THESIS ON NATURAL AND EXACT SCIENCES

Inverse problems to determine non-homogeneous degenerate memory kernels in heat flow

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Declaration:
Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for any degree.

/Enno Pais/

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Contents

Introduction

In the linear theory of heat conduction in materials with memory the constitutive relations between the heat flux and the gradient of temperature and between the internal energy and the temperature involve integral terms over the past history of the material containing time-dependent convolution kernels. These memory terms bring a certain inertia into the heat flow process and enable to model the propagation of heat by finite speed [3, 14, 26, 27, 33, 34]. Models with memory are used also in coupled processes, e.g. thermoelasticity [40] and phase transition [1, 4, 5].

In case the memory kernels and other physical data are known, we can solve the direct problem, i.e. the problem to determine the temperature function. But sometimes it is necessary to solve an inverse problem, i.e. a problem where the kernels are unknown. To recover the kernels in the inverse problems additional measurements of the temperature or heat flux (so-called observation data) are used. Inverse problems serve for two purposes:

1) testing the relevance of the model - solving the inverse problem several times with different data, an approximate coincidence of the solutions proves the relevance of the model, whereas a big difference between the solutions shows the irrelevance;

2) practical determination of the memory kernels for particular materials.

Different inverse problems for memory kernels in heat flow in homogeneous case have been posed and studied in a number of papers [2, 10, 11, 12, 16, 18, 19, 20, 22, 25, 29, 30, 31, 41, 42, 43]. Most of these works deal with the case when the heat equation contains a single memory kernel of the heat flux. However, the papers [11, 20] treat the inverse problems for two kernels, i.e. for the kernels of heat flux m and the internal energy n. The character of the problem for two kernels very much depends on the type of observation conditions. In case temperature observations in two points are given ahead, one determines n with higher smoothness than m [20]. In case both temperature and flux observations in single points are provided, one gets n and m with the same level of smoothness [11, 20]. The third case when purely flux observations are given, was not covered by the papers [11, 20]. This case turns out to be more complicated.

When the material is non-homogeneous, then the memory kernels depend on the space variable(s), too. In this case the inverse problems require more observation data. The most general approach is to make use of a restricted Dirichlet-to-Neumann map to determine the kernel of heat flux [15]. This requires a lot of measurements from different experiments. The situation simplifies in case one possesses additional a priori information on the kernels. For instance, if the body under consideration is stratified or satisfies other symmetry properties, it is possible to recover the kernel of heat flux from measurements obtained from a single experiment [6, 7, 9, 17, 32].

In some contexts the non-homogeneous kernels can be degenerate, i.e. represented as finite sums of products of known space-dependent functions times unknown time-dependent coefficients. This is so when either the medium is piecewise homogeneous or a problem for a general kernel is replaced by a related problem for an approximated kernel. Then the unknown coefficients can be recovered by the measurement of temperature or heat flux in finite number of points over the time. The problem requires only a single physical experiment. In [21, 23, 24] inverse problems of such a type were studied. Again, these papers deal with the problems to recover the kernel of heat flux.

The main task of the present thesis is to work out certain inverse problems to determine both the kernel of heat flux m and the kernel of internal energy n in the non-homogeneous degenerate case. We will be limited to the one-dimensional problems. We pose and study two different problems. The first one is a problem with purely temperature observations. We prove the existence and uniqueness of the solution. In the solvability theorems the solution occurs with one step higher smoothness in n than in m. The results concerning the problem with temperature observations have been published in the cases of first and third kind boundary conditions in the papers of the author [37] and [38], respectively.

The second problem is the inverse problem with purely flux observations. Treatment of this problem is more complicated. First of all it is necessary to establish a certain second order asymptotical relation for the Green function of the corresponding elliptic problem in the Laplace domain. This requires a lot of technical work. However, having this asymptotical relation already, the proof of existence and uniqueness of the solution is somewhat easier than for the problem with temperature observations. The results in the case of flux observations have been published in the papers of the author [35, 36]. Unfortunately, these papers contain a mistake in the estimation of the Green function. Namely, the estimation of higher-order O-terms in proof of Lemma 3 in these papers is not right. This in turn leads to an error in the asymptotical estimate of an integral of G_x. More precisely, the estimates (2.4.7) and (2.5.7) of the thesis occur with the factor $\frac{V'(x)}{\beta(x)}$ instead of $\left(\frac{V(x)}{\beta(x)}\right)'$ in these papers. This means that they are valid only in the case of constant β. This error is improved in the thesis. Moreover, the thesis contains the treatment of the problem with flux observations in the case of boundary conditions of the third kind, not published by the author so far.

The character of the inverse problem with flux observations is worse than the character of the problem with temperature observations. In many combinations of the data the matrix of the problem is singular. Even more, the corresponding homogeneous problem is severely ill-posed.

Summing up, the *main novelties of the thesis* are as follows.

1. First time inverse problems for both the kernels of heat flux and internal energy in the non-homogeneous case are dealt with.

2. First time an inverse problem to determine the kernels of heat flux and internal energy using purely flux observations is studied.

3. First time severe ill-posedness of a memory-identification problem in continuous media is noticed.

The *major results of the thesis are published* in the papers

1. E. Pais, Identification of degenerate time- and space-dependent kernels in heat flow. In: *Proc. of 5-th International Conference on Inverse Problems in Engineering* (Cambridge, 11-15.07.2005), D. Lesnic ed., Leeds Univ. Press, Leeds, 2005, Vol. III, P01, pp. 1–10.

2. E. Pais and J. Janno, Identification of two degenerate time- and space-dependent kernels in a parabolic equation. *Electron. J. Diff. Eqns.* (http://www.emis.de/journals/EJDE) 2005, No. 108, 1–20.

3. E. Pais, Degenerate memory kernels identification problem with flux-type additional conditions. *J. Inv. Ill-Posed Problems* **14** (2006), 397 – 418.

4. E. Pais and J. Janno, Inverse problem to determine degenerate memory kernels in heat flux with third kind boundary conditions. *Math. Model. Anal.* **11** (2006), 427–450.

The *results of the thesis have been presented* in

1. 5-th International Conference on Inverse problems in Engineering, Cambridge, 11-15.07.2005 - poster,

2. seminar "Applied Functional Analysis", Tallinn, 3.03.2006 - lecture,

3. international conference ”Simulation and Optimization in Business and Industry”, Tallinn, 17-20.05.2006 - contributed talk,

4. International Congress of Mathematicians, Madrid, 22-30.08.2006 - short communication.

Let us give an *overview of the content of the thesis*. Chapter 1 is devoted to the formulation of the inverse problems. In Section 1.1 we pose the direct problems in the cases of boundary conditions of the first and third kind and formulate inverse problems with temperature and flux observations. In the inverse problems we assume the kernels to be degenerate, i.e. representable in the forms (1.1.8), where n_k, $k = 1, \dots, K_1$, and m_k, $k = 1, \dots, K_2$, are $K_1 + K_2$ unknown time-dependent coefficients. In Section 2 we apply the Laplace transform to these problems. This section begins with the Subsection 1.2.1 where certain basic properties of the Laplace transform are listed. Further, in Subsections 1.2.2 and 1.2.3 we rewrite the direct problems in the Laplace domain in the form of system (1.2.15), (1.2.18) containing the Green functions of the elliptic operator of the problem. This system is common both for the boundary conditions of the first and third kind. Finally, in Subsection 1.2.4 we formulate the inverse problems in the Laplace domain and define the generalized solutions of the inverse problems in time domain. The latter ones are simply the inverse Laplace transforms of the solutions of the inverse problems in the Laplace domain.

Chapter 2 plays a preparative role. It starts with Section 2.1 where we collect definitions of the functional spaces used in our analysis. Further, in Section 2.2 basic properties and asymptotical representations of the Green functions are derived. These representations contain solutions of the corresponding Cauchy problems for the elliptic operator. Therefore, a preliminary analysis of such Cauchy problems is provided at the beginning of this section. Section 2.3 includes auxiliary results and Sections 2.4, 2.5 contain further properties of the Green functions. More precisely, in Sections 2.4, 2.5 we give estimates of integrals of Green functions and establish asymptotical behavior of quantities of the type

$$\mathcal{Q} = \int_0^1 \mathcal{G}(x, y, p) V(y) dy$$

in the process $\mathrm{Re}\, p \to +\infty$ where V is a given function and $\mathcal{G} = pG, pG_x, G_{xy}$ with G the Green function. These results are necessary for the analysis of the inverse problems. The estimates of integrals of Green functions are stated in Theorems 2.1 and 2.5. Although these theorems were already proved in [23, 24], in view of the shortness of the proofs, we repeat them. Further, the asymptotics of $\mathcal{Q}$ in the cases $\mathcal{G} = pG, G_{xy}$ was already proved in [23, 24]. Therefore, we cite these results without proofs in the form of Theorems 2.2, 2.3, 2.6, 2.7. The case $\mathcal{G} = pG_x$ is not presented in the literature. We prove the asymptotics in this case in Theorems 2.4 and 2.8. Technically complicated proofs of the latter theorems and the preparation for these proofs fill a large part of Chapter 2.

Chapter 3 contains the study of the inverse problem with temperature observations. In Section 3.1 we reduce this problem to a fixed-point form. To this end, we consider the asymptotics of the problem in the process $\mathrm{Re}\, p \to +\infty$. This asymptotics gives us a linear system (3.1.3) for the initial values of the functions n_k. This system is overdetermined. For the existence of a solution, the consistency condition (3.1.4) must hold. Assuming this condition, it is possible to extract a second order asymptotics from the problem to get the desired fixed point system for unknown functions (formulas (3.1.9), (3.1.10)). More precisely, the obtained system is written in terms of the Laplace transforms of the functions n_k' and m_k. The principal part of this system contains a matrix Γ (formula (3.1.7)) that is assumed to be regular. The proved equivalence of the inverse problem in the Laplace domain and the fixed-point system is formulated in Proposition 3.1.

The fixed-point system (3.1.9), (3.1.10) contains operators of the Laplace transforms of the functions n_k' and m_k that correspond to the solution of the direct problem and its derivative. In Section 3.2 auxiliary estimates for these operators are deduced. After that, in Section 3.3 main results about the inverse problem are proved. Firstly, Theorem 3.2 establishes existence and uniqueness for the system (3.1.9), (3.1.10). The proof of this theorem uses fixed-point technique in half-planes $\mathrm{Re}\, p > \sigma$ with sufficiently large σ, because the right-hand side of this system is of lower order in the process $\mathrm{Re}\, p \to +\infty$. Further, in Corollary 3.1 by means of Theorem 3.2 and Proposition 3.1 the existence and uniqueness results are extended to the generalized inverse problem in the time domain. Chapter 3 is finished by Section 3.4 where the assumptions of Theorem 3.2 and Corollary 3.1 are interpreted. Namely, we write sufficient conditions in the time domain for the assumptions of these sentences and give examples when the matrix Γ is regular.

Chapter 4 contains the study of the inverse problem with flux observations. It has the same structure as Chapter 3. In Section 4.1 the inverse problem is transformed to the fixed-point system (4.1.6), (4.1.7) with the matrix Γ of the form (4.1.4), where the unknowns are the laplace transforms of n_k and m_k. Further, in Section 4.2 the operators corresponding to the direct problem are analysed and Section 4.3 contains the main results. In Theorem 4.2 of Section 4.3 we prove the existence and uniqueness for the fixed-point system and in Corollary 4.1 extend these results to the generalized inverse problem in the time domain. Finally, Section 4.4 contains interpretation of assumptions. There sufficient conditions in the time domain for the assumptions of the existence theorem are given and examples for regular Γ are provided. However, it is shown that in some important cases Γ is singular.

Chapter 4 is complemented with the analysis of the case of singular Γ, too. In Section 4.5 we describe the procedure of extracting higher order principal parts from the inverse problems to get the fixed-point systems in cases of singular Γ. However, sometimes even this procedure doesn't work. An example is the problem with flux observations in the homogeneous case. It is shown that then the inverse problem is severely ill-posed.

1. Formulation of direct and inverse problems

1.1 Problems in time domain

In the linear theory of heat flow in a rigid nonhomogeneous bar consisting of a material with thermal memory, the following system of constitutive relations holds [11, 26, 20]

$$e(x,t) = \beta(x)u(x,t) + \int_0^t n(x,t-\tau)u(x,\tau)d\tau, \tag{1.1.1}$$

$$q(x,t) = -\lambda(x)u_x(x,t) + \int_0^t m(x,t-\tau)u_x(x,\tau)d\tau\,. \tag{1.1.2}$$

Here u is the temperature that is assumed to be 0 for $t < 0$, e is the internal energy, q is the heat flux, β is the product of the specific caloric coefficient and the mass density and λ is the heat conduction coefficient. Moreover, n and m are the *memory kernels* of the internal energy and heat flux, respectively. These constitutive relations can be complemented by the heat balance equation

$$e_t(x,t) + q_x(x,t) = r(x,t), \quad x \in (0,1),\ t > 0\ , \tag{1.1.3}$$

with r being the the heat supply. We assume the rod to be of the unit length for a sake of simplicity.

Using (1.1.1) and (1.1.2) in (1.1.3) we arrive at the following integro-differential equation of heat conduction:

$$\beta(x)\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial t}\int_0^t n(x,t-\tau)u(x,\tau)\,d\tau = \frac{\partial}{\partial x}\left(\lambda(x)u_x(x,t)\right)$$
$$-\frac{\partial}{\partial x}\int_0^t m(x,t-\tau)u_x(x,\tau)\,d\tau + r(x,t), \quad x \in (0,1),\ t > 0. \tag{1.1.4}$$

We require that the function $u(x,t)$ satisfies the initial condition

$$u(x,0) = \varphi(x), \quad x \in (0,1) \tag{1.1.5}$$

and either the boundary conditions of the first kind

$$u(0,t) = f_1(t)\,,\ \ u(1,t) = f_2(t)\,, \quad t > 0 \tag{1.1.6}$$

or the boundary conditions of the third kind

$$-q(0,t) = \alpha_1(u(0,t) - f_1(t))\,, \quad q(1,t) = \alpha_2(u(1,t) - f_2(t))\,, \quad t > 0.$$

Here φ and $f_j,\, j = 1,2$ are given functions and $\alpha_j \geq 0,\, j = 1,2$ are given constants. In view of (1.1.2) the boundary conditions of the third kind can be rewritten as

$$\begin{aligned} \lambda(0)u_x(0,t) - \int_0^t m(0,t-\tau)u_x(0,\tau)\,d\tau &= \alpha_1(u(0,t) - f_1(t))\,, \\ -\lambda(1)u_x(1,t) + \int_0^t m(1,t-\tau)u_x(1,\tau)\,d\tau &= \alpha_2(u(1,t) - f_2(t))\,, \quad t > 0. \end{aligned} \tag{1.1.7}$$

Equation (1.1.4) with the initial condition (1.1.5) and the boundary conditions (1.1.6) or (1.1.7) form the *direct problem* for the temperature u.

In this thesis we will be concerned with the determination of the kernels n and m. We will be limited to the case, when these kernels have the following degenerate forms

$$n(x,t) = \sum_{k=1}^{K_1} \nu_k(x) n_k(t), \quad m(x,t) = \sum_{k=1}^{K_2} \mu_k(x) m_k(t)\,, \tag{1.1.8}$$

where $\nu_k,\, k = 1, \ldots, K_1,\, \mu_k,\, k = 1, \ldots, K_2$ are given x-dependent functions and $n_k,\, k = 1, \ldots, K_1\,,\; m_k,\, k = 1, \ldots, K_2$ are unknown time-dependent coefficients.

There are two important cases when the kernels are of the form (1.1.8). In the first case the rod under consideration is piecewise homogeneous. Then denoting by $J_k = (y_{k-1}, y_k)$ with $0 = y_0 < y_1 < \ldots < y_{K_1} = 1$ the homogeneous pieces of such a rod, we can set $K_1 = K_2$ and $\nu_k,\, \mu_k$ to be the characteristic function of J_k. However, in our analysis we have to assume smoothness ν_k and μ_k (see the solvability theorems in next chapters). Therefore, in this case we have to set ν_k and μ_k to be some smooth approximation the characteristic function of J_k. For instance, we can define

$$\begin{aligned} &\nu_k, \mu_k \in C^l(\mathbb{R})\,, \\ &\nu_k(x) \;=\; \mu_k(x) \;=\; \begin{cases} 1 & \text{if } x \in (y_{k-1}+\epsilon, y_k - \epsilon) \\ 0 & \text{if } x \notin (y_{k-1}-\epsilon, y_k + \epsilon) \end{cases}, \; k = 2, \ldots, K_1 - 1, \\ &\nu_1(x) \;=\; \mu_1(x) \;=\; \begin{cases} 1 & \text{if } x \in [0, y_1 - \epsilon) \\ 0 & \text{if } x \notin [0, y_1 + \epsilon) \end{cases}, \\ &\nu_{K_1}(x) \;=\; \mu_{K_1}(x) \;=\; \begin{cases} 1 & \text{if } x \in (y_{K_1-1}+\epsilon, 1] \\ 0 & \text{if } x \notin (y_{K_1-1}-\epsilon, 1] \end{cases} \end{aligned} \tag{1.1.9}$$

with some $l \in \mathbb{N}$ and small $\epsilon > 0$. In the second case the sums in (1.1.8) are certain finite-dimensional approximations of the actual non-degenerate kernels n

and m. Then ν_k and μ_k form a certain basis of x-dependent functions on $[0,1]$. For instance, ν_k and μ_k could be the power basis: $\nu_k(x) = \mu_k(x) = x^{k-1}$ or the weighted power basis:

$$\nu_k(x) = a(x)x^{k-1}, \quad \mu_k(x) = b(x)x^{k-1}.$$

Here a and b are some smooth functions such that $|a(x)|, |b(x)| > 0$ for any $x \in [0,1]$. The latter one is equivalent to the usual power expansions for the functions $\frac{n(x,t)}{a(x)}$ and $\frac{m(x,t)}{b(x)}$. Trigonometrical expansions could also be used for n and m.

In order to determine the unknown functions n_k and m_k we have to specify $K = K_1 + K_2$ additional conditions. We introduce two different types of additional (observation) conditions.

1. Temperature observations

$$u(x_i,t) \;=\; h_i(t)\,, \quad t>0\,,\;\; i=1,\dots,K. \tag{1.1.10}$$

   Here x_i are K different points of measurement in $(0,1)$.

2. Flux observations

$$q(x_i,t) = -\lambda(x_i)u_x(x_i,t) + \int_0^t m(x_i,t-\tau)u_x(x_i,\tau)\,d\tau = h_i(t), \tag{1.1.11}$$
$$t>0\,,\;\; i=1,\dots,K\,,$$

   where x_i are K different points of measurement in $(0,1)$.

Summing up, our inverse problems are as follows.

1. *Inverse problem with temperature observations.* Find n_k, $k = 1,\dots,K_1$ and m_k, $k=1,\dots,K_2$ such that the solution u of the direct problem with kernels n and m of the form (1.1.8) satisfies the conditions (1.1.10).

2. *Inverse problem with flux observations.* Find n_k, $k=1,\dots,K_1$ and m_k, $k = 1,\dots,K_2$ such that the solution u of the direct problem with kernels n and m of the form (1.1.8) satisfies the conditions (1.1.11).

1.2 Problems in Laplace domain

1.2.1 Laplace transform and its basic properties

In this subsection we collect some basic properties of the Laplace transform that we will use in the sequel. These properties can be found e.g. in the book [8].

The Laplace transform of a function $z(t)$, $t>0$, is given by the formula

$$Z(p) \;=\; \mathcal{L}_{t\to p}z(t) \;=\; \int_0^\infty e^{-pt}z(t)dt\,, \quad p\in\mathbb{C}. \tag{1.2.1}$$

A sufficient condition of existence for the Laplace transform of z is

$$z \in \mathscr{E} = \{z : z(t) - \text{piecewise continuous in } (0,\infty) \text{ and } |z(t)| \le Ce^{\sigma t}, \, t \ge 0 \text{ with some } C \ge 0 \text{ and } \sigma \in \mathbb{R}\}. \tag{1.2.2}$$

In case $z \in \mathscr{E}$ the transformed function $Z(p)$ is holomorphic in the half-plane $\operatorname{Re} p > \sigma$ and satisfies the condition

$$|Z(p)| \le \frac{C}{\operatorname{Re} p - \sigma} \quad \text{for } \operatorname{Re} p > \sigma. \tag{1.2.3}$$

Moreover, in this case the inverse Laplace transform is given by the formula

$$z(t) = \mathcal{L}^{-1}_{p\to t} Z(p) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{tp} Z(p)\, dp, \tag{1.2.4}$$

where ξ is an arbitrary number greater than σ.

Let us list some further properties necessary for our analysis in next sections.

1. Let $z^{(j)} \in \mathscr{E}$, $j = 0, \ldots, k$, with some $k \in \mathbb{N}$. Then

   $$\mathcal{L}_{t\to p} z^{(k)}(t) = p^k Z(p) - p^{k-1} z(0) - p^{k-2} z'(0) - \ldots - z^{(k-1)}(0),$$

   where $Z(p) = \mathcal{L}_{t\to p} z(t)$. Thus,

   $$Z(p) = \frac{z(0)}{p} + \frac{z'(0)}{p^2} + \ldots + \frac{z^{(k-1)}(0)}{p^{k-1}} + V(p),$$

   where by (1.2.3)

   $$|V(p)| = \left| \frac{1}{p^k} \mathcal{L}_{t\to p} z^{(k)}(t) \right| \le \frac{C}{|p|^k (\operatorname{Re} p - \sigma)} \quad \text{for } \operatorname{Re} p > \sigma.$$

2. Let $z_1, z_2 \in \mathscr{E}$. Then $\int_0^{\cdot} z_1(\cdot - \tau) z_2(\tau) d\tau \in \mathscr{E}$ and

   $$\mathcal{L}_{t\to p} \int_0^t z_1(t-\tau) z_2(\tau) d\tau = Z_1(p) Z_2(p) \tag{1.2.5}$$

   with $Z_j(p) = \mathcal{L}_{t\to p} z_j(t)$, $j = 1, 2$.

3. Let $Z(p)$ be holomorphic in some half-plane $\operatorname{Re} p > \sigma$ and have the form $Z(p) = \frac{c}{p} + V(p)$ where $|V(p)| \le \text{Const}\, |p|^{-\alpha}$ for $\operatorname{Re} p > \sigma$ with $\alpha > 1$. Then there exists $z \in C[0,\infty) \cap \mathscr{E}$ such that $Z(p) = \mathcal{L}_{t\to p} z(t)$. Moreover, $z(0) = c$.

4. The inverse Laplace transform is unique. Namely, let the Laplace transforms $Z_j(p)$ of functions $z_j \in \mathscr{E}$, $j = 1, 2$ satisfy the equality $Z_1(p) = Z_2(p)$ for $\operatorname{Re} p > \sigma$. Then $z_1(t) = z_2(t)$ a.e. $t \in (0,\infty)$.

Let us deduce a further property. This is related to the fractional derivative of a function $z(t)$ defined by the formula

$$\frac{d^{\varkappa}}{dt^{\varkappa}} z(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\varkappa)} \int_0^t (t-\tau)^{-\varkappa} z(\tau) d\tau \,, \quad 0 < \varkappa < 1 \,.$$

From (1.2.1) we easily derive the relation $\mathcal{L}_{t\to p} \frac{1}{\Gamma(1-\varkappa)} t^{-\varkappa} = p^{\varkappa-1}$ for $\mathrm{Re}\, p > 0$ containing the main branch of the power function. Thus, from the properties 1, 2 and (1.2.3) we infer

5. Let v, $\frac{d^{\varkappa}}{dt^{\varkappa}} v \in \mathscr{E}$ with some $\varkappa \in (0,1)$. Then $\mathcal{L}_{t\to p} \frac{d^{\varkappa}}{dt^{\varkappa}} v(t) = p^{\varkappa} V(p)$ and
$$|V(p)| \le \frac{C}{|p|^{\varkappa}(\mathrm{Re}\, p - \sigma)} \quad \text{for} \quad \mathrm{Re}\, p > \sigma.$$

1.2.2 Direct problem in the case of boundary conditions of the first kind

Assume a priori that

$$n_k, m_k, r, f_1, f_2 \quad \text{as functions of } t \text{ belong to } \mathscr{E} \tag{1.2.6}$$

and suppose that u solving the direct problem has the properties

$$U, U_x, U_t, U_{xx}, U_{tt} \quad \text{as functions of } t \text{ belong to } \mathscr{E}. \tag{1.2.7}$$

Let us apply the Laplace transform to the equation (1.1.4). Denoting

$$U = \mathcal{L}_{t\to p} u, \quad R = \mathcal{L}_{t\to p} r, \quad M_k = \mathcal{L}_{t\to p} m_k, \quad N_k = \mathcal{L}_{t\to p} n_k,$$

observing the initial condition (1.1.5), the representation of the kernels (1.1.8) and the properties 1 and 2 of the Laplace transform from Section 1.2.1 we obtain

$$\begin{aligned} &\beta(x) \left[pU(x,p) - \varphi(x) \right] + p \sum_{k=1}^{K_1} N_k(p) \nu_k(x) U(x,p) \qquad (1.2.8) \\ &= \frac{\partial}{\partial x} \left(\lambda(x) U_x(x,p) \right) - \sum_{k=1}^{K_2} M_k(p) \frac{\partial}{\partial x} \left(\mu_k(x) U_x(x,p) \right) + R(x,p). \end{aligned}$$

The first kind boundary conditions (1.1.6) are transformed to

$$U(0,p) = F_1(p), \quad U(1,p) = F_2(p). \tag{1.2.9}$$

Here $F_j = \mathcal{L}_{t\to p} f_j$, $j = 1, 2$.

We are going to reduce the direct problem for U into a system of integral equations. To this end we represent the equation (1.2.8) in the form

$$\begin{aligned} \left(LU \right)(x,p) &= p \sum_{k=1}^{K_1} N_k(p) \nu_k(x) U(x,p) \qquad (1.2.10) \\ &+ \sum_{k=1}^{K_2} M_k(p) \frac{\partial}{\partial x} \left(\mu_k(x) U_x(x,p) \right) - R(x,p) - \beta(x)\varphi(x) \end{aligned}$$

with the differential operator

$$(LU)(x,p) = \frac{\partial}{\partial x}\left(\lambda(x)U_x(x,p)\right) - \beta(x)pU(x,p), \quad x \in (0,1). \tag{1.2.11}$$

Further, we introduce the green function $G(x,y,p)$ of the operator L corresponding to the first kind boundary conditions. This is the function satisfying the problem

$$\begin{aligned} &L_y G(x,y,p) = \delta(y-x), \quad x \in (0,1),\ y \in (0,1)\,, \\ &G(x,0,p) = G(x,1,p) \ = \ 0\,, \end{aligned} \tag{1.2.12}$$

where L_y stands for the operator L with respect to the variable y.

Then, the solution of (1.2.8), (1.2.9) is given by

$$\begin{aligned} U(x,p) \;&=\; \sum_{k=1}^{K_1} N_k(p) \int_0^1 G(x,y,p)\nu_k(y)pU(y,p)\,dy \\ &+\; \sum_{k=1}^{K_2} M_k(p) \int_0^1 G(x,y,p)\frac{\partial}{\partial y}\left(\mu_k(y)U_y(y,p)\right)\,dy - Q(x,p), \end{aligned} \tag{1.2.13}$$

where

$$\begin{aligned} Q(x,p) \;&=\; \int_0^1 G(x,y,p)\left[\beta(y)\varphi(y) + R(y,p)\right]\,dy \\ &+\; \lambda(0)G_y(x,0,p)F_1(p) - \lambda(1)G_y(x,1,p)F_2(p)\,. \end{aligned} \tag{1.2.14}$$

Integrating the integrals in the second sum of (1.2.13) by parts and observing the homogeneous boundary conditions of G we obtain the following equation for U:

$$\begin{aligned} U(x,p) \;&=\; \sum_{k=1}^{K_1} N_k(p) \int_0^1 pG(x,y,p)\nu_k(y)U(y,p)\,dy \\ &-\; \sum_{k=1}^{K_2} M_k(p) \int_0^1 G_y(x,y,p)\mu_k(y)U_y(y,p)\,dy - Q(x,p). \end{aligned} \tag{1.2.15}$$

It contains U_x in the right-hand side. Therefore, we have to derive an additional equation for U_x, too. To this end we differentiate (1.2.13) with respect to x:

$$\begin{aligned} U_x(x,p) \;&=\; \sum_{k=1}^{K_1} N_k(p) \int_0^1 pG_x(x,y,p)\nu_k(y)U(y,p)\,dy \\ &+\; \sum_{k=1}^{K_2} M_k(p) \int_0^1 G_x(x,y,p)\frac{\partial}{\partial y}\left(\mu_k(y)U_y(y,p)\right)\,dy - Q_x(x,p). \end{aligned} \tag{1.2.16}$$

Thereupon we split the second integral in (1.2.16) into two parts, from 0 to x and from x to 1, and integrate them by parts. Taking into consideration the equalities $G_x(x,0,p) = G_x(x,1,p) = 0,\ 0 < x < 1$, following from (1.2.12), and the jump relation

$$G_x(x,x-0,p) - G_x(x,x+0,p) = \frac{1}{\lambda(x)}\,, \quad 0 < x < 1 \tag{1.2.17}$$

(see [39], p. 169) we obtain the following equation for U_x:

$$\begin{aligned} U_x(x,p) &= \frac{1}{\lambda(x)}\sum_{k=1}^{K_2} M_k(p)\mu_k(x)U_x(x,p) \\ &+ \sum_{k=1}^{K_1} N_k(p)\int_0^1 pG_x(x,y,p)\nu_k(y)U(y,p)\,dy \\ &- \sum_{k=1}^{K_2} M_k(p)\int_0^1 G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy - Q_x(x,p). \end{aligned} \tag{1.2.18}$$

1.2.3 Direct problem in the case of boundary conditions of the third kind

As in the previous subsection we assume (1.2.6), (1.2.7), deduce the equation (1.2.8) in the Laplace domain and rewrite it in the form (1.2.10) where L is given by (1.2.11). The third kind boundary conditions (1.1.7) are transformed to

$$\begin{aligned} \lambda(0)U_x(0,p) = \alpha_1[U(0,p) - F_1(p)] + \sum_{k=1}^{K_2} \mu_k(0)M_k(p)U_x(0,p) \\ -\lambda(1)U_x(1,p) = \alpha_2[U(1,p) - F_2(p)] - \sum_{k=1}^{K_2} \mu_k(1)M_k(p)U_x(1,p)\,, \end{aligned} \tag{1.2.19}$$

with $F_j = \mathcal{L}_{t\to p} f_j,\ j = 1,2.$

Further, let us denote by $G(x,y,p)$ the Green function of operator L with the third kind boundary conditions, i.e.,

$$\begin{aligned} L_y G(x,y,p) = \delta(y-x), \quad x \in (0,1),\ y \in (0,1)\,, \\ \lambda(0)G_y(x,1,p) = \alpha_1 G(x,0,p),\ -\lambda(1)G_y(x,0,p) = \alpha_2 G(x,1,p),\ x \in (0,1), \end{aligned} \tag{1.2.20}$$

where L_y is the operator L with respect to the variable y, as before.

Then, according to the Green representation in the case of the third kind boundary conditions (see [24]), the solution of (1.2.8) is given by

$$\begin{aligned} U(x,p) &= p\sum_{k=1}^{K_1} N_k(p)\int_0^1 G(x,y,p)\nu_k(y)U(y,p)\,dy \\ &+ \sum_{k=1}^{K_2} M_k(p)\int_0^1 G(x,y,p)\frac{\partial}{\partial y}\left(\mu_k(y)U_y(y,p)\right)\,dy - F(x,p), \end{aligned} \tag{1.2.21}$$

where

$$F(x,p) = \int_0^1 G(x,y,p)\,[\beta(y)\varphi(y) + R(y,p)]\;dy$$
$$+G(x,1,p)[\lambda(1)U_x(1,p) + \alpha_2 U(1,p)] - G(x,0,p)[\lambda(0)U_x(0,p) - \alpha_1 U(0,p)].$$

Due to (1.2.19), the latter formula can be rewritten in the form

$$\begin{aligned} F(x,p) &= \int_0^1 G(x,y,p)\,[\beta(y)\varphi(y) + R(y,p)]\;dy \\ &+ \left(\alpha_1 F_1(p) - \sum_{k=1}^{K_2} \mu_k(0) M_k(p) U_x(0,p)\right) G(0,x,p) \\ &+ \left(\alpha_2 F_2(p) + \sum_{k=1}^{K_2} \mu_k(1) M_k(p) U_x(1,p)\right) G(1,x,p)\,. \end{aligned} \tag{1.2.22}$$

Integrating the integrals in the second sum of (1.2.21) by parts and using (1.2.22) as well as the symmetry relations $G(x,1,p) = G(1,x,p), G(x,0,p) = G(0,x,p)$, we obtain the equation (1.2.15) for U, where

$$\begin{aligned} Q(x,p) &= \int_0^1 G(x,y,p)\,[\beta(y)\varphi(y) + R(y,p)]\;dy \\ &+ \alpha_1 F_1(p) G(0,x,p) + \alpha_2 F_2(p) G(1,x,p). \end{aligned} \tag{1.2.23}$$

To derive the additional equation for U_x we differentiate (1.2.21) with respect to x:

$$\begin{aligned} U_x(x,p) &= \sum_{k=1}^{K_1} p N_k(p) \int_0^1 G_x(x,y,p)\nu_k(y) U(y,p)\,dy \\ &+ \sum_{k=1}^{K_2} M_k(p) \int_0^1 G_x(x,y,p) \frac{\partial}{\partial y}\left(\mu_k(y) U_y(y,p)\right)\,dy - F_x(x,p). \end{aligned} \tag{1.2.24}$$

As in the previous subsection we split the second integral in (1.2.21) into two parts, from 0 to x and from x to 1, and integrate them by parts. Noting that for the Green function with the third kind boundary conditions the same jump relation (1.2.17) holds as for the Green function with the first kind boundary conditions (see [24]), we get

$$
\begin{aligned}
U_x(x,p) = \frac{1}{\lambda(x)} \sum_{k=1}^{K_2} M_k(p)\mu_k(x)U_x(x,p) \\
+ \sum_{k=1}^{K_1} pN_k(p) \int_0^1 G_x(x,y,p)\nu_k(y)U(y,p)\,dy \\
- \sum_{k=1}^{K_2} M_k(p) \Bigg(\int_0^1 (G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy - \mu_k(1)U_x(1,p)G_x(x,1,p) \\
+\mu_k(0)U_x(0,p)G_x(x,0,p) \Bigg) - F_x(x,p).
\end{aligned}
$$

Replacing here F_x by (1.2.22) and observing (1.2.23) as well as the symmetry relations $G(x,1,p) = G(1,x,p), G(x,0,p) = G(0,x,p)$, again, we arrive at the equation (1.2.18) for U_x.

1.2.4 Summary. Generalized formulation of inverse problems

Summing up, the *direct problem in the Laplace domain* is reduced to the system of equations (1.2.15) and (1.2.18) for the Laplace transform of the temperature function $U(x,p)$. In the case of boundary conditions of the first kind this system contains the Green function corresponding to the first kind boundary conditions and the function Q is given by (1.2.14). In the case of boundary conditions of the third kind this system contains the Green function corresponding to the third kind boundary conditions and the function Q is given by (1.2.23).

The systems for U in the cases of the first and third kind boundary conditions have the same form. The difference occurs only in the functions G and Q. This similarity enables to treat these systems in a common form in the next chapters.

Assuming that h_i are measurable and exponentially bounded, the observation conditions (1.1.10) and (1.1.11) in the Laplace domain take the forms

$$U(x_i,p) \;=\; H_i(p)\,, \quad i = 1,\dots,K \tag{1.2.25}$$

and

$$-\lambda(x_i)U_x(x_i,p) + \sum_{k=1}^{K_2} M_k(p)\mu_k(x_i)U_x(x_i,p) = H_i(p),\; i = 1,\dots,K, \tag{1.2.26}$$

respectively, where $H_i(p) = \mathcal{L}_{t\to p}h_i$.

Thus, the inverse problems in the Laplace domain are as follows.

1. *Inverse problem with temperature observations in the Laplace domain.* Find the functions $N_k,\ k = 1,\dots,K_1$ and $M_k,\ k = 1,\dots,K_2$ such that the solution U of (1.2.15), (1.2.18) satisfies (1.2.25).

2. *Inverse problem with flux observations in the Laplace domain.* Find the functions N_k, $k = 1, \ldots, K_1$ and M_k, $k = 1, \ldots, K_2$ such that the solution U of (1.2.15), (1.2.18) satisfies (1.2.26).

On the basis of the inverse problems in the Laplace domain we can formulate the generalized inverse problems in time domain.

1. *Generalized inverse problem with temperature observations in the time domain.* Find t-dependent functions n_k, $k = 1, \ldots, K_1$ and m_k, $k = 1, \ldots, K_2$ such that their Laplace transforms satisfy the inverse problem with temperature observations in the Laplace domain.

2. *Generalized inverse problem with flux observations in the time domain.* Find t-dependent functions n_k, $k = 1, \ldots, K_1$ and m_k, $k = 1, \ldots, K_2$ such that their Laplace transforms satisfy the inverse problem with flux observations in the Laplace domain.

From the discussions of this chapter it follows, that in case of sufficiently regular data which guarantee the conditions (1.2.7) for the solution u of the direct problem, any solution n_k, $k = 1, \ldots, K_1$, m_k, $k = 1, \ldots, K_2$ of an inverse problems posed in Section 1.1 solves the corresponding generalized inverse problem, too. The converse assertion is not right, if we consider the direct problem formulated in Section 1.1 in the classical sense. Namely, the system (1.2.15), (1.2.18) related to the generalized inverse problems doesn't require the existence of second derivative U_{xx} in contrast to the equation (1.1.4) related to the classical inverse problems.

2. Functional spaces and properties of Green function

2.1 Functional spaces

To analyse the direct and inverse problems we define the spaces

$$\mathcal{A}_{\gamma,\sigma} = \{V : V(p) \quad \text{is holomorphic on } \mathrm{Re} p > \sigma\,, \ \ \|V\|_{\gamma,\sigma} < \infty\}\,, \qquad (2.1.1)$$

$\gamma, \sigma \geq 0$, where

$$\|V\|_{\gamma,\sigma} = \sup_{\mathrm{Re}p>\sigma} |p|^{\gamma}|V(p)|$$

and

$$(\mathcal{A}_{\gamma,\sigma})^K = \{V = (V_1, \ldots, V_K) \, : \, V_k(p) \in \mathcal{A}_{\gamma,\sigma}, k = 1, \ldots, K\} \qquad (2.1.2)$$

with the norms

$$\|V\|_{\gamma,\sigma} = \sum_{k=1}^{K} \|V_k\|_{\gamma,\sigma}\,, \quad V \in (\mathcal{A}_{\gamma,\sigma})^K\,.$$

These spaces are complete due to the completeness in supremum-norm of sets of holomorphic functions. Thus, they are Banach spaces. Moreover, we note that $\mathcal{A}_{\gamma,\sigma} \subset \mathcal{A}_{\gamma,\sigma'}$, $(\mathcal{A}_{\gamma,\sigma})^K \subset \left(\mathcal{A}_{\gamma,\sigma'}\right)^K$ and $\|\cdot\|_{\gamma,\sigma'} \leq \|\cdot\|_{\gamma,\sigma}$ if $\sigma' > \sigma$.

Further, let us introduce the following spaces of K-component vector functions:

$$\mathcal{M}_{c,\alpha,\sigma} = \left\{Z : Z = \frac{c}{p} + V(p),\, V \in (\mathcal{A}_{\alpha,\sigma})^K\right\}\,.$$

Here $c = (c_1, \ldots, c_K) \in \mathbb{R}^K$ is a given constant vector and $\alpha > 1$ is a fixed number. We remark that any vector function $Z \in \mathcal{M}_{c,\alpha,\sigma}$ has the unique original $z(t) = \mathcal{L}^{-1}_{p\to t}Z(p)$ in the time domain which is continuous for $t \in [0,\infty)$ and $z(0) = c$ (see property 3 in Section 1.2.1).

In the next chapters we will seek for the solutions of the inverse problems such that the vector $(Z_k|_{k=1,\ldots,K_1}, M_k|_{k=1,\ldots,K_2})$ with Z_k being either N_k or a Laplace transform of a derivative n_k belongs to $\mathcal{M}_{c,\alpha,\sigma}$. We will always assume that the parameter α in the definition of $\mathcal{M}_{c,\alpha,\sigma}$ satisfies the condition

$$1 < \alpha < \frac{3}{2}\,. \qquad (2.1.3)$$

Recall that in the previous chapter we reduced the direct problem to a system of integral equations for the pair $(U(x,p), U_x(x,p))$. Therefore, we need suitable spaces for pairs of x- and p-dependent functions. Let us define the following Banach spaces of single functions

$$\hat{\mathcal{B}}_{\gamma,\sigma} = \{F(x,p) : F(x,\cdot) \in \mathcal{A}_{\gamma,\sigma} \text{ for } x \in [0,1],\ F(\cdot,p) \in C[0,1] \text{ for } \mathrm{Re}p > \sigma\}\,,$$

where $\gamma, \sigma \geq 0$ with the norms

$$\|F\|_{\gamma,\sigma} = \max_{0\leq x\leq 1} \sup_{\mathrm{Re}p>\sigma} |p|^{\gamma}|F(x,p)|\,, \quad F \in \hat{\mathcal{B}}_{\gamma,\sigma},$$

and the Banach spaces of pairs of functions:

$$\mathcal{B}_{\gamma,\sigma} = \hat{\mathcal{B}}_{\gamma,\sigma} \times \hat{\mathcal{B}}_{\gamma-\frac{1}{2},\sigma}\,, \quad \gamma \geq \frac{1}{2}\,, \ \ \sigma \geq 0,$$

with the norms

$$\|F\|_{\gamma,\sigma} = \|F_1\|_{\gamma,\sigma} + \|F_2\|_{\gamma-\frac{1}{2},\sigma}\,, \quad F = (F_1, F_2) \in \mathcal{B}_{\gamma,\sigma}. \tag{2.1.4}$$

2.2 Asymptotical representation of Green function.

2.2.1 Asymptotical representation of a solution of Cauchy problem

In this section we prove asymptotical properties of the solution of the Cauchy problem for the operator L defined by (1.2.11). Let $\psi = \psi(x,p)$ solve the Cauchy problem

$$(L\psi)(x,p) = 0,\ x \in (0,1),\ \ \psi(0,p) = \theta_0,\ \psi_x(0,p) = \theta_1 \tag{2.2.1}$$

with the differential operator L defined in (1.2.11) and some given numbers $\theta_0, \theta_1 \in \mathbb{C}$ independent of p. If $\lambda \in C^1[0,1]$, $\beta \in C[0,1]$ and $\lambda(x) > 0,\ x \in [0,1]$, then by the Cauchy theorem the problem (2.2.1) has the unique solution $\psi(\cdot,p) \in C^2[0,1]$.

Lemma 2.1. *Let*

$$\lambda, \beta \in C^2[0,1]\,,\ \lambda(x), \beta(x) > 0\,,\ \ x \in [0,1]. \tag{2.2.2}$$

Then the solution $\psi(x,p)$ *of* (2.2.1) *and its derivative* $\psi_x(x,p)$ *are holomorphic in* $\mathrm{Re}\,p > 0$ *for any* $x \in [0,1]$. *Moreover, the following asymptotical relations are valid:*

$$\left.\begin{aligned}
\psi(x,p) \ =\ & \tfrac{a(0)}{a(x)}\,\theta_0\, ch\ sz + \tfrac{a(0)}{a(x)}\left(\tfrac{\zeta(x)}{2} + l\bar{b}(0)a'(0)\right)\,\theta_0 \tfrac{1}{s} sh\ sz \\
& + \tfrac{l}{a(x)b(0)}\,\theta_1 \tfrac{1}{s} sh\ sz + \tfrac{l\,\zeta(x)}{2a(x)b(0)}\,\theta_1 \tfrac{1}{s^2}\, ch\ sz + \left(|\theta_0| + \tfrac{|\theta_1|}{|s|}\right) O\left(\tfrac{e^{sz}}{s^2}\right)\,, \\
\psi_x(x,p) =\ & \tfrac{b(x)a(0)}{l}\,\theta_0\, s\ sh\ sz + \tfrac{b(x)a(0)}{l}\Big(\tfrac{\zeta(x)}{2} + l\bar{b}(0)a'(0) \\
& - l\bar{b}(x)a'(x)\Big)\theta_0\, ch\ sz + \tfrac{b(x)}{b(0)}\,\theta_1\, ch\ sz \\
& + \tfrac{b(x)}{b(0)}\left(\tfrac{\zeta(x)}{2} - l\bar{b}(x)a'(x)\right)\theta_1 \tfrac{1}{s} sh\ sz + \left(|\theta_0| + \tfrac{|\theta_1|}{|s|}\right) O\left(\tfrac{e^{sz}}{s}\right) \\
& \textit{for } Re\,p \to +\infty \ \textit{ uniformly with respect to } \ x \in [0,1] \ \textit{ and } \ \mathrm{Im}\,p.
\end{aligned}\right\} \tag{2.2.3}$$

Here

$$l = \int_0^1 \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}\, d\tau\,, \tag{2.2.4}$$

$$s = l\sqrt{p} \text{ for } \operatorname{Re} p > 0, \quad \text{where } \sqrt{p} \text{ is the main branch of square root, i.e. it satisfies } \sqrt{p} > 0 \text{ for real } p > 0. \tag{2.2.5}$$

$$a(x) = (\beta(x)\lambda(x))^{1/4},\ b(x) = \beta(x)^{1/4}\lambda(x)^{-3/4},\ \bar{b}(x) = \lambda(x)^{1/4}\beta(x)^{-3/4}, \tag{2.2.6}$$

$$z = z(x) = \frac{1}{l}\int_0^x \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}} d\tau \tag{2.2.7}$$

and

$$\zeta(x) \; = \; \frac{1}{l}\int_0^x a_1(\tau)\sqrt{\frac{\beta(\tau)}{\lambda(\tau)}} d\tau \tag{2.2.8}$$

with

$$a_1(x) = \frac{l^2}{\beta(x)}\left[a(x)\lambda(x)\frac{d}{dx}\left(\frac{a'(x)}{a^2(x)}\right) + \frac{\lambda'(x)a'(x)}{a(x)}\right]. \tag{2.2.9}$$

Proof. We apply the Liouville transform replacing the argument x by z using the formula (2.2.7) and the unknown ψ by v using the relation

$$v(z,p) \; = \; a(x)\psi(x,p), \tag{2.2.10}$$

where $a(x)$ is given by (2.2.6). Then the equation $L\psi = 0$ with $x \in [0,1]$ is equivalent to the equation

$$v_{zz}(z,p) - s^2 v(z,p) \; = \; c(z)v(z,p)\,, \quad z \in [0,1], \tag{2.2.11}$$

where $c(z) = a_1(x)$ with a_1 defined in (2.2.9). The assumptions (2.2.2) imply $c \in C[0,1]$. Note that (2.2.10) with (2.2.6) and (2.2.7) yields the formulas

$$\psi(x,p) \; = \; \frac{1}{a(x)}v(z,p)\,, \quad \psi_x(x,p) \; = \; \frac{b(x)}{l}v_z(z,p) - \frac{a'(x)}{a^2(x)}v(z,p)\,. \tag{2.2.12}$$

Thus, the initial conditions $\psi(0,p) = \theta_0,\ \psi_x(0,p) = \theta_1$ in terms of v take the form

$$v(0,p) = \kappa_0 := a(0)\theta_0,\ v_z(0,p) = \kappa_1 := \frac{l}{b(0)}\left(\frac{a'(0)}{a(0)}\theta_0 + \theta_1\right). \tag{2.2.13}$$

Let us solve the equation (2.2.11) with respect to the left-hand side subject to the initial conditions (2.2.13). We obtain the following Volterra integral equation of the second kind for v:

$$v(z,p) \;=\; \kappa_0\, ch\, sz \;+\; \kappa_1 \tfrac{1}{s} sh\, sz \;+\; \tfrac{1}{s}\int_0^z sh\, s(z-\tau)\, c(\tau) v(\tau,p) d\tau\,, \\ z\in[0,1]. \tag{2.2.14}$$

We are going to prove the assertions of the lemma by means of this relation and the connections (2.2.12) between ψ and v.

We start by proving the holomorphy assertion. Since (2.2.14) is a Volterra equation of the first kind with a bounded kernel, its solution can be expressed in terms of the Neumann series

$$v(z,p) = f(z,p) + \sum_{i=1}^{\infty}\int_0^z g_i(z,\tau,p) f(\tau,p) d\tau, \tag{2.2.15}$$

where

$$f(z,p) = \kappa_0\, ch\, sz \,+\, \kappa_1 \tfrac{1}{s} sh\, sz\,,\;\; g_1(z,\tau,p) = \tfrac{1}{s} sh\, s(z-\tau)\, c(\tau), \\ g_i(z,\tau,p) = \int_\tau^z g_1(z,y,p) g_{i-1}(y,\tau,p) dy,\;\; i=2,3,\ldots. \tag{2.2.16}$$

Let us fix some $0<\gamma_1<\gamma_2$. Observing the definition of s (2.2.5) and (2.2.16) we see that

$$\sup_{\substack{0\le z\le 1\\ \gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2}} |f(z,p)| = T_f < \infty, \qquad \sup_{\substack{0\le z\le 1\\ \gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2}} |g_1(z,p)| = T_g < \infty.$$

Making use of the standard technique of estimation of the Neumann series we deduce the following estimate for the remainder of this series:

$$\sup_{\substack{0\le z\le 1\\ \gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2}} \left| \sum_{i=l+1}^{\infty}\int_0^z g_i(z,\tau,p) f(\tau,p) d\tau\right| \le T_f \sum_{i=l+1}^{\infty} \frac{T_g^i}{i!} \to 0 \quad \text{as} \quad l\to+\infty.$$

This estimate shows that the series (2.2.15) is uniformly convergent in the set $0\le z\le 1$, $\gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2$. Further, observing (2.2.16) we see that every addend in (2.2.15) is a holomorphic function in $\gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2$ for any $z\in[0,1]$. Since a limit of uniformly convergent sequence of holomorphic functions is also holomorphic, the function $v(z,p)$ is holomorphic in $\gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2$ for any $z\in[0,1]$. From (2.2.14) we have the relation for the derivative

$$v_z(z,p) = \kappa_0 s\, sh\, sz + \kappa_1 ch\, sz \,+\, \int_0^z ch\, s(z-\tau)\, c(\tau) v(\tau,p) d\tau,\;\; z\in[0,1].$$

By the proven holomorphy of v and other terms in the right-hand side of this formula, the function $v_z(z,p)$ is holomorphic in $\gamma_1<\mathrm{Re}\sqrt{p}<\gamma_2$ for any $z\in[0,1]$. Finally, since $0<\gamma_1<\gamma_2$ where chosen arbitrarily and $\{p : \mathrm{Re}\, p>0\}\subset\{p:$

$\mathrm{Re}\sqrt{p} > 0\}$, by the proven holomorphy of v and v_z and the relation (2.2.12) we get the holomorphy of $\psi(x,p)$ and $\psi_x(x,p)$ in $\mathrm{Re}\, p > 0$ for any $x \in [0,1]$.

Let us continue proving the relations (2.2.3). To this end we derive an estimate for v. Multiplying (2.2.14) by e^{-sz} we have

$$\begin{aligned} e^{-sz}v(z,p) &= \kappa_0\, e^{-sz} ch\, sz + \kappa_1 \tfrac{1}{s} e^{-sz} sh\, sz \\ &\quad + \tfrac{1}{s}\int_0^z e^{-s(z-\tau)} sh\, s(z-\tau)\, c(\tau) e^{-s\tau} v(\tau,p) d\tau \end{aligned} \tag{2.2.17}$$

for $z \in [0,1]$. Making use of the elementary relations

$$|e^{sz}| = e^{\mathrm{Re}\, sz},\ |ch\, sz| \le ch\,(\mathrm{Re}\, sz),\ |sh\, sz| \le ch\,(\mathrm{Re}\, sz) \tag{2.2.18}$$

and denoting $\|v\|_z = \max\limits_{\tau\in[0,z]} |e^{-s\tau}v(\tau,p)|$ from (2.2.17) we get

$$\begin{aligned} &|e^{-sz}v(z,p)| \le \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) e^{-\mathrm{Re}\, sz} ch\,(\mathrm{Re}\, sz) \\ &+ \tfrac{\|c\|_{C[0,1]}}{|s|}\int_0^z e^{-\mathrm{Re}\, s(z-\tau)} ch\,(\mathrm{Re}\, s(z-\tau)) d\tau\, \|v\|_z = \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) \\ &\times e^{-\mathrm{Re}\, sz} ch\,(\mathrm{Re}\, sz) + \tfrac{\|c\|_{C[0,1]}}{2|s|}\left[1 + \tfrac{1}{2\mathrm{Re}\, s}\left(1 - e^{-2\mathrm{Re}\, s}\right)\right]\|v\|_z. \end{aligned} \tag{2.2.19}$$

Note that due to the inequality

$$\mathrm{Re}\sqrt{p} > \frac{1}{\sqrt{2}}\sqrt{|p|} \quad \text{for} \quad \mathrm{Re}\, p > 0 \tag{2.2.20}$$

and the definition (2.2.5) of s, the formula

$$|s| \ge \mathrm{Re}\, s > \frac{|s|}{\sqrt{2}} = \frac{l}{\sqrt{2}}\sqrt{|p|} \ge \frac{l}{\sqrt{2}}\sqrt{\mathrm{Re}\, p} \quad \text{for} \quad \mathrm{Re}\, p > 0 \tag{2.2.21}$$

is valid. Thus, from (2.2.19) in case of a sufficiently large $\sigma_c > 0$ depending on c we obtain

$$|e^{-sz}v(z,p)| \le 2\left(|\kappa_0| + \frac{|\kappa_1|}{|s|}\right) + \frac{1}{2}\|v\|_z \quad \text{for} \ \ z \in [0,1]\,, \ \ \mathrm{Re}\, p > \sigma_c.$$

This implies $\|v\|_z = \max\limits_{\tau\in[0,z]} |e^{-s\tau}v(\tau,p)| \le 2\left(|\kappa_0| + \frac{|\kappa_1|}{|s|}\right) + \frac{1}{2}\|v\|_z$, hence $\|v\|_z = \max\limits_{\tau\in[0,z]} |e^{-s\tau}v(\tau,p)| \le 4\left(|\kappa_0| + \frac{|\kappa_1|}{|s|}\right)$ for $z \in [0,1]$ and $\mathrm{Re}\, p > \sigma_c$. Thus, we arrive at the estimate

$$|v(z,p)| \le 4\left(|\kappa_0| + \frac{|\kappa_1|}{|s|}\right) e^{\mathrm{Re}\, sz} \quad \text{for} \ \ z \in [0,1]\,, \ \ \mathrm{Re}\, p > \sigma_c\,. \tag{2.2.22}$$

Now we are ready to prove (2.2.3). To this end we plug the iterative formula (2.2.14) for v into itself to get

$$\begin{aligned}
v(z,p) \;=\; &\kappa_0\, ch\, sz \;+\; \tfrac{\kappa_1}{s} sh\, sz \;+\; \tfrac{\kappa_0}{s} \underbrace{\int_0^z sh\, s(z-\tau)\, c(\tau) ch\, s\tau\, d\tau}_{I_1} \\
&+ \tfrac{\kappa_1}{s^2} \underbrace{\int_0^z sh\, s(z-\tau)\, c(\tau) sh\, s\tau\, d\tau}_{I_2} \\
&+ \tfrac{1}{s^2} \underbrace{\int_0^z sh\, s(z-\tau)\, c(\tau) \int_0^\tau sh\, s(\tau-y)\, c(y) v(y,p) dy d\tau}_{I_3} .
\end{aligned} \tag{2.2.23}$$

Differentiating this relation we deduce the formula for the derivative of v, too:

$$\begin{aligned}
v_z(z,p) \;=\; &\kappa_0\, s\, sh\, sz \;+\; \kappa_1 ch\, sz \;+\; \kappa_0 \underbrace{\int_0^z ch\, s(z-\tau)\, c(\tau) ch\, s\tau\, d\tau}_{I_4} \\
&+ \tfrac{\kappa_1}{s} \underbrace{\int_0^z ch\, s(z-\tau)\, c(\tau) sh\, s\tau\, d\tau}_{I_5} \\
&+ \tfrac{1}{s} \underbrace{\int_0^z ch\, s(z-\tau)\, c(\tau) \int_0^\tau sh\, s(\tau-y)\, c(y) v(y,p) dy d\tau}_{I_6} .
\end{aligned} \tag{2.2.24}$$

Let us represent I_1, I_2, I_4 and I_5 in the form

$$I_1 = \frac{1}{2}\int_0^z c(\tau)d\tau\, sh\, sz + \frac{1}{2} \underbrace{\int_0^z sh\, s(z-2\tau)\, c(\tau) d\tau}_{I_7} , \tag{2.2.25}$$

$$I_2 = \frac{1}{2}\int_0^z c(\tau)d\tau\, ch\, sz - \frac{1}{2} \underbrace{\int_0^z ch\, s(z-2\tau)(\tau) d\tau}_{I_8} , \tag{2.2.26}$$

$$I_4 = \frac{1}{2}\int_0^z c(\tau)d\tau\, ch\, sz + \frac{1}{4} I_8 , \quad I_5 = \frac{1}{2}\int_0^z c(\tau)d\tau\, sh\, sz - \frac{1}{4} I_7 . \tag{2.2.27}$$

Observing (2.2.18) and (2.2.21) we estimate:

$$\begin{aligned}
|I_7|,\, |I_8| \;\le\; &\|c\|_{C[0,1]} \textstyle\int_0^z \left[e^{\mathrm{Re}\, s(z-2\tau)} + e^{\mathrm{Re}\, s(2\tau - z)}\right] d\tau \\
&= \tfrac{2\|c\|_{C[0,1]}}{\mathrm{Re}\, s} sh(\mathrm{Re}\, sz) \le \tfrac{\mathrm{Const}}{|s|}\, e^{Resz}
\end{aligned} \tag{2.2.28}$$

for any $z \in [0,1]$ and $\mathrm{Re}\, p > 0$. Moreover, due to (2.2.18), (2.2.21) and (2.2.22), we get

$$\begin{aligned} &|I_3|,\ |I_6| \le 4\|c\|^2_{C[0,1]} \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) \\ &\qquad \times \int_0^z ch\,(\mathrm{Re}\,s(z-\tau)) \int_0^\tau ch\,(\mathrm{Re}\,s(\tau-y))\, e^{\mathrm{Re}\,sy} dy d\tau \\ &\qquad = 4\|c\|^2_{C[0,1]} \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) \tfrac{\mathrm{Re}\,sz(1+\mathrm{Re}\,sz)e^{\mathrm{Re}\,sz} + (2\mathrm{Re}\,sz-1)\,sh\,(\mathrm{Re}\,sz)}{8(\mathrm{Re}\,s)^2} \\ &\qquad \le \mathrm{Const}\left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) e^{Resz} \end{aligned} \tag{2.2.29}$$

for any $z \in [0,1]$ and $\mathrm{Re}\,p > \sigma_c$. Using (2.2.28) in (2.2.25) - (2.2.27) and thereupon (2.2.25) - (2.2.27), (2.2.29) in (2.2.23), (2.2.24) we deduce the following asymptotical relations for v and v_z:

$$\left.\begin{aligned} &v(z,p) \ = \ \kappa_0 \left(ch\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s}\ sh\, sz\right) \\ &\quad +\kappa_1 \left(\tfrac{1}{s} sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s^2}\ ch\, sz\right) + \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) O\left(\tfrac{e^{sz}}{s^2}\right)\ , \\ &v_z(z,p) \ = \ \kappa_0 \left(s\, sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau\, ch\, sz\right) \\ &\quad +\kappa_1 \left(ch\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s}\ sh\, sz\right) + \left(|\kappa_0| + \tfrac{|\kappa_1|}{|s|}\right) O\left(\tfrac{e^{sz}}{s}\right) \\ &\quad \text{for } Re\,p \to +\infty \ \text{ uniformly with respect to } \ z \in [0,1] \ \text{ and } \ \mathrm{Im}\,p. \end{aligned}\right\} \tag{2.2.30}$$

Plugging (2.2.30) into (2.2.12), using the formulas (2.2.13), collecting the terms with θ_0 and θ_1 and simplifying by means of the relation $\frac{1}{a^2b} = \bar{b}$ (see (2.2.6)) we deduce

$$\begin{aligned} &\psi(x,p) \ = \ \tfrac{a(0)}{a(x)} \left[ch\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s}\ sh\, sz \right. \\ &\quad \left. + l\bar{b}(0)a'(0)\left(\tfrac{1}{s} sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s^2}\ ch\, sz\right)\right] \theta_0 \\ &\quad + \tfrac{1}{a(x)} \tfrac{l}{b(0)} \left(\tfrac{1}{s} sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s^2}\ ch\, sz\right) \theta_1 + \left(|\theta_0| + \tfrac{|\theta_0|+|\theta_1|}{|s|}\right) O\left(\tfrac{e^{sz}}{s^2}\right)\ , \\ &\psi_x(x,p) = \tfrac{b(x)a(0)}{l} \left[s\, sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau\, ch\, sz \right. \\ &\quad + \left(l\bar{b}(0)a'(0) - l\bar{b}(x)a'(x)\right)\left(ch\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s}\ sh\, sz\right) \\ &\quad \left. - l^2\bar{b}(0)a'(0)\bar{b}(x)a'(x)\left(\tfrac{1}{s} sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s^2}\ ch\, sz\right)\right] \theta_0 \\ &\quad + \tfrac{b(x)}{b(0)} \left[ch\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s}\ sh\, sz \right. \\ &\quad \left. - l\bar{b}(x)a'(x)\left(\tfrac{1}{s} sh\ sz + \tfrac{1}{2}\int_0^z c(\tau)d\tau \tfrac{1}{s^2}\ ch\, sz\right)\right] \theta_1 \\ &\quad + \left(|\theta_0| + \tfrac{|\theta_0|+|\theta_1|}{|s|}\right) \left[O\left(\tfrac{e^{sz}}{s}\right) + O\left(\tfrac{e^{sz}}{s^2}\right)\right] \end{aligned}$$

for $Re\,p \to +\infty$ uniformly with respect to $x \in [0,1]$ and $\mathrm{Im}\,p$.

Let us extract the terms of first and second order in the process $\mathrm{Re}\, p \to +\infty$ inside the factors before θ_0 and θ_1 and shift the remaining parts into the O-terms. Observing in addition the relations (2.2.21),

$$sh\, sz,\, ch\, sz \;=\; O\left(e^{sz}\right)\,, \quad \left|O\left(\frac{e^{sz}}{s}\right)\right| \le \frac{\mathrm{Const}\, e^{\mathrm{Re}\, sz}}{|s|} = O\left(e^{sz}\right) \tag{2.2.31}$$

and $\int_0^z c(\tau)d\tau \;=\; \zeta(x)$ we obtain (2.2.3). The lemma is proved. □

2.2.2 Basic properties and representation of Green function in case of the first kind boundary conditions

Recall that the Green function of the operator L given by (1.2.11) in the interval $x \in (0,1)$ with the first kind boundary conditions is the function G that satisfies the problem (1.2.12). This function is given by the formula (cf. [39] Section 24, [23])

$$G(x,y;p) = \begin{cases} \frac{1}{\Delta_0(p)\lambda(0)}\psi_2(y,p)\psi_1(x,p) & \text{for} \quad 0 \le x \le y \le 1 \\ \frac{1}{\Delta_0(p)\lambda(0)}\psi_1(y,p)\psi_2(x,p) & \text{for} \quad 0 \le y \le x \le 1, \end{cases} \tag{2.2.32}$$

where $\psi_j(x,p), j \;=\; 1,2$ are the solutions of $L\psi \;=\; 0$ satisfying the initial conditions

$$\psi_1(0,p) = 0\,,\ \psi_{1,x}(0,p) = c_1\,,\ \ \psi_2(1,p) = 0\,,\ \psi_{2,x}(1,p) = c_2 \tag{2.2.33}$$

with arbitrarily chosen numbers $c_1, c_2 \neq 0$ and $\Delta_0(p)$ is the Wronski determinant at zero:

$$\Delta_0(p) = \psi_1(0,p)\psi_{2,x}(0,p) - \psi_{1,x}(0,p)\psi_2(0,p) = -\psi_{1,x}(0,p)\psi_2(0,p). \tag{2.2.34}$$

Here $\Delta(p) \neq 0$ for $\mathrm{Re}\, p > 0$ due to the linear independence of ψ_1 and ψ_2 [39].

We note that ψ_1 satisfies the Cauchy problem (2.2.1) studied in the previous subsection. The function ψ_2 satisfies also the problem (2.2.1) provided we change its argument $\xi = 1 - x$. Thus, using the smoothness property $\psi(\cdot,p) \in C^2[0,1]$, $\mathrm{Re}\, p > 0$, and the assertions about holomorphy of the solution of (2.2.1) in Lemma 2.1 as well as the formula (2.2.32) we deduce the following basic properties of the Green function:

Lemma 2.2. *Assume that* (2.2.2) *holds and G is the Green function of the operator L in $[0,1]$ with first kind boundary conditions. Then $G(x,y,p)$, $G_x(x,y,p)$, $G_y(x,y,p)$ and $G_{xy}(x,y,p)$ are are holomorphic in $\mathrm{Re}\, p > 0$ for any $(x,y) \in [0,1]^2$, $x \neq y$, and continuous and bounded in every strip*

$$S_\sigma = \{(x,y) \in [0,1]^2 \,:\, x \neq y\} \times \{p \in \mathbb{C} \,:\, 0 < \mathrm{Re}\, p < \sigma\}\,, \quad \sigma > 0\,.$$

Moreover, for any function $V(x,p)$ *such that* $V(\cdot,p) \in C[0,1]$, $\mathrm{Re}\,p > \sigma$ *and* $V(x,p)$ *- holomorphic in* $\mathrm{Re}\,p > \sigma$ *for any* $x \in [0,1]$ *with some* $\sigma \geq 0$*, the functions*

$$\int_0^1 G(x,y,p)V(y,p)dy, \quad \int_0^1 G_x(x,y,p)V(y,p)dy,$$
$$\int_0^1 G_y(x,y,p)V(y,p)dy, \quad \int_0^1 G_{xy}(x,y,p)V(y,p)dy$$

belong to $C[0,1]$ *for any* $\mathrm{Re}\,p > \sigma$ *and are holomorphic in* $\mathrm{Re}\,p > \sigma$ *for any* $x \in [0,1]$.

Let us set $c_1 = \frac{b(0)}{l}$ with l and b given by (2.2.4) and (2.2.6). Then ψ_1 solves (2.2.1) with $\theta_0 = 0$, $\theta_1 = \frac{b(0)}{l}$, hence by Lemma 2.1 the function ψ_1 and its first derivative possess the following asymptotic behavior:

$$\left.\begin{array}{l}
\psi_1(x,p) = \frac{1}{a(x)}\frac{1}{s} sh\; sz + \frac{1}{a(x)}\frac{\zeta(x)}{2}\frac{1}{s^2}\, ch\, sz + O\left(\frac{e^{sz}}{s^3}\right) \\
= \frac{1}{a(x)}\frac{1}{s} sh\; sz + O\left(\frac{e^{sz}}{s^2}\right)\;, \\
\psi_{1,x}(x,p) = \frac{b(x)}{l}\, ch\; sz + \frac{b(x)}{l}\left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x)\right)\frac{1}{s} sh\; sz + O\left(\frac{e^{sz}}{s^2}\right) \\
= \frac{b(x)}{l}\, ch\; sz + O\left(\frac{e^{sz}}{s}\right) \\
\text{for } Re\, p \to +\infty \text{ uniformly with respect to } x \in [0,1] \text{ and } \mathrm{Im}\, p
\end{array}\right\} \quad (2.2.35)$$

where a, $\bar{b}$ and ζ are defined in (2.2.6) and (2.2.8) with (2.2.9). In (2.2.33) we set $c_2 = \frac{b(1)}{l}$. Then using the change of variables

$$\xi = 1 - x,\; \psi^\dagger(\xi,p) = \psi_2(x,p),\; \lambda^\dagger(\xi) = \lambda(x),\; \beta^\dagger(\xi) = \beta(x) \qquad (2.2.36)$$

the problem for ψ_2 is transformed to the following Cauchy problem for $\psi^\dagger$:

$$(L\psi^\dagger)(\xi,p) \;=\; 0,\; \xi \in (0,1), \quad \psi^\dagger(0,p) = 0,\; \psi^\dagger_\xi(0,p) = -\frac{b(1)}{l}. \qquad (2.2.37)$$

Using Lemma 2.1 we write the asymptotical formulas (2.2.3) for $\psi^\dagger(\xi,p)$ and $\psi^\dagger_\xi(\xi,p)$. These formulas contain instead of the quantities $x, a, b, \bar{b}, z$ and ζ the quantities $\xi, a^\dagger, b^\dagger, \bar{b}^\dagger, z^\dagger$ and $\zeta^\dagger$, respectively, where

$$\begin{array}{l}
a^\dagger(\xi) = (\beta^\dagger(\xi)\lambda^\dagger(\xi))^{1/4},\; b^\dagger(\xi) = \beta^\dagger(\xi)^{1/4}\lambda^\dagger(\xi)^{-3/4}, \\
\bar{b}^\dagger(\xi) = \lambda^\dagger(\xi)^{1/4}\beta^\dagger(\xi)^{-3/4}, \\
z^\dagger = \frac{1}{l}\int_0^\xi \sqrt{\frac{\beta^\dagger(\tau)}{\lambda^\dagger(\tau)}}d\tau,\; \zeta^\dagger(\xi) = \frac{1}{l}\int_0^\xi a_1^\dagger(\tau)\sqrt{\frac{\beta^\dagger(\tau)}{\lambda^\dagger(\tau)}}d\tau
\end{array} \qquad (2.2.38)$$

and $a_1^\dagger$ is defined by (2.2.9) with a, λ, β replaced by $a^\dagger, \lambda^\dagger, \beta^\dagger$. Observing that

$$\begin{array}{c}
a^\dagger(\xi) = a(x),\; a^{\dagger\prime}(\xi) = -a'(x),\; b^\dagger(\xi) = b(x),\; \bar{b}^\dagger(\xi) = \bar{b}(x), \\
z^\dagger(\xi) = 1 - z(x),\; \zeta^\dagger(\xi) = \zeta(1) - \zeta(x)
\end{array} \qquad (2.2.39)$$

from these formulas we obtain

$$\left.\begin{aligned}
&\psi_2(x,p) = \tfrac{1}{a(x)}\tfrac{1}{s} sh\; s(z-1) - \tfrac{1}{a(x)}\tfrac{\zeta(1)-\zeta(x)}{2}\tfrac{1}{s^2}\, ch\, s(1-z) \\
&\quad +O\left(\tfrac{e^{s(1-z)}}{s^3}\right) = \tfrac{1}{a(x)}\tfrac{1}{s} sh\; s(z-1) + O\left(\tfrac{e^{s(1-z)}}{s^2}\right), \\
&\psi_{2,x}(x,p) = \tfrac{b(x)}{l} ch\; s(1-z) - \tfrac{b(x)}{l}\left(\tfrac{\zeta(1)-\zeta(x)}{2} + l\bar{b}(x)a'(x)\right)\tfrac{1}{s} sh\; s(z-1) \\
&\quad +O\left(\tfrac{e^{s(1-z)}}{s^2}\right) = \tfrac{b(x)}{l}\, ch\, s(1-z) + O\left(\tfrac{e^{s(1-z)}}{s}\right) \\
&\quad \text{for } Re\, p \to +\infty \text{ uniformly with respect to } x \in [0,1] \text{ and } \operatorname{Im} p.
\end{aligned}\right\} \quad (2.2.40)$$

Further, defining

$$d_0(p) \; = s\lambda(0)\Delta_0(p) \qquad (2.2.41)$$

from (2.2.34) by means of (2.2.35), (2.2.40), (2.2.31) and the relation $\lambda(0) = \frac{a(0)}{b(0)}$ we derive the following asymptotic formula for d_0:

$$\begin{aligned}
d_0(p) &= -s\lambda(0)\left[\tfrac{b(0)}{l} + O\left(\tfrac{1}{s}\right)\right]\left[-\tfrac{1}{a(0)}\tfrac{1}{s}\, sh\, s + O\left(\tfrac{e^s}{s^2}\right)\right] \\
&= \tfrac{1}{l} sh\; s + O\left(\tfrac{e^s}{s}\right) \quad \text{for } \operatorname{Re} p \to +\infty \quad \text{uniformly in } \operatorname{Im} p\,.
\end{aligned} \qquad (2.2.42)$$

Now we are in the situation to deduce the asymptotical relations for G and its derivatives. To this end we plug (2.2.35), (2.2.40) and (2.2.41) into (2.2.32). In case we take into consideration only the first-order asymptotics, we get the following representation for G:

$$G(x,y;p) = \frac{1}{d_0(p)}\frac{1}{a(x)a(y)}\frac{1}{s}\begin{cases} sh\; sz \cdot sh\; s(w-1) + O_1 & \text{for } x \le y \\ sh\; sw \cdot sh\; s(z-1) + O_2 & \text{for } y \le x, \end{cases} \qquad (2.2.43)$$

where

$$w = w(y) = \frac{1}{l}\int_0^y \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}\, d\tau \qquad (2.2.44)$$

and

$$\begin{aligned}
&O_1 = O\left(\tfrac{e^{s(1-w+z)}}{s}\right), \quad O_2 = O\left(\tfrac{e^{s(w+1-z)}}{s}\right) \\
&\quad \text{for } \operatorname{Re} p \to +\infty \quad \text{uniformly in } 0 \le y < x \le 1 \text{ and } \operatorname{Im} p.
\end{aligned} \qquad (2.2.45)$$

Similarly, from (2.2.32) by means of (2.2.35), (2.2.40) and (2.2.41) for the derivatives G_x, G_y and G_{xy} we have the representations

$$G_x(x,y;p) = \frac{1}{ld_0(p)}\frac{b(x)}{a(y)}\begin{cases} ch\; sz \cdot sh\; s(w-1) + O_3 & \text{for } x < y \\ sh\; sw \cdot ch\; s(z-1) + O_4 & \text{for } y < x, \end{cases} \qquad (2.2.46)$$

$$G_y(x,y;p) = \frac{1}{ld_0(p)}\frac{b(y)}{a(x)}\begin{cases} sh\, sz \cdot ch\, s(w-1) + O_5 & \text{for} \quad x<y \\ ch\, sw \cdot sh\, s(z-1) + O_6 & \text{for} \quad y<x, \end{cases} \tag{2.2.47}$$

$$G_{xy}(x,y;p) = \frac{b(x)b(y)}{l^2 d_0(p)}\, s\begin{cases} ch\, sz \cdot ch\, s(w-1) + O_7 & \text{for} \quad x<y \\ ch\, sw \cdot ch\, s(z-1) + O_8 & \text{for} \quad y<x, \end{cases} \tag{2.2.48}$$

where the terms O_3, O_5, O_7 behave like O_1 and the terms O_4, O_6, O_8 like O_2, respectively, as $\mathrm{Re}\, p \to +\infty$.

To deal with the inverse problem with flux observations we have to prove certain properties of G_x that require the second-order asymptotic relation for this function. Taking the second order asymptotics in (2.2.35), (2.2.40) into account, from (2.2.32) with (2.2.41) we deduce the formula

$$G_x(x,y;p) = \begin{cases} \frac{1}{ld_0(p)}\frac{b(x)}{a(y)}\Big[\, ch\, sz \cdot sh\, s(w-1) \\ \quad + \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x)\right)\frac{1}{s} sh\, sz \cdot sh\, s(w-1) \\ \quad - \frac{\zeta(1)-\zeta(y)}{2}\frac{1}{s} ch\, sz \cdot ch\, s(w-1) + \frac{1}{s}O_9\Big] & \text{for} \quad x<y, \\ \frac{1}{ld_0(p)}\frac{b(x)}{a(y)}\Big[sh\, sw \cdot ch\, s(z-1) \\ \quad - \left(\frac{\zeta(1)-\zeta(x)}{2} + l\bar{b}(x)a'(x)\right)\frac{1}{s} sh\, sw \cdot sh\, s(z-1) \\ \quad + \frac{\zeta(y)}{2}\frac{1}{s} ch\, sw \cdot ch\, s(z-1) + \frac{1}{s}O_{10}\Big] & \text{for} \quad y<x \end{cases} \tag{2.2.49}$$

where O_9 and O_{10} behave like O_1 and O_2, respectively, as $\mathrm{Re}\, p \to +\infty$.

2.2.3 Basic properties and representation of Green function in case of the third kind boundary conditions

The Green function G of the operator L given by (1.2.11) in the interval $x \in (0,1)$ with the third kind boundary conditions is a solution to the problem (1.2.20). This function is given by the formula (2.2.32), where $\psi_j(x,p), j=1,2$ are the solutions of $L\psi = 0$ satisfying the initial conditions

$$\begin{aligned} &\psi_1(0,p) = c_1\lambda(0),\ \psi'_{1,x}(0,p) = c_1\alpha_1, \\ &\psi_2(1,p) = c_2\lambda(1),\ \psi'_{2,p}(1,p) = -c_2\alpha_2 \end{aligned} \tag{2.2.50}$$

with arbitrarily chosen numbers c_1 and c_2 and $\Delta_0(p)$ again is the Wronski determinant at zero, i.e.

$$\Delta_0(p) = \psi_1(0,p)\psi_{2,x}(0,p) - \psi_{1,x}(0,p)\psi_2(0,p) \tag{2.2.51}$$

(cf. [39] Section 24, [13] Chapter 7.2 and [24]). The assumed inequalities $\alpha_1, \alpha_2 \geq 0$ (Section 1.1) imply that 0 is not an eigenvalue of the operator L with the third kind boundary conditions in case $\mathrm{Re}\, p > 0$. This in turn yields that ψ_1 and ψ_2 are linearly independent [39]. Thus, $\Delta(p) \neq 0$ for $\mathrm{Re}\, p > 0$.

As in the previous subsection we deduce the following basic properties of G:

Lemma 2.3. *Assume that* (2.2.2) *holds and G is the Green function of the operator L in $[0,1]$ with the third kind boundary conditions. Then the assertions of Lemma 2.2 are valid for G.*

Let us set $c_1 = \frac{1}{a(0)\lambda(0)}$ with a and b given by (2.2.6). Then using the assertion (2.2.3) of Lemma 2.1 and observing the relation $\frac{1}{ab} = a\bar{b}$ we derive for the function ψ_1 and its first derivative the following asymptotical formulas:

$$\left.\begin{aligned}
\psi_1(x,p) &= \tfrac{1}{a(x)} ch\, sz + \tfrac{1}{a(x)}\left[\tfrac{\zeta(x)}{2} + \tfrac{l\bar{b}(0)}{\lambda(0)}\left(\lambda(0)a'(0) + \alpha_1 a(0)\right)\right]\tfrac{1}{s}\, sh\, sz \\
&\quad + O\left(\tfrac{e^{sz}}{s^2}\right) = \tfrac{1}{a(x)} ch\, sz + O\left(\tfrac{e^{sz}}{s}\right), \\
\psi_{1,x}(x,p) &= \tfrac{b(x)}{l}\, s\, sh\, sz \\
&\quad + \tfrac{b(x)}{l}\left[\tfrac{\zeta(x)}{2} - l\bar{b}(x)a'(x) + \tfrac{l\bar{b}(0)}{\lambda(0)}\left(\lambda(0)a'(0) + \alpha_1 a(0)\right)\right] ch\, sz \\
&\quad + O\left(\tfrac{e^{sz}}{s}\right) = \tfrac{b(x)}{l}\, s\, sh\, sz + O\left(e^{sz}\right) \\
&\text{for } Re\, p \to +\infty \text{ uniformly with respect to } x \in [0,1] \text{ and } \operatorname{Im} p.
\end{aligned}\right\} \quad (2.2.52)$$

Further, let us set and $c_2 = \frac{1}{a(1)\lambda(1)}$. Using again the change of variables (2.2.36) we get the following Cauchy problem for $\psi^\dagger$:

$$L(\psi^\dagger)(\xi,p) = 0,\ \xi \in (0,1), \quad \psi^\dagger(0,p) = \frac{1}{a(0)},\ \psi^\dagger_\xi(0,p) = \frac{\alpha_2}{a(1)\lambda(1)}. \quad (2.2.53)$$

Writing the asymptotical formulas (2.2.3) for $\psi^\dagger$ and $\psi^\dagger_\xi$ in terms of the quantities (2.2.38) and observing the relations (2.2.39) and $\frac{1}{ab} = a\bar{b}$, again, we arrive at the following representations:

$$\left.\begin{aligned}
\psi_2(x,p) &= \tfrac{1}{a(x)} ch\, s(1-z) \\
&\quad - \tfrac{1}{a(x)}\left[\tfrac{\zeta(1)-\zeta(x)}{2} + \tfrac{l\bar{b}(1)}{\lambda(1)}\left(-\lambda(1)a'(1) + \alpha_2 a(1)\right)\right]\tfrac{1}{s}\, sh\, s(z-1) \\
&\quad + O\left(\tfrac{e^{s(1-z)}}{s^2}\right) = \tfrac{1}{a(x)} ch\, s(1-z) + O\left(\tfrac{e^{s(1-z)}}{s}\right), \\
\psi_{2,x}(x,p) &= \tfrac{b(x)}{l}\, s\, sh\, s(z-1) - \tfrac{b(x)}{l}\left[\tfrac{\zeta(1)-\zeta(x)}{2}\right. \\
&\quad \left. + l\bar{b}(x)a'(x) + \tfrac{l\bar{b}(1)}{\lambda(1)}\left(-\lambda(1)a'(1) + \alpha_2 a(1)\right)\right] ch\, s(1-z) \\
&\quad + O\left(\tfrac{e^{s(1-z)}}{s}\right) = \tfrac{b(x)}{l}\, s\, sh\, s(z-1) + O\left(e^{s(1-z)}\right) \\
&\text{for } Re\, p \to +\infty \text{ uniformly with respect to } x \in [0,1] \text{ and } \operatorname{Im} p.
\end{aligned}\right\} \quad (2.2.54)$$

We define

$$d_1(p) = \frac{1}{s}\lambda(0)\Delta_0(p). \quad (2.2.55)$$

Then from (2.2.51) by means of (2.2.52), (2.2.54) with the first-order asymptotics, (2.2.31) and the relation $\lambda(0) = \frac{a(0)}{b(0)}$ we obtain

$$\begin{aligned} d_1(p) &= \tfrac{1}{s}\lambda(0)\left[\tfrac{1}{a(0)} + O\left(\tfrac{1}{s}\right)\right]\left[-\tfrac{b(0)}{l}s\, sh\, s + O\left(e^s\right)\right] \\ &\quad -\tfrac{1}{s}\lambda(0)\, O\left(1\right)\left[\tfrac{1}{a(0)} ch\, s + O\left(\tfrac{e^s}{s}\right)\right] \\ &= -\tfrac{1}{l} sh\, s + O\left(\tfrac{e^s}{s}\right) \quad \text{for } \operatorname{Re} p \to +\infty \quad \text{uniformly in } \operatorname{Im} p\,. \end{aligned} \tag{2.2.56}$$

Now we can formulate the asymptotical relations for G and its derivatives. Formulas (2.2.32), (2.2.52), (2.2.54) with the first-order asymptotics and (2.2.55) yield

$$G(x,y;p) = \frac{1}{d_1(p)}\frac{1}{a(x)a(y)}\frac{1}{s}\begin{cases} ch\, sz \cdot ch\, s(w-1) + \hat{O}_1 & \text{for} \quad x \le y \\ ch\, sw \cdot ch\, s(z-1) + \hat{O}_2 & \text{for} \quad y \le x, \end{cases} \tag{2.2.57}$$

$$G_x(x,y;p) = \frac{1}{l d_1(p)}\frac{b(x)}{a(y)}\begin{cases} sh\, sz \cdot ch\, s(w-1) + \hat{O}_3 & \text{for} \quad x < y \\ ch\, sw \cdot sh\, s(z-1) + \hat{O}_4 & \text{for} \quad y < x, \end{cases} \tag{2.2.58}$$

$$G_y(x,y;p) = \frac{1}{l d_1(p)}\frac{b(y)}{a(x)}\begin{cases} ch\, sz \cdot sh\, s(w-1) + \hat{O}_5 & \text{for} \quad x < y \\ sh\, sw \cdot ch\, s(z-1) + \hat{O}_6 & \text{for} \quad y < x, \end{cases} \tag{2.2.59}$$

$$G_{xy}(x,y;p) = \frac{b(x)b(y)}{l^2 d_1(p)} s\begin{cases} sh\, sz \cdot sh\, s(w-1) + \hat{O}_7 & \text{for} \quad x < y \\ sh\, sw \cdot sh\, s(z-1) + \hat{O}_8 & \text{for} \quad y < x, \end{cases} \tag{2.2.60}$$

where $w = w(y)$ is defined by (2.2.44) and the terms $\hat{O}_1, \hat{O}_3, \hat{O}_5, \hat{O}_7$ behave like O_1 and the terms $\hat{O}_2, \hat{O}_4, \hat{O}_6, \hat{O}_8$ like O_2, respectively, in the process $\operatorname{Re} p \to +\infty$ (see (2.2.45)). Moreover, using the second order asymptotics in (2.2.52), (2.2.54) we derive the following representation for G_x:

$$G_x(x,y;p) = \begin{cases} \frac{1}{l d_1(p)}\frac{b(x)}{a(y)}\Big[\ sh\, sz \cdot ch\, s(w-1) \\ +\ \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x) + \vartheta_0\right)\frac{1}{s} ch\, sz \cdot ch\, s(w-1) \\ -\left(\frac{\zeta(1)-\zeta(y)}{2} - \vartheta_1\right)\frac{1}{s} sh\, sz \cdot sh\, s(w-1) + \frac{1}{s}\hat{O}_9\Big] \ \text{for } x < y, \\ \frac{1}{l d_1(p)}\frac{b(x)}{a(y)}\Big[ch\, sw \cdot sh\, s(z-1) \\ -\ \left(\frac{\zeta(1)-\zeta(x)}{2} + l\bar{b}(x)a'(x) - \vartheta_1\right)\frac{1}{s} ch\, sw \cdot ch\, s(z-1) \\ +\ \left(\frac{\zeta(y)}{2} + \vartheta_0\right)\frac{1}{s} sh\, sw \cdot sh\, s(z-1) + \frac{1}{s}\hat{O}_{10}\Big] \ \text{for } y < x, \end{cases} \tag{2.2.61}$$

where

$$\vartheta_0 = \frac{l\bar{b}(0)}{\lambda(0)}\left(\lambda(0)a'(0) + \alpha_1 a(0)\right), \ \vartheta_1 = \frac{l\bar{b}(1)}{\lambda(1)}\left(\lambda(1)a'(1) - \alpha_2 a(1)\right) \tag{2.2.62}$$

and $\hat{O}_9$ and $\hat{O}_{10}$ behave like O_1 and O_2, respectively, as $\operatorname{Re} p \to +\infty$.

2.3 Auxiliary results

Lemma 2.4. *Let* $\lambda, \beta \in C[0,1]$ *and* $\lambda, \beta > 0$ *in* $[0,1]$ *and* $\mathcal{G}$ *be a function that is bounded in every strip* S_σ*,* $\sigma > 0$*, and has the following asymptotical representation:*

$$\mathcal{G}(x,y;p) = \frac{1}{\widetilde{d}_0(p)} \begin{cases} \rho_1(x,y)\chi_1(sz)\chi_2(s(w-1)) + \mathcal{O}_1 & \text{for} \quad x < y \\ \rho_2(x,y)\chi_3(sw)\chi_4(s(z-1)) + \mathcal{O}_2 & \text{for} \quad y < x, \end{cases} \tag{2.3.1}$$

where ρ_1, ρ_2 *are some bounded functions,* χ_1*,* χ_2*,* χ_3 *and* χ_4 *are either* sh *or* ch *and the terms* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *behave like* O_1 *and* O_2*, respectively, as* $\operatorname{Re} p \to +\infty$*. Moreover, let*

$$\widetilde{d}_0(p) = \alpha\, sh\, s + O\left(\frac{e^s}{s}\right) \quad \text{for } \operatorname{Re} p \to +\infty \quad \textit{uniformly in } \operatorname{Im} p \tag{2.3.2}$$

with some $\alpha \neq 0$*. Then the estimate*

$$\sup_{\substack{0 \le x \le 1 \\ \operatorname{Re} p > 0}} \sqrt{|p|} \int_0^1 |\mathcal{G}(x,y;p)| dy < \infty \tag{2.3.3}$$

holds.

Proof. First, let us estimate $\widetilde{d}_0(p)$ from below. From (2.3.2) due to (2.2.18) and the relation $|sh\, s| \geq sh(\operatorname{Re} s)$ we have

$$|\widetilde{d}_0(p)| \geq |\alpha|\, sh(\operatorname{Re} s) - O\left(\frac{e^{\operatorname{Re} s}}{|s|}\right) \quad \text{for } \operatorname{Re} p \to +\infty \quad \text{uniformly in } \operatorname{Im} p\,.$$

Since $\operatorname{Re} p \to +\infty \Rightarrow \operatorname{Re} s \to +\infty$ (see (2.2.21)), the term $|\alpha| sh(\operatorname{Re} s)$ dominates over the term $O\left(\frac{e^{\operatorname{Re} s}}{|s|}\right)$ in the process $\operatorname{Re} p \to +\infty$. Thus, in case of sufficiently large σ the estimate

$$|\widetilde{d}_0(p)| \;\geq\; \frac{|\alpha|}{2}\, sh(\operatorname{Re} s) \quad \text{for} \quad \operatorname{Re} p > \sigma$$

is valid. Due to this estimate, (2.2.18) and the relation $s = l\sqrt{p}$ from (2.3.1) we obtain

$$\begin{aligned}
\sqrt{|p|} \textstyle\int_0^1 |\mathcal{G}(x,y;p)| dy &\leq \text{Const} \tfrac{|s|}{sh(\operatorname{Re} s)} \Big[\textstyle\int_0^x ch(\operatorname{Re} s \cdot w) ch \operatorname{Re} s(z-1) dy \\
&\quad + \textstyle\int_x^1 ch(\operatorname{Re} s \cdot z) ch \operatorname{Re} s(w-1) dy \Big] \\
&= \text{Const} \tfrac{|s|}{\operatorname{Re} s \cdot sh(\operatorname{Re} s)} \Big[\textstyle\int_0^x \tfrac{1}{w'(y)} \tfrac{d}{dy} sh(\operatorname{Re} s \cdot w)\, dy \cdot ch \operatorname{Re} s(z-1) \\
&\quad + ch(\operatorname{Re} s \cdot z) \textstyle\int_x^1 \tfrac{1}{w'(y)} \tfrac{d}{dy} sh \operatorname{Re} s(w-1)\, dy \Big], \quad x \in [0,1], \quad \operatorname{Re} p > \sigma\,.
\end{aligned}$$

We note that the relation

$$w'(y) \geq \varkappa > 0\,, \quad y \in [0,1] \tag{2.3.4}$$

holds due to (2.2.44) and the assumptions imposed on λ and β. Using (2.3.4), the positivity of $\frac{d}{dy} sh(\mathrm{Re}\, s \cdot w)$, $\frac{d}{dy} sh\, \mathrm{Re}\, s(w-1)$, the definitions (2.2.7), (2.2.44) of z, w and the relation $s = l\sqrt{p}$ we further obtain

$$\begin{aligned}
&\sqrt{|p|} \int_0^1 |\mathcal{G}(x,y;p)|dy \leq \mathrm{Const} \tfrac{|s|}{\mathrm{Re}\, s \cdot sh(\mathrm{Re}\, s)} \\
&\quad \times \left[\int_0^x \tfrac{d}{dy} sh(\mathrm{Re}\, s \cdot w)\, dy \cdot ch\, \mathrm{Re}\, s(z-1) + ch(\mathrm{Re}\, s \cdot z) \int_x^1 \tfrac{d}{dy} sh\, \mathrm{Re}\, s(w-1)\, dy \right] \\
&\quad = \mathrm{Const} \tfrac{|s|}{\mathrm{Re}\, s} \tfrac{1}{sh(\mathrm{Re}\, s)} \left[sh(\mathrm{Re}\, s \cdot z)\, ch\, \mathrm{Re}\, s(1-z) + ch(\mathrm{Re}\, s \cdot z)\, sh\, \mathrm{Re}\, s(1-z) \right] \\
&\quad = \mathrm{Const} \tfrac{|s|}{\mathrm{Re}\, s} \quad \text{for} \quad x \in [0,1]\,, \quad \mathrm{Re}\, p > \sigma\,.
\end{aligned}$$

Due to (2.2.21) we deduce the inequality

$$\sup_{\substack{0 \leq x \leq 1 \\ \mathrm{Re}\, p > \sigma}} \sqrt{|p|} \int_0^1 |\mathcal{G}(x,y;p)|dy < \infty.$$

Finally, observing that $\mathcal{G}(x,y,p)$ is bounded in S_σ, this supremum can be extended to $0 \leq x \leq 1$, $\mathrm{Re}\, p > 0$. This proves (2.3.3). □

Lemma 2.5. *Let $\lambda, \beta \in C^1[0,1]$ and $\lambda, \beta > 0$ in $[0,1]$. Let $\mathcal{G}$ be a function that has the following asymptotical representation:*

$$\mathcal{G}(x,y;p) = \begin{cases} q_1(x)\rho_1(y)\chi_1(sz)\chi_2'(s(w-1)) + \mathcal{O}_1 & \textit{for} \quad x < y \\ q_2(x)\rho_2(y)\chi_3'(sw)\chi_4(s(z-1)) + \mathcal{O}_2 & \textit{for} \quad y < x, \end{cases} \tag{2.3.5}$$

where q_1, q_2, χ_1, χ_4 are some continuous functions, $\rho_1, \rho_2, \chi_2, \chi_3$ are some differentiable functions and the terms $\mathcal{O}_1$ and $\mathcal{O}_2$ behave like O_1 and O_2, respectively, as $\mathrm{Re}\, p \to +\infty$. Then for any $V \in C^1[0,1]$ the relation

$$\begin{aligned}
&\int_0^1 \mathcal{G}(x,y;p)V(y)dy \\
&\quad = \tfrac{1}{s} \left[q_1(x) \tfrac{\rho_1(y)}{w'(y)} \chi_1(sz)\chi_2(s(w-1)) + \mathcal{O}_3(x,y,p) \right] V(y) \Big|_{y=x}^{y=1} \\
&\quad + \tfrac{1}{s} \left[q_2(x) \tfrac{\rho_2(y)}{w'(y)} \chi_3(sw)\chi_4(s(z-1)) + \mathcal{O}_4(x,y,p) \right] V(y) \Big|_{y=0}^{y=x} \\
&\quad - \int_0^1 \mathcal{T}[V](x,y;p)dy
\end{aligned} \tag{2.3.6}$$

holds, where

$$\begin{aligned}
&\mathcal{T}[V](x,y,p) \\
&\quad = \frac{1}{s} \begin{cases} q_1(x) \left(\frac{\rho_1(y)V(y)}{w'(y)} \right)' \chi_1(sz)\chi_2(s(w-1)) + V'(y)\mathcal{O}_3 & \textit{for} \quad x < y \\[2ex] q_2(x) \left(\frac{\rho_2(y)V(y)}{w'(y)} \right)' \chi_3(sw)\chi_4(s(z-1)) + V'(y)\mathcal{O}_4 & \textit{for} \quad y < x, \end{cases}
\end{aligned} \tag{2.3.7}$$

$$\mathcal{O}_3(x,y,p)=s\int_1^y \mathcal{O}_1(x,\eta,p)d\eta\,,\quad \mathcal{O}_4(x,y,p)=s\int_0^y \mathcal{O}_2(x,\eta,p)d\eta \tag{2.3.8}$$

and $\mathcal{O}_3$, $\mathcal{O}_4$ *behave like* O_1, O_2, *respectively, as* $\mathrm{Re}\,p\to+\infty$.

Proof. Let ρ,χ be some differentiable functions and $0\le c_1,c_2\le 1$. By means of integration by parts we get the following auxiliary relation:

$$\begin{aligned}\int_{c_1}^{c_2}\rho(y)\chi'(sw)dy &= \frac{1}{s}\int_{c_1}^{c_2}\frac{\rho(y)}{w'(y)}\,\frac{d}{dy}\chi(sw)dy\\ &= \frac{1}{s}\frac{\rho(y)}{w'(y)}\chi(sw)\Big|_{y=c_1}^{y=c_2}-\frac{1}{s}\int_{c_1}^{c_2}\left(\frac{\rho(y)}{w'(y)}\right)'\chi(sw)dy.\end{aligned} \tag{2.3.9}$$

Further, in view of the definition (2.3.8) of $\mathcal{O}_3$ and $\mathcal{O}_4$ we have $\mathcal{O}_1=\frac{1}{s}\frac{d}{dy}\mathcal{O}_3$ and $\mathcal{O}_2=\frac{1}{s}\frac{d}{dy}\mathcal{O}_4$. Thus, due to (2.3.5), we can rewrite the integral $\int_0^1\mathcal{G}(x,y,p)V(y)dy$ in the form

$$\begin{aligned}&\textstyle\int_0^1\mathcal{G}(x,y,p)V(y)dy\\ &\quad=q_1(x)\chi_1(sz)\int_x^1\rho_1(y)V(y)\chi_2'(s(w-1))dy+\frac{1}{s}\int_x^1V(y)\frac{d}{dy}\mathcal{O}_3(x,y,p)dy\\ &\quad+q_2(x)\chi_4(s(z-1))\int_0^x\rho_2(y)V(y)\chi_3'(sw)dy+\frac{1}{s}\int_0^xV(y)\frac{d}{dy}\mathcal{O}_4(x,y,p)dy.\end{aligned}$$

In this relation we apply to the integrals containing χ_2' and χ_3' the formula (2.3.9) and integrate the integrals containing $\mathcal{O}_3$ and $\mathcal{O}_4$ by parts. As a result we immediately obtain (2.3.6) with (2.3.7).

To complete the proof it remains to show that $\mathcal{O}_3$ and $\mathcal{O}_4$ behave like O_1 and O_2, respectively, as $\mathrm{Re}\,p\to+\infty$. To this end let us define

$$\xi=\xi(\eta)=\frac{1}{l}\int_0^\eta\sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau\,. \tag{2.3.10}$$

Observing (2.3.8), the behavior of $\mathcal{O}_1$, $\mathcal{O}_2$, (2.2.45) and the definition (2.2.44) of $w(y)$ and (2.3.10) of $\xi(\eta)$ we compute:

$$\begin{aligned}&\left|\frac{e^{s(1-w+z)}}{s}\right|^{-1}|\mathcal{O}_3(x,y,p)|=\left[\frac{e^{\mathrm{Re}\,s(1-w+z)}}{|s|}\right]^{-1}|s|\left|\textstyle\int_1^y\mathcal{O}_1(x,\eta,p)d\eta\right|\\ &\quad\le \mathrm{Const}\left[\frac{e^{\mathrm{Re}\,s(1-w+z)}}{|s|}\right]^{-1}|s|\int_y^1\frac{e^{\mathrm{Re}\,s(1-\xi+z)}}{|s|}d\eta=\mathrm{Const}\,|s|\int_y^1e^{\mathrm{Re}\,s(w-\xi)}d\eta\\ &\quad=-\frac{\mathrm{Const}\,|s|}{\mathrm{Re}\,s}\int_y^1\frac{1}{\xi'(\eta)}\frac{d}{d\eta}e^{\mathrm{Re}\,s(w-\xi)}d\eta\quad\text{for }\ 0\le x<y\le 1\,,\ \mathrm{Re}\,p>\sigma\,,\end{aligned}$$

$$\begin{aligned}&\left|\frac{e^{s(w+1-z)}}{s}\right|^{-1}|\mathcal{O}_4(x,y,p)|=\left[\frac{e^{\mathrm{Re}\,s(w+1-z)}}{|s|}\right]^{-1}|s|\left|\textstyle\int_0^y\mathcal{O}_2(x,\eta,p)d\eta\right|\\ &\quad\le \mathrm{Const}\left[\frac{e^{\mathrm{Re}\,s(w+1-z)}}{|s|}\right]^{-1}|s|\int_y^1\frac{e^{\mathrm{Re}\,s(\xi+1-z)}}{|s|}d\eta=\mathrm{Const}\,|s|\int_0^ye^{\mathrm{Re}\,s(\xi-w)}d\eta\\ &\quad=\frac{\mathrm{Const}\,|s|}{\mathrm{Re}\,s}\int_0^y\frac{1}{\xi'(\eta)}\frac{d}{d\eta}e^{\mathrm{Re}\,s(\xi-w)}d\eta\quad\text{for }\ 0\le y<x\le 1\,,\ \mathrm{Re}\,p>\sigma\end{aligned}$$

with some sufficiently large $\sigma > 0$ independent of x, y. Since $\frac{1}{\xi'(\eta)}$ is bounded, $\frac{d}{d\eta} e^{\mathrm{Re}\, s(w-\xi)} \leq 0$ and $\frac{d}{d\eta} e^{\mathrm{Re}\, s(\xi-w)} \geq 0$ using the inequality (2.2.21) we obtain

$$\left| \frac{e^{s(1-w+z)}}{s} \right|^{-1} |\mathcal{O}_3(x,y,p)| \leq -\frac{\mathrm{Const}\,|s|}{\mathrm{Re}\, s} \int_y^1 \frac{d}{d\eta} e^{\mathrm{Re}\, s(w-\xi)} d\eta$$
$$= \frac{\mathrm{Const}\,|s|}{\mathrm{Re}\, s} \left[1 - e^{\mathrm{Re}\, s(w-1)}\right] \leq \mathrm{Const} \quad \text{for } \ 0 \leq x < y \leq 1\,,\ \mathrm{Re}\, p > \sigma\,,$$

$$\left| \frac{e^{s(w+1-z)}}{s} \right|^{-1} |\mathcal{O}_4(x,y,p)| \leq \frac{\mathrm{Const}\,|s|}{\mathrm{Re}\, s} \int_0^y \frac{d}{d\eta} e^{\mathrm{Re}\, s(\xi-w)} d\eta$$
$$\frac{\mathrm{Const}\,|s|}{\mathrm{Re}\, s} \left[1 - e^{-\mathrm{Re}\, sw}\right] \leq \mathrm{Const} \quad \text{for } \ 0 \leq y < x \leq 1\,,\ \mathrm{Re}\, p > \sigma\,.$$

This proves that $\mathcal{O}_3$ and $\mathcal{O}_4$ behave like O_1 and O_2, respectively, as $\mathrm{Re}\, p \to +\infty$ (cf. (2.2.45)). □

2.4 Asymptotical properties of Green function in case of the first kind boundary conditions

Let us start by proving some estimates of integrals of G and its derivatives.

Theorem 2.1. *Let* λ, β *satisfy* (2.2.2)*. Then the Green function* G *of the operator* L *in* $[0,1]$ *with the first kind boundary conditions satisfies the following estimates:*

$$C_1 = \sup_{\substack{0 \leq x \leq 1 \\ \mathrm{Re}\, p > 0}} |p| \int_0^1 |G(x,y;p)| dy < \infty, \tag{2.4.1}$$

$$C_2 = \sup_{\substack{0 \leq x \leq 1 \\ \mathrm{Re}\, p > 0}} \sqrt{|p|} \int_0^1 |G_x(x,y;p)| dy < \infty, \tag{2.4.2}$$

$$C_3 = \sup_{\substack{0 \leq x \leq 1 \\ \mathrm{Re}\, p > 0}} \sqrt{|p|} \int_0^1 |G_y(x,y;p)| dy < \infty, \tag{2.4.3}$$

$$C_4 = \sup_{\substack{0 \leq x \leq 1 \\ \mathrm{Re}\, p > 0}} \int_0^1 |G_{xy}(x,y;p)| dy < \infty\ . \tag{2.4.4}$$

Theorem 2.1 was proved in [23], but since the proof is short, we will present it here.

Proof. Estimates (2.4.1) - (2.4.4) follow from Lemma 2.4 if we take the relations (2.2.43), (2.2.46) - (2.2.48) for G, G_x, G_y, G_{xy}, the positivity and boundendness of a and b following from (2.2.6), (2.2.2), the assumptions of Theorem 2.1 and the equality $s = l\sqrt{p}$ into account. □

Next we formulate without proofs two theorems proved also in [23].

Theorem 2.2. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the first kind boundary conditions. If* $V \in C^1[0,1]$, *then the estimate*

$$\sup_{\mathrm{Re}\, p>0} \left| \sqrt{|p|} \left(\int_0^1 G_{xy}(x,y,p)V(y)dy \;-\; \frac{V(x)}{\lambda(x)} \right) \right| \le C_5 \|V'\|_{C[0,1]} \qquad (2.4.5)$$

holds for any $x \in [0,1]$, *where* C_5 *is a positive constant.*

Theorem 2.3. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the first kind boundary conditions. If* $V \in C^1[0,1]$, *then the estimate*

$$\sup_{\mathrm{Re}\, p>0} \left| \sqrt{|p|} \left(\int_0^1 pG(x,y,p)V(y)dy \;+\; \frac{V(x)}{\beta(x)} \right) \right| \le C_6(x) \|V\|_{C^1[0,1]} \quad (2.4.6)$$

holds for any $x \in (0,1)$, *where* $C_6(x)$ *is some function bounded in every compact subinterval of* $(0,1)$.

Finally, we prove a similar theorem about the asymptotics of the derivative of $\int_0^1 G(x,y,p)V(y)dy$.

Theorem 2.4. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the first kind boundary conditions. If* $V \in C^2[0,1]$, *then the estimate*

$$\sup_{\mathrm{Re}\, p>0} \left| \sqrt{|p|} \left[\int_0^1 pG_x(x,y,p)V(y)dy + \left(\frac{V(x)}{\beta(x)} \right)' \right] \right| \le C_7(x) \|V\|_{C^2[0,1]} \, (2.4.7)$$

is valid for any $x \in (0,1)$, *where* $C_7(x)$ *is some function bounded in every compact subinterval of* $(0,1)$.

Proof. We begin the proof by deducing some auxiliary relations. Similarly to the estimation of $\widetilde{d}_0$ in the proof of Lemma 2.4, from (2.2.42) we get $|d_0(p)| \ge \frac{1}{2l}\, sh(\mathrm{Re}\, s)$ for $\mathrm{Re}\, p > \sigma$ with some sufficiently large $\sigma > 0$. This relation implies

$$\begin{aligned}
&\left|\tfrac{e^{sz}}{d_0(p)}\right|, \left|\tfrac{sh\, sz}{d_0(p)}\right|, \left|\tfrac{ch\, sz}{d_0(p)}\right| \le \mathrm{Const}\tfrac{|e^{sz}|}{sh(\mathrm{Re}\, s)} = \mathrm{Const}\, e^{\mathrm{Re}\, s(z-1)} \\
&\qquad = \mathrm{Const}\, e^{-\mathrm{Re}\sqrt{p}\int_x^1 \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} \quad \text{for } \mathrm{Re}\, p > \sigma\,, \\
&\left|\tfrac{e^{s(1-z)}}{d_0(p)}\right|, \left|\tfrac{sh\, s(z-1)}{d_0(p)}\right|, \left|\tfrac{ch\, s(z-1)}{d_0(p)}\right| \le \mathrm{Const}\tfrac{\left|e^{s(1-z)}\right|}{sh(\mathrm{Re}\, s)} = \mathrm{Const}\, e^{-\mathrm{Re}\, sz} \\
&\qquad = \mathrm{Const}\, e^{-\mathrm{Re}\sqrt{p}\int_0^x \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} \quad \text{for } \mathrm{Re}\, p > \sigma.
\end{aligned} \qquad (2.4.8)$$

Due to (2.2.20) the following inequalities

$$\begin{gathered}
e^{-\mathrm{Re}\sqrt{p}\int_x^1 \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau},\; e^{-\mathrm{Re}\sqrt{p}\int_0^x \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} \;\le\; \tfrac{C(\kappa,x)}{|p|^\kappa} \\
\text{for any } \; x \in (0,1)\,, \quad \mathrm{Re}\, p > 0\,, \quad \kappa > 0
\end{gathered} \qquad (2.4.9)$$

are valid with some function $C(\kappa, x)$ of $\kappa > 0$ and x. Note that the function $C(\kappa, \cdot)$ is bounded in every compact subinterval of $(0, 1)$. Moreover, by (2.2.42) and (2.2.21) we have

$$\begin{aligned} \frac{sh\,s}{d_0(p)} &= \frac{l\,sh\,s}{sh\,s+O\left(\frac{e^s}{s}\right)} = \frac{l}{1+O\left(\frac{1}{s}\right)} = l+O\left(\frac{1}{s}\right) = l+O\left(\frac{1}{\sqrt{|p|}}\right) \\ \frac{ch\,s}{d_0(p)} &= \frac{l\,ch\,s}{sh\,s+O\left(\frac{e^s}{s}\right)} = \frac{l+O\left(e^{-2s}\right)}{1+O\left(\frac{1}{s}\right)} = l+O\left(\frac{1}{s}\right) = l+O\left(\frac{1}{\sqrt{|p|}}\right) \end{aligned} \tag{2.4.10}$$

as $\operatorname{Re} p \to +\infty$ uniformly in $\operatorname{Im} p$.

Due to the representation (2.2.49), we can decompose G_x as follows:

$$G_x(x, y, p) = \mathcal{G}_1(x, y, p) + \mathcal{G}_2(x, y, p) + \mathcal{G}_3(x, y, p), \tag{2.4.11}$$

where

$$\mathcal{G}_1(x, y, p) = \frac{1}{l d_0(p)} \frac{b(x)}{a(y)} \begin{cases} ch\,sz \cdot sh\,s(w-1) & \text{for} \quad x < y \\ sh\,sw \cdot ch\,s(z-1) & \text{for} \quad x > y, \end{cases} \tag{2.4.12}$$

$$\begin{aligned} \mathcal{G}_2(x, y, p) &= \frac{1}{l d_0(p)} \frac{b(x)}{a(y)} \frac{1}{s} \\ &\times \begin{cases} \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x)\right) sh\,sz \cdot sh\,s(w-1) & \text{for} \quad x < y \\ \left(\frac{\zeta(x)-\zeta(1)}{2} - l\bar{b}(x)a'(x)\right) sh\,sw \cdot sh\,s(z-1) & \text{for} \quad x > y, \end{cases} \end{aligned} \tag{2.4.13}$$

$$\begin{aligned} \mathcal{G}_3(x, y, p) &= \frac{1}{d_0(p)} \frac{1}{s} \\ &\times \begin{cases} \frac{b(x)}{l\,a(y)} \frac{\zeta(y)-\zeta(1)}{2} ch\,sz \cdot ch\,s(w-1) + \mathcal{O}_1 & \text{for} \quad x < y \\ \frac{b(x)}{l\,a(y)} \frac{\zeta(y)}{2} ch\,sw \cdot ch\,s(z-1) + \mathcal{O}_2 & \text{for} \quad x > y \end{cases} \end{aligned} \tag{2.4.14}$$

and $\mathcal{O}_1 = \frac{b(x)}{l\,a(y)} O_9$ and $\mathcal{O}_2 = \frac{b(x)}{l\,a(y)} O_{10}$ behave like O_1 and O_2, respectively, as $\operatorname{Re} p \to +\infty$.

We apply twice Lemma 2.5 to the integral $\int_0^1 \mathcal{G}_1(x, y, p) V(y) dy$ and use the relation $\frac{1}{aw'} = l\bar{b}$ following from (2.2.6), (2.2.44) to get

$$\int_0^1 \mathcal{G}_1(x, y, p) V(y) dy = \mathcal{D}_1[V](x, p) + \int_0^1 \mathcal{T}_1[V](x, y, p) dy \tag{2.4.15}$$

with

$$\begin{aligned}\mathcal{D}_1[V](x,p) = \frac{b(x)}{d_0(p)} \Bigg\{ \frac{1}{s} \Bigg[& ch\, sz \cdot ch\, s(w-1)\bar{b}(y)V(y)\Big|_{y=x}^{y=1} \\ & + ch\, sw \cdot ch\, s(z-1)\bar{b}(y)V(y)\Big|_{y=0}^{y=x} \Bigg] \\ & - \frac{1}{s^2} \Bigg[ch\; sz \cdot sh\; s(w-1) \frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \Big|_{y=x}^{y=1} \\ & + sh\; sw \cdot ch\; s(z-1) \frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \Big|_{y=0}^{y=x} \Bigg] \Bigg\},\end{aligned} \tag{2.4.16}$$

$$\mathcal{T}_1[V](x,y.p) = \frac{b(x)}{s^2 d_0(p)} \left(\frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \right)' \begin{cases} ch\; sz \cdot sh\; s(w-1) & \text{for} \quad x < y \\ sh\; sw \cdot ch\; s(z-1) & \text{for} \quad x > y. \end{cases} \tag{2.4.17}$$

Applying once Lemma 2.5 to the integrals $\int_0^1 \mathcal{G}_2(x,y,p)V(y)dy$ and $\int_0^1 \mathcal{G}_3(x,y,p)V(y)dy$ we similarly get

$$\int_0^1 \mathcal{G}_k(x,y,p)V(y)dy = \mathcal{D}_k[V](x,p) - \int_0^1 \mathcal{T}_k[V](x,y,p)dy, \; k = 2,3, \tag{2.4.18}$$

where

$$\begin{aligned}\mathcal{D}_2[V](x,p) &= \frac{b(x)}{s^2 d_0(p)} \Bigg\{ \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x) \right) sh\; sz \cdot ch\; s(w-1)\, \bar{b}(y)V(y)\Big|_{y=x}^{y=1} \\ &+ \left(\frac{\zeta(x)-\zeta(1)}{2} - l\bar{b}(x)a'(x) \right) ch\; sw \cdot sh\; s(z-1)\, \bar{b}(y)V(y)\Big|_{y=0}^{y=x} \Bigg\},\end{aligned} \tag{2.4.19}$$

$$\begin{aligned}\mathcal{D}_3[V](x,p) &= \frac{b(x)}{s^2 d_0(p)} \Bigg\{ \left[\frac{\zeta(y)-\zeta(1)}{2} \bar{b}(y)\; ch\; sz \cdot sh\; s(w-1) + \mathcal{O}_3(x,y,p) \right] V(y)\Big|_{y=x}^{y=1} \\ &+ \left[\frac{\zeta(y)}{2} \bar{b}(y)\; sh\; sw \cdot ch\; s(z-1) + \mathcal{O}_4(x,y,p) \right] V(y)\Big|_{y=0}^{y=x} \Bigg\},\end{aligned} \tag{2.4.20}$$

$$\begin{aligned}\mathcal{T}_2[V](x,y.p) &= \frac{b(x)}{s^2 d_0(p)} \\ &\times \begin{cases} \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x) \right) \left(\bar{b}(y)V(y)\right)' sh\, sz \cdot ch\, s(w-1), & x < y \\ \left(\frac{\zeta(x)-\zeta(1)}{2} - l\bar{b}(x)a'(x) \right) \left(\bar{b}(y)V(y)\right)' ch\, sw \cdot sh\; s(z-1), & x > y \end{cases}\end{aligned} \tag{2.4.21}$$

$$\mathcal{T}_3[V](x,y,p) = \frac{1}{s^2 d_0(p)} \times \begin{cases} b(x)\Big(\frac{\zeta(y)-\zeta(1)}{2}\bar b(y)V(y)\Big)' ch\, sz \cdot sh\, s(w-1) + V'(y)\mathcal{O}_3, & x<y \\ b(x)\Big(\frac{\zeta(y)}{2}\bar b(y)V(y)\Big)' sh\, sw \cdot ch\, s(z-1) + V'(y)\mathcal{O}_4, & x>y \end{cases} \quad (2.4.22)$$

and $\mathcal{O}_3$, $\mathcal{O}_4$ given by (2.3.8) behave like O_1, O_2, respectively as $\mathrm{Re}\, p \to +\infty$.

Summing up, by (2.4.11), (2.4.15), (2.4.18) we have

$$\int_0^1 G_x(x,y,p)V(y)dy \;=\; \mathcal{D}[V](x,p) + \int_0^1 \mathcal{T}[V](x,y,p)dy \qquad (2.4.23)$$

where $\mathcal{D}[V] = \mathcal{D}_1[V] + \mathcal{D}_2[V] + \mathcal{D}_3[V]$ and $\mathcal{T}[V] = \mathcal{T}_1[V] - \mathcal{T}_2[V] - \mathcal{T}_3[V]$. Let us estimate the quantity $\int_0^1 \mathcal{T}[V](x,y,p)dy$ using Lemma 2.4 for the integrals of functions $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ given by (2.4.17), (2.4.21), (2.4.22). This lemma with the relation $s = l\sqrt{p}$ implies

$$\left|\int_0^1 \mathcal{T}[V](x,y,p)dy\right| \;\le\; \frac{\mathrm{Const}\|V\|_{C^2[0,1]}}{|p|^{3/2}}, \quad x\in[0,1], \quad \mathrm{Re}\, p>0. \quad (2.4.24)$$

Further, computing $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$ from (2.4.16), (2.4.19), (2.4.20), simplifying by means of the relations $w(0)=0, w(1)=1, w(x)=z$ and reordering the terms we obtain

$$\mathcal{D}[V] \;=\; D_{right}[V] + D_{left}[V] + D_{diag}[V] + \widetilde{\mathcal{D}}[V], \qquad (2.4.25)$$

where

$$\begin{aligned} D_{right}[V](x,p) &= \tfrac{b(x)}{sd_0(p)}\bar b(1)V(1)ch\; sz\,, \\ D_{left}[V](x,p) &= -\tfrac{b(x)}{sd_0(p)}\bar b(0)V(0)ch\; s(z-1), \end{aligned} \qquad (2.4.26)$$

$$D_{diag}[V](x,p) \;=\; \frac{b(x)}{s^2 d_0(p)}\, sh\, s\, \left[l(\bar b(x))^2 a'(x)V(x) - \frac{(\bar b(x)V(x))'}{w'(x)}\right], \quad (2.4.27)$$

$$\begin{aligned} &\widetilde{\mathcal{D}}[V](x,p) \\ &= \tfrac{b(x)}{s^2 d_0(p)} \Big\{ \Big[\Big(\tfrac{\zeta(x)}{2} - l\bar b(x)a'(x)\Big)\, sh\; sz\, \bar b(1) + \mathcal{O}_3(x,1,p)\Big]\, V(1) \\ &- \Big[\Big(\tfrac{\zeta(x)-\zeta(1)}{2} - l\bar b(x)a'(x)\Big)\, sh\; s(z-1)\, \bar b(0) + \mathcal{O}_4(x,0,p)\Big]\, V(0) \\ &+ \left(\mathcal{O}_4(x,x,p) - \mathcal{O}_3(x,x,p)\right) V(x) \Big\}. \end{aligned} \qquad (2.4.28)$$

Let $x \in (0,1)$. From the formulas (2.4.26) by means of the inequalities (2.4.8) and (2.4.9) with $\kappa = 1$ we obtain

$$|D_{right}[V](x,p)|,\ |D_{left}[V](x,p)| \le \frac{\mathrm{C}^1(x)\,\|V\|_{C[0,1]}}{|p|^{3/2}} \quad \text{for } \mathrm{Re}\,p > \sigma \tag{2.4.29}$$

with a function C^1 bounded in every compact subinterval of $(0,1)$. Further, by the behavior of $\mathcal{O}_3$ and $\mathcal{O}_4$ (cf. (2.2.45)) we have $\mathcal{O}_3(x,x,p), \mathcal{O}_4(x,x,p) = O\left(\frac{e^s}{s}\right)$ as $\mathrm{Re}\,p \to +\infty$ uniformly in x and $\mathrm{Im}\,p$. Using this relation and (2.4.8), (2.4.9) in (2.4.28) and observing the equality $s = l\sqrt{p}$ and (2.4.9) with $\kappa = 1/2$ we deduce the estimate

$$\begin{aligned} &|\widetilde{\mathcal{D}}[V](x,p)| \\ &\le \frac{\mathrm{Const}}{|p|}\left[e^{-\mathrm{Re}\sqrt{p}\int_x^1\sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau}|V(1)| + e^{-\mathrm{Re}\sqrt{p}\int_0^x\sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau}|V(0)|\right. \\ &\left. + \frac{1}{\sqrt{|p|}}|V(x)|\right] \le \frac{\mathrm{C}^2(x)\,\|V\|_{C[0,1]}}{|p|^{3/2}} \qquad \text{for } \mathrm{Re}\,p > \sigma, \end{aligned} \tag{2.4.30}$$

where C^2 bounded in every compact subinterval of $(0,1)$.

Finally, let us consider $D_{diag}[V](x,p)$. Due to the definitions (2.2.6), (2.2.7), (2.2.44) of $a, b, \bar{b}, z, w$ we can further simplify (2.4.27) as follows: $D_{diag}[V](x,p) = -\frac{l\,sh\,s}{s^2 d_0(p)}\left(\frac{V(x)}{\beta(x)}\right)'$. Thus, by (2.4.10) we have

$$\begin{aligned} &D_{diag}[V](x,p) \\ &= -\frac{l}{s^2}\left[l + O\left(\frac{1}{\sqrt{|p|}}\right)\right]\left(\frac{V(x)}{\beta(x)}\right)' = -\frac{1}{p}\left(\frac{V(x)}{\beta(x)}\right)' + O\left(\frac{1}{|p|^{3/2}}\right)\|V\|_{C^1[0,1]}, \end{aligned} \tag{2.4.31}$$

as $\mathrm{Re}\,p \to +\infty$ uniformly in $x \in [0,1]$ and $\mathrm{Im}\,p$.

Let us sum up. Using (2.4.24), (2.4.25) and (2.4.29) - (2.4.31) in (2.4.23) we obtain the estimate (2.4.7) with the inequality $\mathrm{Re}\,p > \sigma$ instead of $\mathrm{Re}\,p > 0$ under the sign of supremum. But we can replace the inequality $\mathrm{Re}\,p > \sigma$ by $\mathrm{Re}\,p > 0$ there, because owing to the boundedness of G_x in the strip S_σ, the estimate (2.4.7) hold trivially in case the range of the supremum is $0 < \mathrm{Re}\,p < \sigma$. The theorem is proved. □

2.5 Asymptotical properties of Green function in case of the third kind boundary conditions

First we prove estimates of G and its derivatives.

Theorem 2.5. *Let* λ, β *satisfy* (2.2.2). *Then the Green function* G *of the operator* L *in* $[0,1]$ *with the third kind boundary conditions satisfies the estimates* (2.4.1) - (2.4.4). *Moreover, for the boundary values of* G *the relations*

$$\sup_{\mathrm{Re}\,p>0}|p|^\kappa|G(x,1,p)|,\ \sup_{\mathrm{Re}\,p>0}|p|^\kappa|G(x,0,p)| \le C^1_\kappa(x) \tag{2.5.1}$$

hold for any $x \in (0,1)$ *and* $\kappa > 0$*, where* $C^1_\kappa(x)$ *is a function bounded in every compact subinterval of* $(0,1)$.

Proof. The estimates (2.4.1) - (2.4.4) were proved in [24]. They follow from Lemma 2.4, if we take into account the relations (2.2.57) - (2.2.60) for G, G_x, G_y, G_{xy}, the positivity and boundendness of a and b (cf. (2.2.6) and assumptions of Theorem 2.5) as well as the equality $s = l\sqrt{p}$.

To prove (2.5.1) we use the representation (2.2.57) for G. Since $w(1) = 1, w(0) = 0$, from this representation we get

$$\begin{aligned} G(x,1,p) &= \tfrac{1}{sd_1(p)a(x)a(1)}\left[ch\, sz + \hat{O}_1(x,1,p)\right], \\ G(x,0,p) &= \tfrac{1}{sd_1(p)a(x)a(0)}\left[ch\, s(z-1) + \hat{O}_2(x,0,p)\right]. \end{aligned} \tag{2.5.2}$$

Comparing (2.2.56) with (2.2.42) we see that $d_1(p)$ has the same asymptotical behavior as $-d_0(p)$. This implies that the inequalities (2.4.8) hold with $d_0(p)$ replaced by $d_1(p)$ and instead of (2.4.10) we have the formulae

$$\frac{sh\, s}{d_1(p)}, \frac{ch\, s}{d_1(p)} = -\,l + O\left(\frac{1}{\sqrt{|p|}}\right) \quad \text{as } \mathrm{Re}\, p \to +\infty \text{ uniformly in } \mathrm{Im}\, p. \tag{2.5.3}$$

Observing the relations $\hat{O}_1(x,1,p) = O\left(\frac{e^{sz}}{s}\right)$, $\hat{O}_2(x,0,p) = O\left(\frac{e^{s(1-z)}}{s}\right)$ (cf. (2.2.45)), the inequalities (2.4.8) with $d_1(p)$ instead of $d_0(p)$ and the relation $s = l\sqrt{p}$ from (2.5.2) we deduce

$$\begin{aligned} |G(x,1,p)| &\le \tfrac{\mathrm{Const}}{\sqrt{|p|}} e^{-\mathrm{Re}\sqrt{p}\int_x^1 \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau}, \\ |G(x,0,p)| &\le \tfrac{\mathrm{Const}}{\sqrt{|p|}} e^{-\mathrm{Re}\sqrt{p}\int_0^x \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} \end{aligned} \tag{2.5.4}$$

for any $x \in (0,1)$ and $\mathrm{Re}\, p > \sigma$ where $\sigma > 0$ is sufficiently large. These relations with (2.4.9) and boundedness of G in S_σ imply (2.5.1). □

Next, let us formulate two theorems proved in [24].

Theorem 2.6. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the third kind boundary conditions. If* $V \in C^1[0,1]$*, then the estimate*

$$\sup_{\mathrm{Re}\, p>0}\left|\sqrt{|p|}\left(\int_0^1 G_{xy}(x,y,p)V(y)dy\ -\ \frac{V(x)}{\lambda(x)}\right)\right| \le C_8(x)\|V\|_{C^1[0,1]} \tag{2.5.5}$$

holds for any $x \in (0,1)$*, where* $C_8(x)$ *is some function bounded in every compact subinterval of* $(0,1)$.

Theorem 2.7. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the third kind boundary conditions. If* $V \in C^1[0,1]$*, then the estimate*

$$\sup_{\mathrm{Re}\, p>0} \left| \sqrt{|p|} \left(\int_0^1 pG(x,y,p)V(y)dy \; + \; \frac{V(x)}{\beta(x)} \right) \right| \le C_9 \|V\|_{C^1[0,1]} \qquad (2.5.6)$$

holds for any $x \in [0,1]$*, where* C_9 *is a positive constant.*

Finally, we prove the following theorem.

Theorem 2.8. *Let* λ, β *satisfy* (2.2.2) *and* G *be the Green function of the operator* L *in* $[0,1]$ *with the third kind boundary conditions. If* $V \in C^2[0,1]$*, then the estimate*

$$\sup_{\mathrm{Re}\, p>0} \left| \sqrt{|p|} \left[\int_0^1 pG_x(x,y,p)V(y)dy + \left(\frac{V(x)}{\beta(x)} \right)' \right] \right| \le C_{10}(x) \|V\|_{C^2[0,1]} \quad (2.5.7)$$

is valid for any $x \in (0,1)$*, where* $C_{10}(x)$ *is some function bounded in every compact subinterval of* $(0,1)$*.*

Proof is similar to the proof of Theorem 2.4. Due to the representation (2.2.61) we can decompose G_x as (2.4.11), where

$$\mathcal{G}_1(x,y,p) = \frac{1}{ld_1(p)} \frac{b(x)}{a(y)} \begin{cases} sh\, sz \cdot ch\, s(w-1) & \text{for} \quad x < y \\ ch\, sw \cdot sh\, s(z-1) & \text{for} \quad x > y, \end{cases} \qquad (2.5.8)$$

$$\mathcal{G}_2(x,y,p) = \frac{1}{ld_1(p)} \frac{b(x)}{a(y)} \frac{1}{s} \times \begin{cases} \left(\frac{\zeta(x)}{2} - l\bar{b}(x)a'(x) + \vartheta_0 \right) ch\, sz \cdot ch\, s(w-1) & \text{for} \quad x < y \\ \left(\frac{\zeta(x) - \zeta(1)}{2} - l\bar{b}(x)a'(x) + \vartheta_1 \right) ch\, sw \cdot ch\, s(z-1) & \text{for} \quad x > y, \end{cases} \qquad (2.5.9)$$

$$\mathcal{G}_3(x,y,p) = \frac{1}{d_1(p)} \frac{1}{s} \times \begin{cases} \frac{b(x)}{l\,a(y)} \left(\frac{\zeta(y) - \zeta(1)}{2} + \vartheta_1 \right) sh\, sz \cdot sh\, s(w-1) + \mathcal{O}_1 & \text{for} \quad x < y \\ \frac{b(x)}{l\,a(y)} \left(\frac{\zeta(y)}{2} + \vartheta_0 \right) sh\, sw \cdot sh\, s(z-1) + \mathcal{O}_2 & \text{for} \quad x > y, \end{cases} \qquad (2.5.10)$$

the numbers ϑ_0, ϑ_1 are given by (2.2.62) and $\mathcal{O}_1 = \frac{b(x)}{l\,a(y)} \hat{O}_9$ and $\mathcal{O}_2 = \frac{b(x)}{l\,a(y)} \hat{O}_{10}$ behave like O_1 and O_2, respectively, as $\mathrm{Re}\, p \to +\infty$.

Applying twice Lemma 2.5 to the integral $\int_0^1 \mathcal{G}_1(x,y,p)V(y)dy$ and using the relation $\frac{1}{aw'} = l\bar{b}$ we obtain (2.4.15) with

$$\begin{aligned} \mathcal{D}_1[V](x,p) = \tfrac{b(x)}{d_1(p)} \Bigg\{ & \tfrac{1}{s} \Big[sh\, sz \cdot sh\, s(w-1) \bar{b}(y)V(y) \Big|_{y=x}^{y=1} \\ & + sh\, sw \cdot sh\, s(z-1) \bar{b}(y)V(y) \Big|_{y=0}^{y=x} \Big] \\ & - \tfrac{1}{s^2} \Big[sh\, sz \cdot ch\, s(w-1) \tfrac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \Big|_{y=x}^{y=1} \\ & + ch\, sw \cdot sh\, s(z-1) \tfrac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \Big|_{y=0}^{y=x} \Big] \Bigg\}, \end{aligned} \tag{2.5.11}$$

$$\begin{aligned} &\mathcal{T}_1[V](x,y,p) \\ &\quad = \frac{b(x)}{s^2 d_1(p)} \left(\frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)} \right)' \begin{cases} sh\, sz \cdot ch\, s(w-1) & \text{for} \quad x < y \\ ch\, sw \cdot sh\, s(z-1) & \text{for} \quad x > y. \end{cases} \end{aligned} \tag{2.5.12}$$

Using once Lemma 2.5 for integrals $\int_0^1 \mathcal{G}_k(x,y,p)V(y)dy$, $k = 2,3$ we get (2.4.18), where

$$\begin{aligned} &\mathcal{D}_2[V](x,p) = \tfrac{b(x)}{s^2 d_1(p)} \\ &\times \Bigg\{ \left(\tfrac{\zeta(x)}{2} - l\bar{b}(x)a'(x) + \vartheta_0 \right) ch\, sz \cdot sh\, s(w-1)\, \bar{b}(y)V(y) \Big|_{y=x}^{y=1} \\ &+ \left(\tfrac{\zeta(x)-\zeta(1)}{2} - l\bar{b}(x)a'(x) + \vartheta_1 \right) sh\, sw \cdot ch\, s(z-1)\, \bar{b}(y)V(y) \Big|_{y=0}^{y=x} \Bigg\}, \end{aligned} \tag{2.5.13}$$

$$\begin{aligned} &\mathcal{D}_3[V](x,p) = \tfrac{b(x)}{s^2 d_1(p)} \\ &\times \Bigg\{ \Big[\left(\tfrac{\zeta(y)-\zeta(1)}{2} + \vartheta_1 \right) \bar{b}(y)\, sh\, sz \cdot ch\, s(w-1) + \mathcal{O}_3(x,y,p) \Big] V(y) \Big|_{y=x}^{y=1} \\ &+ \Big[\left(\tfrac{\zeta(y)}{2} + \vartheta_0 \right) \bar{b}(y)\, ch\, sw \cdot sh\, s(z-1) + \mathcal{O}_4(x,y,p) \Big] V(y) \Big|_{y=0}^{y=x} \Bigg\}, \end{aligned} \tag{2.5.14}$$

$$\begin{aligned} &\mathcal{T}_2[V](x,y,p) = \frac{b(x)}{s^2 d_1(p)} \\ &\times \begin{cases} \left(\tfrac{\zeta(x)}{2} - l\bar{b}(x)a'(x) + \vartheta_0 \right) \left(\bar{b}(y)V(y) \right)' ch\, sz \cdot sh\, s(w-1), & x < y \\ \left(\tfrac{\zeta(x)-\zeta(1)}{2} - l\bar{b}(x)a'(x) + \vartheta_1 \right) \left(\bar{b}(y)V(y) \right)' sh\, sw \cdot ch\, s(z-1), & x > y, \end{cases} \end{aligned} \tag{2.5.15}$$

$$\begin{aligned} &\mathcal{T}_3[V](x,y.p) = \frac{b(x)}{s^2 d_1(p)} \\ &\times \begin{cases} \left(\left(\tfrac{\zeta(y)-\zeta(1)}{2} + \vartheta_1 \right) \bar{b}(y)V(y) \right)' sh\, sz \cdot ch\, s(w-1) + \tfrac{V'(y)}{b(x)} \mathcal{O}_3, & x < y \\ \left(\left(\tfrac{\zeta(y)}{2} + \vartheta_0 \right) \bar{b}(y)V(y) \right)' ch\, sw \cdot sh\, s(z-1) + \tfrac{V'(y)}{b(x)} \mathcal{O}_4, & x > y \end{cases} \end{aligned} \tag{2.5.16}$$

and $\mathcal{O}_3$, $\mathcal{O}_4$ given by (2.3.8) behave like O_1, O_2, respectively as $\operatorname{Re} p \to +\infty$.

By (2.4.11), (2.4.15), (2.4.18) we have the relation (2.4.23) with $\mathcal{D}[V] = \mathcal{D}_1[V] + \mathcal{D}_2[V] + \mathcal{D}_3[V]$ and $\mathcal{T}[V] = \mathcal{T}_1[V] - \mathcal{T}_2[V] - \mathcal{T}_3[V]$. Applying Lemma 2.4 for the integrals of functions $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ given by (2.4.17), (2.4.21), (2.4.22) and using the relation $s = l\sqrt{p}$ we get for $\mathcal{T}[V]$ the estimate (2.4.24).

Further, computing $\mathcal{D} = \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3$ from (2.5.11), (2.5.13), (2.5.14), simplifying by means of the relations $w(0) = 0, w(1) = 1, w(x) = z$ and $\zeta(0) = 0$ (see (2.2.44), (2.2.8)) and reordering the terms we obtain

$$\mathcal{D}[V] \,=\, D_{diag}[V] + \widetilde{\mathcal{D}}[V], \tag{2.5.17}$$

where

$$D_{diag}[V](x,p) = -\,\frac{b(x)}{s^2 d_1(p)}\, sh\, s\, \left[l(\bar{b}(x))^2 a'(x) V(x) - \frac{(\bar{b}(x)V(x))'}{w'(x)} \right], \tag{2.5.18}$$

$$\begin{aligned} \widetilde{\mathcal{D}}[V](x,p) &= -\,\tfrac{b(x)}{s^2 d_1(p)} \\ &\times \left\{ -\frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)}\Bigg|_{y=1} sh\; sz \;+\; \frac{\left(\bar{b}(y)V(y)\right)'}{w'(y)}\Bigg|_{y=0} sh\; s(z-1) \right. \\ &+ \left[\vartheta_1 \bar{b}(1)\, sh\; sz + \mathcal{O}_3(x,1,p)\right] V(1) \\ &- \left[\vartheta_0 \bar{b}(0) sh\; s(z-1) - \mathcal{O}_4(x,0,p)\right] V(0) \\ &\left. + (\mathcal{O}_4(x,x,p) - \mathcal{O}_3(x,x,p)) V(x) \right\}. \end{aligned} \tag{2.5.19}$$

Let $x \in (0,1)$. Due to the behavior of $\mathcal{O}_3$ and $\mathcal{O}_4$ (cf. (2.2.45)), we have $\mathcal{O}_3(x,1,p) = O\left(\frac{e^{sz}}{s}\right)$, $\mathcal{O}_4(x,0,p) = O\left(\frac{e^{s(1-z)}}{s}\right)$, $\mathcal{O}_3(x,x,p), \mathcal{O}_4(x,x,p) = O\left(\frac{e^{s}}{s}\right)$. Using these relations and (2.4.8) (with $d_1(p)$ instead of $d_0(p)$) in (2.5.19) and observing the equality $s = l\sqrt{p}$ and (2.4.9) with $\kappa = 1/2$ we deduce the estimate

$$\begin{aligned} |\widetilde{\mathcal{D}}[V](x,p) \leq \tfrac{\text{Const}}{|p|} &\left[e^{-\operatorname{Re}\sqrt{p}\int_x^1 \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} (|V(1)| + |V'(1)|) \right. \\ &+ e^{-\operatorname{Re}\sqrt{p}\int_0^x \sqrt{\frac{\beta(\tau)}{\lambda(\tau)}}d\tau} (|V(0)| + |V'(0)|) \\ &\left. + \tfrac{1}{\sqrt{|p|}} |V(x)| \right] \leq \frac{\mathrm{C}^3(x)\, \|V\|_{C^1[0,1]}}{|p|^{3/2}} \qquad \text{for } \operatorname{Re} p > \sigma, \end{aligned} \tag{2.5.20}$$

where C^3 bounded in every compact subinterval of $(0,1)$.

Finally, we consider $D_{diag}[V](x,p)$. As in the proof of Theorem 2.4 we simplify the expression of $D_{diag}[V](x,p)$. From (2.5.18) we get $D_{diag}[V](x,p) = \frac{l\, sh\, s}{s^2 d_1(p)} \left(\frac{V(x)}{\beta(x)}\right)'$. Using here (2.5.3) we obtain (2.4.31).

Summing up, the relations (2.4.24) and (2.5.17), (2.5.20), (2.4.31) with (2.4.23) and the boundedness of G_x in S_σ yield (2.5.7). The proof is complete. □

3. Inverse problem with temperature observations

In this chapter we study the generalized inverse problem with temperature observations.

3.1 Reduction of the inverse problem to a fixed-point form

Here we deduce a fixed-point system for the inverse problem in the Laplace domain and transform further the system for U and U_x. Recall that this problem was formulated as the system (1.2.25) with the additional equations (1.2.15), (1.2.18).

Using the relation (1.2.15) for U in the left-hand side of the conditions (1.2.25) we obtain

$$\begin{aligned}
&\sum_{k=1}^{K_1} pN_k(p) \int_0^1 pG(x_i,y,p)\nu_k(y)pU(y,p)\,dy \\
&-\sum_{k=1}^{K_2} \sqrt{p}M_k(p)\,\sqrt{p}\int_0^1 G_y(x_i,y,p)\mu_k(y)pU_y(y,p)\,dy \qquad (3.1.1)\\
&\qquad = p^2[H_i(p)+Q(x_i,p)]\,, \quad i=1,\ldots,K.
\end{aligned}$$

Conversely, using (1.2.15) in (3.1.1) we deduce (1.2.25). Therefore, the system (1.2.25) with (1.2.15), (1.2.18), is equivalent to the system (3.1.1) with (1.2.15), (1.2.18).

The latter system (3.1.1) is more convenient for our analysis because we can rewrite it in a fixed-point form. The main idea is to separate the principal part of the system in the process $\operatorname{Re} p \to +\infty$. In case such a the principal part contains unknowns N and M, we leave it to the left hand side of the system bringing the remainder to the right-hand side. The remainder is already a term of a lower order, i.e. a contractive mapping of the unknowns in some half-plane $\operatorname{Re} p > \sigma$ with sufficiently large σ. We note that transforming the system (3.1.1) into the fixed-point form is rather a formal procedure and doesn't require regularity assumptions on the data and solutions (except for (3.1.4), $\det \Gamma \neq 0$ and the decomposition (3.1.16) of the function Φ^0 below). However, ideas of extracting this principal part are based on certain proper asymptotics of the system in the process $\operatorname{Re} p \to +\infty$. Such an asymptotics is achieved under the conditions that the system (3.1.1) with (1.2.15), (1.2.18) has a solution $N_k|_{k=1,\ldots,K_1}$, $M_k|_{k=1,\ldots,K_2}$, U that satisfies the following properties:

(1) $n_k(t) = \mathcal{L}^{-1}_{p\to t} N_k(p)$ and n'_k belong to $\mathscr{E}$ implying $pN_k(p) \to n_k(0)$, as $\mathrm{Re}p \to +\infty$ (cf. property 1 of the Laplace transform in Section 1.2.1);

(2) $\sqrt{p}M_k(p) \to 0$, as $\mathrm{Re}p \to +\infty$ (strengthened form of (1.2.3));

(3) for $u(x,t) = \mathcal{L}^{-1}_{p\to t} U(x,p)$ the relations $u(x,\cdot), u_t(x,\cdot), u_x(x,\cdot), u_{xt}(x,\cdot) \in \mathscr{E}$ hold with the initial condition (1.1.5) implying

$$pU(x,p) \to \varphi(x)\,, \quad pU_x(x,p) \to \varphi'(x)\,, \quad \text{as} \quad \mathrm{Re}\, p \to \infty\,. \tag{3.1.2}$$

We emphasize that (1) - (3) are not assumptions. We only use them to explain the ideas of the transformation of the system (3.1.1).

In case (1) - (3) hold by the assertions of Theorems 2.1, 2.3, 2.5, 2.7 we see that the first sum on the left-hand side of (3.1.1) dominates with respect to the second sum in the process $\mathrm{Re}\, p \to +\infty$. As a result in the process $\mathrm{Re}\, p \to +\infty$ from (3.1.1) we obtain the following system for $n^0_k = n_k(0)$:

$$-\sum_{k=1}^{K_1} n^0_k \, \frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)} = \lim_{\mathrm{Re}p\to\infty} p^2 \left[H_i(p) + Q(x_i,p)\right], \; i = 1, \ldots, K. \tag{3.1.3}$$

Since this system doesn't contain the full unknowns $N(p)$ and $M(p)$, we have to compute the second approximation of the system (3.1.1) in the process $\mathrm{Re}\, p \to +\infty$, too. To this end we have to assume that (3.1.3) is solvable. According to the Kronecker theorem the system (3.1.1) has a solution provided

$$\begin{aligned}\mathscr{R} :=\; & \mathrm{rank}\left(\left(\frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}\right)_{k=1,\ldots,K_1}, \lim_{\mathrm{Re}p\to\infty} p^2 \left[H_i(p) + Q(x_i,p)\right] \right)_{i=1,\ldots,K} \\ =\; & \mathrm{rank}\left(\frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}\right)_{\substack{k=1,\ldots,K_1 \\ i=1,\ldots,K}}\,. \end{aligned} \tag{3.1.4}$$

Here the matrix in the left-hand side is formed by placing two matrices left to right.

In the sequel we assume that the solvability condition (3.1.4) for the system (3.1.3) is satisfied. In case $\mathscr{R} = K_1$ the solution of (3.1.3) is even unique.

Let us continue the transformation of (3.1.1). As we remarked, under the conditions (1) - (3) the first sum on the left-hand side of (3.1.1) dominates with respect to the second sum in the process $\mathrm{Re}\, p \to +\infty$. This suggests that the kernels n_k and m_k can be determined simultaneously with higher smoothness in n_k than in m_k. Therefore, we introduce the new unknowns

$$Z = (Z_1, \ldots, Z_K)\,, \; Z_k(p) = \begin{cases} pN_k(p) - n^0_k, & k = 1, \ldots, K_1, \\ M_{k-K_1}(p), & k = K_1 + 1, \ldots, K, \end{cases} \tag{3.1.5}$$

where n^0_k, $k = 1, \ldots, K_1$ solves (3.1.3). To find the second-order approximation of (3.1.1) we multiply it by p. Then, in view of (3.1.3) and (3.1.5), (1) - (3) and

Theorems 2.1, 2.3, 2.5, 2.7 we see that the system (3.1.1) is in the process $\operatorname{Re} p \to +\infty$ is asymptotically equivalent to the system

$$\sum_{k=1}^{K_1} Z_k(p)\frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i) + \sum_{k=K_1+1}^{K} Z_k(p)\frac{1}{\beta(x_i)}\left(\mu_{k-K_1}(x)\varphi'(x)\right)'\Big|_{x=x_i} = 0.$$

The left-hand side of this system represents the principal part of (3.1.1). Thus, separating the principal part we after some elementary transformations rewrite (3.1.1) in the following equivalent form:

$$\begin{aligned}
&\sum_{k=1}^{K_1} Z_k(p)\frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i) + \sum_{k=K_1+1}^{K} Z_k(p)\frac{1}{\beta(x_i)}\left(\mu_{k-K_1}(x)\varphi'(x)\right)'\Big|_{x=x_i} \\
&\quad= \sum_{k=1}^{K_1} Z_k(p)\left[\int_0^1 pG(x_i,y,p)\nu_k(y)\left[pU(y,p)-\varphi(y)\right]dy\right. \\
&\qquad\qquad \left. + \int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\right] \\
&\quad - \sum_{k=K_1+1}^{K} Z_k(p)\left[\int_0^1 pG_y(x_i,y,p)\mu_{k-K_1}(y)\left[pU_y(y,p)-\varphi'(y)\right]\,dy\right. \\
&\qquad\qquad - \int_0^1 pG(x_i,y,p)\left(\mu_{k-K_1}(y)\varphi'(y)\right)'dy - \frac{1}{\beta(x_i)}\left(\mu_{k-K_1}(x)\varphi'(x)\right)'\Big|_{x=x_i} \\
&\qquad\qquad \left. +\mu_{k-K_1}(0)\varphi'(0)pG(x_i,0,p) - \mu_{k-K_1}(1)\varphi'(1)pG(x_i,1,p)\right] \\
&\quad + \sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i,y,p)\nu_k(y)\left[pU(y,p)-\varphi(y)\right]\,dy \\
&\quad - p^2\left[H_i(p)+Q(x_i,p)\right] + \lim_{\operatorname{Re} q\to\infty} q^2\left[H_i(q)+Q(x_i,q)\right],\ i=1,\dots,K.
\end{aligned} \tag{3.1.6}$$

The main reason of transformations in the right-hand side is to bring $pU(x,p)$ and $pU_x(x,p)$ to the form $pU(x,p)-\varphi(x)$ and $pU_x(x,p)-\varphi'(x)$ that converge to zero in the process $\operatorname{Re} p \to +\infty$ (see (3.1.2)).

Further, let us introduce the matrix

$$\Gamma = (\gamma_{ik})_{i,k=1,\dots,K}\,,\ \gamma_{ik} = \begin{cases} \frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}, & k=1,\dots,K_1, \\ \frac{\left(\mu_{k-K_1}(y)\varphi'(y)\right)'\big|_{y=x_i}}{\beta(x_i)}, & k=K_1+1,\dots,K \end{cases} \tag{3.1.7}$$

related to the principal part and assume $\det\Gamma \neq 0$. Moreover, we introduce the functions

$$B^0[Z](x,p) = pU[Z](x,p) - \varphi(x),\ \ B^1[Z](x,p) = pU_x[Z](x,p) - \varphi'(x), \quad (3.1.8)$$

where $U[Z](x,p)$ is the solution of (1.2.15), (1.2.18) with the given vector Z in the form (3.1.5).

Now we are ready to rewrite system (3.1.6) in the equivalent fixed-point form

$$Z \,=\, \Gamma^{-1}\mathcal{F}(Z)\,, \quad (3.1.9)$$

where $\mathcal{F}(Z) = (\mathcal{F}_1(Z), \ldots, \mathcal{F}_K(Z))$,

$$\begin{aligned}
\mathcal{F}_i[Z](p) \,=\, & \sum_{k=1}^{K_1} Z_k(p) \left[\int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy \right. \\
& \left. + \int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i) \right] \\
& + \sum_{k=K_1+1}^{K} Z_k(p) \left[-\int_0^1 pG_y(x_i,y,p)\mu_{k-K_1}(y)B^1[Z](y,p)\,dy \right. \\
& + \int_0^1 pG(x_i,y,p)\left(\mu_{k-K_1}(y)\varphi'(y)\right)'\,dy \,+ \frac{1}{\beta(x_i)}\left(\mu_{k-K_1}(x)\varphi'(x)\right)'\Big|_{x=x_i} \\
& \left. + \mu_{k-K_1}(0)\varphi'(0)pG(x_i,0,p) - \mu_{k-K_1}(1)\varphi'(1)pG(x_i,1,p) \right] \\
& + \sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy + \widehat{\Psi}_i(p),\ \ i = 1,\ldots,K \quad (3.1.10)
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\Psi}_i(p) = & \sum_{k=1}^{K_1} n_k^0 \left[\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i) \right] \\
& - p^2\left[H_i(p) + Q(x_i,p)\right] + \lim_{\mathrm{Re} q\to\infty} q^2\left[H_i(q) + Q(x_i,q)\right]. \quad (3.1.11)
\end{aligned}$$

Since the system (3.1.10) contains the quantities $B^0[Z]$ and $B^1[Z]$ we have to deduce additional equations for these quantities, too. To this end we simply rewrite the system (1.2.15), (1.2.18) for U in terms of these quantities. In view of the definitions of Z, $B^0[Z]$ and $B^1[Z]$ from (1.2.15) we have

$$\begin{aligned} B^0[Z](x,p) = &\sum_{k=1}^{K_1} Z_k(p) \int_0^1 G(x,y,p)\nu_k(y) \left[B^0[Z](y,p) + \varphi(y)\right] dy \\ &- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y) \left[B^1[Z](y,p) + \varphi'(y)\right] dy \\ &+ \sum_{k=1}^{K_1} n_k^0 \int_0^1 G(x,y,p)\nu_k(y) B^0[Z](y,p)\, dy + \Phi^0(x,p) \end{aligned} \tag{3.1.12}$$

with

$$\Phi^0(x,p) = \sum_{k=1}^{K_1} n_k^0 \int_0^1 G(x,y,p)\nu_k(y)\varphi(y)\, dy - pQ(x,p) - \varphi(x) \tag{3.1.13}$$

and from (1.2.18) we get

$$\begin{aligned} B^1[Z](x,p) \;=\; &\sum_{k=1}^{K_1} Z_k(p) \int_0^1 G_x(x,y,p)\nu_k(y) \left[B^0[Z](y,p) + \varphi(y)\right] dy \\ &+ \sum_{k=K_1+1}^{K} Z_k(p) \left[\frac{\mu_{k-K_1}(x)}{\lambda(x)} B^1[Z](x,p) \right. \\ &\qquad \left. - \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y) \left[B^1[Z](y,p) + \varphi'(y)\right] dy \right] \\ &+ \sum_{k=1}^{K_1} n_k^0 \int_0^1 G_x(x,y,p)\nu_k(y) B^0[Z](y,p)\, dy + \sum_{k=K_1+1}^{K} Z_k(p) \frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)} \\ &+ \Phi^1(x,p) \end{aligned} \tag{3.1.14}$$

with

$$\Phi^1(x,p) = \sum_{k=1}^{K_1} n_k^0 \int_0^1 G_x(x,y,p)\nu_k(y)\varphi(y)\, dy - pQ_x(x,p) - \varphi'(x). \tag{3.1.15}$$

For the function $B^0[Z]$, which in contrast to $B^1[Z]$ doesn't contain a space derivative of U, we need a certain higher regularity in the time variable. To this end we assume that the data-dependent term Φ^0 given by (3.1.13) can be decomposed as follows:

$$\Phi^0(x,p) = \tfrac{c_\Phi(x)}{p} + \widetilde{\Phi}^0(x,p) \tag{3.1.16}$$

where c_Φ is some function in $C[0,1]$. The meaning of this decomposition is that the term $\frac{c_\Phi(x)}{p}$ forms the higher-order part of $\Phi^0(x,p)$ in the process $\mathrm{Re}\, p \to +\infty$. Indeed, in the forthcoming sections we will assume that $\widetilde{\Phi}^0 \in \hat{\mathcal{B}}_{\alpha,\sigma_0}$ with some $\alpha > 1$, $\sigma_0 \geq 0$ implying the estimate $|\widetilde{\Phi}^0(x,p)| \leq \frac{\mathrm{Const}}{|p|^\alpha}$, $x \in [0,1]$ $\mathrm{Re}\, p > \sigma_0$. But this moment such properties of $|\widetilde{\Phi}^0(x,p)|$ are not necessary to assume.

Let us split $B^0[Z]$ into the sum

$$B^0[Z](x,p) = \frac{c_\Phi(x)}{p} + \widehat{B}^0[Z](x,p). \tag{3.1.17}$$

From (3.1.12) and (3.1.14) in view of (3.1.13), (3.1.15) and the definitions of Z and $B^1[Z]$ we deduce the following fixed-point equation for the vector $B[Z] = (\widehat{B}^0[Z], B^1[Z])$:

$$B[Z] = A[Z]B[Z] + b[Z]\,, \tag{3.1.18}$$

where $A[Z] = (A^0[Z], A^1[Z])$ is the Z-dependent linear operator of B and $b[Z] = (b^0[Z], b^1[Z])$ is the Z-dependent B-free term with with the following components:

$$\begin{aligned}\left(A^0[Z]B\right)(x,p) &= \sum_{k=1}^{K_1} \left(Z_k(p) + n_k^0\right) \int_0^1 G(x,y,p)\nu_k(y)\widehat{B}^0(y,p)\,dy \\ &- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y)B^1(y,p)\,dy,\end{aligned} \tag{3.1.19}$$

$$\begin{aligned}\left(A^1[Z]B\right)(x,p) &= \sum_{k=1}^{K_1} \left(Z_k(p) + n_k^0\right) \int_0^1 G_x(x,y,p)\nu_k(y)\widehat{B}^0(y,p)\,dy \\ &+ \sum_{k=K_1+1}^{K} Z_k(p) \left[\frac{\mu_{k-K_1}(x)}{\lambda(x)} B^1(x,p)\right. \\ &\left. - \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y)B^1(y,p)\,dy\right],\end{aligned} \tag{3.1.20}$$

$$\begin{aligned}b^0[Z](x,p) &= \sum_{k=1}^{K_1} Z_k(p) \int_0^1 G(x,y,p)\nu_k(y) \left[\frac{c_\Phi(y)}{p} + \varphi(y)\right] dy \\ &- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy + \widehat{\Phi}^0(x,p),\end{aligned} \tag{3.1.21}$$

$$
\begin{aligned}
b^1[Z](x,p) \quad &= \quad \sum_{k=1}^{K_1} Z_k(p) \int_0^1 G_x(x,y,p)\nu_k(y)\left[\frac{c_\Phi(y)}{p}+\varphi(y)\right]dy \\
&+ \quad \sum_{k=K_1+1}^{K} Z_k(p)\Big[\frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)} \\
&- \quad \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy\Big] + \widehat{\Phi}^1(x,p)
\end{aligned}
\tag{3.1.22}
$$

and

$$
\widehat{\Phi}^0(x,p) \quad = \quad \frac{1}{p}\sum_{k=1}^{K_1} n_k^0 \int_0^1 G(x,y,p)\nu_k(y)c_\Phi(y)dy + \widetilde{\Phi}^0(x,p), \tag{3.1.23}
$$

$$
\widehat{\Phi}^1(x,p) = \frac{1}{p}\sum_{k=1}^{K_1} n_k^0 \int_0^1 G_x(x,y,p)\nu_k(y)c_\Phi(y)dy + \Phi^1(x,p).
$$

Summing up, we have proved

Proposition 3.1. *Let the condition* (3.1.4) *hold implying the existence of the solution* n_k^0, $k = 1,\ldots,K_1$ *to the system* (3.1.3)*. Moreover, let* $\det\Gamma \neq 0$ *and* Φ^0 *given by* (3.1.13) *have the representation* (3.1.16)*. Then the inverse problem in the Laplace domain with temperature observations is equivalent to equation* (3.1.9) *with* $\mathcal{F}$ *given by* (3.1.10)*. The solutions of these two equivalent problems are related by* (3.1.5) *with* (3.1.8)*. Moreover, the term* $B^0[Z]$ *in* (3.1.8) *has the form* (3.1.17) *and the vector* $B[Z] = (\widehat{B}^0[Z], B^1[Z])$ *consisting of the function* $\widehat{B}^0[Z]$ *from* (3.1.17) *and the term* $B^1[Z]$ *from* (3.1.8) *satisfies the system* (3.1.18) *with* (3.1.19) - (3.1.24)*.*

Remark. In the case of boundary conditions of the first kind the addends containing the terms $G(x_i,1,p)$ and $G(x_i,0,p)$ in (3.1.10) vanish because in this case $G(x_i,1,p) = G(x_i,0,p) = 0$ (see (1.2.12)).

3.2 Analysis of direct problem system

In this section we study the equation (3.1.18) whose solution $B[Z]$ is connected with the solution U of the direct problem in Laplace domain.

Let us introduce the following basic assumptions on the data:

$$\left.\begin{array}{l}\Phi^0 \text{ given by (3.1.13) admits the decomposition (3.1.16) where}\\ c_\Phi \in C[0,1] \text{ and } \widetilde{\Phi}^0 \in \hat{\mathcal{B}}_{\alpha,\sigma_0} \text{with some}\\ \sigma_0 > 0 \text{ and } \alpha \text{ satisfying (2.1.3);}\\ \Phi^1 \text{ given by (3.1.15) belongs to } \hat{\mathcal{B}}_{\alpha-\frac{1}{2},\sigma_0};\\ \nu_k \in C[0,1],\ k=1,\dots,K_1,\ \ \mu_k \in C[0,1],\ k=1,\dots,K_2;\\ \varphi \in C^1[0,1].\end{array}\right\} \quad (3.2.1)$$

We note that without restriction of generality we can replace the condition $\sigma_0 > 0$ in (3.2.1) by $\sigma_0 \in \mathbb{R}$. Indeed, since $\hat{\mathcal{B}}_{\gamma,\sigma_1} \subset \hat{\mathcal{B}}_{\gamma,\sigma_2}$ for $\sigma_1 < \sigma_2$, the assumption (3.2.1) holds for any $\sigma_0 > 0$ provided it holds for some $\sigma_0 \leq 0$. However, the condition $\sigma_0 > 0$ simplifies the treatment of terms of the form $\frac{1}{\sigma^\gamma}$ for $\sigma \geq \sigma_0$ in forthcoming estimations (see e.g. (3.2.7)).

We start by proving some properties of the free term b of the equation (3.1.18).

Lemma 3.1. *Let the assumptions* (2.2.2), (3.2.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$, *then the vector function* $b[Z] = (b^0[Z], b^1[Z])$, *given by* (3.1.21), (3.1.22), *belongs to* $\mathcal{B}_{\alpha,\sigma_0}$ *and satisfies the estimate*

$$\|b[Z]\|_{\alpha,\sigma} \quad \leq \quad \text{Const}\left[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha}}\left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\right] \quad (3.2.2)$$

with any $\sigma \geq \sigma_0$ *and some constant, where*

$$|c| = \sum_{k=1}^{K} |c_k|. \quad (3.2.3)$$

Moreover, for every $\sigma \geq \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$ *the difference* $b[Z^1] - b[Z^2]$ *fulfils the estimate*

$$\left\|b[Z^1] - b[Z^2]\right\|_{\alpha,\sigma} \leq \text{Const}\,\frac{1}{\sqrt{\sigma}}\left\|V^1 - V^2\right\|_{\alpha,\sigma} \quad (3.2.4)$$

with some constant.

Proof. Let us start with the estimation of $b^0[Z]$. Substituting $\frac{c}{p} + V$ for Z in

(3.1.21), multiplying by $|p|^\alpha$ and estimating we have

$$|p|^\alpha \left|b^0[Z](x,p)\right| \leq \sum_{k=1}^{K_1}\left(|c_k| + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha-1}}\right)|p|\int\limits_0^1 \left|G(x,y,p)\right|dy$$
$$\times \|\nu_k\|_{C[0,1]}\left[\frac{1}{|p|^{3-\alpha}}\max_{0\leq y\leq 1}|c_\Phi(y)| + \frac{\|\varphi\|_{C[0,1]}}{|p|^{2-\alpha}}\right] \qquad (3.2.5)$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k| + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha-1}}\right)\sqrt{|p|}\int\limits_0^1 |G_y(x,y,p)|\,dy\|\mu_{k-K_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{|p|^{\frac{3}{2}-\alpha}}$$
$$+|p|^\alpha\left|\widehat{\Phi}^0(x,p)\right|.$$

From (3.1.23) in view of the assumption (3.2.1), Lemmas 2.2 and 2.3 and the assertion (2.4.1) of Theorems 2.1 and 2.5 we get $\widehat{\Phi}^0 \in \mathcal{B}_{\alpha,\sigma_0}$ for the function $\widehat{\Phi}^0$. Using this relation, the assertions (2.4.1), (2.4.3) of Theorems 2.1, 2.5, the assumption (3.2.1) and the definitions of the norms $\|\cdot\|_{\gamma,\sigma}$ in $\mathcal{A}_{\alpha,\sigma}$ and $\hat{\mathcal{B}}_{\alpha,\sigma}$ we obtain from (3.2.5)

$$|p|^\alpha\left|b^0[Z](x,p)\right| \leq \sum_{k=1}^{K_1}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{|p|^{\alpha-1}}\right)C_1\|\nu_k\|_{C[0,1]}\left[\frac{\|c_\Phi\|_{C[0,1]}}{|p|^{3-\alpha}}+\frac{\|\varphi\|_{C[0,1]}}{|p|^{2-\alpha}}\right]$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{|p|^{\alpha-1}}\right)C_3\|\mu_{k-K_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{|p|^{\frac{3}{2}-\alpha}}+\|\widehat{\Phi}^0\|_{\alpha,\sigma_0}$$

for $\mathrm{Re}p > \sigma$, $\sigma \geq \sigma_0$, $x \in [0,1]$. Taking here the supremum over $\mathrm{Re}p > \sigma$, $x \in [0,1]$ and observing the relation $|p|^\gamma > \sigma^\gamma$ for $\mathrm{Re}p > \sigma$, which holds in the cases $\gamma = \alpha-1,\ 3-\alpha,\ 2-\alpha,\ 3/2-\alpha$ due to (2.1.3), we have

$$\left\|b^0[Z]\right\|_{\alpha,\sigma} \leq \sum_{k=1}^{K_1}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)C_1\|\nu_k\|_{C[0,1]}\left[\frac{\|c_\Phi\|_{C[0,1]}}{\sigma^{3-\alpha}}+\frac{\|\varphi\|_{C[0,1]}}{\sigma^{2-\alpha}}\right]$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)C_3\|\mu_{k-K_1}\|_{C[0,1]}\frac{\|\varphi'\|_{C[0,1]}}{\sigma^{\frac{3}{2}-\alpha}}+\|\widehat{\Phi}^0\|_{\alpha,\sigma_0} \qquad (3.2.6)$$

for $\sigma \geq \sigma_0$. Finally, observing that

$$\sigma^{\gamma'} \geq \kappa\sigma^\gamma \quad \text{for} \quad \sigma \geq \sigma_0 \quad \text{in case} \quad \gamma' \geq \gamma \quad \text{with} \quad \kappa = \sigma_0^{\gamma'-\gamma} > 0 \qquad (3.2.7)$$

because $\sigma_0 > 0$, we can take the minimal exponent of σ in the nominators of the estimate (3.2.6). Thus, in view of the definition of the norm in $(\mathcal{A}_{\alpha,\sigma})^K$ and (3.2.3), we arrive at the relation

$$\left\|b^0[Z]\right\|_{\alpha,\sigma} \leq \frac{\mathrm{Const}}{\sigma^{\frac{3}{2}-\alpha}}\left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) + \|\widehat{\Phi}^0\|_{\alpha,\sigma_0}\,, \quad \sigma \geq \sigma_0 \qquad (3.2.8)$$

with some constant depending on the data of the problem.

Next, we perform similar transformations with $b^1[Z]$ in (3.1.22) multiplying by $|p|^{\alpha-\frac{1}{2}}$ instead of $|p|^{\alpha}$. We have

$$|p|^{\alpha-\frac{1}{2}}\left|b^1[Z](x,p)\right| \le \sum_{k=1}^{K_1}\left(|c_k|+\frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\right)\sqrt{|p|}\int_0^1|G_x(x,y,p)|\,dy$$
$$\times\|\nu_k\|_{C[0,1]}\left[\frac{1}{|p|^{3-\alpha}}\max_{0\le y\le 1}|c_\Phi(y)|+\frac{\|\varphi\|_{C[0,1]}}{|p|^{2-\alpha}}\right]$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\right)\frac{1}{|p|^{\frac{3}{2}-\alpha}}$$
$$\times\left|\frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)}-\int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy\right|+|p|^{\alpha-\frac{1}{2}}\left|\widehat{\Phi}^1(x,p)\right|.$$

From (3.1.24) in view of the assumption (3.2.1), Lemmas 2.2 and 2.3 and the assertion (2.4.2) of Theorems 2.1, 2.5 we get $\widehat{\Phi}^1\in\mathcal{B}_{\alpha-\frac{1}{2},\sigma_0}$ for the function $\widehat{\Phi}^1$. Thus, using the assertions (2.4.2), (2.4.4) of Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p>\sigma$, $x\in[0,1]$ we obtain

$$\left\|b^1[Z]\right\|_{\alpha-\frac{1}{2},\sigma} \le \sum_{k=1}^{K_1}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)C_2\|\nu_k\|_{C[0,1]}\left[\frac{\|c_\Phi\|_{C[0,1]}}{\sigma^{3-\alpha}}+\frac{\|\varphi\|_{C[0,1]}}{\sigma^{2-\alpha}}\right]$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\left(\frac{1}{\min\limits_{x\in[0,1]}\lambda(x)}+C_4\right)\frac{\|\mu_{k-K_1}\varphi'\|_{C[0,1]}}{\sigma^{\frac{3}{2}-\alpha}}+\|\widehat{\Phi}^1\|_{\alpha-\frac{1}{2},\sigma_0}$$

for $\sigma\ge\sigma_0$. Since $\min\limits_{x\in[0,1]}\lambda(x)>0$ by assumption, this estimate due to (3.2.7) yields

$$\|b^1[Z]\|_{\alpha-\frac{1}{2},\sigma}\le\frac{\mathrm{Const}}{\sigma^{\frac{3}{2}-\alpha}}\left(|c|+\frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)+\|\widehat{\Phi}^1\|_{\alpha-\frac{1}{2},\sigma_0},\quad \sigma\ge\sigma_0 \tag{3.2.9}$$

with a constant depending on the data.

In particular, observing the estimates (3.2.8) and (3.2.9) and applying Lemmas 2.2 and 2.3 for the components in (3.1.21), (3.1.22) we see that $b[Z]=(b^0[Z],b^1[Z])\in\mathcal{B}_{\alpha,\sigma}$ for $\sigma\ge\sigma_0$. Moreover, we get (3.2.2). To prove (3.2.4) we denote $Z=Z^1-Z^2$. Observe that in this case $b[Z]=b[Z^1]-b[Z^2]$, where the components $b^0[Z]$ and $b^1[Z]$ of the vector $b[Z]$ are expressed by the formulas (3.1.21) with $\widehat{\Phi}^0=0$ and (3.1.22) with $\widehat{\Phi}^1=0$, respectively. Using the proved estimates (3.2.8) and (3.2.9) for the components of $b[Z]$ and observing that Z has the form $Z=\frac{c}{p}+V$, where $c=0$ and $V=V^1-V^2$, we deduce (3.2.4). The proof is complete. □

We continue proving properties of the operator $A[Z]$ of the equation (3.1.18).

Lemma 3.2. *Let the assumptions* (2.2.2), (3.2.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$, *then the linear operator* $A[Z] = (A^0[Z], A^1[Z])$, *defined by* (3.1.19), (3.1.20), *is well-defined and bounded in* $\mathcal{B}_{\alpha,\sigma}$ *and satisfies the estimate*

$$\|A[Z]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})} \le C_0 \left[\frac{1+|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} \right] \tag{3.2.10}$$

for any $\sigma \ge \sigma_0$ *with some constant* C_0. *Moreover, taking* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$, *the estimate for difference*

$$\left\|A[Z^1] - A[Z^2]\right\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})} \le \widehat{C}_0 \frac{1}{\sigma^\alpha} \left\|V^1 - V^2\right\|_{\alpha,\sigma} \tag{3.2.11}$$

holds for any $\sigma \ge \sigma_0$ *with some constant* $\widehat{C}_0$.

Proof. First we show that the linear operator $A[Z]$, is well-defined and bounded in $\mathcal{B}_{\alpha,\sigma}$. From (3.1.19) by $Z = \frac{c}{p} + V$ we get

$$\begin{aligned} |p|^\alpha \left|\left(A^0[Z]B\right)(x,p)\right| \;\; \le \;\; & \sum_{k=1}^{K_1} \left(\frac{|c_k|}{|p|^2} + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha+1}} + \frac{|n_k^0|}{|p|} \right) \\ & \times |p| \int_0^1 |G(x,y,p)|\, dy \cdot \|\nu_k\|_{C[0,1]} \; |p|^\alpha \max_{0\le y\le 1} \left|\widehat{B}^0(y,p)\right| \\ & + \sum_{k=K_1+1}^{K} \left(\frac{|c_k|}{|p|} + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha+1}} \right) \sqrt{|p|} \int_0^1 |G_y(x,y,p)|\, dy \\ & \times \|\mu_{k-K_1}\|_{C[0,1]} \; |p|^{\alpha-\frac{1}{2}} \max_{0\le y\le 1} \left|B^1(y,p)\right| . \end{aligned}$$

Using Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p > \sigma,\ x \in [0,1]$ we deduce

$$\begin{aligned} \left\|A^0[Z]B\right\|_{\alpha,\sigma} \;\; \le \;\; & \sum_{k=1}^{K_1} \left(\frac{|c_k|}{\sigma^2} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha+1}} + \frac{|n_k^0|}{\sigma} \right) C_1 \|\nu_k\|_{C[0,1]} \|\widehat{B}^0\|_{\alpha,\sigma} + \\ + \;\; & \sum_{k=K_1+1}^{K} \left(\frac{|c_k|}{\sigma} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^\alpha} \right) C_3 \|\mu_{k-K_1}\|_{C[0,1]} \|B^1\|_{\alpha-\frac{1}{2},\sigma} \end{aligned}$$

for $\sigma \ge \sigma_0$. This, due to the relation (3.2.7), implies

$$\left\|A^0[Z]B\right\|_{\alpha,\sigma} \le \text{Const} \left[\frac{|n^0| + |c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} \right] \|B\|_{\alpha,\sigma}, \quad \sigma \ge \sigma_0 \tag{3.2.12}$$

with some constant and

$$|n^0| = \sum_{k=1}^{K_1} |n_k^0| .$$

Further, from (3.1.20) we derive

$$
\begin{aligned}
|p|^{\alpha-\frac{1}{2}} \left|\left(A^1[Z]B\right)(x,p)\right| \;\le\; & \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|^2}+\frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha+1}}+\frac{|n_k^0|}{|p|}\right) \\
& \times \sqrt{|p|}\int_0^1 |G_x(x,y,p)|\,dy\cdot \|\nu_k\|_{C[0,1]}\,|p|^\alpha \max_{0\le y\le 1}\left|\widehat{B}^0(y,p)\right| \\
& + \sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{|p|}+\frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right)\|\mu_{k-K_1}\|_{C[0,1]} \\
& \times\left[\frac{1}{\min\limits_{x\in[0,1]}\lambda(x)}+\int_0^1 |G_{xy}(x,y,p)|\,dy\right]|p|^{\alpha-\frac{1}{2}}\max_{0\le y\le 1}\left|B^1(y,p)\right| .
\end{aligned}
$$

Using Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p>\sigma,\ x\in[0,1]$ we get

$$
\left\|A^1[Z]B\right\|_{\alpha-\frac{1}{2},\sigma} \;\le\; \mathrm{Const}\left[\frac{|n^0|+|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right]\|B\|_{\alpha,\sigma}\,,\quad \sigma\ge\sigma_0 \qquad (3.2.13)
$$

with some constant.

Putting the estimates (3.2.12) and (3.2.13) together we have

$$
\|A[Z]B\|_\sigma \;\le\; \mathrm{Const}\left[\frac{|n^0|+|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right]\|B\|_{\alpha,\sigma}\,,\qquad \sigma\ge\sigma_0\,. \qquad (3.2.14)
$$

Using this relation and Lemmas 2.2, 2.3 for the components of (3.1.19), (3.1.20) we see that $A[Z]$ is well-defined and bounded in $\mathcal{B}_{\alpha,\sigma}$. Moreover, (3.2.14) implies the estimate (3.2.10).

It remains to prove (3.2.11). Denoting $Z=Z^1-Z^2$ the components $A^0[Z]$ and $A^1[Z]$ of the vector $A[Z]=A[Z^1]-A[Z^2]$ are expressed by the formulas (3.1.19) and (3.1.20), respectively, containing $n^0=0$. Using the estimate (3.2.14) for $A[Z]$ and observing that $Z=\frac{c}{p}+V$, where $c=0$ and $V=V^1-V^2$, we deduce (3.2.11). The lemma is proved. □

Now we are in the situation to prove the main result concerning the equation (3.1.18).

Theorem 3.1. *Let the assumptions* (2.2.2), (3.2.1) *hold. Then for any* $\sigma\ge\sigma_0$ *and* $Z=\frac{c}{p}+V\in\mathcal{M}_{c,\alpha,\sigma}$, *satisfying the inequality*

$$
\eta(Z,\sigma) \;:=\; \frac{1+|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\le\frac{1}{2C_0}\,, \qquad (3.2.15)
$$

where C_0 *is the constant from* (3.2.10), *equation* (3.1.18) *has a unique solution* $B[Z]$ *in* $\mathcal{B}_{\alpha,\sigma}$. *This solution satisfies estimate*

$$
\|B[Z]\|_{\alpha,\sigma} \;\le\; \mathrm{Const}\left[1+\frac{1}{\sigma^{\frac{3}{2}-\alpha}}\left(|c|+\frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\right] \qquad (3.2.16)
$$

with some constant. Moreover, for every $\sigma \geq \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$ *such that* $\eta(Z^j, \sigma) \leq \frac{1}{2C_0}$, $j = 1, 2$, *the difference* $B[Z^1] - B[Z^2]$ *fulfils estimate*

$$\left\|B[Z^1] - B[Z^2]\right\|_{\alpha,\sigma} \leq \text{Const}\left\{\frac{1}{\sigma^\alpha}\left[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha}}\left(|c| + \frac{\|V^1\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\right] + \frac{1}{\sqrt{\sigma}}\right\}\left\|V^1 - V^2\right\|_{\alpha,\sigma} \quad (3.2.17)$$

with some constant.

Proof. Due to Lemmas 3.1, 3.2 the equation (3.1.18) is well-defined in $\mathcal{B}_{\alpha,\sigma}$ if $\sigma \geq \sigma_0$ and $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$. Moreover, in case the relation (3.2.15) holds, from (3.2.10) we get $\|A[Z]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})} \leq \frac{1}{2}$. Thus, by the contraction principle equation (3.1.18) has a unique solution $B = B[Z] \in \mathcal{B}_{\alpha,\sigma}$.

Furthermore, from the system (3.1.18) we have

$$\|B[Z]\|_{\alpha,\sigma} \leq \left(1 - \|A[Z]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})}\right)^{-1} \|b[Z]\|_{\alpha,\sigma}.$$

This, in view of the inequality $\|A[Z]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})} \leq \frac{1}{2}$ and (3.2.2), yields the estimate (3.2.16).

Finally, let us prove the estimate (3.2.17). Let $\sigma \geq \sigma_0$ and $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2$ be such that (3.2.15) is valid for V replaced by V^1 and V^2, i.e. $\eta(Z^j, \sigma) \leq \frac{1}{2C_0}$, $j = 1, 2$. Subtracting equation (3.1.18) for $Z = Z^2$ from the corresponding equation for $Z = Z^1$ we have

$$\begin{aligned} B[Z^1] - B[Z^2] &= A[Z^2]\left(B[Z^1] - B[Z^2]\right) + \left(A[Z^1] - A[Z^2]\right)B[Z^1] \\ &+ b[Z^1] - b[Z^2]. \end{aligned}$$

This implies

$$\begin{aligned} \|B[Z^1] - B[Z^2]\|_{\alpha,\sigma} &\leq \left(1 - \|A[Z^2]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})}\right)^{-1} \\ &\times \left[\|A[Z^1] - A[Z^2]\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})}\|B[Z^1]\|_{\alpha,\sigma} + \|b[Z^1] - b[Z^2]\|_{\alpha,\sigma}\right]. \end{aligned}$$

Using in this relation the inequality $\left\|A[Z^2]\right\|_{\mathscr{L}(\mathcal{B}_{\alpha,\sigma})} \leq \frac{1}{2}$ and the estimates (3.2.4), (3.2.11), (3.2.16) we deduce (3.2.17). The proof is complete. □

3.3 Existence and uniqueness for inverse problem

In this section we study the fixed-point equation (3.1.9) with (3.1.10) and thereupon infer results for the generalized inverse problem with temperature observations in the time domain.

Using the decomposition (3.1.17) for $B^0[Z]$ in the term containing n_k^0 in (3.1.10) we extract the Z-free addend $\frac{1}{p}\sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i, y, p)\nu_k(y)c_\Phi(y)dy$ from this term and add it to $\widehat{\Psi}_i(p)$ to obtain the full Z-free term of the operator $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_N)$ in the following form:

$$\Psi = (\Psi_1, \ldots, \Psi_K), \quad \Psi_i(p) = \widehat{\Psi}_i(p) + \frac{1}{p} \sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i, y, p)\nu_k(y)c_\Phi(y)dy. \tag{3.3.18}$$

Recall that $\widehat{\Psi}_i$ is defined by (3.1.11).

Theorem 3.2. *Assume that* (2.2.2), (3.2.1) *hold and*

$$\nu_k \in C^1[0,1],\ k = 1, \ldots, K_1, \quad \mu_k \in C^2[0,1],\ k = 1, \ldots, K_2, \quad \varphi \in C^3[0,1]. \tag{3.3.19}$$

Moreover, let $\det \Gamma \neq 0$ *for* Γ*, given by* (3.1.7)*, and*

$$\Psi = \frac{d}{p} + Y \in \mathcal{M}_{d,\alpha,\sigma_0} \tag{3.3.20}$$

with some $d \in \mathbb{R}^K$*. Then there exists* $\sigma_1 \geq \sigma_0$ *such that equation* (3.1.9) *with* (3.1.10) *has a solution* $Z = \frac{c}{p} + V$ *in the space* $\mathcal{M}_{c,\alpha,\sigma_1}$*, where* $c = \Gamma^{-1}d$*. The solution is unique in the union of spaces* $\bigcup\limits_{\substack{\sigma \geq \sigma_1 \\ c \in \mathbb{R}}} \mathcal{M}_{c,\alpha,\sigma}$.

Proof. Firstly, we prove the existence assertion of theorem. Let us set $c = \Gamma^{-1}d$. Observing (3.1.10), (3.3.18), (3.3.20) we see that equation (3.1.9) for $Z = \frac{c}{p} + V$ in $\mathcal{M}_{c,\alpha,\sigma}$ is equivalent to the following equation for V in $(\mathcal{A}_{\alpha,\sigma})^K$:

$$V = F(V), \tag{3.3.21}$$

where $F = \Gamma^{-1}F_1$,

$$F_1(V) = L_0\Big(\frac{c}{p} + V\,,\ B[Z]\Big) + L_1\Big(\frac{c}{p} + V\Big) + L_2(B[Z]) + Y, \tag{3.3.22}$$

L_0 is the following bilinear operator of $Z \in \mathcal{M}_{c,\alpha,\sigma}$ and $B = (\widehat{B}^0, B^1) \in \mathcal{B}_{\alpha,\sigma}$:

$$(L_0(Z,B))_i(p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG(x_i, y, p)\nu_k(y)\widehat{B}^0(y,p)dy$$
$$- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 pG_y(x_i, y, p)\mu_{k-K_1}(y)B^1(y,p)\,dy,\ i = 1, \ldots, K, \tag{3.3.23}$$

and L_1 and L_2 are the linear operators of $Z \in \mathcal{M}_{c,\alpha,\sigma}$ and $B = (\widehat{B}^0, B^1) \in \mathcal{B}_{\alpha,\sigma}$, given by the formulae

$$(L_1(Z))_i(p) \;=\; \sum_{k=1}^{K_1} Z_k(p)\Bigg\{\int_0^1 G(x_i,y,p)\nu_k(y)c_\Phi(y)dy$$
$$+\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}\Bigg\}$$
$$+\sum_{k=K_1+1}^{K} Z_k(p)\Bigg\{\int_0^1 pG(x_i,y,p)\left(\mu_{k-K_1}(y)\varphi'(y)\right)'dy + \frac{\left(\mu_{k-K_1}(x)\varphi'(x)\right)'\Big|_{x=x_i}}{\beta(x_i)}$$
$$+\mu_{k-K_1}(0)\varphi'(0)pG(x_i,0,p) - \mu_{k-K_1}(1)\varphi'(1)pG(x_i,1,p)\Bigg\},$$
$$i=1,\ldots,K, \tag{3.3.24}$$

and

$$(L_2(B))_i(p) \;=\; \sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i,y,p)\nu_k(y)\widehat{B}^0(y,p)dy,\; i=1,\ldots,K, \tag{3.3.25}$$

respectively.

We will make use of the fixed-point argument for the equation (3.3.21) the following balls:

$$D_{\alpha,\sigma}(\rho) = \Big\{V\in(\mathcal{A}_{\alpha,\sigma})^K : \|V\|_{\alpha,\sigma}\le\rho\Big\}. \tag{3.3.26}$$

To this end we first deduce some estimates for L_0, L_1 and L_2. Multiplying by $|p|^\alpha$ in (3.3.23) and estimating we have

$$|p|^\alpha\left|\Big(L_0\Big(\frac{c}{p}+V,B\Big)\Big)_i(p)\right| \;\le\; \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|}+\frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right)|p|\int_0^1|G(x_i,y,p)|\,dy$$
$$\times\;\|\nu_k(y)\|_{C[0,1]}\,|p|^\alpha\max_{0\le y\le1}\left|\widehat{B}^0(y,p)\right|$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{|p|^\alpha|V_k(p)|}{|p|^{\alpha-1}}\right)\sqrt{|p|}\int_0^1|G_y(x_i,y,p)|\,dy$$
$$\times\;\|\mu_{k-K_1}\|_{C[0,1]}\,|p|^{\alpha-\frac12}\max_{0\le y\le1}\left|B^1(y,p)\right|,\; i=1,\ldots,K.$$

Using assertions (2.4.1), (2.4.3) of Theorems 2.1, 2.5 and the definitions of the norms $\|\cdot\|_{\gamma,\sigma}$ in $\mathcal{A}_{\gamma,\sigma}$ and $\hat{\mathcal{B}}_{\gamma,\sigma}$we obtain

$$|p|^\alpha \left|\Big(L_0\Big(\frac{c}{p}+V,B\Big)\Big)_i(p)\right| \le \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|}+\frac{\|V_k\|_{\alpha,\sigma}}{|p|^\alpha}\right) C_1\,\|\nu_k(y)\|_{C[0,1]}\,\|\widehat{B}^0\|_{\alpha,\sigma}$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k|+\frac{\|V_k\|_{\alpha,\sigma}}{|p|^{\alpha-1}}\right) C_3\,\|\mu_{k-K_1}\|_{C[0,1]}\,\|B^1\|_{\alpha-\frac{1}{2},\sigma}\,,\ i=1,\ldots,K,$$

for $\mathrm{Re}p>\sigma$, $\sigma\ge\sigma_0$. Taking in this relation the supremum over $\mathrm{Re}p>\sigma$ and observing the definition of the norm $\|\cdot\|_{\alpha,\sigma}$ in $(\mathcal{A}_{\gamma,\sigma})^K$ as well as (3.2.3) we have

$$\Big\|L_0\Big(\frac{c}{p}+V,B\Big)\Big\|_{\alpha,\sigma}\le \text{ Const }\bigg[\Big(\frac{|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\Big)\|\widehat{B}^0\|_{\alpha,\sigma}$$
$$+\Big(|c|+\frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\Big)\|B^1\|_{\alpha-\frac{1}{2},\sigma}\bigg]\ ,\ \sigma\ge\sigma_0$$

with some constant. Using here the inequality (3.2.7) with $\gamma=0,\gamma'=1$ and $\gamma=\alpha-1,\gamma'=\alpha$ and observing the definition of the norm $\|\cdot\|_{\alpha,\sigma}$ in $\mathcal{B}_{\gamma,\sigma}$ we deduce the estimate

$$\Big\|L_0\Big(\frac{c}{p}+V,B\Big)\Big\|_{\alpha,\sigma}\le \text{ Const }\Big(|c|+\frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\Big)\|B\|_{\alpha,\sigma}\,,\quad \sigma\ge\sigma_0 \qquad (3.3.27)$$

with some constant.

Similarly, multiplying in (3.3.24) by $|p|^\alpha$, observing the norm $\|\cdot\|_{\alpha,\sigma}$ in $\mathcal{A}_{\alpha,\sigma}$ and estimating we have

$$|p|^\alpha\left|\Big(L_1\Big(\frac{c}{p}+V\Big)\Big)_i(p)\right|$$
$$\le\sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|^{\frac{3}{2}-\alpha}}+\frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}}\right)\left(\frac{1}{\sqrt{|p|}}\,\|\nu_k\|_{C[0,1]}\,|p|\int_0^1|G(x_i,y,p)|dy\right.$$
$$\left.\times\max_{0\le y\le 1}|c_\Phi(y)|+\sqrt{|p|}\left|\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy+\frac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}\right|\right)$$
$$+\sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{|p|^{\frac{3}{2}-\alpha}}+\frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}}\right)\left(\sqrt{|p|}\,\left|\int_0^1 pG(x_i,y,p)\,\big(\mu_{k-K_1}(y)\varphi'(y)\big)'dy\right.\right.$$
$$\left.+\frac{1}{\beta(x_i)}\,\big(\mu_{k-K_1}(x)\varphi'(x)\big)'\Big|_{x=x_i}\right|$$
$$\left.+|\mu_{k-K_1}(0)||\varphi'(0)|\,|p|^{3/2}|G(x_i,0,p)|+|\mu_{k-K_1}(1)||\varphi'(1)|\,|p|^{3/2}|G(x_i,1,p)|\right),$$
$$i=1,\ldots K,$$

for $\mathrm{Re}p>\sigma$, $\sigma\ge\sigma_0$. In the case of boundary conditions of the first kind the terms $G(x_i,0,p)$ and $G(x_i,1,p)$ in this estimate are equal to zero (see (1.2.12)). In the case of boundary conditions of the third kind we make use of the assertion (2.5.1) of Theorem 2.5 with $\kappa = 3/2$ and take the assumption $x_i \in (0,1)$ into account. Then we get

$$|p|^{3/2}|G(x_i,0,p)| \le C^1_{3/2}(x_i)\,, \quad |p|^{3/2}|G(x_i,1,p)| \le C^1_{3/2}(x_i)\,, \quad \mathrm{Re}\, p > 0.$$

Using these relations, the assertion (2.4.1) of Theorems 2.1, 2.5 and the assertions (2.4.6) and (2.5.6) of Theorems 2.3 and 2.7, respectively, we deduce

$$\begin{aligned}
|p|^{\alpha}\Big|\Big(L_1\Big(\frac{c}{p}+V\Big)\Big)_i(p)\Big| &\le \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|^{\frac{3}{2}-\alpha}}+\frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}}\right) \\
&\times\left(\frac{C_1}{\sqrt{|p|}}\,\|\nu_k\|_{C[0,1]}\,\|c_\Phi\|_{C[0,1]}+\widetilde{C}_i\,\|\nu_k\varphi\|_{C^1[0,1]}\right) \\
&+\sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{|p|^{\frac{3}{2}-\alpha}}+\frac{\|V_k\|_{\alpha,\sigma}}{\sqrt{|p|}}\right)\Big[\widetilde{C}_i\|\,\big(\mu_{k-K_1}\varphi'\big)'\,\|_{C^1[0,1]} \\
&+\theta\Big(|\mu_{k-K_1}(0)\varphi'(0)|C^1_{3/2}(x_i)+|\mu_{k-K_1}(1)\varphi'(1)|C^1_{3/2}(x_i)\Big)\Big],\ i=1,\dots K,
\end{aligned}$$

for $\mathrm{Re} p > \sigma$, $\sigma \ge \sigma_0$ where $\widetilde{C}_i = C_6(x_i)$, $\theta = 0$ in the case of boundary conditions of the first kind and $\widetilde{C}_i = C_9$, $\theta = 1$ in the case of boundary conditions of the third kind. Taking here the supremum over $\mathrm{Re} p > \sigma$ and observing (3.2.7) with $\gamma = 0, \gamma' = 1/2$ we obtain

$$\Big\|L_1\Big(\frac{c}{p}+V\Big)\Big\|_{\alpha,\sigma} \le \text{ Const }\left(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}}+\frac{\|V\|_{\alpha,\sigma}}{\sqrt{\sigma}}\right),\quad \sigma\ge\sigma_0 \tag{3.3.28}$$

with some constant.

Finally, from (3.3.25) in view of the assertion (2.2.32) of Theorems 2.1, 2.5 we deduce

$$\|L_2(B)\|_{\alpha,\sigma} \le \text{ Const } \|B\|_{\alpha,\sigma}\,,\ \sigma \ge \sigma_0 \tag{3.3.29}$$

with some constant.

Let us return to the equation (3.3.21) with the operator $F = \Gamma^{-1}F_1$, where F_1 is given by (3.3.22). By means of (3.3.27), (3.3.28), (3.3.29) and the relation $\|Y\|_{\alpha,\sigma} \le \|Y\|_{\alpha,\sigma_0}$ we obtain

$$\begin{aligned}
\|F_1(V)\|_{\alpha,\sigma} \le \text{Const}\left(1+\frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\left[\|B[Z]\|_\sigma+\frac{1}{\sigma^{\frac{3}{2}-\alpha}}\right] \\
+\|Y\|_{\alpha,\sigma_0},\ \sigma\ge\sigma_0
\end{aligned} \tag{3.3.30}$$

with some constant depending on $|c|$. Let us suppose that $V \in D_{\alpha,\sigma}(\rho)$. Then, using Lemmas 2.2 and 2.3 to the components of (3.3.23) - (3.3.25) and observing (3.3.22) and (3.3.30) we see that $F_1(V) \in (\mathcal{A}_{\alpha,\sigma})^K$ for $\sigma \ge \sigma_0$. If in addition, σ, ρ satisfy the relation

$$\eta_0(\rho,\sigma) := \frac{1+|c|}{\sigma}+\frac{\rho}{\sigma^\alpha} \le \frac{1}{2C_0}\,, \tag{3.3.31}$$

then (3.2.15) holds for σ and $Z = \frac{c}{p} + V$. Hence, we can apply estimate (3.2.16) of Theorem 3.1 for $\|B[Z]\|_{\alpha,\sigma}$. Plugging (3.2.16) into (3.3.30) and estimating $\|V\|_{\alpha,\sigma}$ by ρ and observing that $\frac{1}{\sigma^{\frac{3}{2}-\alpha}} \leq \text{Const}$ for $\sigma \geq \sigma_0$ we have

$$\|F_1(V)\|_{\alpha,\sigma} \leq \widehat{\mathrm{C}}\left(1 + \frac{\rho}{\sigma^{\alpha-1}}\right)\left(1 + \frac{\rho}{\sqrt{\sigma}}\right) + \|Y\|_{\alpha,\sigma_0} \tag{3.3.32}$$

with some constant $\widehat{C}$ independent of ρ and σ. From (3.3.31) and (3.3.32), due to the equality $F = \Gamma^{-1}F_1$ and the relation $\alpha > 1$, we see that for every $\rho > \rho_0 := \left|\Gamma^{-1}\right|(\widehat{C} + \|Y\|_{\alpha,\sigma_0})$ there exists $\sigma_2 = \sigma_2(\rho) \geq \sigma_0$ such that the inequalities $\eta_0(\rho,\sigma) \leq \frac{1}{2C_0}$ and $\|FV\|_{\alpha,\sigma} \leq \rho$ hold for any $\sigma \geq \sigma_2(\rho)$.

Consequently, by the definition of $D_{\alpha,\sigma}(\rho)$ we have

$$F : D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \quad \text{for} \quad \rho > \rho_0 \quad \text{and} \quad \sigma \geq \sigma_2(\rho). \tag{3.3.33}$$

Next, we prove that F is a contraction in $D_{\alpha,\sigma}(\rho)$ with suitable parameters ρ and σ. From (3.3.22), due to the bilinearity of L_0 and the linearity of L_1, L_2, we have

$$\begin{aligned} F_1(V) - F_1(\widetilde{V}) &= L_0(V - \widetilde{V}, B[Z]) + L_0\Big(\frac{c}{p} + \widetilde{V}, B[Z] - B[\widetilde{Z}]\Big) \\ &\quad + L_1(V - \widetilde{V}) + L_2(B[Z] - B[\widetilde{Z}]), \end{aligned}$$

where $Z = \frac{c}{p} + V$, $\widetilde{Z} = \frac{c}{p} + \widetilde{V}$. Using here (3.3.27), (3.3.28) and (3.3.29) we get

$$\begin{aligned} \|F_1(V) - F_1(\widetilde{V})\|_{\alpha,\sigma} \leq \text{Const}\Bigg\{&\left(\frac{1}{\sqrt{\sigma}} + \frac{\|B[Z]\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\|V - \widetilde{V}\|_{\alpha,\sigma} \\ &+ \Big(1 + \frac{\|\widetilde{V}\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\Big)\|B[Z] - B[\widetilde{Z}]\|_{\alpha,\sigma}\Bigg\}, \ \sigma \geq \sigma_0 \end{aligned}$$

with some constant depending on $|c|$. Let us suppose that $V, \widetilde{V} \in D_{\alpha,\sigma}(\rho)$, where $\sigma \geq \sigma_0$ and ρ, σ satisfy (3.3.31). Then we can make use of the estimates (3.2.16) and (3.2.17) of Theorem 3.1 to get

$$\begin{aligned} \|F_1(V) - F_1(\widetilde{V})\|_{\alpha,\sigma} \leq \text{Const}\Bigg\{&\frac{1}{\sqrt{\sigma}} + \frac{1}{\sigma^{\alpha-1}}\left(1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha}}\Big(1 + \frac{\rho}{\sigma^{\alpha-1}}\Big)\right) \\ &+ \Big(1 + \frac{\rho}{\sigma^{\alpha-1}}\Big)\left[\frac{1}{\sigma^{\alpha}}\left(1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha}}\Big(1 + \frac{\rho}{\sigma^{\alpha-1}}\Big)\right) + \frac{1}{\sqrt{\sigma}}\right]\Bigg\}\|V - \widetilde{V}\|_{\alpha,\sigma}. \end{aligned}$$

In view of the assumed inequalities $1 < \alpha < \frac{3}{2}$ (cf. (3.2.1), (2.1.3)), the coefficient of $\|V - \widetilde{V}\|_{\alpha,\sigma}$ on the right-hand side of this estimate approaches zero, as $\sigma \to \infty$ for any fixed $\rho > 0$. Hence, for any $\rho > 0$ there exists $\sigma_3 = \sigma_3(\rho) \geq \sigma_0$ such that the inequality $\eta_0(\rho,\sigma) \leq \frac{1}{2C_3}$ holds and $F = \Gamma^{-1}F_1$ is a contraction in the ball $D_{\alpha,\sigma}(\rho)$ for $\rho > 0$ and $\sigma \geq \sigma_3(\rho)$. This together with (3.3.33) shows that the equation (3.3.21) has a unique solution V in every ball $D_{\alpha,\sigma}(\rho)$, where $\rho > \rho_0$ and $\sigma \geq \sigma_4(\rho) = \max\{\sigma_2(\rho); \sigma_3(\rho)\}$. This proves the existence assertion of the theorem with $\sigma_1 = \sigma_4(2\rho_0)$.

It remains to prove the uniqueness assertion of the theorem. We start it proving the uniqueness in the following union: $\bigcup\limits_{\sigma \geq \sigma_1} \mathcal{M}_{c,\alpha,\sigma}$ where $c = \Gamma^{-1}d$. Suppose that (3.1.9) has two solutions $Z^1 = \frac{c}{p} + V^1$ and $Z^2 = \frac{c}{p} + V^2$ in this union. Observing the relation

$$\mathcal{M}_{c,\alpha,\sigma} \subset \mathcal{M}_{c,\alpha,\widetilde{\sigma}} \quad \text{for any} \quad 0 \leq \sigma \leq \widetilde{\sigma},$$

following from the definition of $\mathcal{M}_{c,\alpha,\sigma}$, we see that there exists $\widehat{\sigma} \geq \sigma_1$ such that $Z^1, Z^2 \in \mathcal{M}_{c,\alpha,\widehat{\sigma}}$. This implies that the components V^1 and V^2 of these solutions belong to $\left(\mathcal{A}_{\alpha,\widehat{\sigma}}\right)^N$. Moreover, V^1 and V^2 solve the equation (3.3.21). Let us define

$$\bar{\rho} := \max(2\rho_0; \|V^1\|_{\alpha,\widehat{\sigma}}; \|V^2\|_{\alpha,\widehat{\sigma}}) \quad \text{and} \quad \bar{\sigma} := \max(\widehat{\sigma}; \sigma_4(\bar{\rho})). \tag{3.3.34}$$

From the left equality in this formula we have $\|V^j\|_{\alpha,\widehat{\sigma}} \leq \bar{\rho}$, $j = 1, 2$. Since the norm $\|\cdot\|_{\alpha,\sigma}$ is non-increasing with respect to σ and $\bar{\sigma} \geq \widehat{\sigma}$, we derive $\|V^j\|_{\alpha,\bar{\sigma}} \leq \bar{\rho}$, $j = 1, 2$. Thus, by the definition of $D_{\alpha,\sigma}(\rho)$ we get

$$V^j \in D_{\alpha,\bar{\sigma}}(\bar{\rho}), \quad j = 1, 2.$$

But the uniqueness of the solution of the equation (3.3.21) in the ball $D_{\alpha,\bar{\sigma}}(\bar{\rho})$ has already been shown. This is so, because by (3.3.34) the inequalities $\bar{\rho} > \rho_0$ and $\bar{\sigma} \geq \sigma_4(\bar{\rho})$ are valid. Consequently, $V^1 = V^2$ implying $Z^1 = Z^2$.

To complete the proof of the uniqueness assertion we have to show that (3.1.9) has no solution in any space $\mathcal{M}_{\widetilde{c},\alpha,\sigma}$ where $\widetilde{c} \neq c = \Gamma^{-1}d$ and $\sigma \geq \sigma_1$. Suppose contrary that (3.1.9) has a solution $Z = \frac{\widetilde{c}}{p} + V$ in some of such spaces. Then, observing (3.1.9), (3.1.10), (3.3.18), (3.3.20) we see that component $V \in (\mathcal{A}_{\alpha,\sigma})^K$ of this solution satisfies the equation

$$\Gamma V = L_0\Big(\frac{c}{p} + V\,,\, B[Z]\Big) + L_1\Big(\frac{c}{p} + V\Big) + L_2(B[Z]) + Y + \frac{\widetilde{c} - c}{p}, \tag{3.3.35}$$

where and L_0, L_1, L_2 are given by (3.3.23) - (3.3.25). Observing the definitions of $(\mathcal{A}_{\alpha,\sigma})^K$, $\mathcal{M}_{c,\alpha,\sigma}$, $\|\cdot\|_{\alpha,\sigma}$, the relation $V \in (\mathcal{A}_{\alpha,\sigma})^K$, the assumption (3.3.20) and the estimates (3.3.27), (3.3.28), (3.3.29) we see that all terms in (3.3.35) except for $\frac{\widetilde{c}-c}{p}$ are estimated by quantities of the form $\frac{\mathrm{Const}}{|p|^\alpha}$ in the half-plane $\mathrm{Re}\, p > \sigma_0$, where the constant is independent of p and $\alpha > 1$. This is not possible in the case $\widetilde{c} - c \neq 0$. We reached the contradiction. The uniqueness is completely proved. □

Finally, we deduce the following corollary for the generalized inverse problem in the time domain.

Corollary 3.1. *Let conditions* (3.1.4) *hold with* $\mathscr{R} = K_1$ *implying the existence of the unique solution* n_k^0*,* $k = 1, \dots, K_1$*, to the system* (3.1.3)*. Moreover, let* (2.2.2) *hold and the assumptions* (3.2.1), (3.3.19), (3.3.20) *be satisfied for the functions* λ_k*,* μ_k*,* φ *and the quantities* Φ^0*,* Φ^1*,* Ψ *given in terms of the Laplace transforms* R*,* F_1*,* F_2*,* H_i *of the data* r*,* f_1*,* f_2*,* h_i *by formulas* (3.1.13), (3.1.15), (3.3.18), (3.1.11) *and either* (1.2.14) *or* (1.2.23) *depending on the type of the boundary conditions. Assume that* $\det \Gamma \neq 0$ *for* Γ*, given by* (3.1.7)*.*

Then the generalized inverse problem with temperature observations has a unique solution of the form

$$
\begin{aligned}
n_k(t) &= n_k^0 + c_k t + \tfrac{1}{2\pi i} \int_0^t \int_{\xi - i\infty}^{\xi + i\infty} e^{\tau p} Z_k(p)\, dp\, d\tau\,, \quad k = 1, \dots K_1\,, \\
m_k(t) &= c_{k+K_1} + \tfrac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{tp} Z_{k+K_1}(p)\, dp\,, \quad k = 1, \dots K_2\,,
\end{aligned}
\tag{3.3.36}
$$

where $c = (c_1, \dots, c_K) \in \mathbb{R}^K$*,* $Z = (Z_1, \dots, Z_K) \in (\mathcal{A}_{\alpha,\sigma})^K$*,* $\sigma > 0$*.*

The functions n_k *are continuously differentiable and* m_k *are continuous for* $t \geq 0$*. Moreover, the vector* c *in the formulas* (3.3.36) *is expressed by* $c = \Gamma^{-1} d$*, where* d *is the component of* Ψ *in the assumption* (3.3.20)*. In addition,* $n_k(0) = n_k^0$*,* $n_k'(0) = c_k$*,* $k = 1, \dots, K_1$ *and* $m_k(0) = c_{k+K_1}$*,* $k = 1, \dots, K_2$*.*

Proof. Corollary follows from Theorem 3.2 and Proposition 3.1 together with the inversion formula (1.2.4) of the Laplace transform, the properties 1, 3, 4 of the Laplace transform in Section 1.2.1, the definitions of the spaces $(\mathcal{A}_{\alpha,\sigma})^K$, $\mathcal{M}_{c,\alpha,\sigma}$ and the formulas $\mathcal{L}_{t\to p}(c) = \frac{c}{p}$, $\mathcal{L}_{t\to p}\left(\int_0^t w(\tau) d\tau\right) = \frac{1}{p} \mathcal{L}_{t\to p} w(t)$. □

3.4 Interpretation of assumptions

This section consists of two parts. In the first part we interpret the regularity assumptions of the existence theorem in the time domain. In the second part we analyse the non-vanishing condition $\det \Gamma \neq 0$.

3.4.1 Interpretation of regularity assumptions

The assumptions of Theorem 3.2 and Corollary 3.1 contain assumptions on the Laplace transforms of the data of the inverse problem. Let us give sufficient conditions in the time domain implying these assumptions.

We recall that the condition $\sigma_0 > 0$ in (3.2.1) can be replaced by $\sigma_0 \in \mathbb{R}$ (see the remark after (3.2.1)). Thus, let us start with the following condition in (3.2.1):

$$
\begin{aligned}
&\Phi^0 \text{ given by (3.1.13) satisfies (3.1.16) where } c_\Phi \in C[0,1] \\
&\text{and } \widetilde{\Phi}^0 \in \hat{\mathcal{B}}_{\alpha,\sigma} \text{ with some } \alpha \text{ satisfying (2.1.3).}
\end{aligned}
\tag{3.4.37}
$$

We split $\Phi^0 = \Phi^{0,1} + \Phi^{0,2}$ with $\Phi^{0,1}(x,p) = \sum_{k=1}^{K_1} n_k^0 \int_0^1 G(x,y,p) \nu_k(y) \varphi(y) dy$ and $\Phi^{0,2}(x,p) = -pQ(x,p) - \varphi(x)$. In case $\nu_k, \varphi \in C^1[0,1]$ and (2.2.2) hold

Theorems 2.3 and 2.7 immediately imply

$$\Phi^{0,1}(x,p) = -\frac{1}{p}\sum_{k=1}^{K_1} n_k^0 \frac{\nu_k(x)\varphi(x)}{\beta(x)} + \widetilde{\Phi}^{0,1}(x,p)$$

with $\widetilde{\Phi}^{0,1} \in \hat{\mathcal{B}}_{3/2,0}$. The latter relation yields $\widetilde{\Phi}^{0,1} \in \hat{\mathcal{B}}_{\alpha,\sigma}$ for any α satisfying (2.1.3) and $\sigma > 0$. To study $\Phi^{0,2}$, let us denote by $u^0(x,t)$ the solution of the direct problem in the case $n = m = 0$, i.e.

$$\begin{aligned} &\beta(x)u_t^0(x,t) = \tfrac{\partial}{\partial x}\left(\lambda(x)u_x^0(x,t)\right) + r(x,t), \quad x\in(0,1),\ t>0,\\ &u^0(x,0) = \varphi(x), \quad x \in (0,1), \\ &u^0 \text{ satisfies either (1.1.6) or (1.1.7) with } m=0. \end{aligned} \tag{3.4.38}$$

Assume

$$u^0(x,\cdot), u_t^0(x,\cdot), u_x^0(x,\cdot), u_{xx}^0(x,\cdot) \in \mathscr{E} \ \text{ for any } x\in(0,1) \tag{3.4.39}$$

and denote $U^0(x,p) = \mathcal{L}_{t\to p}u^0(x,t)$. Then U^0 satisfies the equation (1.2.15) with $M_k = N_k = 0$. From this equation we see that $U^0 = -Q$. Thus, by virtue of the property 1 of the Laplace transform in Section 1.2.1 and the initial value $u^0(x,0) = \varphi(x)$, we have $\Phi^{0,2}(x,p) = \mathcal{L}_{t\to p}u_t^0(x,t)$. Applying again the property 1 for $\mathcal{L}_{t\to p}u_{tt}^0(x,t)$ we get $\Phi^{0,2}(x,p) = \frac{u_t^0(x,0)}{p} + \frac{1}{p}\mathcal{L}_{t\to p}u_{tt}^0(x,t)$. Consequently, by the properties 1 and 5 of the Laplace transform the assumptions

$$\begin{aligned} &u_{tt}^0(x,\cdot) \text{ and } \tfrac{d^{\alpha-1}}{dt^{\alpha-1}}u_{tt}^0(x,\cdot) \text{ with some } \alpha \text{ satisfying (2.1.3) belong to } \mathscr{E}\\ &\text{for } x\in[0,1], \text{ where the parameters } C,\ \sigma \text{ of the space } \mathscr{E}\\ &\text{are independent of } x \end{aligned} \tag{3.4.40}$$

yield the relation

$$\Phi^{0,2}(x,p) = \frac{u_t^0(x,0)}{p} + \widetilde{\Phi}^{0,2}(x,p)$$

with $\widetilde{\Phi}^{0,2} \in \hat{\mathcal{B}}_{\alpha,\sigma}$. Summing up, (3.4.39), (3.4.40) with $\nu_k, \varphi \in C^1[0,1]$, $u_t^0(x,0) \in C[0,1]$ and (2.2.2) imply (3.4.37) with

$$c_\Phi(x) = -\sum_{k=1}^{K_1} n_k^0 \frac{\nu_k(x)\varphi(x)}{\beta(x)} + u_t^0(x,0).$$

Secondly, let us consider the following assumption in (3.2.1):

$$\Phi^1 \text{ given by (3.1.15) belongs to } \hat{\mathcal{B}}_{\alpha-\frac{1}{2},\sigma} \text{ with } \alpha \text{ satisfying (2.1.3).} \tag{3.4.41}$$

Comparing (3.1.15) with (3.1.13) we see that $\Phi^1 = \Phi_x^0$. Thus, $\Phi^1 = \Phi_x^{0,1} + \Phi_x^{0,2}$. In case $\nu_k, \varphi \in C^2[0,1]$ Theorems 2.4, 2.7 imply the property (3.4.41) for Φ_x^0.

Since $\Phi_x^{0,2}(x,p) = \mathcal{L}_{t\to p} u_{xt}^0(x,t)$, due to property 5 of the Laplace transform in §1.2.1, the conditions

$$\begin{aligned}&u_{xt}^0(x,\cdot) \text{ and } \tfrac{d^{\alpha-1/2}}{dt^{\alpha-1/2}} u_{xt}^0(x,\cdot) \text{ with some } \alpha \text{ satisfying (2.1.3)}\\ &\text{belong to } \mathscr{E} \text{ for } x\in[0,1], \text{ where the parameters } C, \sigma \text{ of the space } \mathscr{E}\\ &\text{are independent of } x\end{aligned} \tag{3.4.42}$$

yield the property (3.4.41) for $\Phi_x^{0,2}$. Summing up, (3.4.39), (3.4.42) with $\nu_k, \varphi \in C^2[0,1]$ imply (3.4.41).

Further, we deduce a formula in the time domain for the quantity

$$\lim_{\operatorname{Re} p\to+\infty} p^2[H_i(p) + Q(x_i,p)]$$

in the rank condition (3.1.4). In view of property 1 of the Laplace transform and the relations $U^0 = -Q$, $u^0(x,0) = \varphi(x)$, the conditions (3.4.39), (3.4.40) with

$$h_i^{(j)} \in \mathscr{E}\,, \quad j = 0,1,2\,, \quad i = 1,\dots,K, \tag{3.4.43}$$

imply the formula

$$\begin{aligned}p^2[H_i(p) + Q(x_i,p)] &= \mathcal{L}_{t\to p}\left[h_i''(t) - u_{tt}^0(x_i,t)\right] + p\left[h_i(0) - \varphi(x_i)\right]\\ &\quad + h_i'(0) - u_t^0(x_i,0).\end{aligned} \tag{3.4.44}$$

Assuming the consistency conditions

$$h_i(0) = \varphi(x_i)\,, \quad i = 1,\dots,K, \tag{3.4.45}$$

and observing that the Laplace transform vanishes as $\operatorname{Re} p \to +\infty$ (see (1.2.3)) we obtain the formula

$$\lim_{\operatorname{Re} p\to+\infty} p^2[H_i(p) + Q(x_i,p)] \;=\; h_i'(0) - u_t^0(x_i,0)\,, \quad i = 1,\dots,K. \tag{3.4.46}$$

Finally, we interpret the assumption (3.3.20) for the vector $\Psi = (\Psi_1,\dots,\Psi_K)$ given by (3.3.18) with (3.1.11). Let us decompose $\Psi = \Psi^1 + \Psi^2 + \Psi^3$ with $\Psi^j = (\Psi_1^j,\dots,\Psi_K^j)$, where

$$\begin{aligned}\Psi_i^1(p) &= \sum_{k=1}^{K_1} n_k^0 \left[\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)dy + \tfrac{\nu_k(x_i)\varphi(x_i)}{\beta(x_i)}\right],\\ \Psi_i^2(p) &= \sum_{k=1}^{K_1} n_k^0 \int_0^1 pG(x_i,y,p)\nu_k(y)\widetilde{B}^0(y,p)dy,\\ \Psi_i^3(p) &= -p^2[H_i(p) + Q(x_i,p)] + \lim_{\operatorname{Re} q\to+\infty} q^2[H_i(q) + Q(x_i,q)]\end{aligned}$$

and study the components Ψ^1, Ψ^2, Ψ^3 separately. Let us start with Ψ^1. Note that Theorems 2.3 and 2.7 provide estimate of the order $|p|^{-1/2}$ for $\Psi_i^1(p)$ at infinity, which is not enough for our purposes. To get necessary higher estimate we have to assume more regularity about the functions β, ν_k, φ inside Ψ^1. We introduce these regularity assumptions in an implicit form in terms of the solution of a certain direct problem. Namely, let $w(x,t)$ be the solution of the following problem:

$$\begin{aligned} &\beta(x)w_t(x,t) = \tfrac{\partial}{\partial x}\left(\lambda(x)w_x(x,t)\right), \quad x \in (0,1),\ t>0,\\ &w(x,0) = -\sum_{k=1}^{K_1} n_k^0 \tfrac{\nu_k(x)\varphi(x)}{\beta(x)}, \quad x\in(0,1), \\ &w \text{ satisfies the homogeneous first or third kind boundary conditions.}\end{aligned} \tag{3.4.47}$$

Assuming

$$w(x,\cdot), w_t(x,\cdot), w_x(x,\cdot), w_{xx}(x,\cdot) \in \mathscr{E} \ \text{ for any } x\in(0,1) \tag{3.4.48}$$

and denoting $W(x,p) = \mathcal{L}_{t\to p} w(x,t)$ the transformed problem reads

$$\begin{aligned} &\tfrac{\partial}{\partial x}\left(\lambda(x)W_x(x,p)\right) - p\beta(x)W(x,p) = \sum_{k=1}^{K_1} n_k^0 \nu_k(x)\varphi(x), \quad x\in(0,1),\\ &W \text{ satisfies the homogeneous first or third kind boundary conditions.}\end{aligned}$$

Making use of the Green function we can represent the solution of this problem in the form $W(x,p) = \sum_{k=1}^{K_1} n_k^0 \int_0^1 G(x,y,p)\nu_k(y)\varphi(y)dy$. Thus, by the property 1 of the Laplace transform, we get

$$\begin{aligned}\mathcal{L}_{t\to p} w_t(x,t) &= pW(x,p) - w(x,0)\\ &= \sum_{k=1}^{K_1} n_k^0 \left[\int_0^1 pG(x,y,p)\nu_k(y)\varphi(y)dy + \frac{\nu_k(x)\varphi(x)}{\beta(x)}\right].\end{aligned}$$

Consequently,

$$\Psi_i^1(p) = \mathcal{L}_{t\to p} w_t(x_i,t).$$

To get the condition (3.3.20) for Ψ^1, we have to assume sufficient regularity of the solution of (3.4.47) in the interior points $x_i \in (0,1)$. More precisely, due to the property 5, is sufficient to assume

$$\begin{aligned}&w_{tt}(x_i,\cdot) \ \text{ and } \ \tfrac{d^{\alpha-1}}{dt^{\alpha-1}} w_{tt}(x_i,\cdot) \ \text{ with some } \alpha \text{ satisfying (2.1.3)}\\ &\text{belong to } \mathscr{E} \text{ for } i=1,\ldots,K.\end{aligned} \tag{3.4.49}$$

The condition (3.3.20) for Ψ^2 immediately follows from (3.4.37) with the help of assertions of Theorems 2.3 and 2.7. Here the relation (3.4.37) is the consequence of the assumptions (3.4.39), (3.4.40), $\nu_k, \varphi \in C^1[0,1]$, $u_t^0(x,0) \in C[0,1]$ and (2.2.2), as was shown above. To study Ψ^3 we note that under the assumptions (3.4.39), (3.4.40), (3.4.43) and (3.4.45) the formulas (3.4.44) and (3.4.46) yield

$$\Psi_i^3(p) = -\mathcal{L}_{t\to p}\left[h_i''(t) - u_{tt}^0(x_i,t)\right].$$

By virtue of the properties 1 and 5 of the Laplace transform, the sufficient conditions for Ψ^2 to satisfy the condition (3.3.20), are (3.4.40), (3.4.43), (3.4.45) and

$$\begin{aligned}&h_i''' - u^0_{ttt}(x_i,\cdot) \text{ and } \tfrac{d^{\alpha-1}}{dt^{\alpha-1}}[h_i''' - u^0_{ttt}(x_i,\cdot)] \text{ with some } \alpha\\ &\text{satisfying (2.1.3) belong to } \mathscr{E} \text{ for } i=1,\ldots,K.\end{aligned} \tag{3.4.50}$$

Summing up, sufficient conditions for the assumption (3.3.20) for Ψ are $\nu_k, \varphi \in C^1[0,1]$, (2.2.2), (3.4.39), (3.4.40), $u^0_t(x,0) \in C[0,1]$, (3.4.43), (3.4.45), (3.4.48), (3.4.49) and (3.4.50).

We gave the conditions in the time domain in the form of the smoothness of the solutions u^0 and w of the problems (3.4.38) and (3.4.47). Using known results about the regularity of solutions of parabolic problems (see e.g. [28]) it is possible to write these conditions in terms of the data r, φ, f_1 , f_2, ν_k and β of these problems, as well.

3.4.2 *Interpretation of non-vanishing condition*

Let us analyse the non-vanishing condition $\det\Gamma \neq 0$ for the matrix Γ given by (3.1.7) in some particular cases. Firstly, we consider the case of piecewise homogeneous rod where $K_1 = K_2$, $K = 2K_1$ and the functions ν_k and μ_k are given by (1.1.9) with $0 < y_1 < y_2 < \ldots < y_{K_1-1} < 1$ and small $\epsilon > 0$. Let us choose $0 < x_1 < x_2 < \ldots < x_K < 1$ so that

$$\begin{aligned}&x_1, x_2 \in (0, y_1-\epsilon),\; x_3, x_4 \in (y_1+\epsilon, y_2-\epsilon), \ldots,\\ &\qquad x_{K-1}, x_K \in (y_{K_1+1}+\epsilon, 1).\end{aligned} \tag{3.4.51}$$

Then $\nu_k(x_{2k-1}) = \mu_k(x_{2k-1}) = \nu_k(x_{2k}) = \mu_k(x_{2k}) = 1$, $\nu_k(x_i) = \mu_k(x_i) = 0$ for $i \notin \{2k-1; 2k\}$ and $\mu_k'(x_i) = 0$ for any $i = 1, \ldots, K$. From (3.1.7) we get

$$\det\Gamma = \prod_{l=1}^{K} \frac{1}{\beta(x_l)}$$

$$\times \det\begin{pmatrix} \varphi(x_1) & & \ldots & \varphi''(x_1) & & \ldots & \\ \varphi(x_2) & & \ldots & \varphi''(x_2) & & \ldots & \\ & \varphi(x_3) & \ldots & & \varphi''(x_3) & \ldots & \\ & \varphi(x_4) & \ldots & & \varphi''(x_4) & \ldots & \\ & & \cdot\;\cdot\;\cdot\;\cdot & & & & \\ & \ldots & \varphi(x_{K-1}) & & & \ldots & \varphi''(x_{K-1}) \\ & \ldots & \varphi(x_K) & & & \ldots & \varphi''(x_K) \end{pmatrix}$$

Consequently, $\det\Gamma \neq 0$ provided the initial value φ of u satisfies the contition

$$\det\begin{pmatrix} \varphi(x_{2l-1}) & \varphi''(x_{2l-1}) \\ \varphi(x_{2l}) & \varphi''(x_{2l}) \end{pmatrix} \neq 0 \quad \text{for any } l = 1, \ldots, K_1.$$

It is not difficult to define φ so that it satisfies this condition.

Secondly, let us consider the power basis $\nu_k(x) = \mu_k(x) = x^{k-1}$. Then

$$\det\Gamma = \prod_{l=1}^{K} \frac{1}{\beta(x_l)}$$
$$\times \det\left(\left(x_i^k \varphi(x_i)\right)_{k=0,\dots,K_1-1} \; \left(x_i^k \varphi''(x_i) + k x_i^{k-1} \varphi'(x_i)\right)_{k=0,\dots,K_2-1}\right)_{i=1,\dots,K}$$

Here the matrix is formed by placing two matrices left to right. It is possible to choose the initial value φ and points x_i so that the condition $\det\Gamma \neq 0$ is valid. For instance, in case

$$\varphi(x) = \cos\left(\pi K x^{\frac{K_1+2}{2}}\right), \quad x_i = \left(\frac{i}{K}\right)^{\frac{2}{K_1+2}}$$

we can represent $\det\Gamma$ as a constant times the Vandermonde determinant:

$$\det\Gamma = \prod_{l=1}^{K} \frac{1}{\beta(x_l)} (-1)^{\lceil \frac{K}{2} \rceil} \left[-\left(\frac{\pi K(K_1+2)}{2}\right)^2\right]^{K_2} \det\left(x_i^k\right)_{\substack{k=0,\dots,K-1\\ i=1,\dots,K}} \neq 0$$

Here $\lceil x \rceil$ is the smallest integer $\geq x$.

4. Inverse problem with flux observations

In this chapter we consider the generalized inverse problem with flux observations.

4.1 Reduction of the inverse problem to a fixed-point form

Inverse problem in the Laplace domain consist of the system (1.2.26) with additional equations (1.2.15), (1.2.18).

Using the relation (1.2.18) for U_x in the left-hand side of the conditions (1.2.26) we obtain

$$\sum_{k=1}^{K_1} N_k(p) \int_0^1 pG_x(x_i, y, p)\nu_k(y)U(y, p)\, dy \tag{4.1.1}$$
$$-\sum_{k=1}^{K_2} M_k(p) \int_0^1 G_{xy}(x_i, y, p)\mu_k(y)U_y(y, p)\, dy \quad = \quad Q_x(x_i, p) - \frac{H_i(p)}{\lambda(x_i)},$$

for $i = 1, \ldots, N$.

The system (4.1.1) with (1.2.15), (1.2.18) is equivalent to the system (1.2.26) with (1.2.15), (1.2.18). We are going to transform the system (4.1.1) into a fixed-point form extracting the principal part. As in Section 3.1 we use some proper asymptotics of the system in the process $\mathrm{Re}p \to +\infty$. Such an asymptotics is achieved under the conditions that the system (4.1.1) with (1.2.15), (1.2.18) has a solution $N_k\Big|_{k=1,\ldots,K_1}, M_k\Big|_{k=1,\ldots,K_2}, U$ satisfying the following properties

(1) $n_k(t) = \mathcal{L}^{-1}_{p\to t}N_k(p)$ and $m_k(t) = \mathcal{L}^{-1}_{p\to t}M_k(p)$ belong to $\mathscr{E}$ implying $N_k(p) \to 0$, $M_k(p) \to 0$ as $\mathrm{Re}p \to +\infty$ (cf. property 1 in Section 1.2.1);

(2) for $u(x, t) = \mathcal{L}^{-1}_{p\to t}U(x, p)$ the relations $u(x, \cdot), u_t(x, \cdot), u_x(x, \cdot), u_{xt}(x, \cdot) \in \mathscr{E}$ hold with the initial condition (1.1.5) implying (3.1.2).

We emphasize that (1) and (2) are not assumptions. We use them to explain the ideas of transformations.

Let us introduce the new unknowns

$$Z = (Z_1, \ldots, Z_K),\ Z_k(p) = \begin{cases} N_k(p), & k = 1, \ldots, K_1, \\ M_{k-K_1}(p), & k = K_1 + 1, \ldots, K. \end{cases} \tag{4.1.2}$$

Now the system (4.1.1) can be rewritten in the form

$$\begin{aligned}
&\sum_{k=1}^{K_1} Z_k(p)\left(\frac{\nu_k(x)\varphi(x)}{\beta(x)}\right)'\Big|_{x=x_i} + \sum_{k=K_1+1}^{K} Z_k(p)\frac{\mu_{k-K_1}(x_i)\varphi(x_i)}{\lambda(x_i)} \\
&\quad = \sum_{k=1}^{K_1} Z_k(p)\left\{\int_0^1 pG_x(x_i,y,p)\nu_k(y)\left[pU(y,p)-\varphi(y)\right]dy \right. \\
&\qquad \left. + \int_0^1 pG_x(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \left(\frac{\nu_k(x)\varphi(x)}{\beta(x)}\right)'\Big|_{x=x_i}\right\} \\
&\quad - \sum_{k=K_1+1}^{K} Z_k(p)\left\{\int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y)\left[pU_y(y,p)-\varphi'(y)\right]\,dy \right. \\
&\qquad \left. + \int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy - \frac{\mu_{k-K_1}(x_i)\varphi'(x_i)}{\lambda(x_i)}\right\} \\
&\quad -p\left[Q_x(x_i,p)-\frac{H_i(p)}{\lambda(x_i)}\right], \qquad\qquad i=1,\ldots,K.
\end{aligned} \tag{4.1.3}$$

In view of (1), (2) and Theorems 2.2, 2.4, 2.6, 2.8 the left-hand side of (4.1.3) is the principal part of this system in the process $\mathrm{Re}p \to +\infty$. Therefore, we introduce the matrix

$$\Gamma \;=\; (\gamma_{ik})_{i,k=1,\ldots,K}\,,\; \gamma_{ik} = \left\{\begin{array}{ll} \left(\frac{\nu_k(x)\varphi(x)}{\beta(x)}\right)'\Big|_{x=x_i}, & k=1,\ldots,K_1, \\ \frac{\mu_{k-K_1}(x_i)\varphi'(x_i)}{\lambda(x_i)}, & k=K_1+1,\ldots,K, \end{array}\right. \tag{4.1.4}$$

related to the principal part and assume $\det\Gamma \neq 0$.

Next we define the functions

$$\begin{aligned}
B^0[Z](x,p) &= pU[Z](x,p)-\varphi(x)\,, \\
B^1[Z](x,p) &= pU_x[Z](x,p)-\varphi'(x)\,,
\end{aligned} \tag{4.1.5}$$

where $U[Z](x,p)$ is the solution of (1.2.15), (1.2.18) with the given vector Z in the form (4.1.2). The system (4.1.3) can be rewritten now in the fixed-point form

$$Z \;=\; \Gamma^{-1}\mathcal{F}(Z)\,, \tag{4.1.6}$$

where $\mathcal{F}(Z) = (\mathcal{F}_1(Z), \dots, \mathcal{F}_K(Z))$,

$$\mathcal{F}_i[Z](p) = \sum_{k=1}^{K_1} Z_k(p) \left\{ \int_0^1 pG_x(x_i, y, p)\nu_k(y) B^0[Z](y,p)\, dy \right.$$
$$\left. + \int_0^1 pG_x(x_i,y,p)\nu_k(y)\varphi(y)\, dy + \left(\frac{\nu_k(x)\varphi(x)}{\beta(x)} \right)' \Big|_{x=x_i} \right\}$$
$$+ \sum_{k=K_1+1}^{K} Z_k(p) \left\{ - \int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y) B^1[Z](y,p)\, dy \right. \tag{4.1.7}$$
$$\left. - \int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y)\varphi'(y)\, dy + \frac{\mu_{k-K_1}(x_i)\varphi'(x_i)}{\lambda(x_i)} \right\} + \Psi_i(p),$$

and

$$\Psi_i(p) = -p \left[Q_x(x_i,p) - \frac{H_i(p)}{\lambda(x_i)} \right], \quad i = 1, \dots, K. \tag{4.1.8}$$

We need to deduce a fixed-point system for the quantities $B^0[Z]$ and $B^1[Z]$ too. To this end we rewrite the system (1.2.15), (1.2.18) in terms of these quantities:

$$B^0[Z](x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG(x,y,p)\nu_k(y) \left[B^0[Z](y,p) + \varphi(y) \right] dy$$
$$- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y) \left[B^1[Z](y,p) + \varphi'(y) \right] dy + \Phi^0(x,p)$$
$$B^1[Z](x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG_x(x,y,p)\nu_k(y) \left[B^0[Z](y,p) + \varphi(y) \right] dy$$
$$+ \sum_{k=K_1+1}^{K} Z_k(p) \left\{ \frac{\mu_{k-K_1}(x)}{\lambda(x)} B^1[Z](y,p) - \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y) \right.$$
$$\left. \times \left[B^1[Z](y,p) + \varphi'(y) \right] dy \right\} + \sum_{k=K_1+1}^{K} Z_k(p) \frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)} + \Phi^1(x,p)$$

with

$$\begin{aligned} \Phi^0(x,p) &= -pQ(x,p) - \varphi(x), \\ \Phi^1(x,p) &= -pQ_x(x,p) - \varphi'(x) = \Phi^0_x(x,p). \end{aligned} \tag{4.1.9}$$

We obtain the following fixed-point equation for vector $B[Z] = (B^0[Z], B^1[Z])$:

$$B[Z] = A[Z]B[Z] + b[Z]\,, \tag{4.1.10}$$

where $A[Z] = (A^0[Z], A^1[Z])$ is the Z-dependent linear operator of B with the components

$$\left(A^0[Z]B\right)(x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG(x,y,p)\nu_k(y)B^0(y,p)\,dy - \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y)B^1(y,p)\,dy, \tag{4.1.11}$$

$$\left(A^1[Z]B\right)(x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG_x(x,y,p)\nu_k(y)B^0(y,p)\,dy + \sum_{k=K_1+1}^{K} Z_k(p) \left\{ \frac{\mu_{k-K_1}(x)}{\lambda(x)} B^1(x,p) - \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y)B^1(y,p)\,dy \right\} \tag{4.1.12}$$

and $b[Z] = (b^0[Z], b^1[Z])$ is the Z-dependent B-free term with the components

$$b^0[Z](x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG(x,y,p)\nu_k(y)\varphi(y)\,dy - \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_y(x,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy + \Phi^0(x,p), \tag{4.1.13}$$

$$b^1[Z](x,p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG_x(x,y,p)\nu_k(y)\varphi(y)\,dy + \sum_{k=K_1+1}^{K} Z_k(p) \left\{ \frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)} - \int_0^1 G_{xy}(x,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy \right\} + \Phi^1(x,p), \tag{4.1.14}$$

with $\Phi^0(x,p)$ and $\Phi^1(x,p)$ from (4.1.9). Summing up, we have proved

Proposition 4.1. *Let* $\det\Gamma \neq 0$. *Then the inverse problem in the Laplace domain with flux observations is equivalent to equation* (4.1.6) *with* $\mathcal{F}$ *given by* (4.1.7). *The solutions of these two equivalent problems are related by* (4.1.2) *with* (4.1.5). *Moreover, the pair* $B[Z] = (B^0[Z], B^1[Z])$ *in* (4.1.5) *satisfies the system* (4.1.10) *with* (4.1.11) - (4.1.14).

4.2 Analysis of direct problem system

In this section we study the system (4.1.10). Let us introduce the following basic assumptions:

$$\left.\begin{array}{l} \Phi^0 \text{ and } \Phi^1 \text{ given by (4.1.9) belong to } \hat{\mathcal{B}}_{1,\sigma_0} \text{ and } \hat{\mathcal{B}}_{\frac{1}{2},\sigma_0}, \\ \text{respectively, with some } \sigma_0 > 0\,; \\ \nu_k \in C[0,1],\ k = 1,\dots,K_1\,,\ \ \mu_l \in C^1[0,1],\ l = 1,\dots,K_2; \\ \varphi \in C^2[0,1]. \end{array}\right\} \tag{4.2.1}$$

As in Section 3.2 we note that without a restriction of generality the condition $\sigma_0 > 0$ in (4.2.1) can be replaced by $\sigma_0 \in \mathbb{R}$.

We prove first some estimates for vectors $b[Z] = \big(b^0[Z], b^1[Z]\big)$ and $A[Z] = \big(A^0[Z], A^1[Z]\big)$.

Lemma 4.1. *Let the assumptions* (2.2.2), (4.2.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$*, then the vector function* $b[Z] = (b^0[Z], b^1[Z])$*, given by* (4.1.13), (4.1.14)*, belongs to* $\mathcal{B}_{1,\sigma_0}$ *and satisfies the estimate*

$$\|b[Z]\|_{1,\sigma} \le \text{Const}\left(1 + |c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) \tag{4.2.2}$$

with any $\sigma \ge \sigma_0$ *and some constant, where* $|c|$ *is given by* (3.2.3). *Moreover, for every* $\sigma \ge \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$ *the difference* $b[Z^1] - b[Z^2]$ *fulfils the estimate*

$$\left\|b[Z^1] - b[Z^2]\right\|_{1,\sigma} \le \text{Const}\,\frac{1}{\sigma^{\alpha-1}}\left\|V^1 - V^2\right\|_{\alpha,\sigma} \tag{4.2.3}$$

with some constant.

Proof. Taking $Z = \frac{c}{p} + V$ in (4.1.13) and multiplying by $|p|$ we have

$$|p|\left|b^0[Z](x,p)\right| \le \sum_{k=1}^{K_1}\left(|c_k| + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha-1}}\right)|p|\int_0^1 \Big|G(x,y,p)\Big|dy$$
$$\times\|\nu_k\varphi\|_{C[0,1]} + \sum_{k=K_1+1}^{K}\left(|c_k| + \frac{|p|^\alpha |V_k(p)|}{|p|^{\alpha-1}}\right)\sqrt{|p|}\int_0^1 |G_y(x,y,p)|\,dy$$
$$\times\frac{\|\mu_{k-K_1}\,\varphi'\|_{C[0,1]}}{\sqrt{|p|}} + |p|\left|\Phi^0(x,p)\right|.$$

Using here the assertions of Theorems 2.1, 2.5 and definitions of the norms

$\|\cdot\|_{\gamma,\sigma}$ and taking the supremum over $\mathrm{Re}p > \sigma,\ x \in [0,1]$ we obtain

$$\left\|b^0[Z]\right\|_{1,\sigma} \leq \sum_{k=1}^{K_1}\left(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) C_1 \|\nu_k\varphi\|_{C[0,1]}$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) C_3 \frac{\|\mu_{k-K_1}\,\varphi'\|_{C[0,1]}}{\sqrt{\sigma}} + \left\|\Phi^0\right\|_{1,\sigma_0}$$

for $\mathrm{Re}p > \sigma,\ \sigma \geq \sigma_0,\ x \in [0,1]$. From this relation in view of (3.2.7) with $\gamma' = 1/2$, $\gamma = 0$ we get finally

$$\left\|b^0[Z]\right\|_{1,\sigma} \leq \mathrm{Const}\left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) + \left\|\Phi^0\right\|_{1,\sigma_0}, \quad \sigma \geq \sigma_0 \tag{4.2.4}$$

with some constant.

Next, we perform similar transformations with $b^1[Z]$ in (4.1.14) multiplying by $\sqrt{|p|}$ instead of $|p|$. We have

$$\sqrt{|p|}\left|b^1[Z](x,p)\right| \leq \sum_{k=1}^{K_1}\left(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\right)\sqrt{|p|}\int_0^1 |G_x(x,y,p)|\,dy$$
$$\times \|\nu_k\varphi\|_{C[0,1]} + \sum_{k=K_1+1}^{K}\left(|c_k| + \frac{|p|^{\alpha}|V_k(p)|}{|p|^{\alpha-1}}\right)\frac{1}{\sqrt{|p|}}$$
$$\times\left\{\left|\frac{\mu_{k-K_1}(x)\varphi'(x)}{\lambda(x)}\right| + \int_0^1 |G_{xy}(x,y,p)|\,dy\,\|\mu_{k-K_1}\,\varphi'\|_{C[0,1]}\right\}$$
$$+\sqrt{|p|}\left|\Phi^1(x,p)\right|.$$

Using here the assumption (4.2.1), the assertions of Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p > \sigma,\ x \in [0,1]$ we get

$$\left\|b^1[Z]\right\|_{\frac{1}{2},\sigma} \leq \sum_{k=1}^{K_1}\left(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) C_2 \|\nu_k\varphi\|_{C[0,1]}$$
$$+\sum_{k=K_1+1}^{K}\left(|c_k| + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right)\left(\frac{1}{\lambda_0} + C_4\right)\frac{\|\mu_{k-K_1}\varphi'\|_{C[0,1]}}{\sqrt{\sigma}} + \left\|\Phi^1\right\|_{\frac{1}{2},\sigma_0}$$

for $\sigma \geq \sigma_0$, where $\lambda_0 = \min\limits_{x\in[0,1]} \lambda(x)$. Consequently,

$$\|b^1[Z]\|_{\frac{1}{2},\sigma} \leq \mathrm{Const}\left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right) + \left\|\Phi^1\right\|_{\frac{1}{2},\sigma_0}, \quad \sigma \geq \sigma_0 \tag{4.2.5}$$

with some constant.

Observing inequalities (4.2.4), (4.2.5) and using Lemmas 2.2, 2.3 for the terms in (4.1.13), (4.1.14) we see that $b[Z] = (b^0[Z], b^1[Z]) \in \mathcal{B}_{1,\sigma}$ for $\sigma \geq \sigma_0$. Moreover, (4.2.4), (4.2.5) imply (4.2.2). To prove (4.2.3) take $Z = Z^1 - Z^2$. Then the components $b^0[Z]$ and $b^1[Z]$ of the vector $b[Z] = b[Z^1] - b[Z^2]$ are expressed by the formulas (4.1.13) and (4.1.14) with $\Phi^0 = 0$ and $\Phi^1 = 0$, respectively. Using the estimates (4.2.4) and (4.2.5) for the components of $b[Z]$ and observing that $Z^1 - Z^2 = V^1 - V^2$ we deduce (4.2.3). The lemma is proved. □

Lemma 4.2. *Let the assumptions* (2.2.2), (4.2.1) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$, *then the linear operator* $A[Z] = (A^0[Z], A^1[Z])$, *defined by* (4.1.11), (4.1.12), *is well-defined and bounded in* $\mathcal{B}_{1,\sigma}$ *and satisfies the estimate*

$$\|A[Z]\|_{\mathscr{L}(\mathcal{B}_{1,\sigma})} \leq \widetilde{C_0}\left(\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right) \tag{4.2.6}$$

for any $\sigma \geq \sigma_0$ *with a constant* $\widetilde{C_0}$. *Moreover, taking* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$, *the estimate for difference*

$$\left\|A[Z^1] - A[Z^2]\right\|_{\mathscr{L}(\mathcal{B}_{1,\sigma})} \leq \overline{C_0}\frac{1}{\sigma^\alpha}\left\|V^1 - V^2\right\|_{\alpha,\sigma} \tag{4.2.7}$$

holds for any $\sigma \geq \sigma_0$ *with a constant* $\overline{C_0}$.

Proof. Taking $Z = \frac{c}{p} + V$ in (4.1.11) we get

$$\begin{aligned}|p|\left|\left(A^0[Z]B\right)(x,p)\right| \leq \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|} + \frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right)|p|\int_0^1 |G(x,y,p)|\,dy \\ \times \|\nu_k\|_{C[0,1]}\ |p| \max_{0\leq x\leq 1}\left|B^0(x,p)\right| + \sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{|p|} + \frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right) \\ \times\sqrt{|p|}\int_0^1 |G_y(x,y,p)|\,dy\,\|\mu_{k-K_1}\|_{C[0,1]}\sqrt{|p|}\max_{0\leq x\leq 1}\left|B^1(x,p)\right|.\end{aligned}$$

Using Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p > \sigma,\ x \in [0,1]$ we deduce

$$\begin{aligned}\left\|A^0[Z]B\right\|_{1,\sigma} \ \leq\ & \sum_{k=1}^{K_1}\left(\frac{|c_k|}{\sigma} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^\alpha}\right)C_1\|\nu_k\|_{C[0,1]}\|B^0\|_{1,\sigma} + \\ & + \sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{\sigma} + \frac{\|V_k\|_{\alpha,\sigma}}{\sigma^\alpha}\right)C_3\|\mu_{k-K_1}\|_{C[0,1]}\|B^1\|_{\frac{1}{2},\sigma}\end{aligned}$$

for $\sigma \geq \sigma_0$. This implies

$$\left\|A^0[Z]B\right\|_{1,\sigma} \leq \mathrm{Const}\left(\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right)\|B\|_{1,\sigma}, \quad \sigma \geq \sigma_0 \tag{4.2.8}$$

with some constant.

Further, from (4.1.12) we get

$$\sqrt{|p|}\,\left|\left(A^1[Z]B\right)(x,p)\right| \le \sum_{k=1}^{K_1}\left(\frac{|c_k|}{|p|}+\frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right)\sqrt{|p|}\int\limits_0^1 |G_x(x,y,p)|\,dy$$

$$\times\,\|\nu_k\|_{C[0,1]}\,|p|\max_{0\le x\le 1}\left|B^0(x,p)\right| + \sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{|p|}+\frac{|p|^\alpha|V_k(p)|}{|p|^\alpha}\right)$$

$$\times\left(\frac{1}{\lambda_0}+\int\limits_0^1 |G_{xy}(x,y,p)|\,dy\right)\|\mu_{k-K_1}\|_{C[0,1]}\sqrt{|p|}\max_{0\le x\le 1}|B^1(x,p)|\,.$$

Using Theorems 2.1, 2.5 and taking the supremum over $\mathrm{Re}p > \sigma,\ x \in [0,1]$ we have

$$\left\|A^1[Z]B\right\|_{\frac{1}{2},\sigma} \;\le\; \sum_{k=1}^{K_1}\left(\frac{|c_k|}{\sigma}+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^\alpha}\right)C_2\,\|\nu_k\|_{C[0,1]}\,\|B^0\|_{1,\sigma}$$

$$+\sum_{k=K_1+1}^{K}\left(\frac{|c_k|}{\sigma}+\frac{\|V_k\|_{\alpha,\sigma}}{\sigma^\alpha}\right)\|\mu_{k-K_1}\|_{C[0,1]}\left(\frac{1}{\lambda_0}+C_4\right)\|B^1\|_{\frac{1}{2},\sigma}$$

for $\sigma \ge \sigma_0$. Finally, we obtain

$$\left\|A^1[Z]B\right\|_{\frac{1}{2},\sigma} \;\le\; \mathrm{Const}\left(\frac{|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right)\|B\|_{1,\sigma}\,,\quad \sigma \ge \sigma_0 \tag{4.2.9}$$

with some constant.

Putting together (4.2.8) and (4.2.9) we have

$$\|A[Z]B\|_{1,\sigma} \;\le\; \mathrm{Const}\left(\frac{|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha}\right)\|B\|_{1,\sigma}\,,\quad \sigma \ge \sigma_0. \tag{4.2.10}$$

Observing this relation and using Lemmas 2.2, 2.3 for the terms in (4.1.11), (4.1.12) we see that $A[Z]$ is well-defined and bounded in $\mathcal{B}_\sigma$. Moreover, (4.2.10) yields (4.2.6). Denoting $Z = Z^1 - Z^2$ the components $A^0[Z]$ and $A^1[Z]$ of the vector $A[Z] = A[Z^1] - A[Z^2]$ are expressed by the formulas (4.1.11) and (4.1.12) respectively. Using the estimate (4.2.10) for $A[Z]$ and observing that $Z = \frac{c}{p} + V$ with $c = 0$ and $V = V^1 - V^2$ we deduce (4.2.7). The lemma is proved. $\square$

Due to Lemmas 4.1, 4.2 and the contraction principle, the equation (4.1.10) has a unique solution $B = B[Z] \in \mathcal{B}_{1,\sigma}$ provided $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ and $\sigma \ge \sigma_0$ satisfy the relation

$$\widetilde{\eta}(Z,\sigma) \;:=\; \frac{|c|}{\sigma}+\frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} \le \frac{1}{2\widetilde{C_0}}\,. \tag{4.2.11}$$

From (4.1.10) we have $\|B[Z]\|_\sigma \le \left(1 - \|A[Z]\|_{\mathscr{L}(\mathcal{B}_{1,\sigma})}\right)^{-1} \|b[Z]\|_\sigma$. This, taking into consideration (4.2.2), (4.2.6) and (4.2.11), gives the estimate for the solution of (4.1.10)

$$\|B[Z]\|_{1,\sigma} \le \text{Const}\left[1 + |c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right] \tag{4.2.12}$$

with some constant.

Next we will find an estimate for $B[Z^1]-B[Z^2]$. Let $\sigma \ge \sigma_0$ and $Z^1 = \frac{c}{p}+V^1$, $Z^2 = \frac{c}{p} + V^2$ be such that (4.2.11) is valid for V replaced by V^1 and V^2 i.e. $\widetilde{\eta}(Z^j, \sigma) \le \frac{1}{2\widetilde{C}_0}$, $j = 1, 2$. Subtracting equation (4.1.10) for $Z = Z^2$ from the corresponding equation for $Z = Z^1$ we get

$$\begin{aligned} B[Z^1] - B[Z^2] &= A[Z^2]\left(B[Z^1] - B[Z^2]\right) + \left(A[Z^1] - A[Z^2]\right) B[Z^1] \\ &+ b[Z^1] - b[Z^2]. \end{aligned}$$

This implies

$$\begin{aligned} \|B[Z^1] - B[Z^2]\|_{1,\sigma} \le & \left(1 - \|A[Z^2]\|_{\mathscr{L}(\mathcal{B}_{1,\sigma})}\right)^{-1} \\ & \times\Big[\|A[Z^1] - A[Z^2]\|_{\mathscr{L}(\mathcal{B}_{1,\sigma})}\|B[Z^1]\|_{1,\sigma} + \|b[Z^1] - b[Z^2]\|_{1,\sigma}\Big]. \end{aligned}$$

Using this relation and the estimates (4.2.3), (4.2.6), (4.2.7), (4.2.11), (4.2.12) we obtain

$$\begin{aligned} &\left\|B[Z^1] - B[Z^2]\right\|_{1,\sigma} \\ &\quad \le \text{Const}\left\{\frac{1}{\sigma^\alpha}\left[1 + |c| + \frac{\|V^1\|_{\alpha,\sigma}}{\sigma^{\alpha-1}}\right] + \frac{1}{\sigma^{\alpha-1}}\right\} \|V^1 - V^2\|_{\alpha,\sigma} \end{aligned} \tag{4.2.13}$$

with some constant. Summing up, we have proved the following theorem.

Theorem 4.1. *Let the assumptions* (2.2.2), (4.2.1) *hold. Then for any* $\sigma \ge \sigma_0$ *and* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma}$ *, satisfying the inequality* (4.2.11)*, where* $\widetilde{C}_0$ *is the constant from* (4.2.6)*, equation* (4.1.10) *has a unique solution* $B[Z] = \left(B^0[Z], B^1[Z]\right)$ *in* $\mathcal{B}_{1,\sigma}$*. This solution satisfies estimate* (4.2.12)*. Moreover, for every* $\sigma \ge \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$*,* $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\alpha,\sigma}$ *such that* $\widetilde{\eta}(Z^j, \sigma) \le \frac{1}{2\widetilde{C}_0}$*,* $j = 1, 2$*, the difference* $B[Z^1] - B[Z^2]$ *fulfils estimate* (4.2.13)*.*

4.3 Existence and uniqueness for inverse problem

In this section we study the inverse problem with flux observations in the fixed-point form (4.1.6) and thereupon infer a result for the corresponding generalized inverse problem in the time domain.

Theorem 4.2. *Assume that* (2.2.2), (4.2.1) *hold and*

$$\nu_k \in C^2[0,1]\,, \quad k = 1, \ldots, K_1. \tag{4.3.1}$$

Moreover, let $\det\Gamma \neq 0$ *for* Γ*, given by* (4.1.4)*, and the vector* $\Psi = (\Psi_1, \ldots, \Psi_K)$*, defined in* (4.1.8)*, satisfies*

$$\Psi \;=\; \frac{d}{p} + Y \in \mathcal{M}_{d,\alpha,\sigma_0} \tag{4.3.2}$$

with some $d \in \mathbb{R}^K$*. Then there exists* $\sigma_1 \geq \sigma_0$ *such that equation* (4.1.6) *has a unique solution* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\alpha,\sigma_1}$*. Here* $c = \Gamma^{-1}d$*. The solution is unique in the union of spaces* $\bigcup\limits_{\substack{\sigma\geq\sigma_1\\ c\in\mathbb{R}}} \mathcal{M}_{c,\alpha,\sigma}$*.*

Proof. Setting $c = \Gamma^{-1}d$ the problem (4.1.6) in $\mathcal{M}_{c,\alpha,\sigma}$ is equivalent to the following equation for V in $(\mathcal{A}_{\alpha,\sigma})^K$:

$$V = F(V)\,, \tag{4.3.3}$$

where $F = \Gamma^{-1}\widetilde{F}$ and

$$\widetilde{F} = \widetilde{L_0}\left(\frac{c}{p} + V, B[Z]\right) + \widetilde{L_1}\left(\frac{c}{p} + V\right) + Y\,, \tag{4.3.4}$$

$\widetilde{L_0}$ is the bilinear operator of $Z \in \mathcal{M}_{c,\alpha,\sigma}$, $B = (B^0, B^1) \in \mathcal{B}_{1,\sigma}$ given by

$$\left(\widetilde{L_0}(Z,B)\right)_i(p) = \sum_{k=1}^{K_1} Z_k(p) \int_0^1 pG_x(x_i,y,p)\nu_k(y)B^0(y,p)\,dy$$
$$- \sum_{k=K_1+1}^{K} Z_k(p) \int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y)B^1(y,p)\,dy\,, \quad i = 1,\ldots,K, \tag{4.3.5}$$

and $\widetilde{L_1}$ is the linear operator of $Z \in \mathcal{M}_{c,\alpha,\sigma}$ defined by

$$\left(\widetilde{L_1}(Z)\right)_i(p) \;=\; \sum_{k=1}^{K_1} Z_k(p) \left\{ \int_0^1 pG_x(x_i,y,p)\nu_k(y)\varphi(y)\,dy \right.$$
$$\left. + \left(\frac{\nu_k(x)\varphi(x)}{\beta(x)}\right)'\Big|_{x=x_i} \right\} - \sum_{k=K_1+1}^{K} Z_k(p) \left\{ \int_0^1 G_{xy}(x_i,y,p)\mu_{k-K_1}(y)\varphi'(y)\,dy \right.$$
$$\left. - \frac{\mu_{k-K_1}(x_i)\varphi'(x_i)}{\lambda(x_i)} \right\}, \qquad i = 1,\ldots,K. \tag{4.3.6}$$

As in the proof of Theorem 3.2 we make use of the fixed-point argument in the balls $D_{\alpha,\sigma}(\rho)$ defined by (3.3.26).