THESIS ON NATURAL AND EXACT SCIENCES B40

Solitons and solitary waves in media with higher order dispersive and nonlinear effects

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Hereby I declare that this doctoral thesis, my original investigation and achievement, submitted for the doctoral degree at Tallinn University of Technology has not been submitted for any degree or examination.

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VÄITEKIRI LOODUS- JA TÄPPISTEADUSTES

Kõrgemat järku mittelineaarsete ja dispersiivsete efektide mõju solitonide ja üksiklainete tekkele

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Introduction

Solitons in contemporary understanding were first described by N.J. Zabusky and M.D. Kruskal in 1965 and they form now a paradigm in mathematical physics. Soliton is a solitary wave with finite energy and the necessary conditions of its existence include nonlinearity and dispersion. Soliton dynamics is one of the hot topics due to wide applications in hydrodynamics, electronics, solid mechanics, biophysics and other disciplines, and accompanying theoretical deepness (Ablowitz and Clarkson 1991, Fokas and Zakharov 1993). When the classical, so called Korteweg–de Vries (KdV) soliton is based on quadratic nonlinearity and cubic dispersion, then the contemporary wave dynamics (especially in solid mechanics) needs much more complicated mechanisms to be taken into account (Jeffrey and Engelbrecht 1994). In this case the celebrated inverse scattering method elaborated for classical cases cannot be used and numerical methods give the only way to analyse the complicated situations. The pseudospectral methods and the idea of spectral analysis are without any doubt useful due to additional information on the energetical background of wave structures beside their spatial-temporal profiles. This is deeply related to the concept of solitons as the particle-like waves with certain energy. Earlier studies in the Institute of Cybernetics and Tallinn University of Technology have given good results in the spectral analysis of the classical KdV equation and some of its modifications (Engelbrecht 1991, 1995; Salupere 1995, Salupere et al. 1996).

This thesis deals with solitary wave and soliton formation mechanism for wave dynamics in solids with microstructure. These problems are essential in crystalline solids where the dislocations or shape–memory effects are of importance. Such materials are now widely used in contemporary high technology. The elastic potential for such materials can be described by a quartic function with two minima and the microstructure leads to the dispersion described by higher order derivatives. Therefore the studied model equation describing the evolution of the one dimensional wave propagation includes quartic nonlinearity and both, the third and the fifth order dispersive terms. This equation is not integrable, but earlier numerical simulations have proved the existence of soliton–type solutions with certain radiation in some particular cases.

This thesis is organised as follows: Section 1 starts with the description of first observations of solitons made by Scottish engineer S. Russell. It is followed by describing the concept of solitons. Various studies of KdV- and KdV-like equations are also described in Section 1. As proposed model equation is nonintegrable numerical method has been used. Section 2 gives short overview of pseudospectral method and discrete spectral analysis. Section 3 is dedicated to the wave propagation in microstructured solids. At first, the background of the model equation is presented. Secondly, the essence of the higher order dispersive and nonlinear effects is discussed. Thirdly, numerical studies of the model equation are summarised. In addition to three novel problems, which are the main topics of the present thesis, results of previous studies are revisited. Final comments and main results of this thesis have been elaborated in Section 4.

Results of the present thesis are published in six papers. Proposed model equation with harmonic initial condition and normal dispersion is studied in **Publication I** and **Publication II**. Results of this study have been presented by the author in the EUROMECH Colloquium 436 „Nonlinear Waves in Microstructured Solids“ (Tallinn, Estonia, 2001) and in the 15th Nordic Seminar on Computational Mechanics (Aalborg, Denmark, 2002).
The second problem of the proposed model equation includes localized initial condition with normal dispersion. Results of this study have been published in **Publication III**, **Publication IV** and **Publication V** and presented in 21st International Congress of Theoretical and Applied Mechanics, ICTAM04 (Warsaw, Poland, 2004).
Localized initial condition with mixed dispersion has been studied as the third problem. **Publication VI** includes main results of this study. Results of the third study have been presented in the 4th IMACS International Conference "Nonlinear Evolution Equations and Wave Phenomena: Computation and Theory" (Athens, USA, 2005).

The aims of the thesis are related to analysis of the influence of higher-order nonlinearity and dispersion to the wave propagation in microstructured materials. Main goals of this thesis have been:

- to find and analyse numerical solutions for proposed model equation over wide range of dispersion parameters under different initial conditions;
- to define solution types and detect how do solution types depend on dispersion parameters as well as on the amplitude of initial excitation;
- to determine the dispersion parameters of the media and amplitude of the initial excitation permitting formation of stable solitonic structures or propagation of stable solitary waves;
- to examine various properties (recurrence, super-recurrence, periodicity etc.) of the solutions.

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1 Theoretical background

1.1 Short history of KdV- and related equations. Introduction of solitons.

Once Universe was created, solitons were created as well. Like with many other great phenomena of Nature, it took very long time until mankind was able to observe first time those waves, with unique properties. In 1834 Scottish civil engineer, Scott Russell, was observing a boat drawn by a pair of horses along the canal between Edinburgh and Glasgow. Once the boat was stopped, surrounding water was set in motion, Russell describes it as follows – "rolled forward with great velocity assuming the form of a large solitary elevation, a rounded smooth and well-defined heap of water, which continued its course without change of form or diminution of speed". According to Russell the giant wave was about thirty feet long and one-and-a-half feet high. Russell, who was on horseback, rode down the towpath following the wave until it eventually petered out a mile or so further along the canal. The observations made by Scott Russell are considered as first discovery of solitons. However, he called this hump-shaped disturbance 'a great wave of translation', but it soon became known as a „solitary wave". Term „soliton" was brought into practice more than a century later.
Russell was intrigued enough by his solitary wave to carry out some laboratory experiments. He found it is easy to generate a solitary wave by dropping a weight into water at one end of a long rectangular tank. From this he discovered an empirical equation describing the wave: the speed of each wave depends on the depth of the undisturbed water, the maximum height of the wave above the level of the undisturbed water (the amplitude of the wave) and the acceleration due to gravity. He noted that higher waves travel faster than smaller waves. Russell also made several other acute observations which were not well understood for more than a century.
In the 1870's two great physicists, the French mathematical physicist Joseph Boussinesq at the University of Paris and the English Lord Rayleigh, showed independently how Russell's solitary wave arose and also explained it mathematically. They showed that it is a wave of smallish amplitude that can arise in a shallow layer of a frictionless and incompressible liquid - such as water. Boussinesq and Rayleigh deduced Russell's empirical formula for the speed of a solitary wave and related its height to the distance. They also showed that tall waves are narrow, or short, and small waves are wide, or long.

Year 1895 is the second very important mile-stone in solitons' history. Namely, the Dutchmen Diederik Korteweg and G. de Vries found an equation governing waves of small amplitude in shallow water and showed that the wave of Boussinesq and Rayleigh is a solution of this equation, now called the 'KdV equation'. Russell's solitary wave should in theory have travelled unchanged for ever, but in practice the viscosity of the water slowly dissipated the energy of the wave while Russell rode along the towpath of the canal. This odd solution to the KdV equation was dutifully recorded in many books, but it was not well understood for many years.

Another remarkable discovery, which at first sight had nothing to do with solitary waves was made by E. Fermi, J. Pasta and S. Ulam (1955) as they studied the heat transfer problem, that is, the flow of incoherent energy in solid modelled by nonlinear springs. In fact, the one-dimensional monoatomic lattice is possibly the simplest discrete structure with which to study lattice dynamics. It provides a model of quasi-one-dimensional crystals within which a clear understanding of the dynamics is more readily attainable. When the interactions between particles are harmonic, which is, when particle displacements are very small deviations from equilibrium positions, the equations of motion of the particles can be decoupled and the dynamics of the lattice can be described by superpositions of normal modes, represented by sinusoidal waves, which are mutually independent.

In 1955 E. Fermi, J. Pasta and S. Ulam (FPU) intended to verify this assumption by computer simulations on a one-dimensional lattice. They examined the dynamical behaviour of a chain with nonlinear interactions between atoms (mass points), expecting that the initial energy would eventually be shared among all degrees of freedom of lattice. Much to their surprise, the system did not approach energy equipartion, that is, the energy did not spread throughout all the normal modes, but returned almost periodically to the originally excited mode. This remarkable near recurrence phenomena, known nowadays as the FPU problem was confirmed by some other scientists, that the nonlinear terms did not guarantee the approach of the system to thermal equilibrium.

This work inspired Norman Zabusky at Bell Telephone Laboratories and Martin Kruskal at Princeton University in 1965 to study the continuum limit of the FPU problem solving the corresponding nonlinear KdV equation numerically. In doing so, Zabusky and Kruskal made a surprising discovery when they examined the solitary wave solution. When solitary waves meet each other, they move through each other so that they emerge with their original shapes, sizes and speeds. It was almost as if the waves were governed by the principle of superposition associated with linear waves. Zabusky and Kruskal coined the word 'soliton' for the solitary-wave solution of the KdV equation. Solitary waves keep their character after interacting with one another. In fact they seem to be behaving like interacting elementary particles such as electrons or protons, a significant point that we shall come back to later.

The paper by Zabusky and Kruskal started an intense burst of research related to this phenomenon. Soon Kruskal and other colleagues proved the properties of solitons mathematically. With ingenuity and persistence they found a linear problem 'behind' the KdV equation. Gardner et al. (1967) showed that the

problem of solving the nonlinear KdV equation could be broken down into solving two linear problems analogous to those of the scattering of an electron in a one-dimensional electric field, and the inverse problem of finding an electric field from the energy levels and the scattering of electrons. This inverse problem is analogous to the problem of finding the shape and size of a drum when you hear it beaten in different ways, rather than working out the sounds of drumbeats from the shape and size of the drum. The inverse problem gives the name 'inverse scattering transform' to this theory.
One has to mention that it was the first time in history of science when new paradigm (the concept of solitons can be called as the paradigm) was introduced by discoveries of numerical experiments.
Since then two directions have evolved in examining the nonlinear problems (Zabusky 1981):

- Mathematical approach to the generic properties of classical discrete Hamiltonian systems.
- Numerical experiments based on partial differential equations.

1.2 Concept of solitons

Solitons are found in different matters:

i) the one-dimensional (1D) solitons: waves in shallow water, signals in optical fibres;

ii) the two-dimensional (2D) solitons: magnetic flux domains in superconductors, vortex-antivortex in fluids;

iii) the three-dimensional (3D) solitons: magnetic monopoles in gauge theories, skyrmions, i.e. soliton models for protons and neutrons.

Different authors have given different definitions of solitons, most widely are used the following:

- soliton definition given by Scott Russell – solitary wave is localized wave that propagates along one space direction only, with undeformed shape.

- soliton definition given by Zabusky and Kruskal – soliton is a large amplitude coherent pulse or very stable solitary wave, the exact solution of a nonlinear wave equation, whose shape and speed are not altered by collision with other solitary waves.

In the present study the soliton definition given by Drazin (1983) is used. By this definition soliton

- is a solitary wave which conserves its speed and shape;
- can interact with other solitons nonlinearly and after interaction restores its speed and shape, i.e., their interaction is elastic.

These solitons are clearly visible physical entities. However, beside visible solitons, other similar entities exist – hidden solitons. The concept of hidden (virtual) solitons is described in detail in Salupere et al. (1996), Salupere (2000), **Publication I** and **Publication II**. The main characteristics of hidden solitons are as follows: hidden solitons can emerge together with visible solitons (for example from harmonic excitation), hidden solitons have very small energy and amplitude, these interact with visible solitons and cause distinct changes in visible soliton amplitudes and trajectories during interaction, can be detected in wave profiles for a short time interval only when several soliton interactions have taken place, if ever. Physical essence of visible and hidden solitons is the same (Engelbrecht and Salupere 2005).

1.3 Original KdV

For future discussions the nature of the KdV equation must be explained. The KdV equation can be presented in following form

$$u_t + uu_x + du_{xxx} = 0\,, \qquad (1)$$

where indices denote the differentiation and *d* stands for the dispersion parameter. By Ablowitz and Clarkson (1991) one can describe the following properties — first, the KdV equation describes the propagation of solitary waves; second, it describes the emergence of solitary waves from the initial excitation (for example harmonic excitation); third the KdV equation is the simplest nonclassical partial differential equation possessing

- the minimum number of independent variables (2);
- the lowest order of the derivative not considered classically (3);
- the fewest terms of that order (1);
- the simplest such a term (an unmixed derivative);
- the smallest number of terms (1) containing the other derivative which is of the first order;
- the simplest structure for this term (linear);
- the simplest additional term to make the equation nonlinear (quadratic).

As the celebrated KdV equation (1) is an integrable equation its stationary solution in a frame moving with velocity *c* can be found analytically. If to substitute

$$u(x,t) = u(\xi), \text{ with } \xi = x - ct \qquad (2)$$

into equation (1) then for *u* one gets a third-order nonlinear ordinary differential equation (ODE)

$$-cu_\xi + uu_\xi + du_{\xi\xi\xi} = 0. \qquad (3)$$

Here *c* is the phase velocity and *d* the dispersion parameter. The solution of ODE (3) can be found directly by integrating the differential equation (3) three times. In the case of asymptotic boundary conditions

$$u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi} \to 0, \text{ if } \xi \to \pm\infty \qquad (4)$$

the solution can be expressed in the following form

$$u = 3c\,\mathrm{sech}^2\, 0.5\sqrt{\frac{c}{d}}(\xi - \xi_0), \qquad (5)$$

where ξ_0 is an arbitrary constant (cf. Zabusky and Kruskal 1965). Here the quantity $A = 3c$ can be considered as the amplitude of the soliton, i.e. the higher the soliton, the higher its velocity.

1.4 Higher order KdV-like evolution equations

KdV equation is obtained at a certain degree of approximation (higher-order dispersive effects have been neglected), in many cases the physical reality needs better accuracy. Under certain circumstances it may happen that the fifth-order dispersion has a significant role in the wave propagation process. For this reason, beside the KdV equation, the so called reduced fifth-order KdV (RFKdV) with the same nonlinear term as the KdV equation is of importance. I shall call equations which consist of third- and fifth order dispersive terms and nonlinearity as in the KdV, fifth-order KdV equations (FKdV for short). If higher order nonlinear terms have been also included, these equations will be called as fifth-order KdV-like equations (FKdV-like equations).

While studying papers of other authors corresponding to higher order KdV-like evolution equations I focused on following fields of researches:

1. derivation of model equations based on experiments or theoretical approaches;
2. derivation of (analytical) methods for finding any kind of solutions for proposed model equations;
3. analysis of analytically or numerically found solutions; stability of solutions, interaction of solitary waves, etc.

I have paid less attention on studies that are related to derivation of numerical methods for finding solutions for proposed model equations. In the following chapters, overviews of papers are given according to above mentioned categorization.

Reduced fifth order KdV equation

Under certain circumstances it may happen that the coefficient of the third order derivative in the KdV equation becomes very small or even zero and the higher order dispersive terms are accounted for which may balance the nonlinear effect. Keeping the accuracy of quadratic nonlinearity, the same nonlinear term appears like in the celebrated KdV equation.

Kakutani and Ono (1969) showed that when the angle φ between the propagation direction of the magneto-acoustic wave in a cold collision-free plasma and the external magnetic field becomes a critical angle given by $\varphi_c = \tan^{-1}\left(\sqrt{\frac{m_i}{m_e}} - \sqrt{\frac{m_e}{m_i}}\right)$, m_i and m_e being the masses of ion and of electron respectively, then the third order derivative term in the KdV equation vanishes and is replaced by the fifth order dispersive term.

Nagashima (1979) observed solitary waves in the transmission line (see Fig. 1) which is described by the nonlinear equation

$$u_t + uu_x - u_{5x} = 0 . \qquad (6)$$

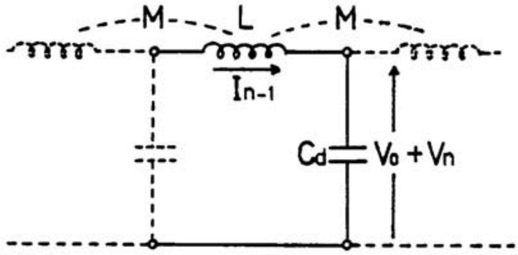

Fig. 1. The n-th section (solid lines) of the transmission line; V_0 and V_n denote the D.C. bias and A.C. (signal) voltage, respectively. L: inductance, M: mutual inductance, C_d : differential capacitance of the nonlinear capacitor.

In the case of this particular study solitary wave is expressed as

$$u = \lambda f\left(\lambda^{\frac{1}{4}}\left(x - \lambda t\right)\right) \tag{7}$$

for an arbitrary value of λ, where the function f describes the shape of the wave. Solitary wave, described by Eqs. (6) and (7) is stable irrespective of the small scattering of the capacitance and the inductance of the line. In paper Nagashima and Kuwahara (1981) computer simulations of equation

$$u_t + uu_x - \gamma^2 u_{5x} = 0, \tag{8}$$

where γ^2 is a constant of a small value, were carried out. It is found that the initial wave breaks into a train of solitary waves, which is similar to solutions of the KdV equation. One solitary wave with oscillatory tails is stable; this is expressed by Eq. (7). The interaction of two solitary waves is classified into two types, according to the relative amplitudes of waves.

Exact solutions for Eq. (6) were found in 1981 by Kano and Nakyama (1981); Yamamoto and Takizawa (1981).

According to Yamamoto and Takizawa, equation

$$u_t + \left(\frac{105}{16}\right)\alpha^2 uu_x - u_{5x} = 0 \tag{9}$$

has solution

$$u = \frac{\beta}{\alpha}\,\mathrm{cn}^4\left(\frac{1}{2\sqrt{2}}(\alpha\beta)^{\frac{1}{4}}(\xi - \xi_0), \sqrt{\frac{1}{2}}\right), \tag{10}$$

where cn(z,k) is the Jacobi cn-function of modulus k, with $\xi = x - \left(\frac{21\alpha\beta t}{8}\right)$, positive constants α, β, γ and any constant ξ_0.

In the case of a small wave amplitude, the recurrence of initial waveform for Eq. (6) is observed in numerical studies by Yoshimura and Watanabe (1982). If the amplitude is increased beyond a certain threshold, the solution depends on the initial condition.

Fifth-order KdV equation

KdV-like equation including both, the third and the fifth order dispersive terms was first proposed by Kakutani and Ono (1969) in their study related to the problem of magneto-acoustic wave in a cold collision-free plasma. The third- and the fifth order dispersive terms appear in the case of propagation near the critical angle.
Steady solutions of the fifth-order KdV (FKdV) equation have been first found numerically by Kawahara (1972). Kawahara examined the FKdV equation

$$u_t + 1.5uu_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0\,. \qquad (11)$$

This is a non-integrable differential equation and therefore one cannot find its solitary wave solutions analytically. Kawahara integrated the evolution equation (11) numerically with respect to the space variable *x* in the case of asymptotic boundary conditions. These "numerical solitons" or solitary waves have a bell-like shape. If the coefficient of the third order derivative is dominant over that of the fifth order, then a monotone solitary wave solution was found. If the fifth order derivative is dominating over the third order one, oscillatory structure of the solitary waves forms. In this case these solitons are called "Kawahara solitons".
Yamamaoto and Takizawa (1981) have found solution for FKdV equation

$$u_t + \left(\frac{105}{16}\right)\gamma^2 uu_x + \left(\frac{13}{4}\right)\delta u_{xxx} - u_{xxxxx} = 0\,. \qquad (12)$$

According to them this equation has solution

$$u = \left(\frac{\delta}{\gamma}\right)^2 \operatorname{sech}^4\left(\frac{1}{4}\delta^{\frac{1}{2}}(\eta - \eta_0)\right), \qquad (13)$$

where $\eta = x - \left(\frac{9\delta^2 t}{4}\right)$, γ is positive constant and η_0 is arbitrary.

Hunter and Schuerle (1988) have developed equation

$$u_t + uu_x + \sigma u_{3x} + u_{5x} = 0 \qquad (14)$$

as a model equation for capillary-gravity waves when the Bond number σ is just less than critical value of $\frac{1}{3}$. They have constructed traveling wave solutions which do not decay at zero as $|x| \to +\infty$. Instead, when *x* is large, these solutions approach small amplitude oscillations. However, there exist branches of traveling wave solutions to the water wave equations, which are perturbations of supercritical elevation solitary waves, and which bifurcate from Froude number 1 and Bond number $\frac{1}{3}$.

Watanabe and Jiang (1993) have used higher-order approximation of the reductive perturbation method in order to find higher order solutions for HKdV equation

$$u_t + uu_x + u_{3x} + \varepsilon u_{5x} = 0. \quad (15)$$

Grimshaw et. al. (1994) have constructed solitary wave solutions for FKdV equation,

$$u_t + 6uu_x + u_{3x} + u_{5x} = 0, \quad (16)$$

where the oscillations decay at infinity. These waves arise as a bifurcation from the linear dispersion curve to that wavenumber where linear phase speed and group velocity coincide.

Jeffrey and Mohamad (1991) have presented a direct method for the construction of travelling wave solutions to FKdV equation

$$u_t + auu_x + bu_{3x} + cu_{5x} = 0. \quad (17)$$

According to them the travelling wave solution can be presented in the following form

$$u(x,t) = -\frac{105b^2}{169ac}\operatorname{sech}^4\left(\pm\frac{1}{2}\left(-\frac{b}{13c}\right)^{\frac{1}{2}}\left(x + \frac{36b^2}{169c}t\right)\right) \quad (18)$$

Behaviour of the equation

$$u_t + uu_x + au_{3x} - bu_{5x} = 0 \quad (19)$$

has been studied by Nagashima (1984). It is found that in this system both regular and chaotic motions exist, the type of the motion is determined by the initial condition and parameters *a,b*. It is also determined that chaos originates because the solitary waves loose their identity in a collision.

Examining the Eq. (19) Kawahara and Takaoka (1988) have found that if the initial configurations of solitons are appropriate, the solitons can form stationary bound states and propagate steadily keeping the inter-pulse distances unchanged. However, increasing the initial value from a fixed point with centre-like singularity, periodic motions show frequency down-shifts and lead to chaotic behaviour.

The relation between the pole (movable singularities) distribution and the steady pulse solution is investigated for FKdV

$$u_t + auu_x + bu_{3x} + cu_{5x} = 0 \quad (20)$$

by Takaoka (1988). Poles are generally found to distribute fractally and to form natural boundary, whereas the steady pulse has a prominent oscillatory tail structure. Periodic distribution of poles is found for some specific cases where the solutions are represented by hyperbolic functions.

Solitary wave stability of FKdV equation

$$u_t + 6uu_x + u_{3x} + \varepsilon^2 u_{5x} = 0 \quad (21)$$

has been studied by Pomeau et. al. (1988). They have shown that the solution ceases to be strictly localized but develops an infinite oscillating tail.

Interactions of waves described by this equation have been studied by Grimshaw and Malomed (1993).

Karpman (1993) has shown that planar solitons, found in equation (11) proposed by Kawahara, are unstable with respect to bending if the coefficient at the fifth-order dispersive term is positive and stable if it is negative.
Hai and Xiao (1995) have constructed a general soliton solution of the FKdV Eq. (11) in the first order approximation for the travelling wave case. The perturbed soliton is lower and narrower than the unperturbed soliton.
By making use of the techniques of exponential asymptotics Grimshaw and Joshi (1995) have studied the Eq. (14) proposed by Hunter and Scheurle. They have found that solutions on this equation form an one-parameter family characterized by the phase shift of the trailing oscillations.
Stability of solitary wave solutions of the FKdV equation

$$u_t + 6uu_x + u_{3x} + u_{5x} = 0 \tag{22}$$

have been studied by Buryak and Champneys (1997), Calvo et al. (2000), Dias and Kuznetsov (1999). Based on asymptotic theory Buryak and Champneys found that half of the two-pulses solutions are stable. The other half develops a mode of instability that causes the wave to split into two simpler waves travelling at different speeds. According to Calvo et al. (2000) the branch of the so-called elevation waves in unstable, whereas the branch of depression wave is stable. Dias and Kuznetsov (1999) showed that the Hamiltonian is bounded from below for fixed momentum. If there exists a solitary wave solution that realizes this minimum, then it is stable with respect to not only small perturbations but also finite ones.

Fifth-order KdV-like equations

Next set of equations consist of both third and fifth-order dispersion and higher order nonlinear terms. In this case nonlinearity is expressed by term $u^p u_x$, where p>0 (if not stated differently in a particular paper).

$$u_t + \alpha u^p u_x + \beta u_{3x} + \gamma u_{5x} = 0 \tag{23}$$

Karpman and Vanden-Broeck (1995) showed numerically that the fifth order derivative term in (23) is of critical importance for the soliton stability at sufficient high p. At $\gamma = 0$ and $p \geq 4$ the soliton solutions are unstable, in agreement with theory of collapse instability. On the other hand, no instability was detected in their calculations at $\gamma < 0$. Finally they suggested that the results obtained can be, in principle, checked experimentally by modelling Eq. (23) in electronic transmission line.
Dey et al. (1996) have obtained exact stationary soliton solutions

$$u(\xi) = \left(\frac{D}{8} \frac{(p+1)(p+4)(3p+4)}{p+2} \right)^{\frac{1}{p}} \operatorname{sec} h^{\frac{4}{p}} \left(\frac{p\xi\sqrt{D(p^2+4p+8)}}{4(p+2)} \right), \tag{24}$$

where $\xi = x - Dt$, $\alpha = \beta = 1$, $D > 0$, $\gamma < 0$ and $\varepsilon \equiv (D|\gamma|^{\frac{1}{2}}) = \dfrac{2(p+2)}{p^2+4p+8} < 1$.

for Eq. (23) in the case for any $p > 0$ in case $\alpha\beta > 0$, $D\beta > 0$, $\beta\gamma < 0$ (where D is the soliton velocity).

According to their study solutions are unstable with respect to small perturbations in case of $p \geq 5$. In particular, it is shown that for any p these solitons are lower and narrower than the corresponding $\gamma = 0$ solitons (KdV-solitons). Finally, for $p = 2$ they have obtained an exact stationary soliton solution even when D, α, β, γ are all > 0.

The stability of the solitons of the Eq. (23) with arbitrary power nonlinearities is studied by Karpman (1996A). It is shown that a sufficient condition of the soliton stability with respect to small perturbations is a minimum of the Hamiltonian, constrained by the constancy of the momentum. Results obtained in this paper demonstrate that higher order dispersion, under certain conditions, stabilizes soliton instabilities, which is in agreement with numerical experiments. In other paper by Karpman (1996B) Eq. (23) is studied by means of Lyapunov approach. From the results obtained it follows that the solitons are stable at $p < 8$ where p is the power of nonlinearity.

Tan et al. (2002) have studied the evolution of perturbed embedded solitons in a general Hamiltonian FKdV-like equation

$$u_t + u_{3x} + u_{5x} + [N(u)]_x = 0, \qquad (25)$$

where $N(u) = \alpha_0 u^2 + \alpha_1 u u_{2x} + \alpha_2 u_x^2 + \alpha_3 u^3$. They have shown that when an embedded soliton is perturbed, it sheds continuous-wave radiation in front of the embedded soliton. The amplitude of this continuous-wave is not minimal in general. Behind the embedded soliton, no flat shelf is created. Conditions under which perturbed embedded soliton will decay or persist are also obtained.

1.5 Summary of higher-order KdV-like equations

Model equations, related to the KdV equation, are rich in their nature. To summarise, one can bring out following properties:

- dependence on the parameters;
- solutions can be stable and unstable;
- solitons can be detected;
- chaotic and periodic regimes may occur;
- recurrence and super-recurrence phenomena can be examined.

In most cases analysed above, the nonlinearity is of the quadratic type. The Eqs. (23) and (25), however demonstrate the growing complexity of the solution for more complicated nonlinearities. In the present thesis wave propagation mictrostructured media is modelled by higher-order KdV-like equation and the attention is paid to higher order nonlinearity that might be balanced by dispersive terms. This analysis serves for testing the methods and comparing the results of our studies.

2 Introduction of numerical method

Proposed model equation is nonintegrable and therefore numerical integration algorithms have to be used in order to find solutions for the equation under investigation. Several numerical methods have been developed for solving nonlinear evolution equations. To name some of these: finite difference method, the Galerkin method, the Hopscotch method, the Fourier expansion method, the split-step Fourier method, the spectral methods, the pseudospectral method, etc. Salupere (1995, 1997) have examined the advantages of pseudospectral method compared with other numerical methods for solving the KdV equation. According to this study the pseudospectral method is adequately accurate and stable for solving the KdV related equations with harmonic initial condition. This is in agreement with works of other authors who have established that the modified pseudospectral method is sufficiently accurate, stable and fast (Fornberg and Sloan 1994, Fornberg 1998).

2.1 The essence of the pseudospectral method

The pseudospectral method was first proposed by Kreiss and Oliger (1972) in the following form. Let the initial condition $u(x,0)$ be given on the interval 2π. The space grid is formed by n points with

$$\Delta x = \frac{2\pi}{n}. \tag{26}$$

The discrete Fourier transform (DFT) is defined by

$$U(\omega,t) = \mathrm{F}u = \sum_{j=0}^{n-1} u(j\Delta x,t)e^{(\frac{-2\pi i j\omega}{N})} \tag{27}$$

and the inverse discrete Fourier transform (IDFT) by

$$u(x,t) = \mathrm{F}^{-1}U = \sum_{\omega} U(\omega,t)e^{(\frac{2\pi i j\omega}{n})} \tag{28}$$

where i is the imaginary unit and

$$\omega = 0, \pm 1, \pm 2, \ldots, \pm(\frac{n}{2}-1), -\frac{n}{2}. \tag{29}$$

In expressions (27) and (28) F denotes the Fourier transform and F^{-1} the inverse Fourier transform. Fast Fourier Transform (FFT) algorithm is applied to find the Fourier transform (Bracewell 1972). Space derivatives are then given by

$$\frac{\partial u}{\partial x} = \mathrm{F}^{-1}(i\omega\mathrm{F}),$$
$$\frac{\partial^2 u}{\partial x^2} = -\mathrm{F}^{-1}(\omega^2\mathrm{F}u), \tag{30}$$
$$\frac{\partial^n u}{\partial x^n} = \mathrm{F}^{-1}\left[(i\omega)^n \mathrm{F}u\right].$$

In time, the finite difference leap-frog (LF) scheme was proposed to use by Kreiss and Oliger (1972). For example, the KdV-like equation (45) with quartic potential (46) leads to the following straightforward pseudospectral approximation

$$u(x,t+\Delta t) = u(x,t-\Delta t) - 2\Delta t(u^3 - u)\mathrm{F}^{-1}(i\omega \mathrm{F}u) + \ldots$$
$$\ldots + 2d\Delta t\mathrm{F}^{-1}(i\omega^3 \mathrm{F}u) - 2b\Delta t\mathrm{F}^{-1}(i\omega^5 \mathrm{F}u) \qquad (31)$$

The LF scheme has a disadvantage. Namely, one has to use a very small time step to get stable results. Furthermore, there are not proper criterions for choosing a suitable (in the sense of stability of the numerical scheme) size for the time step. Runge-Kutta type methods are found to be more stable than the LF scheme (Salupere 1997).

For analyses of numerical results discrete spectral analysis is used, i.e., in order to characterise the space-time behaviour of the solution Fourier transform related spectral quantities are used (Salupere et al. 1996). If Fourier transform is defined by Eq. (27) then spectral amplitudes are defined in the following form:

$$S_\omega(t) = \frac{2|U(\omega,t)|}{N}, \omega = 1,\ldots,\frac{N}{2}-1 \text{ and } S_\omega(t) = \frac{|U(\omega,t)|}{N}, \omega = \frac{N}{2} \qquad (32)$$

These spectral characteristics carry additional information about the internal structure of waves. For example one can detect how the energy is shared between different spectral characteristics. Based on this information one can detect several properties of the solutions – solutions periodicity, recurrence, super-recurrence. In same case the number of solitons in the train of solitons can also be estimated by making use of spectral characteristics (Salupere et al. 1996).

3 Wave propagation in microstructured solids

3.1 Overview of properties of microstructured solids

A special higher order KdV-like evolution equation has been proposed by G.A. Maugin. As this equation is considered to be the model equation in the framework of this thesis, more detailed description of the derivation of the equation is presented (Maugin 1995).

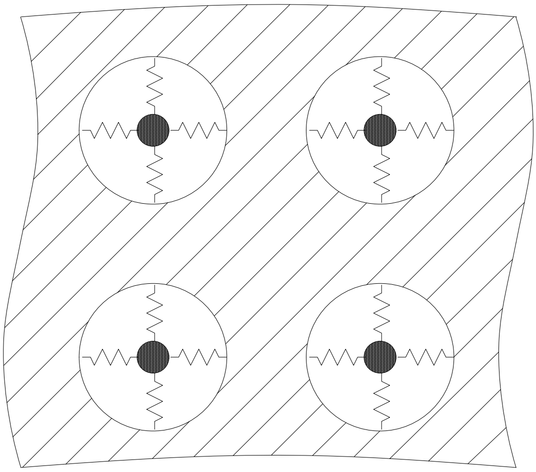

Fig. 2. The Maxwell-Rayleigh model of anomalous dispersion (foreign inclusions linearly or nonlinearly elastically connected to the elastic matrix).

Material description of continuum mechanics is considered in order to accommodate easily nonlinear phenomena. The material point X has for image x such that $x = \chi(X,t)$ where t is time. This defines the deformation of the elastic matrix, of which the displacement is $u(X,t) = x(X,t) - X$. But there is a continuous distribution of "atoms" at each X with relative displacement $\zeta(X,t)$ with respect to the matrix. That is, the instantaneous physical position of these atoms is given by $x_I(X,t) = X + u(X,t) + \zeta(X,t)$. This can be viewed as a microstructure giving rise to a continuum of inclusions. Let ρ and r be the mass densities of the matrix and the "inclusions", respectively. Then the density of kinetic energy is given by

$$K = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}r\left(\frac{\partial u}{\partial t} + \frac{\partial \zeta}{\partial t}\right)^2 . \tag{33}$$

One-dimensional model is considered, which results in a composite lattice in the form of a one-dimensional chain, however it does not allude further to any discrete structure. Each inclusion is supposed to be maintained in its displacement in the matrix by an attractive force $r\omega_0^2\zeta$, where ω_0 is a characteristic frequency, with linear elastic matrix of elasticity coefficient E. Then the density of potential energy is given by

$$V = \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}r\omega_0^2\zeta^2 . \tag{34}$$

The associated Euler-Lagrange equations of motion are:

$$\rho\frac{\partial^2 u}{\partial t^2} + r\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 \zeta}{\partial t^2}\right) - E\frac{\partial^2 u}{\partial x^2} = 0, \tag{35}$$

$$r\left(\frac{\partial^2 \zeta}{\partial t^2} + \frac{\partial^2 u}{\partial t^2}\right) + r\omega_0^2\zeta = 0. \tag{36}$$

By applying the operator $1+\frac{1}{\sqrt{\omega_0}}\left(\frac{\partial^2}{\partial t^2}\right)$ to the first of these and substituting for the second, the internal degree of freedom ζ is eliminated and the following wave equation for the matrix displacement *u* is deduced:

$$(1+\upsilon)u_{tt} - c_0^2 u_{xx} + \omega_0^{-2} u_{tttt} - k_0^{-2} u_{ttxx} = 0\,, \tag{37}$$

wherein $\upsilon = \frac{r}{\rho}$ is the ratio of densities, $c_0 = \sqrt{\frac{E}{\rho}}$ is the characteristic elastic speed, $k_0 = \frac{\omega_0}{c_0}$ is a characteristic wave number, and indices show partial derivates with respect to *t* and *x*. Equation (37) is the Maxwell-Rayleigh equation for anomalous dispersion. It contains two dispersion terms, but either of these would be sufficient to produce the required dispersion. For further comparison, after appropriate scaling it can be rewritten in fully nondimensional form as

$$u_{tt} - u_{xx} + \varepsilon(u_{tttt} - u_{ttxx}) = 0, \tag{38}$$

where ordering parameter ε emphasizes the eventual smallness of dispersion effects.

If one considers the elastic matrix to be weakly nonlinear, then the first contribution in (34) is replaced by

$$V_M = \frac{1}{2}E\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{2}{3}\alpha\left(\frac{\partial u}{\partial x}\right)^3 + \ldots\right],\ \alpha = const. \tag{39}$$

In this case, it can be immediately checked that equation (38) is replaced by an equation of the form:

$$u_{tt} - u_{xx} + \alpha(1+\beta\partial_t^2)(u_x u_{xx}) + \varepsilon(u_{tttt} - u_{ttxx}) = 0 \tag{40}$$

with $\beta = const.$ From this one will essentially retain the following nonlinear generalization of (38) which is sufficient for our purpose:

$$u_{tt} - u_{xx}(1+\alpha u_x) + \varepsilon(u_{tttt} - u_{ttxx}) = 0. \tag{41}$$

This equation still contains the potentiality of a resonance phenomenon, but is directly comparable to the nonlinear (“bad”) Boussinesq (B) equation of crystal physics

$$u_{tt} - u_{xx}(1+\varepsilon u_x) + \varepsilon u_{xxxx} = 0 \tag{42}$$

and to the “good” or “improved” Boussinesq equation proposed by Bogolubsky and others (see references in Maugin 1995):

$$u_{tt} - u_{xx}(1+\varepsilon u_x) - \varepsilon u_{ttxx} = 0 \tag{43}$$

where the same factors of nonlinear and dispersive contributions indicates that these two effects intervene at the same order of magnitude. The Boussinesq equation of fluid mechanics indeed contains a dispersive term of the same type as equation (43). Contrary to (42), Eq. (43) presents good stability properties, hence its qualification of “good”. The same obviously holds good of (41) and its further generalization (40), that does not contain derivatives of order higher than four. Equation (43) is also referred to as the Regularized Long-Wave (RLW)

Boussinesq equation. But it was noticed by Christov and Maugin (1993, 1994), while studying lattice models in ferroelastic crystals and their continuum approximation, that another way to remedy the “bad” dispersion was in fact to continue the expansion and obtain dispersive terms with sixth-order and, perhaps, higher-order space derivatives. Such a model in martensitic alloys prone to phase transitions is given by

$$u_{tt} - u_{xx} - \left[F(u) - \beta u_{xx} + u_{xxxx}\right]_{xx} = 0, \qquad (44)$$

where *F(u)* may be thought of as a polynomial in *u* starting with second degree and $\beta > 0$. While equations (42) and (43) are known to yield the KdV equation after reduction to a one-directional motion, and thus be exactly integrable in the sense of soliton theory, equations (44) and (41) which appear to be good physical models may not, in general, be exactly integrable. Their associated evolution equations are generalizations of the KdV systems, but rather than being interested in the infinite hierarchy of conservation laws exhibited by exactly integrable systems, the basic conservation laws which are still satisfied by some, only nearly integrable, systems, are considered.

The evolution equation that corresponds to the two-solitary wave equation (44) is the following:

$$u_t + \left[F(u)\right]_x + du_{xxx} + bu_{xxxxx} = 0, \qquad (45)$$

where *d* and *b* are the third- and fifth-order dispersion parameters respectively, *F(u)* represents the nonlinearity. Equation (45) together with following nonlinear term

$$F(u) = \left(-\frac{u^2}{2} + \frac{u^4}{4}\right) \qquad (46)$$

has been considered as the model equation in the following studies and will be named as KdV435 equation.

3.2 Nonlinearity and dispersion

Nonlinearity

Formation of solitons takes place because of a certain balance between nonlinearity and dispersion. The first description of the process was given by Zabusky and Kruskal (1965) — by their description initially first two terms of KdV equation (1) — u_t, uu_x, dominate. This is why *u* steepens in regions where it has a negative slope. Secondly, after *u* has steepened sufficiently, the third term — dispersion, becomes important and serves to prevent the formation of a discontinuity. Instead, oscillations of small wavelength develop on the left of the front. The amplitudes of the oscillations grow and finally each oscillation achieves almost steady amplitude. Finally, each such "solitary-wave pulse" or "soliton" begins to move uniformly at a rate which is linearly proportional to its amplitude. Thus, the solitons spread apart. Because of the periodicity (in terms of space periodicity), two or more solitons eventually overlap spatially and interact nonlinearly.

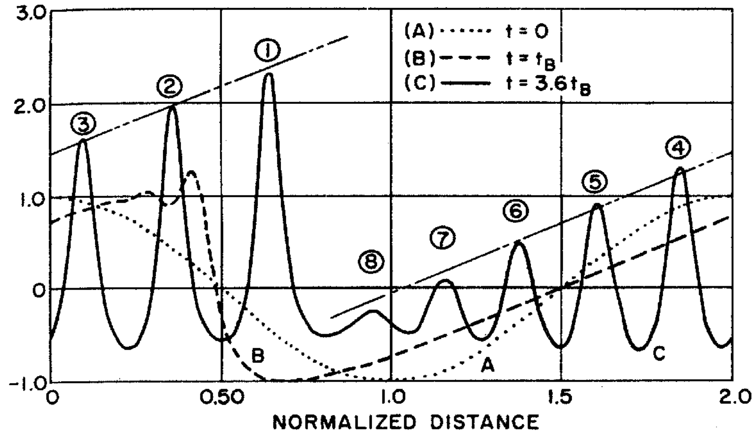

Fig. 3. The temporal development of the wave form *u(x)* (Zabusky and Kruskal 1965). Curve A corresponds to the time moment $t = 0$, curve B corresponds to the time moment $t = 1/\pi$ and curve C corresponds to the time moment $t = 3.6/\pi$.

The quartic nonlinearity (46) in KdV435 equation (45) describes the fourth-order elastic potential possessing two minima, whereas the ordinary KdV equation possesses the quadratic nonlinear terms only. Potentials for the KdV, mKdV (quartic term of equation (46)) and KdV435 equations are shown in Fig. 4.

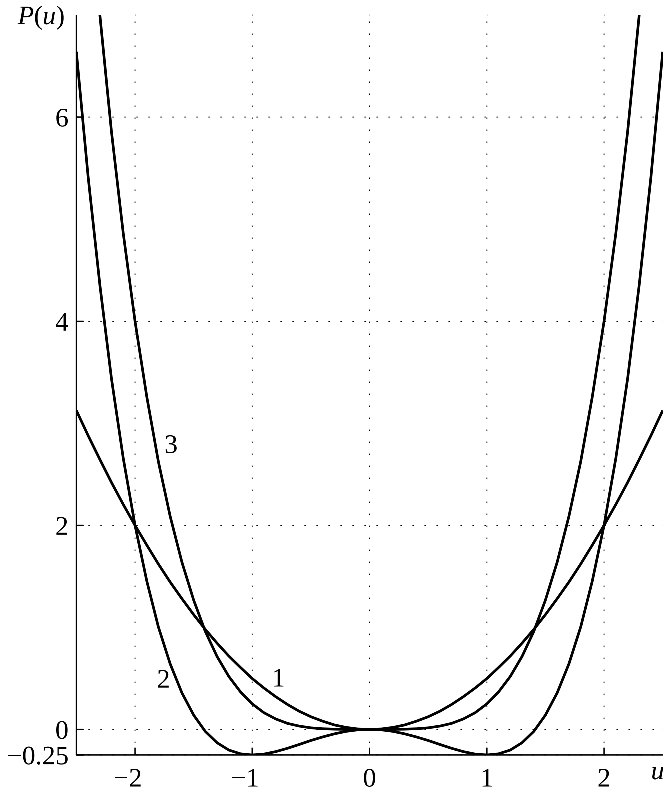

Fig. 4. Potentials *F(u)* versus *u*: (1) KdV equation; (2) KdV435 equation; (3) mKdV equation (Salupere et al. 2001).

The derivative $[F(u)]_u$ is shown in Fig. 5 for case of KdV and KdV435. An essential difference between these two cases is clearly observed. At $u = u_{cr}$ the influence of the potential and the standard KdV one is the same. It can be easily calculated that $u_{cr} = \pm\sqrt{2}$. At $u = u_q = \pm 1$, the character of the quartic potential (46) is changed. For $0 < |u| < |u_q| < |u_{cr}|$, the nonlinear effects due to potential (8) are qualitatively different from KdV case. For $|u| < |u_q| < |u_{cr}|$, the nonlinear effects due to potential (46) are weaker than for the KdV case while for $|u| > |u_{cr}|$ they are stronger.

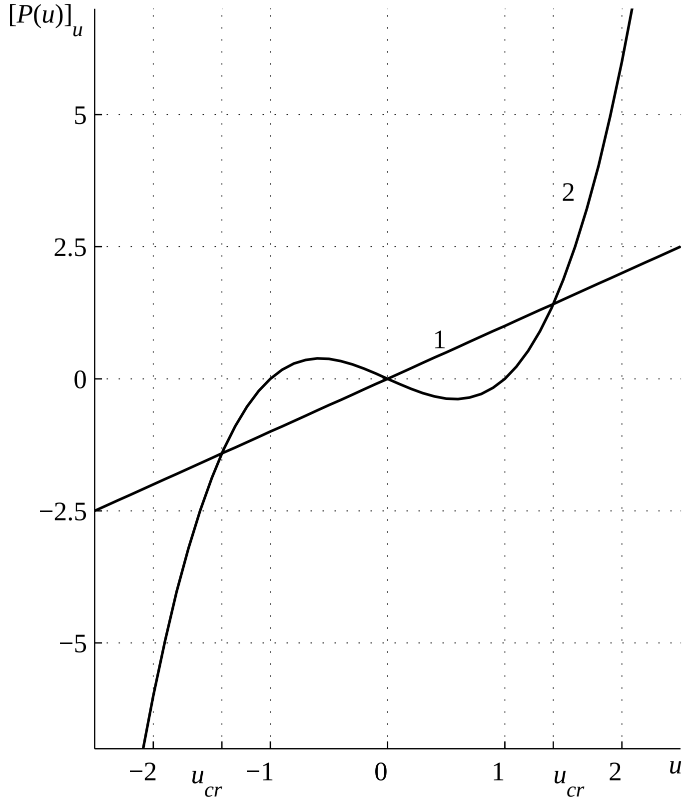

Fig. 5. Derivative of the potential *F(u)* versus *u*: (1) KdV equation; (2) KdV435 equation (Salupere et al. 2001).

The influence of nonlinearities is crucial in the formation process of solitary waves. Zabusky and Kruskal (1965) showed that the train of solitons starts to form in a region where there is the tendency for shock wave formation in the purely nonlinear case (dispersion neglected). In Fig. 6 shock wave profiles are presented for three potentials. In the KdV-type of the quadratic nonlinearity (case (a) of Fig.6), the shock wave from an initial harmonic excitation has the well-known *N*-form. The case (b) of Fig. 6 shows two discontinuities formed at the same region of the wave profile (sign correspondence). Finally, the case (c) of the potential (46) has three discontinuities for a period, two corresponding to the quartic term and one to the quadratic term (note the influence of sign difference). For both the KdV-type and mKdV-type nonlinearities the wave profiles shown in Fig. 6 correspond to t = 1.1, while for the potential (46) the wave profile is shown at t = 2.0. Therefore, in the last case the formation of a discontinuous wave profile lasts about two times longer than the other cases.

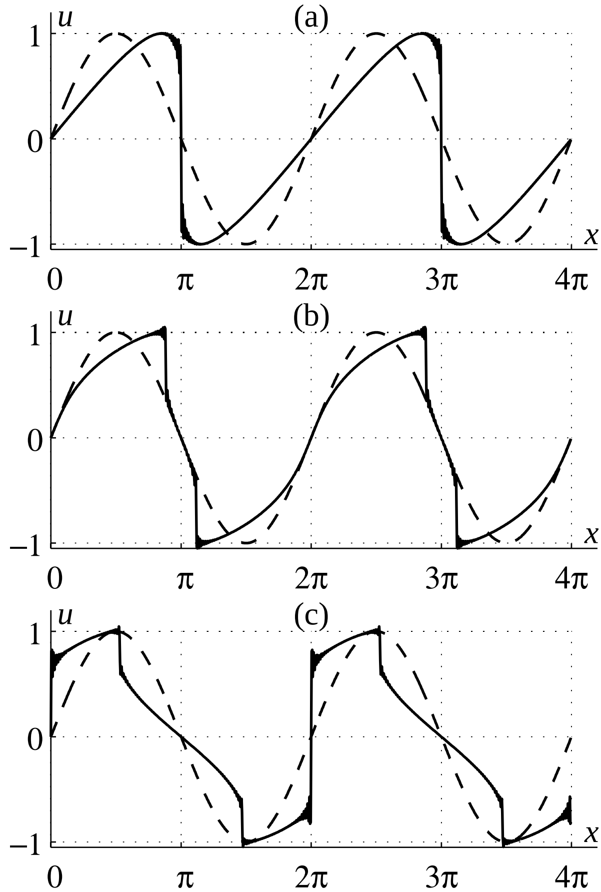

Fig. 6. Shock wave profiles: (a) quadratic potential of the KdV-type; (b) potential involving only the quartic term (mKdV); (c) quartic potential (KdV435) (Salupere et al. 2001).

Dispersion

The properties (nature) of the dispersion depend on the values of the dispersion parameters. It has been shown (Salupere et al. 2001) that if both dispersion parameters have positive values then the dispersion can be normal as well as anomalous.

The linearised version of KdV435 equation has the dispersion relation

$$\omega = k^3(bk^2 - d) \tag{47}$$

for frequency ω and wavenumber k. Evidently, the phase and group velocities are

$$c_{ph} = \frac{\omega}{k} = k^2(bk^2 - d),\ c_{gr} = \frac{d\omega}{dt} = k^2(5bk^2 - 3d) \tag{48}$$

respectively. For the comparison, the KdV equation yields

$$c_{ph} = -dk^2,\ c_{gr} = -3dk^2 \tag{49}$$

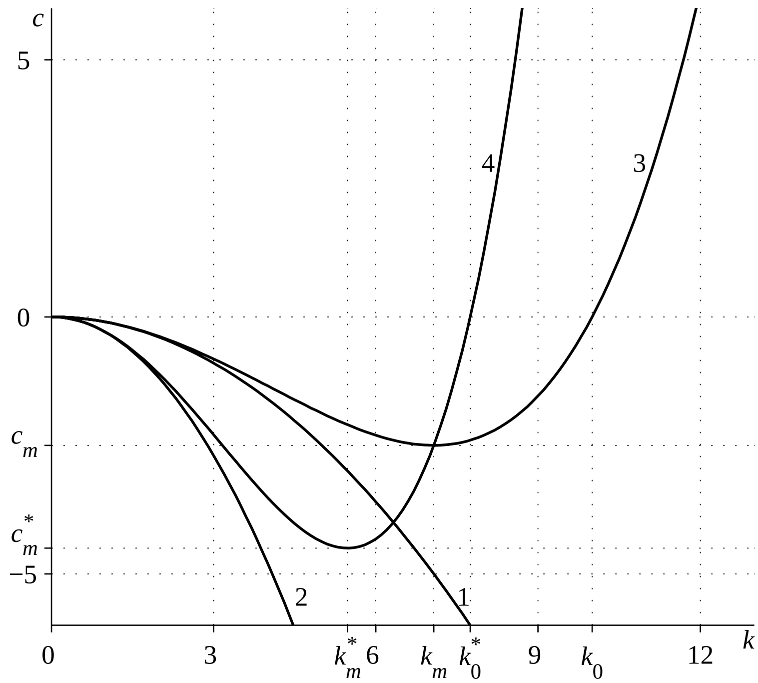

Fig. 7. Phase and group velocities: (1) c_{ph} for the KdV; (2) c_{gr} for the KdV; (3) c_{ph} for the KdV435; (4) c_{gr} for the KdV435. Here $d_l = 1$, $b_l = 3$ (Salupere et al. 2001).

In Fig. 7 phase and group velocities together with corresponding KdV-case dependencies are plotted against k. For the KdV equation dispersion is normal, i.e.

$$c_{gr} < c_{ph} < 0\,, \tag{50}$$

for KdV435 equation, following its dispersion relation, dispersion is either normal or anomalous satisfying inequality

$$c_{gr} > c_{ph}\,. \tag{51}$$

The zeros of phase and group velocities for KdV435 case are

$$k_0 = \sqrt{\frac{d}{b}} = 10^{(1/2)(b_l - d_l)}\,,\ k_0^* = \sqrt{\frac{3}{5}}k_0 \tag{52}$$

respectively. The minimum values of phase and group velocities occur at

$$k_m = \frac{1}{\sqrt{2}}k_0\,,\ k_m^* = \sqrt{\frac{3}{10}}k_0 \tag{53}$$

respectively. For these wavenumbers, the respective values of phase and group velocities are

$$c_m = -\frac{1}{4}\frac{d^2}{b}\,,\ c_m^* = -\frac{9}{20}\frac{d^2}{b}\,. \tag{54}$$

Dispersion relation describes normal dispersion in a certain interval $0 < k < k_e$. The interval can be determined using the condition $c_{ph} = c_{gr}$ that yields $k_e = k_m$. Indeed,

$$c_{gr} = \frac{d\omega}{dk} = c_{ph} + k\frac{dc_{ph}}{dk} \tag{55}$$

and $c_{gr} = c_{ph}$ is satisfied only at $\frac{dc_{ph}}{dk} = 0$. In terms of harmonics, condition $k_e = k_m$ means that the behaviour of harmonics with wavelength $\lambda < 2\pi k_m^{-1}$ corresponds to the anomalous dispersion and with wavelength $\lambda > 2\pi k_m^{-1}$ to the normal dispersion.

3.3 Numerical studies of the KdV435 equation

In my studies I have examined the KdV435 model equation with different kind of initial conditions. Following Section gives overview of the main results.
Notations

$$d_l = -\log d \text{ and } b_l = -\log(\pm b) \tag{56}$$

are introduced for future analysis.
In the case of all experiments wide range of dispersion parameters d_l and b_l:

$$0.8 \le d_l \le 2.4 \text{ and } 1.2 \le b_l \le 4.8 \tag{57}$$

has been examined.

The KdV435 equation with harmonic initial conditions

The KdV435 equation with periodic boundary conditions

$$u(x,t) = u(x + 2n\pi, t), \qquad n = \pm 1, \pm 2, ... \tag{58}$$

and initial excitation

$$u(x,0) = \sin x, 0 \le x \le 2\pi \tag{59}$$

is studied in Salupere et al. (1997, 2001), Ilison (2001), **Publication I** and **Publication II**. Both cases, normal as well as mixed dispersion have been examined.
Main results of the study with mixed dispersion are following (Salupere et al. 1997, 2001):

- Emerging solitonic structures from an initial harmonic excitation were studied and their dependence on dispersion parameters established (see Fig. 8).
- In region 1 (Fig. 8) the normal dispersion dominates, only for very short wavelengths the dispersion is anomalous. Third-order dispersion effects dominate over fifth-order effects. A train of negative solitons forms from the initial harmonic excitation.
- In region 3 (Fig. 8) the anomalous dispersion dominates and the fifth-order dispersive effects take over the third-order effects. A train of positive solitons forms from the initial harmonic excitation.

– Region 2 in Fig. 8 is more complicated and involves several subregions. The main feature is the rivalry between normal and anomalous dispersion that depends also on the wavelength. For long waves (smaller wave numbers) the dispersion is normal while for short waves (larger wave numbers) the dispersion is anomalous. In a train of solitons both situations can occur. The rivalry between normal and anomalous dispersion leads to a situation when both the train of negative and the train of positive solitons might start to form. In subregion 2a the dispersion (caused by both the third- and fifth-order effects) is stronger that yields in multiple solitons. In subregion 2c dispersion is weaker and spatio-temporal chaos can take place. It means the fluctuations in amplitudes (and spectral densities) are irregular within a certain limit. In subregion 2b interaction of multiple solitons and the train of positive solitons takes place.

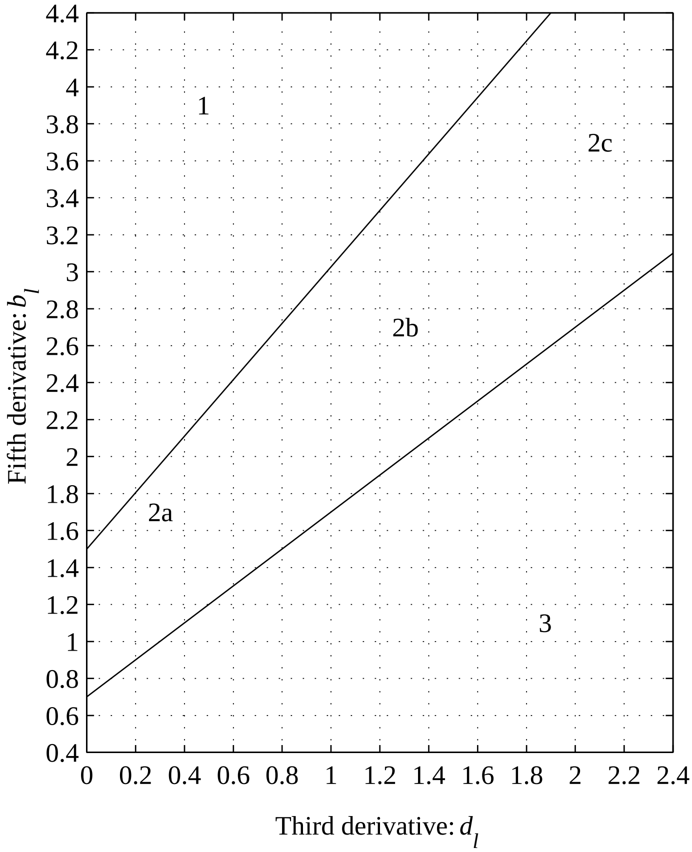

Fig. 8. Regions with different dispersive properties in the d_l — b_l plane (Salupere et al. 2001).

In the case the of normal dispersion two model equations have been studied – the KdV435 equation and the FKdV equation. In the FKdV equation nonlinearity is presented with the same term as in original KdV equation

$$F(u) = \frac{u^2}{2}. \tag{60}$$

Results of the study are presented in **Publication I** and **Publication II.** Based on large number of numerical experiments one can formulate following conclusions:

- In the case of the KdV435 equation a typical solution type is a train of negative solitons (see Fig. 9), except the case of very weak dispersion which results in simultaneous formation of trains of negative as well as positive solitons.
- In the case of FKdV equation, the solution is train of positive solitons (see Fig. 10).
- By making use of spectral characteristics one can detect recurrence and super-recurrence in the case of both model equations.
- There exists at least one hidden soliton in the case of both model equations.

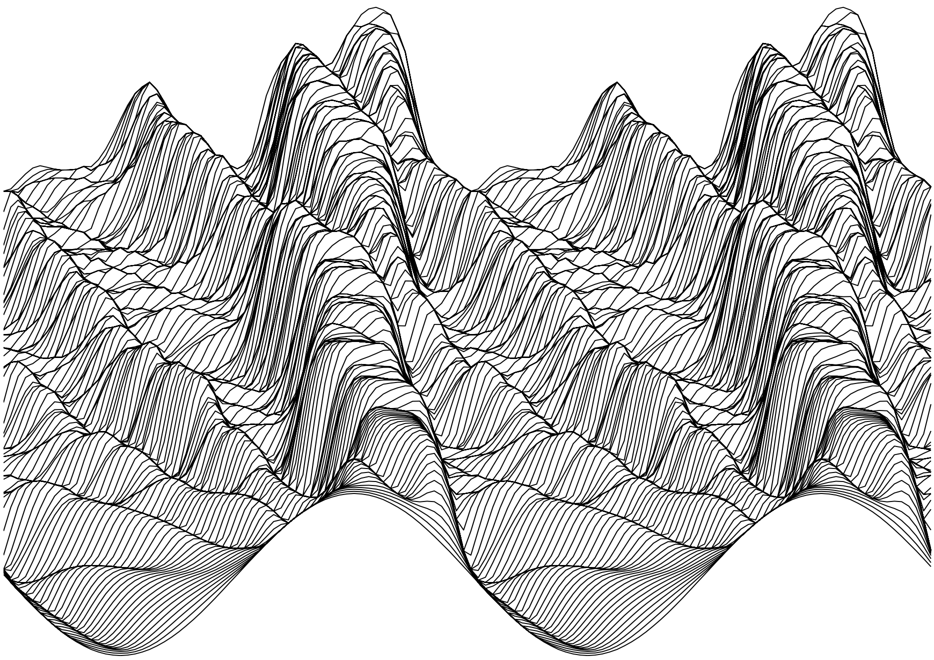

Fig. 9. Train of negative solitons, case d_l = 2.4 and b_l = 2.8, $-u$ instead of u is plotted.

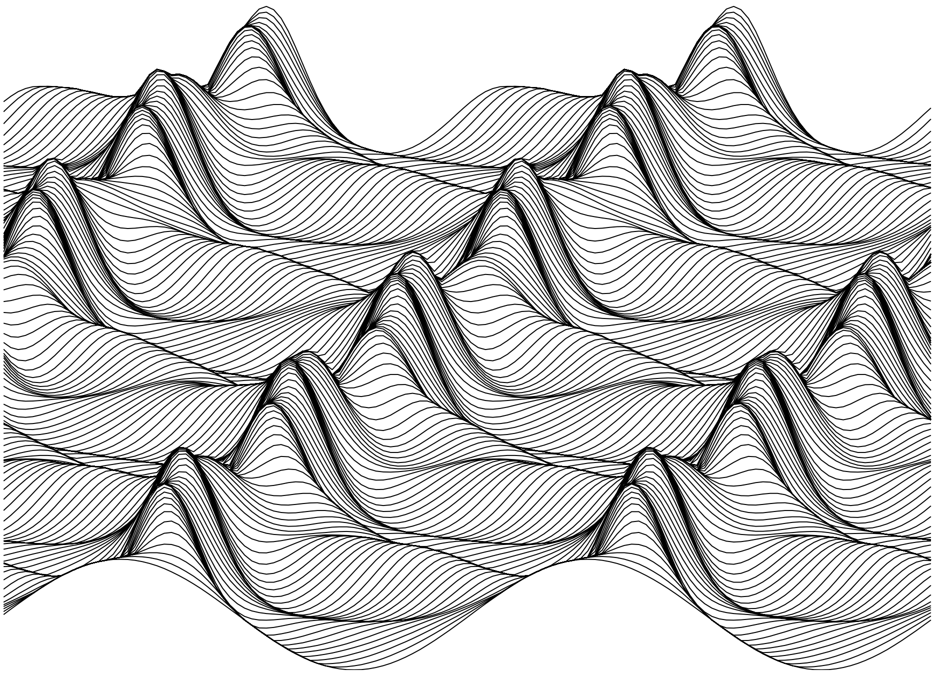

Fig. 10. Train of positive solitons, case $d_l = 1.2$ and $b_l = 2.8$.

The KdV435 equation with asymptotic boundary conditions.

The KdV435 equation (45) with asymptotic boundary conditions

$$u, u_{\xi}, \ldots, u_{\xi\xi\xi\xi\xi} \to 0, \text{ if } \xi \to \pm\infty \qquad (61)$$

is studied in Salupere and Ilison (1998A, 1998B), Ilison (1999), Salupere et al. (1999). In these studies travelling wave solutions under boundary conditions (61) are found numerically and propagation as well as interactions of "numerical" solitary waves are simulated.

Main results here are the following:

- Negative solitary waves as well as positive solitary waves could be detected for the KdV435 equation.
- The shape of the solitary wave depends on the values of dispersion parameters d_l - b_l and phase velocity c.
- Interaction of two positive solitary waves is inelastic, i.e., they do not behave like solitons (Fig 11).
- Interaction of two negative solitary waves is elastic, i.e., they behave like solitons (Fig 12).
- The interaction of a positive and a negative soliton creates quickly the instability.

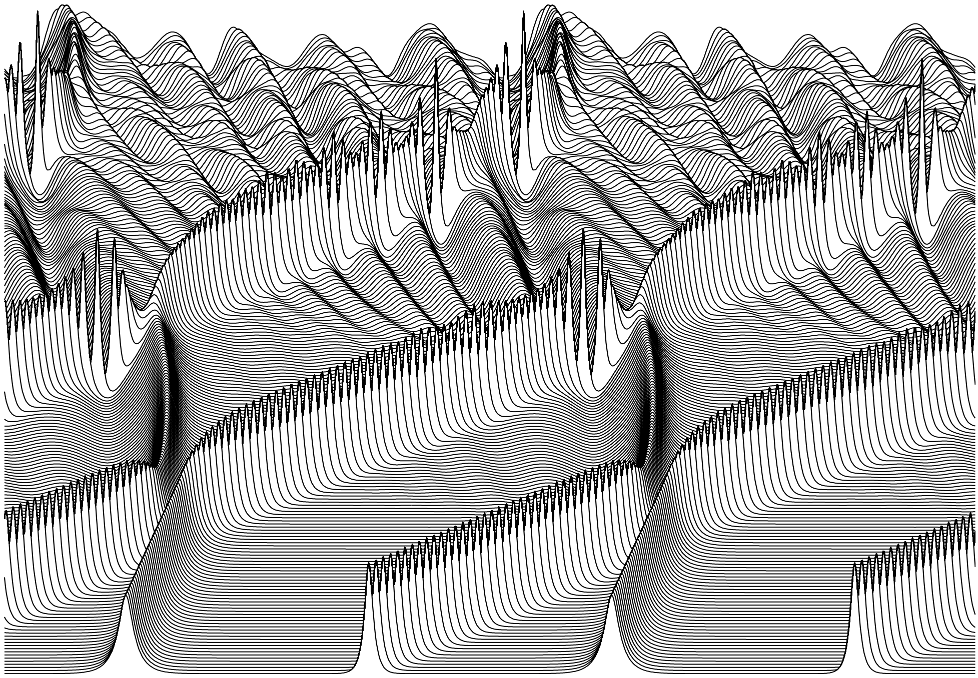
Fig. 11. Interaction of two positive solitary waves.

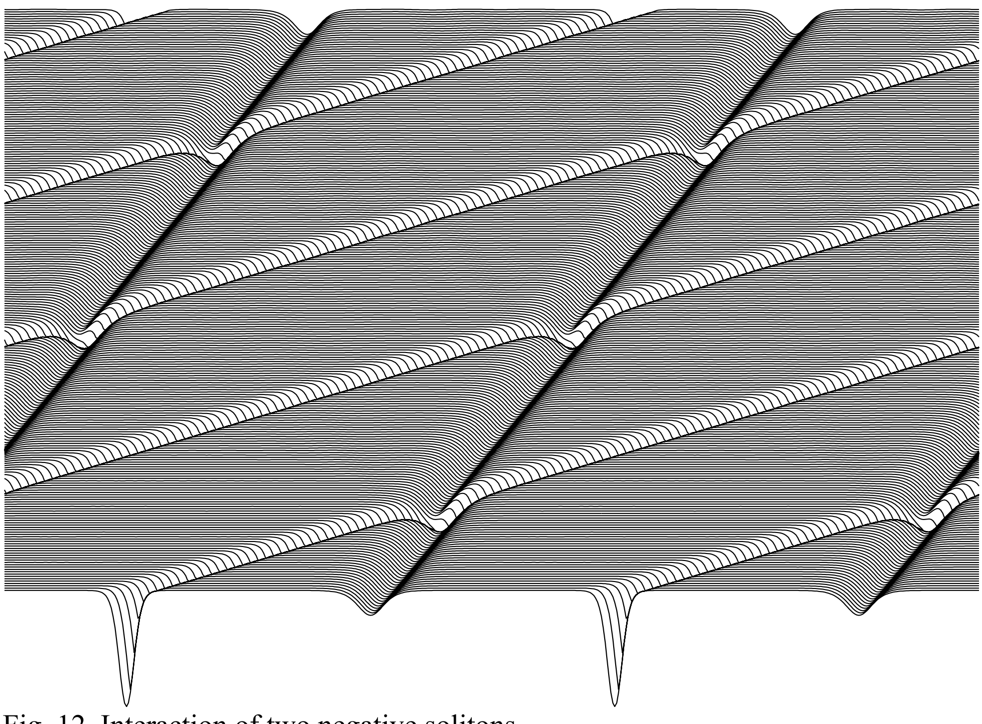
Fig. 12. Interaction of two negative solitons.

The KdV435 equation with localized initial conditions

The KdV435 equation with localized initial conditions has been studied in **Publication III**, **Publication IV**, **Publication V** and **Publication VI**. The initial excitation is given in the form of a localized initial excitation

$$u(x,0) = A\operatorname{sech}^2 \frac{x}{\Delta}, \tag{62}$$

where A is the amplitude and

$$\Delta = \sqrt{\frac{12d}{A}} \tag{63}$$

corresponds to the width of the soliton corresponding to the analytical solution of the KdV equation. In our case model equation (45) and initial excitation (62) are tied through dispersion parameter d. The case of normal dispersion is studied in **Publication III**, **Publication IV** and **Publication V**, mixed dispersion in **Publication VI**, respectively.

From the study of normal dispersion one can bring out following results:

- One can find three solution types. First and third type of the solution can be found for all values of dispersion parameters d_l and b_l.
- In the case of the first type of the solution the initial solitary wave is spread into a train of waves having chaotic behaviour.
- The second type of the solution can be detected for few pairs of dispersion parameters only. In this case a wave-train having periodic behaviour in time emerges.
- In the case of $A>A^*$ the initial solitary wave can travel with minimal disturbances (see Fig. 13). Its speed and amplitude changes by a small extent only during the propagation.
- The critical amplitude A^* depends on the quantity $d_l - 2b_l$. However, the critical amplitude A^* has limit value.
- Interaction of two solitary waves is nearly elastic.

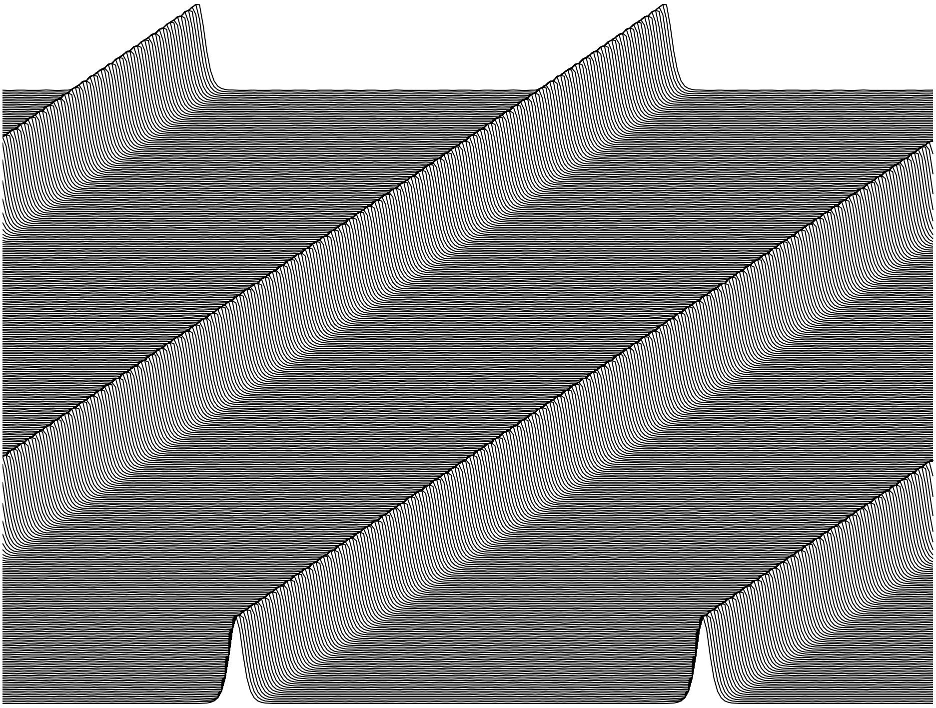

Fig. 13. Propagation of initial excitation, amplitude A=2.09, d_l=2.0 and b_l=4.0.

In **Publication VI** the KdV435 equation (45) with quartic nonlinearity (46), localized initial condition and mixed dispersion (dispersion parameters have same signs) is studied. One can bring out following main results:

- Two solution types were detected — solutions with irregular and regular behaviour.
- The first type, i.e., the irregular solution emerges in the case of small initial amplitude (small initial energy). This results in weak co-operation between dispersive and nonlinear effects and stable solitary wave(s) can not be formed.
- The second type (the regular solution) can have three sub-types:
  a) “plaited” solitons (see Fig. 14),
  b) two solitary waves, and
  c) one solitary wave.
- Generally, the sub-type (a) is not stable — after a certain time interval it is changed to sub-type (b). Only in very few cases the “plaited” solitons lives until the end of simulation.
- In the case of sub-type (b) sequential nonelastic interactions take place and the whole process ends up in one solitary wave (sub-type (c)).
- The latter one was found in our numerical experiments to be stable, i.e., a single solitary wave can propagate with (nearly) constant speed and amplitude over long time intervals.

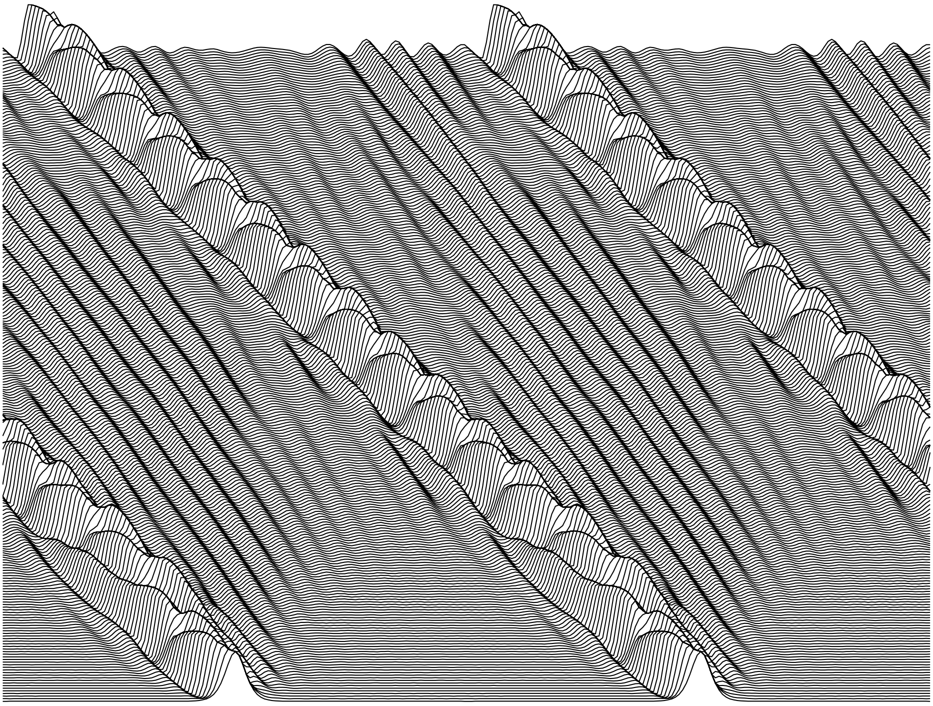

Fig. 14. “Plaited” solitons, amplitude A=0.37, d_l=1.2 and b_l=2.0.

4 Final comments

World of solitons is rich and complex. During last decades numerous authors have found many new cases where there exists balance between dispersive and nonlinear effects, also for nonintegrable systems. The increase of any term in model equations will bring us to the next (and more complex) level of problems. We would be quite hopeless without continuously advancing information technology. Increasing computing power and more advanced numerical methods will enable us to solve problems that seamed to be unsolvable some time ago. Besides good mathematical, physical and IT skills one should have a little bit of prophecy for searching in the right direction.

In this thesis I have studied the KdV435 model with three different sets of parameters. One should bring out main results:

- for the harmonic initial conditions solution is train of negative solitons, i.e., there exists a certain balance between higher order dispersion and quartic nonlinearity that results in solitonic solution.
- for the localized initial conditions with normal dispersion there exists a threshold for the initial amplitude above what solitary waves can travel with minimal disturbances.
- for the localized initial conditions with mixed dispersion “plaited” solitons can be found if value of the initial amplitude is higher than a certain threshold.

Further studies will be carried out for simulating various interactions of solitary waves in order to detect their solitonic behaviour.

The problems studied in this thesis may be considered as steps towards further understanding of the world we are living in.

Kokkuvõte

Töö eesmärk on analüüsida lainelevi protsesse mikrostruktuuriga materjalides, kus olulist rolli omavad dispersioon ja mittelineaarsus. See probleem on seotud kõrgemat järku Kortewegi–de Vriesi tüüpi evolutsioonivõrrandite käitumise uurimisega. Vastavas mudelvõrrandis on dispersiivsed efektid kirjaldatud läbi kolmandat- ja viiendat järku tuletiste, mittelineaarsus aga neljandat järku elastse potentsiaali abil. Antud võrrand kirjeldab lainelevi mikrostruktuuriga keskkondades. Mudelvõrrand ei ole integreeruv, seega pole võimalik leida analüütilisi lahendeid. Kasutatud on numbrilist arvutusmeetodit – pseudospektraalmeetodit. Põhitähelepanu on pööratud dispersiooni ja mittelineaarsuse mõju selgitamisele. Uuritud on mudelvõrrandi lahendeid kolmel erineval juhtumil – normaalne dispersioon ja harmooniline algtingimus; normaalne dispersioon ja lokaliseeritud algtingimus; muutuv dispersioon ja lokaliseeritud algtingimus. Lahendite omaduste uurimiseks on läbi viidud suur hulk numbrilisi eksperimente. On kirjeldatud lahendite evolutsioon ja spektraalkoostis, mille põhjal on määratud lahendite omadused. Olulise osa analüüsist moodustab solitoni-tüüpi lainete formeerumise mehhanismi uurimine harmoonilise algtingimuse korral. Modelleeritud on ühe üksiklaine levi ning kahe üksiklaine interaktsiooni, et tuvastada uuritavate lahendite solitonilist käitumist.

Peamised tulemused tööst on ettekantud mitmel rahvusvahelisel konverentsil ning avaldatud artiklitena eelretsenseeritavates rahvusvahelistes teadusajakirjades.

Abstract

The aim of this thesis is to analyse the wave propagation in microstructured solids where dispersion and nonlinear effects are of importance. This problem is related to the analysis of higher order Korteweg –de Vries type equation. In this model equation dispersive effects are expressed by third- and fifth order derivatives and the fourth-order elastic potential depicts the quartic nonlinearity. This equation is nonintegrable and it is not possible to find analytical solutions. That is why numerical integration algorithms, based on pseudospectral method, have been applied over wide range of dispersion parameters. Numerical results have been obtained for three different sets of parameters – normal dispersion and harmonic initial conditions; normal dispersion and localized initial conditions; mixed dispersion and localized initial conditions. Large number of numerical experiments has been carried out in order to examine the behaviour and properties of the solutions. The main focus of thesis has been on examination of

these numerical solutions. Various properties of the solutions have been detected and discussed. The emergence of soliton-type solutions from the harmonic initial excitation is analyzed in detail. Propagation of a single solitary wave and interaction of two solitary waves have been modelled in order to detect their solitonic behaviour. In two cases solitonic structures have been found, in one case nearly solitonic structure was detected. Main results of the thesis have been presented at the international conferences and published in papers of the CC journals.

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Curriculum Vitae

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6. Professional Employment

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09/1997 – 09/2003	TUT Institute of Cybernetics	Engineer
09/2003 – ...	TUT Institute of Cybernetics	Research fellow

7. Scientific work

Published papers

Salupere, A., Ilison, O. 1998. On the numerical determination of solitary waves for systems with quartics potential and higher order dispersion. In: A. Eriksson and C. Pacoste, (eds). Proc. of the NSCM-11: Nordic Seminar on Computational Mechanics. TRITA-BKN. Bulletin 39, 106-109, Stockholm.

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Salupere, A., Engelbrecht, J., Ilison, O., Maugin G.A. 1999. Numerical analysis of solitonic structures in systems with higher-order dispersion and nonlinearity. Research Report Mech 202/99, Institute of Cybernetics at Tallinn University of Technology.
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Ilison O., Salupere A. 2005. On the Propagation of Solitary Pulses in Microstructured Materials. Research Report Mech 276/05, Institute of Cybernetics at Tallinn University of Technology.

Attended conferences

The 13th Nordic Seminar on Computational Mechanics, October 20 – 21, 2000, Oslo, Norway	Presentation
The EUROMECH colloquium 436, Nonlinear Waves in Microstructured Solids, May 29 – June 1, 2002, Tallinn, Estonia	Presentation

The 15th Nordic Seminar on Computational Mechanics, October 18 – 19, 2002, Aalborg, Denmark	Presentation
The 21st International Congress of Theoretical and Applied Mechanics, August 15 – 21, 2004, ICTAM04, Warsaw, Poland	Presentation
The 4th International Conference on Nonlinear Evolution Equations and Wave Phenomena: Computation and Theory, Aprill 10 –14, 2005, The University of Georgia, Athens, USA	Presentation

Awards received

First price on the competition of the student research of the Estonian Academy of Science	2001

8. Defended thesis

Title of the thesis	Date
Higher-order dispersive effects on solitary waves	1999
Soliton formation in dispersive media with lower and higher order nonlinearity	2001

9. Main research areas

Nonlinear waves in microstructured solids, solitons

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-

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Salupere, A., Ilison, O. 1998. On the numerical determination of solitary waves for systems with quartics potential and higher order dispersion. In: A. Eriksson and C. Pacoste, (eds). Proceedings of the NSCM-11: Nordic Seminar on Computational Mechanics. TRITA-BKN. Bulletin 39, 106-109, Stockholm, KTH.

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Ilison O., Salupere A. 2005. On the Propagation of Solitary Pulses in Microstructured Materials. Research Report Mech 276/05, Tallinna Tehnikaülikooli Küberneetika Instituut.

Osalemine konverentsidel

13th Nordic Seminar on Computational Mechanics, 20 – 21 oktoober 2000, Oslo, Norra	Ettekanne
The EUROMECH colloquium 436 „Nonlinear Waves in Microstructured Solids“ 29 mai – 1 juuni 2002, Tallinn, Eesti	Ettekanne

15th Nordic Seminar on Computational Mechanics, 18 – 19 oktoober 2002, Aalborg, Taani	Ettekanne
21st International Congress of Theoretical and Applied Mechanics, 15 – 21 august 2004, ICTAM04, Varssav, Poola	Ettekanne
The Fourth International Conference on Nonlinear Evolution Equations and Wave Phenomena: Computation and Theory, 10 –14 aprill 2005, The University of Georgia, Athens, USA	Ettekanne

Saadud auhinnad

Eesti Teaduste Akadeemia üliõpilastööde konkursi esimene auhind	2001

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Töö pealkiri	Kaitsmise aasta
Üksiklained kõrgemat järku dispersiooniga keskkondades	1999
Solitonide formeerumine erinevate omadustega materjalides	2001

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Mittelineaarsed lained mikrostruktuuriga materjalides, solitonid