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On the Stress-Strain Relations for Isotropic Materials

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I.

The transformation formulae of strain-components are usually developed independently from those of stress-components. The first set of formulae mentioned is obtained by geometrical considerations from the distortion of a very small tetrahedral portion of any strained isotropic body, and the second as condition of equilibrium for the same portion of the strained body.

In the following it will be shown that basing calculations on Hooke's Law of the linear relation between the stress- and strain-components and Stokes' Principle of independent action of the normal and shearing forces applied on a very small portion of the strained body (Principle of Superposition for small distortions), it is possible to develop both sets of formulae mentioned mutually from each other [1]).

Indeed, the well-known transformation formulae of stress-components are as follows [2]):

$$\left.\begin{aligned}
t_{uu} &= t_{xx} l_1^2 + t_{yy} m_1^2 + t_{zz} n_1^2 + 2 t_{xy} l_1 m_1 + 2 t_{yz} m_1 n_1 + 2 t_{zx} n_1 l_1 \\
t_{vv} &= t_{xx} l_2^2 + t_{yy} m_2^2 + t_{zz} n_2^2 + 2 t_{xy} l_2 m_2 + 2 t_{yz} m_2 n_2 + 2 t_{zx} n_2 l_2 \\
t_{ww} &= t_{xx} l_3^2 + t_{yy} m_3^2 + t_{zz} n_3^2 + 2 t_{xy} l_3 m_3 + 2 t_{yz} m_3 n_3 + 2 t_{zx} n_3 l_3 \\
t_{uv} &= t_{xx} l_1 l_2 + t_{yy} m_1 m_2 + t_{zz} n_1 n_2 + t_{xy} (l_1 m_2 + l_2 m_1) + \\
&\qquad + t_{yz} (m_1 n_2 + m_2 n_1) + t_{zx} (n_1 l_2 + n_2 l_1) \\
t_{vw} &= t_{xx} l_2 l_3 + t_{yy} m_2 m_3 + t_{zz} n_2 n_3 + t_{xy} (l_2 m_3 + l_3 m_2) + \\
&\qquad + t_{yz} (m_2 n_3 + m_3 n_2) + t_{zx} (n_2 l_3 + n_3 l_2) \\
t_{wu} &= t_{xx} l_3 l_1 + t_{yy} m_3 m_1 + t_{zz} n_3 n_1 + t_{xy} (l_3 m_1 + l_1 m_3) + \\
&\qquad + t_{yz} (m_3 n_1 + m_1 n_3) + t_{zx} (n_3 l_1 + n_1 l_3)
\end{aligned}\right\} \quad (1)$$

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[1]) For the twodimensional state of stress this sentence was proved: O. Maddison, *Tehniline mehaanika*, Vol. I_2, Tallinn, 1926, pp. 185—187.

[2]) A. E. H. Love, *A Treatise of the Mathematical Theory of Elasticity*, Cambridge, 1934, p. 80.

By $t_{xx}, t_{yy}, t_{zz}, t_{xy}, t_{yz}, t_{zx}$ are marked here the six components of stress at any point based on an orthogonal system of axes X, Y, Z and by $t_{uu}, t_{vv}, t_{ww}, t_{uv}, t_{vw}, t_{wu}$ — the six stress-components at the same point referred to a new orthogonal system of axes U, V, W. This new system of axes is determined by direction cosines, corresponding to the following orthogonal scheme:

Axes	X	Y	Z
U	l_1	m_1	n_1
V	l_2	m_2	n_2
W	l_3	m_3	n_3

In reference to Hooke's Law and Stokes' Principle the strain-components $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ depend on the correspondent stress-components $t_{xx}, t_{yy}, t_{zz}, t_{xy}, t_{yz}, t_{zx}$ in the following manner[3]):

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[t_{xx} - \eta\,(t_{yy} + t_{zz})\right] \\
\varepsilon_y &= \frac{1}{E}\left[t_{yy} - \eta\,(t_{zz} + t_{xx})\right] \\
\varepsilon_z &= \frac{1}{E}\left[t_{zz} - \eta\,(t_{xx} + t_{yy})\right] \\
\gamma_{xy} &= \frac{t_{xy}}{G} \\
\gamma_{yz} &= \frac{t_{yz}}{G} \\
\gamma_{zx} &= \frac{t_{zx}}{G}
\end{aligned}\right\} \quad (2)$$

where E denotes the Young's Modulus,
G „ „ Modulus of Rigidity
and η „ „ Poisson's Ratio.

[3]) Ph. Frank und R. v. Mises, *Die Differential- und Integralgleichungen der Mechanik und Physik, II,* Braunschweig, 1935, p. 254.

For the new strain-components $\varepsilon_u, \varepsilon_v, \varepsilon_w, \gamma_{uv}, \gamma_{vw}, \gamma_{wu}$ we have similar expressions depending on the correspondent stress-components $t_{uu}, t_{vv}, t_{ww}, t_{uv}, t_{vw}, t_{wu}$:

$$\left.\begin{aligned}
\varepsilon_u &= \frac{1}{E}\left[t_{uu} - \eta\,(t_{vv} + t_{ww})\right] \\
\varepsilon_v &= \frac{1}{E}\left[t_{vv} - \eta\,(t_{ww} + t_{uu})\right] \\
\varepsilon_w &= \frac{1}{E}\left[t_{ww} - \eta\,(t_{uu} + t_{vv})\right] \\
\gamma_{uv} &= \frac{t_{uv}}{G} \\
\gamma_{vw} &= \frac{t_{vw}}{G} \\
\gamma_{wu} &= \frac{t_{wu}}{G}
\end{aligned}\right\} \quad (2\,\text{bis})$$

Determining the quantities $t_{xx}, \ldots$ from the systems (2) and (2 bis) of simultaneous equations we find for the stress-components the following set of expressions:

$$\left.\begin{aligned}
t_{xx} &= \frac{E}{1+\eta}\left[\varepsilon_x + \frac{\eta}{1-2\eta}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
t_{yy} &= \frac{E}{1+\eta}\left[\varepsilon_y + \frac{\eta}{1-2\eta}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
t_{zz} &= \frac{E}{1+\eta}\left[\varepsilon_z + \frac{\eta}{1-2\eta}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
t_{xy} &= G\,\gamma_{xy} \\
t_{yz} &= G\,\gamma_{yz} \\
t_{zx} &= G\,\gamma_{zx}
\end{aligned}\right\} \quad (3)$$

and for the new stress-components $t_{uu}, t_{vv}, t_{ww}, t_{uv}, t_{vw}, t_{wu}$ similarly:

$$\left.\begin{aligned}
t_{uu} &= \frac{E}{1+\eta}\left[\varepsilon_u + \frac{\eta}{1-2\eta}(\varepsilon_u+\varepsilon_v+\varepsilon_w)\right]\\
t_{vv} &= \frac{E}{1+\eta}\left[\varepsilon_v + \frac{\eta}{1-2\eta}(\varepsilon_u+\varepsilon_v+\varepsilon_w)\right]\\
t_{ww} &= \frac{E}{1+\eta}\left[\varepsilon_w + \frac{\eta}{1-2\eta}(\varepsilon_u+\varepsilon_v+\varepsilon_w)\right]\\
t_{uv} &= G\gamma_{uv}\\
t_{vw} &= G\gamma_{vw}\\
t_{wu} &= G\gamma_{wu}
\end{aligned}\right\} \text{(3 bis)}$$

By substituting into (1) these expressions of stress-components (3) and (3 bis) and taking account that:

$$E = 2(1+\eta)G \qquad (4)$$

and

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_u + \varepsilon_v + \varepsilon_w, \qquad (5)$$

we find the well-known expressions of transformation of strain-components [4]):

$$\left.\begin{aligned}
\varepsilon_u &= \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \varepsilon_z n_1^2 + \gamma_{xy} l_1 m_1 + \gamma_{yz} m_1 n_1 + \gamma_{zx} n_1 l_1\\
\varepsilon_v &= \varepsilon_x l_2^2 + \varepsilon_y m_2^2 + \varepsilon_z n_2^2 + \gamma_{xy} l_2 m_2 + \gamma_{yz} m_2 n_2 + \gamma_{zx} n_2 l_2\\
\varepsilon_w &= \varepsilon_x l_3^2 + \varepsilon_y m_3^2 + \varepsilon_z n_3^2 + \gamma_{xy} l_3 m_3 + \gamma_{yz} m_3 n_3 + \gamma_{zx} n_3 l_3\\
{}^1\!/_2\gamma_{uv} &= \varepsilon_x l_1 l_2 + \varepsilon_y m_1 m_2 + \varepsilon_z n_1 n_2 + {}^1\!/_2\gamma_{xy}(l_1 m_2 + l_2 m_1) +\\
&\quad + {}^1\!/_2\gamma_{yz}(m_1 n_2 + m_2 n_1) + {}^1\!/_2\gamma_{zx}(n_1 l_2 + n_2 l_1)\\
{}^1\!/_2\gamma_{vw} &= \varepsilon_x l_2 l_3 + \varepsilon_y m_2 m_3 + \varepsilon_z n_2 n_3 + {}^1\!/_2\gamma_{xy}(l_2 m_3 + l_3 m_2) +\\
&\quad + {}^1\!/_2\gamma_{yz}(m_2 n_3 + m_3 n_2) + {}^1\!/_2\gamma_{zx}(n_2 l_3 + n_3 l_2)\\
{}^1\!/_2\gamma_{wu} &= \varepsilon_x l_3 l_1 + \varepsilon_y m_3 m_1 + \varepsilon_z n_3 n_1 + {}^1\!/_2\gamma_{xy}(l_3 m_1 + l_1 m_3) +\\
&\quad + {}^1\!/_2\gamma_{yz}(m_3 n_1 + m_1 n_3) + {}^1\!/_2\gamma_{zx}(n_3 l_1 + n_1 l_3)
\end{aligned}\right\} (6)$$

In the same manner going in the opposite direction we obtain the set of transformation formulae for the stress-components (1) starting by (6).

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[4]) Love, ibidem, p. 43.

II.

Cauchy has developed the stress-strain relations for isotropic solid bodies by means of the assumption that the principal planes of stress are normal to the principal axes of strain, that is, that the principal directions of stress coincide with the principal directions of strain [5]).

It will be shown that Cauchy's assumption represents a result of Hooke's Law and the availability of Stokes' Principle.

Let the principal stresses at any point of a strained isotropic body be denoted by t_1, t_2, t_3 and let the principal directions of stress at this point be determined by the direction cosines, corresponding to the following orthogonal scheme:

Axes	X	Y	Z
t_1	l_{01}	m_{01}	n_{01}
t_2	l_{02}	m_{02}	n_{02}
t_3	l_{03}	m_{03}	n_{03}

Considering at any point of the strained body the equilibrium conditions for a very small tetrahedral portion which fourth plane is normal to one of the principal stresses, say normal to the first principal stress t_1, we find: [6])

$$\left.\begin{aligned} t_1\, l_{01} &= t_{xx}\, l_{01} + t_{yx}\, m_{01} + t_{zx}\, n_{01} \\ t_1\, m_{01} &= t_{xy}\, l_{01} + t_{yy}\, m_{01} + t_{zy}\, n_{01} \\ t_1\, n_{01} &= t_{xz}\, l_{01} + t_{yz}\, m_{01} + t_{zz}\, n_{01} \end{aligned}\right\} \quad (7)$$

[5]) Cauchy's *Exercices de mathématique,* Paris, 1827 (28) (cf. Love, ibidem, Introduction, p. 8, footnote).

[6]) Frank-Mises, ibidem, p. 243.

From these formulae and those concerning the two other principal stresses t_2 and t_3 we deduce the following expressions determining the directions of the principal stresses:

$$\left.\begin{aligned}
\frac{t_{xx} l_{01} + t_{yx} m_{01} + t_{zx} n_{01}}{l_{01}} &= \frac{t_{xy} l_{01} + t_{yy} m_{01} + t_{zy} n_{01}}{m_{01}} = \\
&= \frac{t_{xz} l_{01} + t_{yz} m_{01} + t_{zz} n_{01}}{n_{01}} \\
\frac{t_{xx} l_{02} + t_{yx} m_{02} + t_{zx} n_{02}}{l_{02}} &= \frac{t_{xy} l_{02} + t_{yy} m_{02} + t_{zy} n_{02}}{m_{02}} = \\
&= \frac{t_{xz} l_{02} + t_{yz} m_{02} + t_{zz} n_{02}}{n_{02}} \\
\frac{t_{xx} l_{03} + t_{yx} m_{03} + t_{zx} n_{03}}{l_{03}} &= \frac{t_{xy} l_{03} + t_{yy} m_{03} + t_{zy} n_{03}}{m_{03}} = \\
&= \frac{t_{zx} l_{03} + t_{yz} m_{03} + t_{zz} n_{03}}{n_{03}}
\end{aligned}\right\} \quad (8)$$

Denoting the principal strains at the same point of the strained body by ε_1, ε_2, ε_3, let the principal directions of strain at this point be determined by direction cosines corresponding to the following scheme:

Axes	X	Y	Z
ε_1	l'_{01}	m'_{01}	n'_{01}
ε_2	l'_{02}	m'_{02}	n'_{02}
ε_3	l'_{03}	m'_{03}	n'_{03}

Then the directions of the principal axes of strain have to satisfy the relations [7]):

[7]) Love, ibidem, p. 42.

$$\left.\begin{aligned}
\frac{\varepsilon_x l'_{01}+{}^1/_2\gamma_{yx}m'_{01}+{}^1/_2\gamma_{zx}n'_{01}}{l'_{01}}&=\frac{{}^1/_2\gamma_{xy}l'_{01}+\varepsilon_y m'_{01}+{}^1/_2\gamma_{zy}n'_{01}}{m'_{01}}=\\
&=\frac{{}^1/_2\gamma_{xz}l'_{01}+{}^1/_2\gamma_{yz}m'_{01}+\varepsilon_z n'_{01}}{n'_{01}}\\
\frac{\varepsilon_x l'_{02}+{}^1/_2\gamma_{yx}m'_{02}+{}^1/_2\gamma_{zx}n'_{02}}{l'_{02}}&=\frac{{}^1/_2\gamma_{xy}l'_{02}+\varepsilon_y m'_{02}+{}^1/_2\gamma_{zy}n'_{02}}{m'_{02}}=\\
&=\frac{{}^1/_2\gamma_{xz}l'_{02}+{}^1/_2\gamma_{yz}m'_{02}+\varepsilon_z n'_{02}}{n'_{02}}\\
\frac{\varepsilon_x l'_{03}+{}^1/_2\gamma_{yx}m'_{03}+{}^1/_2\gamma_{zx}n'_{03}}{l'_{03}}&=\frac{{}^1/_2\gamma_{xy}l'_{03}+\varepsilon_y m'_{03}+{}^1/_2\gamma_{zy}n'_{03}}{m'_{03}}=\\
&=\frac{{}^1/_2\gamma_{xz}l'_{03}+{}^1/_2\gamma_{yz}m'_{03}+\varepsilon_z n'_{03}}{n'_{03}}
\end{aligned}\right\}\quad(9)$$

Comparing this set of formulae with (8) the question arises, in what relation each to other are the directions of principal strain and the planes of principal stress?

By the following it will be shown that the directions of principal strain are normal to the principal planes of stress, that is, the directions of principal stress coincide with the principal directions of strain.

Indeed, on substituting for (9) the expressions of strain-components (2) and taking into account (3) we can transform the first line of (9) as follows:

$$\frac{\varepsilon_x l'_{01}+\dfrac{G(1+\eta)}{E}\gamma_{yx}m'_{01}+\dfrac{G(1+\eta)}{E}\gamma_{zx}n'_{01}+\dfrac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)\,l'_{01}}{l'_{01}}=$$

$$=\frac{\dfrac{G(1+\eta)}{E}\gamma_{xy}l'_{01}+\varepsilon_y m'_{01}+\dfrac{G(1+\eta)}{E}\gamma_{zy}n'_{01}+\dfrac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)m'_{01}}{m'_{01}}=$$

$$=\frac{\dfrac{G(1+\eta)}{E}\gamma_{xy}l'_{01}+\dfrac{G(1+\eta)}{E}\gamma_{yz}m'_{01}+\varepsilon_z n'_{01}+\dfrac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)n'_{01}}{n'_{01}}$$

Further transformation gives:

$$\frac{\frac{E}{1+\eta}\left[\varepsilon_x+\frac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)\right]l'_{01}+G\gamma_{yx}m'_{01}+G\gamma_{zx}n'_{01}}{l'_{01}}=$$

$$=\frac{G\gamma_{xy}l'_{01}+\frac{E}{1+\eta}\left[\varepsilon_y+\frac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)\right]m'_{01}+G\gamma_{zy}n'_{01}}{m'_{01}}=$$

$$=\frac{G\gamma_{xz}l'_{01}+G\gamma_{yz}m'_{01}+\frac{E}{1+\eta}\left[\varepsilon_z+\frac{\eta}{1-2\eta}(\varepsilon_x+\varepsilon_y+\varepsilon_z)\right]n'_{01}}{n'_{01}}.$$

Taking now into consideration (2) and transforming in the same manner the second and third line of (9), we finally obtain the following set of formulae:

$$\left.\begin{aligned}
\frac{t_{xx}l'_{01}+t_{yx}m'_{01}+t_{zx}n'_{01}}{l'_{01}}&=\frac{t_{xy}l'_{01}+t_{yy}m'_{01}+t_{zy}n'_{01}}{m'_{01}}=\\
&=\frac{t_{xz}l'_{01}+t_{yz}m'_{01}+t_{zz}n'_{01}}{n'_{01}}\\
\frac{t_{xx}l'_{02}+t_{yx}m'_{02}+t_{zx}n'_{02}}{l'_{02}}&=\frac{t_{xy}l'_{02}+t_{yy}m'_{02}+t_{zy}n'_{02}}{m'_{02}}=\\
&=\frac{t_{xz}l'_{02}+t_{yz}m'_{02}+t_{zz}n'_{02}}{n'_{02}}\\
\frac{t_{xx}l'_{03}+t_{yx}m'_{03}+t_{zx}n'_{03}}{l'_{03}}&=\frac{t_{xy}l'_{03}+t_{yy}m'_{03}+t_{zy}n'_{03}}{m'_{03}}=\\
&=\frac{t_{xz}l'_{03}+t_{yz}m'_{03}+t_{zz}n'_{03}}{n'_{03}}
\end{aligned}\right\}\quad(10)$$

Comparing the set of formulae obtained with (8) we see that:

$$\left.\begin{aligned}
l'_{01}=l_{01},\ m'_{01}=m_{01},\ n'_{01}=n_{01}\\
l'_{02}=l_{02},\ m'_{02}=m_{02},\ n'_{02}=n_{02}\\
l'_{03}=l_{03},\ m'_{03}=m_{03},\ n'_{03}=n_{03}
\end{aligned}\right\}\quad(11)$$

The principal directions of strain coincide therefore with the principal directions of stress.

Hence it follows that Cauchy's assumption concerning the principal directions of strain and those of stress, represents a consequence of Hooke's Law and Stokes' Principle.

Let a_0 and a_0' denote the angles between the axis of coordinate X and a principal direction of stress respectively strain. The formulae (8) and (9) correspondingly in the case of a twodimensionally strained isotropic body then get the following shape:

$$\left.\begin{aligned} \frac{t_{xx}\cos a_0 + t_{yx}\cos\left(\frac{\pi}{2}-a_0\right)}{\cos a_0} &= \frac{t_{xy}\cos a_0 + t_{yy}\cos\left(\frac{\pi}{2}-a_0\right)}{\cos\left(\frac{\pi}{2}-a_0\right)} \\ \frac{\varepsilon_x\cos a_0' + {}^1/_2\,\gamma_{yx}\cos\left(\frac{\pi}{2}-a_0'\right)}{\cos a_0'} &= \frac{{}^1/_2\,\gamma_{xy}\cos a_0' + \varepsilon_y\cos\left(\frac{\pi}{2}-a_0'\right)}{\cos\left(\frac{\pi}{2}-a_0'\right)} \end{aligned}\right\} \quad (12)$$

or, after some transformations:

$$\left.\begin{aligned} \tan 2\,a_0 &= \frac{2\,t_{xy}}{t_{xx}-t_{yy}} \\ \tan 2\,a_0' &= \frac{\gamma_{xy}}{\varepsilon_x-\varepsilon_y} \end{aligned}\right\} \quad (12 \text{ bis})$$

By substituting into (12 bis) the expressions of strain-components:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}(t_{xx}-\eta\,t_{yy}) \\ \varepsilon_y &= \frac{1}{E}(t_{yy}-\eta\,t_{xx}) \\ \gamma_{xy} &= \frac{t_{xy}}{G} \end{aligned}\right\} \quad (13)$$

we find[8]):

[8]) O. Maddison, ibidem, pp. 187—188.

$$\tan 2\alpha_0' = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{t_{xy}}{\frac{G}{E}\left[(t_{xx} - t_{yy}) + \eta\,(t_{xx} - t_{yy})\right]} =$$

$$= \frac{t_{xy}}{\frac{G(1+\eta)}{E}(t_{xx} - t_{yy})} = \frac{2\,t_{xy}}{t_{xx} - t_{yy}} = \tan 2\alpha_0,$$

which gives:

$$\alpha_0' = \alpha_0 \quad \text{or} \quad \alpha_0' = \alpha_0 + \frac{\pi}{2} \qquad (14).$$

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