\Omega}_u$, then by Lemma 2.6, system (2.34) is disturbance decoupled. □

Since for SISO system, $\Omega = \Omega_u$, the conditions of Theorem 2.4 are also necessary.

Corollary 2.4. *For SISO systems, the conditions of Theorem 2.4 are necessary and sufficient.*

Proof. It remains to prove the necessity. Since the closed-loop system is disturbance decoupled, by Lemma 2.6

$$\mathrm{d}y^{[\bar{r}]} \in \bar{\Omega}_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}, \tag{2.41}$$

where by $\bar{r}$ is denoted the relative degree of y with respect to v and by $\bar{\Omega}_u$ the subspace Ω_u for the closed-loop system, respectively. Choose $s \geq 1$ such that $\bar{r} = r + s - 1$.

Next, since now $\Omega = \Omega_u$, one can show, as in the proof of Theorem 2.4, that $\bar{\Omega}_u = \Omega + \cdots + \Omega^{[s-1]}$. Now, one can find the 1-form $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$, with rank 1, such that from (2.41) one gets

$$\mathrm{d}y^{[r+s-1]} - \omega \in \Omega + \cdots + \Omega^{[s-1]}.$$

Let $\omega = \beta \mathrm{d}\alpha$ for some functions $\beta, \alpha \in \mathcal{K}$. Clearly, the feedback that solves the DDP, also linearizes strictly the function α, since $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$. Thus, the conditions (i) and (ii) of Theorem 2.4 are satisfied. □

Remark 2.4. Taking $s = 1$ in Theorem 2.4, one gets solvability conditions for the DDP by *static* measurement feedback. In this case strict linearizability of functions α_j, $j = 1, \ldots, l$, means that the system of equations $\alpha_j(y_*, u) = v_j$, $j = 1, \ldots, l$, is solvable for u.

Example 2.5. Consider the system

$$\begin{aligned} x_1^{[1]} &= u_1 \\ x_2^{[1]} &= x_3 u_3 + x_2 x_4 u_2 - x_1 \\ x_3^{[1]} &= u_2 \\ x_4^{[1]} &= x_1 w \\ x_5^{[1]} &= u_1 u_2 x_4 + x_2 \\ y_1 &= x_2 \\ y_2 &= x_5 \\ y_* &= x_4. \end{aligned} \tag{2.42}$$

First, note that the relative degrees r_1 and r_2 of outputs y_1 and y_2 with respect to u are both 1. One can also compute the subspaces $\Omega = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5\}$ and $\Omega_u = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_5\}$. Clearly, $\mathrm{d}y_i^{[1]} \notin \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$ for $i = 1, 2$. Therefore, system (2.42) is not disturbance decoupled.

To find the 1-forms ω_i, defined in (i) of Theorem 2.4, calculate $\mathrm{d}y_i^{[r_i+s-1]}$ for $s = 1, 2, \ldots$, until

$$\begin{aligned} \mathrm{d}y_i^{[r_i+s-1]} &\in \Omega + \cdots + \Omega^{[s-1]} \\ &+ \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}. \end{aligned}$$

For system (2.42), one gets

$$\begin{aligned} \mathrm{d}y_1^{[1]} &= u_3\mathrm{d}x_3 - \mathrm{d}x_1 + y_*u_2\mathrm{d}x_2 + x_3\mathrm{d}u_3 + x_2\mathrm{d}(y_*u_2) \\ &\notin \Omega + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y_*\} \\ \mathrm{d}y_2^{[1]} &= \mathrm{d}x_2 + \mathrm{d}(u_1u_2y_*) \in \Omega + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y_*\}. \end{aligned}$$

Thus, $s \neq 1$. Compute $\Omega + \Omega^{[1]} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}x_5^{[1]}\}$. Now,

$$\begin{aligned} \mathrm{d}y_1^{[2]} &= \mathrm{d}(u_3^{[1]}u_2 - u_1) + y_*^{[1]}u_2^{[1]}\mathrm{d}x_2^{[1]} + x_2^{[1]}\mathrm{d}(y_*^{[1]}u_2^{[1]}) \\ &\in \Omega + \Omega^{[1]} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}u^{[1]}, \mathrm{d}y_*, \mathrm{d}y_*^{[1]}\} \\ \mathrm{d}y_2^{[2]} &= \mathrm{d}x_2^{[1]} + \mathrm{d}(u_1^{[1]}u_2^{[1]}y_*^{[1]}) \in \Omega + \Omega^{[1]} + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}u^{[1]}, \mathrm{d}z, \mathrm{d}z^{[1]}\}. \end{aligned}$$

meaning that $s = 2$. Next, one can choose the 1-forms ω_i as

$$\begin{aligned} \omega_1 &= \mathrm{d}(u_3^{[1]}u_2 - u_1) + x_2^{[1]}\mathrm{d}(y_*^{[1]}u_2^{[1]}) \\ \omega_2 &= \mathrm{d}(u_1^{[1]}u_2^{[1]}y_*^{[1]}), \end{aligned}$$

so that they satisfy condition (i) of Theorem 2.4. Obviously, $\operatorname{rank}\omega_1 = 2$ and $\operatorname{rank}\omega_2 = 1$. It remains to check whether the functions $\alpha_{1,1} = u_3^{[1]}u_2 - u_1$, $\alpha_{1,2} = y_*^{[1]}u_2^{[1]}$ and $\alpha_{2,1} = u_1^{[1]}u_2^{[1]}y_*^{[1]}$ are strictly linearizable. One can check, that the dynamic measurement feedback

$$\begin{aligned} \eta_1^{[1]} &= \tfrac{y_*(\eta_2v_1+\eta_3)}{\eta_2^2} \\ \eta_2^{[1]} &= v_2 \\ \eta_3^{[1]} &= v_3 \\ u_1 &= \tfrac{\eta_3}{\eta_2} \\ u_2 &= \tfrac{\eta_2}{y_*} \\ u_3 &= \eta_1, \end{aligned} \tag{2.43}$$

linearizes the functions $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$ and also decouples the disturbances from the outputs y_1 and y_2. Really, in the closed-loop system

$$\begin{aligned} y_1^{[2]} &= v_1 + x_2^{[1]}v_2 \\ y_2^{[2]} &= v_3 + x_2^{[1]} \end{aligned}$$

and since $\bar{\Omega}_u = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}\eta_2\}$, the conditions of Lemma 2.6 are satisfied. This means that the closed-loop system is disturbance decoupled.

2.4 Conclusions

In this chapter the problems of i/o linearization, i/o decoupling and DDP were studied for discrete-time systems. The main result of this chapter is Theorem 2.1, which gives the necessary and sufficient conditions for solvability of the i/o linearization problem by dynamic output feedback. The theorem depends on certain functions $\phi_{i,l}^k$, which can be computed step-by-step from the system equations (1.3). The main difficulties of the solution are in computing the required functions $\phi_{i,l}^k$. More precisely, one needs to compute the minimal integrable vector space of 1-forms, which contains a given 1-form, and this is in general a difficult task.

The results on the i/o decoupling problem and the DDP are generalizations of previous results [73, 80, 52] and the novelty comes from the improved i/o linearization procedure.

Chapter 3

Input-State Linearization

In Chapter 2 the goal was to linearize the input-output (i/o) equations of system (1.3) by a dynamic output feedback. In this Chapter, feedback linearization of the state equations (1.1) is studied. For systems (1.3) which are realizable in the state-space form (1.1), the results of this chapter relax the linearizability conditions from Chapter 2, since the state feedback is more general than output feedback. The dynamic state feedback and the state transformation, which was not used in Chapter 2, are used here. For instance, the state equations

$$\begin{aligned} x_1(t+1) &= x_2(t) + u_1(t) \\ x_2(t+1) &= x_3(t)u_1(t) \\ x_3(t+1) &= x_3(t)u_2(t) \\ x_4(t+1) &= x_4(t) + u_1(t) \\ y_1(t) &= x_2(t) \\ y_2(t) &= x_1(t) - x_4(t) \end{aligned} \tag{3.1}$$

can be linearized by state transformation and dynamic state feedback, see [6], but the corresponding i/o equations

$$\begin{aligned} y_1(t+2) &= \tfrac{u_1(t+1)y_1(t+1)u_2(t)}{u_1(t)} \\ y_2(t+2) &= y_1(t+1) + y_2(t+1) - y_1(t) - u_1(t) \end{aligned} \tag{3.2}$$

are not linearizable by dynamic output feedback, according to the results of Section 2.1.

The traditional methods for studying discrete-time nonlinear control systems assume that the functions, describing the system dynamics, are smooth or even analytic. In Section 3.1 the static state feedback linearization of *possibly non-smooth* systems is studied. The method called functions' algebra (see Section 1.3) is used to solve the problem. The Section 3.2 is devoted to the study of flatness property of discrete-time nonlinear control systems. In particular, one looks for a constructive algorithm to compute the flat output of a given system, based on which a dynamic endogenous state feedback and a state transformation can be found, that linearize the system equations.

3.1 Static Solution for Non-Smooth Systems

The results of this section generalize the results of [62] to multi-input case. The generalization is not direct, since multiple inputs create more complexity. Like in [62], the necessary and sufficient conditions for the existence of static state feedback and coordinate transformation are given in terms of certain finite sequence of vector functions δ^i. Finally, it is shown that these results are related to those from [6] on static feedback linearization when systems are described by meromorphic functions.

The discrete-time systems of the following form

$$x(t+1) = f(x(t), u(t)), \tag{3.3}$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$ and f is possibly non-smooth, are considered. It is assumed that the inputs influence the system equations (3.3) independently.

3.1.1 Necessary and Sufficient Condition

Problem statement. One searches for a state transformation $z = \varphi(x)$ and a regular static state feedback $u = G(x, v)$ that transform the system equations (3.3) into the form

$$\begin{aligned} z_{i,j}(t+1) &= z_{i,j+1}(t) \\ z_{i,k_i}(t+1) &= v_i(t), \end{aligned} \tag{3.4}$$

where $z(t) \in Z \subseteq \mathbb{R}^n$, $i = 1, \ldots, m$, $j = 1, \ldots, k_i - 1$ and Z is an open and dense subset of the range of φ.

The solution of the feedback linearization problem will be expressed in terms of a sequence

$$\delta^0 \leq \delta^1 \leq \delta^2 \leq \cdots \leq \delta^i \leq \cdots \tag{3.5}$$

of the vector functions δ^i, defined as follows, see [62]. Let $\delta^0 = x$ and δ^1 be the minimal vector function such that its forward shift $(\delta^1)^{[1]}$ does not depend on the input u. For $i \geq 1$ define

$$\delta^{i+1} = \delta^i \oplus \mathbf{m}(\delta^i). \tag{3.6}$$

The sequence δ^i, $i \geq 1$, converges, see [62]. Denote the limit by δ and let k be such that $\delta^k \neq \delta$, $\delta^{k+1} = \delta$.

Another possibility to compute the sequence δ^i is by using the following Lemma below. Note that unlike (3.6), the relation (3.7) below is also true for $i = 0$.

Lemma 3.1. *The vector functions δ^i satisfy the relations*

$$(\delta^{i+1})^{[1]} = \delta^i \oplus (\delta^i)^{[1]} \tag{3.7}$$

for $i \geq 0$.

Proof. Since $\mathbf{m}(\alpha) = ((\alpha \times u) \oplus f)^{[-1]}$, then one gets from (3.6)

$$\begin{aligned}(\delta^{i+1})^{[1]} &= (\delta^i)^{[1]} \oplus (\mathbf{m}(\delta^i))^{[1]} \\ &= (\delta^i)^{[1]} \oplus (\delta^i \times u) \oplus f \\ &= (\delta^i)^{[1]} \oplus (\delta^i \times u).\end{aligned}$$

The last equivalence comes from the facts that $(\delta^i)^{[1]} = \delta^i(f)$, $\delta^i(f) \geq f$ and thus $(\delta^i)^{[1]} \oplus f = (\delta^i)^{[1]}$. Now,

$$(\delta^i)^{[1]} \oplus (\delta^i \times u) = (\delta^i)^{[1]} \oplus \delta^i$$

by the properties of $\oplus$ and $\times$ and the fact that $(\delta^i)^{[1]}$ does not depend on u.

If $i = 0$, then $\delta^0 \oplus (\delta^0)^{[1]} = x \oplus f$, which is exactly the shift of δ^1. □

Definition 3.1. The relative degree of a vector function $\alpha = [\alpha_1, \ldots, \alpha_k]^T$ is defined as minimal number r such that $\alpha_j^{[r]}$ depends on system inputs for some $j \in \{1, \ldots, k\}$.

Another property of the sequence (3.5) is the following.

Lemma 3.2. *The relative degree of δ^i is $i + 1$, for $i \geq 0$.*

Proof. The proof is done by induction over i. For $i = 0$, the relative degree of $\delta^0 = x$ is clearly 1. Now, assume that the claim is also true for δ^i, $i = 0, \ldots, p$. Then, by Lemma 3.1, $(\delta^{p+1})^{[1]} = \gamma(\delta^p)$ for some vector function γ. Since the relative degree of δ^p is $p + 1$, therefore the relative degree of δ^{p+1} is $p + 2$. □

In the rest of this section, $|\alpha|$ denotes the number of independent non-constant elements of the vector α.

Theorem 3.1. *The system (3.3) can be transformed into the form (3.4) by a state transformation $z = \varphi(x)$ and static state feedback $u = G(x, v)$ if and only if $\delta = \mathbf{1}$ and*

$$\sum_{r=1}^{k+1} (|\delta^{r-1}| - |\delta^r \times (\delta^r)^{[1]}|) = m, \tag{3.8}$$

where k is defined such that $\delta^{k+1} = \delta$.

Proof. Necessity. First, note that the sequence of functions δ^i, $i \geq 1$, is invariant with respect to the state transformation and static state feedback, see [62].

Consider the ith subsystem of (3.4) and compute the vector functions δ_i^r, $r = 1, \ldots, k_i$, defined by (3.6), for this subsystem:

$$\begin{aligned}
\delta_i^1 &= [z_{i,j}; j = 1, \ldots, k_i - 1]^T \\
&\vdots \\
\delta_i^r &= [z_{i,j}; j = 1, \ldots, k_i - r]^T \\
&\vdots \\
\delta_i^{k_i - 1} &= z_{i,1} \\
\delta_i^{k_i} &= \mathbf{1}.
\end{aligned}$$

Since

$$(\delta_i^r)^{[1]} = [z_{i,j}; j = 2, \ldots, k_i - r + 1]^T$$

one gets

$$\begin{aligned}
\delta_i^r \times (\delta_i^r)^{[1]} &= [z_{i,j}; j = 1, \ldots, k_i - r + 1]^T = \delta_i^{r-1} \quad \text{if } r < k_i \\
\delta_i^r \times (\delta_i^r)^{[1]} &= \mathbf{1} \quad \text{if } r = k_i.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\delta_i^{r-1}| - |\delta_i^r \times (\delta_i^r)^{[1]}| &= 0, \quad \text{if } r < k_i \\
|\delta_i^{k_i - 1}| - |\delta_i^{k_i} \times (\delta_i^{k_i})^{[1]}| &= |z_{i,1}| = 1.
\end{aligned}$$

Now, $\delta^r = \delta_1^r \times \cdots \times \delta_m^r$ and

$$\begin{aligned}
\sum_{r=1}^{k+1} (|\delta^{r-1}| - |\delta^r \times (\delta^r)^{[1]}|) &= \sum_{i=1}^{m} \sum_{r=1}^{k_i} (|\delta_i^{r-1}| - |\delta_i^r \times (\delta_i^r)^{[1]}|) \\
&= \sum_{i=1}^{m} 1 = m.
\end{aligned}$$

Sufficiency. Because $\delta^{r-1} \leq \delta^r$ and by (3.7), $\delta^{r-1} \leq (\delta^r)^{[1]}$, one gets $\delta^{r-1} \leq \delta^r \times (\delta^r)^{[1]}$. Then for every $r = 1, \ldots, k+1$ there exists a vector function φ_r (possibly equal to $\mathbf{1}$) such that

$$\delta^{r-1} = \delta^r \times (\delta^r)^{[1]} \times \varphi_r, \tag{3.9}$$

where

$$|\varphi_r| = |\delta^{r-1}| - |\delta^r \times (\delta^r)^{[1]}|.$$

Let $|\varphi_r| = \rho_r$ for $r = 1, \ldots, k+1$ and

$$\varphi_r = [\varphi_{r,1}, \ldots, \varphi_{r,\rho_r}]^T.$$

Then, by (3.9)

$$\begin{aligned}
\delta^0 &= [\delta^1, (\delta^1)^{[1]}, \varphi_{1,1}, \ldots, \varphi_{1,\rho_1}]^T \\
\delta^1 &= [\delta^2, (\delta^2)^{[1]}, \varphi_{2,1}, \ldots, \varphi_{2,\rho_2}]^T \\
&\vdots \\
\delta^k &= [\delta^{k+1}, (\delta^{k+1})^{[1]}, \varphi_{k+1}]^T = [\varphi_{k+1,1}, \ldots, \varphi_{k+1,\rho_{k+1}}]^T.
\end{aligned}$$

Substituting step by step δ^r and $(\delta^r)^{[1]}$ into δ^{r-1} for $r = 1, \ldots, k$, one gets

$$\delta^0 = [(\varphi_{i,l}(x))^{[j]}; i = 1, \ldots, k+1; j = 0, \ldots, i-1; l = 1, \ldots, \rho_i]^T. \quad (3.10)$$

The elements $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i',l'}(x))^{[j']}$, $i \neq i'$, are independent by definition and since $\delta = \mathbf{1}$, then the elements $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i,l}(x))^{[j']}$, $j \neq j'$, are also independent. Really, if $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i,l}(x))^{[j']}$ were dependent, then there would exist a function γ, such that $(\varphi_{i,l}(x))^{[j]} = \gamma((\varphi_{i,l}(x))^{[j']})$ (assume that $j < j'$). This would mean that the relative degree of $(\varphi_{i,l}(x))^{[j]}$ is infinite and therefore $\delta \neq \mathbf{1}$.

Because of (3.8), $\sum_{r=1}^{k+1} |\varphi_r| = m$, and there exist exactly m non-constant functions $\varphi_{i,j}$, $i = 1, \ldots, k+1$, $j = 1, \ldots, \rho_i$. Let ϕ_i, $i = 1, \ldots, m$, be these functions. Then (3.10) becomes

$$\delta^0 = [(\phi_i(x))^{[j]}; i = 1, \ldots, m; j = 0, \ldots, k_i - 1]^T \quad (3.11)$$

for some k_i. Define the state transformation

$$\begin{aligned}
z_{i,1} &= \phi_i(x) \\
&\vdots \\
z_{i,k_i} &= \phi_i(x)^{[k_i-1]}
\end{aligned} \quad (3.12)$$

for $i = 1, \ldots, m$. Equations (3.12) really define a state transformation (i.e. a one-to-one correspondence) since by (3.11), $z = [z_{i,1}, \ldots, z_{i,k_i}; i = 1, \ldots, m]^T$ is equivalent to δ^0, which is equivalent to x.

Now, in the new coordinates the system equations (3.3) become

$$\begin{aligned}
z_{i,j}^{[1]} &= z_{i,j+1} \\
z_{i,k_i}^{[1]} &= K_i(z, u),
\end{aligned} \quad (3.13)$$

for $i = 1, \ldots, m$, $j = 1, \ldots, k_i - 1$ and where K_i is the forward-shift of $z_{i,k_i} = \phi_i(x)^{[k_i-1]}$, i.e. $K_i = \phi_i(x)^{[k_i]}$. Finally, since the input variables are independent, then $v = K(z,u) = [K_1(z,u),\dots,K_m(z,u)]^T$ is solvable in u. This gives a static state feedback which takes the system into the form (3.4). □

Example 3.1. Consider the discretized system of a simplified model of the underwater vehicle moving on a vertical plane (see [29]):

$$\begin{array}{rcl} x_1^{[1]} & = & x_1 + x_2, \\ x_2^{[1]} & = & x_2 + \frac{1}{J}(M_0\sin(x_1) + \frac{1}{2}\rho x_3\,\mathrm{abs}(x_3)(m_1\sin(2(x_1-x_4)) \\ & + & \frac{x_2}{x_3}m_2V^{3/4})), \\ x_3^{[1]} & = & x_3 + \frac{1}{m_x}(-\frac{1}{2}\rho x_3\,\mathrm{abs}(x_3)(r_{x0} + r_{x1}\cos(x_1-x_4) \\ & + & r_{x2}\sin(2(x_1-x_4))) + P\sin(x_4) + u_1\cos(x_1-x_4) \\ & - & u_2\sin(x_1-x_4)), \\ x_4^{[1]} & = & x_4 + \frac{1}{m_y x_3}(\frac{1}{2}\rho x_3\,\mathrm{abs}(x_3)(r_y\sin(2(x_1-x_4)) + \frac{x_2}{x_3}CV) \\ & + & P\cos(x_4) + u_2\cos(x_1-x_4) - u_1\sin(x_1-x_4)), \\ x_5^{[1]} & = & x_5 + x_3\sin(x_4). \end{array} \tag{3.14}$$

The model is developed under the assumption that the control moment is insignificant. Here x_1 and x_2 are the trim angle of the vehicle and its velocity, respectively, x_3 is the linear speed of the vehicle, x_4 is the the bank angle, and x_5 is the depth of plunge; J, M_0, ρ, m_1, m_2, m_x, m_y, r_{x0}, r_{x1}, r_{x2}, r_y, V, P, and C are some constant coefficients, characterizing construction of the vehicle.

By Lemma 3.2, it becomes clear that $\delta^1 = (\,x_1 \quad x_2 \quad x_5)^T$, $\delta^2 = x_1$, $\delta^3 = \mathbf{1}$ for system (3.14). Then

$$(\delta^1)^{[1]} = \begin{pmatrix} x_1 + x_2 \\ x_2^{[1]} \\ x_5 + x_3\sin(x_4) \end{pmatrix}, \qquad (\delta^2)^{[1]} = x_1 + x_2.$$

Therefore,

$$\begin{array}{l} \sum_{r=1}^{3}(|\delta^{r-1}| - |\delta^r \times (\delta^r)^{[1]}|) = (|\delta^0| - |\delta^1 \times (\delta^1)^{[1]}|) \\ \quad +(|\delta^1| - |\delta^2 \times (\delta^2)^{[1]}|) + (|\delta^2| - |\delta^3 \times (\delta^3)^{[1]}|) \\ \quad = (5-5) + (3-2) + (1-0) = 2 = m. \end{array}$$

Thus, by Theorem 3.1, the system (3.14) can be linearized by static state feedback and a change of coordinates.

Set $z_{1,1} := x_1$, $z_{2,1} := x_5$, $z_{1,2} := x_1 + x_2$, $z_{1,3} := x_1 + 2x_2 + \frac{1}{J}(M_0\sin(x_1) + \frac{1}{2}\rho x_3\,\mathrm{abs}(x_3)(m_1\sin(2(x_1-x_4)) + \frac{x_2}{x_3}m_2V^{3/4}))$ and $z_{2,2} := x_5 + x_3\sin(x_4)$. These relations define the change of coordinates and by solving the equations

$$\begin{array}{rcl} v_1 & = & z_{1,3}^{[1]} =: K_1(x,u) \\ v_2 & = & z_{2,2}^{[1]} =: K_2(x,u) \end{array}$$

in u_1 and u_2, one gets the static state feedback. The expressions for K_1 and K_2, as well as the feedback itself, are rather complex and are thus omitted.

3.1.2 Comparison

Next, it will be shown that the conditions of Theorem 3.1 are equivalent to those in [6]. To make the comparison, assume that the vector function f in (3.3) is meromorphic.

First, recall the conditions from [6]. Define the sequence $\mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots$ of subspaces of $\mathcal{E}$ as

$$\begin{array}{rcll}\mathcal{H}_0 & = & \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\} & \\ \mathcal{H}_k & = & \operatorname{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_{k-1} \mid \delta\omega \in \mathcal{H}_{k-1}\}, & k \geq 1.\end{array} \tag{3.15}$$

This is a non-increasing sequence and thus it converges. Let k^* be such that $\mathcal{H}_{k^*} \neq \mathcal{H}_{k^*+1}$, but $\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2}$ and define $\mathcal{H}_\infty := \mathcal{H}_{k^*+1}$.

Theorem 3.2. *[6] The system (3.3), where f consists of meromorphic functions, is transformable into the form (3.4) by a state transformation and a static state feedback if and only if $\mathcal{H}_\infty = 0$ and subspaces $\mathcal{H}_i$, $i = 1, \ldots, k^*$, are integrable.*

Note that it has been shown in [6], that if $\mathcal{H}_\infty = 0$ then

$$\begin{array}{rcl}\mathcal{H}_{k^*} & = & \operatorname{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_{\rho_1}\} \\ \mathcal{H}_{k^*-1} & = & \operatorname{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_{\rho_1}, \omega_1^{[1]}, \ldots, \omega_{\rho_1}^{[1]}, \omega_{\rho_1+1}, \ldots, \omega_{\rho_2}\} \\ & \vdots & \\ \mathcal{H}_1 & = & \operatorname{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_{\rho_1}, \ldots, \omega_1^{[k^*-1]}, \ldots, \omega_{\rho_1}^{[k^*-1]}, \ldots, \omega_{\rho_{k^*}}, \ldots, \omega_m\},\end{array} \tag{3.16}$$

where $1 \leq \rho_1 < \rho_2 < \cdots < \rho_{k^*} \leq m$. Thus,

$$\sum_{i=1}^{k^*} \left[\dim \mathcal{H}_i - \dim(\mathcal{H}_{i+1} \cup \mathcal{H}_{i+1}^{[1]})\right] = m. \tag{3.17}$$

In [62] it has been shown that δ^i, defined by (3.6), corresponds to the maximal integrable subspace $\hat{\mathcal{H}}_{i+1}$ of $\mathcal{H}_{i+1}$ for $i \geq 0$, i.e. $\hat{\mathcal{H}}_{i+1} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\delta^i\}$. Also, since $\mathcal{H}_\infty$ is always integrable, see [6], then $\delta = \mathbf{1}$ is equivalent to the condition $\mathcal{H}_\infty = 0$.

Now, if subspaces $\mathcal{H}_i$, $i = 1, \ldots, k^*$, are integrable, then by (3.17), the condition (3.8) is satisfied for δ^i, $i \geq 0$. The opposite is also true: if the condition (3.8) is satisfied, then by (3.17) the subspaces $\mathcal{H}_i$, $i = 1, \ldots, k^*$, must be integrable.

3.2 Flatness

In this Section the flatness property of discrete-time nonlinear control systems will be addressed. It is well-known that flatness property of continuous-time nonlinear control system is equivalent to the existence of a dynamic state feedback and a state transformation that linearize the system equations [31]. Similar equivalence was proved in [55] for the discrete-time case, except in this case the flatness property is equivalent to the existence of *an endogenous* dynamic state feedback and a state transformation that linearize the system equations. Thus, one can study the two problems - flatness and endogenous feedback linearization - as one.

The necessary and sufficient condition for the existence of a dynamic endogenous state feedback, that linearizes the given discrete-time system, was given already 20 years ago [6]. Unfortunately, the condition in [6] depends on the existence of certain polynomial matrix and therefore, is not constructive. Attempts have been made to construct the desired matrix, but the procedures in [64, 63, 53] are not finite, since the upper bound for the degree of the polynomial matrix is unknown.

Here, a different approach is used to find constructive necessary and sufficient conditions to check whether a given nonlinear discrete-time system is flat or not. The idea behind this approach is to transform the system equations into a certain form, which allows to eliminate some of the state equations. If one repeats the process, then at some point either one can eliminate no more equations or one eliminates all the equations. It will be proved that the latter case is necessary and sufficient for flatness.

Note that by the Definition 3.2 below, any system for which $m \geq n$ is flat. Thus, from now on it is assumed that $n > m$. In this section, the systems of the form (1.1) without the output are considered, and denoted by Σ_0:

$$x^{[1]} = f(x, u). \tag{3.18}$$

Additionally to the submersivity assumption (1.2), it is also assumed that

$$\operatorname{rank}_{\mathcal{K}} \Big[\frac{\partial f(x,u)}{\partial u} \Big] = m, \tag{3.19}$$

which is not restrictive. If condition (3.19) is not true, then one can always eliminate some of the input variables by an input transformation, such that (3.19) becomes true for the transformed system. Consider also the subspaces $\mathcal{H}_i$, $i = 0, \dots, k^*$, defined by (3.15). As before, it is assumed that $\mathcal{H}_\infty = 0$, which guarantees that system (3.18) is accessible, see [6]. Note that the accessibility is necessary condition for flatness and thus, not restrictive.

3.2.1 Definition

Next, the flatness property of system (3.18) is defined and some properties of flat systems are proved. One possibility to define the property of flatness for systems of the form (3.18) is the following.

Definition 3.2. [6] An output function

$$y = h(x, u, \ldots, u^{[l]}), \qquad l \geq 0 \tag{3.20}$$

($y \in \mathbb{R}^m$) is called a flat output[1] of system (3.18) if it satisfies the following properties:

(i) y defines an invertible system;

(ii) $\dim_{\mathcal{K}}(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \bigcap \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{[k]}; k \geq 0\}) = n$.

When system (3.18) has a flat output, then it is said to be flat.

The condition (*ii*) of Definition 3.2 guarantees that the state x of a flat system can be represented as a function of a flat output y and a finite number of its forward shifts. The condition (*i*) of Definition 3.2 guarantees the same for system input u and moreover, that the flat outputs and their shifts are functionally independent.

Because of (3.19), one can write the inputs u in terms of x and $x^{[1]}$. It means that for finding flat outputs of system (3.18), it is enough to find m independent functions φ such that the states x can be written in terms of φ and a finite number of its forward-shifts.

When the state variables x can be written in terms of the flat output y and its forward shifts, then there exists a matrix $G \in \mathcal{K}[\vartheta]^{n\times m}$ such that $\mathrm{d}x = G\mathrm{d}y$. Now, if there exists a unimodular matrix $L \in \mathcal{U}_m[\vartheta]$ such that $\mathrm{d}y = L\mathrm{d}\hat{y}$ for some functions $\hat{y}$, then clearly $\hat{y}$ is also a flat output of the given system, since $\mathrm{d}x = GL\mathrm{d}\hat{y}$, i.e. one can write the state x in terms of $\hat{y}$ and its forward-shifts. In particular, any exact basis of $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y\}$ is a flat output of a given system. Since, there exist infinitely many such exact bases, any flat system has infinitely many different flat outputs. Flat systems are often grouped according to the highest shift of u one needs in (3.20). So, when a system (3.18) has a flat output, depending only on x, then the system is said to be 0-flat. When all the flat outputs depend on x and u, then the system is said to be 1-flat etc.

Next, it is proved that unlike in the continuous-time case, a discrete-time system (3.18) is flat if and only if it is 0-flat. To prove it, the following lemma is used.

[1] In [6] the term linearizing output was used.

Lemma 3.3. *Whenever there exists a flat output of system (3.18) depending on x and u then there exists another flat output of system (3.18) depending only on x.*

Proof. Let $y = (y_1, \ldots, y_m)$ be a flat output of system (3.18) depending on x and u. For simplicity, assume, that $y_k = h(x, u_k)$ depends on u_k for some $1 \le k \le m$. The case when multiple outputs depend on u can be handled similarly.

Because a flat system is invertible, one has from the inversion algorithm $\tilde{y}_0 = y_k$ and by Lemma 1.2 that for some $\rho \in \mathbb{N}$

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(y_k, \tilde{y}_1^{[1]}, \ldots, \tilde{y}_\rho^{[\rho]})}{\partial u} = m.$$

Also, since for a flat system, one can represent the states x in terms of the flat outputs y and their shifts, then

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\bar{y}_0, \ldots, \bar{y}_{\rho-1}^{[\rho-1]})}{\partial x} = n. \tag{3.21}$$

In the following, the notations $\tilde{\mathcal{Y}} = \operatorname{span}_{\mathcal{K}}\{d\tilde{y}_p^{[p+l]}; p \ge 1; l \ge 0\}$ and $\bar{\mathcal{Y}} = \operatorname{span}_{\mathcal{K}}\{d\bar{y}^{[p]}; p \ge 0\}$ are used.

Because $d\bar{y}_{i,\lambda}^{[i]} \in \operatorname{span}_{\mathcal{K}}\{dx, d\tilde{y}_p^{[p+l]}; p \ge 1; l \ge 0\}$ for $i = 0, \ldots, \rho - 1$, $\lambda = 1, \ldots, \dim \bar{y}_i^{[i]}$, there exist 1-forms $\omega_{i,\lambda} \in \operatorname{span}_{\mathcal{K}}\{dx\}$ and $\tilde{\omega}_{i,\lambda} \in \tilde{\mathcal{Y}}$ such that $d\bar{y}_{i,\lambda}^{[i]} = \omega_{i,\lambda} + \tilde{\omega}_{i,\lambda}$. For the rest of this proof $i = 0, \ldots, \rho - 1$, $\lambda = 1, \ldots, \dim \bar{y}_i^{[i]}$. Let $\gamma_{i,\lambda}$ be the rank of the 1-form $\omega_{i,\lambda}$, then one can write the 1-forms $\omega_{i,\lambda}$ as

$$\omega_{i,\lambda} = \sum_{j=1}^{\gamma_{i,\lambda}} a_{i,\lambda,j} d\varphi_{i,\lambda,j}.$$

Now, one has

$$d\bar{y}_{i,\lambda}^{[i]} = \sum_{j=1}^{\gamma_{i,\lambda}} a_{i,\lambda,j} d\varphi_{i,\lambda,j} + \tilde{\omega}_{i,\lambda}. \tag{3.22}$$

By (3.21)

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\varphi_{i,\lambda,j}; j = 1, \ldots, \gamma_{i,\lambda})}{\partial x} = n.$$

Choose n independent rows of $\frac{\partial(\varphi_{i,\lambda,j})}{\partial x}$ and denote the corresponding 1-forms as $d\varphi_{i,\lambda}$. Note that one can do that, since y and its shifts are independent, $\sum_{i=0}^{\rho-1} \dim \bar{y}_i^{[i]} = n$. Now, (3.22) can be rewritten as

$$d\bar{y}_{i,\lambda}^{[i]} = \sum_{j=1}^{n} \sum_{l=1}^{\dim \bar{y}_j^{[j]}} b_{i,\lambda,j,l} d\varphi_{j,l} + \tilde{\omega}_{i,\lambda}. \tag{3.23}$$

Since in (3.23) there are n equations and n exact 1-forms $\mathrm{d}\varphi_{j,l}$, then (3.23) can be rewritten as

$$\mathrm{d}\bar{y}_{i,\lambda}^{[i]} = e_{i,\lambda}\mathrm{d}\varphi_{i,\lambda} + \bar{\omega}_{i,\lambda} + \hat{\omega}_{i,\lambda}, \tag{3.24}$$

where $\bar{\omega}_{i,\lambda} \in \bar{\mathcal{Y}}$ and $\hat{\omega}_{i,\lambda} \in \tilde{\mathcal{Y}}$.

Let i_* be minimal number such that $\bar{y}_{i_*,\lambda_*}^{[i_*]}$ depends on y_k for some λ_*. Such i_* always exists, because of (3.19) and the fact that y_k depends on u_k. Then define an output $\hat{y} = (y_1, \ldots, y_{k-1}, \varphi_{i_*,\lambda_*}, y_{k+1}, \ldots, y_m)$. Next, it is shown that $\hat{y}$ is also a flat output of system (3.18). It is enough to show that all the states x can be written in terms of $\hat{y}$ and the forward-shifts. Really, the rank condition (3.21) is satisfied for the output $\hat{y}$, since, by (3.24) and the construction of the inversion algorithm, when replacing $\bar{y}_{i_*,\lambda_*}^{[i_*]}$ and its shifts by φ_{i_*,λ_*} and its shifts in (3.21), the rank must remain full. □

Now, the following theorem can be proved.

Theorem 3.3. *A discrete-time system (3.18) is flat if and only if it is 0-flat.*

Proof. One has to show that when there exists a flat output depending on $x, u, \ldots, u^{[k]}$, then there exists another flat output, which depends only on x. The proof is by induction over k. For $k = 1$, the claim is true by Lemma 3.3. Now, assume that the claim is true for $i = 1, \ldots, k-1$. Next, it will be shown that then it is also true for k.

Assume that (3.18) has flat output depending on the variables $x, u, \ldots, u^{[k]}$. Then, the system

$$\begin{array}{rcl} x^{[1]} & = & f(x,u) \\ u^{[1]} & = & v \end{array} \tag{3.25}$$

has flat output depending on the variables $x, u, \ldots, u^{[k-1]}$. By assumption, there exists a flat output of system (3.25) depending only on x and u. Clearly, this is also a flat output of system (3.18) and thus, by Lemma 3.3 a flat output of system (3.18) exists, which depends only on x. Therefore, system (3.18) is 0-flat. □

3.2.2 Construction of flat outputs

As said before, the essential part of checking flatness property for system (3.18) is transforming it to certain form specified below by a state transformation and a static state feedback. First, the existence of such transformations is studied. So, one looks for a state transformation $z = \varphi(x)$ and a static state feedback $u = \alpha(x, \tilde{u})$ such that system (3.18) is transformed into the form

$$\begin{array}{rcl} z_1^{[1]} & = & g_1(z, \tilde{u}_1) \\ z_2^{[1]} & = & \tilde{u}_2, \end{array} \tag{3.26}$$

where $z = (z_1, z_2)^T$, $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)^T$ and $\dim z_2 = \dim u_2 =: q$ is as large as possible[2]. Consider also the subsystem

$$\Sigma_1 : \quad z_1^{[1]} = g_1(z, \tilde{u}_1) \tag{3.27}$$

that has state variables z_1 and input variables $z_2, \tilde{u}_1$. Observe that in general $\operatorname{rank}_{\mathcal{K}} \frac{\partial g_1}{\partial(z_2, \tilde{u}_1)} \leq m$. If this is the case, one can always find a static state feedback that eliminates some variables $z_2, \tilde{u}_1$ such that the rank condition (3.19) is satisfied for system Σ_1.

Theorem 3.4. *System (3.18) can be transformed into the form (3.26), where $m \geq q \neq 0$, by a state transformation and a static state feedback if and only if there exists a $(n-q)$-dimensional integrable subspace $\mathcal{A}$ that satisfies the condition*

$$\mathcal{H}_2 \subseteq \mathcal{A} \subset \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}. \tag{3.28}$$

Proof. Necessity. It will be shown that $\mathcal{A} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z_1\}$ satisfies condition (3.28). Clearly, $\mathcal{A} \subset \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$ is true. It remains to show that $\mathcal{A}$ contains $\mathcal{H}_2$. Note that $\mathcal{H}_2$ is invariant with respect to state transformations and static state feedback and thus a 1-form $\omega = a_1 \mathrm{d}z_1 + a_2 \mathrm{d}z_2$ is an element of $\mathcal{H}_2$ if and only if $\delta\omega \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z\}$. Since

$$\delta\omega = a_1^{[1]} \mathrm{d}g_1 + a_2^{[1]} \mathrm{d}\tilde{u}_2$$

the latter implies that $a_2^{[1]} = 0$ and thus a_2 is a zero vector. Therefore, the elements of $\mathcal{H}_2$ are in the form $\omega = a_1 \mathrm{d}z_1$, which yields that $\mathcal{H}_2 \subseteq \mathcal{A}$.

Sufficiency. Let $\mathcal{A} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z_1\}$. Note that one can always extend z_1 by z_2 such that $(z_1, z_2)^T = z = \varphi(x)$ defines a state transformation. The subspace $\mathcal{A}$ has a dimension $n - q$ and thus there exists a $(m-q)$-dimensional subspace $\mathcal{B}$ such that $\mathcal{A} = \mathcal{H}_2 \oplus \mathcal{B}$. Therefore, denoting $\mathcal{A}^{[1]} := \operatorname{span}_{\mathcal{K}}\{\delta(\mathrm{d}z_1)\}$, one has

$$\dim(\operatorname{span}_{\mathcal{K}}\{\mathrm{d}z\} \cap \mathcal{A}^{[1]}) = n - m$$

and the subspace $\mathcal{A}_1 := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z, \delta(\mathrm{d}z_1)\}$ has dimension $n + n - q - (n - m) = n + m - q$. Thus, $\operatorname{span}_{\mathcal{K}}\{\mathrm{d}z\} \subseteq \mathcal{A}_1 \subset \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z, \mathrm{d}u\}$ and one can find $\tilde{u}_1 = \alpha_1(z, u)$ such that $\mathcal{A}_1 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}z, \mathrm{d}\tilde{u}_1\}$. Finally, one can take $\tilde{u}_2 = z_2^{[1]}$. □

The condition of Theorem 3.4 depends on the existence of an integrable subspace $\mathcal{A}$. For this reason, another condition is given under which one

[2]Note that there always exist a state transformation and a static state feedback, that transform system (3.18) into the form (3.26), since one can always take $z_1 = x$, $\tilde{u}_1 = u$ and $z_2, \tilde{u}_2$ as empty vectors.

can transform system (3.18) into the form (3.26), where $q \neq 0$. For this, let γ be the dimension of a minimal integrable subspace, that contains $\mathcal{H}_2$ of system (3.18).

Theorem 3.5. *System (3.18) can be transformed into the form (3.26), where $q \neq 0$, by a state transformation and a static state feedback if and only if $\gamma < n$.*

Proof. This is a direct consequence of Theorem 3.4, when one takes $\mathcal{A}$, in Theorem 3.4, equal to the minimal integrable subspace containing $\mathcal{H}_2$. □

Note that Theorem 3.5 allows to check whether a transformation to the form (3.26) is possible, but it does not give a hint on how to find this transformation. To the contrary, Theorem 3.4 depends on the existence of an integrable subspace, but once this is known, one can compute the necessary state and input transformations.

Next, some important properties of system Σ_1 are proved.

Lemma 3.4. *(i) If system Σ_1 is flat, then system Σ_0 is flat.*

(ii) If system Σ_0 is flat, then system Σ_1 is flat.

(iii) If system Σ_0 is flat, then one can transform Σ_0 into the form (3.26), where $q \neq 0$, by a state transformation and a static state feedback.

Proof. (i) Let the number of inputs in system Σ_1 be $m_* \leq m$. Now, complete the m_* flat outputs of system Σ_1 by $m - m_*$ variables $z_2, \tilde{u}_1$, which do not occur in Σ_1, to define the flat output of system Σ_0. Clearly, the variables $z_1, z_2, \tilde{u}_1$ can be written in terms of these flat outputs. Also, from the last q equations of (3.26) one gets that $\tilde{u}_2 = z_2^{[1]}$ and $\tilde{u}_2$ can be represented by the flat outputs of system Σ_1 and $m - m_*$ variables $z_2, \tilde{u}_1$, which do not occur in Σ_1. Finally, one can inverse the state transformation $z = \varphi(x)$ and static state feedback $u = \alpha(x, \tilde{u})$ and express the variables x, u by these flat outputs.

(ii) Assume, by contradiction, that system Σ_1 is not flat. Extending system Σ_1 by $z_2^{[1]} = \tilde{u}_2$ gives system (3.26), i.e. system Σ_0. Adding $z_2^{[1]} = \tilde{u}_2$ to system Σ_1 can not make the extended system flat (one only applies a dynamic feedback), which means that Σ_0 is not flat. This is a contradiction and thus Σ_1 must be flat.

(iii) Since system Σ_0 is flat, then it is invertible with respect to the flat output y[3] and by Lemma 1.2 one has for some $\rho \in \mathbb{N}$ that

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\tilde{y}_1^{[1]}, \ldots, \tilde{y}_\rho^{[\rho]})}{\partial u} = m.$$

[3]Which, by Theorem 3.3, can be chosen such that they depend only on x.

Also, since for a flat system, one can represent the states x in terms of the flat outputs y and their shifts, then

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\bar{y}_0, \ldots, \bar{y}_{\rho-1}^{[\rho-1]})}{\partial x} = n. \tag{3.29}$$

In the following, the notations $\tilde{\mathcal{Y}} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\tilde{y}_p^{[p+l]}; p \geq 1; l \geq 0\}$ and $\bar{\mathcal{Y}} = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}\bar{y}^{[p]}; p \geq 0\}$ are used.

Because $\mathrm{d}\bar{y}_{i,\lambda}^{[i]} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\tilde{y}_p^{[p+l]}; p \geq 1; l \geq 0\}$ for $i = 0, \ldots, \rho - 1$, $\lambda = 1, \ldots, \dim \bar{y}_i^{[i]}$[4], then there exist 1-forms $\omega_{i,\lambda} \in \operatorname{span}_{\mathcal{K}}\{\mathrm{d}x\}$ and $\tilde{\omega}_{i,\lambda} \in \tilde{\mathcal{Y}}$ such that $\mathrm{d}\bar{y}_{i,\lambda}^{[i]} = \omega_{i,\lambda} + \tilde{\omega}_{i,\lambda}$. Let $\gamma_{i,\lambda}$ be the rank of the 1-form $\omega_{i,\lambda}$, then one can write the 1-forms $\omega_{i,\lambda}$ as

$$\omega_{i,\lambda} = \sum_{j=1}^{\gamma_{i,\lambda}} a_{i,\lambda,j} \mathrm{d}\varphi_{i,\lambda,j}.$$

Now, one has

$$\mathrm{d}\bar{y}_{i,\lambda}^{[i]} = \sum_{j=1}^{\gamma_{i,\lambda}} a_{i,\lambda,j} \mathrm{d}\varphi_{i,\lambda,j} + \tilde{\omega}_{i,\lambda}. \tag{3.30}$$

By (3.29)

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial(\varphi_{i,\lambda,j})}{\partial x} = n.$$

Choose n independent rows of $\frac{\partial(\varphi_{i,\lambda,j})}{\partial x}$ and denote the corresponding 1-forms as $\mathrm{d}\varphi_{i,\lambda}$. Note that one can do that, because, since y and its shifts are independent, $\sum_{i=0}^{\rho-1} \dim \bar{y}_i^{[i]} = n$. Now, (3.30) can be rewritten as

$$\mathrm{d}\bar{y}_{i,\lambda}^{[i]} = \sum_{j=1}^{n} \sum_{l=1}^{\dim \bar{y}_j^{[j]}} b_{i,\lambda,j,l} \mathrm{d}\varphi_{j,l} + \tilde{\omega}_{i,\lambda}. \tag{3.31}$$

Since in (3.31) there are n equations and n exact 1-forms $\mathrm{d}\varphi_{j,l}$, then (3.31) can be rewritten as

$$\mathrm{d}\bar{y}_{i,\lambda}^{[i]} = e_{i,\lambda} \mathrm{d}\varphi_{i,\lambda} + \bar{\omega}_{i,\lambda} + \hat{\omega}_{i,\lambda}, \tag{3.32}$$

where $\bar{\omega}_{i,\lambda} \in \bar{\mathcal{Y}}$ and $\hat{\omega}_{i,\lambda} \in \tilde{\mathcal{Y}}$. Then, $\varphi(x) = (\varphi_{i,\lambda}(x))$ defines the state transformation.

Divide the input vector u into two parts $\bar{u}$ and u_* such that both matrices

$$\frac{\partial(\tilde{y}_1^{[1]}, \ldots, \tilde{y}_{\rho-1}^{[\rho-1]})}{\partial \bar{u}} \tag{3.33}$$

[4]For the rest of this proof $i = 0, \ldots, \rho - 1$, $\lambda = 1, \ldots, \dim \bar{y}_i^{[i]}$

and

$$\frac{\partial \tilde{y}_{\rho}^{[\rho]}}{\partial u_*} \tag{3.34}$$

are of full rank over $\mathcal{K}$. Let $q := \dim \tilde{y}_{\rho}^{[\rho]} = \dim \bar{y}_{\rho-1}^{[\rho-1]} = \dim u_*$. By (3.34) the forward-shifts of functions $\varphi_{\rho-1,\lambda}$, $\lambda = 1, \ldots, \dim \bar{y}_{\rho-1}^{[\rho-1]}$, depend on u_*. Next, it is shown that there exists an input transformation which eliminates u_* from the forward-shifts of $\varphi_{i,\lambda}$, $i = 0, \ldots, \rho-2$. Assume that the forward-shift of $\varphi_{i,\lambda}$, $i = 0, \ldots, \rho-2$, depends on u_*. Note that $\bar{y}_{i,\lambda}^{[i+1]}$ can be part of either $\tilde{y}_{i+1}^{[i+1]}$ or $\bar{y}_{i+1}^{[i+1]}$, $i = 0, \ldots, \rho-2$. By construction and (3.32), there must exist a static state feedback, which eliminates u_* from the forward-shift of $\varphi_{i,\lambda}$. Otherwise the rank (3.33) would not be full. Therefore, one can eliminate the input u_* from the forward-shifts of $\varphi_{i,\lambda}$, $i = 0, \ldots, \rho-2$.

Finally, one can transform a flat system Σ_0 into the form (3.26), where $q \neq 0$. □

If one allows q to be zero, then it is always possible to transform system Σ_0 into the form (3.26) and define the system Σ_1. Define a sequence of systems

$$\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_i, \ldots, \tag{3.35}$$

where every system Σ_i is subsystem Σ_1 of previous system Σ_{i-1}, and where every time q is chosen as large as possible. The sequence (3.35) converges, since $\dim \Sigma_i = \dim \Sigma_{i+1}$ yields $\dim \Sigma_i = \dim \Sigma_j$ for $j \geq i+1$.

Theorem 3.6. *System Σ_0 is flat if and only if $\dim \Sigma_{i_*} = 0$ for some $i_* \in \mathbb{N}$.*

Proof. Necessity. By (*ii*) of Lemma 3.4, every system Σ_i is flat. Now, by (*iii*) of Lemma 3.4, one has $\dim \Sigma_i < \dim \Sigma_{i-1}$ for $i \geq 1$, which yields that $\dim \Sigma_{i_*} = 0$ for some $i_* \in \mathbb{N}$.

Sufficiency. Let i_* be such that $\dim \Sigma_{i_*-1} \neq 0$, but $\dim \Sigma_{i_*} = 0$. Then, since for every transformation one eliminates q state variables and $q \leq m_i$, where m_i is the number of inputs of system Σ_i, then system Σ_{i_*-1} has more (or equal number) inputs than states. Thus, system Σ_{i_*-1} is flat and by (*i*) of Lemma 3.4 all the systems Σ_i, $i \geq 0$, are flat. □

To check whether the system (3.18) is flat, one must compute the sequence of subsystems Σ_i. In principle one could compute system Σ_i until $\dim \Sigma_i = 0$ or $\dim \Sigma_i = \dim \Sigma_{i+1}$. However, in practice there is no need to continue when it becomes clear that the subsystem Σ_i is flat. In particular, since it is easy to check whether the system (3.18) is static state feedback linearizable, one can stop whenever Σ_i is such system.

Example 3.2. Consider the nonlinear discrete-time system

$$\begin{aligned} x_1^{[1]} &= x_2 \\ x_2^{[1]} &= x_3 + x_2(u_1 - x_1u_2) \\ x_3^{[1]} &= x_4 + x_1x_2u_2 \\ x_4^{[1]} &= x_1(u_1 - x_1u_2) \\ x_5^{[1]} &= x_2x_4u_3. \end{aligned} \tag{3.36}$$

Compute the system Σ_1 for (3.36). Note that since

$$\mathcal{H}_2 = \text{span}_{\mathcal{K}}\{\text{d}x_1, x_1^{[-1]}\text{d}x_2 - x_1\text{d}x_4\}$$

for system (3.36), $\mathcal{H}_2$ is not integrable. Clearly, the minimal integrable subspace containing $\mathcal{H}_2$ is $\text{span}_{\mathcal{K}}\{\text{d}x_1, \text{d}x_2, \text{d}x_4\}$, which has dimension $3 < 5 = n$. By Theorem 3.5, one can transform the system (3.36) into the form (3.26), where $q > 0$. By the proof of Theorem 3.4, $z_1 = (x_1, x_2, x_4)$ and thus $z_2 = (x_3, x_5)$, which yield the state transformation

$$\begin{aligned} z_{1,1} &= x_1 \\ z_{1,2} &= x_2 \\ z_{1,3} &= x_4 \\ z_{2,1} &= x_3 \\ z_{2,2} &= x_5. \end{aligned} \tag{3.37}$$

For the input transformation, take, like in the proof of Theorem 3.4, $\tilde{u}_{2,1} = x_3^{[1]} = x_4 + x_1x_2u_2$ and $\tilde{u}_{2,2} = x_5^{[1]} = x_2x_4u_3$, which can be solved for u_2 and u_3. To define $\tilde{u}_1$, note that

$$\text{span}_{\mathcal{K}}\{\text{d}x, \delta(\text{d}x_1), \delta(\text{d}x_2), \delta(\text{d}x_4)\} = \text{span}_{\mathcal{K}}\{\text{d}x\} \oplus \text{span}_{\mathcal{K}}\{\text{d}\alpha_1(x,u)\},$$

where $\alpha_1 = u_1 - x_1u_2$. Take $\tilde{u}_1 = \alpha_1$, which gives the input transformation

$$\begin{aligned} u_1 &= \tilde{u}_1 + \frac{\tilde{u}_{2,1} - x_4}{x_2} \\ u_2 &= \frac{\tilde{u}_{2,1} - x_4}{x_1x_2} \\ u_3 &= \frac{\tilde{u}_{2,2}}{x_2x_4}. \end{aligned} \tag{3.38}$$

The state transformation (3.37) and static state feedback (3.38) transform the system (3.36) into the form

$$\begin{aligned} z_{1,1}^{[1]} &= z_{1,2} \\ z_{1,2}^{[1]} &= z_{2,1} + z_{1,2}\tilde{u}_1 \\ z_{1,3}^{[1]} &= z_{1,1}\tilde{u}_1 \\ z_{2,1}^{[1]} &= \tilde{u}_{2,1} \\ z_{2,2}^{[1]} &= \tilde{u}_{2,2}. \end{aligned}$$

Therefore, system Σ_1 is

$$\begin{aligned} z_{1,1}^{[1]} &= z_{1,2} \\ z_{1,2}^{[1]} &= z_{2,1} + z_{1,2}\tilde{u}_1 \\ z_{1,3}^{[1]} &= z_{1,1}\tilde{u}_1, \end{aligned} \tag{3.39}$$

where $z_{2,1}$ and $\tilde{u}_1$ are the input variables. Obviously, system (3.39) is static state feedback linearizable and thus flat. By (i) of Lemma 3.1, system (3.36) is also flat. The flat outputs for system (3.39) are $z_{1,1}$ and $z_{1,3}$. The flat outputs for system (3.36) are the flat outputs of system (3.39)[5] and additionally $z_{2,2}$, since this did not appear in system Σ_1. In original state variables the flat outputs of system (3.36) are $y_1 = x_1$, $y_2 = x_4$ and $y_3 = x_5$.

3.3 Conclusions

In this chapter the feedback linearization problem of discrete-time nonlinear systems described by state equations, was studied. First, a solution by a static state feedback and a state transformation was given in terms of the functions' algebra. The advantage of this method over other existing methods is that the functions' algebra can also handle non-smooth functions. The condition itself is a direct consequence of the condition in [6], but stated in terms of tools of functions' algebra. Although, in principle, one can utilize the results of Section 3.1 for systems described by non-smooth functions, at the moment no general formulas for computations exist for such case.

Second, the flatness property of nonlinear discrete-time system was studied. More precisely, a method to compute a flat output of a given system was proved. Note that based on the flat output, the dynamic endogenous state feedback and the state transformation can be found, which linearize the system. The method requires computing a sequence of systems, initialized by the given system, by transforming every previous system of the sequence into the form (3.26), where the last q equations can be eliminated. The original system is flat if and only if all the equations can be eliminated. Computationally, this method is much easier than the one described in [64, 63, 53, 3]. The only difficult part is finding a state transformation and a static state feedback which transform a system into the form (3.26). This requires computing the minimal integrable vector space that contains $\mathcal{H}_2$, defined by (3.15), of the given system.

[5]One has to apply the inverse of the state transformation (3.37).

Chapter 4

Time-Delay Systems

In this chapter continuous time-delay systems are considered. First, the integrability of a set of 1-forms, defined as in Section 1.2, is extended to this class of systems. Two different notions of integrability - weak and strong - are defined, which reduce to the standard integrability, when there are no delays. These concepts correspond to two possible generalizations of the standard integrability notion. Also, conditions are found to check whether a set of 1-forms is strongly or weakly integrable. It is shown that the new notions are more natural than the one recalled in Section 1.2, for studying time-delay systems.

Second, in Section 4.2 the disturbance decoupling problem is studied. The problem is solved for the case when a pure shift dynamic feedback is used. For single-input single-output (SISO) systems the dynamic feedback solution is also given. Finally, in Section 4.3 necessary and sufficient conditions are found to decouple system outputs and inputs by static state feedback, which is causal, i.e. does not depend on future values of the states and/or inputs.

4.1 Integrability of 1-forms

Although there exist a small number of papers addressing integrability issues for the nonlinear time-delay systems [72, 69, 18], no general theory exists. In [72] and [69] integrability problem was tackled for a single 1-form. Necessary and sufficient conditions of integrability were given. However, note that the definitions of integrability were different in [72] and [69]. The one in [72] corresponds to weak integrability, as defined below, and the definition of [69] corresponds to strong integrability below.

In this section, the notions of integrable 1-forms are developed for the case when time-delay systems are studied. Although everything that is written in Section 1.2 about integrability of 1-forms is true also for the time-delay case, it is not the best way for studying such systems, since the action of the delay operator δ is not taken into account. For this reason, in this section, two more general notions of integrability are defined and characterized. In the case of no delays, these new notions of integrability reduce to the standard integrability notion, defined in Section 1.2.

4.1.1 Definition of Integrability

In this section, a set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is considered. The latter means that, there is no *non-zero linear combinations* over the ring $\mathcal{K}[\vartheta]$ which vanish. That is, the 1-forms are looked as elements of the module $\mathcal{M}$ rather than elements of the vector space $\mathcal{E}$. For the 1-forms, which are looked as elements of the vector space $\mathcal{E}$, there exists only one notion of integrability, defined in Section 1.2. However, as shown hereafter, considering the 1-forms as elements of $\mathcal{M}$ naturally leads to two different notions of integrability.

In the delay-free case, if the set of 1-forms $\{\omega_1 \ldots, \omega_k\}$ is considered over $\mathcal{K}$, then the set is said to be integrable if there exists an *invertible* matrix $A \in \mathcal{K}^{k\times k}$ and a vector function $\varphi = (\varphi_1, \ldots, \varphi_k)^T$, such that $\omega = A\mathrm{d}\varphi$. The invertibility of A is guaranteed by the full rank of A, since $\mathcal{K}$ is a field. Instead, if the 1-forms $\{\omega_1 \ldots, \omega_k\}$ are viewed as elements of the module $\mathcal{M}$, then the matrix A has to belong to $\mathcal{K}[\vartheta]^{k\times k}$. Since $A(\vartheta)$ may be of full rank but not unimodular (i.e. invertible in $\mathcal{K}[\vartheta]^{k\times k}$), it is necessary to distinguish between the two cases.

Example 4.1. For example, the matrix

$$A(\vartheta) = \begin{pmatrix} 1 & x_2(t-1)\vartheta \\ \vartheta & 1 + x_2(t-2)\vartheta^2 \end{pmatrix}$$

is unimodular, since the matrix

$$A(\vartheta)^{-1} = \begin{pmatrix} 1 + x_2(t-1)\vartheta^2 & -x_2(t-1)\vartheta \\ -\vartheta & 1 \end{pmatrix}$$

is such that $A(\vartheta)A(\vartheta)^{-1} = A(\vartheta)^{-1}A(\vartheta) = I_2$. However, there is no polynomial inverse for $(1+\vartheta)$.

Therefore, one has two definitions of integrability.

Definition 4.1.
- A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is said to be *strongly integrable* if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that

$$\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}.$$

- A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is said to be *weakly integrable* if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that

$$\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\} \subseteq \text{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}.$$

If the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly (respectively weakly) integrable, then the submodule $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be strongly (respectively weakly) integrable.

Clearly, strong integrability yields weak integrability. Also, the set of 1-forms expressed as a column vector $\omega = (\omega_1, \ldots, \omega_k)^T$ is weakly integrable if and only if there exists a matrix $A(\vartheta) \in \mathcal{K}[\vartheta]^{k\times k}$ with full rank and functions $\varphi = (\varphi_1, \ldots, \varphi_k)^T$ such that $\omega = A(\vartheta)\mathrm{d}\varphi$. If in addition the matrix $A(\vartheta)$ can be chosen to be unimodular, then the 1-forms ω are also strongly integrable.

Remark 4.1. By definition of the closure of a submodule, definitions of weak and strong integrability are equivalent for a closed submodule $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. As a consequence, since any submodule $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is closed in the case of delay-free 1-forms ω_i, $i = 1, \ldots, k$, the notions of strong and weak integrability coincide in such case.

4.1.2 Strong Integrability

The conditions that allow to check whether a set of k independent 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly integrable are given in terms of certain sequence of integrable (in the sense of the definition in Section 1.2) vector spaces. To compute these vector spaces, one uses the Derived Flag Algorithm (DFA). Starting from a given vector space I_0 in $\mathcal{E}$, the DFA computes

$$I_i = \text{span}_{\mathcal{K}}\{\omega \in I_{i-1} \mid \mathrm{d}\omega = 0 \mod I_{i-1}\}. \tag{4.1}$$

The sequence (4.1) converges as it defines a strictly non-increasing sequence of vector spaces I_i and by the Frobenius Theorem, the limit I_∞ has an exact basis, which represents the largest integrable vector space contained in I_0. The DFA will be used to compute the largest integrable subspaces of a sequence of vector spaces:

$$I_0^p = \text{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_k, \ldots, \delta^p\omega_1, \ldots, \delta^p\omega_k\}, \tag{4.2}$$

for $p \geq 0$. Clearly, for every $p \geq 0$, $I_0^p \subseteq \text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. By Definition 4.1, if the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly integrable, then there exist k linearly independent exact 1-forms $\mathrm{d}\varphi_i$, $i = 1, \ldots, k$, that belong to $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. Here, the DFA is used to compute the exact 1-forms $\mathrm{d}\varphi_i$, $i = 1, \ldots, k$, by computing the limit of (4.1), initialized by

(4.2). More precisely, the sequence I_i^p, defined by (4.1), converges to an integrable vector space

$$I_\infty^p = \text{span}_{\mathcal{K}}\{\text{d}\varphi_{1,p}, \ldots, \text{d}\varphi_{\gamma_p,p}\} \tag{4.3}$$

for all $p \geq 0$ and some $\gamma_p \geq 0$. By definition, $\text{d}\varphi_{i,p} \in \text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ for $i = 1, \ldots, \gamma_p$ and $p \geq 0$.

The exact 1-forms $\text{d}\varphi_{i,p}$, $i = 1, \ldots, \gamma_p$, are independent over $\mathcal{K}$, but may not be independent over $\mathcal{K}[\vartheta]$. It remains to be checked whether there exist k 1-forms among $\text{d}\varphi_{i,p}$, $i = 1, \ldots, \gamma_p$, that form a basis for $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. Below, Theorem 4.1 gives necessary and sufficient conditions whether such k 1-forms exist.

To simplify the presentation, let $\omega_i \in \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}x\}$ for $i = 1, \ldots, k$. Let $\omega = (\omega_1, \ldots, \omega_k)^T$ and $\text{d}\varphi^p$ be the vector of independent (over $\mathcal{K}[\vartheta]$) basis elements of I_∞^p, such that $\text{span}_{\mathcal{K}[\vartheta]}\{\text{d}\varphi^p\}$ contains I_∞^p. One can always find matrices $M(\vartheta)$ and $N_p(\vartheta)$, for every $p \geq 0$, such that[1]

$$\begin{array}{rcl} \omega & = & M(\vartheta)\text{d}x \\ \text{d}\varphi^p & = & N_p(\vartheta)\text{d}x. \end{array} \tag{4.4}$$

A necessary and sufficient condition for strong integrability of the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is given by the following theorem expressed in terms of the matrices $M(\vartheta)$ and $N_p(\vartheta)$.

Theorem 4.1. *A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}[\vartheta]$, is strongly integrable if and only if the Jacobson forms of matrices $M(\vartheta)$ and $N_p(\vartheta)$, defined by (4.4), are equal for some $p \geq 0$.*

Proof. Necessity. Let Λ_1 be the Jacobson form of the matrix $M(\vartheta)$. Since the set of 1-forms $\omega = (\omega_1, \ldots, \omega_k)^T$ is strongly integrable, there exist an unimodular matrix $A(\vartheta)$ and a vector function $\varphi = (\varphi_1, \ldots, \varphi_k)^T$, such that $\omega = A(\vartheta)\text{d}\varphi$. Define matrix $\bar{N}(\vartheta)$ such that $\text{d}\varphi = \bar{N}(\vartheta)\text{d}x$. Now, $M(\vartheta) = A(\vartheta)\bar{N}(\vartheta)$ and

$$\Lambda_1 = V(\vartheta)M(\vartheta)U(\vartheta) = V(\vartheta)A(\vartheta)\bar{N}(\vartheta)U(\vartheta)$$

for some unimodular matrices $V(\vartheta), U(\vartheta)$. Since the product of unimodular matrices $V(\vartheta)A(\vartheta)$ is also unimodular, then Λ_1 is also the Jacobson form of matrix $\bar{N}(\vartheta)$. Also, because $\text{d}\varphi = A^{-1}(\vartheta)\omega$, then $\text{d}\varphi = \text{d}\varphi^p$ for some $p \geq 0$. Thus, $\bar{N}(\vartheta) = N_p(\vartheta)$ and the condition of the theorem is satisfied.

Sufficiency. Let the condition of the theorem be satisfied for $p = p_*$ and denote $\text{d}\varphi := \text{d}\varphi^{p_*}$, $N(\vartheta) := N_{p_*}(\vartheta)$. By construction, the set of 1-forms $\text{d}\varphi = N(\vartheta)\text{d}x$ belongs to $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. Next, it is shown that

[1]Note that matrices $N_p(\vartheta)$ are not unique since $\text{d}\varphi^p$ are not unique.

$\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ for $i = 1, \ldots, k$. Since the Jacobson forms of $M(\vartheta)$ and $N(\vartheta)$ are equal, there exist unimodular matrices $V_1(\vartheta), V_2(\vartheta), U_1(\vartheta), U_2(\vartheta)$ such that

$$V_1(\vartheta)M(\vartheta)U_1(\vartheta) = V_2(\vartheta)N(\vartheta)U_2(\vartheta). \tag{4.5}$$

Because $N(\vartheta) = A(\vartheta)M(\vartheta)$ for some full rank matrix $A(\vartheta)$, then one can take $U_1(\vartheta) = U_2(\vartheta)$ in (4.5). Now, from (4.5) one gets $M(\vartheta) = V_1^{-1}(\vartheta)V_2(\vartheta)N(\vartheta)$. The matrix $V_1^{-1}(\vartheta)V_2(\vartheta) =: A^{-1}(\vartheta)$ is unimodular and $\omega = M(\vartheta)\mathrm{d}x = A^{-1}(\vartheta)N(\vartheta)\mathrm{d}x = A^{-1}(\vartheta)\mathrm{d}\varphi$. Therefore $\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$ for $i = 1, \ldots, k$ and by Definition 4.1, the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is strongly integrable. □

The condition of Theorem 4.1 can be checked step-by-step, increasing the index p at every step. The sequence $\mathrm{d}\varphi^p$, $p \geq 0$, converges, because there can be only up to k independent 1-forms in $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$, and the limit $\mathrm{d}\varphi$ defines the largest strongly integrable submodule, denoted by $\bar{\mathcal{A}}$ contained in $\mathcal{A} := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$. Unfortunately, one does not know an upper bound for the index p in Theorem 4.1, which makes application of the theorem, to verify strong integrability of a set of 1-forms, a difficult task.

Remark 4.2. In general, a good choice for p in Theorem 4.1 is $s(k-1)$, where s is the largest delay (in coefficients or differentials) that appears in the given set of k 1-forms. Note that $s(k-1)$ reduces to correct bound for p in case of no delays ($s = 0$) or when $k = 1$. In fact, up to the knowledge of the author of the thesis, there are no examples, when bigger value of p is needed.

Example 4.2. Given the set of 1-forms

$$\begin{aligned} \omega_1 &= \mathrm{d}x_1 + x_3\mathrm{d}x_2^{[-1]} \\ \omega_2 &= \mathrm{d}x_2^{[-2]} \end{aligned} \tag{4.6}$$

check its strong integrability. Compute by the DFA

$$\begin{aligned} I_\infty^0 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2^{[-2]}\} \\ I_\infty^1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2^{[-2]}, \mathrm{d}x_2^{[-3]}, \mathrm{d}x_1^{[-1]}\} \end{aligned}$$

which yield $\mathrm{d}\varphi^0 = \mathrm{d}x_2^{[-2]}$ and $\mathrm{d}\varphi^1 = (\mathrm{d}x_1^{[-1]}, \mathrm{d}x_2^{[-2]})^T$. Next, compute the matrices $M(\vartheta)$ and $N_1(\vartheta)$:

$$\begin{aligned} \omega &= \begin{pmatrix} 1 & x_3\vartheta & 0 \\ 0 & \vartheta^2 & 0 \end{pmatrix} \mathrm{d}x =: M(\vartheta)\mathrm{d}x \\ \mathrm{d}\varphi^1 &= \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & \vartheta^2 & 0 \end{pmatrix} \mathrm{d}x =: N_1(\vartheta)\mathrm{d}x. \end{aligned}$$

It is easy to find that the Jacobson forms of $M(\vartheta)$ and $N_1(\vartheta)$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \vartheta^2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \vartheta & 0 & 0 \\ 0 & \vartheta^2 & 0 \end{pmatrix}$$

respectively. Thus, the condition of Theorem 4.1 is not satisfied and the set of 1-forms (4.6) is not strongly integrable. The largest strongly integrable submodule contained in $\text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \omega_2\}$ is $\text{span}_{\mathcal{K}[\vartheta]}\{\text{d}x_1^{[-1]}, \text{d}x_2^{[-2]}\}$.

Example 4.3. Check the strong integrability of the 1-forms

$$\begin{aligned} \omega_1 &= \text{d}x_2 \\ \omega_2 &= x_3\text{d}x_1 + \text{d}x_2^{[-1]} \\ \omega_3 &= x_3^{[-1]}\text{d}x_1^{[-1]} + x_5\text{d}x_4. \end{aligned} \tag{4.7}$$

Compute for $p = 0, 1, 2$[2] the subspaces I_∞^p:

$$\begin{aligned} I_\infty^0 &= \text{span}_{\mathcal{K}}\{\text{d}x_2\} \\ I_\infty^1 &= \text{span}_{\mathcal{K}}\{\text{d}x_2, \text{d}x_2^{[-1]}, \text{d}x_1\} \\ I_\infty^0 &= \text{span}_{\mathcal{K}}\{\text{d}x_2, \text{d}x_2^{[-1]}, \text{d}x_2^{[-2]}, \text{d}x_1, \text{d}x_1^{[-1]}, \text{d}x_4\}. \end{aligned}$$

Clearly, $\text{d}\varphi^2 = (\text{d}x_1, \text{d}x_2, \text{d}x_4)^T$. Now,

$$\begin{aligned} \omega &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x_3 & \vartheta & 0 & 0 & 0 \\ x_3^{[-1]}\vartheta & 0 & 0 & x_5 & 0 \end{pmatrix} \text{d}x =: M(\vartheta)\text{d}x \\ \text{d}\varphi^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{d}x =: N_2(\vartheta)\text{d}x. \end{aligned}$$

One can check that the Jacobson form of both matrices $M(\vartheta)$ and $N_2(\vartheta)$ is $(I_3, 0)$. Thus, the set of 1-forms (4.7) is strongly integrable.

4.1.3 Weak Integrability

The conditions for weak integrability are expressed via strong integrability of a set of independent 1-forms.

Lemma 4.1. *A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is weakly integrable if and only if the closure of the submodule, generated by $\{\omega_1, \ldots, \omega_k\}$, is (strongly) integrable.*

[2] Note that $s(k-1) = 2$ in this example.

Proof. Necessity. By definitions of weak integrability and closure, there exist functions $\varphi = (\varphi_1, \ldots, \varphi_k)^T$ and a matrix $A(\vartheta)$, such that $\mathrm{d}\varphi = A(\vartheta)\bar{\omega}$, where $\bar{\omega}$ is the basis of the closure of the submodule, generated by $\{\omega_1, \ldots, \omega_k\}$. Choose $\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$ such that for $i = 1, \ldots, k$

$$\mathrm{d}\varphi_i \neq a\mathrm{d}\phi + \sum_{j=1; j\neq i}^{k} b_j(\vartheta)\mathrm{d}\varphi_j \tag{4.8}$$

for any $\phi \in \mathcal{K}$ and $b_j(\vartheta) \in \mathcal{K}[\vartheta]$. It remains to show that one can choose φ such that $A(\vartheta)$ is unimodular, i.e. $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ for $i = 1, \ldots, k$.

Assume, by contradiction, that no φ exist such that $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$. Then, for some q, $\bar{\omega}_q \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ and also $\delta^j\bar{\omega}_q \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ for $j \geq 1$ and any $\mathrm{d}\varphi$ satisfying (4.8). In fact, if one assumes by contradiction that

$$\delta^j\bar{\omega}_q = \sum_{i=1}^{k} c_i(\vartheta)\mathrm{d}\varphi_i, \tag{4.9}$$

then, since on the left-hand side of (4.9) everything is delayed at least j times, everything that is delayed less than j times on the right-hand side should vanish. Therefore, one is able to find functions $\phi_i, \psi_i \in \mathcal{K}$, $i = 1, \ldots, k$, such that $\mathrm{d}\varphi_i = \mathrm{d}\phi_i + \mathrm{d}\psi_i$ and

$$c_i(\vartheta)\mathrm{d}\phi_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x^{[-j]}\} \quad \sum_i c_i(\vartheta)\mathrm{d}\psi_i = 0.$$

Now, because of (4.8), $\psi_i = 0$, $\phi_i = \varphi_i$ for $i = 1, \ldots, k$ and thus $\delta^j\bar{\omega}_q = \delta^j \sum_i \bar{c}_i(\vartheta)\mathrm{d}\varphi_i^{[j]}$ which yields $\bar{\omega}_q = \sum_i \bar{c}_i(\vartheta)\mathrm{d}\varphi_i^{[j]}$. Clearly, 1-forms $\mathrm{d}\varphi_i^{[j]}$ have to belong to $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\bar{\omega}\}$, because $\mathrm{d}\varphi_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\bar{\omega}\}$. Now, one has a contradiction and therefore $\delta^j\bar{\omega}_q \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ for $j \geq 1$.

By construction, $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\} \subset \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_i; i = 1, \ldots, k; i \neq q\}$, which is impossible. Thus, the assumption that no φ exists such that $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi\}$ must be wrong.

Sufficiency. Follows directly from the definitions of strong and weak integrability. □

By Lemma 4.1, the algorithm below can be used to check whether a set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is weakly integrable.

Algorithm 4.1 Let $\{\omega_1, \ldots, \omega_k\}$ be linearly independent over $\mathcal{K}[\vartheta]$.

1. Compute the closure $cl_{\mathcal{K}[\vartheta]}(\mathcal{A})$ of $\mathcal{A} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$.

2. Check whether $cl_{\mathcal{K}[\vartheta]}(\mathcal{A})$ is strongly integrable. If yes, then the 1-forms $\{\omega_1, \ldots, \omega_k\}$ are weakly integrable, otherwise not.

Example 4.4. (Continuation of Example 4.2) Check whether the 1-forms (4.6) are weakly integrable. Compute first the closure of $\mathcal{A} := \text{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \omega_2\}$. This can be done by finding the right-kernel of matrix $M(\vartheta)$, defined in Example 4.2, and then computing the left-kernel of that kernel. The resulting matrix defines the basis elements of the closure. The kernel of

$$M(\vartheta) = \begin{pmatrix} 1 & x_3\vartheta & 0 \\ 0 & \vartheta^2 & 0 \end{pmatrix}$$

is $(0,0,1)^T$, which has a kernel $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Therefore, $cl_{\mathcal{K}[\vartheta]}(\mathcal{A}) = \text{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1, \mathrm{d}x_2\}$, which is clearly strongly integrable. Thus, the 1-forms (4.6) are weakly integrable.

4.1.4 Applications of Strong and Weak Integrability

In this subsection, it is shown that accessibility of nonlinear time-delay system (1.5) can be characterized through integrability of a certain submodule. The accessibility property of system (1.5) is defined via the concept of autonomous element, like in cases of delay-free systems [25] or linear time-delay systems [34].

An autonomous element of system (1.5) is defined similarly to the delay-free systems, see [25].

Definition 4.2. A nonzero function $\varphi \in \mathcal{K}$ is said to be an autonomous element of system (1.5) if there exist an integer ν and a nonzero function $F \in \mathcal{K}$ such that

$$F(\varphi, \dot{\varphi}, \ldots, \varphi^{(\nu)}) = 0. \tag{4.10}$$

Definition 4.3. System (1.5) is said to be accessible if it does not admit any autonomous element.

The autonomous elements are described through their relative degree, which is defined similarly to the relative degree of an output function.

Definition 4.4. A 1-form $\omega \in \text{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ is said to have relative degree r if r is the smallest integer such that $\omega^{(r)} \notin \text{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$. A function $\varphi \in \mathcal{K}$ is said to have relative degree r if the 1-form $\mathrm{d}\varphi$ has relative degree r.

Define a sequence of submodules $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots$ of the module $\mathcal{M}$ of 1-forms as follows:

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\} \\ \mathcal{H}_i &= \text{span}_{\mathcal{K}[\vartheta]}\{\omega \in \mathcal{H}_{i-1} \mid \dot{\omega} \in \mathcal{H}_{i-1}\}. \end{aligned} \tag{4.11}$$

Since $\mathcal{H}_1$ has finite rank and all the submodules $\mathcal{H}_i$ are closed, sequence (4.11) converges (see [94]). Let $\mathcal{H}_\infty$ denote its limit. A submodule $\mathcal{H}_i$ contains all the 1-forms with relative degree greater or equal to i. Thus, the limit $\mathcal{H}_\infty$ contains all the 1-forms with infinite relative degree.

Lemma 4.2. *Function $\varphi \in \mathcal{K}$ is an autonomous element of system (1.5) if and only if it has infinite relative degree.*

Proof. Necessity. Let φ be an autonomous element of system (1.5) and assume, by contradiction, that it has finite relative degree. Then,

$$\dim(\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi, \dots, \mathrm{d}\varphi^{(k-1)}\}) = k$$

for all $k \geq 1$. Because of (4.10), the equality above is not satisfied for $k = \nu + 1$, which is a contradiction. Thus, function φ has infinite relative degree.

Sufficiency. Let φ be a nonzero function with infinite relative degree. Then the 1-forms $\mathrm{d}\varphi^{(j)} \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ for $j = 0, \dots, n$. Since there are $n+1$ 1-forms $\mathrm{d}\varphi^{(j)}$, $j = 0, \dots, n$, but the rank of the module $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ is n, then the 1-forms $\mathrm{d}\varphi, \dots, \mathrm{d}\varphi^{(n)}$ are dependent over the ring $\mathcal{K}[\vartheta]$. Thus, there exist $a_i \in \mathcal{K}[\vartheta]$, $i = 0, \dots, n$, and at least one of them is non-zero, such that

$$\omega := a_0\mathrm{d}\varphi + \cdots + a_n\mathrm{d}\varphi^{(n)} = 0. \tag{4.12}$$

Let γ be the smallest integer such that

$$\omega = b_1\mathrm{d}\alpha_1(\varphi, \dots, \varphi^{(n)}) + \cdots + b_\gamma\mathrm{d}\alpha_\gamma(\varphi, \dots, \varphi^{(n)}), \tag{4.13}$$

where $0 \neq b_i \in \mathcal{K}[\vartheta]$, $\alpha_i \in \mathcal{K}$, $i = 1, \dots, \gamma$. Then, from (4.12) and (4.13) one gets $\alpha_i(\varphi, \dots, \varphi^{(n)}) = 0$ for $i = 1, \dots, \gamma$. By Definition 4.2, the function φ is an autonomous element of system (1.5). □

One can characterize accessibility of system (1.5) by the next theorem.

Theorem 4.2. *System (1.5) is accessible if and only if the largest integrable submodule of $\mathcal{H}_\infty$, denoted by $\bar{\mathcal{H}}_\infty$, is 0.*

Proof. Necessity. If system (1.5) is accessible, then by Lemma 4.2 it does not admit any non-constant function in $\mathcal{K}$ with infinite relative degree. Therefore, there are no exact non-zero 1-forms in $\mathcal{H}_\infty$ and thus $\bar{\mathcal{H}}_\infty = 0$ must be true.

Sufficiency. The submodule $\mathcal{H}_\infty$ contains all the 1-forms with infinite relative degree. Since $\bar{\mathcal{H}}_\infty = 0$, there is no non-constant exact 1-forms with infinite relative degree and therefore, by Lemma 4.2, there is no autonomous elements. □

Example 4.5. Consider the model of the JAK-STAT signaling pathway in the cell [91]. The model describes certain signal transduction from membrane receptors to gene activation in the nucleus and is described by the equations

$$\begin{aligned}\dot{x}_1 &= -k_1 x_1 u + 2k_4 x_3^{[-1]} \\ \dot{x}_2 &= k_1 x_1 u - k_2 x_2^2 + 2k_5 x_3 \\ \dot{x}_3 &= -k_3 x_3 + \frac{k_2 x_2^2}{2} - k_5 x_3 \\ \dot{x}_4 &= k_3 x_3 - k_4 x_3^{[-1]},\end{aligned} \tag{4.14}$$

where

- u is the amount of activated EPO-receptors;
- x_1 is the amount of unphosphorylated monomeric STAT-5 (a member of the STAT (signal transduction and activator of transcription) family of transcription factors);
- x_2 is the amount of phosphorylated monomeric STAT-5;
- x_3 is the amount of phosphorylated dimeric STAT-5 in the cytoplasm;
- x_4 is the amount of phosphorylated dimeric STAT-5 in the nucleus

and $k_1, \ldots, k_5$ are parameters belonging to $\mathbb{R}$. For more information, see [91].

To check whether the system (4.14) is accessible, one has to compute the sequence $\mathcal{H}_i$ of submodules and examine whether the largest integrable submodule contained in the limit $\mathcal{H}_\infty$ is zero or not. Compute:

$$\begin{aligned}\mathcal{H}_2 &= \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}(x_1 + x_2), \text{d}x_3, \text{d}x_4\} \\ \mathcal{H}_3 &= \text{span}_{\mathcal{K}[\vartheta]}\{\text{d}(x_1 + x_2 + 2x_3), \text{d}x_4\} \\ \mathcal{H}_4 &= \mathcal{H}_\infty = \{0\}.\end{aligned}$$

Therefore, the system (4.14) is accessible.

In the rest of this chapter, two control problems - the disturbance decoupling problem and input-output decoupling problem - will be solved, where one makes use of the new integrability notions. More precisely, the weak integrability of a single 1-form will be used.

4.2 Disturbance Decoupling

The disturbance decoupling problem (DDP) of system time-delay is considered in this section. More precisely, one looks for state feedback, which eliminates the effects of disturbances from the system output. It is important to stress that one is looking for a causal feedback, not depending on the future state values. Causality is not an issue in delay-free case and that makes the time-delay disturbance decoupling more challenging problem. Necessary and sufficient conditions are derived for solvability of the DDP by pure shift dynamic compensator. The results below fix the incorrect results from [68] and also generalize those for SISO systems [72] to the MIMO case. Additionally, necessary and sufficient conditions are given for the problem solvability via general dynamic compensator for SISO systems.

In this Section, systems of the form (1.5), where the function f also depends on the disturbance input vector w and its delays, are considered. More precisely, these systems are in the form

$$\begin{array}{rcl} \dot{x} & = & f(x, \ldots, x^{[-D]}, u, \ldots, u^{[-D]}, w, \ldots, w^{[-D]}) \\ y & = & h(x, \ldots, x^{[-D]}), \end{array} \tag{4.15}$$

where w is the disturbance, i.e. uncontrollable input, of the system.

Since in the case of time-delay systems the 1-forms are looked as elements of the module $\mathcal{M}$, first, the definition of the rank of a 1-form is generalized to this case.

Definition 4.5. [42] A one-form ω is said to have rank γ, if γ is the minimal number, such that

$$\omega = a_1 \mathrm{d}\varphi_1 + \cdots + a_\gamma \mathrm{d}\varphi_\gamma$$

for some $a_i \in \mathcal{K}[\vartheta]$ and $\varphi_i \in \mathcal{K}$, $i = 1, \ldots, \gamma$.

Clearly, a one-form ω is weakly integrable if and only if its rank is equal to 1. Since a 1-form, which generates a closed submodule, is integrable if and only if the Frobenius theorem is satisfied, then the following procedure can be applied to compute the rank of a 1-form $\omega \in \mathcal{M}$. Find the basis element $\bar{\omega}$ of the closure of the submodule, generated by ω, and compute the rank of $\bar{\omega}$ by Definition 1.6. The result is also the rank of ω.

Another concept, which will be used in this section, is the invariant submodule with respect to the dynamics (4.15). Given a submodule $\mathcal{A} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_k\}$ of $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$, by $\dot{\mathcal{A}}$ one means the submodule

$$\dot{\mathcal{A}} = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\dot{\omega}_1, \ldots, \dot{\omega}_k\}.$$

Then, an invariant submodule can be defined as follows.

Definition 4.6. [68] A submodule $\mathcal{A} \subseteq \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ is said to be invariant with respect to the dynamics (4.15) if

$$\dot{\mathcal{A}} \subseteq cl_{\mathcal{K}[\vartheta]}(\mathcal{A}) + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u\}. \tag{4.16}$$

4.2.1 Disturbance decoupling by pure shift dynamic feedback

Since in time-delay case two operators, the time differentiation and time delay operators, act on system equations, one can construct several classes of feedback for such systems. The pure shift dynamic feedback, introduced in [72], is defined as feedback of the form

$$\begin{array}{rcl} z^{[1]} & = & M(x^{[-i]}, z^{[-i]}, v^{[-i]}; i = 0, \dots, \sigma) \\ u & = & G(x^{[-i]}, z^{[-i]}, v^{[-i]}; i = 0, \dots, \sigma), \end{array} \tag{4.17}$$

for some delay $\sigma > 0$ and where z is a vector of new system variables. The additional variable z allows to handle situations, when an inverse of a polynomial is needed to construct a feedback. For example, $v = u + u^{[-1]}$ is not a standard static feedback, since one can not state the input u in terms of v and possibly its shifts. In this case a new variable $z := u^{[-1]}$ is defined, which allows to write u in terms of new variables v and z.

It is also assumed that the compensator (4.17) is regular. By regularity of (4.17) is meant that all the new system variables, i.e. z and v, can be expressed as functions of old system variables. Thus, there exists $\tilde{G}, K \in \mathcal{K}$ such that $v = \tilde{G}(x^{[-i]}, u^{[-i]}; i = 0, \dots, D)$ and $z = K(x^{[-i]}, u^{[-i]}; i = 0, \dots, D)$. Another way to express a regular compensator (4.17) is the following:

$$P(\vartheta)\mathrm{d}u = Q(\vartheta)\mathrm{d}v + R(\vartheta)\mathrm{d}x, \tag{4.18}$$

where the matrix $Q(\vartheta)$ is unimodular[3] and $P_0 \in \mathcal{K}^{m\times m}$ is of full rank, where P_0 is defined by $P(\vartheta) = \sum_j P_j \vartheta^j$, $P_j \in \mathcal{K}^{m\times m}$. If the matrix $P(\vartheta)$ is also unimodular, then the compensator (4.18) is said to be *compatible*, which means that dim $z = l = 0$.

Here, one looks for a compensator (4.17) (or equivalently (4.18)), such that for the closed-loop system, $y^{(k)}$ does not depend on the disturbance input w (nor its delays) for all $k \in \mathbb{N}$, i.e. $\partial y^{(k)}/\partial w^{[-i]} \equiv 0$ for all $k, i \in \mathbb{N}$.

The following Lemma gives a condition, whether a system (4.15) is disturbance decoupled or not.

Lemma 4.3. *System (4.15) is disturbance decoupled if and only if there exists an invariant submodule $\mathcal{A} \subset \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ with respect to system dynamics, such that $\mathrm{d}y_i \in \mathcal{A}$, $i = 1, \dots, p$.*

Proof. Necessity. Since system (4.15) is disturbance decoupled, then

$$\mathrm{d}y_i^{(k)} \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{(k-1)}\}$$

[3]This guarantees the regularity.

for $i = 1, \ldots, p$ and all $k \geq 1$. This means that there must exist a submodule $\mathcal{A} \subset \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$ such that

$$\mathcal{A}^{(k)} \subset \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{(k-1)}\}.$$

Clearly, $\mathcal{A}$ must be invariant with respect to the dynamics (4.15).

Sufficiency. The invariant submodule $\mathcal{A}$, that contains $\mathrm{d}y_i$ for $i = 1, \ldots, p$, satisfies (4.16). Note that when $\mathcal{A}$ is invariant with respect to dynamics (4.15), then so is its closure $cl_{\mathcal{K}[\vartheta]}(\mathcal{A})$. Thus, since $\mathrm{d}y_i \in \mathcal{A}$ for $i = 1, \ldots, p$, one has

$$\mathrm{d}y_i^{(k)} \in cl_{\mathcal{K}[\vartheta]}(\mathcal{A}) + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u, \ldots, \mathrm{d}u^{(k-1)}\}$$

for $i = 1, \ldots, p$ and all $k \geq 0$. Since $cl_{\mathcal{K}[\vartheta]}(\mathcal{A}) \subset \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$, then the outputs y_i do not depend on w nor its shifts. □

In [68] necessary and sufficient solvability conditions were given for DDP by feedback of the form (4.18).

Theorem 4.3. *[68] The DDP admits a solution by compensator (4.18) if and only if there exists an integrable submodule* Ω *such that*

(i) $\mathrm{d}y_i \in \Omega$ *for* $i = 1, \ldots, p$*;*

(ii) there exist functions $\varphi_1, \ldots, \varphi_\rho$ *such that*

$$cl_{\mathcal{K}[\vartheta]}(\Omega + \dot{\Omega}) = \Omega \oplus \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_\rho\}$$

(iii) $\mathrm{rank}_{\mathcal{K}} \frac{\partial(\varphi_1, \ldots, \varphi_\rho)}{\partial u} = \rho$.

Next, it is shown on the counterexample that the conditions of Theorem 4.3 are only sufficient.

Example 4.6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 u_1^{[-1]} + u_2^{[-1]} \\ \dot{x}_2 &= w + u_2 \\ y &= x_1, \end{aligned} \tag{4.19}$$

being disturbance decoupled, since the submodule $\Omega := \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1\}$ is invariant, though the conditions of Theorem 4.3 are not satisfied. Really,

$$cl_{\mathcal{K}[\vartheta]}(\Omega + \dot{\Omega}) = \Omega \oplus \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega\},$$

where $\omega = x_1 \mathrm{d}u_1^{[-1]} + \mathrm{d}u_2^{[-1]}$. By condition *(ii)* of Theorem 4.3 the 1-form ω should be weakly integrable, but it is not since its rank is clearly equal to 2. Therefore, the conditions of Theorem 4.3 are not satisfied.

The theorem below gives necessary and sufficient solvability conditions of the DDP by compensator (4.17). This theorem generalizes the results of [72] to the MIMO case and also corrects the results of [68].

Theorem 4.4. *The DDP is solvable by compensator (4.17) if and only if there exists a submodule Ω such that the following conditions are satisfied:*

(i) $\mathrm{d}y_i \in \Omega$ *for* $i = 1, \ldots, p$*;*

(ii) there exist 1-forms $\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u\}$*,* $i = 1, \ldots, \rho$*, with rank* γ_i *such that*

$$cl_{\mathcal{K}[\vartheta]}(\Omega + \dot{\Omega}) = \Omega \oplus \mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_\rho\}$$

and $\omega_i = a_{i,1}\mathrm{d}\varphi_{i,1} + \cdots + a_{i,\gamma_i}\mathrm{d}\varphi_{i,\gamma_i}$*, then*

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\varphi_{i,j})}{\partial u} = \xi \tag{4.20}$$

where ξ *is the number of independent (over* $\mathcal{K}[\vartheta]$*) exact 1-forms* $\mathrm{d}\varphi_{i,j}$*),* $i = 1, \ldots, \rho$ *and* $j = 1, \ldots, \gamma_i$*.*

Proof. Sufficiency: Let $\mathrm{d}\varphi_l$, $l = 1, \ldots, \xi \leq m$, be the independent (over $\mathcal{K}[\vartheta]$) exact 1-forms $\mathrm{d}\varphi_{i,j}$). By (4.20), the system of equations

$$\varphi_l(x(\cdot), u(\cdot)) = v_l \tag{4.21}$$

is solvable in u, which gives a feedback of the form (4.18). Under this feedback $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}\varphi_{i,j}\} \subseteq \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}v\}$ and thus the submodule Ω is invariant. Then, because of (i), the DDP is solved.

Necessity: If the disturbance decoupling problem is solved, then by Lemma 4.3 there exists a closed submodule Ω, which is invariant in the closed-loop system, such that $\mathrm{d}y_i \in \Omega$ for $i = 1, \ldots, p$. Thus condition (i) is satisfied. Since Ω is invariant, it satisfies

$$\dot{\Omega} \subseteq \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}v\}.$$

If the 1-forms $\omega_1, \ldots, \omega_\rho$ are defined as in (ii), then clearly $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_1, \ldots, \omega_\rho\} \subseteq \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}v\}$. This means that (4.21) must be solvable in u and thus (4.20) must be satisfied. □

Remark 4.3. Note that one can find a compatible feedback (4.17), that solves the DDP, if the functions $\varphi = (\varphi_1, \ldots, \varphi_\lambda)^T$, in the proof of Theorem 4.4, satisfy the condition: the matrix $\sum_i \frac{\partial \varphi}{\partial u^{[-i]}}\vartheta^i$ is unimodular.

In general, the choice of the 1-forms ω_i and the functions $\varphi_{i,j}$ (even if ω_i are fixed) is not unique and different choices may yield different results regarding the solvability of the DDP.

The difficulty in application of Theorem 4.4 is finding the submodule Ω. Clearly, since one wants that $\mathrm{d}y_i \in \Omega$ for $i = 1, \ldots, p$, Ω should satisfy the condition

$$cl_{\mathcal{K}[\vartheta]}(\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(r_i-1)}; i = 1, \ldots, p\}) \subseteq \Omega$$

where r_i is the relative degree of the output y_i.
Example 4.7. Consider the system

$$\begin{array}{rcl}\dot{x}_1 & = & x_2(u_1^{[-1]} + x_3^{[-2]} - x_1 u_1^{[-2]} - x_1 x_3^{[-3]}) \\ \dot{x}_2 & = & u_2^{[-1]} + x_1^{[-1]} u_2^{[-2]} + x_2(u_3^{[-2]} - x_3^{[-2]}) \\ \dot{x}_3 & = & x_2 w^{[-2]} \\ y_1 & = & x_1 \\ y_2 & = & x_2^{[-1]}.\end{array} \tag{4.22}$$

It is easy to see, that $r_1 = r_2 = 1$ and the relative shifts $\mu_1 = 1$, $\mu_2 = 2$ in this example, since $\dot{y}_1$ depends on $u_1^{[-1]}$ and $\dot{y}_2$ depends on $u_2^{[-2]}$. By the discussion above, the subspace Ω one looks for, has to include

$$\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1, \mathrm{d}x_2\}.$$

Therefore, one may start, by taking $\Omega = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1, \mathrm{d}x_2\}$. To find the 1-forms $\omega_1, \ldots, \omega_s$ in condition (ii) of Theorem 4.4, compute

$$\begin{array}{rcl}\mathrm{d}\dot{x}_1 & = & (-x_2(u_1^{[-2]} + x_3^{[-3]}))\mathrm{d}x_1 \\ & + & (u_1^{[-1]} + x_3^{[-2]} - x_1 u_1^{[-2]} - x_1 x_3^{[-3]})\mathrm{d}x_2 + x_2 \mathrm{d}u_1^{[-1]} \\ & + & x_2 \mathrm{d}x_3^{[-2]} - x_2 x_1 \mathrm{d}u_1^{[-2]} - x_2 x_1 \mathrm{d}x_3^{[-3]} \\ \mathrm{d}\dot{x}_2 & = & \mathrm{d}u_2^{[-1]} + u_2^{[-2]}\mathrm{d}x_1^{[-1]} + x_1^{[-1]}\mathrm{d}u_2^{[-2]} + x_2 \mathrm{d}u_3^{[-2]} \\ & - & x_2 \mathrm{d}x_3^{[-2]} + (u_3^{[-2]} - x_3^{[-2]})\mathrm{d}x_2.\end{array}$$

One can choose, for example

$$\begin{array}{rcl}\omega_1 & = & x_2 \mathrm{d}u_1^{[-1]} + x_2 \mathrm{d}x_3^{[-2]} - x_2 x_1 \mathrm{d}u_1^{[-2]} - x_2 x_1 \mathrm{d}x_3^{[-3]} \\ \omega_2 & = & \mathrm{d}u_2^{[-1]} + x_1^{[-1]}\mathrm{d}u_2^{[-2]} + x_2 \mathrm{d}u_3^{[-2]} - x_2 \mathrm{d}x_3^{[-2]}.\end{array}$$

These one-forms can be written as

$$\begin{array}{rcl}\omega_1 & = & \left(x_2\vartheta - x_1 x_2 \vartheta^2\right)\mathrm{d}(u_1 + x_3^{[-1]}) \\ \omega_2 & = & \left(\vartheta + x_1^{[-1]}\vartheta^2\right)\mathrm{d}u_2 + x_2\vartheta^2 \mathrm{d}(u_3 - x_3),\end{array}$$

meaning that the ranks of ω_1 and ω_2 are 1 and 2, respectively. Thus, $\varphi_{1,1} = u_1 + x_3^{[-1]}$, $\varphi_{2,1} = u_2$ and $\varphi_{2,2} = u_3 - x_3$. Now, clearly

$$\dim\left(\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}(u_1 + x_3^{[-1]}), \mathrm{d}u_2, \mathrm{d}(u_3 - x_3)\}\right) = 3$$

and

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\varphi_{1,1}, \varphi_{2,1}, \varphi_{2,2})^T}{\partial u} = \mathrm{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$$

and so the condition (4.20) is satisfied. The feedback can be obtained by solving the equations

$$\begin{aligned} v_1 &= u_1 + x_3^{[-1]} \\ v_2 &= u_2 \\ v_3 &= u_3 - x_3 \end{aligned}$$

in u:

$$\begin{aligned} u_1 &= v_1 - x_3^{[-1]} \\ u_2 &= v_2 \\ u_3 &= v_3 + x_3. \end{aligned}$$

Observe that this feedback is compatible.

Next, consider the other choice of 1-forms, which satisfy condition (ii) of Theorem 4.4:

$$\begin{aligned} \omega_1 &= x_2 \mathrm{d}u_1^{[-1]} + x_2 \mathrm{d}x_3^{[-2]} - x_2 x_1 \mathrm{d}u_1^{[-2]} - x_2 x_1 \mathrm{d}x_3^{[-3]} \\ \omega_2 &= \mathrm{d}u_2^{[-1]} + u_2^{[-2]} \mathrm{d}x_1^{[-1]} + x_1^{[-1]} \mathrm{d}u_2^{[-2]} + x_2 \mathrm{d}u_3^{[-2]} - x_2 \mathrm{d}x_3^{[-2]}, \end{aligned}$$

which can be written as

$$\begin{aligned} \omega_1 &= \big(x_2 \vartheta - x_1 x_2 \vartheta^2\big) \mathrm{d}(u_1 + x_3^{[-1]}) \\ \omega_2 &= \vartheta \mathrm{d}(u_2 + x_1 u_2^{[-1]}) + x_2 \vartheta^2 \mathrm{d}(u_3 - x_3). \end{aligned}$$

Thus $\varphi_{1,1} = u_1 + x_3^{[-1]}$, $\varphi_{2,1} = u_2 + x_1 u_2^{[-1]}$ and $\varphi_{2,2} = u_3 - x_3$. Now, clearly (4.20) is satisfied and the feedback is found by solving the equations

$$\begin{aligned} v_1 &= u_1 + x_3^{[-1]} \\ v_2 &= u_2 + x_1 u_2^{[-1]} \\ v_3 &= u_3 - x_3 \end{aligned}$$

in u. The obtained feedback

$$\begin{aligned} z^{[1]} &= v_2 - x_1 z \\ u_1 &= v_1 - x_3^{[-1]} \\ u_2 &= v_2 - x_1 z \\ u_3 &= v_3 + x_3. \end{aligned}$$

is not compatible, since dim $z \neq 0$.

4.2.2 Dynamic disturbance decoupling for SISO systems

Next, the DDP is solved for systems of the form (4.15), where $m = p = 1$, using the dynamic state feedback. The goal is to find a regular (i.e. invertible) dynamic compensator of the form

$$\begin{aligned} \dot{\eta} &= F(x^{[-i]}, \eta^{[-i]}, z^{[-i]}, v^{[-i]}; i = 0, \dots, \sigma) \\ z^{[1]} &= M(x^{[-i]}, \eta^{[-i]}, z^{[-i]}, v^{[-i]}; i = 0, \dots, \sigma) \\ u &= G(x^{[-i]}, \eta^{[-i]}, z^{[-i]}, v^{[-i]}; i = 0, \dots, \sigma), \end{aligned} \tag{4.23}$$

where η is the state of the compensator, such that for the closed-loop system, $y^{(k)}$ does not depend on the disturbance input w for all $k \in \mathbb{N}$, i.e. $\partial y^{(k)}/\partial w^{[-q]} \equiv 0$ for all $k, q \in \mathbb{N}$. Denote by r the relative degree and by μ the relative shift of the output y, respectively.

To solve the dynamic disturbance decoupling problem (DDDP), a submodule[4] Ω of $\mathcal{M}$ is defined as follows

$$\begin{aligned} \Omega \;=\; & cl_{\mathcal{K}[\vartheta]}(\{\omega \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\} \mid \forall k \in \mathbb{N} \\ & \omega^{(k)} \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}y^{(r)}, \dots, \mathrm{d}y^{(r+k-1)}\}\}). \end{aligned} \tag{4.24}$$

This definition also yields

$$\mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y, \dots, \mathrm{d}y^{(r-1)}\} \subseteq \Omega.$$

If a 1-form ω belongs to the submodule Ω, clearly $\dot{\omega} \in \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(r)}\}$. Thus, the submodule Ω can be computed as the limit of the algorithm:

$$\begin{aligned} \Omega^0 &= \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\} \\ \Omega^{k+1} &= \{\omega \in \Omega^k \mid \dot{\omega} \in \Omega^k + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(r)}\}\}. \end{aligned}$$

The latter also yields that $\dot{\Omega} \subseteq \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(r)}\}$.

The following lemma gives a condition to check whether a SISO system is disturbance decoupled or not.

Lemma 4.4. *The SISO time-delay system (4.15) is disturbance decoupled if and only if*

$$\mathrm{d}y^{(r)} \in \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u\}. \tag{4.25}$$

Proof. Necessity. Since r is the relative degree of output y with respect to the input u,

$$\mathrm{d}y^{(r)} = \omega_0 + b(\vartheta)\mathrm{d}u,$$

where $b(\vartheta) \in \mathcal{K}[\vartheta]$ and $\omega_0 \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$. Next, it is shown that $\omega_0 \in \Omega$. Assume, by contradiction, that $\omega_0 \notin \Omega$. Then there exists $s \in \mathbb{N}$ such that

$$\omega_0^{(s)} \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{(s-1)}\}.$$

[4] Note that this is also one possible choice of submodule Ω in Theorem 4.4.

This means that the 1-form ω_0 is not disturbance decoupled and thus $\mathrm{d}y^{(r)}$ also is not disturbance decoupled. This is a contradiction and therefore, $\omega_0 \in \Omega$, meaning that (4.25) is satisfied.

Sufficiency. If (4.25) is true, then $\dot{\Omega} \subseteq \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u\}$, since $\dot{\Omega} \subseteq \Omega + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(r)}\}$. Thus, Ω is invariant with respect to the system dynamics and since $\mathrm{d}y \in \Omega$, the system is disturbance decoupled. □

The Theorem 4.5 below is a generalization of Theorem 11 in [72], where static solutions were considered.

Theorem 4.5. *The DDDP is solvable for SISO time-delay systems (4.15) if and only if there exist $k+1$ weakly integrable 1-forms $\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\dot{\omega}_{i-1}, \mathrm{d}x^{[-\tau]}, \mathrm{d}u^{[-\tau]}; \tau \geq \mu\}$, $i = 0, \ldots, k$, such that*

$$\mathrm{d}y^{(r+j)} - \omega_j \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}y^{(r)}, \ldots, \mathrm{d}y^{(r+j-1)}\}$$

for $j = 0, \ldots, k-1$ and

$$\mathrm{d}y^{(r+k)} - \omega_k \in \Omega + \cdots + \Omega^{(k)}.$$

Proof. Necessity: Since the closed-loop system is disturbance decoupled,

$$\mathrm{d}y^{(r+k)} \in \Omega_{cl} + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}v\},$$

by Lemma 4.4, where Ω_c is the subspace Ω for the closed-loop system. Since in the closed-loop system the relative degree of output y is $r+k$, it can be shown, similarly as in the proof of Theorem 2.2, that $\Omega_c = \Omega + \cdots + \Omega^{(k)}$. The weakly integrable 1-form ω_k can always be taken as $\omega_k = a_k(\vartheta)\mathrm{d}v$, where $a_k(\vartheta) \in \mathcal{K}[\vartheta]$ is such that

$$\mathrm{d}y^{(r+k)} - a_k(\vartheta)\mathrm{d}v \in \Omega + \cdots + \Omega^{(k)}.$$

Now, assume by contradiction that there are no integrable 1-forms ω_j, $j = 0, \ldots, k-1$, satisfying the conditions of Theorem 4.5. Then either some $y^{(r+j)}$ depend on the disturbance w (which is a contradiction) or some 1-forms ω_j are not weakly integrable. In the latter case

$$\mathrm{d}y^{(r+j)} \notin \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}\eta\},$$

which is also a contradiction. Thus, there exist integrable 1-forms ω_j, $j = 0, \ldots, k-1$, that satisfy the conditions of Theorem 4.5. Finally, since the feedback is causal $\omega_i \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\dot{\omega}_{i-1}, \mathrm{d}x^{[-\tau]}, \mathrm{d}u^{[-\tau]}; \tau \geq \mu\}$, $i = 0, \ldots, k$.

Sufficiency: Let $\omega_i = a_i(\vartheta)\mathrm{d}\varphi_i(\dot{\varphi}_{i-1}, x^{[-j]}, u^{[-j]}; j = 0, \ldots, \sigma)$ for $i = 0, \ldots, k$ and construct the system of equations

$$\begin{aligned} \eta_j &= \varphi_j \\ v &= \varphi_k, \end{aligned}$$

where $j = 0, \ldots, k-1$. This system is solvable in variables $\{\dot{\eta}_j, u\}$ $j = 0, \ldots, k-1$ yielding a feedback, such that $\omega_k = a_k(\vartheta)\mathrm{d}v$ in the closed-loop system. Also, the relative degree of y of the closed-loop system is $r+k$. Thus $\mathcal{A} := \Omega + \cdots + \Omega^{(k)} \subseteq \Omega_c$. Really, since

$$\dot{\mathcal{A}} \subseteq \mathcal{A} + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}y^{(r+k)}\},$$

$\mathcal{A} \subseteq \Omega_c$ must be true. Therefore $\mathrm{d}y^{(r+k)} \in \Omega_c + \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}v\}$ and by Lemma 4.4 the SISO system (4.15) is disturbance decoupled. □

Theorem 4.5 yields a solution by a pure shift dynamic feedback (4.17) for SISO systems of the form (4.15), given already in [72].

Corollary 4.1. *The DDP is solvable for SISO time-delay systems (4.15) by compensator (4.17) if and only if there exist a weakly integrable 1-form* $\omega \in \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x^{[-\tau]}, \mathrm{d}u^{[-\tau]}; \tau \geq \mu\}$*, such that*

$$\mathrm{d}y^{(r)} - \omega \in \Omega.$$

Example 4.8. This example demonstrates that unlike in delay-free case, for time-delay systems the existence of dynamic feedback which solves the DDP, does not yield that there also exists a static solution. Consider a nonlinear time-delay system

$$\begin{aligned} \dot{x}_1 &= x_2^{[-1]}u^{[-1]} + x_3 \\ \dot{x}_2 &= w \\ \dot{x}_3 &= x_2^{[-1]} \\ y &= x_1. \end{aligned} \tag{4.26}$$

Clearly, system (4.26) can not be disturbance decoupled by static feedback. This happens because $\dot{y}$ depends on x_3, whose delay is smaller than the relative shift $\mu = 1$, and thus can not be compensated. But there exists a dynamic feedback which solves the problem. For system (4.26) $\Omega = \mathrm{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x_1\}$ and 1-forms ω_i in Theorem 4.5 are

$$\begin{aligned} \omega_0 &= \mathrm{d}(x_2^{[-1]}u^{[-1]}) \\ \omega_1 &= \mathrm{d}\ddot{y} = \dot{\omega}_0 + \mathrm{d}x_2^{[-1]}. \end{aligned}$$

A feedback can be found by solving the equations

$$\begin{aligned} \eta &= x_2 u \\ v &= \dot{\eta} + x_2 \end{aligned}$$

in variables $\dot{\eta}, u$:

$$\begin{aligned} \dot{\eta} &= v - x_2 \\ u &= \tfrac{\eta}{x_2}. \end{aligned} \tag{4.27}$$

In the closed-loop system

$$\begin{aligned}\dot{x}_1 &= \eta^{[-1]} + x_3\\ \dot{x}_2 &= w\\ \dot{x}_3 &= x_2^{[-1]}\\ \dot{\eta} &= v - x_2\\ y &= x_1,\end{aligned}$$

$\dot{y} = \eta^{[-1]} + x_3$ and $\ddot{y} = v^{[-1]}$. Therefore, output y and its derivatives do not depend on the disturbance explicitly.

4.3 Input-Output Decoupling

In this section systems of the form (1.5) with $p = m$ are considered. The goal is to find a feedback of the form (4.17), such that every output variable of system (1.5) depends exactly on one distinct input variable for all time instants. The problem is similar to the SISO DDP, except that here one allows the feedback to depend also on the "disturbance", i.e. on the other input variables.

One says that the system (1.5) is input/output (i/o) decoupled if possibly after reordering the inputs, one has

$$dy_i^{(k)} \in \operatorname{span}_{\mathcal{K}[\vartheta]}\{dx, du_i, d\dot{u}_i, \ldots, du_i^{(k-1)}\} \tag{4.28}$$

for $i = 1, \ldots, m$ and for all $k \geq 0$. One looks for a regular feedback (4.17), such that after applying (4.17) to the system (1.5), the closed-loop system is i/o decoupled.

To solve the problem, define the submodules Ω_i, $i = 1, \ldots, m$, as follows

$$\begin{aligned}\Omega_i \;=\; & cl_{\mathcal{K}[\vartheta]}(\{\omega \in \operatorname{span}_{\mathcal{K}[\vartheta]}\{dx\} \mid \forall k \in \mathbb{N}, \omega^{(k)} \in \operatorname{span}_{\mathcal{K}[\vartheta]}\{dx,\\ & dy_i^{(r_i)}, \ldots, dy_i^{(r_i+k-1)}\}\}),\end{aligned}$$

where r_i is the relative degree of the output y_i.

A lemma similar to Lemma 4.4, can be given, to check whether a given system is already i/o decoupled.

Lemma 4.5. *System (1.5) is i/o decoupled if and only if for* $i = 1, \ldots, m$

$$dy_i^{(r_i)} \in \Omega_i + \operatorname{span}_{\mathcal{K}[\vartheta]}\{du_i\}. \tag{4.29}$$

Proof. Necessity. Since r_i is the relative degree of the output y_i,

$$dy_i^{(r_i)} = \omega_i + b_i(\vartheta)du_i,$$

where $b_i(\vartheta) \in \mathcal{K}[\vartheta]$ and $\omega_i \in \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x\}$. Next, it will be shown that $\omega_i \in \Omega_i$. Assume, by contradiction, that $\omega_i \notin \Omega_i$. Then there exists $s \in \mathbb{N}$ such that

$$\omega_i^{(s)} \notin \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}y_i^{(r_i)}, \ldots, \mathrm{d}y_i^{(r_i+s-1)}\}$$

and thus

$$\omega_i^{(s)} \notin \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{(s-1)}\}.$$

This means that (4.28) is not satisfied for $k = r_i + s$ and thus $\mathrm{d}y_i^{(r_i)}$ is not i/o decoupled. This is a contradiction and therefore $\omega_i \in \Omega_i$, meaning that (4.29) is satisfied.

Sufficiency. It is shown that $\mathrm{d}y_i^{(r_i+j)}$ satisfies (4.28) for all $j \geq 0$. For $j = 0$, condition (4.28) is satisfied by (4.29). Now, assume, that $\mathrm{d}y_i^{(r_i+j)}$ satisfies (4.28) for $j = 0, \ldots, s-1$, and show that then condition (4.28) is also satisfied for $j = s$. By definition of Ω_i

$$\Omega_i^{(j)} \subseteq \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}y_i^{(r_i)}, \ldots, \mathrm{d}y_i^{(r_i+j-1)}\}$$

for all $j \geq 0$. Thus, one has

$$\begin{aligned} \mathrm{d}y_i^{(r_i+s)} &\in \Omega_i + \cdots + \Omega_i^{(s)} + \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{(s)}\} \\ &\in \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}y_i^{(r_i)}, \ldots, \mathrm{d}y_i^{(r_i+s-1)}\} \\ &\quad + \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{(s)}\} \end{aligned}$$

and because $\mathrm{d}y_i^{(r_i+j)}$ satisfies (4.28) for $j = 0, \ldots, s-1$,

$$\mathrm{d}y_i^{(r_i+s)} \in \operatorname{span}_{\mathcal{K}[\vartheta]}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{(s)}\}.$$

□

Theorem 4.6. *System (1.5) with $p = m$ can be i/o decoupled by a feedback (4.17) if and only if there exist 1-forms ω_i, $i = 1, \ldots, m$, that satisfy the following conditions for $i = 1, \ldots, m$:*

(i)
$$\mathrm{d}y_i^{(r_i)} \in \Omega_i + \operatorname{span}_{\mathcal{K}[\vartheta]}\{\omega_i\}$$

(ii) the 1-forms ω_i are weakly integrable, i.e. $\omega_i = p_i(\vartheta)\mathrm{d}\varphi_i$ for some $p_i(\vartheta) \in \mathcal{K}[\vartheta]$;

(iii) the matrix $\frac{\partial(\varphi_1,\ldots,\varphi_m)}{\partial u}$ has full rank.

Proof. Necessity. By Lemma 4.5, $\omega_i = \mathrm{d}v_i = \mathrm{d}\tilde{G}(x^{[-i]}, u^{[-i]}; i = 0, \ldots, \sigma)$ satisfies the conditions.

Sufficiency. Let $\omega_i = p_i(\vartheta)\mathrm{d}\varphi_i$ satisfy condition (i). Define, $v_i = \varphi_i$. Denote $v = (v_1, \ldots, v_m)^T$ and $\varphi = (\varphi_1, \ldots, \varphi_m)^T$, then

$$\mathrm{d}v = P(\vartheta)\mathrm{d}u + R(\vartheta)\mathrm{d}x, \tag{4.30}$$

where

$$P(\vartheta) = \sum_j \frac{\partial \varphi}{\partial u^{[-j]}}\vartheta^j.$$

By (iii), matrix $\frac{\partial \varphi}{\partial u}$ has full rank, and thus (4.30) defines the feedback of the form (4.18). Under this feedback, by (i) the condition of Lemma 4.5 is satisfied and thus the closed-loop system is i/o decoupled. □

Remark 4.4. If one wants to solve the i/o decoupling problem by compatible feedback (4.17), then the condition (iii) in Theorem 4.6 should be replaced by

(iii') the matrix $P(\vartheta) := \sum_j \frac{\partial(\varphi_1, \ldots, \varphi_m)}{\partial u^{[-j]}}\vartheta^j$ is unimodular.

Note that, if $\omega_i = p_i(\vartheta)\bar{\omega}_i$ satisfies condition (i), then so does the 1-form $\bar{\omega}_i$. It means that ω_i can be always chosen such that $\mathrm{span}_{\mathcal{K}[\vartheta]}\{\omega_i\}$ is closed, and thus the condition (ii) can be substituted by

$$(ii') \;\; \mathrm{d}\omega_i \wedge \omega_i = 0.$$

The most obvious choice for ω_i is $\omega_i = \mathrm{d}y_i^{(r_i)}$. For this choice, conditions (i) and (ii) are always satisfied and one only has to check the condition (iii) for $\varphi_i = y_i^{(r_i)}$. But, this is only sufficient, because, for example, when $\dot{y} = yu^{[-1]}$, then condition (iii) is not satisfied. But, when there is no delay in u, then one can always take $\omega_i = \mathrm{d}y_i^{(r_i)}$. Therefore, one has the corollary.

Corollary 4.2. *If there is no delay in the input u, then system (1.5) with $p = m$ can be i/o decoupled by a feedback (4.17) if and only if*

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(y_1^{(r_1)}, \ldots, y_m^{(r_m)})}{\partial u} = m. \tag{4.31}$$

There is an another way to write the conditions for solvability of the i/o decoupling problem. Let $y^{(r)} = (y_1^{(r_1)}, \ldots, y_m^{(r_m)})^T$.

Theorem 4.7. *System (1.5) with $p = m$ can be i/o decoupled by a feedback (4.17) if and only if*

$$\mathrm{d}y^{(r)} = Q(\vartheta)[P(\vartheta)\mathrm{d}u + L(\vartheta)\mathrm{d}x] + K(\vartheta)\mathrm{d}x,$$

where $Q(\vartheta)$ is a diagonal matrix, $\mathrm{d}[P(\vartheta)\mathrm{d}u + L(\vartheta)\mathrm{d}x] = 0$, $(K(\vartheta)\mathrm{d}x)_i \in \Omega_i$ for $i = 1, \ldots, m$ and matrix P_0, where $P(\vartheta) = \sum_i P_i\vartheta^i$, has full rank.

Proof. This is a direct consequence of Theorem 4.6, when one takes ω_i, $i = 1, \ldots, m$, such that $\omega = Q(\vartheta)[P(\vartheta)\mathrm{d}u + L(\vartheta)\mathrm{d}x]$, where $\omega = (\omega_1, \ldots, \omega_m)^T$. □

4.4 Conclusions

In this chapter nonlinear continuous time-delay systems were considered. For such systems the standard integrability notion for a set of 1-forms is too restrictive, because it does not take into consideration the effects of time-delays. Since the integrability of 1-forms is essential in the differential algebraic method to solve various control problems, a more general notion of integrability is needed. In the the first section of this chapter, such generalization is done. In fact two possible generalizations are considered and named as weak and strong integrability. In the case of strong integrability, certain matrix over a polynomial ring $\mathcal{K}[\vartheta]$ has to be unimodular, while in the case of weak integrability, this matrix has to have only full rank. Therefore, the strong integrability property always yields the weak integrability property. The main result of this chapter is Theorem 4.1, which gives necessary and sufficient conditions to check strong integrability property of a set of 1-forms. Unfortunately, this theorem depends on a sequence of matrices $N_p(\vartheta)$, $p \geq 0$, and to check the strong integrability property, one has to know a bound for index p. A suggestion for such bound is made in Remark 4.2, but the proof of this seems to be rather difficult. The weak integrability of a set of 1-forms can be checked by using the conditions for strong integrability. More precisely, one has to check whether the closure of a submodule, defined by the given 1-forms, is strongly integrable.

The study of different problems has shown that in most cases weak integrability of a set of 1-forms is enough to get good solutions. In the second part of the chapter, the weak integrability was used to characterize the accessibility property of time-delay systems and to solve the DDP by pure shift dynamic feedback or by dynamic state feedback for SISO systems. Finally, the weak integrability notion was applied to prove some preliminary results on i/o decoupling problem for time-delay systems.

Conclusions

In the thesis algebraic methods are used to solve various design problems for nonlinear discrete-time and continuous time-delay systems. There are three main contributions: the full solution to the input-output (i/o) linearization problem by dynamic output feedback for nonlinear discrete-time systems, the necessary and sufficient conditions to check the flatness property of nonlinear discrete-time control systems and the characterization of integrable 1-forms in the case of nonlinear time-delay systems. Besides these results, the thesis unifies the study of time-delay and delay-free systems. It is shown (in Chapter 1) how similar mathematical tools are used in the study of different system classes.

First, the i/o linearization problem by dynamic output feedback is studied for the class of nonlinear discrete-time systems. Based on the system equations, a set of functions is computed, which characterizes the nonlinearities of the system. The obtained necessary and sufficient condition for the existence of the linearizing feedback is given in terms of the previously computed set of functions. It guarantees that a certain set of algebraic equations is solvable, after which the required feedback can be found by solving the found equations. While in the previous results [52, 54], the sufficient solvability conditions depended on the existence of certain functions, here a different set of functions is computed first, and then the condition is given in terms of these functions. Then, the i/o linearization is applied to solve the i/o decoupling problem and to give a sufficient solvability condition for the disturbance decoupling problem by a dynamic measurement feedback. These results are extensions of previous works and the novelty comes from the improved i/o linearization conditions.

Secondly, the state feedback linearization of nonlinear discrete-time systems is presented. It is known that the linearization by a dynamic endogenous feedback is equivalent to the flatness property, see [55]. First, an important result is proved, which makes the study of flatness property different for discrete- and continuous-time systems. Namely, it is proven that a discrete-time system is flat if and only if it is 0-flat, i.e. the flat outputs depend only on system's states. For the study of flatness of discrete-time systems, a sequence of systems is defined, which is initialized by a given system. It is proved that the original system is flat if and only if the sequence converges to a system which has dimension 0. The novelty of this work is not exactly in the necessary and sufficient condition itself, but in the procedure for checking flatness property and computing the flat output. Compared to the results of [6, 64, 63, 55, 53], here the computations needed to verify flatness property of a given system are much simpler. Another algebraic approach is applied to address the feedback linearization problem for systems, which are described by possibly non-smooth functions. A solution by static state feedback is given and compared to the existing results, obtained for analytic systems.

The third main contribution is devoted to integrability problem of the set of differential 1-forms in the case of time-delay systems. Two concepts of integrability are defined, strong and weak integrability, which, in the case of no delays, reduce both to the known integrability concept. The characterization of integrability property is different in delay-free and time-delay cases, since in the first case the 1-forms are looked as elements of a vector space, while in the other as elements of a module. The different properties of modules and vector spaces yield the need for more general integrability notion for time-delay case. The strong and weak integrability notions define two possible extensions. The conditions for checking the strong or weak integrability of a given set of 1-forms are derived in the thesis. It is also shown how the obtained results on integrability can be applied to study the accessibility property of a time-delay system and to solve decoupling problems.

There are three possible directions for future work. The first, and the easiest to obtain, is the implementation of the results of the thesis in the *Mathematica* environment, which can be included to the software package NLControl [92]. The most important and possibly the most difficult task is to implement the computations of the maximal integrable subspace, contained in a given vector space of 1-forms and the minimal integrable vector space that contains a given vector space of 1-forms. All solutions to the main problems, considered in the thesis, require at least one of the mentioned computations. The second possible research direction is to extend the results of Chapters 2 and 3 from discrete-time case to the continuous-time case. The extensions are not direct. In the case of the i/o linearization problem, the computation of the functions, in terms of which the necessary and sufficient condition is formulated, is different in the continuous-time case, since the derivative operator acts differently than the forward shift operator. As for the flatness property, the continuous-time case is much more complicated. The procedure, given in this thesis, can be generalized for the continuous-time case, but this would yield only sufficient solvability conditions. This is a consequence of the fact that not every continuous-time flat system is 0-flat, which is the case for discrete-time systems. The third possible future research direction is the use of the obtained integrability concepts to generalize the well known results from the delay-free case to the time-delay case. The possible problems to be encountered include causality, which is unavoidable when considering time-delay systems, but also other difficulties show up. For instance, the construction of a state transformation becomes more challenging. When in the delay-free case one has $n-k$ linearly independent functions, one can always add k functions, to define the state transformation. In the time-delay case, this is not true anymore, since invertibility of a state transformation is not guaranteed only by the independence of the functions, but the module their differentials generate, must also be closed.

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Kokkuvõte

Diskreetsete ja hilistumistega mittelineaarsete juhtimissüsteemide süntees

Käesolevas väitekirjas on lahendatud mitmeid mittelineaarsete diskreetsete ja hilistumistega juhtimissüsteemide struktuurse sünteesi ülesandeid, rakendades laialt kasutatavat algebralist lähenemist, mis põhineb süsteemi poolt defineeritud diferentskorpusel ning diferentsiaalvormidel. Töö keskendub mittelineaarsete diskreetaja (muuhulgas mittesiledate) süsteemide lineariseerimisele (dünaamilise) tagasisidega ning mõningatele viimastega seotud probleemidele. Valitud metoodika ei põhine otseselt süsteemi kirjeldavatel võrranditel, vaid nende diferentsiaalidel. See tähendab, et töötatakse diferentsiaalvormide vektorruumidega (mõningatel juhtudel moodulitega), mis on defineeritud üle meromorfsete funktsioonide korpuse, mitte funktsioonidega. Oluline antud lähenemises on sõltumatute diferentsiaalsete 1-vormide hulga integreeruvuse mõiste, mis võimaldab 1-vormide abil saadud lahendid esitada funktsioonide kaudu. Hilistumisteta süsteemide jaoks saab integreeruvust kontrollida Frobeniuse teoreemi abil, aga hilistumistega süsteemide jaoks ei ole 1-vormide integreeruvust palju uuritud. See on peamine takistus üldistamaks antud algebralist lähenemist hilistumistega süsteemidele.

Väitekirjas on üldistatud diferentsiaalsete 1-vormide integreeruvuse mõiste hilistumistega süsteemide jaoks ning leitud konstruktiivsed tingimused integreeruvuse kontrollimiseks. Üldisemat definitsiooni on rakendatud mitmete probleemide uurimisel, muuhulgas häiringute kompenseerimise ülesande lahendamisel ja juhitavuse omaduse kontrollimisel.

Integreeruvuse probleemi lahendus hilistumistega süsteemide jaoks ja diskreetaja süsteemide lineariseerimine dünaamilise olekutagasisidega taanduvad matemaatiliselt sarnasele ülesandele. Mõlemal juhul otsitakse teatud 1-vormide mooduli eksaktset baasi, kusjuures moodul on defineeritud üle mittekommutatiivsete polünoomide ringi. Olekutagasisidega lineariseerimise probleem on omakorda ekvivalentne süsteemi nn. lameduse omadusega. Väitekirjas on leitud algoritm kontrollimaks, kas etteantud diskreetaja juhtimissüsteem on lame. Lisaks võimaldab algoritm leida ka süsteemi lamedad väljundid, mille kaudu on omakorda leitav nii tagasiside kui ka olekuteisendus, mis lineariseerivad antud süsteemi.

On uuritud ka võimalusi diskreetaja süsteemi lineariseerimiseks dünaamilise väljundtagasisidega. Töös on leitud tarvilikud ning piisavad tingimused probleemi lahenduvuseks. Antud lahendust on kasutatud ka häiringu kompenseerimiseks ning süsteemi dekomponeerimiseks ühe sisendi ja ühe väljundiga alamsüsteemideks dünaamilise mõõdetavatest väljunditest sõltuva tagasiside abil.

Abstract

Advanced Design of Nonlinear Discrete-time and Delayed Systems

In this thesis a well-known algebraic approach is used to solve several structural design problems for nonlinear discrete-time and delayed control systems. In particular, the thesis focuses on (dynamic) feedback linearization (including the case of non-smooth systems) and some of the related problems. In the chosen algebraic approach, the controller design is based on the global linearized system description instead of difference/differential equations, describing the system. This means that one works with the vector spaces (or sometimes modules) of differential forms over the difference/differential field of meromorphic functions and not directly with the functions themselves. The key factor of the approach is the integrability property of a set of independent differential 1-forms, which allows to write the obtained solutions again in terms of functions. For delay-free control systems, the conditions for integrability are well established and given by the Frobenius theorem, but for the time-delay case, the integrability problem is not much studied. This is the main obstacle in extending the algebraic formalism for time-delay systems.

In the thesis, the concept of integrability of 1-forms is generalized for the time-delay case and conditions to check this property are given. Moreover, it is shown that the introduced concept is useful in solution of several problems, including the decoupling problems and checking the accessibility property.

Characterization of integrability property in the time-delay case is mathematically closely linked to the problem of dynamic feedback linearization in case of discrete-time systems. In both cases, one searches for an exact bases for some module of 1-forms, defined over a non-commutative polynomial ring. The feedback linearization problem is equivalent to the system property called flatness. In the thesis, an algorithm for checking flatness property of the discrete-time system is developed, which allows also to find the flat outputs. Based on flat outputs, the linearizing feedback and the

state transformation can be computed.

A more restrictive dynamic output feedback linearization problem is also studied to linearize the input-output (i/o) equations of a nonlinear discrete-time system. A full solution to the i/o linearization problem is given. Later, this solution is used to solve the decoupling problems via dynamic measurement feedback for nonlinear discrete-time systems.

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Input–output linearization of discrete-time systems by dynamic output feedback

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ABSTRACT

The paper addresses the input–output linearization problem by dynamic output feedback for multi-input multi-output nonlinear systems, described by a set of higher order difference equations. Necessary and sufficient solvability conditions are given together with the constructive procedure to check the conditions and compute the feedback.

1. Introduction

The output feedback solutions to different control problems like disturbance decoupling [1,7,18], I/O decoupling [6,14,16] and I/O linearization [8,15,17], are much less studied than the state feedback solutions, especially in the nonlinear case. Moreover, output feedback solutions are much more difficult to obtain, and in many cases (disturbance decoupling, input–output linearization) full solutions are still missing, even in the single-input single-output (SISO) case. However, not all systems, described by a set of higher order I/O difference (or differential) equations, are realizable in the state space form, and in such a case the use of output feedback is the only choice. Even when the state equations are given, it may happen that not all states are directly available for measurement. In such a case one may rely on observers which are, in general, not easy to construct [19] or use the output feedback.

For continuous-time nonlinear systems the I/O linearization problem by static output feedback has been completely solved in [13,17]. As for dynamic output feedback, sufficient conditions were given in [5], but only for SISO systems. For discrete-time SISO systems, the problem has been addressed in [7,15]. In the discrete-time case, similar mathematical tools and concepts, as in [5,17] were used, though the computations are different. In the multi-input multi-output (MIMO) case, there are no results for continuous-time systems (except static case) and only [8] for discrete-time systems.

The goal of this paper[1] is to obtain the necessary and sufficient conditions of I/O linearizability by dynamic output feedback for MIMO discrete-time systems, described by the set of higher order I/O difference equations. This paper generalizes the results of [7] into the MIMO case. Compared with earlier extension [8], the main theorem of this paper is not algorithm-dependent as that of [8], but depends on certain functions, which can be computed easily for all systems. Moreover, the condition itself is now necessary and sufficient.[2] Under the additional assumption of right invertibility of the system these conditions are also necessary and sufficient for solvability of the most general problem statement. The same algebraic approach of differential one-forms as in [7,15] is used in this paper to obtain the results. The main difference between the results of this paper and the results for SISO systems is that instead of checking whether certain one-forms are integrable, we find integrable spaces of minimal dimension where these one-forms belong to. After that, it remains to check if one can construct the feedback, that solves the problem, based on these integrable spaces. Finally, the main result of this paper is specified for the additive NARX (ANARX) systems.

While the I/O linearization problem is an important problem itself, it plays also a key role in the solution of other control problems. In particular, in [18,7], the I/O linearization is used to develop sufficient conditions for the disturbance decoupling problem by dynamic measurement feedback, for continuous- and discrete-time cases, respectively. Moreover, I/O linearization is also used in the solution of the I/O decoupling problem [16].

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[1] Being the extension and improvement of the conference paper [8].
[2] Only sufficient condition was given in [8].

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The paper is organized in the following manner. In Section 2, mathematical tools and key definitions are given. The main results are stated in Section 3, including the necessary and sufficient condition for solvability of the I/O linearization problem. Three examples are given in Section 4 to characterize the computations of the feedback. Finally some conclusions are given in Section 5.

2. Preliminaries

Throughout the paper[3] we assume that $i, \tau = 1, \ldots, p$ and $j = 1, \ldots, m$. Also, we write y or $y^{[0]}$ for $y(t)$ and $y^{[k]}$ for $y(t+k)$, $k \geq 1$. Similar notations are used for the other variables. Consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the set of difference equations:

$$y_i^{[n_i]} = \Phi_i(y_\tau, \ldots, y_\tau^{[n_{i\tau}]}, u_j, \ldots, u_j^{[q_{ij}]}) \quad (1)$$

where the functions Φ_i are supposed to be analytic and the indices in (1) satisfy the relations

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_p, \quad n_{i\tau} < n_\tau \\ &n_{i\tau} < n_i, \quad \tau \leq i \\ &n_{i\tau} \leq n_i, \quad \tau > i. \end{aligned} \quad (2)$$

The conditions (2) mean that Eq. (1) are assumed to be in the so-called doubly (row- and column) reduced form. Note that whenever the system is well-defined, one can always transform an arbitrary set of I/O equations, at least locally, under mild rank conditions, into the form (1), see [10]. The advantage of the form (1) is that the forward-shift operator, associated with the control system, is explicitly defined, see below.

Definition 1. The relative degree r_i of the ith output component y_i of the system (1) is defined as $r_i := n_i - q_i$.

Also, we assume, like in the majority of papers addressing discrete-time nonlinear systems, that the system (1) is submersive, i.e. the map $\Phi = (\Phi_1, \ldots, \Phi_p)^T$ satisfies generically[4] the condition

$$\text{rank}\left[\frac{\partial \Phi}{\partial(y, u)}\right] = p,$$

where $y = (y_1, \ldots, y_p)$ and $u = (u_1, \ldots, u_m)$.

Recall the algebraic structures from [2] to be used in this paper. Let $\mathcal{K}$ be the field of meromorphic functions in variables y, u and a finite number of their independent forward shifts, i.e. variables from the set $\mathcal{C} = \{y_i, \ldots, y_i^{[n_i-1]}, u_j^{[k]}; k \geq 0\}$. Next, we define the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ associated with system (1) as follows. In particular, $\delta y_i^{[n_i-1]} = \Phi_i(\cdot)$, meaning that $y_i^{[n_i]}$ as a dependant variable has to be replaced by $\Phi_i(\cdot)$ from (1). For the remaining elements of $\mathcal{C}$ the forward shift operator is defined in a standard manner, i.e. $\delta y_i^{[\alpha]} := y_i^{[\alpha+1]}$, $\alpha = 0, \ldots, n_i - 2$, $\delta u_j^{[\beta]} := u_j^{[\beta+1]}$, $\beta \geq 0$. Then the shift of an arbitrary function just requires applying forward-shift to all its arguments. For example, if function ϕ depends on variables $\{y_i, \ldots, y_i^{[n_i-1]}, u_j, \ldots, u_j^{[k]}\}$, then

$$\delta[\phi(y_i, \ldots, y_i^{[n_i-1]}, u_j, \ldots, u_j^{[k]})] := \phi(y_i^{[1]}, \ldots, \Phi_i(\cdot), u_j^{[1]}, \ldots, u_j^{[k+1]}).$$

Under the assumption that the system (1) is submersive, the pair $(\mathcal{K}, \delta)$ is an algebraic object called difference field, which we denote simply by $\mathcal{K}$. In general, $\mathcal{K}$ is not inversive, i.e. there does not exist inverse of operator δ, but there always exists an overfield $\mathcal{K}^*$ of field $\mathcal{K}$, called inversive closure of $\mathcal{K}$. Because $\mathcal{K}^*$ is inversive, there exists an operator δ^{-1}, satisfying $\delta\delta^{-1} = \delta^{-1}\delta = id$, which will be interpreted as the backward-shift operator. By δ^k and δ^{-k} we denote the k-fold application of operators δ and δ^{-1}, respectively. The detailed explanation for the construction of $\mathcal{K}^*$ (the rule for computation of δ^{-1}) is given in [11] for the special case of SISO systems. The MIMO case, though technically more involved, can be handled in a similar manner. The crucial point is the choice of the new independent variables of the field extension. These new variables are advisable to be chosen in such a manner that the further computations are as simple as possible. From now on, we denote the field $\mathcal{K}^*$ simply by $\mathcal{K}$.

[3] Except in Section 3.2.

[4] Almost everywhere, except on the set of zero measure.

Define the vector spaces $\mathcal{U} = \text{span}_\mathcal{K}\{du_j^{[k]}; k \geq 0\}$, $\mathcal{Y} = \text{span}_\mathcal{K}\{dy_i^{[k]}; k = 1, \ldots, n_i - 1\}$ and the vector space of one-forms $\mathcal{E} = \mathcal{Y} + \mathcal{U}$. So, the elements of $\mathcal{E}$ are the linear combinations over the field $\mathcal{K}$ of the standard basis elements from the set $d\mathcal{C} := \{dy_i, \ldots, dy_i^{[n_i-1]}, du_j^{[k]}; k \geq 0\}$. A one-form ω is said to be exact if it is a differential of some function, i.e. $\omega = d\varphi$, $\varphi \in \mathcal{K}$. Also, define $\mathcal{E}^k := \text{span}_\mathcal{K}\{dy_i, \ldots, dy_i^{[k-1]}, du_j, \ldots, du_j^{[k-1]}\}$ for any $k \in \mathbb{N}$. The forward-shift of a one-form $\omega = \sum_s a_s \, d\phi_s$ is defined by

$$\delta\left(\sum_s a_s d\phi_s\right) := \sum_s \delta(a_s) \, d(\delta\phi_s),$$

where $a_s, \phi_s \in \mathcal{K}$.

Though a one-form $\omega \in \mathcal{E}$ is, in general, given as a linear combination of the elements of $d\mathcal{C}$, it is often possible to find a linearly independent set of exact one-forms with less elements than those from $d\mathcal{C}$ in terms of which ω can be expressed.

Definition 2 (*Rank of a one-form Choquet-Bruhat et al. [4]*). We say that γ is the rank of a one-form ω, if γ is minimal number of linearly independent exact one-forms necessary to express a one-form ω.

If the rank γ of a one-form ω is one, then $\omega = \xi \, d\alpha (\xi, \alpha \in \mathcal{K})$ is clearly integrable. Thus the rank of a one-form generalizes the concept of integrability of a one-form.

Example 1. A one-form $\omega = u^{[1]} \, dy + y \, du^{[1]} + y^{[1]} \, du^{[2]}$ is a linear combination of three standard basis elements $\{dy, du^{[1]}, du^{[2]}\}$. However, one can express ω as a linear combination of two exact one-forms $d(yu^{[1]})$ and $du^{[2]}$, i.e. $\omega = d(yu^{[1]}) + y^{[1]} du^{[2]}$. Thus, one says that the rank of ω is 2.

3. I/O linearization

3.1. Problem statement

Given a control system of the form (1), we are searching for a regular dynamic output feedback of the form:

$$\begin{aligned} \eta^{[1]} &= F(\eta, y, v) \\ u &= H(\eta, y, v), \end{aligned} \quad (3)$$

where $\eta \in \Delta \subset \mathbb{R}^\rho$ and $v \in V \subset \mathbb{R}^m$ are the state and the input of the compensator (3), respectively, such that the differentials of the input–output equations of the closed-loop system satisfy the relations

$$\begin{aligned} &dy_i^{[n_i]} \in \text{span}_\mathbb{R}\{dy_\tau^{[n_{i\tau}]}, \ldots, dy_\tau, dv_j\} \\ &dy_i^{[n_i]} \notin \text{span}_\mathbb{R}\{dy_\tau^{[n_{i\tau}]}, \ldots, dy_\tau\}, \end{aligned} \quad (4)$$

This means that, for the closed-loop system, $dy_i^{[n_i]}$ is equal to a linear combination of the elements from $\{dy_\tau^{[n_{i\tau}]}, \ldots, dy_\tau, dv_j\}$ over $\mathbb{R}$. Then, after integrating, which is always possible, one gets that $y_i^{[n_i]}$ is a linear function of the variables $\{y_\tau^{[n_{i\tau}]}, \ldots, y_\tau, v_j\}$.

If there exists such a feedback, then we say that system (1) is input–output linearizable. Finally, note that we call the compensator (3) regular, if it is right-invertible. For more information, see [9].

3.2. Necessary and sufficient condition

To present a necessary and sufficient I/O linearizability condition via dynamic output feedback, we first define certain[5] one-forms, from which we find a set of functions, in terms of which the condition is formulated.

Let

$$\tilde{\omega}_i := \mathrm{d}y_i^{[n_i]} \quad \mathrm{mod}\ \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \dots, \mathrm{d}y_\tau\}.$$

Doing so, we ignore in the constructions that will follow the terms of the right-hand side of Eq. (1) that already depend linearly[6] on outputs and their forward shifts. For solvability of the I/O linearization problem, it is necessary that[7]

$$\tilde{\omega}_i \in \mathcal{E}^{n_i - r_i + 1}, \tag{5}$$

since otherwise nonlinearities appear before the input u starts to affect the output y_i. The goal of the method, described below, is to find a feedback of the form (3), such that in the closed-loop system $\mathrm{span}_{\mathbb{R}}\{\tilde{\omega}_i\} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$.

First, let ω_i, $i = 1, \dots, p_1$, be the basis elements[8] of $\mathrm{span}_{\mathbb{R}}\{\tilde{\omega}_i\}$. In the rest of this section assume that $i, \tau = 1, \dots, p_1$ and $j = 1, \dots, m$.

Let σ_i be such that

$$\omega_i \in \mathcal{E}^{\sigma_i}.$$

Next, define the one-forms

$$\overline{\omega}_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{[\sigma_i - l]}, \dots, \mathrm{d}y^{[\sigma_i - 1]}, \mathrm{d}u^{[\sigma_i - l]}, \dots, \mathrm{d}u^{[\sigma_i - 1]}\},$$

where $l = 1, \dots, \sigma_i - 1$, such that

$$\omega_i - \overline{\omega}_{i,l} \in \mathcal{E}^{\sigma_i - l} \tag{6}$$

and

$$\overline{\omega}_{i,\sigma_i} := \omega_i. \tag{7}$$

It means that the one-forms $\overline{\omega}_{i,l}$ depend on the $(\sigma_i - l)$ th and higher order terms of the one-forms ω_i. Let $\gamma_{i,l}$ be the rank of a one-form $\overline{\omega}_{i,l}$ for $l = 1, \dots, \sigma_i$. Then there exist $\gamma_{i,l}$ functions $\tilde{\phi}_{i,l}^k(y^{[\sigma_i - l]}, \dots, y^{[\sigma_i - 1]}, u^{[\sigma_i - l]}, \dots, u^{[\sigma_i - 1]})$ such that

$$\overline{\omega}_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\tilde{\phi}_{i,l}^1, \dots, \mathrm{d}\tilde{\phi}_{i,l}^{\gamma_{i,l}}\}.$$

Finally, define functions $\phi_{i,l}^k$ as a $\sigma_i - l$ step backward shift of functions $\tilde{\phi}_{i,l}^k$, i.e.

$$\phi_{i,l}^k := (\delta^{-1})^{\sigma_i - l}\tilde{\phi}_{i,l}^k = \delta^{l - \sigma_i}\tilde{\phi}_{i,l}^k$$

for $l = 1, \dots, \sigma_i$ and $k = 1, \dots, \gamma_{i,l}$.

Theorem 1. *Under the assumption (5) the system (1) is input–output linearizable by dynamic output feedback of the form (3) if and only if*

$$\dim(\mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}\phi_{i,l}^k\right\}) = \mathrm{rank}_{\mathcal{K}}\frac{\partial \phi_{i,l}^k}{\partial(u, \delta\phi_{i,l^*}^k)}, \tag{8}$$

for $l = 1, \dots, \sigma_i$, $l^* = 1, \dots, \sigma_i - 1$, $k = 1, \dots, \gamma_{i,l}$ and functions ϕ_{i,σ_i}^1 are independent from all the other functions.

Proof (*Sufficiency*). Construct the feedback that solves the input–output linearization problem in the following way. Take all the independent functions $\phi_{i,l}^k$, $l = 1, \dots, \sigma_i - 1$, $k = 1, \dots, \gamma_{i,l}$, as the states of the compensator (3), i.e.

$$\eta_{i,l,k} := \phi_{i,l}^k. \tag{9}$$

Also, let

$$v_i := \phi_{i,\sigma_i}^1. \tag{10}$$

By (8) the system of equations (9) and (10) is solvable with respect to the variables $\{u, \eta_{i,l,k}^{[1]}\}$. Note that if $p_1 < m$, the number of equations is less than that of variables, and so $m - p_1$ variables are free. Take these free variables equal to the new input v_π, $\pi = p_1 + 1, \dots, m$. Solution of the equations (9) and (10), with respect to variables $\{u, \eta_{i,l,k}^{[1]}\}$, results in a feedback of the form (3). This feedback yields, because of (7) and (10), $\omega_i = \mathrm{d}v_i$. From the definition of the one-forms ω_i and $\tilde{\omega}_i$, one concludes $\mathrm{d}y_i^{[n_i]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \dots, \mathrm{d}y_\tau, \mathrm{d}v\}$, i.e. the system (1) is input–output linearized.

Necessity: To prove the necessity of condition (8), we use the following one-forms: $\psi_{i,l} := \delta^{l - \sigma_i}\overline{\omega}_{i,l}$, $l = 1, \dots, \sigma_i$. These one-forms can be recursively computed as

$$\begin{aligned}
\psi_{i,1} &= \overline{\psi}_{i,1} \\
\psi_{i,2} &= \delta\psi_{i,1} + \overline{\psi}_{i,2} \\
&\ \vdots \\
\psi_{i,\sigma_i - 1} &= \delta\psi_{i,\sigma_i - 2} + \overline{\psi}_{i,\sigma_i - 1} \\
\psi_{i,\sigma_i} &= \delta\psi_{i,\sigma_i - 1} + \overline{\psi}_{i,\sigma_i},
\end{aligned} \tag{11}$$

where $\overline{\psi}_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y\}$, $l = 1, \dots, \sigma_i$. Also, it is obvious from the definition of one-forms $\psi_{i,l}$ that $\psi_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,l}^k\}$, where $l = 1, \dots, \sigma_i$ and $k = 1, \dots, \gamma_{i,l}$.

Because of (4), in the closed-loop system one has $\omega_i = \mathrm{d}v_i$. Since $\omega_i = \overline{\omega}_{i,\sigma_i} = \psi_{i,\sigma_i}$, one gets that $\psi_{i,\sigma_i} = \mathrm{d}v_i$. Thus, to find a feedback, that guarantees $\omega_i = \mathrm{d}v_i$, one has to take $\psi_{i,\sigma_i} = \mathrm{d}v_i$ in (11) and solve the set of equations in $\mathrm{d}u$ and $\delta\psi_{i,l}$, $l = 1, \dots, \sigma_i - 1$. Now, instead of integrating the one-forms $\psi_{i,l}$, we use the concept of rank of a one-form. Choose the state coordinates η of a feedback as the integrals of the basis elements of a one-forms $\psi_{i,l}$, i.e. $\psi_{i,l} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\eta\}$ like in (9). Since the given system is feedback linearizable, the system of equations (9) and (10) must be solvable with respect to the variables $\{u, \eta_{i,l,l}^{[1]}\}$. This means that (8) must be satisfied. □

3.3. Solution for ANARX systems

Consider a special subclass of MIMO systems, the so-called ANARX systems, which are described by the equations of the form

$$y_i^{[n_i]} = \sum_{s=1}^{n_i} \varphi_{i,s}(y_\tau^{[n_i - s]}, u_j^{[n_i - s]}; n_i - s < n_\tau). \tag{12}$$

Note that $n_i - s < n_\tau$ in (12) means that the functions $\varphi_{i,s}$ depend only on the independent variables of the field $\mathcal{K}$. In ANARX model the restrictions are imposed on the structure (1), not allowing coupling of shifts of different orders in the same term (function $\varphi_{i,s}$). The choice of an appropriate (restricted) structure is a typical approach in control to guarantee that the restricted system structure will satisfy certain properties, important for feedback construction.

Consider the case when

$$\omega_i = \tilde{\omega}_i. \tag{13}$$

Then for ANARX systems (12) the computation of functions $\phi_{i,l}^k$ in Theorem 1 is simplified. In this case, $\sigma_i = n_i - r_i + 1$ and the one-

[5] Not necessarily integrable.

[6] This is the reason that we take here span over $\mathbb{R}$ and not over $\mathcal{K}$ as below.

[7] Note that if $r_i = 1$, then the condition (5) is always satisfied.

[8] These basis elements are exact, since one-forms $\tilde{\omega}_i$ are exact.

forms $\overline{\omega}_{i,l}$ are given by

$$\overline{\omega}_{i,l} = \sum_{s=r_i}^{r_i+l-1} \mathrm{d}\varphi_{i,s},$$

where $l = 1, \ldots, n_i - r_i + 1$. Note that these one-forms are all exact, which means that $\gamma_{i,l} = 1$. Thus, functions $\phi_{i,l}^1$ are defined by

$$\phi_{i,l}^1 = \sum_{k=r_i}^{r_i+l-1} \delta^{r_i - n_i + l - 1} \varphi_{i,k}. \tag{14}$$

Note that one can alternatively define these functions recursively

$$\begin{aligned} \phi_{i,1}^1 &= \delta^{r_i - n_i} \varphi_{i,r_i} \\ \phi_{i,l}^1 &= \delta \phi_{i,l-1}^1 + \delta^{r_i - n_i + l - 1} \varphi_{i,r_i+l-1} \quad l = 2, \ldots, n_i - r_i + 1. \end{aligned} \tag{15}$$

Next, we give a simple sufficient condition for the solvability of the linearization problem for systems of the form (12).

Corollary 1. *Under the assumptions* (5) and (13), *the system* (12) *is I/O linearizable by dynamic output feedback if*

$$\dim\left(\mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}(\delta^{r_i-n_i}\varphi_{i,r_i})\right\}\right) = \mathrm{rank}_{\mathcal{K}} \frac{\partial(\delta^{r_i-n_i}\varphi_{i,r_i})}{\partial u} = p. \tag{16}$$

Proof. By (16), the functions $\phi_{i,1}^1$ are independent and (15) implies the independence of all the functions $\phi_{i,l}^1$. Also, it is easy to see that $\mathrm{rank}_{\mathcal{K}} \partial \phi_{i,l}^1 / \partial(u, \delta\phi_{i,l^*}^1)$, $l = 1, \ldots, n_i - r_i + 1$, $l^* = 1, \ldots, n_i - r_i$, is equal to the number of functions $\phi_{i,l}^1$. Thus, the condition of Theorem 1 is satisfied and therefore, system (12) is I/O linearizable by dynamic output feedback. □

By renumbering the inputs u_j, if necessary, the dynamic output feedback that linearizes the system (12) is given by

$$\begin{aligned} \eta_{i,l}^{[1]} &= \eta_{i,l+1} - \delta^{l+r_i-n_i} \varphi_{i,l+r_i}(\cdot) \\ \eta_{i,n_i-r_i}^{[1]} &= v_i - \varphi_{i,n_i}(\cdot) \\ u_i &= \delta^{r_i - n_i} \varphi_{i,r_i}^{-1}(\cdot) \\ u_s &= v_s, \end{aligned} \tag{17}$$

where $l = 1, \ldots, n_i - r_i - 1$, $s = p+1, \ldots, m$ and inverse of φ_{i,r_i} is taken with respect to the argument $u_i^{[n_i - r_i]}$.

Remark 1. In case of the SISO ANARX systems, conditions (16) and (13) are always satisfied, since by the definition of relative degree, $\delta^{r_i-n_i}\varphi_{1,r_i}$ depends on the control variable u. Then it remains only to check whether the condition (5) is satisfied.

3.4. Generalized problem statement

In this section, we generalize the problem statement of I/O linearization and then show that under the assumption that system (1) is right-invertible the conditions of Theorem 1 are also necessary and sufficient for the solvability of generalized problem.

Now, we are looking for a regular feedback of the form (3) such that the closed-loop system satisfies

$$\begin{aligned} \mathrm{d}y_i^{[n_i]} &\in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v_j^{[n_i-1]}, \ldots, \mathrm{d}v_j\} \\ \mathrm{d}y_i^{[n_i]} &\notin \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{i\tau}]}, \ldots, \mathrm{d}y_\tau\}. \end{aligned} \tag{18}$$

The difference with relations (4) is that now we also allow in the closed-loop system $y_i^{[n_i]}$ to depend on the forward-shifts of v.

Lemma 1. *Assume that system (1) is right-invertible. Then there exists a feedback of the form (3), such that (18) is satisfied for the closed-loop system if and only if the conditions of Theorem 1 are satisfied.*

Proof (*Necessity*). Assume that there exists a regular feedback such that (18) is satisfied for the closed-loop system. We show that then there exists another regular feedback, such that (4) is satisfied for the closed-loop system. The latter means that the conditions of Theorem 1 are satisfied. Clearly, since we apply regular feedback and system (1) is right-invertible, the closed-loop system is right-invertible. Next, we show that every right-invertible system satisfying (18) satisfies the conditions of Theorem 1. Since the closed-loop system is linear,

$$\begin{aligned} \phi_{i,1} &= \psi_{i,1}(u) \\ \phi_{i,l} &= \delta\phi_{i,l-1} + \psi_{i,l}(u) \end{aligned}$$

for $l = 2, \ldots, \sigma_i$, $i = 1, \ldots, p_1$ and some functions $\psi_{i,l}(\cdot)$.[9] Therefore

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,l}\}) = \mathrm{rank}_{\mathcal{K}} \frac{\partial \phi_{i,l}}{\partial(u, \delta\phi_{i,l^*})}$$

for $i = 1, \ldots, p_1$, $l = 1, \ldots, \sigma_i$ and $l^* = 1, \ldots, \sigma_i - 1$. The right-invertibility guarantees that functions ϕ_{i,σ_i} are independent from all the other functions $\phi_{i,l}$, i.e. one can define the system of equations (9) and (10). Thus, the conditions of Theorem 1 are satisfied.

Sufficiency: This is obvious. □

4. Examples

Example 2. Consider a system given by the set of I/O equations

$$\begin{aligned} y_1^{[4]} &= y_1^{[3]} + u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1 \\ y_2^{[2]} &= y_1^{[1]} u_1^{[1]} + u_3 y_2. \end{aligned} \tag{19}$$

For this system, sufficient conditions given in [8] are not satisfied, but as we show here, the conditions of Theorem 1 are satisfied. Note that relative degrees $r_1 = 2$ and $r_2 = 1$. We check the condition of Theorem 1 for system (19). Define the one-forms $\tilde{\omega}_1 = \mathrm{d}(u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1)$ and $\tilde{\omega}_2 = \mathrm{d}y_2^{[2]}$. It is easy to see that the condition (5) is satisfied in both cases. Note that for this example $\omega_i = \tilde{\omega}_i$, $i = 1,2$. Next we compute the one-forms $\overline{\omega}_{i,l}$, $i = 1,2$, $l = 1, \ldots, n_i$:

$$\begin{aligned} \overline{\omega}_{1,1} &= u_1^{[1]} \mathrm{d}(y_1^{[2]} u_1^{[2]}) \\ \overline{\omega}_{1,2} &= \mathrm{d}(u_1^{[1]} y_1^{[2]} u_1^{[2]}) + y_2 \mathrm{d}u_2^{[1]} \\ \overline{\omega}_{1,3} &= \mathrm{d}(u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1) \\ \overline{\omega}_{2,1} &= \mathrm{d}(y_1^{[1]} u_1^{[1]}) \\ \overline{\omega}_{2,2} &= \mathrm{d}(y_1^{[1]} u_1^{[1]} + u_3 y_2). \end{aligned} \tag{20}$$

From (20) it is easy to see that $\gamma_{1,2} = 2$ and $\gamma_{1,1} = \gamma_{1,3} = \gamma_{2,1} = \gamma_{2,2} = 1$. One can define the functions $\phi_{i,l}^k$, $i = 1,2$, $l = 1, \ldots, n_i$, $k = 1, \ldots, \gamma_{i,l}$ as follows:

$$\begin{aligned} \phi_{1,1}^1 &= y_1 u_1 \\ \phi_{1,2}^1 &= u_1 y_1^{[1]} u_1^{[1]} \quad \phi_{1,2}^2 = u_2 \\ \phi_{1,3}^1 &= u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1 \\ \phi_{2,1}^1 &= y_1 u_1 \\ \phi_{2,2}^1 &= y_1^{[1]} u_1^{[1]} + u_3 y_2. \end{aligned}$$

Now, since $\phi_{1,1}^1 = \phi_{2,1}^1$ and all other functions depend on some different independent variables:

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{1,1}^1, \mathrm{d}\phi_{1,2}^1, \mathrm{d}\phi_{1,2}^2, \mathrm{d}\phi_{1,3}^3, \mathrm{d}\phi_{2,1}^1, \mathrm{d}\phi_{2,2}^1\}) = 5.$$

Also,

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(\phi_{1,1}^1, \phi_{2,1}^1, \phi_{1,2}^1, \phi_{1,2}^2, \phi_{2,2}^1, \phi_{1,3}^1)^T}{\partial(u, \delta\phi_{1,1}^1, \delta\phi_{2,1}^1, \delta\phi_{1,2}^1, \delta\phi_{1,2}^2)}$$

[9] Note that here we write $\phi_{i,l}$ instead of $\phi_{i,l}^k$ since k is equal to 1 for all the functions.

$$= \text{rank}_{\mathcal{K}} \begin{pmatrix} y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta\phi^1_{1,1} & 0 & 0 & u_1 & u_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 1 & 1 & 0 & 0 \\ y_2 & 0 & 0 & 0 & 0 & 1 & y_2 \end{pmatrix} = 5,$$

i.e. everywhere except when $y_1 = 0$, $y_2 = 0$ or $u_1 = 0$.[10] Thus, the condition (8) is satisfied. The feedback of the form (3) can be found by taking $\eta_{i,l,k}$ and new input v as follows:

$$\eta_{1,1,1} = \phi^1_{1,1} = y_1 u_1$$
$$\eta_{1,2,1} = \phi^1_{1,2} = u_1 y_1^{[1]} u_1^{[1]}$$
$$\eta_{1,2,2} = \phi^2_{1,2} = u_2$$
$$v_1 = \phi^1_{1,3} = u_1^{[1]} y_1^{[2]} u_1^{[2]} + y_2 u_2^{[1]} + y_2 u_1$$
$$v_2 = \phi^1_{2,2} = y_1^{[1]} u_1^{[1]} + u_3 y_2.$$

This set of equations has to be solved with respect to variables $\{\eta^{[1]}_{1,1,1}, \eta^{[1]}_{1,2,1}, \eta^{[1]}_{1,2,2}, u_1, u_2, u_3\}$. Since there are five equations, but six unknowns, then one unknown, for example $\eta^{[1]}_{1,2,2}$, will remain free. This variable will be taken equal to the new input v_3. To conclude, the feedback

$$\eta^{[1]}_{1,1,1} = \frac{y_1 \eta_{1,2,1}}{\eta_{1,1,1}}$$
$$\eta^{[1]}_{1,2,1} = v_1 - y_2 v_3 - \frac{y_2 \eta_{1,1,1}}{y_1}$$
$$\eta^{[1]}_{1,2,2} = v_3$$
$$u_1 = \frac{\eta_{1,1,1}}{y_1}$$
$$u_2 = \eta_{1,2,2}$$
$$u_3 = \frac{v_2 \eta_{1,1,1} - y_1 \eta_{1,2,1}}{y_2 \eta_{1,1,1}}$$

solves the input–output linearization problem for system (19).

Example 3. Consider the 'Ball and Beam' system, given for example in [5].

$$\left(\frac{J}{R^2} + m\right)\ddot{y} + mg \sin u - my\dot{u}^2 = 0, \tag{21}$$

where output y is the position of the ball, input u is the angle between the beam and horizontal plane. The parameters of the system have the following meaning: J, R and m are the inertia, radius and mass of the ball, respectively, and g is the gravitational constant.

Eq. (21) can be written as

$$\ddot{y} = c_1 \sin u + c_2 y \dot{u}^2, \tag{22}$$

where the parameters c_1 and c_2 are certain functions of system parameters. Time-discretization of (22) yields the following equations:

$$y^{[2]} = 2y^{[1]} - y + c_1 \sin u + c_2 y((u^{[1]})^2 - 2uu^{[1]} + u^2). \tag{23}$$

Note that by the results of [12] the model (23) is not transformable into the state space form. Next, we find the feedback of the form (3), that linearizes Eq. (23).. First, define

$$\omega = (2c_2 y u^{[1]} - 2c_2 y u)\, du^{[1]} + (c_1 \cos u - 2c_2 y u^{[1]} + 2c_2 y u)\, du + (c_2 (u^{[1]})^2 - 2c_2 u u^{[1]} + c_2 u^2)\, dy.$$

[10] Note that $\delta\phi^1_{1,1}$ is zero only if $y_1 = 0$ or $u_1 = 0$.

Since the relative degree of the output y is 1, condition (5) is automatically satisfied for ω. Define, according to (6) and (7) two one-forms

$$\varpi_1 = (2c_2 y u^{[1]} - 2c_2 y u)\, du^{[1]}$$
$$\varpi_2 = \omega.$$

Obviously, the rank of both one-form is one and $\varpi_1 \in \text{span}_{\mathcal{K}}\{du^{[1]}\}$, $\varpi_2 \in \text{span}_{\mathcal{K}}\{d(c_1 \sin u + c_2 y((u^{[1]})^2 - 2uu^{[1]} + u^2)\}$. Therefore, integrating ϖ_1 and ϖ_2, one obtains $\tilde{\phi}^1_{1,1} = u^{[1]}$, thus $\phi^1_{1,1} = u$, and $\tilde{\phi}^1_{1,2} = \phi^1_{1,2} = c_1 \sin u + c_2 y(u^{[1]} - u)^2$. It is easy to see that the condition (8) is satisfied for $\phi^1_{1,1}$ and $\phi^1_{1,2}$. Following the procedure in the proof of Theorem 1, denote

$$\eta := u$$
$$v := c_1 \sin \eta + c_2 y(\eta^{[1]} - \eta)^2$$

that results in the feedback

$$\eta^{[1]} = \pm\sqrt{\frac{v - c_1 \sin \eta}{c_2 y}} + \eta$$
$$u = \eta.$$

Note that, to be able to calculate the square root, one has to guarantee that the position of the ball $y > 0$. Then $v - c_1 \sin \eta = c_2 y(\eta^{[1]} - \eta)^2 > 0 (c_2 = mR^2/(J + mR^2) > 0)$ and the expression under the square root is also positive. The equation of the closed-loop system is

$$y^{[2]} = 2y^{[1]} - y + v. \tag{24}$$

Example 4. As an example of ANARX system, consider the model of the liquid level system of interconnected tanks [3], defined by the I/O equation

$$y^{[3]} = 0.43y^{[2]} + 0.681y^{[1]} - 0.149y + 0.396u^{[2]} + 0.014u^{[1]} - 0.071u - 0.351y^{[2]}u^{[2]} - 0.03(y^{[1]})^2 - 0.135y^{[1]}u^{[1]} - 0.027(y^{[1]})^3 - 0.108(y^{[1]})^2 u^{[1]} - 0.099(u^{[1]})^3. \tag{25}$$

Since it is a SISO system, by Remark 1, the only condition one has to check is (5). The latter is satisfied, because the relative degree of y is 1. Now, according to (17) one can write down the equations of the compensator

$$\eta_1^{[1]} = \eta_2 - 0.681y + 0.03y^2 + 0.027y^3 - (0.014 - 0.135y - 0.108y^2)\frac{\eta_1 - 0.43y}{0.396 - 0.351y} + \left(\frac{0.463\eta_1 - 0.199y}{0.183 - 0.163y}\right)^3$$
$$\eta_2^{[1]} = v + 0.149y + \frac{0.071\eta_1 - 0.031y}{0.028 - 0.025y}$$
$$u = \frac{\eta_1 - 0.43y}{0.396 - 0.351y} \tag{26}$$

yielding the closed-loop system equation $y^{[3]} = v$.

5. Conclusions

The necessary and sufficient conditions were given to linearize the set of higher order nonlinear I/O difference equations by dynamic output feedback. The main results are specified for a subclass of ANARX systems. The future goal is to apply the obtained results to solve the disturbance decoupling and I/O decoupling problems by dynamic output feedback.

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Input-output decoupling of discrete-time nonlinear systems by measurement feedback

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***Abstract*— This paper addresses the input-output decoupling problem for discrete-time nonlinear systems by measurement feedback. Necessary and sufficient conditions are given to solve the problem by static or dynamic measurement feedback, respectively. Since the dynamic measurement solution presented here depends on the solution of the input-output linearization problem, a sufficient condition for linearizability of certain functions is also given. Finally, the derived conditions are specified to solve the problem by output feedback.**

I. Introduction

The necessary and sufficient conditions for solvability of the input-output (i/o) decoupling problem by state feedback were given already in [10], [11] for continuous-time systems and in [7], [9] for discrete-time systems. The purpose of this paper is to consider a case when all the states are not available for measurement. Then one has to consider a different feedback, an output feedback or a measurement feedback, where only some functions of states are measured.

In the nonlinear case only few papers address the i/o decoupling problem by measurement or output feedback. Necessary and sufficient conditions have been given to solve the problem for continuous-time systems by static measurement feedback in [3] and for discrete-time systems by static output feedback in [8]. Theorem 1 below is the analogue of the conditions in [3], whereas in [8] the special case was studied when the controlled and measured outputs coincide. To our best knowledge, dynamic measurement feedback solution is looked for only in [12], where solvability conditions are given that depend explicitly on linearizability property of certain functions. These conditions are only sufficient since linearizability property is specified in the theorem via sufficient linearizability conditions.

The goal of this paper is to solve the i/o decoupling problem by dynamic measurement feedback for discrete-time systems. In this paper we use similar approach and the same mathematical tools as in [3], [8], [12]. However, compared to [12], weaker linearizability conditions are used in this paper. Necessary and sufficient solvability conditions of i/o decoupling problem are given, which can be specified to the dynamic output feedback case. The problem of i/o linearization is also briefly mentioned in the paper, since it is an important part of the solution, given in this paper. Unlike in state feedback case, here one can not use all the states in feedback. Thus, we divide some shifts of controlled outputs into two parts: the one, which has to be compensated and the one, which does not. Finally, a feedback is found, that compensates the part which needs to be compensated, by linearizing certain functions, whenever it is possible. Notice that the obtained necessary and sufficient condition can be generalized to continuous-time systems very easily.

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II. Preliminaries

Consider a discrete-time nonlinear system, described by the equations

$$\begin{aligned} x(t+1) &= f(x(t),u(t)) \\ y_*(t) &= h_*(x(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the input, $y_*(t) \subset Y \in \mathbb{R}^m$ is the controlled output and $y(t) \subset Z \in \mathbb{R}^q$ is the measured output. It is assumed that the functions f, h_* and h are meromorphic. Also, we assume, that the system (1) is submersive, meaning that generically, i.e. everywhere except on a set of measure zero,

$$\operatorname{rank}\Big[\frac{\partial f}{\partial(x(t),u(t))}\Big] = n. \tag{2}$$

In this paper, the following notations are used. Instead of $x(t)$ and $x(t+k)$ ($k \geq 1$) we use x and $x^{[k]}$, respectively. The same notations are used for the other variables. Throughout the paper it is assumed that $i = 1,\dots,m$.

Extend the map $f : (x,u) \mapsto x^{[1]}$ to the map $\tilde{f} : (x,u) \mapsto (x^{[1]}, z)$, where $z = \chi(x,u)$, $z \in \mathbb{R}^m$, such that $\tilde{f}$ is generically invertible. Let $\mathcal{K}$ be the field of meromorphic functions in finite number of variables from the set $\mathcal{C} = \{x, u^{[k]}, z^{[-l]}; k \geq 0, l > 0\}$. Introduce the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$, defined by equations (1); in particular $\delta x = f(x,u)$. Moreover, $\delta u^{[k]} = u^{[k+1]}$ ($k \geq 0$), $\delta z^{[-1]} = \chi(x,u)$, $\delta z^{[-l]} = z^{[-l+1]}$ ($l > 1$) and

$$\delta\varphi(x,u,\dots,u^{[k]},z^{[-1]},\dots,z^{[-l]}) = \varphi(f(x,u),u^{[1]},\dots,u^{[k+1]},\chi(x,u),\dots,z^{[-l+1]}).$$

Since $\tilde{f}$ is invertible, one can also define inverse of operator δ, called backward-shift operator δ^{-1}, as $\delta^{-1}x = \tilde{f}^{-1}(x,z^{[-1]})$, $\delta^{-1}u^{[k]} = u^{[k-1]}$ ($k \geq 0$), $\delta^{-1}z^{[-l]} = z^{[-l-1]}$ ($l > 1$) and

$$\delta^{-1}\varphi(x,u,\dots,u^{[k]},z^{[-1]},\dots,z^{[-l]}) = \varphi(\tilde{f}^{-1}(x,z^{[-1]}),u,\dots,u^{[k-1]},z^{[-2]},\dots,z^{[-l-1]}).$$

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Since the operator δ is an automorphism of the field $\mathcal{K}$, the pair $(\mathcal{K}, \delta)$ is an inversive difference field, see [1], and denoted simply by $\mathcal{K}$.

Introduce the set of symbols $\mathrm{d}\mathcal{C} = \{\mathrm{d}x, \mathrm{d}u^{[k]}, \mathrm{d}z^{[-l]}; k \geq 0, l > 0\}$. Let $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{C}\}$ be the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$. The elements of $\mathcal{E}$, i.e.

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i + \sum_{k\geq 0}\sum_{j=1}^{m} b_{kj}\mathrm{d}u_j^{[k]} + \sum_{l>0}\sum_{\rho=1}^{m} c_{l\rho}\mathrm{d}z_\rho^{[-l]}$$

where only finite number of coefficients $a_i, b_{kj}, c_{l\rho} \in \mathcal{K}$ are nonzero, are called one-forms. A one-form ω is called exact if it is a differential of some function $\varphi \in \mathcal{K}$, i.e. $\omega = \mathrm{d}\varphi$. The operators δ and δ^{-1} are extended to $\mathcal{E}$ by the rules

$$\begin{aligned} \delta\big(\sum_j a_j \mathrm{d}\varphi_j\big) &= \sum_j \delta(a_j)\mathrm{d}(\delta\varphi_j) \\ \delta^{-1}\big(\sum_j a_j \mathrm{d}\varphi_j\big) &= \sum_j \delta^{-1}(a_j)\mathrm{d}(\delta^{-1}\varphi_j), \end{aligned}$$

where $a_j, \varphi_j \in \mathcal{K}$. Let $y_* = (y_{*1}, \ldots, y_{*m})$ be the controlled output vector of the system (1). The relative degree r_i of an output y_{*i} is defined by $r_i := \min\{k \in \mathbb{N} \mid \mathrm{d}y_{*i}^{[k]} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}\}$. If there does not exist such integer k, then set $r_i := \infty$. We also define some subspaces of $\mathcal{E}$, i.e. $\mathcal{E}^k = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[k-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[k-1]}\}$ for every $k \in \mathbb{N}$ and $\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$.

In general, a one-form ω is a linear combination over $\mathcal{K}$ of certain number of standard basis elements of $\mathcal{E}$, i.e. $\mathrm{d}\mathcal{C}$. However, it is often possible to find a linearly independent set of exact one-forms, with less elements than those basis elements of $\mathcal{E}$, in terms of which ω can be expressed.

Definition 1: A minimal number $\gamma \in \mathbb{N}$ so that there exist γ linearly independent exact one-forms such that ω is a linear combination of these exact one-forms, is called the rank of a one-form ω.

If the rank of a one-form $\omega \in \mathcal{E}$ is equal to 1, then by definition 1 there exist $\lambda, \varphi \in \mathcal{K}$ such that $\omega = \lambda\mathrm{d}\varphi$, i.e. the one-form ω is integrable.

We say that system (1) is right-invertible with respect to the output y_* if there exist $j_i \in \mathbb{N}$ such that

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(h_{*1}(x^{[j_1]}), \ldots, h_{*m}(x^{[j_m]}))^T}{\partial u} = m, \tag{3}$$

where $h_*(x) = (h_{*1}(x), \ldots, h_{*m}(x))$, see [2] for more information. Also, let $j_{max} := \max\{j_1, \ldots, j_m\}$.

As in [12], we define for each output component y_{*i} a subspace Ω_i of $\mathcal{X}$ in the following way:

$$\begin{aligned} \Omega_i = \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \delta^k\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \\ \ldots, \mathrm{d}y_{*i}^{[r_i+k-1]}\}\}. \end{aligned} \tag{4}$$

The subspaces Ω_i are essential to solve the input-output decoupling problem. It is because the forward-shifts of the elements of Ω_i do not depend on the input u explicitly. Suppose $\Omega_i = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_l\}$. We define the forward-shift of subspace Ω_i elementwise by $\Omega_i^{[1]} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1^{[1]}, \ldots, \mathrm{d}\theta_l^{[1]}\}$. Denote $\Omega_i^{[0]} := \Omega_i$, and $\Omega_i^{[k]} := (\Omega_i^{[k-1]})^{[1]}$. The following lemma gives a procedure for computing the subspaces Ω_i.

Lemma 1: [4] The subspace Ω_i may be computed as the limit of the following algorithm:

$$\begin{aligned} \Omega_i^0 &= \mathcal{X} \\ \Omega_i^{k+1} &= \{\omega \in \Omega_i^k \mid \delta\omega \in \Omega_i^k + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{*i}^{[r_i]}\}\}. \end{aligned} \tag{5}$$

III. MAIN RESULTS

A. Problem statement

One says that system (1) is i/o decoupled if every controlled output y_{*i} depends on exactly one input variable u_i, i.e. the relative degrees r_i are finite and

$$\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-r_i]}\} \quad k \geq r_i.$$

The next lemma gives the necessary and sufficient condition for a system to be i/o decoupled.

Lemma 2: Under the assumption that $r_i < \infty$, for $i = 1, \ldots, m$, the system (1) is input-output decoupled iff

$$\mathrm{d}y_{*i}^{[r_i]} \in \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i\}. \tag{6}$$

Proof: Necessity. If the system (1) is input-output decoupled, then

$$\mathrm{d}y_{*i}^{[r_i]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i\}.$$

Thus, there exists $\omega_i \in \mathcal{X}$ and $\lambda_i \in \mathcal{K}$ such that $\mathrm{d}y_{*i}^{[r_i]} = \omega_i + \lambda_i\mathrm{d}u_i$. We will show that $\omega_i \in \Omega_i$. Note that

$$\delta^\sigma\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\} \tag{7}$$

for every $\sigma \in \mathbb{N}$. Since $\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-r_i]}\}$ for $k \geq 0$, then

$$\begin{aligned} \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\} = \\ \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+\sigma-1]}\}. \end{aligned} \tag{8}$$

Thus, (7) and (8) give

$$\delta^\sigma\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+\sigma-1]}\}$$

for every $\sigma \in \mathbb{N}$, which means that $\omega_i \in \Omega_i$.

Sufficiency. By Lemma 1 and (6), one gets

$$\Omega_i^{[1]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{*i}^{[r_i]}\} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i\}.$$

Thus, $\Omega_i^{[k]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-1]}\}$ and therefore,

$$\begin{aligned} \mathrm{d}y_{*i}^{[r_i+k]} &\in \Omega_i^{[k]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i^{[k]}\} \\ &\subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\} \\ &\subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\}, \end{aligned}$$

which means, that system (1) is i/o decoupled. ■

The input-output decoupling problem can be formulated as follows. Find a regular dynamic measurement feedback of the form

$$\begin{aligned} \eta^{[1]} &= F(\eta, z, v) \\ u &= H(\eta, z, v), \end{aligned} \tag{9}$$

where $v \subset V \in \mathbb{R}^m$ is the new input and $\eta \subset \Delta \in \mathbb{R}^\rho$ is the state of the feedback, such that the closed-loop system (1),(9) satisfies the following conditions:

(i) the relative degree $\bar{r}_i$ of output y_{*i} of the closed-loop system is finite;
(ii) $\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v_i, \ldots, \mathrm{d}v_i^{[k-\bar{r}_i]}\}$ for $k \geq \bar{r}_i$.

Condition (ii) guarantees that outputs y_{*i} are decoupled, i.e. different outputs y_{*i} are affected by different inputs v_i at every time instant $k \geq \bar{r}_i$. By regularity of feedback we mean that (9) defines generically the (z, η)-dependent one-to-one correspondence between the variables v and u. Feedback (9) is called static if $\rho := \dim \eta = 0$.

Since the main Theorem of this paper, given below, depends on the solution of the i/o linearization problem, we give first the problem statement for the i/o linearization. For more information, see [5].

B. Input-output linearization

In this section, we consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the set of i/o difference equations

$$y_l^{[n_l]} = \Phi_l(y_\tau, \ldots, y_\tau^{[n_{l\tau}]}, u_i, \ldots, u_i^{[n_l-1]}) \tag{10}$$

for $l, \tau = 1, \ldots, q$, where Φ_l are supposed to be meromorphic functions of their arguments and the indices in (10) satisfy the relations

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_q, \quad n_{l\tau} < n_\tau \\ &n_{l\tau} < n_l, \quad \tau \leq l \\ &n_{l\tau} \leq n_l, \quad \tau > l. \end{aligned} \tag{11}$$

Like above, we assume, that system (10) is submersive, which for the i/o model means that the map $\Phi = (\Phi_1, \ldots, \Phi_q)^T$ satisfies generically the condition

$$\mathrm{rank}\Big[\frac{\partial \Phi}{\partial(y, u)}\Big] = q,$$

where $y = (y_1, \ldots, y_q)$ and $u = (u_1, \ldots, u_m)$.

One says that equations (10) are linearizable by regular dynamic output feedback of the form (9), if the differentials $\mathrm{d}y_l^{[n_l]}$, defined by the input-output equations of the closed-loop system, satisfy the relations

$$\mathrm{d}y_l^{[n_l]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{l\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v\} \tag{12}$$

for $l, \tau = 1, \ldots, q$. In the case, when

$$\mathrm{d}y_l^{[n_l]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}v\}$$

for $l = 1, \ldots, q$, equations (10) are said to be strictly linearizable.

One says that the set of functions $\varphi_i(z, \ldots, z^{[s-1]}, u, \ldots, u^{[s-1]})$, $i = 1, \ldots, m$, are linearizable (strictly linearizable) if the system of equations

$$y_i^{[s]} = \varphi_i(y, \ldots, y^{[s-1]}, u, \ldots, u^{[s-1]})$$

is linearizable (strictly linearizable).

C. Input-output decoupling

First, we give a solution to the input-output decoupling problem by static measurement feedback. Let $h_*(x) = (h_{*1}(x), \ldots, h_{*m}(x))$.

Theorem 1: Let the relative degrees r_i of outputs y_i be finite. Then the system (1) is input-output decoupable by static measurement feedback iff

(i)
$$\mathrm{rank}_{\mathcal{K}}\Big[\frac{\partial(h_{*1}(x^{[r_1]}), \ldots, h_{*m}(x^{[r_m]}))^T}{\partial u}\Big] = m;$$

(ii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}u\}$ with rank 1, such that $\mathrm{d}y_{*i}^{[r_i]} - \omega_i \in \Omega_i$.

The proof of Theorem 1 is given in [8] for the case when $y = y_*$. The proof of the general case is similar.

In Theorem 2 below, the necessary and sufficient conditions for solvability of the input-output decoupling problem by dynamic measurement feedback are given, relaxing the conditions of Theorem 1.

Theorem 2: The system (1) is input-output decoupable by dynamic measurement feedback (9) iff the following conditions are satisfied:

(i) the system (1) is right-invertible with respect to the controlled outputs y_{*i};

(ii) there exists $s \geq j_{max} - r_i + 1$ such that[1]
$$\mathrm{d}y_{*i}^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\};$$

(iii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ with rank 1 such that
$$\mathrm{d}y_{*i}^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]};$$

(iv) for $\omega_i = \lambda_i \mathrm{d}\varphi_i$, $\lambda_i, \varphi_i \in \mathcal{K}$, the functions $\varphi_i(y, \ldots, y^{[s-1]}, u, \ldots, u^{[s-1]})$ are strictly linearizable by dynamic feedback.

Proof. Necessity. Let $s \geq 1$ be such that in the closed-loop system the relative degree $\bar{r}_i$ of output y_{*i} is $\bar{r}_i = r_i + s - 1$. By Lemma 2 and the fact that the closed-loop system is i/o decoupled,

$$\mathrm{d}y_{*i}^{[\bar{r}_i]} \in \bar{\Omega}_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}, \tag{13}$$

where $\bar{\Omega}_i$ is the subspace Ω_i for the closed-loop system. Next, we show that $\bar{\Omega}_i = \Omega_i + \cdots + \Omega_i^{[s-1]}$. From the definition of the subspace Ω_i,

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+s-2]}\}.$$

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}. \tag{14}$$

Thus,

$$\begin{aligned} \Omega_i + \cdots + \Omega_i^{[s-1]} &= \{\bar{\omega} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ &\bar{\omega}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_{*i}^{[r_i+s-1]}, \ldots, \mathrm{d}y_{*i}^{[r_i+s-k-2]}\}\} \\ &= \bar{\Omega}_i. \end{aligned}$$

[1]Note that, one can, in principle, search, instead of the joint index s, a separate s_i that satisfies $s_i \geq j_{max} - r_i + 1$. Then s can be taken as $s = \max_i\{s_i\}$.

The last equality comes from the definition (4) of the subspace $\bar{\Omega}_i$. Therefore, (13) becomes

$$\mathrm{d}y_{*i}^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}.$$

Then one can define the one-forms $\omega_i = \lambda_i \mathrm{d}v_i$ such that $\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]}$. Now, conditions (ii) and (iii) must be satisfied, since otherwise v_i would depend on x', where $\mathrm{d}x' \in \mathcal{X}$ and $\mathrm{d}x' \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y\}$, i.e. the feedback would not be measurement feedback. Since conditions (ii) and (iii) are satisfied, $\omega_i = \lambda_i \mathrm{d}\varphi_i(u, \ldots, u^{[s-1]}, y, \ldots, y^{[s-1]})$ for some functions φ_i. Note that under the feedback $\omega_i = \lambda_i \mathrm{d}v_i$, i.e. the functions φ_i are strictly linearizable.

Sufficiency. We show that the feedback that linearizes strictly functions φ_i in (iv), solves the i/o decoupling problem.

Since for the closed-loop system $\mathrm{d}\varphi_i = \mathrm{d}v_i$, the relative degree of output y_{*i} is $r_i + s - 1$. Thus

$$\mathrm{d}y_{*i}^{[r_i+j]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \tag{15}$$

for $j = 0, \ldots, s-2$. From the definition (4) of the subspace Ω_i one concludes $\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}$.

Now, as in the necessity part, one can show that $\Omega_i + \cdots + \Omega_i^{[s-1]} = \bar{\Omega}_i$, where $\bar{\Omega}_i$ is the subspace Ω_i for the closed-loop system. Therefore, by (ii), (iii) and (iv), $\mathrm{d}y_{*i}^{[r_i+s-1]} \in \bar{\Omega}_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}$. By Lemma 2, system (1) is i/o decoupled. ■

Next, we give a simple sufficient condition for the strict linearizability of functions φ_i in (iv) of Theorem 2. For general input-output linearization problem, see [6].

Theorem 3: Functions φ_i in (iv) of Theorem 2 are strictly linearizable by dynamic measurement feedback if there exist functions $\phi_{i,j} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[j-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[j-1]}\}$ for $j = 1, \ldots, s$, such that

$$\begin{aligned} \mathrm{d}\varphi_i &= \mathrm{d}\phi_{i,s}(\cdot, \delta\phi_{\mu,\nu}, y, u; \nu = 1, \ldots, s-1) \\ &\quad \circ \delta\phi_{i,s-1}(\cdot, \delta\phi_{\mu,\nu}, y, u; \nu = 1, \ldots, s-2) \circ \cdots \\ &\quad \cdots \circ \delta\phi_{i,1}(y, u) \end{aligned} \tag{16}$$

where $\mu = 1, \ldots, m$ and

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,j}, 1 \le j \le j_i\}) = m. \tag{17}$$

Proof: By condition (i) of Theorem 2 the indices j_i, defined by (3), are finite. Then set[2]

$$\eta_{i,\tau} = \phi_{i,\tau}(\cdot) \tag{18}$$
$$v_i = \phi_{i,s}(\cdot) \tag{19}$$

for $\tau = j_i, \ldots, s-1$. Because of (17), one can find a dynamic measurement feedback by solving (18), (19) with respect to the variables u and $\eta_{i,\tau}^{[1]}$, $\tau = j_i, \ldots, s-1$. Then, from (19) and (16) one concludes that in the closed-loop system $\mathrm{d}\varphi_i = \mathrm{d}v_i$. Thus, the functions φ_i are strictly linearizable. ■

[2]Note that here $\tau = j_i, \ldots, s-1$. This is because functions $\phi_{i,j}$, $j = 1, \ldots, j_i - 1$ depend by (3) and (17) on functions ϕ_{i,j_i}.

The main difficulty in checking the conditions of Theorem 3 is related to finding the functions $\phi_{i,j}$, $j = 1, \ldots, s$. Below an algorithm is given for searching such functions whenever they exist. The algorithm is based on the input-output linearization algorithm, introduced in [5]. The main difference is that the one-forms $\bar{\omega}_{i,p}$ are defined differently to make the Algorithm more transparent, and the number of functions φ_i equals to the number of inputs u_i.

Algorithm.

Step 0. Find the one-forms ω_i, defined in condition (iii) and (ii) of Theorem 2.

Step 1. Let

$$\begin{aligned} \bar{\omega}_{i,1} &= \sum_{\mu=1}^{q} \alpha_{i,1,\mu} \mathrm{d}y_\mu^{[s-1]} + \sum_{j=1}^{m} \beta_{i,1,j} \mathrm{d}u_j^{[s-1]} \\ &\quad \alpha_{i,1,\mu}, \beta_{i,1,j} \in \mathcal{K} \end{aligned}$$

be such that $\omega_i - \bar{\omega}_{i,1} \in \mathcal{E}^{s-1}$. Check whether $\gamma_{i,1} := \mathrm{rank}\ \bar{\omega}_{i,1} = 1$. If not, then stop, the conditions of Theorem 3 are not satisfied. Otherwise, let $\phi_{i,1}$ be such that $\bar{\omega}_{i,1} = \pi_i \mathrm{d}(\delta^{s-1}\phi_{i,1})$ for some $\pi_i \in \mathcal{K}$.

Step p. ($p = 2, \ldots, s-1$) Let

$$\begin{aligned} \bar{\omega}_{i,p} &= \sum_{\mu=1}^{q} \alpha_{i,p,\mu} \mathrm{d}y_\mu^{[s-p]} + \sum_{j=1}^{m} \beta_{i,p,j} \mathrm{d}u_j^{[s-p]} + \sum_{l=1}^{p-1} \bar{\omega}_{i,l}, \\ &\quad \alpha_{i,p,\mu}, \beta_{i,p,j} \in \mathcal{K} \end{aligned}$$

be such that $\omega_i - \bar{\omega}_{i,p} \in \mathcal{E}^{s-p}$. Check whether

$$\mathrm{d}\bar{\omega}_{i,p} \wedge \bar{\omega}_{i,p} \wedge \mathrm{d}(\delta^k \phi_{j,l}) = 0,$$

where $j = 1, \ldots, m$, $l = 1, \ldots, p-1$ and $k = s-p, \ldots, s-1$. If not, then stop. Otherwise, there exist $\gamma, \theta_{j,l} \in \mathcal{K}$ such that

$$\hat{\omega}_{i,p} := \gamma\big(\bar{\omega}_{i,p} + \sum_{j=1}^{m}\sum_{l=1}^{p-1} \theta_{j,l} \mathrm{d}(\delta^k \phi_{j,l})\big)$$

is exact ($k = s-p, \ldots, s-1$). Then define $\phi_{i,p}$ such that $\hat{\omega}_{i,p} = \mathrm{d}(\delta^{s-p}\phi_{i,p})$.

Step s. Define $\phi_{i,s} = \varphi_i$, i.e. such that $\omega_i = \lambda_i \mathrm{d}\phi_{i,s}$ for some $\lambda_i \in \mathcal{K}$. End of the algorithm.

In the case when the controlled output y_* is measurable, i.e. $y_* = y$, one gets from Theorems 2 and 3 the following two corollaries.

Corollary 1: Under the assumption that the relative degrees r_i of outputs y_{*i} are finite, the system of the form (1) is i/o decoupable by dynamic output feedback iff the following conditions are satisfied

(i) the system (1) is right-invertible with respect to controlled outputs y_{*i};

(ii) there exists $s \ge j_{max} - r_i + 1$ such that
$$\begin{aligned} \mathrm{d}y_{*i}^{[r_i+s-1]} &\in \Omega_i + \cdots + \Omega_i^{[s-1]} \\ &\quad + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}; \end{aligned}$$

(iii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ with rank 1 such that
$$\mathrm{d}y_{*i}^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]};$$

(iv) for $\omega_i = \lambda_i \mathrm{d}\varphi_i$, the functions $\varphi_i(y_*, \ldots, y_*^{[s-1]}, u, \ldots, u^{[s-1]})$ are strictly linearizable by dynamic feedback.

Corollary 2: Functions φ_i in (iv) of Corollary 1 are strictly linearizable by dynamic output feedback if there exist functions $\phi_{i,j} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[j-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[j-1]}\}$, $j = 1, \ldots, s$, such that

$$\begin{aligned} \mathrm{d}\varphi_i &= \mathrm{d}\phi_{i,s}(\cdot, \delta\phi_{\mu,\nu}, y_*, u; \nu = 1, \ldots, s-1) \\ &\quad \circ \delta\phi_{i,s-1}(\cdot, \delta\phi_{\mu,\nu}, y_*, u; \nu = 1, \ldots, s-2) \circ \cdots \\ &\quad \cdots \circ \delta\phi_{i,1}(y_*, u) \end{aligned}$$

where $\mu = 1, \ldots, m$ and

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,j}, 1 \le j \le j_i\}) = m.$$

Consider the system (1) without measured output y. In [7], it has been shown that such system can be i/o decoupled by state feedback if and only if it is right invertible, i.e condition (3) is satisfied. Next, we explain briefly that, when we take $y = x$, the conditions of Theorem 2 become necessary and sufficient condition for i/o decoupling problem by state feedback. For this, we show, that in the case of $y = x$, the conditions (ii), (iii) and (iv) of Theorem 2 are always satisfied.

Note that, when $y = x$, then $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ and since $\mathrm{d}y_{*i}^{[r_i+s-1]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ for $i = 1, \ldots, m$, the condition (ii) is always satisfied. Also, one can take $\omega_i = \mathrm{d}y_{*i}^{[r_i+s-1]}$, then condition (iii) of Theorem 2 is satisfied. It also means that $\varphi_i = y_{*i}^{[r_i+s-1]}$ in condition (iv) of Theorem 2 and these functions are linearizable if and only if the given system is right-invertible. Thus, condition (iv) of Theorem 2 is always satisfied if system (1) with $y = x$ is right-invertible.

IV. Example

Consider a system described by the difference equations

$$\begin{aligned} x_1^{[1]} &= (x_3 + x_4)u_1 - x_2 \\ x_2^{[1]} &= \frac{u_1 x_5}{x_4} + x_1 \\ x_3^{[1]} &= x_1 x_3 \\ x_4^{[1]} &= (x_3 + x_4)u_1 x_5 \\ x_5^{[1]} &= \frac{u_2 x_5}{x_4} \\ y_{*1} &= x_1, \quad y_{*2} = x_4 \\ y_1 &= x_3 + x_4 \quad y_2 = \frac{x_5}{x_4}. \end{aligned} \tag{20}$$

We check if the conditions of Theorem 2 are satisfied for system (20). First, note that the relative degrees of outputs y_1 and y_2 are $r_1 = r_2 = 1$. Since

$$\begin{aligned} y_{*1}^{[1]} &= (x_3 + x_4)u_1 - x_2 \\ y_{*2}^{[2]} &= \left(y_{*1}^{[2]} + x_1 + \frac{u_1 x_5}{x_4}\right)\frac{u_2 x_5}{x_4}, \end{aligned}$$

one gets $\mathrm{rank}_{\mathcal{K}} \frac{\partial (y_{*1}^{[1]}, y_{*2}^{[2]})^T}{\partial u} = 2$. Therefore, the system (20) is right-invertible and $j_1 = 1$, $j_2 = 2$. The subspaces Ω_i are, according to Lemma 1, $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_3\}$ and $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_4\}$.

Since s has to satisfy the inequalities $s \ge j_{max} - r_i + 1$ for $i = 1, 2$, the first choice for s is $s = 2$. Compute

$$\begin{aligned} \mathrm{d}y_{*1}^{[2]} &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} - y_2\mathrm{d}u_1 - u_1\mathrm{d}y_2 - \mathrm{d}x_1 \\ &\in \Omega_1 + \Omega_1^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y, \mathrm{d}u^{[1]}, \mathrm{d}y^{[1]}\} \\ \mathrm{d}y_{*2}^{[2]} &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]} + y_2 u_1^{[1]} y_1^{[1]}\mathrm{d}u_2 \\ &\quad + u_2 u_1^{[1]} y_1^{[1]}\mathrm{d}y_2 \\ &\in \Omega_2 + \Omega_2^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y, \mathrm{d}u^{[1]}, \mathrm{d}y^{[1]}\} \end{aligned}$$

and really, condition (ii) of Theorem 2 is satisfied. Choosing

$$\begin{aligned} \omega_1 &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} - y_2\mathrm{d}u_1 - u_1\mathrm{d}y_2 \\ &= \mathrm{d}(y_1^{[1]}u_1^{[1]} - y_2 u_1) \\ \omega_2 &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]} + y_2 u_1^{[1]} y_1^{[1]}\mathrm{d}u_2 \\ &\quad + u_2 u_1^{[1]} y_1^{[1]}\mathrm{d}y_2 = \mathrm{d}(u_1^{[1]} y_1^{[1]} u_2 y_2), \end{aligned}$$

then the condition (iii) is also satisfied. To check the condition (iv), we use Theorem 3. For that, apply Algorithm to the one forms ω_1 and ω_2.

Step 1. Take

$$\begin{aligned} \bar{\omega}_{1,1} &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} \\ \bar{\omega}_{2,1} &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]}. \end{aligned}$$

It is obvious that the ranks of these one-forms are 1, because $\bar{\omega}_{1,1} = \mathrm{d}(y_1^{[1]}u_1^{[1]})$ and $\bar{\omega}_{2,1} = y_2 u_2 \mathrm{d}(y_1^{[1]}u_1^{[1]})$. Thus, one takes $\phi_{1,1} = \phi_{2,1} = y_1 u_1$.

Step 2. Since this is the last step, one takes $\phi_{1,2} = y_1^{[1]}u_1^{[1]} - y_2 u_1 = \delta\phi_{1,1} - u_1 y_2$ and $\phi_{2,2} = u_1^{[1]} y_1^{[1]} u_2 y_2 = \delta\phi_{2,1} u_2 y_2$.

It is easy to check that $\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{1,1}, \mathrm{d}\phi_{2,1}, \mathrm{d}\phi_{2,2}\}) = 2 = m$. Therefore, conditions of Theorem 3 are satisfied.

Using the functions $\phi_{i,j}$ from Algorithm 1, one can find the feedback that decouples the system, as suggested in the proof of Theorem 3. For that, take

$$\begin{aligned} \eta_{1,1} &= \phi_{1,1} = y_1 u_1 \\ v_1 &= \phi_{1,2} = \eta_{1,1}^{[1]} - u_1 y_2 \\ v_2 &= \phi_{2,2} = \eta_{1,1}^{[1]} u_2 y_2. \end{aligned} \tag{21}$$

Solving equations (21) in terms of u_1, u_2 and $\eta_{1,1}^{[1]}$, one gets the decoupling feedback

$$\begin{aligned} \eta_{1,1}^{[1]} &= v_1 + \frac{\eta_{1,1} y_2}{y_1} \\ u_1 &= \frac{\eta_{1,1}}{y_1} \\ u_2 &= \frac{v_2 y_1}{y_2(y_1 v_1 + \eta_{1,1} y_2)}. \end{aligned} \tag{22}$$

For the closed-loop system one gets $\bar{r}_1 = \bar{r}_2 = r_i + s - 1 = 2$ and

$$\begin{aligned} \mathrm{d}y_{*1}^{[2]} &= \mathrm{d}v_1 - \mathrm{d}x_1 \in \bar{\Omega}_1 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_1\} \\ \mathrm{d}y_{*2}^{[2]} &= \mathrm{d}v_2 \in \bar{\Omega}_2 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_2\}, \end{aligned}$$

which means that by Lemma 2, the closed-loop system is i/o decoupled.

V. Conclusion

A necessary and sufficient conditions for the solvability of i/o decoupling problem by static and dynamic measurement feedback was given in this paper. The solution, given in this paper, depends on the linearization of certain functions by output feedback. A sufficient condition was given to linearize a set of functions, defined in the solution of the i/o decoupling problem. The dynamic output feedback solution was also given, based on the dynamic measurement feedback solution.

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Disturbance decoupling by measurement feedback

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Abstract: The paper addresses the disturbance decoupling problem for MIMO discrete-time nonlinear systems. A sufficient conditions are derived to solve the problem by dynamic measurement feedback, i.e. the feedback that depends on measurable outputs only. The solution to the disturbance decoupling problem, described in this paper, is based on the input-output linearization, which is used to linearize certain functions. Two examples are added to illustrate the results.

1. INTRODUCTION

The disturbance decoupling problem (DDP) is one of the fundamental problems in control theory. There are a lot of papers, that solve the problem by state feedback, see Aranda-Bricaire and Kotta [2001, 2004], Fliegner and Nijmeijer [1994], Grizzle [1985], Monaco and Normand-Cyrot [1984] for nonlinear discrete-time systems and Conte et al. [2007], Isidori [1995], Nijmeijer and van der Schaft [1990] for nonlinear continuous-time systems. For output or measurement feedback, the problem lacks the full solution.

The first paper that applied measurement feedback to solve the DDP was Isidori et al. [1981], where sufficient solvability conditions were given for continuous-time systems, and the feedback that was used was restricted to the so-called pure dynamic measurement feedback. In Kaldmäe et al. [2013], similar results as in Isidori et al. [1981] were given for discrete-time systems (though, more general feedback was used), using algebraic approach (lattice theory), that is able to address also certain type of non-smooth systems. A more general feedback, where the state of the compensator is not a function of the state of the system, but can be chosen independently of it, was used in Xia and Moog [1999] and Kaldmäe and Kotta [2012b], where sufficient conditions for the solvability of the problem by dynamic measurement feedback were given for continuous- and discrete-time SISO systems, respectively. For static measurement feedback solutions see Pothin et al. [2002] and Kaldmäe and Kotta [2012a].

In this paper, we extend the results of Kaldmäe and Kotta [2012b] for MIMO discrete-time systems [1]. However, the extension is not direct since we relax certain integrability conditions. The result of this paper depends heavily on the solution of the input-output linearization problem, see Kaldmäe and Kotta [2014]. We show that a feedback that linearizes certain functions also solves the disturbance decoupling problem. It is our conjecture that our results can be generalized directly for continuous-time systems, though the computations are different because the differential operator and forward-shift operator act differently on the set of functions.

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[1] Note that there are no solutions for MIMO continuous-time systems.

2. PRELIMINARIES

2.1 Algebraic tools

In this paper, x stands for $x(t)$ and for $k \geq 1$, $x^{[k]}$ stands for kth-step forward time shift of x, defined by $x^{[k]} := x(t+k)$. Similar notations are used for the backward shift and the other variables.

Consider a nonlinear system, described by the equations

$$\begin{aligned} x^{[1]} &= f(x,u,w) \\ y &= h_*(x) \\ z &= h(x), \end{aligned} \qquad (1)$$

where $x \in X \subset \mathbb{R}^n$ is the state, $u \in U \subset \mathbb{R}^m$ is the controlled input, $w \in W \subset \mathbb{R}^\iota$ is the disturbance input, $y \subset Y \in \mathbb{R}^p$ is the controlled output and $z \subset Z \in \mathbb{R}^q$ is the measured output. It is assumed that the functions f, h_* and h are meromorphic. Also, we assume, that the system (1) is submersive, meaning that generically, i.e. everywhere except on a set of measure zero,

$$\text{rank}\Big[\frac{\partial f}{\partial(x(t),u(t))}\Big] = n. \qquad (2)$$

Also, throughout the paper it is assumed that $i = 1,\dots,p$.

Let $\mathcal{K}$ denote the field of meromorphic functions which depend on finite number of variables from the set $\{x, u^{[k]}, w^{[k]}; k \geq 0\}$. Introduce the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$, defined by the equations (1); in particular

$$\delta x := f(x,u,w)$$

and for $k \geq 0$, $\delta u^{[k]} := u^{[k+1]}$, $\delta w^{[k]} := w^{[k+1]}$. Moreover,

$$\delta\varphi(x,u,w,\dots,u^{[k]},w^{[s]}) := \varphi(f(x,u,w),u^{[1]},w^{[1]},\dots,u^{[k+1]},w^{[s+1]})$$

for $\varphi \in \mathcal{K}$. Under the submersivity assumption (2), the pair $(\mathcal{K},\delta)$ is a difference field. In general, this difference field is not inversive, i.e. the operator δ is not inversive in $\mathcal{K}$. However, one can always find an overfield $\mathcal{K}^*$ of $\mathcal{K}$, called the inversive closure of $\mathcal{K}$, which is inversive. See Aranda-Bricaire et al. [1996], Aranda-Bricaire and Kotta [2004] for details how to compute $\mathcal{K}^*$. From now on, we assume that difference field $(\mathcal{K},\delta)$ is inversive and denote it by $\mathcal{K}$. Note that then there exists an operator δ^{-1}, which is called backward-shift operator. By δ^k and δ^{-k} we denote the k-fold application of operators δ and δ^{-1}, respectively.

Define the vector space of one-forms as $\mathcal{E} = \mathrm{span}_\mathcal{K}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. Also, define $\mathcal{X} := \mathrm{span}_\mathcal{K}\{\mathrm{d}x\}$, $\mathcal{W} := \mathrm{span}_\mathcal{K}\{\mathrm{d}w^{[k]}, k \geq 0\}$. The operators δ and δ^{-1} are extended to $\mathcal{E}$ by the rules

$$\delta\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) = \sum_j \delta(a_j)\mathrm{d}(\delta\varphi_j)$$
$$\delta^{-1}\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) = \sum_j \delta^{-1}(a_j)\mathrm{d}(\delta^{-1}\varphi_j),$$

where $a_j, \varphi_j \in \mathcal{K}$. A one-form ω is called exact, if it is a differential of some function $\xi \in \mathcal{K}$, i.e $\omega = \mathrm{d}\xi$. Let $y = (y_1,\dots,y_p)$ be the controlled output vector of the system (1). The relative degree r_i of an output y_i with respect to input u is defined by $r_i := \min\{k \in \mathbb{N} \mid \mathrm{d}y_i^{[k]} \notin \mathcal{X} + \mathcal{W}\}$. If there does not exist such integer k, then set $r_i := \infty$.

In general, a one-form ω is a linear combination over $\mathcal{K}$ of finite number of standard basis elements of $\mathcal{E}$, i.e. $\{\mathrm{d}x, \mathrm{d}u^{[k]}, \mathrm{d}w^{[k]}; k \geq 0\}$. However, it is often possible to find a linearly independent set of exact one-forms with less elements than those basis elements of $\mathcal{E}$ in terms of which ω can be expressed.

Definition 1. A number $\gamma \in \mathbb{N}$ is called the rank of a one-form ω, if γ is minimal number of linearly independent exact one-forms necessary to express a one-form ω. The set of these exact one-forms is called the basis of ω.

Next we define two subspaces Ω and Ω_u of $\mathcal{X}$ in the following way:

$$\Omega = \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N}: \delta^k\omega \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]},\dots,\mathrm{d}y_i^{[r_i+k-1]}\}\}. \tag{3}$$

and

$$\Omega_u = \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N}: \delta^k\omega \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{[k-1]}, \mathrm{d}y_i^{[r_i]},\dots,\mathrm{d}y_i^{[r_i+k-1]}\}\}. \tag{4}$$

By definitions, $\Omega \subseteq \Omega_u$. For SISO systems $\Omega = \Omega_u$, since $\mathrm{d}u$ can be written as a linear combination of $\mathrm{d}x$ and $\mathrm{d}y^{[r]}$, where r is the relative degree of output y with respect to input u.

Following lemmas give procedures for computing subspaces Ω and Ω_u.

Lemma 1. Kaldmäe and Kotta [2012a] The subspace Ω may be computed as the limit of the following algorithm:

$$\Omega^0 = \mathcal{X} \tag{5}$$
$$\Omega^{k+1} = \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_\mathcal{K}\{\mathrm{d}y_i^{[r_i]}\}\}.$$

Lemma 2. The subspace Ω_u may be computed as the limit of the following algorithm:

$$\Omega^0 = \mathcal{X} \tag{6}$$
$$\Omega^{k+1} = \{\omega \in \Omega^k \mid \delta\omega \in \Omega^k + \mathrm{span}_\mathcal{K}\{\mathrm{d}u, \mathrm{d}y_i^{[r_i]}\}\}.$$

Suppose $\Omega = \mathrm{span}_\mathcal{K}\{\mathrm{d}\theta_1,\dots,\mathrm{d}\theta_s\}$. Next define the k-time forward-shift of subspace Ω elementwise by $\Omega^{[k]} = \mathrm{span}_\mathcal{K}\{\mathrm{d}\theta_1^{[k]},\dots,\mathrm{d}\theta_s^{[k]}\}$ for $k \geq 1$.

2.2 Problem statement

The DDP by measurement feedback can be stated as follows. Find a dynamic measurement feedback of the form

$$\eta^{[1]} = F(\eta, z, v) \tag{7}$$
$$u = H(\eta, z, v),$$

where $\eta \in \mathbb{R}^\rho$ and $v \in \mathbb{R}^m$, such that controlled outputs y_i of the closed-loop system do not depend on disturbance w at any time instant, i.e.

$$\mathrm{d}y_i^{[k]} \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}\eta\} \quad k < \tilde{r}_i$$
$$\mathrm{d}y_i^{[k]} \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v, \dots, \mathrm{d}v^{[k-\tilde{r}_i]}\} \quad k \geq \tilde{r}_i,$$

where $\tilde{r}_i$ is the relative degree of output y_i of the closed loop system with respect to u.

Lemma 3. If the relative degrees r_i of outputs y_i with respect to u are finite then system (1) is disturbance decoupled if and only if

$$\mathrm{d}y_i^{[r_i]} \in \Omega_u + \mathrm{span}_\mathcal{K}\{\mathrm{d}u\}. \tag{8}$$

Proof: Necessity. Since r_i is the relative degree of output y_i with respect to input u,

$$\mathrm{d}y_i^{[r_i]} = \omega_0 + \sum_{j=1}^m b_{i,j}\mathrm{d}u_j,$$

where $b_{i,j} \in \mathcal{K}$ and $\omega_0 \in \mathrm{span}_\mathcal{K}\{\mathrm{d}x\}$. We show that $\omega_0 \in \Omega_u$. Assume contrary that $\omega_0 \notin \Omega_u$. Then there exists $k \in \mathbb{N}$ such that

$$\delta^k\omega_0 \notin \mathrm{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}u, \dots, \mathrm{d}u^{[k-1]}\}.$$

This means that one-form ω_0 is not disturbance decoupled and thus y_i also is not disturbance decoupled. This is a contradiction and thus $\omega_0 \in \Omega_u$.

Sufficiency. If (8) is true, then by Lemma 2 $\Omega_u^{[1]} \subseteq \Omega_u + \mathrm{span}_\mathcal{K}\{\mathrm{d}u\}$. Thus, Ω_u is invariant with respect to the system dynamics and since $\mathrm{d}y \in \Omega_u$, the system is disturbance decoupled. ■

3. MAIN RESULTS

3.1 Input-output linearization

Since our solution of the DDP depends on the solution of the input-output (i/o) linearization problem, we start with the statement of the i/o linearization problem. For more information, see Kaldmäe and Kotta [2014]. In this section, let $l = 1, \dots, q$.

Consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the difference equations

$$z_l^{[n_l]} = \Phi_l(z_\tau, \dots, z_\tau^{[n_{l\tau}]}, u_j, \dots, u_j^{[n_l-1]}) \tag{9}$$

for $\tau = 1, \dots, q$, $j = 1, \dots, m$, where Φ_l are supposed to be meromorphic functions of their arguments and the indices in (9) satisfy the relations

$$\begin{aligned} &n_1 \le n_2 \le \cdots \le n_q, \quad n_{l\tau} < n_\tau \\ &n_{l\tau} < n_l, \quad \tau \le l \\ &n_{l\tau} \le n_l, \quad \tau > l. \end{aligned} \tag{10}$$

Also, we assume, that system (9) is submersive, i.e. the map $\Phi = (\Phi_1, \dots, \Phi_q)^T$ satisfies generically the condition

$$\operatorname{rank}\Big[\frac{\partial \Phi}{\partial(z,u)}\Big] = q,$$

where $z = (z_1, \dots, z_q)$ and $u = (u_1, \dots, u_m)$.

In this section, let $\mathcal{K}$ be the field of meromorphic functions in variables z, u and a finite number of their independent forward shifts, i.e. variables from the set $\mathcal{C} = \{z_l, \dots, z_l^{[n_l-1]}, u_j^{[k]}; k \ge 0\}$. Also, let $\mathcal{E}^k := \operatorname{span}_\mathcal{K}\{\mathrm{d}z_l, \dots, \mathrm{d}z_l^{[k-1]}, \mathrm{d}u_j, \dots, \mathrm{d}u_j^{[k-1]}\}$ for any $k \in \mathbb{N}$ and r_l denotes the relative degree of the output z_l with respect to the input u.

Given a discrete-time MIMO nonlinear control system of the form (9), we say that system (9) is i/o linearized by feedback (7), if the differentials of the input-output equations of the closed-loop system satisfy the relations

$$\mathrm{d}z_l^{[n_l]} \in \operatorname{span}_\mathbb{R}\{\mathrm{d}z_\tau^{[n_{l\tau}]}, \dots, \mathrm{d}z_\tau, \mathrm{d}v\} \tag{11}$$

for $\tau = 1, \dots, q$. In case when

$$\mathrm{d}z_l^{[n_l]} \in \operatorname{span}_\mathbb{R}\{\mathrm{d}v\},$$

system (9) is said to be strictly i/o linearized.

We say that functions $\varphi_l(z, \dots, z^{[s-1]}, u, \dots, u^{[s-1]})$ are linearizable (strictly linearizable) if the system

$$z_l^{[s]} = \varphi_l(z, \dots, z^{[s-1]}, u, \dots, u^{[s-1]})$$

is i/o linearizable (strictly i/o linearizable).

Let

$$\tilde{\omega}_l := \mathrm{d}z_l^{[n_l]} \mod \operatorname{span}_\mathbb{R}\{\mathrm{d}z_\tau^{[n_{l\tau}]}, \dots, \mathrm{d}z_\tau\},$$

where $\tau = 1, \dots, q$. [2] For solvability of the i/o linearization problem, it is necessary that [3]

$$\tilde{\omega}_l \in \mathcal{E}^{n_l - r_l + 1}, \tag{12}$$

since otherwise nonlinearities appear before the input u starts to affect the output y_i.

First, let ω_{l_*}, $l_* = 1, \dots, q_*$, be the basis elements of $\operatorname{span}_\mathbb{R}\{\tilde{\omega}_l\}$. In the rest of this section assume that $l_*, \tau = 1, \dots, q_*$ and $j = 1, \dots, m$.

Let σ_{l_*} be such that

$$\omega_{l_*} \in \mathcal{E}^{\sigma_{l_*}}.$$

Next, define the one-forms

$$\bar{\omega}_{l_*,\lambda} \in \operatorname{span}_\mathcal{K}\{\mathrm{d}z^{[\sigma_{l_*}-\lambda]}, \dots, \mathrm{d}z^{[\sigma_{l_*}-1]}, \mathrm{d}u^{[\sigma_{l_*}-\lambda]}, \dots, \mathrm{d}u^{[\sigma_{l_*}-1]}\},$$

where $\lambda = 1, \dots, \sigma_{l_*} - 1$, such that

$$\omega_{l_*} - \bar{\omega}_{l_*,\lambda} \in \mathcal{E}^{\sigma_{l_*}-\lambda} \tag{13}$$

and

$$\bar{\omega}_{l_*,\sigma_{l_*}} := \omega_{l_*}. \tag{14}$$

It means that the one-forms $\bar{\omega}_{l_*,\lambda}$ depend on the $(\sigma_{l_*}-\lambda)$th and higher order terms of the one-forms ω_{l_*}. Let $\gamma_{l_*,\lambda}$ be the rank of a one-form $\bar{\omega}_{l_*,\lambda}$ for $\lambda = 1, \dots, \sigma_{l_*}$. Then there exist $\gamma_{l_*,\lambda}$ functions $\tilde{\phi}^k_{l_*,\lambda}(z^{[\sigma_{l_*}-\lambda]}, \dots, z^{[\sigma_{l_*}-1]}, u^{[\sigma_{l_*}-\lambda]}, \dots, u^{[\sigma_{l_*}-1]})$ such that

$$\bar{\omega}_{l_*,\lambda} \in \operatorname{span}_\mathcal{K}\{\mathrm{d}\tilde{\phi}^1_{l_*,\lambda}, \dots, \mathrm{d}\tilde{\phi}^{\gamma_{l_*,\lambda}}_{l_*,\lambda}\}.$$

Finally, define the function $\phi^k_{l_*,\lambda}$ as a $(\sigma_{l_*} - \lambda)$ step backward shift of the function $\tilde{\phi}^k_{l_*,\lambda}$, i.e.

$$\phi^k_{l_*,\lambda} := (\delta^{-1})^{\sigma_{l_*}-\lambda}\tilde{\phi}^k_{l_*,\lambda} = \delta^{\lambda-\sigma_{l_*}}\tilde{\phi}^k_{l_*,\lambda}$$

for $\lambda = 1, \dots, \sigma_{l_*}$ and $k = 1, \dots, \gamma_{l_*,\lambda}$.

Theorem 1. Kaldmäe and Kotta [2014] Under the assumption (12) the system (9) is input-output linearizable by dynamic output feedback of the form (7) if and only if

$$\dim(\operatorname{span}_\mathcal{K}\{\mathrm{d}\phi^k_{l_*,\lambda}\}) = \operatorname{rank}_\mathcal{K}\frac{\partial \phi^k_{l_*,\lambda}}{\partial(u, \delta\phi^k_{l_*,\lambda^*})}, \tag{15}$$

for $\lambda = 1, \dots, \sigma_{l_*}$, $\lambda^* = 1, \dots, \sigma_{l_*} - 1$, $k = 1, \dots, \gamma_{l_*,\lambda}$ and functions $\phi^1_{l_*,\sigma_{l_*}}$ are independent from all the other functions.

3.2 Sufficient conditions for solvability of the DDP

The theorem below gives sufficient solvability conditions of the DDP by dynamic measurement feedback.

Theorem 2. Under the assumption that all the relative degrees r_i of outputs y_i with respect to u are finite, the DDP by dynamic measurement feedback is solvable for system (1), if

(i) there exist one-forms $\omega_i \in \operatorname{span}_\mathcal{K}\{\mathrm{d}z, \dots, \mathrm{d}z^{[s-1]}, \mathrm{d}u, \dots, \mathrm{d}u^{[s-1]}\}$ with rank $\omega_i =: \gamma_i$ such that
$$\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega + \cdots + \Omega^{[s-1]}$$
for some $s \ge 1$;

(ii) for $\omega_i = \sum_{j=1}^{\gamma_i} \beta_{i,j}\mathrm{d}\alpha_{i,j}(z, \dots, z^{[s-1]}, u, \dots, u^{[s-1]})$ from (i), the functions $\alpha_{i,j}$ are strictly linearizable by dynamic measurement feedback.

Proof: We show that the feedback that linearizes strictly the functions $\alpha_{i,j}$ in (ii), solves the disturbance decoupling problem.

Note that the relative degree of y_i with respect to input v is $\bar{r}_i = r_i + s - 1$. Since for the closed-loop system $\omega_i \in \operatorname{span}_\mathcal{K}\{\mathrm{d}v\}$, one gets from (i) that

$$\mathrm{d}y_i^{[\bar{r}_i]} \in \Omega + \cdots + \Omega^{[s-1]} + \operatorname{span}_\mathcal{K}\{\mathrm{d}v\}.$$

Next, we show that $\bar{\Omega} = \Omega + \cdots + \Omega^{[s-1]}$, where $\bar{\Omega}$ is the subspace Ω for the closed-loop system. From the definition of the subspace Ω,

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \operatorname{span}_\mathcal{K}\{\mathrm{d}x, \mathrm{d}y_i^{[r_i]}, \dots, \mathrm{d}y_i^{[r_i+s-2]}\}.$$

[2] In the case of strict linearizability, one has to take $\tilde{\omega}_l := \mathrm{d}z_l^{[n_l]}$.
[3] Note that if $r_l = 1$, then the condition (12) is always satisfied.

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega + \cdots + \Omega^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}.$$

Thus,

$$\begin{aligned}\Omega + \cdots + \Omega^{[s-1]} &= \{\bar{\omega} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ &\quad \bar{\omega}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_i^{[r_i+s-1]}, \ldots, \mathrm{d}y_i^{[r_i+s-k-2]}\}\} \\ &= \bar{\Omega}.\end{aligned}$$

The last equality comes from the definition (3) of the subspace $\bar{\Omega}$.

Since $\bar{\Omega} \subseteq \bar{\Omega}_u$, then by Lemma 3, system (1) is disturbance decoupled. ■

Corollary 1. For SISO systems, the conditions of Theorem 2 are necessary and sufficient.

Proof: It remains to prove the necessity. By Lemma 3, since the closed-loop system is disturbance decoupled,

$$\mathrm{d}y^{[\bar{r}]} \in \bar{\Omega}_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}, \quad (16)$$

where $\bar{r}$ is the relative degree of y in the closed-loop system with respect to the new input v and $\bar{\Omega}_u$ is the subspace Ω_u for the closed-loop system. We choose $s \geq 1$ such that $\bar{r} = r + s - 1$.

Since for single input systems $\Omega = \Omega_u$, one can show, as in the proof of Theorem 2, that $\bar{\Omega}_u = \Omega + \cdots + \Omega^{[s-1]}$. Now, one can find the one-form $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$, with rank 1, such that we get from (16)

$$\mathrm{d}y^{[r+s-1]} - \omega \in \Omega + \cdots + \Omega^{[s-1]}.$$

Assume that $\omega = \beta \mathrm{d}\alpha$ for some functions $\beta, \alpha \in \mathcal{K}$. Clearly, the feedback that solves the disturbance decoupling problem, also linearizes strictly function α, since for the closed-loop system $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v\}$. Thus conditions (i) and (ii) of Theorem 2 are satisfied. ■

Note that if we take $s = 1$ in Theorem 2, we get solvability conditions for DDP by static measurement feedback. In this case the strict linearizability of functions $\alpha_{i,j}$ means that system of equations $\alpha_{i,j}(z, u) = v_\mu$, $\mu = 1, \ldots, m$, is solvable in u.

4. EXAMPLES

Example 1. Consider the system

$$\begin{aligned} x_1^{[1]} &= u_1 \\ x_2^{[1]} &= x_3 u_3 + x_2 x_4 u_2 - x_1 \\ x_3^{[1]} &= u_2 \\ x_4^{[1]} &= x_1 w \\ x_5^{[1]} &= u_1 u_2 x_4 + x_2 \\ y_1 &= x_2 \\ y_2 &= x_5 \\ z &= x_4. \end{aligned} \quad (17)$$

First, note that the relative degrees r_1 and r_2 of outputs y_1 and y_2 with respect to u are both 1. One can also computes subspaces $\Omega = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5\}$ and $\Omega_u = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_5\}$. Clearly, $\mathrm{d}y_i \notin \Omega_u + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u\}$ for $i = 1, 2$. Therefore, system (17) is not disturbance decoupled.

To find the one-forms ω_i, defined in (i) of Theorem 2, we calculate $\mathrm{d}y_i^{[r_i+s_i-1]}$ for $s_i = 1, 2, \ldots$, until

$$\begin{aligned}\mathrm{d}y_i^{[r_i+s_i-1]} &\in \Omega + \cdots + \Omega^{[s_i-1]} \\ &\quad + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z, \ldots, \mathrm{d}z^{[s_i-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s_i-1]}\}.\end{aligned}$$

For system (17), we calculate

$$\begin{aligned}\mathrm{d}y_1^{[1]} &= u_3 \mathrm{d}x_3 - \mathrm{d}x_1 + z u_2 \mathrm{d}x_2 + x_3 \mathrm{d}u_3 + x_2 \mathrm{d}(z u_2) \\ &\notin \Omega + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}z\} \\ \mathrm{d}y_2^{[1]} &= \mathrm{d}x_2 + \mathrm{d}(u_1 u_2 z) \\ &\in \Omega + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}z\}.\end{aligned}$$

Thus, $s_2 = 1$. Compute $\Omega + \Omega^{[1]} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}x_5^{[1]}\}$. Now,

$$\begin{aligned}\mathrm{d}y_1^{[2]} &= \mathrm{d}(u_3^{[1]} u_2 - u_1) + z^{[1]} u_2^{[1]} \mathrm{d}x_2^{[1]} \\ &\quad + x_2^{[1]} \mathrm{d}(z^{[1]} u_2^{[1]}) \\ &\in \Omega + \Omega^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}u^{[1]}, \mathrm{d}z, \mathrm{d}z^{[1]}\},\end{aligned}$$

meaning that $s_1 = 2$. Next, we can choose the one-forms ω_i as

$$\begin{aligned}\omega_1 &= \mathrm{d}(u_3^{[1]} u_2 - u_1) + x_2^{[1]} \mathrm{d}(z^{[1]} u_2^{[1]}) \\ \omega_2 &= \mathrm{d}(u_1 u_2 z).\end{aligned}$$

Obviously, $\mathrm{rank}\,\omega_1 = 2$ and $\mathrm{rank}\,\omega_2 = 1$. It remains to check whether the functions $\alpha_{1,1} = u_3^{[1]} u_2 - u_1$, $\alpha_{1,2} = z^{[1]} u_2^{[1]}$ and $\alpha_{2,1} = u_1 u_2 z$ are linearizable. One can find, that the dynamic feedback

$$\begin{aligned}\eta_1^{[1]} &= \frac{z(\eta_2 v_1 + v_3)}{\eta_2^2} \\ \eta_2^{[1]} &= v_2 \\ u_1 &= \frac{v_3}{\eta_2} \\ u_2 &= \frac{\eta_2}{z} \\ u_3 &= \eta_1,\end{aligned} \quad (18)$$

linearizes functions $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$ and also decouples disturbances from the controlled outputs y_1 and y_2. Really, in the closed-loop system

$$\begin{aligned}y_1^{[2]} &= v_1 + x_2^{[1]} v_2 \\ y_2^{[1]} &= v_3 + x_2\end{aligned}$$

and since $\bar{\Omega}_u = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}x_2^{[1]}, \mathrm{d}\eta_2\}$, the conditions of Lemma 3 are satisfied. This means that the closed-loop system is disturbance decoupled.

Example 2. The next example is taken from Kaldmäe et al. [2013]. The system in Figure 1 is a typical subsystem in many applications and consists of linear subsystems $W_1 = k_1/(1 + T_1 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_2 = k_2/(1 + T_2 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_3 = k_3 T_3 \frac{\mathrm{d}}{\mathrm{d}t}/(1 + T_3 \frac{\mathrm{d}}{\mathrm{d}t})$, $W_4 = k_4 / \frac{\mathrm{d}}{\mathrm{d}t}$ and saturation operation,

$$\sigma(x) = \begin{cases} x, \text{ if } |x| \leq x_0 \\ x_0 \mathrm{sign}\ x, \text{ if } |x| > x_0 \end{cases}$$

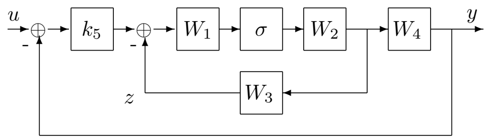

Fig. 1. System with saturation operation.

that corresponds to the amplifier. Here $k_1, \ldots, k_5$, are real coefficients, T_1, T_2 are certain time constants and T_3 may be considered as unknown function of disturbance w because of the unexpected changes in the feedback loop.

After the Euler discretization, one gets a system described by the equations:

$$\begin{aligned} x_1^{[1]} &= k_4 x_2 + x_1 \\ x_2^{[1]} &= \frac{k_2}{T_2}\sigma(x_3) + x_2(1 - \frac{1}{T_2}) \\ x_3^{[1]} &= \frac{1}{T_1}(k_1 k_5 (u - x_1) - k_1 k_3 (x_2 - x_4)) + x_3(1 - \frac{1}{T_1}) \\ x_4^{[1]} &= \frac{1}{T_3(w)} x_2 + x_4(1 - \frac{1}{T_3(w)}) \\ y &= x_1 \\ z &= k_3(x_2 - x_4). \end{aligned} \tag{19}$$

In Kaldmäe et al. [2013], a dynamic measurement feedback is found that solves the DDP for system (19). However, note that the problem statement of Kaldmäe et al. [2013] is somewhat different from that in this paper. Namely, in Kaldmäe et al. [2013] the state η of a compensator is assumed to be a function of state x, i.e. $\eta = \phi(x)$.

Below we solve the DDP for system (19) using the method described in this paper. Since our method assumes all functions to be meromorphic, we take $\sigma(x_3) = x_3$ in (19), i.e. $|x_3| \leq x_{3,0}$ for some $x_{3,0} \in \mathbb{R}$. Note that if $|x_3| > x_{3,0}$, one can show by Lemma 3 that the system (19) is already disturbance decoupled.

The relative degree of output y with respect to input u is $r = 3$. Next, we have to find, by Lemma 1, the subspace Ω. Compute $\Omega = \Omega^1 = \text{span}_{\mathcal{K}}\{dx_1, dx_2, dx_3\}$. Since

$$\begin{aligned} y^{[3]} &= \big(1 - \frac{k_1 k_2 k_4 k_5}{T_1 T_2}\big)x_1 + \big(3k_4 - \frac{3k_4}{T_2} + \frac{k_4}{T_2^2}\big)x_2 \\ &+ \big(\frac{3k_2 k_4}{T_2} - \frac{k_2 k_4}{T_2^2} - \frac{k_2 k_4}{T_1 T_2}\big)x_3 + \frac{k_1 k_2 k_4}{T_1 T_2}\big(k_5 u - z\big), \end{aligned}$$

one can choose $\omega = k_5 du - dz$. Then condition (i) of Theorem 2 is satisfied for $s = 1$. The rank of the one-form ω is obviously 1 and $\alpha = k_5 u - z$. By taking $v = k_5 u - z$, one gets $u = \frac{1}{k_5}(v + z)$. This static measurement feedback solves the DDP for system (19).

The reason, why we get static solution in this paper, but dynamic solution in Kaldmäe et al. [2013], is that the selection of one-form ω, in Theorem 2, is more restricted, than the selection of certain function, based on which the solution is computed, in Kaldmäe et al. [2013]. In the latter case the choice of a function that leads to static solution is not obvious.

5. CONCLUSION

This paper addressed the DDP by dynamic measurement feedback. Using algebraic methods, sufficient solvability conditions were given. For SISO systems, the conditions are also necessary. The key point of the solution is linearization of certain functions by measurement feedback. It is shown that this feedback also solves the disturbance decoupling problem. The future work will include finding necessary and sufficient solvability conditions for MIMO systems. Two examples were given to illustrate the theory.

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–Brief Paper–

DISTURBANCE DECOUPLING OF TIME DELAY SYSTEMS

A. Kaldmäe and C. H. Moog

ABSTRACT

A necessary and sufficient condition is derived for the solvability of the disturbance decoupling problem by a pure shift dynamic compensator. It corrects previously obtained results for multi-input multi-output (MIMO) systems. Also, necessary and sufficient conditions are given for the solvability of the problem by a dynamic compensator for single input single output (SISO) systems.

Key Words: Time delay systems, nonlinear systems, disturbance decoupling.

I. INTRODUCTION

Time delay systems are natural in many areas, like telecommunications, remote control, and biological systems [1].

The disturbance decoupling problem (DDP) has a nice solution for delay-free systems [2], but, for time delay systems, the full solution is missing. For linear time delay systems, the DDP has been studied using the so-called geometric approach [3,4], and for nonlinear time delay systems, the problem has been considered in [5–7].

In this paper, the same mathematical approach as in [6–8] is used to study the DDP for nonlinear time delay systems. First, a counterexample is given to show that the necessary and sufficient conditions in [7] are not necessary. The mistake comes from the fact that certain one-forms are assumed to be integrable, which is shown to be too restrictive for necessity. Then, the correct conditions are given to solve the DDP by pure shift dynamic feedback. The key point of this new solution is the use of rank of a one-form, which generalizes the notion of integrability. Finally, the DDP by dynamic feedback is considered for single input single output (SISO) systems and necessary and sufficient conditions are given to solve the problem.

The paper is organized in the following manner. In Section II, the mathematical tools and preliminary definitions are given. In Section III, pure shift dynamic feedback is considered while in Section IV, dynamic feedback is considered to solve the DDP for SISO systems. In Section V, two examples are considered.

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II. PRELIMINARIES

Consider a nonlinear time delay system of the form:

$$\begin{aligned} \dot{x}(t) &= f(x(\cdot), u(\cdot), q(\cdot)) \\ y(t) &= h(x(\cdot)), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $q \in \mathbb{R}^r$ is the disturbance input, $y \in \mathbb{R}^p$ is the output of the system, and $\xi(\cdot) := (\xi(t), \xi(t-1), \ldots)$ for $\xi \in C = \{x, u, \ldots, u^{(k)}, q, \ldots, q^{(k)}; k \in \mathbb{N}\}$. Also, functions f and h are assumed to be meromorphic.

Let $\mathcal{K}$ denote the field of meromorphic functions that depend on a finite number of variables from the set $C = \{x(\cdot), u(\cdot), \ldots, u^{(k)}(\cdot), q(\cdot), \ldots, q^{(k)}(\cdot); k \in \mathbb{N}\}$. Also, denote by $\mathcal{E}$ the vector space spanned by the symbols $\mathrm{d}C = \{\mathrm{d}x(\cdot), \mathrm{d}u(\cdot), \ldots, \mathrm{d}u^{(k)}(\cdot), \mathrm{d}q(\cdot), \ldots, \mathrm{d}q^{(k)}(\cdot); k \in \mathbb{N}\}$ over the field $\mathcal{K}$ (in case of linear systems over the field $\mathbb{R}$). The elements of $\mathcal{E}$ are called one-forms. If a one-form $\omega = \mathrm{d}\varphi$, for some $\varphi \in \mathcal{K}$, one says that ω is an exact one-form.

Consider the shift operator $\delta : \mathcal{K} \to \mathcal{K}$ defined as $\delta\varphi(\xi(t-i); i \in \mathbb{N}) := \varphi(\xi(t-i-1); i \in \mathbb{N})$, where $\xi \in C$. The shift operator is extended to the vector space $\mathcal{E}$ by applying δ to all functions appearing in a given one-form. Denote by $\mathcal{K}(\vartheta]$ the non-commutative ring of polynomials in ϑ over the field $\mathcal{K}$. The elements of this ring are in the form $a_0 + a_1\vartheta + \cdots + a_s\vartheta^s$ for some finite $s \in \mathbb{N}$. Addition is defined on this ring as usual, but the rule for multiplication is $\vartheta \cdot \varphi = \delta(\varphi) \cdot \vartheta$ for $\varphi \in \mathcal{K}$. Also, let $\mathcal{K}(\vartheta]^{p\times r}$ be the ring of matrices over $\mathcal{K}(\vartheta]$. A matrix $A(\vartheta) \in \mathcal{K}(\vartheta]^{p\times p}$ is said to be unimodular, if it has an inverse in the same ring.

Let $\mathcal{M}$ be the formal module over the ring $\mathcal{K}(\vartheta]$, *i.e.*

$$\mathcal{M} = \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\zeta \mid \zeta \in \mathcal{K}\}.$$

For a given set of one-forms $\{\omega_1, \dots, \omega_s\} \in \mathcal{E}$, let $\text{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \dots, \omega_s\}$ represent the submodule of $\mathcal{M}$ generated by $\{\omega_1, \dots, \omega_s\}$.

The next definition generalizes the definition of a closure of a module, given in [3].

Definition 1. Let $\mathcal{A}$ be a submodule of $\mathcal{M}$ with rank s. The closure of $\mathcal{A}$ over a ring $\mathcal{K}(\vartheta]$ is defined as the largest submodule of $\mathcal{M}$ containing $\mathcal{A}$ and having rank s. The closure of $\mathcal{A}$ is denoted by $cl_{\mathcal{K}(\vartheta]}(\mathcal{A})$.

Definition 2 [7]. A submodule $\mathcal{A} \subseteq \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t)\}$ is said to be invariant with respect to the dynamics (1) if $\dot{\mathcal{A}} \subseteq \mathcal{A} + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}u(t)\}$.

A very important definition in nonlinear control is the definition of integrable one-form.

Definition 3. A one-form ω is said to be integrable if there exist a polynomial $p(\vartheta) \in \mathcal{K}(\vartheta]$ and a function $\varphi \in \mathcal{K}$, such that $\omega = p(\vartheta)\mathrm{d}\varphi$.

Note that this definition differs from the definition of the integrable one-form in the case of delay-free nonlinear systems. This happens because the set of one-forms is considered as a module here, not a vector space. Thus, the integrating factor belongs to a ring $\mathcal{K}(\vartheta]$ instead of a field $\mathcal{K}$. This also means that the traditional methods to check the integrability of a one-form, *i.e.* the Frobenius theorem, are no longer valid. For more information about integrability conditions in the case of time delay systems, see [6,9].

In general, integrability of a one-form is a restrictive notion, but it is necessary in nonlinear control to obtain a solution to different control problems. In this paper, we define and use the rank of a one-form instead. It weakens the definition of integrable one-form, but still allows us to move from working with one-forms to functions.

Similarly, as in [10], we introduce the following definition.

Definition 4. A one-form ω is said to have rank γ if γ is the minimal number, such that $\omega = a_1\mathrm{d}\varphi_1 + \cdots + a_\gamma\mathrm{d}\varphi_\gamma$ for some $a_i \in \mathcal{K}(\vartheta]$ and $\varphi_i \in \mathcal{K}, i = 1, \dots, \gamma$.

A one-form ω is integrable if and only if its rank is equal to 1.

To compute the rank of a one-form ω, apply the following procedure. First, write ω as $\omega = \sum_{i=1}^{k} p_i(\vartheta)\mathrm{d}\xi_i(t)$, where $p_i(\vartheta) \in \mathcal{K}(\vartheta]$ and $\xi_i \in C$. Next, let $p(\vartheta)$ be the greatest common left factor of polynomials $p_i(\vartheta)$. This means that ω can be written as $\omega = p(\vartheta)\omega_0$ for some $\omega_0 \in \mathcal{E}$. It follows that the rank of ω is equal to the rank of ω_0. By Pfaff-Darboux theorem, see [10], the rank γ of ω_0 is the smallest integer k such that $(\mathrm{d}\omega_0)^k \wedge \omega_0 = 0$, where $(\mathrm{d}\omega_0)^k$ is a k-times wedge product of $\mathrm{d}\omega_0$.

Next, we define the relative degrees of system outputs $y_i(t), i = 1, \dots, p$.

Definition 5 [6]. The relative degree ρ_i of output $y_i(t)$ with respect to the control input $u(t)$ is defined as $\rho_i = \min\left\{k \in \mathbb{N} \mid \exists\tau \in \mathbb{N} \frac{\partial y_i^{(k)}(t)}{\partial u_j(t-\tau)} \neq 0\right\}$ for some $j \in \{1, \dots, m\}$.

In a similar manner, by replacing u with q in the above definition, one can define the relative degree σ_i of output $y_i(t)$ with respect to the disturbance input $q(t)$. It is also useful to characterize the minimal shift of $u(t)$, that $y_i^{(k)}(t)$ depend on.

Definition 6 [6]. The relative shift μ_i of $y_i(t)$ is defined as $\mu_i = \min\left\{\tau \in \mathbb{N} \mid \frac{\partial y_i^{(\rho_i)}(t)}{\partial u_j(t-\tau)} \neq 0\right\}$ for some $j \in \{1, \dots, m\}$.

III. DISTURBANCE DECOUPLING FOR MIMO TIME-DELAY SYSTEMS

3.1 Problem statement

In this paper, the so called pure shift dynamic compensators of the form

$$\begin{aligned} z(t+1) &= M(x(\cdot), z(\cdot), v(\cdot)) \\ u(t) &= G(x(\cdot), z(\cdot), v(\cdot)) \end{aligned} \tag{2}$$

are considered as introduced in [6]. It is also assumed that the compensator (2) is regular. By regularity of the compensator (2), we mean that all the new system variables, *i.e.* $z(t)$ and $v(t)$, can be expressed as a function of old system variables. Thus, there exists $\tilde{G}, K \in \mathcal{K}$ such that $v(t) = \tilde{G}(x(\cdot), u(\cdot))$ and $z(t) = K(x(\cdot), u(\cdot))$. A regular compensator (2) also can be written as $P(\vartheta)\mathrm{d}u(t) = Q(\vartheta)\mathrm{d}v(t) + R(\vartheta)\mathrm{d}x(t)$, where the matrix $Q(\vartheta)$ is unimodular. (This guarantees the regularity.) If the matrix $P(\vartheta)$ is also unimodular, then the compensator (2) is said to be compatible, which means that $\dim(z(t)) = 0$.

The disturbance decoupling problem considered in this paper is the following. Find a compensator (2) such that, for the closed-loop system, $y^{(k)}(t)$ does not depend on the disturbance input $q(\cdot)$ for all $k \in \mathbb{N}$, *i.e.* $\partial y^{(k)}(t)/\partial q(\cdot) \equiv 0$ for all $k \in \mathbb{N}$.

Lemma 1. System (1) is disturbance decoupled if and only if there exists an invariant submodule $\mathcal{A} \subset \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x\}$ with respect to system dynamics, such that $\mathrm{d}y_i \in \mathcal{A}, i = 1, \ldots, p$.

3.2 Problem solution

First, we show that the necessary and sufficient condition given in [7] for the solvability of the DDP by compatible compensator is only sufficient.

Example 1. Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)u_1(t-1) + u_2(t-1) \\ \dot{x}_2(t) &= q(t) + u_2(t) \\ y(t) &= x_1(t). \end{aligned} \tag{3}$$

This system is disturbance decoupled, since the submodule $\Omega = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x_1(t)\}$ is invariant. Nevertheless, the conditions of theorem 1 in [7] are not satisfied. Really, if

$$cl_{\mathcal{K}(\vartheta]}(\Omega + \dot{\Omega}) = \Omega \oplus \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega\},$$

then ω cannot be chosen such that the conditions (*ii*) and (*iii*) of theorem 1 in [7] are satisfied. When we take $\omega = x_1(t)\mathrm{d}u_1(t-1) + \mathrm{d}u_2(t-1)$, then it is not integrable since its rank is equal to 2. If $\omega = \mathrm{d}(x_1(t)u_1(t-1) + u_2(t-1))$, then the condition (*iii*) is not satisfied. Therefore, the conditions of Theorem 1 in [7] cannot be satisfied.

The following theorem generalizes the results of [6] to multi-input multi-output (MIMO) systems and corrects the results of [7].

Theorem 1. The disturbance decoupling problem is solvable by a compensator (2) if and only if there exists a submodule Ω such that the following conditions are satisfied:

(i) $\mathrm{d}y \in \Omega$;
(ii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}u(t)\}, i = 1, \ldots, s$, with rank γ_i such that

$$cl_{\mathcal{K}(\vartheta]}(\Omega + \dot{\Omega}) = \Omega \oplus \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_s\}$$

and $\omega_i = a_{i,1}\mathrm{d}\varphi_{i,1} + \cdots + a_{i,\gamma_i}\mathrm{d}\varphi_{i,\gamma_i}$, then

$$\dim(\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_{i,j}\}) = \mathrm{rank}_{\mathcal{K}} \frac{\partial(\varphi_{i,j})}{\partial u(t)} \tag{4}$$

where $i = 1, \ldots, s$ and $j = 1, \ldots, \gamma_i$.

Proof. *Sufficiency.* Let $\mathrm{d}\varphi_l, l = 1, \ldots, \dim(\mathrm{span}_{\mathcal{K}(\vartheta]} \{\mathrm{d}\varphi_{i,j}\})$, be the basis of $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_{i,j}\}$. By (4), the system of equations

$$\varphi_l(x(\cdot), u(\cdot)) = v_l(t) \tag{5}$$

is solvable in $u(t)$, and we get $u(t) = L(x(\cdot), v(t), u(t-k); k \geq 1)$. Now, define $z_q(t) := u(t-q)$. Thus, we get a feedback of the form (2). Under this feedback $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_{i,j}\} \subseteq \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}v(t)\}$ and the submodule Ω is invariant. Then, because of (*i*), the problem is solved.

Necessity. If the disturbance decoupling problem is solved, then, by Lemma 1, there exists a submodule Ω, which is invariant in the closed-loop system, such that $\mathrm{d}y \in \Omega$. Thus, condition (*i*) is satisfied. Since Ω is invariant, it satisfies $\dot{\Omega} \subseteq \Omega + \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}v(t)\}$. If one-forms $\omega_1, \ldots, \omega_s$ are defined as in (*ii*), then $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_s\} \subseteq \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}v(t)\}$. This means that (5) must be solvable in $u(t)$ and (4) must be satisfied.

In general, the choice of one-forms ω_i and functions $\varphi_{i,j}$ (even if ω_i are fixed) is not unique and different choices may yield different results regarding the solvability of the problem.

The difficulty of Theorem 1 is finding a correct submodule Ω. Clearly, since one wants that $\mathrm{d}y \in \Omega$, it should satisfy

$$cl_{\mathcal{K}(\vartheta]}\left(\mathrm{span}_{\mathcal{K}(\vartheta]}\left\{\mathrm{d}y_i(t), \ldots, \mathrm{d}y_i^{(\rho_i-1)}(t)\right\}\right) \subseteq \Omega \subseteq \mathrm{span}_{\mathcal{K}(\vartheta]}\left\{\frac{\partial f}{\partial q(\cdot)}\right\}^{\perp},$$

where $i = 1, \ldots, p$.

IV. DYNAMIC DISTURBANCE DECOUPLING FOR SISO TIME DELAY SYSTEMS

In this section, we consider SISO systems, *i.e.* systems of the form (1), where $m = p = 1$. The goal is to find a regular (*i.e.* invertible) dynamic compensator of the form

$$\begin{aligned} \dot{\eta}(t) &= F(x(\cdot), \eta(\cdot), z(\cdot), v(\cdot)) \\ z(t+1) &= M(x(\cdot), \eta(\cdot), z(\cdot), v(\cdot)) \\ u(t) &= G(x(\cdot), \eta(\cdot), z(\cdot), v(\cdot)), \end{aligned} \tag{6}$$

such that, for the closed-loop system, $y^{(k)}(t)$ does not depend on the disturbance input $q(\cdot)$ for all $k \in \mathbb{N}$, *i.e.* $\partial y^{(k)}(t)/\partial q(\cdot) \equiv 0$ for all $k \in \mathbb{N}$.

To solve the dynamic disturbance decoupling problem, we define submodule Ω of $\mathcal{M}$ as follows (note that this can also be one possible choice of submodule Ω in Theorem 1):

$$\Omega = cl_{\mathcal{K}(\vartheta]}(\{\omega \in \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t)\} \mid \forall k \in \mathbb{N} \\ \omega^{(k)} \in \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}y^{(\rho)}(t), \dots, \\ \mathrm{d}y^{(\rho+k-1)}(t)\}\}), \tag{7}$$

where ρ is the relative degree of output $y(t)$ with respect to input $u(t)$. Note that this definition yields $\text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y(t), \dots, \mathrm{d}y^{(\rho-1)}(t)\} \subseteq \Omega$. If a one-form ω belongs to the submodule Ω, clearly $\dot{\omega} \in \Omega + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y^{(\rho)}(t)\}$. Thus, the submodule Ω can be computed as the limit of the following algorithm:

$$\Omega^0 = \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t)\} \\ \Omega^{k+1} = \{\omega \in \Omega^k \mid \dot{\omega} \in \Omega^k + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y^{(\rho)}(t)\}\}.$$

The latter also yields that $\dot{\Omega} \subseteq \Omega + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y^{(\rho)}(t)\}$.

Lemma 2. The SISO time delay system (1) is disturbance decoupled if and only if

$$\mathrm{d}y^{(\rho)}(t) \in \Omega + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}u(t)\}. \tag{8}$$

Proof. *Necessity.* Since ρ is the relative degree of output $y(t)$ with respect to input $u(t)$, $\mathrm{d}y^{(\rho)}(t) = \omega_0 + b\mathrm{d}u(t)$, where $b \in \mathcal{K}(\vartheta]$ and $\omega_0 \in \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t)\}$, we show that $\omega_0 \in \Omega$. Assume by contradiction that $\omega_0 \notin \Omega$. Then, there exists $s \in \mathbb{N}$ such that $\omega_0^{(s)} \notin \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}u(t), \dots, \mathrm{d}u^{(s-1)}(t)\}$. This means that one-form ω_0 is not disturbance decoupled and that $\mathrm{d}y^{(\rho)}(t)$ also is not disturbance decoupled. This is a contradiction; therefore $\omega_0 \in \Omega$, meaning that (8) is satisfied.

Sufficiency. If (8) is true, then $\dot{\Omega} \subseteq \Omega + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}u(t)\}$, since $\dot{\Omega} \subseteq \Omega + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y^{(\rho)}(t)\}$. Thus, Ω is invariant with respect to the system dynamics. Moreover, since $\mathrm{d}y(t) \in \Omega$, the system is disturbance decoupled.

The next theorem gives necessary and sufficient conditions for the solvability of the dynamic disturbance decoupling problem for SISO systems. It is a generalization of theorem 11 in [6], where static solutions were considered.

Theorem 2. The DDP is solvable for SISO time delay systems by the compensator (6) if and only if there exist $k+1$ integrable one-forms $\omega_i \in \text{span}_{\mathcal{K}(\vartheta]}\{\dot{\omega}_{i-1}, \mathrm{d}x(t-\tau), \mathrm{d}u(t-\tau); \tau \geq \mu\}$ (Here μ is relative shift of $y(t)$), $i = 0, \dots, k$, such that

$$\mathrm{d}y^{(\rho+j)}(t) - \omega_j \in \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}y^{(\rho)}(t), \\ \dots, \mathrm{d}y^{(\rho+j-1)}(t)\} \\ \mathrm{d}y^{(\rho+k)}(t) - \omega_k \in \Omega + \cdots + \Omega^{(k)} \tag{9}$$

for $j = 0, \dots, k-1$.

Proof. *Necessity.* Since the closed-loop system is disturbance decoupled, by Lemma 2 $\mathrm{d}y^{(\rho+k)}(t) \in \Omega_{cl} + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}v(t)\}$, where Ω_{cl} is the subspace Ω for the closed-loop system. Since, in the closed-loop system, the relative degree of output $y(t)$ is $\rho+k$, it can be shown that $\Omega_{cl} = \Omega + \cdots + \Omega^{(k)}$. The existence of integrable one-form ω_k is clear, since it can be taken as $\omega_k = a_k\mathrm{d}v(t)$, where $a_k \in \mathcal{K}(\vartheta]$ is such that $\mathrm{d}y^{(\rho+k)}(t) - a_k\mathrm{d}v(t) \in \Omega + \cdots + \Omega^{(k)}$.

Now, assume by contradiction that there are no integrable one-forms $\omega_j, j = 0, \dots, k-1$, satisfying the conditions of Theorem 2. Then, either some $y^{(\rho+j)}(t)$ depend on the disturbance $q(t)$ (which is a contradiction) or some one-forms ω_j are not integrable. In the latter case, $\mathrm{d}y^{(\rho+j)}(t) \notin \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}\eta(t)\}$, which is also a contradiction. Thus, there exist integrable one-forms $\omega_j, j = 0, \dots, k-1$, that satisfy the conditions of Theorem 2. Finally, since the feedback is causal $\omega_i \in \text{span}_{\mathcal{K}(\vartheta]}\{\dot{\omega}_{i-1}, \mathrm{d}x(t-\tau), \mathrm{d}u(t-\tau); \tau \geq \mu\}, i = 0, \dots, k$.

Sufficiency. Let $\omega_i = a_i\mathrm{d}\varphi_i(\dot{\varphi}_{i-1}, x(\cdot), u(\cdot))$ for $i = 0, \dots, k$, and construct the system of equations $\eta_j = \varphi_j, v = \varphi_k$, where $j = 0, \dots, k-1$. One can see that this system is solvable in variables $\{\dot{\eta}_j, u(t)\} j = 0, \dots, k-1$. This will yield a feedback under which $\omega_k = a_k\mathrm{d}v(t)$. Also, the relative degree of $y(t)$ of the closed-loop system is $\rho + k$. Thus, $\mathcal{A} = \Omega + \cdots + \Omega^{(k)} \subseteq \Omega_{cl}$, where Ω_{cl} is the subspace Ω for the closed-loop system. Really, since $\dot{\mathcal{A}} \subseteq \mathcal{A} + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}y^{(\rho+k)}(t)\}$, $\mathcal{A} \subseteq \Omega_{cl}$ must be true. Therefore, $\mathrm{d}y^{(\rho+k)}(t) \in \Omega_{cl} + \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}v(t)\}$ and by Lemma 2 the system is disturbance decoupled.

V. EXAMPLES

Example 2 (Continuation of Example 1)**.** Consider again system (3). We show that the conditions of Theorem 1 are satisfied for this system. Let $\Omega = \text{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x_1(t)\}$. Then, a one-form $\omega_1 = x_1(t)\mathrm{d}u_1(t-1) + \mathrm{d}u_2(t-1)$ satisfies condition (*ii*) of Theorem 1. Thus, $\varphi_{1,1} = u_1(t), \varphi_{1,2} = u_2(t)$. One can see that condition (4) is satisfied.

Example 3. This example demonstrates that, unlike in the delay-free case, for time delay systems, the existence of a dynamic feedback, which solves the DDP, does not yield that there also exists a static solution. Consider a nonlinear time delay system:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1)u(t-1) + x_3(t) \\ \dot{x}_2(t) &= q(t) \\ \dot{x}_3(t) &= x_2(t-1) \\ y(t) &= x_1(t). \end{aligned} \tag{10}$$

This system cannot be disturbance decoupled by static feedback. This happens because $\dot{y}(t)$ depends on $x_3(t)$, whose shift is smaller than the relative shift $\mu = 1$, and thus cannot be compensated. Nevertheless, there exists a dynamic feedback that solves the problem. For system (10) $\Omega = \text{span}_{\mathcal{K}(\vartheta]}\{dx_1(t)\}$ and one-forms ω_i in Theorem 2 are $\omega_0 = d[x_2(t-1)u(t-1)]$ and $\omega_1 = d\ddot{y}(t) = \dot{\omega}_0 + dx_2(t-1)$. A feedback can be found by solving the equations $\eta(t) = x_2(t)u(t)$, $v(t) = \dot{\eta}(t) + x_2(t)$ in variables $\dot{\eta}(t), u(t)$:

$$\begin{aligned} \dot{\eta}(t) &= v(t) - x_2(t) \\ u(t) &= \frac{\eta(t)}{x_2(t)}. \end{aligned} \tag{11}$$

In the closed-loop system $\dot{y}(t) = \eta(t-1) + x_3(t)$ and $\ddot{y}(t) = v(t-1)$. Therefore, the output $y(t)$ and its derivatives do not depend on the disturbance explicitly.

VI. CONCLUSIONS

Necessary and sufficient conditions have been derived for the solvability of the DDP for nonlinear time-delay systems by a pure shift dynamic compensator. The new results were obtained using the rank of a one-form instead of integrability. Necessary and sufficient conditions for the solvability of the problem by dynamic compensator were also given for SISO systems. Research perspectives include the search for a general dynamic feedback solution for MIMO systems and the study of stability of the closed-loop system.

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Integrability for nonlinear time-delay systems

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***Abstract*—In this paper the notion of integrability is defined for 1-forms defined in the time–delay context. While in the delay-free case, a set of 1-forms defines a vector space, it is shown that 1-forms computed for time-delay systems have to be viewed as elements of a module over a certain non-commutative polynomial ring. Two notions of integrability are defined, strong and weak integrability, which coincide in the delay-free case. Necessary and sufficient conditions are given to check if a set of 1-forms is strongly or weakly integrable. To show the importance of the topic, integrability of 1-forms is used to characterize the accessibility property for nonlinear time-delay systems. The possibility of transforming a system into a certain normal form is also considered.**

***Index Terms*—Time-delay systems, algebraic methods, accessibility**

I. Introduction

Time-delay systems are used in many important areas, like telecommunications, remote control and biological systems (see [1] and the references therein). The great success of algebraic [2] and differential geometric [3], [4] methods for delay-free systems has motivated many authors to generalize the approaches to the time-delay case [5], [6], [7], [8], [9], [10], [11], [12]. Of major importance in these approaches is the notion of integrability of codistributions (or distributions). In the delay-free case, the integrability is fully characterized by the so-called Frobenius Theorem. The class of time-delay systems is a special class of infinite dimensional systems though, it was shown in [6] that Frobenius Theorem is still valid to derive specific results. In [13] and [5], integrability was tackled in the case of one-dimensional submodules and a necessary and sufficient condition was derived. A sufficient condition for the general case was also given in [5]. A different approach was used in [6], where the integrability was characterized using the extended Lie brackets.

At this point, there is no general theory about integrability of 1-forms/codistributions in the case of time-delay systems. The main goal in this paper is to clarify those notions of integrability of 1-forms and which are not fully captured by the integrability of vector fields. In [5], the existence of an exact basis is defined for a module, while in [13] as the existence of an exact basis is defined for the closure of a module.

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In this paper, we use the notion of closure of a submodule [14] to define two notions of integrability - strong integrability and weak integrability - and give necessary and sufficient conditions to check both these properties for a set of 1-forms. The relationship between the obtained results and the dual results of [6] is also discussed. Then, two problems are considered, where the integrability of 1-forms plays a key role. Accessibility of nonlinear time-delay systems is characterized through the integrability of a certain submodule and conditions are found under which a given system can be transformed into a certain normal form. Preliminary results and examples can be found in [15].

The paper is organized in the following manner. In Section II, basic mathematical notions are given, which will be used in the paper. In Section III, the main results are presented. The integrability of 1-forms is defined and the condition is given, together with two algorithms, to check integrability. In section IV, the connection between the results of section III and [6] is argued. Applications of integrability of 1-forms are considered in Section V. The paper ends with some conclusions.

II. Preliminaries

Non-commutative algebra is used to define the integrability of 1-forms and to find the necessary and sufficient conditions to get integrability (which is done in Section III). More precisely, the proposed method refers to modules over non-commutative rings (see [13], [7]). In this section, the mathematics and definitions beyond this method are introduced.

Let $\mathcal{K}$ denote the field of meromorphic functions that depend on a finite number of variables from the set $\{x(t-i); i \in \mathbb{N}\}$, $\dim(x(t)) = n$. Also, denote by $\mathcal{E}$ the vector space spanned by the differentials $\{\mathrm{d}x(t-i); i \in \mathbb{N}\}$ over the field $\mathcal{K}$. The elements of $\mathcal{E}$ are called 1-forms.

Consider the time shift acting over functions $\delta : \mathcal{K} \to \mathcal{K}$ defined as $\delta f(x(t-i); i \in \mathbb{N}) := f(x(t-i-1); i \in \mathbb{N})$. On the 1-form $\omega = \sum_{i=1}^{n} \sum_{j=0}^{k} a_i \mathrm{d}x_i(t-j)$, one gets that the time shift $\delta\omega$ of ω is given by

$$\delta\omega =: \omega^- = \sum_{i=1}^{n} \sum_{j=0}^{k} \delta(a_i) \mathrm{d}x_i(t-j-1).$$

Accordingly, $\omega^{-p} := \delta\omega^{-p+1}$. Furthermore, ω is said to be exact if there exists $\varphi \in \mathcal{K}$ such that $\omega = \mathrm{d}\varphi$. The use of exterior differentiation and of the wedge product allows to state in a concise manner both Poincaré Lemma and Frobenius Theorem [16]:

- the 1-form ω is locally exact if and only if $\mathrm{d}\omega = 0$;
- the codistribution $\mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_q\}$ is integrable if and only if the $q+2$-forms $\mathrm{d}\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_q$ are zero for $i = 1, \ldots, q$, where $\wedge$ denotes the wedge product of differential forms [16].

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The following notation is also used:

$$\mathrm{d}\omega = 0 \mod \mathrm{span}_{\mathcal{K}}\{\bar{\omega}_1, \ldots, \bar{\omega}_q\}$$

means that $\mathrm{d}\omega \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_q = 0$.

Next, the non-commutative ring of polynomials $\mathcal{K}(\vartheta]$ is constructed. The elements of this ring are polynomials in the form $a_0 + a_1\vartheta + \cdots + a_s\vartheta^s$ for some finite $s \in \mathbb{N}$ and $a_i \in \mathcal{K}$, $i = 0, \ldots, s$. Addition is defined on this ring as usual, but the rule for multiplication is $\vartheta\psi = \delta(\psi)\vartheta$ for some $\psi \in \mathcal{K}$. Similarly, $\vartheta(\omega) = \delta\omega$, and when no confusion arises, $\vartheta(\omega)$ will be denoted $\vartheta\omega$. The set of matrices $\mathcal{K}(\vartheta]^{k\times s}$ is also used in this paper.

Definition 1: [17] A matrix $A(\vartheta) \in \mathcal{K}(\vartheta]^{k\times k}$ is unimodular if it is invertible within the ring of polynomial matrices, *i.e.* if there exists a $B(\vartheta) \in \mathcal{K}(\vartheta]^{k\times k}$ such that $A(\vartheta)B(\vartheta) = B(\vartheta)A(\vartheta) = I_k$.

Example 1: The matrix

$$A(\vartheta) = \begin{pmatrix} 1 & x_2(t-1)\vartheta \\ \vartheta & 1 + x_2(t-2)\vartheta^2 \end{pmatrix}$$

is unimodular, since the matrix

$$A(\vartheta)^{-1} = \begin{pmatrix} 1 + x_2(t-1)\vartheta^2 & -x_2(t-1)\vartheta \\ -\vartheta & 1 \end{pmatrix}$$

is such that $A(\vartheta)A(\vartheta)^{-1} = A(\vartheta)^{-1}A(\vartheta) = I_2$. Note that while any unimodular matrix has full rank, the converse is not true. For example, there is no polynomial inverse for $(1+\vartheta)$.

Let us now note that the set of 1-forms $\mathcal{E}$ has the structure of a vector space over the field $\mathcal{K}$. However, it has also the structure of a module, denoted $\mathcal{M}$, over the ring $\mathcal{K}(\vartheta]$, *i.e.*

$$\mathcal{M} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t)\}.$$

Example 2: The 1-forms $\mathrm{d}x_1(t)$ and $\mathrm{d}x_1(t-1)$ are independent over the field $\mathcal{K}$, but dependent over the ring $\mathcal{K}(\vartheta]$, since $\vartheta\mathrm{d}x_1(t) - \mathrm{d}x_1(t-1) = 0$. This simple example shows that the action of time-delay is taken into account in $\mathcal{M}$, but not in $\mathcal{E}$. This motivates the definition of the module $\mathcal{M}$.

A left-submodule of $\mathcal{M}$ consists of all possible linear combinations of given 1-forms (or row-vectors) $\{\omega_1, \ldots, \omega_k\}$ over the ring $\mathcal{K}(\vartheta]$, *i.e.* linear combinations of row-vectors. A left-submodule, generated by $\{\omega_1, \ldots, \omega_k\}$, is denoted by $\mathcal{A} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\}$. A right-submodule of $\hat{\mathcal{M}}$ [6] consists of all possible linear combinations of column-vectors $q_1, \ldots, q_l$, $q_i \in \mathcal{K}(\vartheta]^{n\times 1}$, and is denoted by $\Delta = \mathrm{span}_{\mathcal{K}(\vartheta]}\{q_1, \ldots, q_l\}$.

Definition 2: The left closure of a left-submodule $\mathcal{A}$ of $\mathcal{M}$, denoted by $cl_{\mathcal{K}(\vartheta]}(\mathcal{A})$, is defined as $cl_{\mathcal{K}(\vartheta]}(\mathcal{A}) = \{\omega \in \mathcal{M} \mid \exists p(\vartheta) \in \mathcal{K}(\vartheta],\ p(\vartheta)\omega \in \mathcal{A}\}$.

By definition, the left closure of the left-submodule $\mathcal{A}$ is the largest left-submodule, containing $\mathcal{A}$, with the same rank as $\mathcal{A}$.

Definition 3: The right closure of a right-submodule Δ of $\hat{\mathcal{M}}$, denoted by $cl_{\mathcal{K}(\vartheta]}(\Delta)$, is defined as $cl_{\mathcal{K}(\vartheta]}(\Delta) = \{X \in \hat{\mathcal{M}} \mid \exists q(\vartheta) \in \mathcal{K}(\vartheta],\ Xq(\vartheta) \in \Delta\}$.

The right closure of the right-submodule Δ is the largest right-submodule, containing Δ, with the same rank as Δ.

Consider a left-submodule $\mathcal{A}$ of $\mathcal{M}$ and let the 1-forms ω be the basis of $\mathcal{A}$. These 1-forms can be written as $\omega = P(\vartheta)\mathrm{d}x(t)$ for some matrix $P(\vartheta) \in \mathcal{K}(\vartheta]^{k\times n}$.

Definition 4: The right-kernel (right-annihilator) of the left-submodule $\mathcal{A}$ is the right-submodule Δ containing all vectors $q(\vartheta) \in \hat{\mathcal{M}}$ such that $P(\vartheta)q(\vartheta) = 0$.

From Definition 4, the right-kernel is necessarily closed. Consider a right-submodule $\Delta = \mathrm{span}_{\mathcal{K}(\vartheta]}\{q_1(\vartheta), \ldots, q_l(\vartheta)\}$.

Definition 5: The left-kernel (left-annihilator) of Δ is the left-submodule $\mathcal{A}$ containing all 1-forms $\omega(\vartheta) \in \mathcal{M}$ such that $\omega(\vartheta)\Delta = 0$.

Again, from Definition 5, the left-kernel is necessarily closed. Finally, it is straightforward to prove the following.

Lemma 1: The right-kernels of the left-submodules $\mathcal{A}$ and $cl_{\mathcal{K}(\vartheta]}(\mathcal{A})$ are equal. The left-kernels of the right-submodules Δ and $cl_{\mathcal{K}(\vartheta]}(\Delta)$ are equal.

III. Results on integrability of 1-forms

In the present section a set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ independent over $\mathcal{K}(\vartheta]$ is considered (that is, there is no non zero linear combination over the ring $\mathcal{K}(\vartheta]$ which vanishes). As it will be shown hereafter, the fact of considering 1-forms as elements of $\mathcal{M}$ naturally leads to two different notions of integrability. If 1-forms are considered as elements of vector space $\mathcal{E}$, there is only one single notion of integrability.

In fact, as it happens in the delay-free case, if the set of 1-forms $\{\omega_1 \ldots, \omega_k\}$ are considered over $\mathcal{K}$, then they are said to be integrable if there exists an invertible matrix $A \in \mathcal{K}^{k\times k}$ and functions $\varphi = (\varphi_1, \ldots, \varphi_k)^T$, such that $\omega = A\mathrm{d}\varphi$. The full rank of A guarantees the invertibility of A, since $\mathcal{K}$ is a field. Instead, if the 1-forms $\{\omega_1 \ldots, \omega_k\}$ are viewed as elements of the module $\mathcal{M}$, then the matrix $A \in \mathcal{K}(\vartheta]^{k\times k}$ instead of $\mathcal{K}^{k\times k}$. Since $A(\vartheta)$ may be of full rank but not unimodular, it is necessary to distinguish two cases. Accordingly, one has the following two definitions of integrability.

Definition 6: A set of k 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}(\vartheta]$, is said to be strongly integrable if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that

$$\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}.$$

A set of k 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}(\vartheta]$, is said to be weakly integrable if there exist k independent functions $\{\varphi_1, \ldots, \varphi_k\}$, such that

$$\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\} \subseteq \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}.$$

If the 1-forms $\omega = (\omega_1, \cdots, \omega_k)^T$ are strongly (respectively weakly) integrable, then the left-submodule $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\}$ is said to be strongly (respectively weakly) integrable.

Clearly, strong integrability yields weak integrability. Also, the 1-forms ω are weakly integrable if and only if there exists a matrix $A(\vartheta) \in \mathcal{K}(\vartheta]^{k\times k}$ with full rank and functions $\varphi = (\varphi_1, \ldots, \varphi_k)^T$ such that $\omega = A(\vartheta)\mathrm{d}\varphi$. If in addition the matrix $A(\vartheta)$ can be chosen to be unimodular, then the 1-forms ω are also strongly integrable.

Example 3: The 1-form $\omega_1 = \mathrm{d}x(t) + x(t-1)\mathrm{d}x(t-1)$ is weakly integrable since $\omega_1 = (1 + x(t-1)\vartheta)\mathrm{d}x(t)$. It is also strongly integrable as $\omega_1 = \mathrm{d}(x(t) + 1/2x(t-1)^2)$. Instead, the 1-form $\omega_2 = \mathrm{d}x_1(t) + x_2(t)\mathrm{d}x_1(t-1) = (1 + x_2(t)\vartheta)\mathrm{d}x_1(t)$

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is weakly integrable, but not strongly integrable, because the polynomial $1+x_2(t)\vartheta$ is not invertible.

Remark 1: Note that integrability of a closed left-submodule $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1,\dots,\omega_k\}$ always implies strong integrability. As a consequence, the two notions of strong and weak integrability coincide in case of delay-free 1-forms.

Integrability of a set of k 1-forms $\{\omega_1,\dots,\omega_k\}$ is tested thanks to the so-called

Derived Flag Algorithm (DFA):
Starting from a given I_0 the algorithm computes

$$I_i = \mathrm{span}_{\mathcal{K}}\{\omega \in I_{i-1} \mid \mathrm{d}\omega = 0 \mod I_{i-1}\}. \quad (1)$$

The sequence (1) converges as it defines a strictly decreasing sequence of vector spaces I_i and by the standard Frobenius Theorem, the limit I_∞ has an exact basis, which represents the largest integrable codistribution contained in I_0.

In order to define I_0 one has to note that when considering a set of k 1-forms $\{\omega_1,\cdots,\omega_k\}$, some shifts of ω_i are required for integration. It follows that the inizialization

$$I_0^p = \mathrm{span}_{\mathcal{K}}\{\omega_1,\dots,\omega_k,\dots,\omega_1^{-p},\dots,\omega_k^{-p}\}, \quad (2)$$

allows to compute the smallest number of time shifts required for the given 1-forms for the maximal integration of the submodule. More precisely, the sequence I_i^p defined by (1) converges to an integrable vector space

$$I_\infty^p = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi_1^p,\dots,\mathrm{d}\varphi_{\gamma_p}^p\} \quad (3)$$

for some $\gamma_p \geq 0$. By definition, $\mathrm{d}\varphi_i^p \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1,\dots,\omega_k\}$ for $i=1,\dots,\gamma_p$ and $p\geq 0$. The exact 1-forms $\mathrm{d}\varphi_i^p$, $i=1,\dots,\gamma_p$, are independent over $\mathcal{K}$, but may not be independent over $\mathcal{K}(\vartheta]$. A basis for $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1^p,\dots,\mathrm{d}\varphi_{\gamma_p}^p\}$ is obtained by computing a basis for

$$I_\infty^0 \cup \bigcup_{i=1}^{p} \left[I_\infty^i \mod(I_\infty^{i-1}, \delta I_\infty^{i-1})\right]$$

as $I_\infty^i + \delta I_\infty^i \subset I_\infty^{i+1}$.

Remark 2: A different initialization of derived flag algorithm is

$$\begin{aligned}\tilde{I}_0^p \;=\;& \mathrm{span}_{\mathcal{K}}\{\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1,\cdots,\omega_k\}\cap \\ & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t),\cdots,\mathrm{d}x(t-p)\}\}. \end{aligned} \quad (4)$$

which allows to compute for each $p\geq 0$, the exact differentials contained in the given submodule and which depend on $x(t),\dots,x(t-p)$ only. Both initialization allow the algorithm to converge towards the same integrable submodule over $\mathcal{K}(\vartheta]$, but follow different steps, as shown in the next example.

Example 4: Let $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x(t-2)\}$. On one hand, initialization (2) is completed for $p=0$ as no time-shift of $\mathrm{d}x(t-2)$ is required for its integration. On the other hand, initialization (4) yields a 0 limit for $p=0$ and $p=1$ as the exact differential involves larger delays than $x(t)$ and $x(t-1)$. The final result is obtained for $p=2$.

Assume that the maximum delay that appears in $\{\omega_1,\dots,\omega_k\}$ (either in the coefficients or differentials) is s. The necessary and sufficient condition for strong integrability of 1-forms $\{\omega_1,\dots,\omega_k\}$ is given by the following theorem in terms of the limit I_∞^p.

Theorem 1: A set of 1-forms $\{\omega_1,\dots,\omega_k\}$, independent over $\mathcal{K}(\vartheta]$, is strongly integrable if and only if there exists an index $p\leq s(k-1)$ such that starting from I_0^p defined by (2), the derived flag algorithm (1) converges to I_∞^p given by (3) with

$$\omega_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1^p,\dots,\mathrm{d}\varphi_{\gamma_p}^p\} \quad (5)$$

for $i=1,\dots,k$.

Proof. Necessity. If a set of 1-forms $\{\omega_1,\dots,\omega_k\}$, independent over $\mathcal{K}(\vartheta]$, is strongly integrable, then there exist k functions φ_i, $i=1,\dots,k$, such that $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1,\dots,\omega_k\} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_k\}$. Thus $\omega_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_k\}$ and $\mathrm{d}\varphi_i \in \mathrm{span}_{\mathcal{K}}\{\omega_1,\dots,\omega_k,\dots,\omega_1^{-p},\dots,\omega_k^{-p}\}$ for $i=1,\dots,k$ and some $p\geq 0$. Clearly, $\mathrm{d}\varphi_i\in I_\infty^p$ and the condition (5) is satisfied for $\gamma_p=k$.

It remains to show that $p\leq s(k-1)$. Note that there exist infinitely many pairs $(A(\vartheta),\varphi)$, that satisfy $\omega = A(\vartheta)\mathrm{d}\varphi$. Since the degree of unimodular matrices $A(\vartheta)$ has a lower bound, then one can find a pair $(A(\vartheta),\varphi)$, where the degree of matrix $A(\vartheta)$ is minimal among all possible pairs. Let $A(\vartheta)$ be such a unimodular matrix for some functions $\varphi=(\varphi_1,\dots,\varphi_k)^T$. Note that $A(\vartheta)$ and φ are not unique.

We show that the degree of $A(\vartheta)$ is less or equal to s. By contradiction, assume that the degree of $A(\vartheta)$ is larger than s, for example $s+1$. Then for some i

$$\omega_i = a_1^i(\vartheta)\mathrm{d}\varphi_1 + \cdots + a_k^i(\vartheta)\mathrm{d}\varphi_k, \quad (6)$$

where $a_j^i(\vartheta)\in\mathcal{K}(\vartheta]$, $j=1,\dots,k$, and at least one polynomial $a_j^i(\vartheta)$ has degree $s+1$.

Let $a_j^i(\vartheta)=\sum_{l=0}^{s+1} a_{j,l}^i\vartheta^l$, $j=1,\dots,k$. From (6) one gets

$$\omega_i = \sum_{j=1}^{k}\sum_{\ell=0}^{s+1} a_{j,\ell}^i \mathrm{d}\varphi_j^{-i}, \quad (7)$$

where at least one coefficient $a_{j,s+1}^i\in\mathcal{K}$ is non-zero. For simplicity assume that $a_{1,s+1}^i\neq 0$ and $a_{\gamma,s+1}^i=0$ for $\gamma=2,\dots,k$.[1] We have assumed that the maximum delay in ω_i is s, but the maximum delay in $\mathrm{d}\varphi_1^{-s-1}$ is at least $s+1$.

Note that $\mathrm{d}\varphi_1,\dots,\mathrm{d}\varphi_1^{-s-1},\dots,\mathrm{d}\varphi_k^{-s-1}$ are independent over $\mathcal{K}$. Therefore, to eliminate $\mathrm{d}\varphi_1^{-s-1}$ from (7),

$$\mathrm{d}\varphi_1^{-s-1} = \sum_{j=1}^{k} b_j \mathrm{d}\varphi_j^{-l_j} + \bar{\omega} \quad (8)$$

for some coefficients $b_j\in\mathcal{K}$, delays $l_j\leq s$ and the 1-form $\bar{\omega}\in\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x,\mathrm{d}x^-,\dots,\mathrm{d}x^{-s}\}$. Let $l:=\min\{l_j\}$. For clarity, let $l=l_2$. We show that $\bar{\omega}$ can be chosen such that it is integrable. By contradiction, assume that $\bar{\omega}$ can not be chosen integrable. Then, the coefficients of $\bar{\omega}$ must depend on higher delays than s. Since $\bar{\omega}$ is not integrable, then the coefficients of $a_{1,s+1}^i\bar{\omega}$ depend also on higher delays than s. Now, substitute $a_{1,s+1}^i\mathrm{d}\varphi_1^{-s-1}$ to (7). One gets that ω_i depends on $a_{1,s+1}^i\bar{\omega}$

[1] If there are multiple non-zero coefficients $a_{\gamma,s+1}^i$, then one would have multiple equations like (8) below.

and thus also on higher delays than s. This is a contradiction and thus $\bar{\omega}$ can be chosen integrable.

Let $\bar{\omega} = a\mathrm{d}\phi^{-l}$ for some $a, \phi \in \mathcal{K}$. Then $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1,\cdots,\mathrm{d}\varphi_k\} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1, \mathrm{d}\phi, \mathrm{d}\varphi_3,\cdots,\mathrm{d}\varphi_k\}$ and there exists an unimodular matrix $\bar{A}(\vartheta)$ with smaller degree than $A(\vartheta)$, and functions $\bar{\varphi} = (\varphi_1, \phi, \varphi_3, \ldots, \varphi_k)^T$ such that $\omega = \bar{A}(\vartheta)\mathrm{d}\bar{\varphi}$, which leads to a contradiction. Thus the degree of $A(\vartheta)$ must be less than or equal to s and the degree of $A^{-1}(\vartheta)$ is less or equal to $s(k-1)$, i.e. $p \le s(k-1)$.

Sufficiency. Let $I^p_\infty = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi\}$, where $p \le s(k-1)$. By construction $I^p_\infty \subset \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\}$ and by (5) $\omega_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$ for $i = 1, \ldots, k$. Thus, $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$. ■

Since $I^p_\infty \subseteq I^{p+1}_\infty$ for any $p \ge 0$, one can check the condition (5) step-by-step, increasing the value of p every step. When for some $p = \bar{p}$ the condition (5) is satisfied, then it is satisfied for all $p > \bar{p}$.

Given the set of 1-forms $\{\omega_1, \ldots, \omega_k\}$, independent over $\mathcal{K}(\vartheta]$, the basis of vector space $I^{s(k-1)}_\infty$ defines the basis for the largest integrable left-submodule contained in $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_k\}$.

Lemma 2: A set of 1-forms $\{\omega_1, \ldots, \omega_k\}$ is weakly integrable if and only if the left closure of the left-submodule, generated by $\{\omega_1, \ldots, \omega_k\}$, is (strongly) integrable.

Proof: Necessity. By definitions of weak integrability and left closure, there exist functions $\varphi = (\varphi_1, \ldots, \varphi_k)^T$ such that $\mathrm{d}\varphi = A(\vartheta)\bar{\omega}$, where $\bar{\omega}$ is the basis of the closure of the left-submodule, generated by $\{\omega_1, \ldots, \omega_k\}$. Choose $\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$ such that for $i = 1, \ldots, k$

$$\mathrm{d}\varphi_i \neq a\mathrm{d}\phi + \sum_{j=1;j\neq i}^{k} b_j(\vartheta)\mathrm{d}\varphi_j \tag{9}$$

for any $\phi \in \mathcal{K}$ and $b_j(\vartheta) \in \mathcal{K}(\vartheta]$. It remains to show that one can choose φ such that $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$.

By contradiction, assume that one can not choose φ such that $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$. Then $\bar{\omega}_k \notin \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$ and also $\bar{\omega}_k^{-j} \notin \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$ for $j \ge 1$ and any φ. Really, if

$$\bar{\omega}_k^{-j} = \sum_i c_i(\vartheta)\mathrm{d}\varphi_i, \tag{10}$$

then, since on the left-hand side of (10) everything is delayed at least j times, everything that is delayed less than j times on the right-hand side should cancel out. Therefore, one is able to find functions $\phi_i, \psi_i \in \mathcal{K}$, $i = 1, \ldots, k$, such that $\mathrm{d}\varphi_i = \mathrm{d}\phi_i + \mathrm{d}\psi_i$ and

$$c_i(\vartheta)\mathrm{d}\phi_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x^{-j}\} \qquad \sum_i c_i(\vartheta)\mathrm{d}\psi_i = 0.$$

Now, because of (9), $\psi_i = 0$, $\phi_i = \varphi_i$ for $i = 1, \ldots, k$ and thus $\delta^j\bar{\omega}_k = \delta^j \sum_i \bar{c}_i(\vartheta)\mathrm{d}\varphi_i^{+j}$ which yields $\bar{\omega}_k = \sum_i \bar{c}_i(\vartheta)\mathrm{d}\varphi_i^{+j}$. Clearly, 1-forms $\mathrm{d}\varphi_i^{+j}$ have to belong to $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\bar{\omega}\}$, because $\mathrm{d}\varphi_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\bar{\omega}\}$. Now, one has a contradiction and therefore $\bar{\omega}_k^{-j} \notin \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$ for $j \ge 1$. Then, by construction $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\} \subset \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \ldots, \omega_{k-1}\}$, which is impossible. Thus, the assumption that one can not choose φ such that $\bar{\omega}_i \in \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}\varphi\}$ must be wrong.

Sufficiency. Sufficiency is satisfied directly by the definitions of strong and weak integrability. ■

Example 5: Consider the following 1-forms

$$\begin{aligned}\omega_1 &= x_3(t-1)\mathrm{d}x_2(t) + x_2(t)\mathrm{d}x_3(t-1) + x_2(t-1)\mathrm{d}x_1(t-1)\\ \omega_2 &= x_3(t-2)\mathrm{d}x_2(t-1) + x_2(t-1)\mathrm{d}x_3(t-2)\\ &\quad + \mathrm{d}x_1(t) + x_2(t-2)\mathrm{d}x_1(t-2).\end{aligned}$$

One gets for $s(k-1) = 2$:

$$I^2_\infty = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1(t), \mathrm{d}x_1(t-1), \mathrm{d}(x_2(t)x_3(t-1))\}.$$

When one eliminates the basis elements, which are dependent over $\mathcal{K}(\vartheta]$, one gets that the rank of $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x_1(t), \mathrm{d}x_1(t-1), \mathrm{d}(x_2(t)x_3(t-1))\}$ is 2. To check the condition (5), one has to check whether there exists a matrix $A(\vartheta)$ such that $\omega = A(\vartheta)\mathrm{d}\varphi$, where $\omega = (\omega_1, \omega_2)^T$, $\varphi = (\varphi_1, \varphi_2)^T$, $\varphi_1 = x_2(t)x_3(t-1)$, $\varphi_2 = x_1(t)$. In fact, $\omega = A(\vartheta)\mathrm{d}\varphi$, where the unimodular matrix $A(\vartheta)$ is defined in Example 1. Thus, the 1-forms (11) are strongly integrable.

Example 6: Consider the following 1-forms:

$$\begin{aligned}\omega_1 &= \mathrm{d}x_2(t)\\ \omega_2 &= x_4(t-1)\mathrm{d}x_1(t) + x_2(t)\mathrm{d}x_2(t-1) + x_1(t)\mathrm{d}x_4(t-1)\\ \omega_3 &= x_3(t)x_4(t)\mathrm{d}x_2(t) + x_2(t)x_4(t)\mathrm{d}x_3(t)\\ &\quad + x_3(t-1)\mathrm{d}x_2(t-1) + x_2(t-1)\mathrm{d}x_3(t-1).\end{aligned} \tag{11}$$

For $s(k-1) = 2$: $I^2_\infty = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2(t), \mathrm{d}(x_4(t-1)x_1(t)), \mathrm{d}x_2(t-1), \mathrm{d}x_2(t-2), \mathrm{d}(x_4(t-2)x_1(t-1))\}$. Now, $\omega_1 \in I^2_\infty$ and $\omega_2 \in I^2_\infty$, but $\omega_3 \notin I^2_\infty$. Thus, 1-forms (11) are not strongly integrable, and $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x_2(t), \mathrm{d}(x_4(t-1)x_1(t))\}$ is the largest integrable left-submodule, contained in $\mathcal{A} = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \omega_2, \omega_3\}$.

Now, one can check if 1-forms (11) are weakly integrable. For that, one has to compute the left closure of $\mathcal{A}$ and check if it is strongly integrable. In practice, the left closure of a left-submodule $\mathcal{A}$ can be computed as the left-kernel of its right-kernel Δ. Thus, the right-kernel of $\mathcal{A}$ is $\Delta = \mathrm{span}_{\mathcal{K}(\vartheta]}\{q(\vartheta)\}$, where $q(\vartheta) = (x_1(t)\vartheta, 0, 0, -x_4(t))^T$. The left-kernel of Δ is

$$cl_{\mathcal{K}(\vartheta]}(\mathcal{A}) = \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}x_2(t), \mathrm{d}x_3(t), \mathrm{d}(x_4(t-1)x_1(t))\}.$$

Therefore, the 1-forms (11) are weakly integrable.

IV. Integrability of right-submodules

Since the left annihilator of a right submodule is by construction closed, the integrability of a right submodule refers only to weak integrability. Consider the right-submodule

$$\Delta = \mathrm{span}_{\mathcal{K}(\vartheta]}\{q_1(\vartheta), \ldots, q_k(\vartheta)\},$$

where $q_i(\vartheta)$ are the $n \times 1$ column vectors.

Definition 7: The right-submodule Δ is said to be integrable if the left-kernel of Δ admits an exact basis.

Define a matrix $Q(\vartheta) = (q_1(\vartheta), \cdots, q_l(\vartheta))$ and let $Q(\vartheta) = Q_0 + Q_1\vartheta + \cdots + Q_s\vartheta^s$ for some $s \ge 0$ and matrices $Q_j \in \mathcal{K}^{n\times k}$, $j = 0, \ldots, s$. Assume, that the ranks of matrices

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$Q(\vartheta)$ and Q_0 are k. Consider the distributions Δ_i defined on $\mathbb{R}^{(i+s+1)n}$,

$$\Delta_i := \mathrm{span}_{\mathcal{K}} \left\{ \begin{array}{cccccc} Q_0 & \cdots & Q_s & 0 & \cdots & \cdots \\ 0 & \ddots & \cdots & \ddots & 0 & \cdots \\ \vdots & \ddots & \delta^i Q_0 & \cdots & \delta^i Q_s & 0 \\ 0 & \cdots & & \cdots & 0 & I_{ns} \end{array} \right\}, i \geq 0$$

Theorem 2: [6] The right-submodule Δ is integrable if and only if there exists an integer γ such that, locally around some point $x^0(\cdot)$, $\dim(\bar{\Delta}_\gamma) - \dim(\bar{\Delta}_{\gamma-1}) = k$.

The integrability of right-submodules and 1-forms are connected by the following corollary, which follows from Corollary 2 and Lemma 1.

Corollary 1: Weak integrability of 1-forms is equivalent to the integrability of its right-kernel.

To show more explicitly how the integrability of right-submodules and weak integrability of 1-forms are related, consider the Algorithm (1) inizialized with (4). The left-kernel of Δ_i, defined above, is equal to I^i_∞, where I^i_∞ is computed with respect to the closure of a given submodule.

The next example shows, that in some cases, one can not use the results of Section IV to check the integrability of 1-forms. In that case, one has to use the results of Section III.

Example 7: Consider the 1-forms

$$\begin{aligned} \omega_1 &= x_1(t-1)\mathrm{d}x_1(t) + x_1(t)\mathrm{d}x_1(t-1) \\ &\quad -x_3(t)\mathrm{d}x_2(t-1) + \mathrm{d}x_3(t-1) \\ \omega_2 &= \mathrm{d}x_2(t) + x_3(t)\mathrm{d}x_2(t-1). \end{aligned} \tag{12}$$

The 1-forms $\omega = (\omega_1, \omega_2)^T$ can be written as

$$\omega = \begin{pmatrix} x_1(t-1) + x_1(t)\vartheta & -x_3(t)\vartheta & \vartheta \\ 0 & 1 + x_3(t)\vartheta & 0 \end{pmatrix} \mathrm{d}x(t).$$

The right-kernel of the left-submodule $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \omega_2\}$ is not causal (*i.e.* one needs forward-shifts of variables $x(t)$ to represent it), thus one can not use Theorem 2 to check the weak integrability of 1-forms (12). But, one can check by using Corollary 2 and Theorem 1, that $\mathrm{span}_{\mathcal{K}(\vartheta]}\{\omega_1, \omega_2\} \subset \mathrm{span}_{\mathcal{K}(\vartheta]}\{\mathrm{d}(x_1(t)x_1(t-1) + x_3(t-1)), \mathrm{d}x_2(t)\}$ and thus, 1-forms (12) are weakly integrable.

V. Applications of integrability

In this Section, two problems are considered, where integrability of 1-forms is used. First, it is shown that accessibility of nonlinear time-delay systems can be characterized through integrability of a certain left-submodule. Secondly, necessary and sufficient conditions are given to transform a nonlinear time-delay system into the form (17) below.

Consider the nonlinear time-delay system

$$\dot{x}(t) = f(x(t-i), u(t-i); i = 0, \ldots, d_{max}), \tag{13}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Also, assume that the function f is meromorphic. To simplify the presentation, the following notation is used: $x(\cdot) := (x(t), x(t-1), \ldots)$. The notation $\varphi(x(\cdot))$ means that function φ can depend on $x(t), \ldots, x(t-i)$ for some finite $i \geq 0$. The same notation is used for other variables.

In this section $\mathcal{K}_u$ denotes the field of meromorphic functions that depend on a finite number of variables from the set $\mathcal{C} = \{x(\cdot), u(\cdot), \ldots, u^{(k)}(\cdot); k \in \mathbb{N}\}$. Also, denote by $\mathcal{E}_u$ the vector space spanned by the symbols $\mathrm{d}\mathcal{C} = \{\mathrm{d}x(\cdot), \mathrm{d}u(\cdot), \ldots, \mathrm{d}u^{(k)}(\cdot); k \in \mathbb{N}\}$ over the field $\mathcal{K}_u$ and $\mathcal{M}_u = \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t), \mathrm{d}u^{(k)}(t); k \geq 0\}$ is the corresponding module spanned over the ring $\mathcal{K}_u(\vartheta]$.

Definition 8: A 1-form $\omega \in \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t)\}$ has relative degree r, if r is the smallest integer such that $\omega^{(r)} \notin \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t)\}$. A function $\varphi \in \mathcal{K}_u$ is said to have relative degree r if the 1-form $\mathrm{d}\varphi$ has relative degree r.

Define a sequence of left-submodules $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots$ of $\mathcal{M}_u$ as follows:

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t)\} \\ \mathcal{H}_i &= \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\omega \in \mathcal{H}_{i-1} \mid \dot{\omega} \in \mathcal{H}_{i-1}\}. \end{aligned} \tag{14}$$

Since $\mathcal{H}_1$ has finite rank and all the left-submodules $\mathcal{H}_i$ are closed, sequence (14) converges (see [7]). Let $\mathcal{H}_\infty$ be the limit of sequence (14). By $\hat{\mathcal{H}}_i$ one denotes the largest integrable left-submodule contained in $\mathcal{H}_i$. A left-submodule $\mathcal{H}_i$ contains all the 1-forms with relative degree i or bigger. Thus, $\mathcal{H}_\infty$ contains all the 1-forms which have infinite relative degree.

A. Accessibility

In this subsection the accessibility property of system (13) is characterized using the notion of autonomous element, as is done in [2] for delay-free systems, or in [18] for linear time-delay systems through the notion of torsion elements.

Definition 9: A nonzero function $\varphi \in \mathcal{K}_u$ is said to be an autonomous element of system (13) if there exist an integer ν and a nonzero function $F \in \mathcal{K}_u$ such that

$$F(\varphi, \dot{\varphi}, \ldots, \varphi^{(\nu)}) = 0. \tag{15}$$

Now, accessibility of system (13) can be defined as non-existence of autonomous elements.

Definition 10: System (13) is said to be accessible if there does not exist any autonomous element.

Lemma 3: Function $\varphi \in \mathcal{K}_u$ is an autonomous element of system (13) if and only if it has infinite relative degree.

Proof: Necessity. Let φ be an autonomous element of system (13) and assume it has finite relative degree. Then, $\dim(\mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}\varphi, \ldots, \mathrm{d}\varphi^{(k-1)}\}) = k$ for all $k \geq 1$. Because of (15), the last equality is not satisfied for $k = \nu + 1$, which is a contradiction. Thus, φ has infinite relative degree.

Sufficiency. Let φ be a nonzero function with infinite relative degree. Then 1-forms $\mathrm{d}\varphi, \ldots, \mathrm{d}\varphi^{(n)}$ are dependent over the ring $\mathcal{K}_u(\vartheta]$. Thus, there exist $a_i \in \mathcal{K}_u(\vartheta]$, $i = 0, \ldots, n$, where at least one of them is nonzero, such that

$$\omega := a_0\mathrm{d}\varphi + \cdots + a_n\mathrm{d}\varphi^{(n)} = 0. \tag{16}$$

Then, there exists a delay differential equation as $\alpha(\delta, \varphi, \ldots, \varphi^{(n)}) = 0$. By Definition 9 function φ is an autonomous element of system (13). ∎

Now, one can characterize accessibility of system (13) in the following way.

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Theorem 3: System (13) is accessible if and only if $\hat{\mathcal{H}}_\infty = \emptyset$.

Proof: Necessity. If system (13) is accessible, then by Lemma 3 there does not exist any non constant function in $\mathcal{K}_u$ with infinite relative degree. Therefore, there can not be any exact nonzero 1-form in $\mathcal{H}_\infty$ and thus $\hat{\mathcal{H}}_\infty = \emptyset$ must be true.

Sufficiency. The left-submodule $\mathcal{H}_\infty$ contains all the 1-forms with infinite relative degree. Since $\hat{\mathcal{H}}_\infty = \emptyset$, there is no non constant exact 1-form with infinite relative degree and therefore, by Lemma 3, there is no autonomous element. ∎

B. Normal form

In this subsection, one considers the possibility of transforming (13), with one single input ($m = 1$), into the form

$$\begin{aligned} \dot{z}^1(t) &= f_1(z^1(\cdot), u(\cdot)) \\ \dot{z}^2(t) &= f_2(z^1(\cdot), z^2(\cdot)), \end{aligned} \tag{17}$$

where the dynamics corresponding to $z^1(t)$ is accessible, by a state transformation $z(t) = \varphi(x(\cdot))$ and a regular static feedback $u(t) = \alpha(x(\cdot), v(\cdot))$.

To solve the above mentioned problem, first, we define invariant and controlled invariant left-submodules. For that, consider a left-submodule $\mathcal{A} = \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\omega_1, \ldots, \omega_k\}$ and let $\dot{\mathcal{A}} = \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\dot{\omega}_1, \ldots, \dot{\omega}_k\}$.

Definition 11: A left-submodule $\mathcal{A} \subseteq \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t)\}$ is said to be invariant if $\dot{\mathcal{A}} \subseteq cl_{\mathcal{K}_u(\vartheta]}(\mathcal{A}) + \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}u(t)\}$.

Definition 12: A left-submodule $\mathcal{A} \subseteq \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}x(t)\}$ is said to be controlled invariant if there exists a regular feedback $u(t) = \alpha(x(\cdot), v(\cdot))$ such that $\dot{\mathcal{A}} \subseteq cl_{\mathcal{K}_u(\vartheta]}(\mathcal{A}) + \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}v(t)\}$.

Theorem 4: System (13), where $m = 1$, can be transformed into the form (17), where $\dim\ z^1(t) = k$, by a state transformation $z(t) = \varphi(x(\cdot))$ and a regular static feedback $u(t) = \alpha(x(\cdot), v(\cdot))$ if and only if

(i) $\mathrm{rank}\ \hat{\mathcal{H}}_2 \geq n - k$
(ii) there exists a weakly integrable controlled invariant left-submodule $\mathcal{A}$ with rank k such that $\mathcal{A} \cap \hat{\mathcal{H}}_\infty = \emptyset$ and $\mathcal{A}$ contains $\mathcal{H}_1/\hat{\mathcal{H}}_2$.

Proof: Necessity. Since $\dim\ z^1(t) = k$, then $\dim\ z^2(t) = n - k$. Because the first order time derivatives of $z^2(t)$ do not depend on the input variable, one gets that $\mathrm{d}z_j^2(t) = \mathrm{d}\varphi_j^2(x(\cdot)) \in \hat{\mathcal{H}}_2$, $j = 1, \ldots, n - k$. Therefore, since z_j^2, $j = 1, \ldots, n - k$ are independent, condition (i) is satisfied.

Let $\mathcal{A} = \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}z^1\}$. Clearly, this left-submodule satisfies the condition (ii) of Theorem 4.

Sufficiency. Because $\mathcal{A}$ is weakly integrable one has $cl_{\mathcal{K}_u(\vartheta]}(\mathcal{A}) = \mathrm{span}_{\mathcal{K}_u(\vartheta]}\{\mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_k\}$. Define $z_i^1 = \varphi_i$, $i = 1, \ldots, k$. Since $\mathcal{A}$ contains $\mathcal{H}_1/\hat{\mathcal{H}}_2$ and $\mathrm{rank}\ \hat{\mathcal{H}}_2 \geq n-k$, one can find $z^2 = \varphi^2(x(\cdot))$ such that $\mathrm{d}z^2 \in \mathcal{H}_2$ and $z(t) = (z^1, z^2)^T$ is a state transformation [19]. Because $\mathcal{A}$ is controlled invariant, there exists a feedback $u(t) = \alpha(x(\cdot), v(\cdot))$ which makes $\mathcal{A}$ invariant. Finally, condition $\mathcal{A} \cap \hat{\mathcal{H}}_\infty = \emptyset$ guarantees accessibility of z^1. ∎

VI. Conclusion

The integrability of 1-forms, which plays an important role in the analysis of time-delay systems, was characterized. Necessary and sufficient conditions were given to check if a set of 1-forms is strongly (weakly) integrable, together with two algorithms to compute the largest integrable left-submodule, which is contained in the (closure of) left-submodule generated by the given 1-forms. It was also shown that accessibility of nonlinear time-delay systems can be characterized through integrability of certain left-submodule.

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   -

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