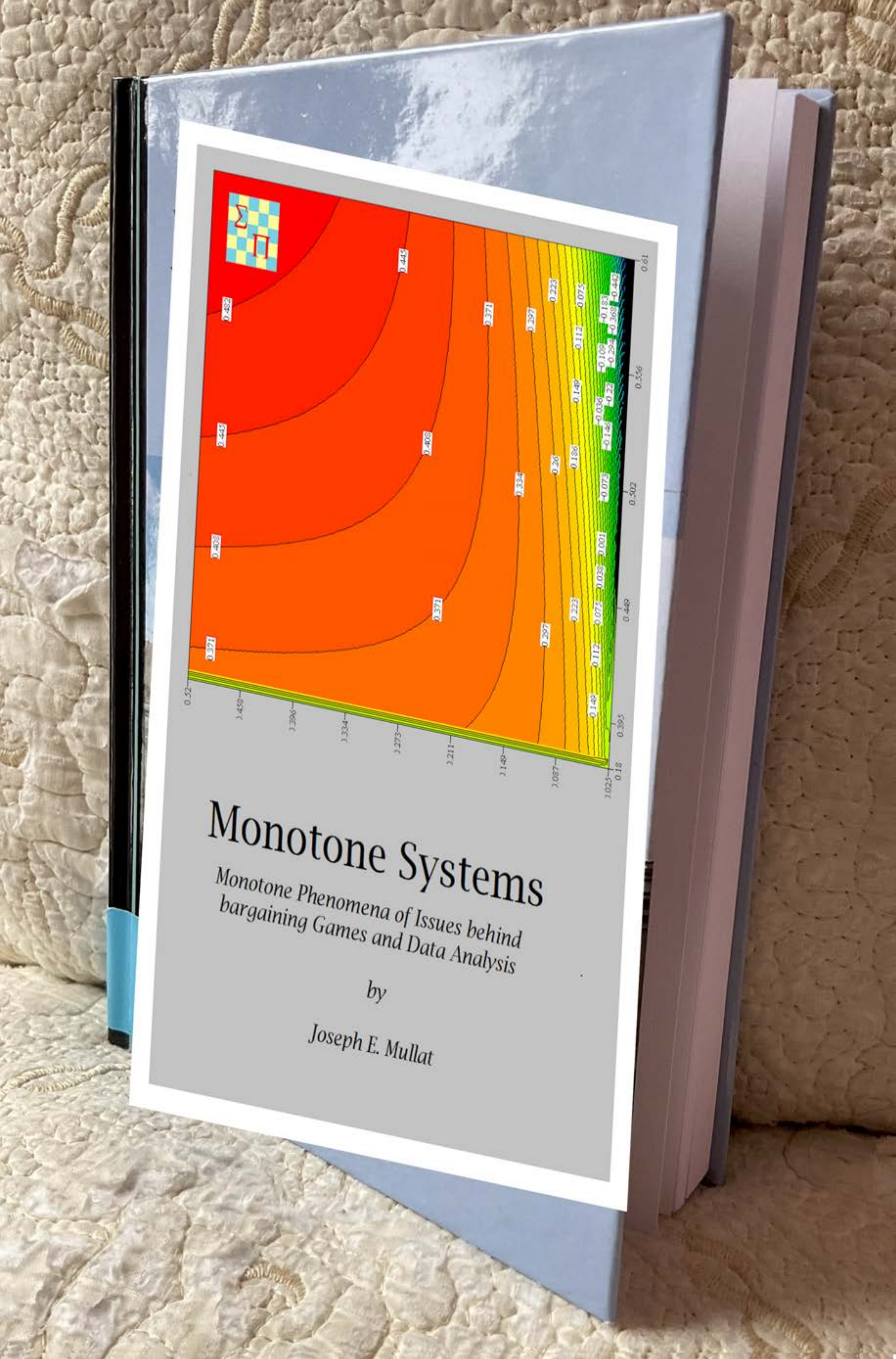

Monotone Systems

Monotone Phenomena of Issues behind bargaining Games and Data Analysis

by

Joseph E. Mullat

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The book discusses the implementation of monotonic systems in two contexts.

1. Political Mechanism Design: Monotonic systems were introduced to reflect the adjustment of negotiating power in bargaining situations, particularly in negotiations between left and right political parties. The objective is to elucidate the framework of political mechanism design. When a political goal of tax minimization is declared, monotonicity in this context implies that as a party's political power grows and taxes decrease, its portion of negotiated tax resources also grows. Conversely, if a party's political power diminishes while taxes increase, its share of negotiated tax resources also diminishes. Monotonicity is a desirable trait in political mechanism design, ensuring fairness and stability in the bargaining process.
2. Data Analysis: Monotonic systems have also been applied in data analysis. In this context, stable sets are used to provide a unifying perspective for virtual experiments. By assigning certain certificates to the elements in a stable set, virtual experiments can be performed to test the stability of the set under different conditions. This provides a basis for stability or equilibrium in the data, as opposed to volatility or fuzziness. Monotonicity in data analysis ensures that if a variable's value increases, then the outcome of the experiment also increases, and vice versa.

Overall, the idea of stable or "stable lists" of elements in sets or topologies is central to the application of monotonic systems in both political mechanism design and data analysis. These stable sets provide a basis for stability and equilibrium, and monotonicity ensures that changes in bargaining power or variable values result in predictable and fair outcomes.

The Monotone Phenomena

This collection of scholarly works elucidates the concept of a monotonic or monotonic system, depicting a structured approach in which subsets of system elements are deployed to arrange and prioritise indicators or credentials that represent characteristics or qualifications of subset elements. Qualifications of indicators have a monotonous quality that harmonizes with the ever-changing nature of reality in a large number of examples. Specifically manifested as real numbers, these indicators show a tendency to either increase or decrease according to the hierarchical order induced by the inclusion of subsets from a wider pool of indicators. Consequently, the Monotone Systems framework serves to formalize and extend the intrinsic understanding of how elements within subsets are sequenced, organized, and arranged. The theory, which originated in 1971, underwent continuous refinement, culminating in its publication in the Russian journal MAIK in 1976. Subsequently, the dissemination of this pioneering theory application to the retail chain network was facilitated by Plenum Publishing Corporation, which introduced it to an English-speaking audience in 1977.

Concise Glossary of Mathematical Nomenclature

W — A common or general set of indicators, elements, objects, etc
Γ_j, X_i, H_i, H^j, H_1, H_2 … — Subsets of the General Set W
For $i, j = 1,2,..,n$ instead we sometimes use short notation $i, j = \overline{1,n}$
α, β, γ, μ, τ,… — Greek letters as elements of W, H_i, Γ,…
Credential $\pi(\alpha, H_j)$ assigned to an element $\alpha \in H_j$ of the subset H_j
Type $\oplus$ and type $\ominus$ operations on elements $\alpha,\beta,...$
$\overline{\alpha}$, $\overline{\beta}$, — Sequences or sets $\langle \alpha_i \rangle$, $\langle \beta_j \rangle$, of ordered elements α_i, β_j,…
$H = \varnothing$, $\subset$, $\supset$, $\subseteq$, $\supseteq$, $H \subseteq W$, $W \supseteq \Gamma$,… – Pairwise relations
$H_1 \cup H_2$, $H_1 \cap H_2$, $W \setminus \overline{\alpha}$,… — Pairwise operations
Π^+H, Π^-H — Collections or arrays of general set W subsets
$\{\Pi^-H \mid H \subseteq W\}$ — This means that $\{\Pi^-H \mid$, where $H \subseteq W\}$, etc.
$\overline{X}$ — Denotes the complement $W \setminus X$ of a set X to the set W
$\forall$ — Generality quantifier and $\exists$ is existential quantifier

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PREFACE

MONOTONE PHENOMENA OF ISSUES BEHIND BARGAINING GAMES AND DATA ANALYSIS

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Introduction

In the social sciences, humane language is often used to describe phenomena related to numerical data. This approach can lead to predictive problems when the descriptions don't accurately reflect reality. On the other hand, in the natural sciences, numerical data are used to describe and predict phenomena, whether they are natural or artificially created. Even in the natural sciences, relying solely on the rigorous language of mathematical assumptions or postulates may not fully capture the complexity of the phenomena being studied. Essentially, most phenomena in the social sciences differ from the natural sciences in the way natural language and numerical data are used. This significant contrast highlights the problems inherent in both approaches and suggests that a detailed understanding of phenomena is crucial for accurate predictions or descriptions of reality.

The problem of forecasting may not primarily lie in the mathematics itself, but in how well defined and appropriate the mathematical methods used are. This is analogous to choosing between window shopping and going to a store when buying something interesting. Thus, to truly understand what mathematics predicts, we must first explain the subject using descriptive language rather than relying solely on numerical analysis. This approach ensures thorough understanding and helps prevent misdirection due to misuse of mathematics. However, this process can take a long time, often taking years or even decades to study and adequately refine mathematical models. Additionally, it is important to note that innovative research into human phenomena often begins without direct support from mathematics.

Navigating the uncertainty of research direction becomes particularly challenging when the subject is diffuse, the path ahead unclear, and identifying a suitable framework daunting. In such situations, uncovering hidden connections among seemingly disparate subjects can offer a way forward. This process involves seeking normatively challenging topics that align with the inherent coherence of natural language. Additionally, descriptions of the phenomena under study should be articulated in simple terms to facilitate synthesis. While definitive answers to these questions may remain elusive, prioritizing intellectually stimulating subjects that resonate with linguistic coherence can make the research journey more compelling for the investigator. Søren Kierkegård's insight from his 1840 master's thesis emphasized the importance of describing any subject in a manner comprehensible to a child. It's worth noting that during his era, master's thesis presentations and defenses in open sessions lasted around 7-8 hours. This rigorous examination required candidates to be thoroughly prepared to address the panel's inquiries on various phenomena. Inspired by their dedication, we'll endeavor to similarly delve into a broad spectrum of topics, striving for clarity and accessibility in our discourse.

Starting with a visual or pedagogical exhibit may initially appear trivial, yet it serves as a powerful tool to convey the essence of a concept. Through allegory, we can unveil the hidden meanings of reality, making it easier to propose novel ideas. We introduce this simple example as a precursor to delving into theoretical discussions. The reader will encounter this passage again, presented in a more precise mathematical form, in a subsequent article within the text. This approach aims to facilitate understanding and pave the way for deeper exploration.

Wine Menu

The ubiquity of order permeates our daily lives, manifesting in various forms. From forming orderly queues at checkout counters to relying on chronological or lexicographical order in our iPhone contact lists, we navigate through life with a sense of structured arrangement. Similarly, we rely on tables of contents to navigate books and catalogs effortlessly. In academic literature, cited works are typically organized chronologically or in lexicographic order, serving the purpose of clarity and accessibility. These instances underscore the importance of order in facilitating efficiency and comprehension. The exploration of order extends beyond these examples and continues to intrigue us.

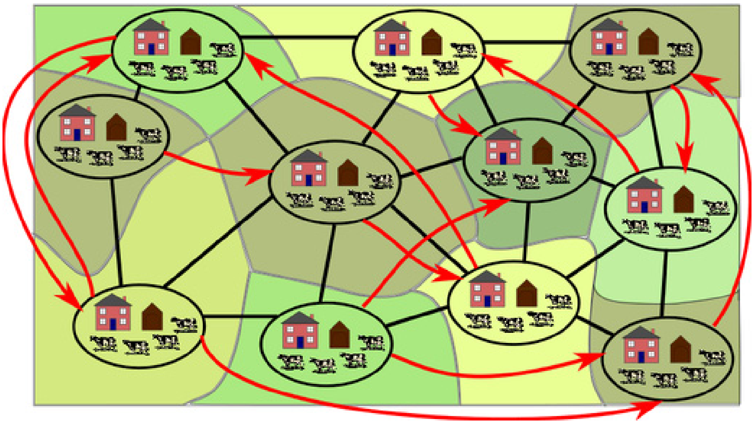
When a restaurant's sommelier informs guests that certain relatively inexpensive wines, or even the cheapest options, are temporarily unavailable, it can influence their choices in interesting ways. The absence of affordable wines on the list may prompt guests to explore other inexpensive options already approved by the sommelier. Conversely, if no initially approved inexpensive wines are available, the sommelier might recommend pricier alternatives among the available options, even though there are other good and cheaper wines to choose from. This dynamic highlights how the availability of wines, particularly within certain price ranges, can shape guests' preferences and ultimately their dining experience.

Certainly, the world of wine is both fascinating and diverse, and price often plays a significant role in consumers' decision-making process. While taste is subjective and not always correlated with price, we'll focus on price as our primary parameter for the sake of simplicity and consistency in our analysis. This approach will allow us to explore how pricing influences perceptions and choices within the realm of wine consumption.

In the wine list ordering process, wines are arranged in descending order of price, with each wine assigned a "price credential" based on its position in the list. The price of the most expensive wine is multiplied by 1, the next by 2, and so forth. These numbers represent the price credentials or moments. When selecting a wine, the guest considers the local maximum of credentials and the corresponding wine price. The guest accepts the price of the wine at the local credential's maximum as an acceptable level of price significance for choosing wines of equal or higher price. This sequence of price credentials is termed the defining sequence. The defining sequence follows a single-peaked pattern, with the peak representing the kernel of a monotonic system, as described by Mullat (1971–1995). Each credential (momentum) is calculated as the price multiplied by the number of different wines in the specific sub-list to which a wine belongs. This definition of credentials effectively organizes nested subsets of wines within the wine list.

Graphs

We will continue our exploration by depicting various phenomena through graphs. A graph is a visual representation of relations between points connected by lines. They are akin to picture books aimed at young children, who are required to join numbered points to reveal the final image. In natural language, we also encounter nodes even if we are not aware of it. When their order is unimportant, they are connected by lines/edges on the graph; otherwise arcs are used as illustrated below. The other form of graph representation is given by quadrangle matrices, i.e., matrices with an equal number of rows and columns comprising items with either 0 or 1 value, thus denoting Boolean tables. In such case, rows represent arcs pointing from vertices/nodes, i.e., out from nodes into other vertices, while columns pertain to arcs pointing into the nodes. A graph given in a Boolean table form is also a binary relation. In the discussions that follow, graphs will be explained in terms of rows and columns.



Summing up all 1-s in each row and all 1-s in each column allows forming so-called “credentials” of rows and columns in graphs. In other words, credentials represent the frequencies of 1-s in rows and columns, as they are equivalent to the total number of incoming and outgoing arcs from any particular node within the graph. Credentials can also be assigned to cells in binary tables by summing up or multiplying credentials of rows and columns in a pair wise fashion. Alternatively, using various types of arithmetic composites can further extend these credentials. These composites, as combined credentials, may characterize graphs, allowing analysis to progress in a desirable direction. This approach is particularly useful for emphasizing the dynamic nature of graph architecture — its monotone phenomena. Indeed, simply eliminating an item assigned a value of 1 from a Boolean table representing the graph would always result in decreasing our credentials values. In other words, it is irrelevant whether we employ composite or simple credentials. Similarly, replacing 0 with 1 would result in increasing credentials, creating reverse dynamics. While this may seem rather complex, in essence, credentials of graph elements are nothing but frequencies of items filled with 1-s. This is the foundation of the theory of Monotone Systems orderings.[1]

Indicators

Indicators are the preferred tools for statisticians, physicists, natural scientists and economists. Think of different metrics, average incomes, taxes, and many other areas where numbers and values are helpful. Nevertheless, despite the apparent diversity, all of these examples obey the same lexicographic or chronological ordering rules. Indeed, upon closer examination, it becomes apparent that any part, subset or sub-list of the lexicographic ordering, regardless of whether they are in ascending or descending order, again, regardless of the original, so-called general or *grand ordering*, are subject to the same ordering lexicographic or chronological rule.

Let us examine an example of grand ordering of items and select two items from the list, denoting them as Item A and Item B. We can always establish that either $\mathrm{A} \prec \mathrm{B}$ or $\mathrm{B} \prec \mathrm{A}$, otherwise $\mathrm{A} \approx \mathrm{B}$. It is very easy to form these relations when the Grand Ordering is available. However, attempting to organize the Grand Ordering with the knowledge of relations between only a various items is problematic. Indeed, suppose that given a line of items $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$ we can only say which one of these three relations $\prec, \succ, \approx$ holds for any pair. Is it possible to arrange the items in this list using some numeric indicator in harmony with these rules? This was the question that von Neumann and Morgenstern [2] attempted to answer. In their pioneering work, they provided some very strong formal axioms for rules allegedly applicable to pairs of items, denoted as the axioms of pairwise relations between items. The authors further posited that these rules must be obeyed to guarantee the desired ordering prop-

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[1] It is was originally published by Mullat (1971) in the article of Tallinn Technical University Proceedings, Очерки по Обработке Информфции и Функциональному Фнализу, Seria A, No. 313, pp. 37-44 (in Russian), and (1972) in the article extension "Ühest Neelavate Markovi Ahelate Klassist," On Absorbing Class of Markov Chains in EESTI NSV Teaduste Akadeemia Toimetised, Füüsika Matemaatika, vol. 21, No. 3, in Russian.

[2] John von Neumann and Oscar Morgenstern, (1953) Theory of Games and Economic Behavior, Princeton University Press.

erty of some numerical indicators, or what they referred to as utilities. Von Neumann and Morgenstern rigorously proved that the existence of such orderings confirmed axioms' validity, and thus established that these can be applied to order the items in accordance with the increase or decrease in their corresponding utilities. Their work was complemented by the famous theorem put forth by John Forbes Nash Jr. He provided its proof in the form of axiomatic approach to the bargaining situations, confirming that the solution of the bargaining problem based on utility orderings, as a prerequisite, is unique given that the axioms reflect the phenomena of the bargaining adequately.[3]

All orderings discussed thus far followed some usual numerical rules. However, Arrow [4], relative to those proposed by Von Neumann and Morgenstern, suggested much simpler rules, in relation to voting schemes. Unfortunately, when ordering axioms presupposing democracy were applied separately, although seemingly reasonable approach, this resulted in a paradox, as it was not possible to satisfy the same axioms applied simultaneously. This led to the conclusion, expressed in barmaid language, that democracy does not exist. Still, it is worthwhile exploring these axioms using more complex examples in which obvious coherence is employed to explain various phenomena more precisely.

Surveys

Polls are a common form of gathering the views and opinions of large groups of people and are used in many contexts. Government agencies, commissions, product market analysts, etc., conduct surveys to identify the true incentives of people. Typically, research results are presented in tabular form because it is a convenient way to visualize data and store it in databases. In fact, overview or observation tables are extensions of charts that range from a quadrilateral to a rectangle. The only difference is that instead of binary (1 and 0) inputs, the elements of such tables usually consist of codes or labels $(A, B, C, \ldots)$ called attributes, measured on a nominal scale. The nominal scale is nothing but a coded form of words or sentences reflecting some properties of products, i.e., a predetermined attitude of respondents towards the media, etc., usually accompanied by some personal data.

When analyzing such data, pie charts are commonly utilized to provide a clear visualization of the frequency of various responses at a glance. In cases where datasets are complex and encompass numerous inputs, multiple diagrams are generated to enable analysts to explore the subject from various perspectives based on their objectives. This mode of presentation essentially offers a visual depiction of the frequency density distribution associated with different

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[3] Nash J.F. (1950) The Bargaining Problem, Econometrica, Vol. 18, No. 2, 155-162.

[4] Arrow, K. (1948) The Possibility of a Universal Social Welfare Function, The Rand Corporation, Objective Analysis, Effective Solution, 20pp., https://www.rand.org/content/dam/rand/pubs/papers/2013/P41.pdf .

responses. As previously mentioned, employing a nominal frequency scale allows for arranging respondents' answers in accordance with specific classifications using personal data such as gender, age, and education. However, it's important to acknowledge that categorizing responses on a nominal scale could inadvertently lead to the ranking of respondents themselves based on their response rates. This phenomenon is evident in the ranking of entities like universities, car manufacturers, and rating scales.

Some researchers believe that this implementation of the nominal scale leads to the so-called *conforming scale*, which actually provides the truth [5,6]. However, we can discover something new by implementing the nominal scale in the form of a *defining ordering/sequence.* [7]

To proceed with the discussion, it is prudent to first explain the defining ordering through an example. Let us assume existence of a Grand Ordering of items $A_1, B_2, A_3, A_4, C_5, D_6, C_7, E_8$. Our goal is to reorganize the sequence according to their frequencies, i.e., frequencies $3,1,2,1,1$ of A, B, C, D, E. The indices $1,2,3,4,5,6,7,8 \equiv \overline{1,8}$ assigned to the items A, B, C, D, E in the sequence above denote their respective occurrences. The lowest frequencies are associated with B_2, D_6 and E_8. Let us eliminate these items from the sequence. After eliminating B_2, D_6, E_8, we eliminate C_5, C_7, as these now have the lowest frequencies, and then A_1, A_3, A_4. This results in $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$, referred to as the Grand defining sequence, highlighting the frequencies of items in different order. Namely, in contrast to its original form, the new sequence lists items in increasing/decreasing order of frequencies $1,1,1,2,1,3,2,1$. We can immediately observe upward and downward changes in frequencies, e.g., from 2 to 1, but also sliding frequencies, such as $3,2,1$. In the collection of our papers, these hikes are designated by Greek letters $\Gamma_1, \Gamma_{2,} ...$ and are thus referred to as Γ-hikes, reflecting the dynamic nature of such lists. In fact, when subsets of respondents or their survey answers/attributes are explored, it is always possible to arrange them into such dynamic lists, reflecting decreasing/increasing order of their corresponding frequencies. As a consequence, in line with representing Monotone Systems

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[5] Karin Juurikas, Ants Torim and Leo Võhandu. (2000) “Mitmemõõtmeliste andmete visualiseerimine isoleeritud majandusruumis, kasutades monotoonsete süsteemide konformismiskaalat: Uurimus Hiiumaa näitel,” (Article: Multivariate Data Visualization in Social Space using Monotone Systems conforming Scale: Case study on Hiiumaa Data).

[6] Tõnu Tamme, Leo Võhandu, and Ermo Täks. (2014) A Method to Compare the Complexity of Legal Acts, NaiL, 2nd International WorkShop on “Network Analysis in Low,” December 5, Amsterdam.

[7] Joseph E. Mullat. (1976) Extremal Subsystems of Monotonic Systems, I, Translated from Avtomatica i Telemekhanika, No. 5, pp. 130 – 139.

through graphs, the frequencies scale is equivalent to the Indicator of matching responses to the survey questions. It is important to emphasize, however, a fundamental property of the defining sequence. Namely, irrespective of which subset, sub-list, or subsequence we take from the Grand Ordering, we have independently arranged the subsequence by applying our defining rule, whereby its defining properties are in harmony with the Grand defining sequence arrangement, from which the subsequence was initially extracted.

Indeed, let us extract a subsequence A_1, C_5, A_4, C_7 form the list given earlier. Arranging the items independently, in accordance with the defining sequence rule, we obtain the frequencies $2{,}1{,}2{,}1$. It is irrelevant whether we eliminated A_1, A_4 before C_5, C_7 or vice versa — C_5, C_7 first, followed by A_1, A_4. Whichever path we take, we arrive at $2{,}1{,}2{,}1$ as the order of the frequencies. This is equivalent to generating the sequence C_5, C_7, A_1, A_4 in accordance with the Grand defining sequence $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$ arrangement.

Many natural phenomena follow well-defined rules and sequences, such as Fibonacci F_n series, in which any subsequent element F_{n+2} is the sum $F_{n+1}+F_n$ of two previous items ($1,2,3,5,8,13,...$), with $F_{n+1}\backslash F_n=1.618$ as its limit. This value is also known as the golden ratio, indicating that the relationship between two quantities is the same as the ratio of their sum to the greater of the two. The Golden Ratio is widespread in nature, from the proportions of the human body to the arrangement of leaves, spiral shells, pinecones, etc. Therefore, we can say that our defining sequence obeys the Fibonacci principle, which states that the characteristics of a part reflect the characteristics of the whole.

Using the information presented above, we can apply the Grand defining sequence to a lexicographical or chronological order of words. It is important to recall that, when some items have been eliminated, similar to the exercise above in which frequencies were presented on a nominal scale, the value of frequencies/credentials decreases. The process starts with searching for items that have the lowest credential values on the credentials scale, followed by those that are next in increasing/decreasing order, while recalculating the remaining credentials as we proceed with item replacement. This is a best-explained using survey table.

Usually, survey tables are used to present respondents' answers reflecting their attitudes or views on a specific topic. For the sake of simplicity, when answering survey questions, respondents are usually required to select one of the options provided, and can thus be represented by $A, B, C, \ldots$, denoting their choice. Now, instead of presenting these items in a straight line, we can proceed with elimination, taking two directions. Respondents, like nodes with outgoing arcs, are presented in the rows of survey tables, while columns, like ingoing arcs in graphs, denote their responses to the survey questions, coded as $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$. Some credentials composed from the corresponding frequencies of items can characterize the rows related to individual respondents $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$. Alternatively, credentials of columns can be characterized by the same or distinct compositions of frequencies using more sophisticated composites of credentials compiling, for example, arithmetic/numerical expressions as products.[8] In applying the compositions of credentials to rows and columns summing up matching answers, it is essential to ensure that the composition functions remain non-decreasing.

Now, aiming to build the defining sequence of the respondents, we can proceed in the same way with credentials of respondents, credentials of their answers, or even combining these two types of credentials (the row and column credentials). First, we must identify a cell with the lowest composition, indicating the most unreliable answer type, suggesting that the respondents are unwilling (for whatever reason) to answer the particular question truthfully. Such unreliable respondents should be eliminated, along with their unreliable answers, before recalculating the credentials of the remaining respondents and their answers. Once this is accomplished, we search for the cell that now has the lowest credentials composite and, in line with the above, remove the respondent (and his/her responses) from any further consideration. As before, we make adjustments in the credentials among all other frequencies of item ($\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$) occurrences. We proceed in the same manner until no items in the survey table remain, as all respondents and answers will be removed. Note that, due to the nature of credentials, the dynamic is always decreasing. It is rather intuitive to conclude that, as the removal procedure progresses, the remaining respondents and their answers will assume increasing positions on the credentials scale — with the lowest credentials presented first — just because we move upwards while building the defining sequence. However, once we reach the peak, the credentials start to decline, indicating that the scale is single peaked. Indeed, it can be demonstrated that the respondents' credentials values will first show the tendency to grow, and once they reach a certain point, their values will start to decline. This pattern corresponds to a typical single-peakedness of the defining sequence. Therefore, the defining sequence does not only provide an ordinary order of the respondents, but also allows identifying the conditions under which the credentials reach the peak — the highest point on the scale.

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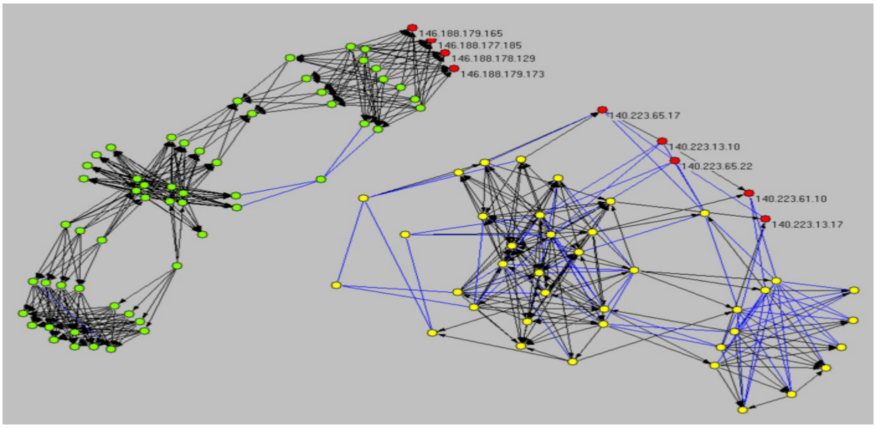
[8] An example of such type arithmetic may be found in L.K. Võhandu. (1980) Some Methods to Order Objects and Variables in Data Systems, Proceedings of Tallinn Technical University, No. 482, pp. 43 – 50..

Owing to this property, the defining sequence of credentials is a double-folded order — as the values of its elements first increase until the peak is reached, after which they start decreasing. In this respect, the defining sequence formation is akin to the Greedy type algorithms, aimed to improving some criteria.[9] Such algorithms are simple to use and are thus suitable for programming. However, it must be ascertained ***a priori*** that the result is an optimal solution, referred to as the Kernels. It is thus fortunate that the optimality of a defining sequence can be rigorously proved. This gives us confidence that we are not only proceeding in the right direction but have also chosen a suitable vehicle for our journey. This will be demonstrated through some significant examples below.

Cellular Networks

In particular, in the narrow sense of the term, *"A cellular network or mobile network is a communication network where the link to and from end nodes is wireless. The network is distributed over land areas called "cells", each served by at least one fixed-location transceiver (typically three cell sites or base transceiver stations). These base stations provide the cell with the network coverage, which can be used for transmission of voice, data, and other types of content…"* this paragraph is quoted from open sources.

In a broad context, cellular networks play a pivotal role in fostering media diversity and revolutionizing our reading behaviors. Yet, the intricate mechanisms facilitating communication with friends through platforms like Facebook, Linkedin,… exploring diverse online content, and swiftly accessing information on our areas of interest often remain obscure to many users. Cellular networks embody a complexity that eludes comprehension for most individuals, making it challenging to grasp their operational intricacies. However, the forthcoming explanation, in conjunction with the accompanying visual aid, aims to illuminate the remarkable technological marvel that is the cellular network.



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[9] Advances in Greedy Algorithms, Edited by Witold Bednorz. Published by In-Tech (2008). In-Tech is Croatian branch of I-Tech Education and Publishing KG, Vienna, Austria, ISBN 978-953-7619-27-5.

In the past, when personal computers were relatively rare, users could only interact with the system through the Disk Operating System (DOS). Some of these DOS commands can still be accessed using the C:\ command prompt. For instance, typing the command "PING www.microsoft.com" usually yields a response time of around 25 milliseconds, confirming the site's activity. If the response takes longer than 25 milliseconds or no response is received at all, it indicates a potential issue with the Internet connection. Such commands serve to verify whether a data packet sent from our PC has successfully reached the designated server. The PING command can establish connections between any two Internet locations, testing the reachability of websites. Similarly, the "TRACERT www.microsoft.com" command provides information about any packet delivery failures encountered en route to the final destination. The path of these packets can be traced as they traverse through cells or locations to their ultimate destination. The first cell in this path is typically occupied by the Gateway cell on the local subnet — the initial router in the chain responsible for packet delivery. Each subsequent router acts as a cell, akin to a post office, responsible for routing and stamping packets with delivery or transit receipts. Consequently, if direct communication cannot be established, pinpointing the location of the error becomes feasible. As cellular network designs accommodate such malfunctions by offering alternative paths, issues in one path or cell can adversely affect total network throughput for other locations. Conversely, enhancing a direct connection in one part of the cellular network can also improve overall throughput.

The outlined process facilitates the assignment of indicators representing the average number of attempts made by packets traversing the network, including cells without direct connections, to reach the destination cell from the source cell. Given the extensive number of cells within the network and the resulting multitude of possible pairwise connections, as per our previous nomenclature, this number corresponds to the total items in the table of rows and columns— a standard form of network representation. Within this table, certain items will remain empty, indicating the absence of direct connections between these cells.

Undoubtedly, the primary characteristic of cellular networks is their dynamic nature. The average number of packet deliveries, representing the attempts to reach the destination, is contingent upon the current network structure, which has the potential to influence these averages. At a higher level of abstraction, the Markov Chain satisfies certain postulates of packet deliveries and can be utilized to describe the delivery processes necessary for packets to reach their destinations. Indicators or metrics derived from the analysis of these Markov Chains can provide insights into this process. Indeed, the following excerpt from Wikipedia may offer valuable insights:"

> ***A Markov chain (discrete-time Markov chain or DTMC), named after Andrey Markov, is a random process that undergoes transitions from one state to another on a state space. It must possess a property that is usually characterized as "memorylessness": the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it. This specific kind of "memory lessness" is called the Markov property. Markov chains have many applications as statistical models of real-world processes.***[10]

While the assumption that the pertinent information of the preceding states is implicitly included in the current state is an important property of Markov Chains is highly beneficial, its dynamic nature is of primary importance for the present discussion.

This principle can be applied to the cellular networks as the most common form of communication network. We will try to elucidate what the dynamics might represent in this context. In a real Web communication network, the cellular networks can be depicted as a collection of routers or switches that are "alive." For the network to function, it is necessary to conduct periodic repairs, reconstruction or extensions, whereby some cells might be removed or replaced. Malfunctions are also a common occurrence due to the vastness and complexity of the network. So, what affect all these changes have on the network performance? Intuitively, malfunctions compromise the communication network abilities, while repairs enhance the quality of services. New communication units bring about better throughput, while removing the cells requires that the traffic be restructured. Similarly, traffic protocols are in place, allowing the packets along open routes to be rerouted in order to reach their destinations automatically.

This is where the notion of "The Monotone System" is evident in its full power. In case of positive actions (repairs/extensions), network performance in enhanced, as the components and processes become more reliable. Conversely, negative actions (malfunctions) exert negative effects, whereby network performance worsens. However, in many cases, this level of abstraction is overly simplistic. In nature, we do not expect localized improvements to result in benefits to all elements and processes. Indeed, in any system, some elements will remain unaffected, or even experience worsening. As mathematics is an exact discipline, it is sometimes necessary to introduce some simplifications when describing such complex systems. Thus, for the sake of the discussions that follow, we will further postulate that the system performance as a whole is improving (worsening) when an improvement or worsening occurs locally.

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[10] Mullat J.E. (1979) An article was published on Markov Chain analysis in the spirit of this lines in Tallinn Technical University Proceedings, Data Processing, Compiler Writing, Programming, Анализ Данных, Построение Трансляторов, Вопросы Программирования, No. 464, pp. 71–84.

This assumption prompts a very reasonable question. What does this view contribute to our understanding, explained above, of the communication networks functioning? It can, for example, allow us to proceed with optimal design of communication networks, as it renders the design process more precise.

Still, we will first revisit our Grand Ordering of items $A_1, B_2, A_3, A_4,\dots$ $\dots, C_5, D_6, C_7, E_8$ when constructing the main, i.e., the Grand defining sequence $B_2, D_6, E_8, C_5, C_7, A_1, A_3, A_4$ and its defining subsequence C_5, C_7, A_1, A_4. Let us examine the removed items B_2, D_6, A_3, E_8 more closely, in the context of constructing the sequence C_5, C_7, A_1, A_4 — as a result of which, the items B_2, D_6, A_3, E_8 and their credentials are removed. We can take an opposite approach and try to include these items back into the sequence C_5, C_7, A_1, A_4. We can first consider B_2 and then try with D_6, then with A_3 and finally E_8. In so doing, we can recreate the individual credentials for all items (B_2, D_6, A_3, E_8) even if they are not included in the existing sequence C_5, C_7, A_1, A_4. In fact, using this strategy would result in the following values: 1 for B_2, 1 for D_6, 3 for A_3 and 1 for E_8. If the objective was to increase credentials' values, we can conclude from the above that only the addition of item A_3 to the sequence C_5, C_7, A_1, A_4 will have ***a posteriori*** a positive effect, as in all other cases the credentials decline below 2. In other words, inclusion of items B_2, D_6 and E_8 will worsen the situation, because the frequencies/credentials decrease from 2 to 1, whereas addition of A_3 does not change the value of credentials, which remain equal to 2. Formally, including items into subsequence can be viewed as a destabilization, or mapping of subsequences of items. It can be shown that, in spite of the destabilization factor, the defining sequence, however, at same point cannot be extended without worsening its quality. In that case, we can say that it has reached a stable or steady state condition.

This has beneficial implications for building a desirable network via some mappings explorations. The nomenclature of these mappings is very similar to the fixed-point approach.[11] It is also evident that, attempting to map a sequence

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[11] Mullat J.E. (October 1979) Fixed point searching was first introduced in "Stable Coalitions in Monotonic Games," Translated from Avtom i Telemekh., No. 10, pp. 84–94, in the form of sequences, in accordance with parameter values upon which the mapping was constructed. Later (July 1981), the mapping technique was explained in greater detail in "Counter Monotonic Systems in the Analysis of the Structure of multivariate Distributions," Translated from Avtom. i Telemekh., No. 7, pp. 167–175.

C_5,C_7,A_1,A_4 to C_5,C_7,A_1,A_3,A_4, we have concluded that the sequence expanded by the addition of item A_3 has reached its most optimal condition. In other words, nothing can be added without worsening its state. Actually, in the discussions that follow, this fixed-point approach will be used to explain some mappings, rather than relying on a defining sequence. Thus, the communication networks analysis below will employ this fixed-point line of reasoning.

When designing a relatively simple communication network, one of the objectives might be to guarantee some throughput, such as stipulating that all packets must reach their destination in a 25 ms interval. As previously noted, the cells of the communication networks consist of routers or switches, responsible for redistributing and conducting packet movements from their source points, via temporary locations, to their final destinations. Switches are superior to routers as they learn about packets' temporary destinations, i.e., the path that must be taken when transmitting the packets, thereby significantly improving the throughput. A potential geographical layout of these extremely sophisticated and expensive devices is usually planned in the initial phase of the network design.

When determining whether to deploy a router or a switch at a particular geographic site, numerous factors must be carefully evaluated [12]. While the addition of either device can enhance throughput, it also introduces increased network maintenance, potentially uncertain operational expenses, and heightened installation costs. Ultimately, an insufficient number of these sophisticated devices may result in inadequate throughput, while an excessive number leads to escalated costs. This dilemma is addressed through a compromise necessitating multilevel optimization in the design of communication networks.

It seems intuitive that the aforementioned fixed-point search can help to solve, at least in some cases, the problem. It is also advantageous to conduct Markov Chain analysis by building the net with a desirable property to maintain the throughput above a certain level. Thus, given a Markov Chain of potential network structure in tabular form, we can proceed by adding further cells or

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[12] In this direction, an extensive study, also based on the theory of "Monotone Systems" with cellular networks, was carried out by О. А. Шорин (2006), генеральный директор ЗАО «НИРИТ», д. т. н., профессор, кафедра радиотехнических систем, Московский Технологический Университет Связи и Информации; by Р. С. Токарь (2014), технический специалист ОАО «МТС», "Elektrosvjaz," No.1, pp. 45-48, , in Russian; Р.С. Аверьянов, директор по производственной деятельности ООО «НСТТ»; Г.О. Бокк (2017), директор по науке ООО «НСТТ», д.т.н., and А.О. Шорин, технический директор ООО «НСТТ»,, "*Optimizing the size of the ring antenna and the rule formation of territorial clusters for cellular network McWILL*", "Elektrosvjaz," No.1, pp. 22-27,, *Method of "Adaptive Distribution of Bandwidth Resource"*, Russian Federation, Federal Service for Intellectual Property, RU 2 640 030 C1, Application 2017112131, in Russian,

communication lines, and analyze the outcome. While it is likely that this process will improve the performance initially, at some point, further additions will be too costly for the benefits they provide. The problem thus reduces to finding the most optimal arrangement of lines and cells in the communication network, which guarantee the best throughput, such as 25 ms stipulated above. In doing so, we have the opportunity to convert the throughput credentials into some sort of effective credentials of packets' pass characteristics, representing average number of pair wise hits between cells within the communication network obeying the monotonic property in line with that applied to items $A, B, C, \ldots$

Highly effective procedures already exist, the aim of which is to find the best stable solutions — the fixed points of Monotone Systems mappings. In these procedures, the defining sequence is constructed by means other than those previously described. However, irrespective of the methodology applied, the outcome is still the defining sequence characterized by single peakedness. Most importantly, the point at which the maximum/minimum is reached will still represent our optimal solution. This is one of the examples of solving NP hard problems with polynomial P-NP complexity.

Economy

Our next item for discussion is the implementation of Monotone Systems, specifically within the framework of retail networks. In economics, this methodology is commonly utilized in bilateral agreements between various agents involved in the delivery or production of goods. This process entails the creation of an economic network, which can be conceptualized through graphical representations showcasing potential agreements. Each node within this network symbolizes an agent, while the connections between them represent contractual arrangements such as bilateral delivery agreements or requests. It's important to highlight that when engaging in the exchange of goods and commodities, factors such as expenses, prices, and maintaining profitability are of utmost concern.

Let's delve into an illustration: Imagine a scenario where a prospective client seeks to secure a parking slot at the airport for a certain fee during their vacation period. It's crucial to note that this request implies their intention to commute to and from the airport by car, necessitating the inclusion of expenses like fuel and toll charges in the overall rental cost. This cost analysis becomes pivotal when juxtaposed with the alternative modes of transportation, such as taxi services or public transit. The decision-making process hinges on fluctuations in prices, underscoring the dynamic nature of the economic ecosystem. Moreover, within this network, each participant retains autonomy in choosing with whom to enter into contractual agreements. From a game theory perspective, these decisions manifest themselves as strategic choices described in lists detailing available agents, the services they offer, and their associated costs.

Undoubtedly, the structure of any economic network remains in a state of flux—new contracts materialize while older ones may fade into obscurity. This perpetual evolution mirrors the dynamics observed within communication networks, reinforcing the application of a Monotone System framework. In bilateral agreements, the non-fulfillment of certain actions within the retail chain can reverberate negatively throughout its entirety. Conversely, the establishment of fresh agreements typically yields positive outcomes. Yet, in practical scenarios, the addition of a new contract might inadvertently yield adverse repercussions, a risk some enterprises willingly undertake in anticipation of future gains. Hence, for the sake of simplicity, we posit that, in general, the introduction of new bilateral relations within the network tends to exert a favorable influence.

When examining economic networks, a critical aspect to explore is their capacity to withstand the so-called market volatility, which arises when prices of commodities, raw materials, or currency exchange rates fluctuate. This volatility introduces additional challenges in reconfiguring the architecture of the network to adapt to changing market conditions. An essential consideration in understanding network dynamics is the concept of transaction cost. This parameter allows for the classification of transactions based on the costs incurred in executing them. By categorizing transactions according to their associated costs, we gain insights into the efficiency and effectiveness of the network's operations. Moreover, transaction cost analysis enables us to establish a sequence of bilateral credentials, facilitating the evaluation of profit indicators concerning the design and structure of the network. Thus, by delving into transaction costs and their implications, we can better comprehend how economic networks navigate market volatility and optimize their performance.

Fixed-Point Technique

In the realm of economic network design, the fixed-point technique can be likened to a quest for equilibrium, where bilateral agreements within the network stabilize, enabling the system to withstand economic fluctuations or volatility. Once this equilibrium is attained, introducing new contracts becomes challenging without overhauling the entire network structure. The concept of single peakedness within the defining sequence aids in identifying network segments or separations that are resilient to volatility, facilitating efficient decision-making regarding commodities delivery and raw material procurement. These advantages prove invaluable in endeavors such as expanding customer base, restructuring existing networks to enhance service offerings, and exploring new avenues for improvement.

Thus far, we have considered Monotonic Systems consisting of atomic items. In other words, it was always possible to count how many items belong to the system, i.e., the number of items was finite. That was the case with lexicographical or chronological ordering of some items, whereby the credentials of items were chosen as frequencies. In such cases, the available items were presented sequentially and were clearly distinguished from others. The communication networks that were considered in the preceding discussions were also atomic, as the aim was to maximize the packet throughput from source to destination (i.e., minimize the delivery time). The same was the case in economic networks, where the network structure was only viable if it was profitable, as measured by transaction costs. In all these examples, our aim was to build a defining sequence in order to find the peak — the kernel of the ordering, because such a sequence was single peaked. It was also emphasized that the aim was to find a fixed point at which the structure design is optimal, whether we chose to design a communication or economic network.

Extending the defining sequence notion to analytical functions defined on various types of topologies is impossible because the resulting defining sequence will be infinite. Instead, we will apply the standard perspective when examining analytical single-peaked functions, aiming to find the peak of these functions. There is nothing new in this approach. The novelty, however, stems from the single-peaked phenomena, akin to the bargaining games. In such cases, one side has single-peaked preferences, and thus exhibits non-conforming behavior, while the second player, aims to maximize his/her benefits. In such scenario, the first player's preferences increase until they reach the peak, after which they start to decrease. In contrast, while the first player is moving along his/her single-peaked preferences, the second player's preferences always increase. The reader may benefit from exploring this further in the context of a sugar-pie game scheme, which is a suitable example of such analytical preferences.[13] In the present discussion, it is important to appreciate the extension of the single-peaked preferences representing the family of single-peaked functions, as this is the main advantage of this fixed-point approach. However, its application requires finding roots of some equations in order to identify stable states, inclusive of those credentials located at the peak of the credentials scale. A good example of such approach can be found in welfare economics, where the credentials of our scheme actually represent the level of transfer payments for those in need.

Mechanisms Design

Rather than analyzing and predicting agents' economic or political behavior based on established norms, our approach involves creating a conducive environment where agents, acting as rational players, can make reasoned decisions autonomously. This strategy relies on agents' rational behavior or actions to lead to reasonable outcomes. An illustrative example is the "Sugar-Pie game," which demonstrates a reversal of the typical trading model. Here, the focus shifts from determining participants' characteristics to achieving a fair distribution of resources among all players. For instance, in a scenario involving two players, an equitable division of a pie into two halves is considered a desirable objective. This approach emphasizes fairness and equitable outcomes rather than solely relying on individual characteristics or predetermined standards.

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[13] Joseph E. Mullat. (2014) "The Sugar-Pie Game: The Case of Non-Conforming Expectations", Walter de Gruyter, Mathematical Economic Letters 2, 27–31.

On the other hand, we may wish to predict the characteristics of participants a posteriori, i.e., after making this particular fair division, proclaimed as the best solution. This solution should also be understood as a design of partners' trading skills in such a way that the determination of the effective solution will be found to pursue this objective. However, it must be noted that this is the objective of the designer, rather than the goal of rational participants. Here, it must also be emphasized that we are not engaged in a symmetrical trading model, but rather the trading model characterized by so-called non-conforming interests of the participants. In fact, a standard economic situation involving company owners and company employees is not always 100% antagonistic with respect to wage negotiations. Frequently, the interests of the workers and the owners are not in conflict, even if this seems counterintuitive based on the well-known principle of scissors.

The resolution to the sweet pie division dilemma becomes more complex when additional costs are factored in. If both parties opt to enlist the services of lawyers, they will incur fees, which can be determined based on their bargaining power. For instance, in the Sugar-Pie game, these fees are illustrated as €230 and €770. Should any negotiator seek a larger share of the pie, they must be prepared to invest more in legal representation, necessitating a proportional adjustment of costs to reflect the desired outcome. Essentially, this entails establishing certain parameters on the credentials scale. Notably, the proposed sweet cake scheme can serve as a model for devising equitable political solutions during negotiations concerning the allocation of tax revenue collected by the state.

Certainly, the dynamic of citizens contributing a portion of their income as taxes underscores the vital role taxation plays in funding essential social services, including support for those in need. As the demand for assistance grows, necessitating increased transfer payments, taxes naturally rise, thereby reducing citizens' after-tax incomes. However, when unemployment rates decline and more individuals enter the workforce, tax revenues experience an upsurge, ultimately resulting in enhanced after-tax benefits for all members of society.

This scenario exemplifies a fundamental concept in the realm of economic mechanisms, illustrating how adjustments in taxation policies can yield profound effects on overall economic stability and welfare. Moreover, this paradigm can serve as a blueprint for designing political systems with desirable attributes. One such attribute is the identification of fixed points within the system, indicative of an optimal state where taxation reaches its minimum threshold. Once this equilibrium is attained, the political system can stabilize in the face of economic volatility, provided that necessary adjustments to taxation rules and norms are implemented accordingly. Thus, by leveraging insights from economic mechanisms, policymakers can strive to create political frameworks that foster stability and prosperity for the populace.

The image entitled "Bargaining Games" in the top half shows two people shaking hands over a contract, suggesting negotiation or agreement. The bottom half shows a Euro coin with its front and back sides, further emphasizing the financial or economic aspect of the negotiations. In the upper left corner, there is a mathematical symbol indicating a possible connection to game theory and mathematical analysis.

The Sugar-Pie Game:
A Case of Non-Conforming Expectations [*]

Abstract. The bargaining game involves two players negotiating for a fair share of the sugar-pie. The first player, not very keen on sweets, emphasizes quality over quantity, indicating a non-conforming expectation compared to the typical desire for more sweets. On the other hand, the second player has an open attitude towards all sweet options, regardless of their specific preferences, which also contrasts with conventional expectations. Despite their differing expectations, both players aim for an equal division of the pie, each wanting to receive half of the available sweets. The paper seeks to analyze the negotiating power of the first player in achieving this equal division, considering their emphasis on quality and the shared goal of equal distribution. In this context, "non-conforming expectations" refer to the players' divergent views or attitudes regarding the sugar-pie and their preferences for sweets.

Keywords: game theory; bargaining power; non-conforming expectations

1. INTRODUCTION

When bargaining, the players are usually trying to split an economic surplus in a rational and efficient manner. In the context of this paper, the main problem the players are trying to solve during negotiations is the slicing of the pie. Slicing depends upon characteristics and expectations of the bargainers. For example, while moving along the line at the z-axis (the size), the u-axis in Fig. 1 displays single-peaked expectations of player No. 1. In comparison, concave expectations of player No. 2 are shown in Fig. 2. The elevated single-peaked $^5/_6$ -slice curve in Fig. 1 corresponds to the lower, but adversely increasing, concave $^1/_6$ curve of expectations in Fig. 2, and for the other sugar-pie allotment $^1/_9$, $^8/_9$.

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[*] Mullat J.E. (2014) The Sugar-Pie Game: The Case of Non-Conforming Expectations, Walter de Gruyter." *Mathematical Economic Letters* 2 , 27–31.

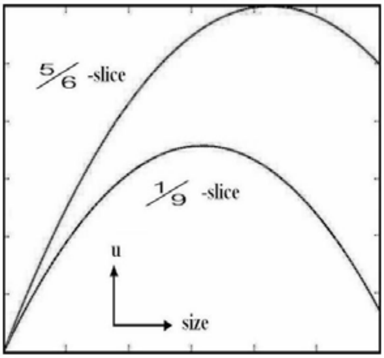

Figure 1. Player No. 1 expectations

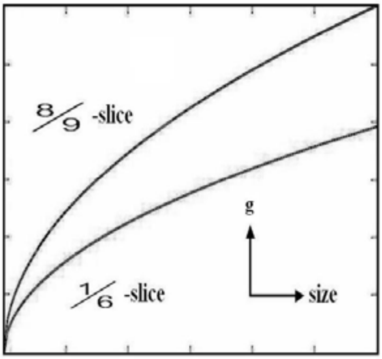

Figure 2. Player No. 2 expectations

Given that the players' expectations are non-conforming,[1] as shown in Fig 1., and Fig. 2, splitting a pie no longer represents any traditional bargaining procedure. Instead of dividing the slices, the procedures can be resettled. Thus, we can proceed at distinct levels of one parameter — parametrical interval of the size, which turns to be the scope of negotiations. In fact, Cardona and Ponsattí (2007, p. 628) noticed that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all the parties in the bargaining process have the same, conforming expectations. This is even true when the expectations of the second player are principally non-conforming, i.e., concave, rather than single-peaked. Indeed, in the case of non-conforming expectations, the scope of negotiations — also known as "*well defined bargaining problem*" or *"bargaining set"* related to individual rationality (Roth, 1977) — allows for dropping the axiom of "Pareto efficiency." Thus, combined with the *breakdown* point, the well-defined problem, instead of slices, can be solved inside parametrical interval of the pie size.

With these remarks in mind, any procedure of negotiating on slices accompanied by sizes can be perceived as two sides of the same bargain portfolio. Therefore, it is irrelevant whether the players are bargaining on slices of the pie, or trying to agree on their size. This highlights the main advantage of the parametric procedure — it brings about a number of different patterns of interpretations of outcomes in the game. For example, it can link an outcome of an economy to a suitable size of production, scarcity of resources, etc. — all of which are indicators of most desirable solutions. Indeed, our initiative could serve to unify the theoretical structure of economic analysis of productivity problem. Leibenstein (1979, p. 493) emphasized that *"…the situation need not be a zero sum game. Tactics, that determine the division can affect the size of the pie.*" Clarifying these guidelines, Altman (2006, p. 149) wrote:

> "*There are two components to the productivity problem: one relates to the determination of the size of the pie, while the second relates to the division of the pie. Looked upon independently, all agents can jointly gain by increasing the pie size, but optimal pie size is determined by the division of pie size.*"

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[1] We say also interpersonally incompatible, i.e., impossible to match through a monotone transformation (Narens & Luce, 1983).

2. THE GAME

The game demonstrates how a sugar-pie is fairly sliced between two players. The first player, denoted as HE, is a soft negotiator, not very keen on sweets, and would not accept a pie if the size of the pie is too small or too large. In HIS view, too small or too large sugar-pies are not of reasonable quality. The second player, hereafter referred to as SHE, is a tough negotiator and prefers obtaining sweets, whatever they are.[2]

The axiomatic bargaining theory finds the asymmetric Nash solution by maximizing the product of players' expectations above the disagreement point $d = \langle d_1, d_2 \rangle$: $\arg\max_{0 \le x+y \le 1} f(x,y,\alpha) = (u(x) - d_1)^{\alpha} \cdot (g(y) - d_2)^{1-\alpha}$, the asymmetric variant (Kalai, 1977).

Although the answer may be known to the game theory purists, the questions often asked by many include: *What are* x, y, α, $u(x)$ *and* $g(y)$*? What does the point* $\langle d_1, d_2 \rangle$ *mean? How is the* $\arg\max$ *formula used?* The simple answer can be given as:

- x *is HIS slicing of the pie, and* α *is HIS bargaining power,* $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ *is HIS expectation, for example* $u(x) \equiv x$, *of HIS* x *slicing of the pie;*
- y *is HER slicing of the pie, and* $1 - \alpha$ *is HER bargaining power,* $0 \le y \le 1$;
- $g(y)$ *is HER expectation, for example* $g(y) \equiv \sqrt{y}$, *of HER* y *slicing of the pie.*

Based on the widely accepted nomenclature, we call $s = \langle u(x), g(y) \rangle$ the utility pair. The disagreement point $d = \langle d_1, d_2 \rangle$ denotes what HE and SHE collect if they disagree on how to slice the pie. The sugar-pie disagreement point is $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$, whereby the players collect nothing. Further, we believe that expectations from the pie are more valuable for HER, indicating HER desire $g(½) = \sqrt{½} = 0.707$ for sweets, which is greater than HIS desire $u(½) = 0.5$. Considering the argmax formula $f(x,y,\alpha)$, one may ask a new question: *What is the standard that will help to redesign bargaining power* α *facilitating* HIS *negotiations to obtain a desired half of the pie?* SHE may only accept or reject the proposal. A technical person can shed light on the solution. We can start by replacing $u(x)$ with x, $y = 1 - x$, $g(y)$ with $\sqrt{1-x}$, and taking the derivative of the result $f(x, 1-x, \alpha)$ with respect to the variable x by evaluating $f'_x(x, 1-x, \alpha)$. Finally, with $x = ½$, the equation $f'_x(½, ½, \alpha) = 0$ can be solved for α; indeed, $\alpha = 1/3$ provides a solution to the equation $f'_x(½, ½, \alpha) = 0$.

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[2] Note that, for the purpose of the game, we do not ignore the size of the pie but put this issue temporarily aside.

In general, one might feel comfort in the following judgment: "*Even in the face of the fact that SHE is twice as tough a negotiator,*[3] *to count on the half of the pie is a realistic attitude toward HIS position of negotiations. Surely, rather sooner than later, since HE revealed that SHE prefers sweets whatever they are, HE would have HER agree to a concession.*" This attitude might well be the standard of redesigning the power of HIS negotiation abilities if half of the pie is desirable as a specific outcome of negotiations.

Returning to the pie size issue, it will be assumed that, in the background of HIS judgment, the quality of the pie first increases, when the size is small. On the other hand, as the size increases, the quality eventually reaches the peak point, after which it starts to decline with the increasing size. Thus, the quality is single-peaked with respect to the size. For HER, the pie is always desirable. To handle the situation, we assume that HE possesses all the relevant skills of the pie slicing. Nonetheless, based on the aforementioned assumptions, for HIM, the slicing may, in some cases, not be worth the effort at all. If the slicing does not meet its goal, as just emphasized, HE can promote HIS own understanding of how to slice the pie properly. HE can enforce decisions, or effectively retaliate for breaches — recruiting for example "enthusiastic supporters," (Kalai, 1977, p.131). SHE, on the other hand, lacks slicing abilities, knowledge, skills or competence. Thus, if interests of both players in the final agreement are sometimes different or sometimes not, SHE cannot fully control HIS actions and intentions. In these circumstances, SHE might show a willingness to agree with HIS pie division, or at least not resist HIS privileges to make arrangements upon the size of the pie. Hence, from HER own critical point of view, by acting in common interest, SHE may admit HER lack of knowledge and skill. This clarifies HIS and HER asymmetric power dynamics.

Whether HE is committed or not is irrelevant for his decision to accept HER recommendation regarding the size z. HE is committed, however, only to slice x aligned in eventual agreement. The above can be restated, then, with the condition that HE seeks an efficient size z of the pie determined by the slice x. Let, e.g., the utility pair $\langle \mathrm{u}, \mathrm{g} \rangle$ of HIS and HER expectations be given by:

$$\mathrm{u}(\mathrm{z},\mathrm{x}) = \mathrm{z} \cdot [(1 + \mathrm{x}/2) - \mathrm{z}] \; ; \; \mathrm{g}(\mathrm{z},\mathrm{y}) = \mathrm{z} \cdot \sqrt{\mathrm{y}} \,, \; \mathrm{z} \in [0,1], \; \mathrm{x},\mathrm{y} \in [0,1].$$

The root $\mathrm{z} = \frac{1}{2}$ resolves $\langle \mathrm{u}'_{\mathrm{z}}(\mathrm{z},\mathrm{x})|_{\mathrm{x}=0} \rangle = 0$ for z, and the root $\mathrm{z} = \frac{3}{4}$ resolves $\langle \mathrm{u}'_{\mathrm{z}}(\mathrm{z},\mathrm{x})|_{\mathrm{x}=1} \rangle$ accordingly. We can thus define efficient slices, relative to the size z, as a curve $\mathrm{x}(\mathrm{z})$, which solves $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},\mathrm{x}) = 0$ for x. Evaluating x from $\mathrm{u}'_{\mathrm{z}}(\mathrm{z},\mathrm{x}) = 0$ and subsequently replacing $\mathrm{x}(\mathrm{z})$ into $\mathrm{u}(\mathrm{z},\mathrm{x})$ and $\mathrm{g}(\mathrm{z},\mathrm{x})$, yields $\mathrm{u}(\mathrm{z}) = \mathrm{z}^2$ and $\mathrm{g}(\mathrm{z}) = \mathrm{z} \cdot \sqrt{3 - 4 \cdot \mathrm{z}}$. Now, given the scope $\mathrm{z} \in [\frac{1}{2}, \frac{3}{4}] \subset [0,1]$ of the negotiations, the bargaining problem $\langle \boldsymbol{S}, \mathrm{d} \rangle$ passes then into parametric space $\boldsymbol{S}_{\mathrm{z}} = \langle \mathrm{u}(\mathrm{z}), \mathrm{g}(\mathrm{z}) \rangle$. In HIS view, the pie must fit the size requirements, since outside the interval $[\frac{1}{2}, \frac{3}{4}] \subset [0,1]$ the size z is

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[3] Let us say that SHE pays HER solicitor twice as much as HE does.

inefficient — too small and thus not useful at all, or too large and of inferior quality. Therefore, the disagreement occurs at $d = \langle u(\frac{1}{2}), g(\frac{3}{4}) \rangle$, $d = \langle \frac{1}{4}, 0 \rangle$. The Nash symmetric solution to the game is found at $z = 0.69$, $x = 0.74$. On the other hand, HIS asymmetric power 0.21 is adequate for negotiating with HER about receiving half of the pie. The size $z = 0.62$, for example, in HIS view, fits the necessary capacities of a *stovetop* for provision of high quality sugar-pie.

Once again, to find the Nash symmetric solution, a technically minded person must resolve the equation $f'_z(z, \alpha) = 0$ for z, where $f(z, \alpha) = (u(z) - \frac{1}{4})^{\alpha} \cdot g(z)^{1-\alpha}$ when $\alpha = ½$; $z = 0.69$ provides a solution to the equation. Thus, solving the equation $u'_z(0.69, x) = 0$ for x yields $x = 0.74$. To find the power of asymmetric solution, we first solve the equation $u'_z(z, \frac{1}{2}) = 0$ for z, $z = 0.62$, $x = \frac{1}{2}$. Then, we solve $f'_z(0.62, \alpha) = 0$ for α and find that HIS power matches $\alpha = 0.21$, which is adequate for negotiating with HER when an equal slicing of the pie is desirable, i.e., both HE and SHE receive ½ of the pie.

3. BARGAINING PROCEDURE

The strategic bargaining game operates as a game of alternating offers. Given some light conditions, it is well known that, when players partaking in this type of game are willing to make concessions during the negotiations, they are likely to embrace the axiomatic solution. That is the reason why we continue our discussion in terms of a procedure similar to the strategic approach.

To recall, there are two players in our game — HE, with emphasis on quality, and SHE, with no specific preferences. A precondition for the agreement was that the expectations of negotiators solely depend on HIS framework of how to set the size parameter, rather than the slice. As a consequence of this dependence, efficient sizes provide a fundamental correspondence to crucial slices. Accepting the precondition, SHE will only propose efficient sizes, as HE will reject all other choices.

Nonetheless, it is realistic that SHE would — by negligence, mistake or some other reason — recommend an inefficient size, which HE would mistakenly accept. On the contrary, it is also realistic that HE has an intention to disregard an efficient recommendation. This will be irrational handling as, in any agreement, no matter what is going on, both players are committed by proposals to slices. Therefore, making a new proposal, HER recommendation on sizes makes a rational argument that HE must accept or reject in a standard way. Such an account, instead of an agreement upon slices, as we believe, explains that the outcome of the bargaining game might be a desirable size $z^{\circ} \in [z_1, z_2]$. Hereby, only the interval, named also the scope $[z_1, z_2]$ of negotiations, bids proposals, which now, by default, are binding efficient sizes with slices x. Consequently, the bargaining game performs exclusively in the interval $[z_1, z_2]$. Hence, $[z_1, z_2]$ is the scope of HIS efficient sizes of most trusted sugar-pie platforms for negotiations, where players would choose sizes, accepting or rejecting proposals. The negotiators' expectations, depending on $[z_1, z_2]$, arrange a bargaining frontier $\boldsymbol{S}_z$ as a way to assemble the bargain portfolio. Therefore, the negotiators may focus on making the size proposals. If rejected, the roles of actors change and a new proposal is submitted. The game continues in a traditional way, i.e., by alternating offers.

Observation. *In the alternating-offers sugar-pie game, the functions* $(u(z) - d_1)^{\alpha}$ *and* $(g(z) - d_2)^{1-\alpha}$ *imply HIS and HER expectations, respectively, over the pie size* $z \in [z_1, z_2]$. *With the risk* $1 >> q > 0$ *of negotiations to collapse prematurely into disagreement point* $d = [d_1, d_2]$, *the solution* z^o *of well-defined bargaining problem* $\langle \boldsymbol{S}_z, d \rangle$ *is enclosed into the interval* $[z', z''] \subset [z_1, z_2]$, $z^o \in [z', z'']$. *The margins* z', z'' *are solving the equations*

$$(1-q)\cdot(u(z^1) - d_1)^{\alpha} = (u(z^2) - d_1)^{\alpha}, \ (1-q)\cdot(g(z^2) - d_2)^{1-\alpha} = (g(z^1) - d_2)^{1-\alpha}$$

for variables z^1, z^2 (cf. Rubinstein 1998, p. 75).

In our example, when $x = ½$ *(the half of the pie is a desirable (ex-ante) solution), HIS negotiation power* 0.21 *leads to the asymmetric solution* $z = 0.62$. *Let the risk factor of the premature collapse of negotiators be* $q = 0.05$. *Then, the interval* $[0.61, 0.64] \subset [0, 1]$ *sets up pie sizes providing the desirable solution, whereby the pie will be divided equally.*

4. CONCLUSION

In view of the above, a pretext for the analysis of the domain and the extent of bargain portfolio for two fictitious negotiators, denoted as HE and SHE, were established. The portfolio was supposed to account for the players having nonconforming expectations. Instead of slicing the sugar-pie, such an account allowed for the inclusion of a guide on how the eventual consensus ought to be analyzed and interpreted within the scope of negotiations upon the size of the pie. Players' bargaining power indicators specified by the bargaining problem solution were used in compliance with their respective desired visions and ambitions.

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The Game of
Wellness Club Formation

Wellness Club/Coalition Formation by Bargaining Based on Boolean Tables

Joseph E. Mullat [1]

Abstract. This study explores the nexus between the Nash Bargaining Problem and club/coalition formation, elucidating the derivation of utility functions using Boolean tables. It appeals to specialists in social sciences and economics by integrating bargaining and rational choice mechanisms. The paper delineates the interrelation between these concepts within the framework of general choice theory, emphasizing the formalization of choice acts through internal and external descriptions via binary relations. In the context of the "Well-Being" company, the CEO's initiative to reduce disability compensations by promoting wellness events among employees underscores the practical application of the Nash Bargaining Problem. Through a survey aimed at discerning employees' preferences, the CEO seeks to optimize the selection of wellness activities based on their varying levels of interest. This scenario epitomizes the integration of rational choice mechanisms and bargaining concepts in addressing organizational challenges, aligning with the theoretical framework discussed in the paper. By leveraging scalar optimization principles, the CEO aims to derive a solution that maximizes employee engagement while minimizing company losses, thus exemplifying the real-world implications of the theoretical foundations presented.

Key words: coalition; game; bargaining; algorithm; monotonic system

Concise Glossary of Mathematical Notations

Matrix $W = \left\| \alpha_{ij} \right\|_n^m$ signifies the Boolean Table, where $\alpha_{ij} = 1$ or $\alpha_{ij} = 0$ denotes one of its Boolean elements holding the value of 1 and 0, respectively. For players' joint expectations, we use the notation (x, y), where $x \in 2^N, N = \{1,...,i,...,n\}$, x – subset of rows N, and $y \in 2^M$, $M = \{1,..., j,...,m\}$, y – subset of columns M. Sub-table H or block denotes the players' joint crossing of rows x and columns y, whereas notation $|H|$ indicates the number $\sum \alpha_{ij} \in H$ entries that are equal to 1 in the sub-table H. In addition to the conventional pairwise operations – $L \cup G$, $L \cap G$, $H = \varnothing$, $L \subset G$, $\supset$, $L \subseteq G$, $\supseteq$, $H \subseteq W$, and $W \supseteq \Gamma$ – we sometimes use the notations $H + i$ and $H + j$, and similarly for $H \setminus i$, $H \setminus j$ we use $H - i$, $H - j$ for $i \in x$, $j \in y$ and $H + i$, $H + j$, for $H \cup i$, $H \cup j$. The notation $\langle \alpha_1, \alpha_2, ... \rangle$ represents an ordered set of elements while $\{\alpha_1, \alpha_2, ...\}$ represents an unordered set.

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1. INTRODUCTION

Science distinguishes itself through its pursuit of empirical knowledge, utilizing systematic observation, experimentation, and reasoning based on evidence (Popper, 2002; Ponterotto, 2005; Abhary et al. 2009). Scientific goals focus on understanding natural phenomena using rigorous methods to minimize bias and subjectivity. Science motives prioritize curiosity and discovery, aiming for objective truths. The conditions involve peer review, reproducibility, and the constant refinement of theories, setting it apart from other human endeavors. Its implementation involves understanding and consolidating the theory, as well as a careful selection of the methodology, approaches, and techniques employed in the investigations, and the reproducibility of the obtained results.

Usually, a theoretical or practical contribution to the theory, or practice of applying a theory, consists of expanding existing categories, concepts, models, simplifications, etc. with the aim of obtaining new theoretical facts or solving unsolved problems. However, there is another approach to extracting new knowledge from old and well-established categories, which necessitates the capacity to recognize new relationships or links hidden between the existing fundamental categories. This innovation that consists of taking aspects that already exist and putting them together in a new way is the main motive behind this article, as a part of which a comparison with, or rather an interpretation of, the well-known Nash Bargaining Scheme based on the theoretical provisions of the coalition game as applied to Boolean Tables. The application of this provision lies in the fact that Boolean Tables facilitate the calculation of the utility functions of the coalition game, thereby allowing the individual division of the total payoff or revenue to be determined for each player separately. For this purpose, we have developed an example involving a “wellness club” offered to company employees to illustrate what such an innovation can do in terms of the use of Boolean Tables for addressing the Bargaining Problem and solving coalition games.

Therefore, the specific type of game situation depicted via Boolean Tables, as it seems to us, does not limit, but rather enriches the theory and provides additional tools, which in practice affect the socioeconomic stability of most societies and open up opportunities for data analysis, both in social networks and for the interpretation of rational behavior of participants in the network models of modern economy.

In conclusion, we were also able to illustrate various utility functions of coalition games (so-called supermodular/concave functions) that are actually responsible for the coalition motivation of players when receiving payouts (supermodular functions) or when the collective behavior of the players loses its appeal (super-concave functions).

Since the publication of “The Bargaining Problem” by John F. Nash Jr. in 1950, the framework proposed within has been developed in different directions. For example, in their *Bargaining and Markets* monograph, Martin Osborn and Ariel Rubinstein (1990) extended the “axiomatic” concept initially developed by Nash to incorporate a “strategic” bargaining process pertinent to everyday life. The authors posited that the “time shortage” is the major factor encouraging agreement between bargainers. Various bargaining problem varieties emerged in the decades following Nash's pioneering work, prompting game theoreticians to seek their solutions, most of which did not necessarily comply with all Nash axioms. Beyond any doubt, the "Nonsymmetrical Solution" proposed by Kalai (1977); Harsanyi's (1967) "Bargaining under Incomplete Information"; "Experimental Bargaining", which was later proposed by Roth (1985); and the "Bargaining and Coalition" paper published by Hart (1985) are among some notable contributions to this field, confirming the fundamental importance of bargaining theory.

Bargaining and rational choice mechanisms are interrelated concepts and are treated as such in this work. In the context of general choice theory, the choice act can be formalized through internal and external descriptions, which requires the use of binary relations and the theoretical approach, respectively. Thus, both description modes apply to the same object, albeit from different perspectives. The Nash Bargaining Problem and its solution express exactly the same phenomenon. Given a list of axioms—such as "Pareto Efficiency" or "Independence of Irrelevant Alternatives"—in terms of binary relations the rational actors must follow, the solution is reached through scalar optimization applied to the set of alternatives. Indeed, the scalar optimization is at the core of the Nash's axiomatic approach and is the reason for its success in facilitating the bargaining solution derivation. In this respect, the motive of this work is also to present a "derivation" of a bargaining solution based on large Boolean Tables and some theoretical foundations offered by the method. Unfortunately, in following Nash's scenario, numerous difficulties emerged.

Boolean table representation transforms the "cacophonous" real-life scenario into a relatively simple scalar index, rendering it more understandable (Malik and Zhang, 2009). However, given the ambiguity of scalar optimization, this representation makes the picture more complicated. Indeed, we consider in this work a purely atomic object that does not intuitively satisfy the "invariance under the change of scale of utilities" postulate typically assumed in the proofs. From the researcher's point of view, the issue stems from the incertitude pertaining to the most optimal choice based on the scalar criteria. The Nash axiomatic approach suggests that employing the product of utilities is the most appropriate, thus removing any uncertainty from further discussion. Nevertheless, in the context of the method presented here, it is posited that a reasonable solution might emerge, while game-analysts would be advised to include the method in a wider range of applicable game analysis tools.

We provide a basic example of our bargaining game in the following section. In the appendix, we also illustrate another negotiation scheme between a coalition and its manager based on Boolean Tables where we adopt some of the usual utility functions. It is worth noting that some elements of the main example, such as signals or distortions, are not the main topic of our discussion and merely serve to illustrate the complexity of the negotiation process.

In Section 3, we attempt to explain our intentions in a more rigorous manner. Accordingly, we formulate our "Bargaining Problem Based on Boolean Tables" as pure strategies, thus providing the foundation for the discussions presented in Section 4, where we exploit our pure Pareto Frontier in terms of so-called Monotonic Systems chain-nested alternatives – the Frontier Theorem.

In order to implement the Nash theorem for nonsymmetrical solution (Kalai, 1977), in Section 5, we introduce what we deem to be an acceptable, albeit complex, algorithm in general form. Even though lottery is not permitted in the treatment of Boolean Tables subsets representing pure strategies, as this approach does not necessary produce the typical convex collection of feasible alternatives, we claim that the algorithm will yield an acceptable solution. Finally, in Section 6, we present an elementary attempt to formulate a regular approach to coalition formation under the guidance of a coalition formation supervisor, which we denoted as the manager's structure. This attempt is depicted in Figure 2, which also provides the notation nomenclature of chain-nested alternatives adopted in our Monotonic Systems theory discussed in Section 4. In Section 7, we summarize and discuss the entire analysis, while also providing an independent heuristic interpretation, before concluding the study in Section 8.

2. WELLNESS GAME DEPICTED AS A BOOLEAN TABLE

The Chief Executive Officer (CEO) of the “Well-Being” company wishes to encourage the employees to partake in wellness-promoting events or activities, as this is expected to reduce company losses arising from disability compensations. To identify the employees’ preferred activity types, the CEO has initiated a survey. According to the survey results, the five proposed wellness events would attract varying degrees of interest, as shown in Table 1.

Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 1		x	x			2
Em. No. 2	x	x		x	x	4
Em. No. 3		x	x	x		3
Em. No. 4	x	x		x	x	4
Em. No. 5	**Heavy smoker**	**Clumsy swimmer**	x	x		2
Em. No. 6	x	x	x	x	x	5
Em. No. 7		x	x			2
Total	3	6	5	5	3	22

Table 1 Employee preferences pertaining to the company-sponsored wellness-promoting events

While the staff responses should serve as an indication that they are ready to participate in their chosen events, knowing the precariousness of human nature, the CEO is not sure that they will keep their promises. Accordingly, the CEO decides to reward all employees who actually participate in recreational events, which will be organized in the "Wellness Club". The CEO has found a sponsor who is willing to issue 12 Bank Notes to cover the cost of the project. Upon closer examination of the rewards policy, the CEO realized that many obstacles had to be overcome in order to put the project into practice.

First, organizing events in which only a few employees would partake is neither practical nor cost-effective. Accordingly, it is necessary to stipulate a minimum number of employees that must subscribe to each wellness event. On the other hand, it is desirable to promote all events, encouraging the employees to attend them in greater numbers. For this initiative to be effective, instructions (as a set of rules full of twists and turns) regarding the rewards regulations should be fair and concise. Usually, in such situations, someone (in this case a manager) must be in charge of the club formation and reward allocation. As the CEO is responsible for financing the wellness events, he/she should retain control of all processes. Thus, the CEO proposes to write down the **First Club Regulation**: *The CEO rewards one Bank Note to an employee participating in at least* k *different events* (where k is determined by the CEO).

Determining the most optimal value of the parameter k is not a straightforward task, as it is not strictly driven by employees' preferences for participation in specific events. In fact, this task is in the manager's jurisdiction, while also being dependent on the employees' decisions, as they act as the club members. The goal is to prohibit some club members from dropping out from the wellness events preferred by other members and joining other events, thus requiring too many different events to be organized. This issue can be avoided by the inclusion of the **Second Club Regulation**: *If a member of the club being organized expresses an intention to participate in fewer than* k *events in favor of receiving a reward, none of the members of the future club is rewarded*. By instituting this regulation, the goal is to eliminate events that would not have a sufficient number of participants. This objective is reflected in the **Third Club Regulation**: *manager's reward basket will be equal to the lowest number of actual participants per event.* Accordingly, the manager might decide to exclude an event with the lowest number of participants and distributing them to other clubs to increase the reward value. However, the third reward regulation does not address the situation in which a club member declines to attend an event, allowing an individual outside the club to participate instead. In such a case, the club "events list" may become shorter than that presented in Table 1, and would determine the reward size.

In this case, it cannot be ruled out that the manager when communicating with CEO will misrepresent the preferences of club members. Indeed, let the CEO makes a decision $\mathrm{k} = 1$, which for some reason became available to the manager. Knowing this, the manager's actions can be predicted in according to club rules. Using employees' responses, the manager can identify the most "popular" wellness event, as well as those who intend to take part in it. From the above provision it follows that the manager will always receive the maximum reward by convincing all employees of the newly organized club to participate only in this specific, that is, the most popular event. Otherwise, the manager will definitely receive less than this maximum. Rational Club members will also definitely agree to this offer, since regardless of their participation in any other event, their reward will still be guaranteed. The same logic obviously applies for $\mathrm{k} > 1$ as well.

The essence of establishing fair rules is related to the determination of the manager's leadership. If no rewards are offered to the manager, the formation of a grand coalition is guaranteed, as all employees will become members of the club. This is the case because participating in any event guarantees that all employees will receive a reward. However, due to the lack of interest in events with a small number of participants, the formation of a club with a large number of participants under the leadership of a manager is not always feasible.

As previously noted, the manager might receive a minor reward if a "curious" employee decides to take part in an "unpopular event". Indeed, the third club regulation stipulates that the number of participants in the most "unpopular event" governs the manager's reward size. Being aware of the potential manipulation of the regulations, and being a rational actor, the company CEO will thus strive to keep the decision k confidential. It is also reasonable to believe that all parties involved – the club members, the manager and the CEO – will have their own preferences regarding the value of k. Therefore, an explanation based on the salon game principles is applicable to this scenario. Using this analogy, let us assume that the CEO has chosen a card k and has hidden it from the remaining players. Let us also assume that the manager and the club members have reached an agreement on their own card choice in line with the three aforementioned club regulations. The game terminates and rewards are paid out only if their chosen card is higher than that selected by the CEO. Otherwise, no rewards will be paid out, despite taking into consideration the club formation.

Not all factors affecting the outcome have been considered above. Indeed, the positive effect, f_k, which the CEO hopes to achieve, depends on the decision k. For some reason, we have to expect a single $\cap$-peakedness of the effect function. As a result, this function separates the region of k values into what we call prohibitive and normal range. In the prohibitive range, which includes the low k values, the effect has not yet reached its maximum value. On the other hand, when the k value is high (i.e., in the normal range), the f_k limit is exceeded. Therefore, in the prohibitive range, the CEO's and the manager's interests compete with each other, making it reasonable to assume that the CEO would keep his/her decision a secret. In the normal range, they might cooperate, as neither benefits from very high k values, given that both can lose their payoffs. Consequently, using the previous card game analogy, in the normal range, it is not in the CEO's best interests to hide the k card.

Given the arguments presented above, the game scenario can be illustrated more precisely. Using the data presented in Table 1, and assuming that a reward will be granted at $k = 1,2$, the CEO may count upon all seven employees to become the club members. If all employees participate in all events, each would receive a Bank Note, and the manager's basket size would be equal to 3. It would be beneficial for the manager to entice to the club members to decline participation in *"No Smoking"* and *"Fattening Diet"* events, as this would increase his/her own reward to 5. As all club members will still preserve their rewards, they have no reason not to support the manager's suggestion, as shown in Table 2.

Table 2

Wellness events	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Total*
Em. No. 1	x	x		2
Em. No. 2	x		x	2
Em. No. 3	x	x	x	3
Em. No. 4	x		x	2
Em. No. 5		x	x	2
Em. No. 6	x	x	x	3
Em. No. 7	x	x		2
Total	6	5	5	16

Table 3

Swimming Pool	*Total*
x	1
x	1
x	1
x	1
	0
x	1
x	1
6	6

In this scenario, the sponsor would have to issue 12 Bank Notes, which can be treated as expenses associated with organizing the club. The sponsor may also conclude that $k = 1$ is undesirable based on the previous observation that the manager can deliberately misrepresent the members' preferences for personal gain.[2] Indeed, the manager can offer one Bank Note to an employee when the CEO makes a decision $k = 1$. Knowing that $k = 1$, the manager may suggest to the club members to subscribe to the ***"Swimming Pool"*** event only. However, in the opinion of the potential swimming club members, the manager must compensate the heavy smoker and clumsy swimmer No. 5 for the losses sustained by forfeiting his/her initial choices. Employee No. 5 may otherwise report the manager to lobbyists of the company's board, as this particular employee, while continuing to smoke, would be able to demand compensation from the manager for not disclosing his/her "fraudulent activities". In this case, following the regulations in force (see Table 3), manager's reward will be equal to 4 (1 deducted for the signal and 1 for clumsy swimmer). This would still exceed the value indicated in Table 1. Thus, in order to decrease project expenses or avoid misrepresentations, the company board may follow the swimming club advice and propose $k \geq 3$.

It could be argued that $k \geq 3$ results in decreased participation in wellness events because Employees No. 1, 5 and 7 will be excluded from the club and will immediately cease to partake in any of their initially chosen events. Based on Table 4, it can also be noted that, in that case, the remaining employees (i.e., 2,3,4 and 6) will still participate in heath events and will still be rewarded.

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[2] The more complex case of misrepresentation follows, as promised.

Table 4

Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 2	x	x		x	x	4
Em. No. 3		x	x	x		3
Em. No. 4	x	x		x	x	4
Em. No. 6	x	x	x	x	x	5
Total	3	4	2	4	3	16

Now, the manager's reward basket is equal to 2, since only Employees No. 3 and 6 would take part in *"Fitness Exercises"*. Consequently, the sponsor expenses decrease from 10 to 6. In this case, the CEO may decide to allow the manager to retain his/her reward of 3 by eliminating *"Fitness Exercises"* from the event list, as organizing it for two participants only is not justified, as shown in Table 5. Note that, due to this decision, Employee No. 3 must be excluded from the club list, in line with the second club regulation (cf. the suggestion above to eliminate the *"No Smoking"* and *"Fattening Diet"* events).

Table 5

Wellness events	*No Smoking*	*Swimming Pool*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 2	x	x	x	x	4
Em. No. 4	x	x	x	x	4
Em. No. 6	x	x	x	x	4
Total	3	3	3	3	12

This decision does not seem reasonable, given that the aim of the initiative was to motivate the employees to partake in fitness programs and improve their wellness. Thus, let us assume that $k = 5$ was the board proposal. This result would only concern Employee No. 6 who is willing to participate in all the wellness events offered, as shown in Table 6.

Table 6

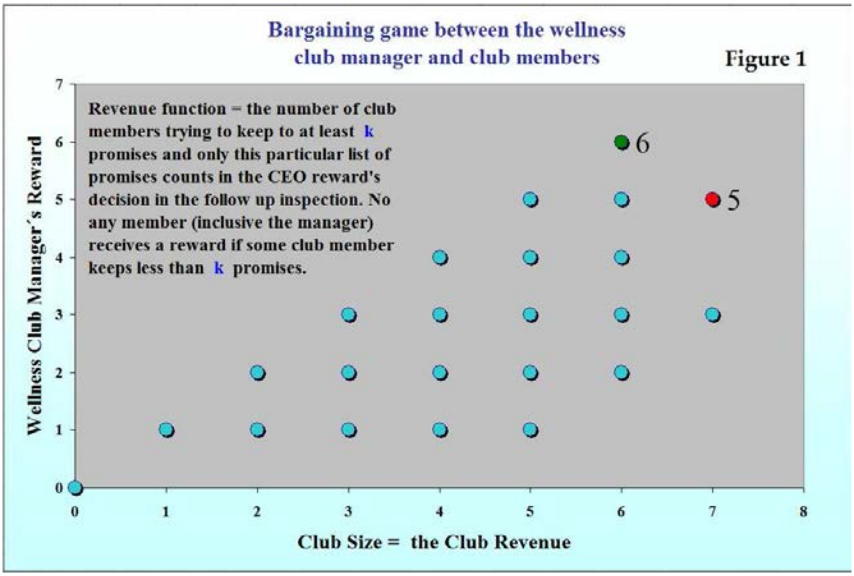
Wellness events	*No Smoking*	*Swimming Pool*	*Fitness Exercises*	*Moderate Alcohol*	*Fattening Diet*	*Total*
Em. No. 6	x	x	x	x	x	5
Total	1	1	1	1	1	5

The manager may decide not to organize the club, as this would result in a reward equal to only one Bank Note. Similarly, the CEO is not incentivized to promote all five events if only one employee would take part in each one. As a result, at the board meeting, the CEO would vote against the proposal $k = 5$. In sum, the CEO's dilemma pertains to the alternative k choices based on the information given in Table 7.

Table 7

	Club Members	*Club Manager*	*Compen-sation*	*Signal*	*Bank Notes Used*	*Bank Notes Left*
T. 1, $k=2$	7	3	0	0	10	2
T. 2, $k=2$	7	5	0	0	12	0
T. 3, $k=1$	6	4	1	1	12	0
T. 4, $k=4$	3	1	0	0	4	8
T. 5, $k=4$	3	3	0	0	6	6
T. 6, $k=5$	1	1	0	0	2	10

To clarify the situation presented in tabular form, it would be helpful to visualize the CEO's dilemma using the bargaining game analogy, where 12 Bank Notes are shared between the manager and the club members.



The decision on the most optimal k value taken at the board meeting will be revealed later using rigorous nomenclature, as only a closing topic is necessary to interrupt our pleasant story for a moment.[3]

Let us assume that three actors are engaged in the bargaining game: N employees, a manager in charge of club formation, and the CEO. Certain employees from this set $N = \{1,...,i,...n\}$ – the potential members of the club x, $x \in 2^N$, have expressed their willingness to participate in events y, $y \in 2^M$, $W = \| \alpha_{ij} \|_n^m$. Let a Boolean Table $W = \| \alpha_{ij} \|_n^m$ reflect the survey results per-

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[3] Those unwilling to continue with the discussions on bargaining presented in the subsequent sections should nonetheless pay attention to this closing remark.

taining to the employees' preferences, whereby $a_{ij} = 1$ if employee i has promised to participate in event j, and $a_{ij} = 0$ otherwise. In addition, the set 2^M of all subsets of columns M denotes the allegedly subsidized events, whereby $y \in 2^M$ have been examined: $M = \{1,..., j,...m\}$.

We can calculate the manager payoff $F_k(H)$ using a sub-table H formed by crossing the entries of the rows x and columns y in the original table W by further selection of a column with the least number $F_k(H)$ from the list y. The number of 1-entries in each column belonging to y determines the payoff $F_k(H)$. The family of utility functions $\upsilon^k(x,y) \equiv \upsilon^k(H)$, $k \in \{1,...,k,...,k_{max}\}$, on N is typically used for analyzing the coalition games. In this particular case, for every pair $L \subset G$, $L, G \in 2^N \times 2^M$, we suppose that $\upsilon^k(L) \leq \upsilon^k(G)$. Further assuming that the CEO payoff function $f_k(H)$ has a single $\cap$-peakedness, in line with the decisions $\langle 1,...,k,...,k_{max} \rangle$, $f_k(H)$ reflects some kind of positive effect on the company deeds. In this case, sponsor expenses will be equal to $\upsilon^k(H) + f_k(H)$.

Finally, it is appropriate to share some ideas regarding a reasonable solution to our game. The situation presented above is similar to the Nash Bargaining Problem first introduced in 1950, where two parties – the club members and the manager – are striving to reach a fair agreement. It is possible to find the Bargaining Solution $S_k \in \{H\} = 2^N \times 2^M$ for each particular decision k, as explained in the sections that follow. The choice of the number k is not straightforward, as previously discussed. For example, $k = 4, 5$ may be useful based on some *ex ante* reasoning, whereas maximum payoffs are guaranteed for the club members when $k = 1$. As that decision is irrational, because only one event will be organized and, even though it will attract the maximum number of participants, it would fail to yield a positive effect $f(S_k)$ on their wellness, which was the primary objective of instituting this initiative. The choice of higher k was previously shown to be counterproductive as too many events will be offered, but would be attended by only a few employees, yet the sponsor would benefit from issuing fewer rewards. For example, for $k = k_{max}$, an employee with the largest number of preferred k_{max} events might become the only member of the club. This is akin to the median voter scheme discussed by Barbera et al. (1993). A further consultation in this "white field" is thus necessary.

3. BARGAINING GAME APPLIED TO BOOLEAN TABLES

Suppose that employees who intend to participate in wellness events have been interviewed in order to reveal their preferences. The resulting data can be arranged in a $n \times m$ table $W = \|\alpha_{ij}\|_n^m$, where $\alpha_{ij} = 1$ indicates that an employee i has promised to participate in event j, otherwise $\alpha_{ij} = 0$. In this respect, the primary table W is a collection of Boolean columns, each of which comprises Boolean elements related to one specific event. In the context of the bargaining game, we can discuss an interaction between the wellness club and the manager. The club choice X is a subset of rows $\langle 1,...,i,...,n \rangle$ denoting the newly recruited club members, whereby a subset y of columns $\langle 1,...,j,...m \rangle$ is the manager's choice – the list of available events. The result of the interaction between the club and the manager can thus represent a sub-table H or a block, denoting the players' joint expectation. In this scenario, there are only two players, with Player No. 1 denoting the club and Player No. 2 the manager, and both parties are driven by the desire to receive the rewards. Let us assume that all employees have approved our three reward regulations.[4] While both players are interested in wellness events, their objectives are different. Player No. 1 might aim to motivate each club member to agree to partake in a greater number of company-sponsored events. Player No. 2, the manager, might desire to subscribe maximum number of participants for each event arranged by the company. Let a pair of utilities (υ, F) denote the players' No. 1 and No. 2 payoffs, whereby both players will bargain considering all possible expected outcomes (x, y) in the form of sub-tables H of table W.

Our intention in developing a theoretical foundation for our story was to follow the Nash's (1950) axiomatic approach. Unfortunately, as previously observed, some fundamental difficulties arise when adopting a similar strategy. Below, we summarize each of these difficulties, and propose a suitable equivalent. When proceeding in this direction, we first formulate the Nash's axioms in their original nomenclature before reexamining their essence in our own nomenclature. This approach would allow us to provide the necessary proofs in the sections that follow.

As noted by Nash (1950), "we may define a two-person expectations as a combination of two one-person expectation. ... A probability combination of two two-person expectations is defined by making the corresponding combinations for their components" (p. 157). Readers are also advised to refer to Sen Axiom 8*1 on page 127, or sets of axioms, as well as to consult the work of Luce and Raiffa (1958), Owen (1968) and von Neumann and Morgenstern (1947), with the latter being particularly relevant for utility index interpretation. Rigorously speaking, the compactness and convexity of a feasible set $\boldsymbol{S}$ of

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[4] We recall the main regulation that none of the club members, inclusive of the manager, receive their rewards if a certain club member participates in fewer than k events.

utility pairs ensure that any continuous and strictly convex function on $\boldsymbol{S}$ reaches its maximum, while convexity guarantees the maximum point uniqueness.

Let us recall the other Nash axioms. The solution must comply with the INV (invariance under the change of scale of utilities), IIA (independence of the irrelevant alternatives), and PAR (Pareto efficiency) postulates. Note that, following PAR, the players would object to an outcome s when an outcome s' that would make both of them better off exists. We expect that the players would act from a *strong individual rationality* (SIR) principle. An arbitrary set $\boldsymbol{S}$ of the utility pairs $s=(s_1,s_2)$ can thus be the outcome of the game. A disagreement arises at the point $d=(d_1,d_2)$ where both players obtain the lowest utility they can expect to realize – the *status quo* point. A *bargaining problem* is a pair $\langle \boldsymbol{S},d\rangle$[5] and there exists $s\in\boldsymbol{S}$ such that $s_i>d_i$ for $i=1,2$ and $d\in\boldsymbol{S}$. A *bargaining solution* is a function $f(\boldsymbol{S},d)$ that assigns to every bargaining problem $\langle \boldsymbol{S},d\rangle$ a unique element of $\boldsymbol{S}$. The bargaining solution f satisfies SIR if $f(\boldsymbol{S},d)>0$ for every bargaining problem $\langle \boldsymbol{S},d\rangle$.

The advantage of our approach, which guarantees the same properties, lies in the following. We define a feasible set $\boldsymbol{S}$ of expectations, or in more convenient nomenclature, a feasible set $\boldsymbol{S}$ of alternatives as a collection of table W blocks: $\boldsymbol{S}\subseteq 2^W$. Akin to the disagreement or point of contention in the Nash scheme, we define an empty block $\varnothing$ as a *status quo* option in any set of alternatives $\boldsymbol{S}$, which we call “the refusal of choice”. Next, given any two alternatives H and H' in $\boldsymbol{S}$, an alternative $H\cup H'$ belongs to $\boldsymbol{S}$. In other words, in our case, the set $\boldsymbol{S}$ of feasible alternatives always forms an upper semi-lattice. If an alternative $H\in\boldsymbol{S}$, it follows that all its subsets meet the condition $2^H\subseteq\boldsymbol{S}$. Although these arguments do necessitate further discussion, at this juncture, we will state that this is our equivalent to the convex property and will play the same role in proofs as it does in the Nash scheme.

The Nash theorem asserts that there is a unique bargaining solution $f(\boldsymbol{S},d)$ for every bargaining problem $\langle \boldsymbol{S},d\rangle$, which maximizes the product of the players’ gains in the set $\boldsymbol{S}$ of utility pairs $(s_1,s_2)\in\boldsymbol{S}$ over the disagreement outcome $d=(d_1,d_2)$. This is a so-called symmetric bargaining solution, which satisfies INV, IIA, PAR, and SYM, which the players symmetrically identify if and only if

$$f(\boldsymbol{S},d)=\arg\max_{(d_1,d_2)\le(s_1,s_2)}(s_1-d_1)\cdot(s_2-d_2). \tag{1}$$

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[5] We use the bold notifications $\boldsymbol{S}$ to comply with the original nomenclature. Notification S is also preserved for the stable point, which is introduced later in the paper.

It is difficult to make an *ad hoc* assertion regarding properties that can guarantee the uniqueness of similar solution based on Boolean Tables. Nevertheless, in the next section, we claim that our bargaining problem on $\boldsymbol{S} \subseteq 2^{W}$ has the same symmetrical or nonsymmetrical shape:

$$f(\boldsymbol{S},\varnothing) \equiv f(\boldsymbol{S}) = \arg\max_{H\in\boldsymbol{S}} \upsilon(H)^{\theta} F(H)^{1-\theta} \qquad (2)$$

for some $0 \le \theta \le 1$ provided that Nash axioms hold.

4. THEORETICAL ASPECTS OF THE BOOLEAN GAME

Henceforth, the table $W = \left\|\alpha_{ij}\right\|_{n}^{m}$ will denote the Boolean Table discussed in the preceding section, representing employees' pledges to attend wellness events. At this juncture, it is beneficial to examine the H rows x, symbolizing the arrival of new members to the club, each of whom is committed to participating in at least k events. The offered events form the event list in column y, $k = 2,3,\ldots$, where k represents the reward decision. For each event in the event list y, at least $F(H)$ of club members intend to fulfill their promises. For example, let us consider the number of rows in H pertaining to the gain $\upsilon(H)$ of Player No. 1 (e.g., the club member's x common gain $\upsilon(H) = |x|$), while the gain of Player No. 2 (the manager's reward) is represented by $F(H)$.

Let us look at the bargaining problem in conjunction with the players' preferences. The expectations of the incoming club members $i \in x$ towards the event list y can easily be "raised" by $r_i = \sum_{j\in y} \alpha_{ij}$ if $r_i \ge k$, and $r_i = 0$ if $\sum_{j\in y} \alpha_{ij} < k$, $i \in x$, $j \in y$. Similarly, the manager's expectation to the event list y can be "accumulated" by means of table H as $c_j = \sum_{i\in x} \alpha_{ij}$, $j \in y$.

We now consider the Bargaining Game scenario depicted in the Boolean Table in more rigorous mathematical form. Below, we use the notation $H \subseteq W$. The block or sub-table H contained in W will be understood in an ordinary set-theoretical nomenclature, where the Boolean Table W is a set of its Boolean 1-elements, whereby all 0-elements will be eliminated from the consideration. Thus, H as a binary relation is also a subset of W. Henceforth, when referring to an element, we assume that it is a Boolean 1-element.

For an element $\alpha \equiv \alpha_{ij} \in W$ in the row i and column j, we use the similarity index $\pi_{ij} = c_j$, counting only on the Boolean elements belonging to H, $i \in x$ and $j \in y$. As the value of $\pi_{ij} = c_j$ depends on each subset $H \subseteq W$, we may write $\pi_{ij} \equiv \pi \equiv \pi(\alpha, H)$, where the set H represents the π-function parameter. It is evident that our similarity indices π_{ij} may only increase with the "expansion" and decrease with the "shrinking" of the parameter H. This yields the following fundamental definitions:

Definition 1. *Basic monotone property. Monotonic System will be understood as a family* $\{\pi(\alpha,H): H\in 2^W\}$ *of* π *-functions, such that the set* H *is a parameter with the following monotone property: for two particular values* $L,G\in 2^W$, $L\subset G$ *of the parameter* H, *the inequality* $\pi(\alpha,L)\le\pi(\alpha,G)$ *holds for all elements* $\alpha\in W$. *In ordinary nomenclature, the* π *-function with the definition area* $W\times 2^W$ *is monotone on* W *with regard to the second parameter on* 2^W.

Definition 2. *Using a given arbitrary threshold* u *for a non-empty subset* $H\subseteq W$ *let* $V(H)$ *be the subset* $V(H)=\{\alpha\in W:\pi(\alpha,H)\ge u\}$. *The non-empty* H *-set indicated by* S *is called a stable point with reference to the threshold* u *if* $S=V(S)$ *and there exists an element* $\xi\in S$, *where* $\pi(\xi,S)=u$. *For a comparable concept, see Mullat (1979, 1981). Stable point* $S=V(S)$ *has some important properties, which will be discussed later.*

Definition 3. *By Monotonic System kernel we understand a stable point* $S^*=S_{\max}$ *with the maximum possible threshold value* $u^*=u_{\max}$.

Libkin et al. (1990), Genkin et al. (1993), Kempner et al. (1997), and Mirkin et al. (2002) have investigated similar properties of Monotonic Systems and their kernels. With regard to the current investigation, it is noteworthy that, given a Monotonic System in general form, without any reference to any kind of “interpretation mechanism”, one can always consider a bargaining game between a coalition H – Player No. 1, with utility function $\upsilon(H)$, and Player No. 2 with the payoff function $F(H)=\min_{\alpha\in H}\pi(\alpha,H)$. In line with the Nash theorem, a symmetrical solution has to be found in form (1). We will prove below that our solution has to be found in the symmetrical or nonsymmetrical form (2).

Definition 4. *Let* d *be the number of Boolean 1-entries in table* W. *An ordered sequence* $\overline{\alpha}=\langle\alpha_0,\alpha_1,...,\alpha_{d-1}\rangle$ *of distinct elements in the table* W *is called a defining sequence if there exists a sequence of sets* $W=\Gamma_0\supset\Gamma_1\supset\ldots\supset\Gamma_p$ *such that:*

A. Let the set $H_k=\{\alpha_k,\alpha_{k+1},...,\alpha_{d-1}\}$. *The value* $\pi(\alpha_k,H_k)$ *of an arbitrary element* $\alpha_k\in\Gamma_j$, *but* $\alpha_k\notin\Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$, $j=0,1,...,p-1$.

B. *There does not exist in the set* Γ_p *a proper subset* L *that satisfies the strict inequality* $F(\Gamma_p)<F(L)$.

Definition 5. *A defining sequence is complete, if for any two sets* Γ_j *and* Γ_{j+1} *it is impossible to find* Γ' *such that* $\Gamma_j \supset \Gamma' \supset \Gamma_{j+1}$ *while* $F(\Gamma_j) < F(\Gamma') < F(\Gamma_{j+1})$, $j = 0,1,...,p-1$.

It has been established that, in an arbitrary Monotonic System, one can always find a complete defining sequence (see Mullat, 1971, 1976). Each set Γ_j is the largest stable set with reference to the threshold $F(\Gamma_j)$. This allows us to formulate our Frontier Theorem.

Frontier Theorem. *Given a Bargaining Game on Boolean Tables with an arbitrary set* $\boldsymbol{S}$ *of feasible alternatives* $H \in \boldsymbol{S}$, *the expectations points* $\left(v(\Gamma_j), F(\Gamma_j)\right)$, $j = 0,1,...,p$, *of a complete defining sequence* $\overline{\alpha}$ *arrange a Pareto frontier in* $\Re^2$.

Proof. Let $W^s \in \boldsymbol{S}$ be the largest set in $\boldsymbol{S}$ containing all other sets $H \in \boldsymbol{S}$: $H \subseteq W^s$. Let a complete defining sequence $\overline{\alpha}$[6] exist for W^s. Let the set H^c be the set containing all such sets $V(H)$, where $V(H) = \{\alpha \in W : \pi(\alpha, H) \geq F(H)\}$. Note that $H \subseteq V(H^c)$ and $F(H^c) \geq F(H)$. Now, for accuracy, we must distinguish three situations:

(a) In the sequence $\overline{\alpha}$ one can find an index j such that $F(\Gamma_j) \leq F(H^c) < F(\Gamma_{j+1})$ $j = 0,1,...,p-1$;

(b) The case of $F(H^c) < F(W) = F(\Gamma_0)$;

(c) $F(H) > F(\Gamma_p)$. This case is impossible because, on the set Γ_p, the function $F(H)$ reaches its global maximum.

In case of (b), the expectation $\left(\upsilon(\Gamma_0), F(\Gamma_0)\right)$, $\Gamma_0 = W$, is more beneficial than $\left(\upsilon(H), F(H)\right)$, which concludes the proof. In case of (a), let $F(\Gamma_j) < F(H^c)$, otherwise the equality $F(\Gamma_j) = F(H^c)$ is the statement of the theorem (when reading the sentence after the next, the index $j+1$ should be replaced by j). In this case, the set H^c must coincide with Γ_{j+1}, $j = 0,1,...,p-1$, otherwise the defining sequence $\overline{\alpha}$ is incomplete. Indeed, looking at the first element $\alpha_k \in H^c$ in the sequence $\overline{\alpha}$, it can be ascertained that, if $\Gamma_{j+1} = H^c$ does not hold, the set $H_k = H^c$ because it is the largest stable

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[6] We are not going to use any new notifications to distinguish between Boolean Tables W and W^s.

set up to the threshold $F(H^c)$. Hence, the set H_k represents an additional Γ-set in the sequence $\overline{\alpha}$ with the property A of a complete defining sequence. The inequalities $F(\Gamma_{j+1}) = F(H^c) \geq F(H)$, $\upsilon(\Gamma_{j+1}) = \upsilon(H^c) \geq \upsilon(H)$, due to $\Gamma_{j+1} = H^c \supseteq H$ and the basic monotonic property, are true. Thus, the point $\left(\upsilon(\Gamma_{j+1}), F(\Gamma_{j+1})\right)$ is more advantageous than $\left(\upsilon(H), F(H)\right)$. ■

5. ALGORITHM FOR SOLVING THE BARGAINING PROBLEM

To summarize the scenario presented above, the discussion that follows will be governed by the Nash bargaining scheme. Some reservations (see, for example, Luce and Raiffa, 1958, 6.6) hold as usual because our bargaining game on Boolean Tables is purely atomic, i.e., it does not permit lotteries (which are an important element of any bargaining scenario). Given this restriction, the uniqueness of the Nash solution cannot be immediately guaranteed. It is important to note that the Nash solution of $\langle \boldsymbol{S}, d \rangle$ depends only on the disagreement point d and the Pareto frontier of $\boldsymbol{S}$. The compactness and convexity of $\boldsymbol{S}$ are important only insofar as they ensure that the Pareto frontier of $\boldsymbol{S}$ is well defined and concave. Rather than starting with the set $\boldsymbol{S}$, we could have imposed our axioms on a problem defined by a non-increasing concave function (and disagreement point d, as argued by Osborn and Rubinstein, 1990, p. 24). In our case, $\left(\upsilon(\Gamma_j), F(\Gamma_j)\right)$, $j = 0,1,...,p$, represents an atomic Pareto frontier. Therefore, it is possible to provide the proof of non-symmetrical solution (see Kalai, 1977, p. 132), as well as perform the derivation with the product of utility gains in its asymmetrical form (2).[7] The problem of maximizing the product is primarily of technical nature. In the discussions that follow, we will introduce an algorithm for that purpose. We will first comment on the individual algorithm step in relation to the definitions.

As shown below, the algorithm's last iteration through the step **T** detects the largest kernel $\overline{K} = S^*$[8] (Mullat, 1995). The original version (Mullat, 1971) of the algorithm aimed to detect the largest kernel and is akin to a greedy inverse serialization procedure (Edmonds, 1971). The original version of the algorithm produces a complete defining sequence, which is imperative for finding the bargaining solution aligned with the Frontier Theorem. In the context of the current version, however, it fails to produce a complete defining sequence. Rather, it only detects some thresholds u_j, and some stable set $\Gamma_j = S_j$. The sequence $u_0, u_1, ...$ is monotonically increasing (i.e., $u_0 < u_1 < ...$) while the sequence $\Gamma_0, \Gamma_1, ...$ is monotonically shrinking (i.e., $\Gamma_0 \supset \Gamma_1 \supset ...$), whereby

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[7] There are many techniques that guarantee the uniqueness of the product of utility gains. We are not going to discuss this matter here, because this case is an exemption rather than a rule.

[8] It is possible that some smaller kernels exist as well.

the set $\Gamma_0 = W$ is stable towards the threshold $u_0 = F(W) = \min\limits_{(i,j)\in W} \pi_{ij}$. Hence, the original algorithm is always characterized by higher complexity. For finding the bargaining solution, we can still implement a less complex algorithm, which would require modification of the indices $\pi_{ij} = c_j$.

Let us consider the problem of identifying the players' joint choice $H_{\max}$ representing a block $\arg\max_{H\in\mathcal{S}} \upsilon(H)^{\theta} F(H)^{1-\theta}$ of the rows x and columns y in the original table W satisfying the property $\sum\limits_{j\in y} \alpha_{ij} \geq k$, $i \in x$. Let an index $\pi_{ij} = r_i \cdot \upsilon^{\theta} \cdot c_j^{1-\theta}$[9]. The following algorithm solves the problem.

Algorithm.

I. Set the initial values.

1i. Assign the table parameter H to be identical with W, $H \Leftarrow W$. Set the minimum and maximum bounds a, b for the threshold u imposed on the $\pi_{ij} \in H$ values.

A. Establish that the next step (Step **B)** produces a non-empty sub-table H. Remember the current status of table H by creating a temporary table H°: $H^{\circ} \Leftarrow H$.

1a. Test u as $\frac{1}{2} \cdot (a + b)$ using Step **B**. If it succeeds, replace a by u, otherwise replace b by u and H by H°: $H \Leftarrow H^{\circ}$ – "regret action".

2a. Go to **1a**.

B. Check if minimum $\pi_{ij} \in H$ over i, j can be equal to or greater than u.

1b. Delete all rows in H where $r_i = 0$. This step fails if all rows in H must be deleted, in which case proceed to **2b**. The table H is shrinking.

2b. Delete all elements in columns where $\pi_{ij} \leq u$. This step fails if all columns in H must be deleted, in which case proceed to **3b**. The table H is shrinking.

3b. Perform Step **T** if no deletions were made in **1b** and **2b**; otherwise go to **1b**.

T. Test whether the global maximum is found. Table H has halted its shrinking.

1t. Among numbers $\pi_{ij} \in H$, find the minimum $\min \leftarrow \pi_{ij}$ and then perform Step **B** with new value $u = \min$. If it succeeds, set $a = \min$ and return to Step **A**; otherwise, terminate the algorithm.

6. BARGAINING GAME – MODIFICATION OF COOPERATIVE ASPECTS

As mentioned earlier, we consider the game of two persons given as the choice of Player No. 1 as a subset of rows x in the table $W = \left\|\alpha_{ij}\right\|_n^m$ and player No. 2 as the choice y of columns. Thus, a joint choice (x, y) is made in the form of a sub-table H or block. Below, we consider this choice as expectation (x, y)

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[9] This index obeys the basic monotone property as well.

in the set-theoretic sense as a subset H of elements of the table W at the intersection of rows x and columns y. The coalition associated with the choice of H in this case is the set of rows. The utility function $\upsilon = \upsilon(H)$ of such a coalition is ambiguous and depends on the Player No. 2's choice y.

A cooperative game is a pair (N, υ), where N symbolizes a set of participants and υ is the game utility function. Function υ is called a supermodular if $\upsilon(L) + \upsilon(G) \leq \upsilon(L \cup G) + \upsilon(L \cap G)$ whereas it is submodular if the inequality sign $\leq$ is replaced by $\geq$, $L, G \in 2^N$. Various properties of supermodular set functions are specified (see Cherenin et al. 1948 and Shapley, 1971, among others). In the appendix, we illustrate a game, which is neither supermodular nor submodular, but rather a mixture of the two, where single and pairwise participants do not receive extra rewards. On the other hand, it is obvious that all properties of supermodular functions υ are also applicable for the submodular $-\upsilon$ utility function and vice versa.

Let the utility function υ of our game when forming a coalition and the manager's choice is represented by some set-theoretic function $\upsilon(H)$. Particularly, it is useful take $\upsilon(H) = |H|$, or some polynomial function p of its argument like $p(|H|)$. The joint marginal contribution to the coalition x of the participant $i \in x$ and, in particular, the marginal expectation of the manager $j \in y$ (the marginal utilities of the participants) can be represented as $\pi(\alpha_{ij}, H) = \frac{\partial \upsilon(H)}{\partial i} \cdot \alpha_{ij} \cdot \frac{\partial \upsilon(H)}{\partial j}$ for $\frac{\partial \upsilon(H)}{\partial i} = \upsilon(H + i) - \upsilon(H)$ if $i \notin x$. When participant $i \in x$ leaves the coalition x, then $\frac{\partial \upsilon(H)}{\partial i} = \upsilon(H) - \upsilon(H - i)$. Marginal notation is valid for any $H \in 2^W$, which is denoted below as $H \cup i \equiv H + i$, and $H \setminus i \equiv H - i$. For manager expectation y, $j \in y$, similar notation $H \pm j$ is used, i.e., $\frac{\partial \upsilon(H)}{\partial j}$. We will not distinguish between the situations when the participant $i \in x$ joins a coalition or leaves the coalition x, or the manager counts on the expectation $j \in y$ or does not count on j when $j \notin y$. We hope that such actions of the participants in our game clearly emphasize the importance of forming a coalition, as well as that of the manager's choice when a participant is already a member of a coalition or when someone only intends to join the coalition. Exactly the same consideration applies to manager expectations.

Suppose that the interest of a participant i to join the coalition x equals the participant's marginal contribution. A coalition game is convex (concave) if for any pair L and G of coalitions $\mathrm{L} \subseteq \mathrm{G} \subseteq \mathrm{x}$ the inequality $\frac{\partial \upsilon(\mathrm{L})}{\partial \mathrm{i}} \leq \frac{\partial \upsilon(\mathrm{G})}{\partial \mathrm{i}}$ $\left(\frac{\partial \upsilon(\mathrm{L})}{\partial \mathrm{i}} \geq \frac{\partial \upsilon(\mathrm{G})}{\partial \mathrm{i}}\right)$ holds for each participant $\mathrm{x} \in \mathrm{W}$. A similar statement can be made regarding the manager's choice $\mathrm{j} \in \mathrm{y}$.[10]

Theorem. *For the bargaining/coalition game to be convex (concave) it is necessary and sufficient for its utility function to be a supermodular (submodular) set function.* Extrapolated from Nemhauser et al. (1978).

Now, in view of the theorem, marginal utilities of participants in the supermodular game motivate them to form coalitions in certain cases. In a modular game, where the utility function is both supermodular and submodular, marginal utilities are indifferent to collective rationality because entering a coalition would not allow anybody to win or lose their respective payments. In contrast, it can be shown that collective rationality is sometimes counterproductive in submodular games. Therefore, in supermodular games, formation of too many coalitions might be unavoidable, resulting in, for example, the grand coalition. In such cases, in Shapley's (1971) words, this leads to a "snowballing" or "band-wagon" effect. On the other hand, submodular games are less cooperative. In order to counteract these "bad motives" of participants in both supermodular and submodular games, we introduce below a second actor – the manager. Hence, we consider a bargaining game between the coalition and the manager.

Convex game induces an accompanying bargaining game with the utility pair $\left(\upsilon(\mathrm{H}), \mathrm{F}(\mathrm{H})\right)$, where $\mathrm{F}(\mathrm{H}) = \min_{\mathrm{i} \in \mathrm{x}} \frac{\partial \upsilon(\mathrm{H})}{\partial \mathrm{i}}$, whereas concave game induces utility pair, where $\mathrm{F}(\mathrm{H}) = \max_{\mathrm{i} \in \mathrm{x}} \frac{\partial \upsilon(\mathrm{H})}{\partial \mathrm{i}}$. Here, the coalition assumes the role of Player No. 1 with the utility function $\upsilon(\mathrm{H})$. The coalition manager, the Player No. 2, expects the reward $\mathrm{F}(\mathrm{H})$.

Proposition. *The solution* $\mathrm{f}(\boldsymbol{S}, \varnothing)$ *of a Nash's Bargaining Problem* $\langle \boldsymbol{S}, \varnothing \rangle$*, which accompanies a convex (concave) coalition game with utility function* v*, lies on its Pareto frontier* $\Gamma_0 \supset \Gamma_1 \supset \ldots \supset \Gamma_p$ *maximizing (minimizing) the product* $\upsilon(\Gamma_j)^{\theta} \cdot \frac{\partial \upsilon(\Gamma_j)}{\partial \alpha}^{1-\theta}$ *for some* $\mathrm{j} = 0,1,\ldots,\mathrm{p}$*, and* $0 \leq \theta \leq 1$. This statement is a clear corollary from the Frontier Theorem. ■

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[10] Shapley (1971) recognized this condition as equivalent, whereby similar marginal utilities in their investigation of some optimization problems (Nemhauser et al., 1978) have been proposed. Muchnik and Shvartser (1987) also pointed to the link between submodular set functions and the Monotonic Systems, as outlined by Mullat (1971).

In accordance with the basic monotonic property (see above), given some monotonic function $\pi(i,H) = \frac{\partial \upsilon(H)}{\partial i}$ on $N \times 2^N$, it is not immediately apparent that there exists some utility function $\upsilon(H)$ for which the function $\pi(i,H)$ constitutes a monotonic marginal utility $\frac{\partial \upsilon(H)}{\partial i}$. The following theorem, guided by the work of Muchnik and Shvartser (1987), addresses this issue.

The existence conjecture. *For the function* $\pi(i,H)$ *to represent a monotonic marginal utility* $\frac{\partial \upsilon(H)}{\partial i}$ *of some supermodular (submodular) function* $\upsilon(H)$*, it is necessary and sufficient that*

$$\frac{\partial}{\partial k}\frac{\partial \upsilon(H)}{\partial i} \equiv \pi(i,H) - \pi(i,H-k) = \pi(k,H) - \pi(k,H-i) \equiv \frac{\partial}{\partial i}\frac{\partial \upsilon H}{\partial k}$$

holds for $i,k \in x \subseteq N$. The interpretation of this condition is left for the reader.

7. DISCUSSION

We start this discussion with a heuristic interpretation of the arguments presented in the preceding sections. Following the apparatus of monotonic systems adopted in data mining (Mullat, 1971), it is reasonable to find the Pareto frontier in terms of the game theory as well. The potential manager's bargaining strategy is presented next. First, in the grand coalition $N \equiv \Gamma_0$, the manager identifies the participants with the least marginal utility $u_0 = F(N) = \min_{i \in N} \frac{\partial \upsilon(N)}{\partial i}$. The manager will advise these individuals to stay in line and wait for their rewards. All participants that have joined the line will be temporarily disregarded in any coalition formation. Following the game convexity, one of the remaining participants (i.e., those still engaged in the coalition formation process) must find themselves worse off owing to the participants in line being excluded from the process. Manager would thus suggest to these participants to also join the line and wait for their rewards. As the manager continues the line construction in the same vein, a scenario will emerge in which all-remaining participants Γ_1 (outside the line) are better off than u_0, i.e., better off than those waiting in line for their rewards. Now, the manager repeats the entire procedure upon participants $\Gamma_1, \Gamma_2, ...$ until all participants from N have agreed to wait in line to obtain their rewards. Manager keeps a record of the events $0,1,...$ and is aware when the marginal utility thresholds increase from u_0 to u_1, etc. It is obvious that the increments are always positive: $u_0 < u_1 < ... < u_p$.

What is the outcome of this process? Participants staying in line arrange a nested sequence of coalitions $\left\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \right\rangle$, whereby the most powerful marginal participants, those present when the last event p occurs, form a coalition Γ_p. The next powerful coalition will be Γ_{p-1}, etc., coming back once again to the starting event 0, when the participants arrange the grand coalition $N = \Gamma_0$. Our Frontier theorem guarantees that such a manager bargaining strategy, in convex games, classifies a Pareto frontier $\left\langle (\upsilon(\Gamma_0), u_0), (\upsilon(\Gamma_1), u_1), ..., (\upsilon(\Gamma_p), u_p) \right\rangle$ for a bargaining game between the manager and coalitions under formation.[11] Thus, the game ends when a bargaining agreement is reached between the manager and the coalition. However, some participants might still stay in line, waiting in vain for their rewards, because the manager might not agree to allow them to partake in coalition formation. Indeed, due to the existence of those marginal participants, the manager may lose a large portion of his/her reward $F(\Gamma_k)$, for some $k's \in \langle 1, ..., p \rangle$.[12]

Only the last issue is relevant to our bargaining solution $\Gamma = f(\boldsymbol{S}, \varnothing)$ to the supermodular bargaining game. The coalition Γ is a stable point with reference to the threshold value $u = F(\Gamma) = \min_{i \in \Gamma} \dfrac{\partial \upsilon(\Gamma)}{\partial i}$. This coalition guarantees a gain $u = F(\Gamma)$ to Player No. 2. Therefore, this player can prevent anyone $i \notin \Gamma$ outside the coalition $\Gamma \in \boldsymbol{S}$ from becoming a new participant of the coalition because the outsider's marginal contribution $\dfrac{\partial \upsilon(\Gamma)}{\partial i}$ reduces his/her guaranteed gain. The same incentive governing the behavior of Player No. 2 will prevent some members $i \in \Gamma$ from leaving the coalition. The unconventional interpretation given below might help elucidate this situation.

Let us observe a family of functions on $N \times 2^N$ monotonic towards the second set variable H, $H \in 2^N$. Let it be a function $\pi(i; H) \equiv \dfrac{\partial \upsilon H}{\partial i}$. We already cited Shapley (1971), who introduced the convex games, with the marginal

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[11] This positioning of players/elements in line arranges a so-called defining sequence in data mining process.

[12] We refer to similar behaviour of players in "Left- and Right-Wing Political Power Design: The Dilemma of Welfare Policy with Low-Income Relief" as political parties' bargaining game with agents registered under the social security administration.

utility $\frac{\partial \upsilon H}{\partial i} = \upsilon(H) - \upsilon(H - i)$, which is the one of many exact utilizations of the monotonicity $\pi(i, L) \leq \pi(i, G)$ for $i \in L \subseteq G$. Authors of some extant studies, including this researcher, refer to these marginal $\upsilon(H) - \upsilon(H - i)$ set functions as the marginal of supermodular functions $\upsilon(H)$. By inverting the inequalities, we obtain submodular set functions.

Convex coalition game, referring to Shapley (1971) once again, can have a “snowballing” or “band-wagon” effect of cooperative rationality; i.e., in a supermodular game, the cooperative rationality suppresses the individual rationality. In contrast, in submodular games with the inverse property $\pi(i, L) \geq \pi(i, G)$ (an extrapolation this time), the individual rationality suppresses the collective rationality. Indeed, according to the rules of the game, the manager’s reward will depend on the least marginal utility $u = F(H) = \min_{i \in H} \frac{\partial \upsilon(H)}{\partial i}$ of some of weakest members of the coalition H under formation. Indeed, according to the rules of the game, the manager's reward will depend on the lowest marginal utility of some of the weakest members of the resulting coalition. If the utility function is submodular, the positive effect of the health club members' cooperation disappears. Now, we can focus on a two-person game to be played out between the manager and the coalition without consideration of cooperation.

8. CONCLUSION

To sum up our efforts, they were made possible by a category called "Monotonic System", which is a kind of quintessence of the monotonous phenomenon of reality, linking two separate categories—"The problem of bargaining" and "Coalition game"—by one guiding thread. Nash bargaining solution being understood as a point on the Pareto frontier in Monotonic System might be an acceptable convention in the framework of “fast” calculation. The corresponding algorithm for finding the solution is characterized by a relatively few operations and can be implemented by applying known computer programming “recursive techniques” to tables. From a purely theoretical perspective, we believe that our technique is a valuable addition to the repertoire presently at the disposal of the game theoreticians. Our bargaining solution is presently not fully grounded in validated scientific facts established in game theory. Consultations with specialists in the field are thus necessary to develop our work further. In our view, our portrayal of coalition formation games is sufficiently clear and does not require specific economic interpretations. Nevertheless, all our arguments need to be confirmed through additional fundamental studies.

APPENDIX. An Illustration of Bargaining in Club Formation Based on Neither Supermodular nor Submodular Utility Functions.

Recall the wellness club formation game from Section 2. Given the utility function $\upsilon(x)$, although whether the club members actually arrive at individual payoffs or not is irrelevant, the club formation is still of interest. Let the game participants $N = \{1,2,3,4,5,6,7\}$ try to organize a club. Let the utility (revenue) function comply with the pledges made by the individual employees to participate in the offered wellness events in accordance with their survey responses shown in Table 1. We demand that all five wellness events be materialized and thus define:

$$\upsilon(x) = |x| + \sum_{i \in x} \sum_{j=1}^{5} \alpha_{ij}\text{, where } x \subseteq N = \{1,2,3,4,5,6,7\}.$$

In other words, a promise fulfilled by the club member contributes a Bank Note to the participant. In addition to all the promises fulfilled, a side payment per capita is available. According to this rule, $\upsilon(\{1\}) = 3$, $\upsilon(\{2\}) = 5$,… Nonetheless, we have changed the side payments rule, so that the game transforms into neither supermodular nor submodular game. Note that

$$\sum_{i}^{7} \upsilon(\{ i\}) = 22 < \upsilon(N) = \upsilon(\{1,2,3,4,5,6,7\}) = 29,$$

which renders it a non-essential game. If the CEO makes a decision $k = 2$, each member of the wellness club Γ_0, according to the rules of the game, receives one basic Bank Note, while a side payment of 7 Bank Notes will allow this player to double the reward if the grand coalition Γ_0 is formed. However, the club manager will not be interested in organizing club Γ_0, since the CEO's reward for organizing clubs Γ_1, Γ_2 and Γ_3 with fewer participants—like on the Pareto frontier (shown in Figure 1−3)—according to the rules of the game, will yield higher rewards.

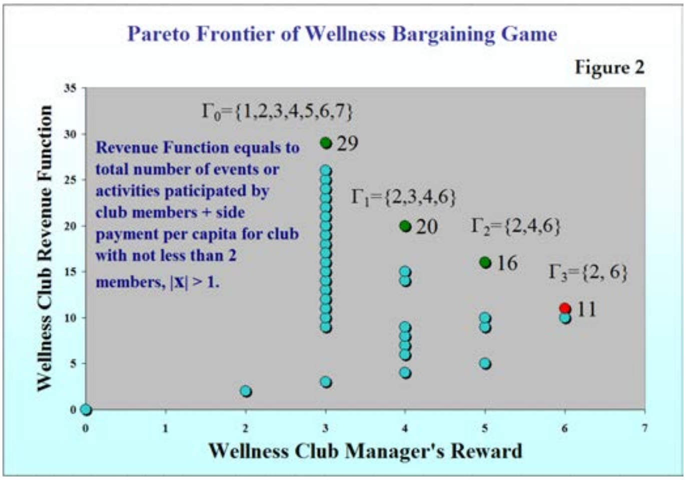
Indeed, whether they choose to cooperate or not, the employees will be discouraged from forming a club which would provide them with the same gains. To change the situation into that similar to "*the real life cacophonous*" scenario, let the side payment per capita be removed for single and pairwise participants while keeping the rewards intact for all other coalitions for which the size exceeds 2. Thus $\upsilon(\{1\}) = 2$, $\upsilon(\{2\}) = 4$, $\upsilon(\{1,2\}) = 6$, $\upsilon(\{3,6\}) = 5$, $\upsilon(\{2,3,5\}) = 12$, etc. The gain, which was defined as $F(x) = \min_{i \in x} \frac{\partial x}{\partial i} \equiv (\upsilon(x) - \upsilon(x - i)$, makes the employees' "cooperative behavior close to grand coalition" less profitable for the manager, as explained above.

Therefore, the manager would benefit from encouraging the employees to enter the club of a “reasonable size”. In Table 8, we examine this phenomenon using different manager gain $F(x)$ values.

Table 8.

Wellness Club List							Marginal Utilities per capita							υ	F
1	2	3	4	5	6	7	1	2	3	4	5	6	7	υ(H)	F(H)
*							2							2	2
	*							4						4	4
*	*						2	4						6	2
		*							3					3	3
*		*					2		3					5	2
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
		*		*					3		2			5	2
*		*		*			5		6		5			10	5
	*	*		*				7	6		5			12	5
*	*	*		*			3	5	4		3			15	3
			*	*						4	2			6	2
*			*	*			5			7	5			11	5
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	.	*	*	*	*	*		.	4	5	3	6	3	21	3
*	.	*	*	*	*	*	3	.	4	5	3	6	3	24	3
.	*	*	*	*	*	*	.	5	4	5	3	6	3	26	3
*	*	*	*	*	*	*	3	5	4	5	3	6	3	29	3

At last, we illustrate the bargaining game in the graph below and make some comments.



N.B. Observe that utility pairs $(29,3)$, $(20,4)$, $(16,5)$ and $(11,6)$ constitute the Pareto frontier of bargaining solutions for the bargaining problem involving the manager as Player No. 1 and coalitions as Player No. 2. Accordingly, given the grand coalition $\mathrm{N} = \Gamma_0 = \{1,2,3,4,5,6,7\}$, three proper coalitions—$\Gamma_1 = \{2,3,4,6\}$, $\Gamma_2 = \{2,4,6\}$ and $\Gamma_3 = \{2,6\}$—exist. Solutions $\upsilon(\Gamma_1) = 20$, $\mathrm{F}(\Gamma_1) = 4$ and $\upsilon(\Gamma_2) = 16$, $\mathrm{F}(\Gamma_2) = 5$, maximize the product of participants' gains over the disagreement point $(0,0)$ at $20 \cdot 4 = 16 \cdot 5 = 80$. More specifically, as noted at the beginning of the paper, the solution might not be unique and some external considerations may need to be taken into account. For example, the sponsor expenses for $(20,4)$ are equal to 24, while those pertaining to $(16,5)$ are equal to 21, which might be decisive. That is the case when the bargaining power $\theta = ½$ of the coalitions Γ_1, Γ_2 and the manager are in balance. Otherwise, choosing the coalition bargaining power $\theta < ½$, the manager will be better off materializing the solution $(5,16)$. Conversely, coalition Γ_2 will be better off if $\theta > ½$.

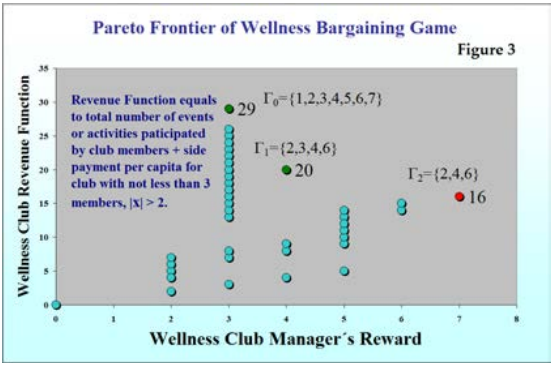

Pareto Frontier of Wellness Bargaining Game

Figure 3

NB. Comparison with Figure 2 reveals that coalition $\Gamma_3 = \{2,6\}$ is no longer located on the Pareto frontier.

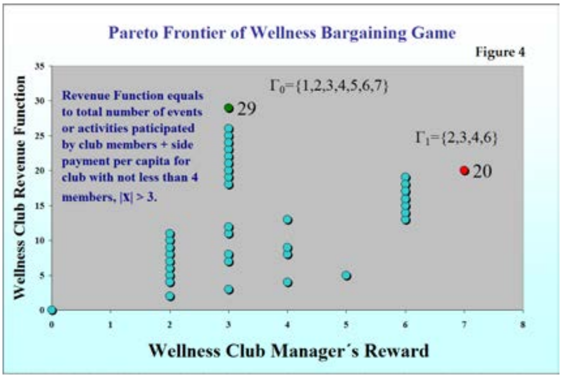

Pareto Frontier of Wellness Bargaining Game

Figure 4

N.B. Comparison with Figure 3 indicates that coalition $\Gamma_2 = \{2,4,6\}$ no longer lies on the Pareto frontier.

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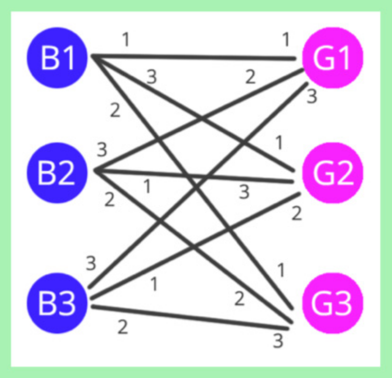


Partial Matching in the Marketing Game: Reassessing Incompatibility Indicators

Abstract. The game under scrutiny serves as a sophisticated model mirroring the intricacies of real-world scenarios within marketing agencies, where the allocation of clients among employees undergoes a continuous series of assessments and prioritizations. This allocation process, termed "matching" in economic discourse, unfolds through a sequential chain of reflections, with each decision influencing subsequent steps. However, the dynamic nature of this environment can result in mismatches between clients and employees, leading to marketing instability. To mitigate this instability and address the inherent fuzziness in marketing, we propose employing indicators of inadequacy or incompatibility to identify when an employee may not be the best fit for a client. By regularly reassessing these metrics throughout the marketing process, our aim is not merely to minimize failures but to optimize outcomes and minimize compensation requirements.

Keywords: marketing game; core; rational choice; cooperation; matching
JEL classification: C71; C78

1. INTRODUCTION

Roommate problem [1] proposed by Gale and Shapley in 1962 was also considered by Bergé (1958, [2]) and has since become the canon for various forms of economic stability. The canonical solution assumes a complete agreement or grand matching for all the members of economic community consisting of an even number of agents. One of the difficulties that we have encountered is expressed in the triplicity of the interests of the clients and marketing agency. Yes, it is true that both clients and employees of an agency have individual interests that may sometimes conflict. It is also true that in any organization, the interests of the staff as a whole can also arise. Even more confusing, however, is the paired interest of clients in what we have called the "*marketing game*." The dynamic maneuvering approach proposed by Lefebvre and Smolyan in 1968, [3], can indeed be a useful tool in situations dealing with many players, which have competing interests in a dynamic and multi-stage marketing environment. The quasi-core concept presented, referring to a partial matching that is optimal for all parties involved, can provide a valuable basis for achieving mutually beneficial results. In a complete match situation, both parties may be satisfied with the outcome, but more often than not, there will be areas of inconsistency. In such cases, partial matching can be useful in determining solutions that are optimal for everyone, although not necessarily ideal. Indeed, the judgment was made that "*the best old client is still the best.*"

We refer to partial stability in which the “rewards and compensations” paid to clients and agency cannot be increased further, despite attempts to improve the situation. The partial stability in this scenario indicates that marketing reservations has reached an optimal state, and further changes or attempts to improve the situation will result in negative effects for the clients and the agency. The quasi-core concept refers to a solution that is considered to be stable, but not necessarily optimal in the game theory context. The situation we are referring to is known as a "forbidden set". In the recent articles by Richter and Rubinstein [4] suggest that there may be a set of matches "Z" that cannot be realized. This can result in ending the game prematurely, similar to what can happen in a university environment during the early years of higher education.

Indeed, soon after the start of their studies, many university and college students are trying to change the nature of their studies or prefer other tasks. Students, in their own opinion, may choose incompatible educational programs, while the composition of the students themselves in a particular program may also not be optimal. Students and programs may not be compatible with each other. So-called discrepancies in mutual rankings have been one of the reasons (Leo Võhandu, LV, 2010, [5]) for the unacceptably high percentage of Estonian students who drop out in their first years of study, wasting their time and entitlement to state support. However, a better matching between students and educational programs can mitigate this problem.

The problem being discussed is a variation of the stable matching (Bergé; Roth & Sotomayor, 1990, [6]), where the goal is to match pairs of agents (in this case, students and programs) in a way that satisfies certain preferences while avoiding blocking pairs. To solve the problem, it was proposed to introduce a "coincident total" as the sum of "matching rankings" selected in two directions—the horizontal rankings of students involved with programs and the vertical rankings of programs matching to students. According to LV, the best solution among all possible horizontal and vertical sums of rankings is where the sum reaches its minimum.

Finding the "coinciding minimum" is a difficult task. Instead, LV suggested a Greedy type workaround. According to LV, the best solution to the problem of matching students and programs would be a fairly close (cf. Cormen et al, 2001, [7]) accumulation of the sum when moving along the direction of mutual matching in a non-decreasing order of rankings. Apparently, the approach of LV to the solution of the problem was drawn up in the spirit of classical utilitarianism, when the sum of utilities should be maximized or minimized (Bentham, Principles of morality and legislation, 1789, [8]; Sidgwick, Methods, Ethics, London, 1907, [9]). The reader studying matching problems may also find information on these issues, where a number of ways to construct an optimal matching strategy have been discussed (Veskioja, 2005, [10]).

The setting of the marketing game will be presented with an attempt to explain by an example a rather complex intersection of interests, where readers must be prepared to engage in a reality masquerade in order to understand the basic concept of the coalition game (Gillies, 1959, p. [11]; also noted as the core by John von Neumann and Morgenstern, 1953, [12]). In particular, we hope to shed light on the dynamic or multi-stage nature of client and agency staff ranking re-evaluation during the game. It should be emphasized that although the game primitives are a separate mathematical entity in a completely different context, we "borrowed" the idea of LV-s ranking to evaluate the rewards of matching. For this reason, we have changed the nomenclature of payments for mutually incompatible agreements, i.e., "imputation", or "imputed compensations" in order to introduce a payment scale that has a monotone character. The scale is organized as incompatibility indicators in the form of a "Monotone System."

N.B. The Monotone system (MS, see also "Monotonic Link Functions", Seiffarth et al, 2021, [15]) is used to reassess the risk indicators of entering into agreements that are not compatible by considering the mutual rankings of clients and employees. The indicators have a monotonic property, which allows for dynamic adjustments to be made in response to changes in rankings ensuring that they remain in synchrony. The system implements the concept of partial matching by ordering the indicators through a process caused by the inclusion of subsets taken from a general set of indicators. The Monotone system formalizes and generalizes the concept of ordering, sequencing, or arrangement of elements in subsets, providing a structured and systematic approach to assessing incompatibility risk in various contexts. The theory was initiated by Mullat (1971, [16]) and since then was further developed and published in Russian periodical of MAIK in 1976. Plenum Publishing Corporation originally distributed it in English. Without the use of the MS, the analysis of marketing game scenarios would be limited and potentially inaccurate, as the system provides a clear framework for understanding the relationships between parties on marketing platform and their impact on each other. Perhaps Monotone Systems provide a framework for analyzing the properties of specific multi-stage dynamic games.

Roadmap. The rest of the paper will be structured as follows. In Section 2, the primitives and notations used in the paper are explained. Section 3 endows with a detailed explanation of the marketing game and its analysis, including basic definitions and the non-traditional theory of quasi-core stability. The main body of the paper ends with Section 4, which contains the conclusions and suggestions for future work. The Appendix provides additional information and computational algorithms for the reader to better understand the concepts discussed in the paper. This includes the explanation of a claim that "*the best old client is still the best*" in A1; compatibility indicators reassessment algorithms in A2; the basic concept of canonical stability in A3; and computational algorithms visualization in A4, A5, and A6. An Excel spreadsheet is also provided to help with the technical details.

2. THE GAME PRIMITIVES AND NOTATIONS

We use a two-sided marketing platform, where clients and agency staff both play an active role in the matching process. The game is played in discrete time slices or reflections k, with an increasing k as the game progresses through the periods. Clients and employees of the agency enter into contracts or deals

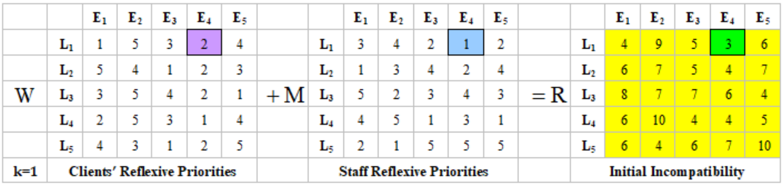
α_k during the period k after which the products and services of the agency prescribed in contracts are considered reserved. It is assumed that the participants may enter into agreements or matches that are not well suited for them, or that may not be compatible with other agreements or matches they have made. This can create dynamic changes in the willingness of participants to enter into agreements or matches describing a multi-stage reflexive process in which the list of matches expands and the list of potential opportunities is gradually narrowed down over time. The game can end at any point at the request of clients or the marketing agency, and it can end with a partial matching or a complete match.

Having said that, we are talking about matchmaking or partnering event or activity where participants are matched based on compatibility. If no participants have been able to find a suitable partner, then it may be difficult to continue offering rewards or compensations. In such a scenario, the marketing agency staff and clients may need to re-evaluate their approach and criteria for matching participants, or consider whether to continue the event at all. It is important to carefully consider the potential risks and drawbacks of offering compensations in situations where the results of the matchmaking process are uncertain or unreliable. Ultimately, the decision on whether or not to continue the marketing effort should be based on a careful assessment of the risks and benefits involved.

2.1. Visualization example

Five clients and five employees decided to attend the marketing game. Clients will be asked to prioritize employees; while agency staff will be asked to prioritize eligible clients accordingly from the agency's point of view. This information to match clients with eligible employees and vice versa employees with clients will be used to reassess indicators of incompatibility. Clients and agency staff providing this information have been promised to collect boxes of "*goodies*" and are henceforth referred to as *participants*, while others are marked as *blanks* by default and cannot participate in the game.

Game participants who find a suitable partner will be rewarded, while failing that receive compensations or cheering payoffs for bad luck. In order to cover the expenses, the marketing fee is set at -50€ per participant. Thus, the amount of +500€ will be at the disposal of the cashier. The tables $W = \|w_{i,j}\|$ and $M = \|m_{i,j}\|$, Table-1&2, are used to represent the dynamic ranking of clients and marketing agency, respectively; also known as strict ranking or linear order. There are $\{1,...,i,...5\}$ clients and $\{1,..., j,...5\}$ staff employees. The $w_{i,j}$ cells indicate clients $i = \overline{1,5}$ who revealed their rankings positioned in the rows of table W towards employees as horizontal permutations of numbers $\langle 1,2,3,4,5\rangle$. Similarly, agency staff revealed its idea on clients ordering in $m_{i,j}$ cells $j = \overline{1,5}$, as vertical permutations in the columns of table M, relative to the employees. The numbers $\left\langle \overline{1,5} = 1,2,3,4,5 \right\rangle$ can be repeated in the columns of table W and in the rows of table M. More than one client may prefer the same employer at priority level $w_{i,j}$. Multiple employees, accordingly, may be well suited to the same client at the level $m_{i,j}$ by the staff reflexive idea. When rankings have been revealed, they can form two 5×5 tables, resulting in $2 \times 5 \times 5$ combinations. The table $R = \left\| r_{i,j} \right\|$, Table-3, shows the matching of clients and employees, who have mutual risks $r_{i,j} = w_{i,j} + m_{i,j}$ of incompatible agreements.



		E₁	E₂	E₃	E₄	E₅			E₁	E₂	E₃	E₄	E₅			E₁	E₂	E₃	E₄	E₅
	L₁	1	5	3	2	4		L₁	3	4	2	1	2		L₁	4	9	5	3	6
	L₂	5	4	1	2	3		L₂	1	3	4	2	4		L₂	6	7	5	4	7
W	L₃	3	5	4	2	1	$+M$	L₃	5	2	3	4	3	$=R$	L₃	8	7	7	6	4
	L₄	2	5	3	1	4		L₄	4	5	1	3	1		L₄	6	10	4	4	5
	L₅	4	3	1	2	5		L₅	2	1	5	5	5		L₅	6	4	6	7	10
k=1	Clients' Reflexive Priorities							Staff Reflexive Priorities							Initial Incompatibility					

2.2. Rawlsian postulate and compensations' arithmetic

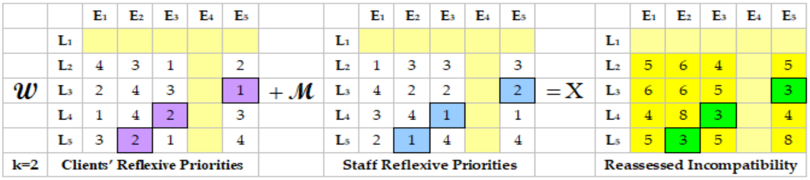
The Rawlsian postulate argues that *"institutions"* should be organized in such a way as to benefit the least advantaged members of community: *"The welfare of the worst-off individual is to be maximized before all others, and the only way inequalities can be justified is if they improve the welfare of this worst-off individual or group…"* Public Choice III, D.C. Mueller, p.600, [14]. Based on this postulate, players may have the following ideas of how the game can continue.

Indeed, let the compensations sums, even if this is impractical postulate, are set proportionally to $\frac{1}{2} r_{i,j} \times 10$€; in such a case, the participants profit can reach 50€ for free! Instead, we try to design the game by encouraging clients and agency employees to follow Rawls' "high of the least" second principle of justice [13]. Some of participants signed deals, while others tend to dynamically reassess the risks $r_{i,j}$ of incompatible agreements. These lucky dealers $\sigma = \left[\boldsymbol{i}_\sigma, \boldsymbol{j}_\sigma \right]$, or $\sigma = \left[L_\sigma, E_\sigma \right]$, were promised rewards. Unsuccessful participants, those who have not yet signed a deal given that only matchings with high level of mutual risks $r_{i,j}$ remained, can claim compensations. On initial reflection, let the expected rewards of all participants are proportional to $\min_{i,j=\overline{1,5}} r_{i,j}$. In Table 3, the lowest mutual risk is $r_{1,4} = 3$. All participants in the game are paid 10€ for goodies if the game ends immediately; in the opposite situation, when the game continues until the complete matching—the situation is the same—the participants still lose, in contrast to the partial matching. The losses of all participants in both cases will be −40€. We assume that participants in the pair $\sigma = \left(\boldsymbol{w}_1, \boldsymbol{m}_4 \right)$ receive $\boldsymbol{w}_1, \boldsymbol{m}_4$ =+30€ each, since by Rawls' principal rule, the *argmin* $r_{1,4} = 3$, viz., 3×10€ = +30€. The other 8 participants, now according to the compensation rule, will receive half, $\frac{1}{2} \cdot r_{1,4} \times 10$€ =

+15€. Everyone benefits from the matching $\sigma = [1,4]$. Indeed, partners $[1,4]$ will be able to reduce their losses $\boldsymbol{w}_1$, $\boldsymbol{m}_4$ to −10€, since their gain according to the rules of the game will be +30€ $= \mathrm{r}_{1,4} \times 10$€. Considering the +10€ cost of goodies, other 8 participants will also reduce their losses, but only to −25€, since −50€ was paid as an entry fee will be reduced by +15€ from the compensation sums.

What happens if the participants $\sigma = [1,4]$ decide to sign the agreement in the initial time period $k = 1$? The entire table R must be dynamically reassessed into sub-block X to reflect that participants $[1,4]$ have been matched. Indeed, the clients $\{2,3,4,5\}$ and the agency staff employees $\{1,2,3,5\}$ can no longer rely on their latent partners $[1,4]$. The ranking's scale $\langle 1,2,3,4,5 \rangle$ is narrowed dynamically to $\langle 1,2,3,4 \rangle$, which leads to a decrease in risks $\mathrm{r}_{i,j}$.

To reflect this, Table 1–3 have been reassessed to Table 4–6. The yellow cells determine the sub-block $\mathrm{X} = \mathrm{R} \div \sigma$; $\mathrm{C(X)} = \{\arg\min \alpha \in \mathrm{X}\}$ determine the green cells choice operator, where the partners $\sigma = [1_\sigma, 4_\sigma]$:



		E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5			E_1	E_2	E_3	E_4	E_5
	L_1							L_1							L_1					
	L_2	4	3	1		2		L_2	1	3	3		3		L_2	5	6	4		5
$\mathcal{W}$	L_3	2	4	3		1	$+\mathcal{M}$	L_3	4	2	2		2	$=\mathrm{X}$	L_3	6	6	5		3
	L_4	1	4	2		3		L_4	3	4	1		1		L_4	4	8	3		4
	L_5	3	2	1		4		L_5	2	1	4		4		L_5	5	3	5		8
k=2	Clients' Reflexive Priorities							Staff Reflexive Priorities							Reassessed Incompatibility					

The compensation sum has not changed, and is still +15€. The balance −50€+10€+2×15€ = −10€ of the pair $[1,4]$ improves; L_1, E_4 each receive, $\boldsymbol{w}_\sigma, \boldsymbol{m}_\sigma$ = +30€, $\sigma = (\boldsymbol{w}_1, \boldsymbol{m}_4)$ as rewards for matching based on the rule that it is equal to twice of the minimum compensation. For those not yet matched, the individual balance remains negative, viz., −25€.

Inclusive goodies, the cashier balance 500€−2×(10€+30€)−8×(10€+15€) falls to 500€ $-\left(\sum \boldsymbol{w}_i + \sum \boldsymbol{m}_j\right)$ =220€, $i, j = \overline{1,5}$. We refer to the list $\mathrm{D}\boldsymbol{x} = \langle \sigma \rangle$ as $\langle \sigma \rangle = \mathrm{R} \div \mathrm{X}$, or $\mathrm{X} = \mathrm{R} \div \mathrm{D}\boldsymbol{x}$; cf. Table 4-6 $\mathcal{W}, \mathcal{M}$ & X. The list of matching pairs $\mathrm{D}\boldsymbol{x}$ is also the "complement list" $\overline{\mathrm{D}}\boldsymbol{x}$ of possible unmatched pairs in the sub-block X to R. Further removal of pairs $\alpha, \ldots$ from X will be denoted by $\mathrm{X} \div \{\alpha\}$.

Based on the information provided, the matching that would best represent the common interests of all clients and agency staff is one that maximizes the least compensation sum, while maintaining the acceptable risk of incompatibility. What should be the matching that will represent the common interests of all clients and staff employees?

3. CONCEPT OF A QUASI-CORE—THE KERNELS

In coalition game theory, imputations refer to allocations of rewards that satisfy certain conditions, such as individual rationality, meaning that each player gets at least as much as they could have obtained on their own. However, marketing cannot be seen as game in traditional sense with a well-defined set of rules and a characteristic function. The concept of marketing presented so far as a game was just a framework for thinking in various directions at the marketing platform.

In view of "monotone system" (Mullat, 1971-1995) exactly as in Shapley's convex games, the basic requirement of our model validity emerges from an inequality of monotonicity $\pi(\alpha, X \div \{\sigma\}) \leq \pi(\alpha, X)$. This means that, by eliminating an element/match σ from X, the utilities (risks) on the rest will decline or remain the same. In particular, a class of monotone systems is called **p**-monotone (Kuznetsov et al, 1982, 1985, [17-18]), where the ordering $\langle \pi(\alpha, X) \rangle$ on each subset X of utilities follows the initial ordering $\langle \pi(\alpha, R) \rangle$ on the table R. The decline of the utilities on **p**-monotone system does not change the ordering of utilities on any subset X. Greedy type (serialization) technique on **p**-monotone system might be effective. Behind a **p**-monotone system lays the fact that an application of Greedy framework can accommodate the structure of all subsets $X \subset R$. For various reasons, many will probably argue that **p**-monotone systems are rather simplistic and cannot be compared with the serialization method. However, many economists, including Narens and Luce (1983, [19]), certainly, will point out that subsets X of **p**-monotone systems *perform* on interpersonally compatible scales.

An inequality $F(X_1 \cup X_2) \geq \min\langle F(X_1), F(X_2) \rangle$ holds for real valued set function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$, referred to as quasi-convexity (Malishevski, 1998, [20]). We observed monotone systems here, which we consider important to distinguish. The system is non-quasi-convex when there are two sub-blocks X_1, X_2 contradicting the last inequality. We consider such systems as non-quasi-convex.

The order of incompatibility risks in marketing games may not be preserved within an arbitrary sub-block X . In these systems, the initial risks order $\langle R = r_{i,j} \rangle$ may not necessarily be true for some order on $\langle X \rangle = \| \pi(\alpha, X) \|$. Unlike $\langle R = r_{i,j} \rangle$, as agency staff employees search for an client for a marketing, and vice versa, the order of risks on $\langle X \rangle = \| \pi(\alpha, X) \|$ can be opposite to the order on $\langle R = r_{i,j} \rangle$ for some pairwise pairs α and β of participants, i.e. as $\pi(\alpha, R) > \pi(\beta, R)$, but $\pi(\alpha, X) \leq \pi(\beta, X)$ and the like. In that case, the ordering of two partners' mutual risks can turn "upside down" while the risks scale is dynamically narrowed down compared to the original ordering $\langle \mathrm{R} \rangle$. This means that the scale of mutual risks is not necessarily interpersonally compatible. The interpersonal incompatibility of the risk scale in the marketing environment is significantly different, leading to difficulties in finding a solution using the Greedy framework and the incremental chain algorithm. This difference became apparent when the monotone system was found to be non-quasi-convex, making it impossible to find a solution using our traditional method (Mullat, 1971). Understanding the essence of the problem is essential before delving into the formal intricacies of the issue.

Definition 1 We call a sub-block $\mathcal{K} \in \arg\max_{X \subseteq \mathcal{P}} F(X)$ by a kernel sub-block; $\{\mathcal{K}\}$ is the set of all kernels.

Recalling the main properties of a chain of increments (a sequence of elements of a monotone system) it is possible to arrange the partners $\alpha \in \mathcal{P}$, i.e., the matchings $\alpha \in \mathcal{P}$ of agents by a Greedy type incremental sequence $\overline{\alpha} = \langle \alpha_1, ..., \alpha_k \rangle$, time slices $k = \overline{1, |\mathcal{P}|}$. The sequence $\overline{\alpha}$ follows the lowest risk ordering in each period k corresponding to sequence of sub-blocks $\langle H_k \rangle$, $H_1 = R$, $H_{k+1} \leftarrow H_k \div \{\alpha_k\}$, $\alpha_k = \arg\min_{\alpha \in H_k} \pi(\alpha, H_k)$. One of the properties of the incremental sequence (cf. ***defining***, Mullat, 1971a) is that $F(H_k)$ is single-peaked. This means that within a peaked sub-block Γ_p for some time slice $k = p$ there does not exist a proper sub-block X' on which the function $F(X')$ would reach a greater value than on Γ_p, i.e., the inequality $F(X') > F(\Gamma_p)$ does not take place. Therefore, under the contrary assumption that such a set X' exists, X' must have a non-empty intersection with the sequence $\overline{\alpha}$ with some α_t in previous time slice; α_t will presumably be at the leftmost position $t < p$ in $\overline{\alpha}$ (or one of those $\overline{\alpha}$ entries α_t in X' that will appear as $\overline{\alpha}$ is constructed). However, complementing the pairs in X' with all those pairs that do not belong to X', so that starting from some t both X' and Γ_p lie entirely in the sequence $\overline{\alpha}$, we do not arrive to contradiction as expected while constructing $\overline{\alpha}$. Otherwise the sequence $\overline{\alpha}$ could potentially be used for finding the largest kernel $\mathcal{K}'$. The reason is, that incremental constructing the sequence $\overline{\alpha}$ is not an exclusion of matchings $\alpha_k \in H_k$, given that the participant $\alpha_k = [i, j]$ is about to match but rather an exclusion of all adjacent partners α in $[i,*]'$-s row and $[*, j]'$-s column. We denote this exclusion or dynamically reassessing of rows and columns by $H_{k+1} \leftarrow H_k \div \{\alpha_k\}$ and by $D_{k+1} \leftarrow D_k + \{\alpha_k\}$.

In conclusion, we note once again that, despite the preservation of the properties of a monotone system, the Greedy algorithm constituting Mullat's defining sequence $\overline{\alpha}$, the sequence cannot guarantee the extraction of the supposedly largest kernel $\mathscr{K}'$, especially in the form given by Kempner et al (2008, [21]). Thus, we need to employ special tools for finding the solution. To move further in this direction, we are ready to formulate some propositions for finding kernels $\mathscr{K}$ by branch and bound algorithm types.

The next argument will require a modified variant of imputation (Owen, 1982, [22]). We define an imputation as the outcome connected to the marketing game. More specifically, the outcome is given as a $|\mathscr{P}|$-vector (a list) of payoffs to all unmatched participants who make up the sub-block X, and matched participants as partners in pairs $\alpha = [\mathrm{L}_\alpha/\text{client}, \mathrm{E}_\alpha/\text{employee}] \notin \mathrm{X}$ representing the list $\mathrm{D}\boldsymbol{x}$. In case the game ends prematurely, at the request of the agency or the clients themselves, for all in $\mathrm{D}\boldsymbol{x}$ who have found a partner a reward $\mathrm{F(X)}$ will be paid; $\mathrm{F(X)} = \min \mathrm{r}_{\mathrm{i,j}}$ among cells $\alpha = [\mathrm{i}, \mathrm{j}] \in \mathrm{X}$, cf. Table 3 and Table 6. For everyone who has not yet found a partner, under the current rules, they will receive $½\mathrm{F(X)}$. The concept of outcome (payoffs) in this form is not generally accepted as a form of imputation of a multi-persons game, since the amount that all participants can now claim is not fixed, but will be dynamically re-evaluated. Thus, it is likely that participants will fail to reach an understanding, and will claim payoffs obtaining less than entrance fees $(\mathrm{n}+\mathrm{m})\cdot 50\,€$ of the cashier. However, the cashier balance, in contrast, when participants will claim more than entrance fees, is also conceivable.

Any sub-block X induces a $|\mathscr{P}|$-vector $\boldsymbol{x} = \langle \boldsymbol{x}_\sigma \rangle$ as an outcome $\boldsymbol{x}$ may be organized in a sequence of payoffs $\langle \boldsymbol{w}_\sigma, \boldsymbol{m}_\sigma \rangle$. Further, we follow the rule that capital letters represent sub-blocks $\mathrm{X}, \mathrm{Y}, \ldots, \mathscr{K}, \mathscr{N}, \ldots$ while lowercase letters $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{k}, \boldsymbol{n} \ldots$ represent outcomes induced by these sub-blocks.

$$\boldsymbol{x}_\sigma = \left(\begin{array}{l} \boldsymbol{w}_\sigma, \boldsymbol{m}_\sigma = 1 + \mathrm{F(X)} \text{ if } \sigma \in \mathrm{D}\boldsymbol{x}, \\ \boldsymbol{w}_\sigma, \boldsymbol{m}_\sigma = 1 + ½\mathrm{F(X)} \text{ if } \sigma \notin \mathrm{D}\boldsymbol{x}. \end{array} \right.$$

The vector $\boldsymbol{x}$ designates an imputation in the terminology of many persons′ games, **1** stands for goodies:

$$\sum_{\sigma \in \mathscr{P}} \boldsymbol{x}_\sigma = \mathrm{F(X)} \cdot \left[|\mathrm{D}\boldsymbol{x}| + ½(|\mathscr{P}| - |\mathrm{D}\boldsymbol{x}|) \right] + |\mathscr{P}|.$$

This definition of the partial matchings $\mathrm{D}\boldsymbol{x} \subseteq \mathscr{P}$ is used later, adapting the concept of the quasi-core for the purpose of the marketing game. We say that an arbitrary sub-block X induces an outcome $\boldsymbol{x}$. Computed and prescribed by sub-block X, the components of $\boldsymbol{x}$ consist of two distinct values $1 + \mathrm{F(X)}$ and $1 + ½\mathrm{F(X)}$. Participants $\sigma \in \mathrm{X}$ could not sign a deal, while participants $\sigma \in \mathrm{D}\boldsymbol{x}$ were able to match. We will also use the notation $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$ emphasizing a mixture for marketing matchings $\mathrm{D}\boldsymbol{x}$.

Before moving on, let's try to justify our mixed notation $\overline{X}$. Although the cells $\alpha \notin X$, whereas α is located in the compliment $\overline{X}$ of X to R, the $D\boldsymbol{x}$ uniquely defines both those $D\boldsymbol{x}$ among participants $\mathscr{P}$ who signed deals, and those $X = R \div D\boldsymbol{x}$ who did not; the cells in $\overline{X}$ does not specifically indicate matched participants. In contrast, using the notation $D\boldsymbol{x}$, we indicate participants in $D\boldsymbol{x}$ who are matched, whereas $\sigma \in D\boldsymbol{x}$ also indicates an individual decision how to match. More specifically, this annotation represents all agency staff employees and all clients in $D\boldsymbol{x}$ like standing in line facing each other at the marketing platform. However, any agreement or matching among participants belonging to $D\boldsymbol{x}$, or whatever matches are formed in $D\boldsymbol{x}$, does not change the payoffs $\boldsymbol{x}_\sigma$ valid for the outcome $\boldsymbol{x}$. In other words, each particular matching $D\boldsymbol{x}$ induces the same outcome $\boldsymbol{x}$. Decisions in $D\boldsymbol{x}$ with respect to how to match provide an example of individual rationality, while the matching $D\boldsymbol{x}$ formation, as a whole, is an example of collective rationality. Therefore, in accordance with payoffs $\boldsymbol{x}$, the notation $D\boldsymbol{x}$ subsumes two different types of rationality—the individual and the collective rationality. In that case, the outcome $\boldsymbol{x}$ accompanying $D\boldsymbol{x}$ represents the result of a partial matching of participants $\mathscr{P}$. Propositions below somehow bind the individual rationality with the collective rationality.

The feasibility issue of induced sub-blocks $X \subset R$ is considered not only in the context of the blocks themselves, but also in the context of the totality $2^{\mathscr{P}}$ of matchings $D \in 2^{\mathscr{P}}$ in relation to special sets of matchings $\mathscr{F} \subset 2^{\mathscr{P}}$. The matching chain $\langle \alpha_k \rangle$ adding participants period-wise in the period k, starting with the empty set $\varnothing$, can, in principal, access any matching $D \in \mathscr{F}$, by removing the participants starting with the grand ordering $\mathscr{P}$ —so called upwards or downwards accessibility.

Definition 2 *Given matching* $D \subseteq \mathscr{P}$, *where* $\mathscr{P}$ *is the Grand Coalition; we call the collection of pairs* $C(X) = \{\arg\min_{\alpha \in X} \pi(\alpha, X)\}$ *naming* $C(X)$ *as best latent participants, which can be matched with a minimum risk of mutual incompatibility in the matching* D.

Consider the formation of the chain $D_{k+1} \leftarrow D_k + \{\alpha_k\}$ of matchings D_k generated during in the periods $k = \overline{1,n}$. Let $X_1 = R$, $X_k = R \div D_k$, where D_k are participants trying to match; by Definition 2, these $C(X_k)$ are participants with the lowest risk of mutual incompatibility among participants D_k that do not yet matched in previous periods $k < k+1$, $D_{n+1} = \varnothing$. In the time slices $D_{k+1} = D_k \div \{\alpha_k\}$ the matching is arranged after the rows and columns, indicated by the matching or partners α_k, which have been removed from W, M and R. Mutual incompatibility risks $R = \|r_{i,j}\|$ have been recalculated accordingly.

Definition 3 *Given the sequence* $\langle \alpha_1,...,\alpha_k \rangle$ *of matched participants, where* $X_1 = R$, $X_{k+1} = X_k \div \{\alpha_k\}$, *we say that matching* $D\mathscr{x} \equiv \overline{X} \equiv R \div X$ *of matched (as well as* X *of not yet matched) participants is feasible, when the chain* $\langle X_1,...,X_{k+1} = X \rangle$ *complies with the rational succession* $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$. *We call the outcome* $\mathscr{x}$, *induced by sequence* $\langle \alpha_1,...,\alpha_k \rangle$, *a feasible payoff, or a feasible outcome.*

Proposition 1 *The succession rationality necessarily emerges from the condition that, under formation of the matching* D_k *partners in* α_k *does not decrease the payoffs of participants* $\langle \alpha_1,...\alpha_{k-1} \rangle$ *formed in previous periods.*

The accessibility or feasibility of matching $D\mathscr{x}$ formation offers a reinforcing interpretation. Indeed, the feasibility of matching $D\mathscr{x}$ means that the matching can be formed by bringing into it a positive increment of rankings to all participants $\mathscr{P}$, or by improving the position of existing participants having already formed the matching when new participants enter the matching in subsequent periods. We argue that in the subsequent periods, matching can be extended via hereditary-rational choice. In the addendum, we outline the heredity rationality in the form suitable for visualization.

The proposition states that, in matches, the individual decisions are also rational in a collective sense only when all participants in $D\mathscr{x}$ individually find a suitable partner. We can use different techniques to meet the individual and collective rationality by matching all participants only in $D\mathscr{x}$, which is akin to the stable marriage procedure (ibid [1], Gale & Shapley). In contrast, the algorithm below provides an optimal outcome/payoff accompanied by partial matching only—i.e., only matching some of participants in $\mathscr{P}$ as participants of $D\mathscr{x}$; once again, this is in line with the Greedy type matching framework. At last, we are ready to focus on our main concept.

Proposition 2 *The set* $\{\mathscr{N}\}$ *of kernels in the marketing game arranges feasible* matchings $\{D\mathscr{n}\}$. *Any outcome* $\mathscr{n}$ *induced by a kernel* $\mathscr{N} \in \{\mathscr{N}\}$ *is feasible.*

Definition 4 *Given a pair of outcomes* $\mathscr{x}$ *and* $\mathscr{y}$, *induced by sub-blocks* X *and* Y, *an outcome* y *dominates the outcome* $\mathscr{x}$ *by* S, $\mathscr{x} \prec_S \mathscr{y}$:

(i) $\exists S \subseteq X \cap Y \big| \forall \sigma \in S \rightarrow \mathscr{x}_\sigma < \mathscr{y}_\sigma$, (ii) the outcome $\mathscr{y}$ is feasible.

Condition (i) states that participants/partners $\sigma \in S$ receiving payoffs $\mathscr{x}_\sigma$ can break the initial matching and instead of merging into $D\mathscr{x} + \sigma$ and establish new matches will try to unite into $D\mathscr{y} + \sigma$. This means that, some partners in X, i.e., not yet matched participants in S, can find suitable partners amid participants in S, so that their compensations may be higher than their rewards in x. Thus, by receiving y_σ instead of x_σ the participants belongings to S are guaranteed to improve their positions. This interpretation of the condition (ii) is obvious. Thus, the relation $x \prec_S y$ indicates that participants in S can cause a split (bifurcation) of $\mathrm{D}x$, or are likely to undermine the outcome x.

Definition 5 *The proper kernel* $\mathcal{N} \in \{\mathcal{K}\}$ *minimal by inclusion, or what is the same: a proper* $\mathrm{D}n$ *, maximal by super-matchings´ induced by* $\mathcal{N}$ *, is called a core kernel or matching.*

Proposition 3 *The set* $\{n\}$ *of outcomes, induced by core kernels in* $\{\mathcal{N}\}$*, arranges a quasi-core of the marketing game. Outcomes in* $\{n\}$ *are non-dominant upon each other i.e.,* $n \prec_S n'$ *, or* $n \succ_S n'$ *are false for any* $S \subset \mathcal{N} \cap \mathcal{N}'$. Thus, the quasi-core is internally stable.

The proposition above indicates that the concept of internal stability is based on "pair comparisons" (binary relation) of outcomes. The traditional solution of marketing games recognizes a more challenging stability, known as *NM* solution, which, in addition to the internal stability, demands external stability. External stability ensures that any outcome X of the game outside *NM*-solution cannot be realized because there is an outcome $n \in \{\mathcal{N}\}$, which is not worse for all, but it is necessarily better for some participants in $\mathrm{D}x$. Therefore, most likely, only the outcomes n that belong to *NM*-solution might be realized. The disadvantage of the marketing scenario is that it is impossible to specify how this can happen. In contrast, we can define how the dynamic or multi-stage reassessment of one matching to another takes place, namely, only along feasible sequence of matchings of partners. However, it may happen that for some matchings $\mathrm{D}x$ outside the quasi-core $\{\mathcal{N}\}$, "feasible sequence" may come to deadlock unable to reach any better outcome that n, whereby starting at $\mathrm{D}x$ the quasi-core is feasibly unreachable. This is a significant difference with respect to the traditional *NM*-solution.

4. CONCLUSIONS

By using mismatch or incompatibility indicators as metrics, we can identify cases where a partial matching may be more beneficial than a complete matching. For example, if a staff member has a high level of expertise in a certain area, but may not be a perfect matching for a particular client, it may be better to assign them to that client anyway, rather than risking a less experienced staff member who is a better matching. By reassessing these metrics throughout the marketing process, we suppose that participants can see, i.e., to reflect all the consequences of their partial matchings as well as actions of their partners to achieve better results overall. This approach may result in a higher total reward than a complete or grand matching, which may not always be feasible or desirable in practice.

The marketing game dynamically develops in time. The scenario is described by multi-stage decision process from current reflection k to the next reflection $k+1$ in the form of a dynamic reassessment of indicators about the willingness to take the risk of entering into incompatible agreements. The objection being raised is that the model presented is not a strict strategic interaction, but rather a "game" in the colloquial sense. On the contrary, it is noted that at each reflection, agents have multiple options and time to consider their moves, including the option to leave the game and receive a payoff or to continue in the hope of obtaining a better outcome. This allows for a more flexible and nuanced approach to cooperative game theory, which is more in line with mathematical standards. The model uses scalar optimization based on the Rawlsian principle of "maximum welfare of the worst-off". In summary, the design of the marketing game should prioritize the promotion of services to clients and benefits to staff, while also providing an engaging and challenging experience for players.

The uniqueness of the marketing game lies in dynamic reassessment of clients and agency staff employees on each other's risks to make deals. As a result, along with the individual and pair rankings, the collective ranking is also subject to reassessment. Indeed, the agreements or matchings indicate the collective action that each agent (clients or staff employees) must take to prepare a suitable deal at each reflection k of the game. This situation is manifested by the construction of an appropriate sequence of risks that increase at the starting periods $1<..k..<<$ and then dynamically decrease in game closing periods $1<<..k<..$ The sequence of risks of incomparability of matchings finally, albeit in the most unfavorable case, converges to a "single point" at the end of the game. The reassessment of risks has a monotonic character, which made it possible to build a game based on the so-called Monotone system (MS).

One disadvantage of the MS is the challenge in aligning the results of the analysis with a realistic interpretation. The quasi-core extraction process may require additional adjustments for proper interpretation. However, the idea of using mismatch or incompatibility indicators as metrics can help to identify latent issues before they arise, and allow marketing agency staff employees to proactively manages the situation. By measuring compatibility between a marketing agency employee and a client, the agency can make more informed decisions about who to assign to each client, ultimately leading to improved customer satisfaction and reduced marketing volatility and fuzziness. It is important to note that $r_{i,j}$ metrics could be used in conjunction with other factors, such as the time frame as explained in Osborne and Rubinstein (2020, [23]).

The question being raised is why a partial matching (in this case, a pairwise matchings of agents) is preferable to a complete matching. In constructing "Greedy-type" the multi-stage time sequences becomes a single-peaked. As a result, a partial matching in the form of quasi-core imputations is more preferable or efficient than a complete matching.

The concept of the core in cooperative game theory refers to the sets of feasible payoffs that can be achieved by the players through cooperation. Finding the exact payoffs, associated with the core, is difficult problem meaning that it may not be solvable using current computing power. The problem becomes unclear also because, among other things, it is not known whether the core is empty. The existence of non-empty payoff sets, similar to the core, called quasi-cores, is guaranteed in marketing game. A quasi-core is defined as a stable sets determined by the marginal values of supermodular utility functions, in accordance with Rawls' second principle of justice. These sets can be identified using a version of the P-NP problem that uses the branch and bound heuristic, which is an optimization algorithm that combines a systematic search in the solution space with checking upper and lower bounds of the remaining subtasks. The heuristic can be visualized using spreadsheets such as Microsoft Excel where an optimization problem can be modeled and solved with a combination of formulas and algorithms. The branch and bound heuristic can give approximate solutions in a relatively efficient way, allowing a rough estimate of the quasi-core.

The quasi-core concept in marketing game refers to a fundamental idea or principal that guides marketing activities. It can be applied to marketing to evaluate the stability of marketing comparisons and determine whether a given marketing strategy is feasible. The stability of the marketing depends on how well it is able to account for externalities, such as the actions of competitors, changes in consumer preferences, and the impact of technology. By analyzing the stability of matchings in the context of the quasi-core, marketing professionals can gain insights into the likelihood that their strategy will be successful and be able to make informed decisions about how to adjust their approach as necessary. In this sense, the quasi-core concept can be seen as a tool for promoting the long-term viability of marketing initiatives.

APPENDIX

A1. Addendum

To understand what is proposed below, the situation is such that the game can be viewed as a dynamic or multi-stage reassessment of rankings by reflections $.,k\,,k+1,.,$ as a chain shrinking sub-blocks from $.,X_k$ to $X_{k+1},..$ When participants as pairs or partners α signed an agreement and reserved their services and products, the sub-blocks $X_k \supset X_{k+1}$ are reassessed or narrowed down. If among best matches $C(X_{k+1})$ in X_{k+1} there are matches from X_k, then, in this new situation X_{k+1}, these best pairs $C(X_k)$ from X_k should be present in their former role as the best choice, especially true for matchings in the quasi-core.

One circumstance must be kept in mind here. On the one hand, we are dealing with matchings, but on the other hand, the considered matchings are also a certain set of cells or sub-blocks X embedded into $n\times m$ tables in our marketing game, and therefore it is quite appropriate to consider matchings from the point of view of Boolean set theory, where the usual operations of inclusion, intersection of table cells as pairs of participants, etc. are allowed.

As the sub-block-formation chain X_k shrinks $\langle X_k \supset X_{k+1} \rangle$ the proposition below can be verified by least-risk $F(X_k) = \min_{\sigma \in X_k} \pi(\sigma, X_k)$ generating choices $C(X_k)$ as a list $\langle \alpha_k = \arg\min_{\sigma \in X_k} \pi(\sigma, X_k) \rangle$ of potential participants $\alpha_k \in X_k$ at risks levels $F(X_k)$. The list $C(X_k)$ represents matchings $\overline{\alpha} = \langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$ that participants $\overline{\alpha}$ decide to match. Partners $\sigma \in X_{k+1}$ now in the role some $\alpha_{k+1} = \sigma$ will try to realize their latent relations. While the chain X_k has been formed, due to the fact that all participants in $\overline{\alpha}$ no longer are available (reserved) for new matching, in the new reflection $k+1$, all eventual partners/cells in X_{k+1}, must reconsider to whom they prefer to match, as their favored σ. Based on the remarks above, the following can be stated.

Proposition 5. *In the marketing game, the participants of the game move from the best choice* $C(X_k)$ *on previous period of the game to the next best choice* $C(X_{k+1})$ *on succeeding period. If it turns out that in succeeding period* X_{k+1} *the old bests* $C(X_k)$ *are still present, i.e.,* $C(X_k) \cap X_{k+1} \neq \varnothing$ *, then* $C(X_{k+1}) \supseteq C(X_k) \cap X_{k+1}$. *These* $C(X_k)$ *"old best clients" will continue to be the best for marketing by the same staff employees of the agency, provided that the reward payments* $F(X_k)$ *will not increase:* $F(X_k) = F(X_{k+1})$.

The proposition somehow revises a rational mechanism of so-called heredity succession choice $C(X)$; Postulate 4, Chernoff (1954, [24]), condition α of Sen (1970, [25]), or fuzzy form [26], cf. Arrow Axiom (1959, [27]; cf. also Malishevski [20]. The proof may be explained in the basic terms. It is possible to reach an arbitrary sub-block X not yet matched participants by sequence $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$, $X_1 = R$, $X_{k+1} = X_k \div \{\alpha_k\}$ $X = X_{k+1}$, starting from the initial reflection R of the game, where nobody has been matched yet. The sequence will improve payoffs x_k on previous periods $\langle \alpha_1, ..., \alpha_k \rangle$ in accordance with non-decreasing values $F(X_k)$.

The statement of the proposition can be verified by observation of all priority tables and all matchings X that emerged from all $n \times m$ tables, when both n and m are small integers. For higher n and m values, it is NP-hard problem. Second, consider an arbitrary sub-block X of the $n \times m$ -game. While the anti-sub-block $\overline{X} \equiv Dx$ includes all participants signed a deal; all participants in X are still unmatched. We can thus always find partners $\alpha_1 \in \overline{X}$ such that $F(R) \le F(R \div \{\alpha_1\})$. Consider $(n-1)\times(m-1)$ -game, which can be arranged from $n \times m$ -game by declaring the partners signed a deal α_1 as blank agents, $\boldsymbol{i}_{\alpha_1}, \boldsymbol{j}_{\alpha_1} \notin \mathcal{P}$.

By the induction scheme, there exists a sequence of matchings $\langle \alpha_1, \alpha_2 ..., \alpha_k \rangle$ with required quality of improving the payoffs X_k starting from $X_1 = R \div \{\alpha_1\}$. Restoring the blank attendees α_1 to the role of clients and agency staff employees in the $n \times m$ -game, we can, in particular, ensure the required quality of the sequence $\langle \alpha_1, \alpha_2, ..., \alpha_k \rangle$. The statement of the proposition is obviously the corollary of the claim above. However, ensured by its logic, the claim is a more general statement than the statement of the proposition. The first part of the statement is self-explanatory. The matching $\mathcal{N}$ stops being a proper subset among kernels $\{\mathcal{K}\}$ as soon as the payoff function $F(\mathcal{N})$ do not allow improving the outcome $\boldsymbol{n}$. The second part of the proposition is the same statement, worded differently. Nonetheless, we consider it necessary to provide complete proofs of all statements, since proofs are presented here only in a concise form.

A2. Finding the quasi-core

In general, algorithms like Greedy improve the solution dynamically through reassessment. However, in the case of the marketing game, this approach is complicated by the fact that local improvements may not necessarily lead to the best outcome or payoff for all agents. The best outcomes for all agents make up the quasi-core of the marketing game, and there may be numerous best compensations. Finding the core in the conventional sense is NP-hard because the number of operations increases exponentially with the number of participants. In the marketing scenario and other marketing games, there is a large family of subsets that make up the traditional basis of imputations. While it may be possible to find all payoff vectors induced by kernels, it is impractical to do so. Therefore, we suggest finding some admissible matchings belonging to the quasi-core and the payoffs induced by these matchings are sufficient.

This can be achieved by applying a strong payoff improvement procedure and several rolling procedures that do not worsen the position of the agents when forming the matching. In some situations, known as succession rationality, Definition 3, the strong improvement procedure cannot find anything. On the contrary, using rolling procedures, we can move forward in one of the promising directions to find payoffs that do not worsen the result. Experiments performed using our polynomial algorithm show that by using a combination of improvement procedures and rolling procedures both with a rational succession, it is possible to use a backtracking search strategy and find possible payoffs belonging to the quasi-core.

We use five procedures in total—one improvement procedure and four variants of rolling procedures. Combining these procedures, the algorithm below is given in a more general form. While we do not aim to explain in detail how to implement these five procedures, in relation to rational succession, it will be useful to explain beforehand some specifics of the procedures because a visual interaction is best way to implement the algorithm.

In the algorithm, we can distinguish two different situations that will determine in which direction to proceed. The first direction promises an improvement in case the attendees $\alpha \in X$ decides to match or sign a deal. We call the situation when $C(X \div \{\alpha\}) \cap C(X) = \varnothing$ as a latent improvement situation. Otherwise, when $C(X \div \{\alpha\}) \cap C(X) \neq \varnothing$, it is a latent rolling direction. Let $CH(X)$ be the set of rows $C(X)$, the horizontal routes in R Table 3 & 6, which contain the set $C(X)$. By analogy $CV(X)$ represents the vertical routes, the set of columns, $C(X) \subseteq CH(X) \times CV(X)$. To apply our strategy upon X, we distinguish four cases of four non-overlapping blocks in the mutual risk $R = \left\| r_{i,j} \right\|$ Table 3 & 6: $CH(X) \times CV(X)$; $CH(X) \times \overline{CV(X)}$; $\overline{CH(X)} \times CV(X)$; $\overline{CH(X)} \times \overline{CV(X)}$.

Proposition 4 *An improvement in payoffs for all participants in the marketing game may occur only when partners* $\alpha \in X$ *comply with the latent improvement situation in relation to the sub-block* X *, the case of* $C(X \div \{\alpha\}) \cap C(X) \neq \varnothing$ *. The attendees* $\alpha \in X$ *are otherwise in a latent rolling situation.*

The following algorithm represents a heuristic approach to finding payoffs $\boldsymbol{n}$ induced by kernels $\{\mathcal{N}\}$ of the marketing game. Recall that R is the notation for the table of mutual risks. Build the mutual risks Table 3 & 6, $R = W + M$—a simple operation in Excel spreadsheet.

Input Set $k \leftarrow 1$, $X \leftarrow R$ in the role of not yet matched participants, i.e., as agents available for latent matching**.** In contrast to the set X, allocate indicating by $D\boldsymbol{x} \leftarrow \varnothing$ the initial status of matched participants.

Do: **S**, Find a match $\alpha_k \in CH(X) \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) < F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k + 1$, else *Track Back.*

Rolling: **D**, Find a match $\alpha_k \in CH(X) \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k + 1$, else *Track Back.*

JumpF **F**, Find a match $\alpha_k \in CH(X) \times \overline{CV(X)}$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k + 1$, else *Track Back.*

```
JumpG   G, Find a match $\alpha_k \in \overline{CH(X)} \times CV(X)$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that
        $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else
        Track Back.
JumpH   H, Find a match $\alpha_k \in \overline{CH(X)} \times \overline{CV(X)}$, $D\boldsymbol{x} \leftarrow D\boldsymbol{x} + \{\alpha_k\}$, such that
        $F(X) = F(X \div \{\alpha_k\})$, $X \leftarrow X \div \{\alpha_k\}$, $X_k = X$, $k = k+1$, else
        Track Back.
Loop Until no participants can be found in accordance with macros S, D, F, G and H.
Output  The set $D\boldsymbol{x}$ forms $D\boldsymbol{x} = \langle \alpha_1,...,\alpha_k \rangle$. The row-column removal of $D\boldsymbol{x}$
        from $R$, $\mathcal{N} = R \div D\boldsymbol{x}$, represent the technical framework of the game
        while the payoff $\boldsymbol{n}$ induced by $\mathcal{N}$ belongs to the quasi-core.
```

In closing, it is worth noting that a technically minded reader would likely observe that matchings X_k are of two types. The first case is $X \leftarrow X \div \{\alpha_k\}$ operation when the mismatch compensation for bad luck increases, i.e., $F(X_k) < F(X_k \div \{\alpha_k\})$. The second case occurs when rolling along the compensation $F(X_k) = F(X_k \div \{\alpha_k\})$. In general, independently of the first or the second type, there are, as said, five different directions in which a move ahead can proceed. In fact, this poses a question—in which sequence of participants α_t should be selected in order to facilitate the generation of the *sequence* $D\boldsymbol{x} = \langle \alpha_1,...,\alpha_k \rangle$ of matchings? We solved the problem for marketing games underpinning our solution by backtracking. It is often clear in which direction to move ahead by selecting improvements, i.e., either a strict improvement by **s)** or rolling procedures though **d)**, **f)**, **g)** or **h)**. However, a full explanation of backtracking is out of the scope of our current investigation. Thus, for more details, one may refer to similar techniques, which effectively solve the problem (Dumbadze, 1989, [28]).

A3. Conventional stability

In order to demonstrate the shortcomings, at least in one particular case, of using traditional game theory concepts such as the core, below we use a mixture of common game theory terms and try to show that the standard core does indeed give a rather poor solution as the core consists of a single imputation in the form of complete or grand matching. This suggests that alternative approaches may be required to solve the marketing game effectively.

The marketing game arrangement is expanded to a more general case. There are $n+m$ participants n of which are clients $\langle 1,...,i,...,n \rangle$, and m are agency staff employees $\langle 1,...,j,...,m \rangle$. Some of the participants expressed their willingness to participate in the game and have revealed their rankings. Those who refused are referred to as *blanks*, while others who agreed to play the game will be arranged by default into the Grand Matching $\mathcal{P}$, $|\mathcal{P}| \le n+m$.

Indices i, j annotate the participants of the game. Participants in $\mathscr{P}$ are regarded as *players*, whereas partners $\alpha = [\mathrm{i}, \mathrm{j}]...$ or $[\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha]...$ are designated to as $\alpha,...,\sigma$. This differentiation helps making notations short.

Marketing game focuses on the participants $\mathrm{D} \subseteq \mathscr{P}$ that are matched. Having formed their rankings, participants in D have the power and ability to assert their rankings. Participants in D can convince all those in $\overline{\mathrm{D}}$ who are not already in D to opt out of the game without a partner and thus be compensated. Given the tables W, M and R, the situation, in contrast to D, which lists matched pairs, i.e., those who made deals, can be represented as a sub-block $\mathrm{X} = \mathrm{R} \div \mathrm{D}$ consisting of rows and columns from $\overline{\mathrm{D}}$.

It is realistic to assume that enforcing the interests of the participants in D is not always possible. Regardless of their participation in D those in the $\mathrm{D}' \subset \mathrm{D}$, whose interests are affected (suppressed), will still be able to receive as much as they receive in D. Sometimes it is convenient for D' to exclude this opportunity, since it is better that the D' matching cannot be implemented simultaneously with D and be its direct competitor.

N.B. It should be emphasized here that the D matching are those participants who have signed deals, and the X sub-block are those who prefer to continue. Matching D and sub-block X characterize the game multi-stage situation achieved in period k, when the participants imitating each other actions must decide on the further course of the game, whether to move to reflection $\mathrm{k}+1$ or not. Each agent identified by the rows and columns in X receives 50% of the rewards of the agents in D in the event of the game is over. A realistic situation may occur when all participants in $\mathscr{P}$ are matched, $\mathrm{D} = \mathscr{P}$, or, in contrast, no one decides to match, $\mathrm{D} = \varnothing$ hereby after revealing their rankings, all might decide not to proceed with the game at all.

Among all matchings D, rational matchings are usually singled out. A participant, entering into the matching D, derives from the interaction in the matching a reward that satisfies $\alpha \in \mathrm{D}$. We assume that the rewards and compensations are strictly dependent on pairwise matchings in D, which in turn were caused by sub-block X. Using the matchings $\mathrm{D} \subseteq \mathscr{P}$, we can always construct a payoff $\boldsymbol{x}$ to all participants $\mathscr{P}$, i.e., we can quantify the positions of all participants. The inverse is also true. Given a payoff $\boldsymbol{x}$, it is easy to establish which participant belongs to the matching D and identify those belonging to block $\mathrm{X} = \mathrm{R} \div \mathrm{D}$. We label this fact also as $\mathrm{D}\boldsymbol{x}$. Recall that participants of the matching $\mathrm{D}\boldsymbol{x}$ receive a reward to match, which is equal to the double amount of the “mismatch” compensation. Thus moving to better positions, the list of participants $\mathrm{D}\boldsymbol{x}$ may provide an opportunity for some participants $\sigma \in \mathscr{P}$ to start, or initiate, new matches. We will soon see that, while the best positions induced by special sub-blocks $\mathscr{K}$, called the kernel block, have been reached, this movement will be impossible to realize. Our terminology is unconventional in this connection.

The concept of stability in matching games refers to the inability of agents to move to better positions by making pairwise comparisons. In the work "Cores of Convex games" by Shapley (1971, [39]) convex games were studied, which are games that have a non-empty core. The core is a convex set of end-points (imputations), representing the available payoffs to all agents in a multidimensional octahedron. The core stability in these games ensures that no agent has an incentive to move from their current position to a better one, leading to a stable solution. Below, despite the agents' asymmetry with respect to $\mathrm{D}\boldsymbol{x} = \mathrm{R} \div \mathrm{X}$, we focus on their payoffs driving their collective behavior as participants $\boldsymbol{\mathscr{P}}$ to form the matching $\mathrm{D}\boldsymbol{x}$, $\mathrm{D}\boldsymbol{x} \subseteq \boldsymbol{\mathscr{P}}$; $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$ is and anti-sub-block to X ; $\overline{\mathrm{X}}$ designates deleted rows and columns.

In contrast to individual payoffs improving or worsening the positions of participants, when playing the marketing game, the total payment to the matching $\mathrm{D}\boldsymbol{x}$ as a whole is referred to the utility function $\boldsymbol{h}(\mathrm{X}) > 0$. In classical cooperative game theory, the payment $\boldsymbol{h}(\mathrm{X})$ to matching $\mathrm{D}\boldsymbol{x}$ is known with certainty, whereby the variance $\boldsymbol{h}(\mathrm{X}) - \boldsymbol{h}(\mathrm{X} \div \{\sigma\})$ provides a marginal utility $\pi(\sigma,\mathrm{X})$. Inequality $\pi(\alpha,\mathrm{X} \div \{\sigma\}) \le \pi(\alpha,\mathrm{X})$ of the scale of risks of incompatible agreements expresses a monotonic decrease (increase) in marginal utilities $\pi(\alpha,\mathrm{X})$ for $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in \mathrm{X}$. This monotonicity is equivalent to supermodularity $\boldsymbol{h}(\mathrm{X}_1) + \boldsymbol{h}(\mathrm{X}_2) \le \boldsymbol{h}(\mathrm{X}_1 \cup \mathrm{X}_2) + \boldsymbol{h}(\mathrm{X}_1 \cap \mathrm{X}_2)$, Nemhauser et al, 1978, [30]. Any utility function $\boldsymbol{h}(\mathrm{X})$, payments for which are built on a scale of risks of incompatible agreements, due to monotonicity, is supermodular. Supermodular functions have been used to solve many combinatorial problems (Petrov & Cherenin 1948, [31]; Emonds 1970, [32]; Bai & Bilmes, 2018, [33]). In general, a supermodular guarantee cannot be given.

Recall that we eliminated all rows and columns X in tables $\mathrm{W} = \|\mathrm{w}_{\mathrm{i,j}}\|$, $\mathrm{M} = \|\mathrm{m}_{\mathrm{i,j}}\|$ in line with $\overline{\mathrm{X}} \equiv \mathrm{D}\boldsymbol{x}$. Table $\|\mathrm{w}_{\mathrm{i,j}}(\mathrm{X}) + \mathrm{m}_{\mathrm{i,j}}(\mathrm{X})\|$ or $\|\pi(\alpha,\mathrm{X})\|$, where $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in \mathrm{X}$ imitates the dynamic outcome of dynamically reassessing rankings $\mathrm{w}_{\mathrm{i,j}}$, $\mathrm{m}_{\mathrm{i,j}}$ when some participants $\sigma \in \overline{\mathrm{X}}$ have been matched and signed a deal. Rankings $\mathrm{w}_{\mathrm{i,j}}$ and $\mathrm{m}_{\mathrm{i,j}}$ are consequently decreasing. Given in the form of utility function, e.g., the value $\boldsymbol{h}(\mathrm{X}) = \sum_{\alpha \in \mathrm{X}} \pi(\alpha,\mathrm{X})$ sets up the marketing game. An imputation for the game $\boldsymbol{h}(\mathrm{X})$ is defined by a $|\boldsymbol{\mathscr{P}}|$-vector fulfilling two conditions: (i) $\sum_{\alpha \in \boldsymbol{\mathscr{P}}} (\boldsymbol{w}_\alpha + \boldsymbol{m}_\alpha) = \boldsymbol{h}(\boldsymbol{\mathscr{P}})$, (ii) individual rationality $\boldsymbol{w}_\alpha, \boldsymbol{m}_\alpha \ge \boldsymbol{h}(\{\alpha\})$, for all $\alpha \in \boldsymbol{\mathscr{P}}$. Condition (ii) stems from repetitive use of monotonic inequality $\pi(\alpha,\mathrm{X} \div \{\sigma\}) \le \pi(\alpha,\mathrm{X})$.

A significant shortcoming of the canonical cooperative theory is related to its inability to define stable matchings (the core is empty) or consisting of only one—the grand matching. At first glance, this shortcoming seems inevitable. Indeed, the lower is the risk $\pi(\alpha, X)$ of incompatible matching $\alpha \in X$, the more reliable the matching $\alpha = [\boldsymbol{i}_\alpha, \boldsymbol{j}_\alpha] \in X$ will be. Let we set up as an exercise a popularity index u_i of client i among agency staff employees $D\boldsymbol{x}$ as $u_i(X) = \sum_{j \in X} m_{i,j}$; accordingly, the index u_j of an employee j popularity among clients will be given by $u_j(X) = \sum_{i \in X} w_{i,j}$. Let we intend to redistribute the payment $\boldsymbol{h}(\mathcal{P})$ of the complete matching $\mathcal{P}$ in proportion to the components of the vector $u(\mathcal{P}) = \langle u_i(\mathcal{P}), u_j(\mathcal{P}) \rangle$. Hereby we can prove, owing to monotonic inequality, that the payoffs in imputation $u(\mathcal{P})$ cannot be improved for any $\alpha \in \mathcal{P}$ inside any partial matching $D\boldsymbol{x} \subset \mathcal{P}$ induced by the sub-block X . Therefore, the game solution, among popularity indices, will be the only imputation $u(\mathcal{P})$ —popularity indices core of the cooperative game consists of only one point $u(\mathcal{P})$. In other words, for matching all participants, any matching using any algorithm (in particular, ibid. Roth & Sotomayor) will be the best matching in terms of cooperative game using the only imputation $u(\mathcal{P})$.

A4. Visualization

Recall that, the input to the algorithm presented in the main body of the paper contains three tables (cf. Table 1-6): $W = \|w_{i,j}\|$ —rankings table w_i where the client specify with the respect to the characteristics the agency staff employees should possess, in the form of permutations of numbers $\overline{1,n}$ in rows; $M = \|m_{i,j}\|$ —visa versa, rankings m_j where staff employees specify the characteristics in the form of clients permutations of numbers $\overline{1,m}$ in columns; and $R = \|w_{i,j} + m_{i,j}\|$. These tables, and tabular information in general, are well suited for use in Excel spreadsheets that feature calculation, graphing tools, pivot tables, and a macro programming language called VBA—Visual Basic for Applications.

A spreadsheet http://datalaundering.com/download/marketings_game.xls (accessed December 23, 2021) was designed to visually represent our idea of finding the quasi-core $\overline{\alpha} = \langle \alpha_1, \ldots, \alpha_{12} \rangle$ of the marketing game, including the stable matchings that belong to the quasi-core. It was compiled by macro-activated rendering capabilities of Excel.

A5. Spreadsheet layout specification

Three tables are available: the Pink table W —client's rankings, the Blue M — agencies' rankings. The Yellow R —table consists of mutual risks $\mathrm{r}_{\mathrm{i,j}} = \mathrm{w}_{\mathrm{i,j}} + \mathrm{m}_{\mathrm{i,j}}$ of matchings incompatibility. The rows and columns, which represent those who ceased the game, will be highlighted with a gray shadow. According to this representation, the yellow sub-block X in R will represent all potential or new opportunities of the matching $\sigma = [\mathrm{i}_\sigma, \mathrm{j}_\sigma] \in \mathrm{X}$. The global $\mathrm{F(X)} = \min \to \mathrm{r}_{\sigma\in\mathrm{X}}$ occupy the cell in the lower right corner of the table R. The line on the right to X shows the minimum risk in the row $\mathrm{i}_\sigma \in \mathrm{X}$, and the horizontal line below X shows the minimum risk in the column $\mathrm{j}_\sigma \in \mathrm{X}$. The green cells in the yellow sub-block X visualize the choice operator $\mathrm{C(X)} = \{\arg\min \to \mathrm{r}_{\sigma\in\mathrm{X}}\}$. The cells **[V24:AO25]** and **[V26:AO26]** contain the sequence $\overline{\alpha} = ..., \mathrm{X}_\mathrm{k} \supset \mathrm{X}_{\mathrm{k+1}}, ...$ of the game generated in periods $..., \mathrm{k}, \mathrm{k}+1, ...$ together with the risks of matching associated by the sequence. The agents' balance of payoffs occupies the cells **[V31:AO32]**. Some cells reflecting the ***state of finances*** of cashier are located below, in the cells **[AP34:AP44]**. Cells in row-**1** and column-**A** contain the participants' labels. We use these labels in all macros.

A6. Extracting the quasi-core of the game

We came closer to the goal of our visualization, where we visually demonstrate the main features of the theoretical model of the game by example. Generating the matching sequence, which is performed in a period-wise fashion, constitutes the framework of the theory. In each period, to the right side of the sequence generated in the preceding periods, we add partners found by one of the macros Case**S**, Case**D**, Case**G** and Case**H**, i.e., partners that has decided to match. This process is repeated until the marketing risks of incompatibility matching reach the level 6. When using these macros one can easily verify that, risks initially increase, and then decline towards the end in case we proceed further with these macros. This marketing $\cap$-peakedness is a consequence of the mutual risks of matching monotonicity $\pi(\alpha, \mathrm{H} \div \{\sigma\}) \leq \pi(\alpha, \mathrm{H})$. Indeed, recall that matching' levels are recalculated after each matching. With the proviso of recommendations in our heuristic algorithm, see above, due to the recalculation, the priority scales will "shrink" or "pack together", as only not yet matched participants remain. The sequence $\overline{\alpha}$ can be generated by macros: Case**S**, Case**D**..., Case**H**. The output will occupy the cells **[V24:O28]**. The initial reflection of the table can be restored with macros: Ctrl+**o**, Ctrl+**b** and Ctrl+**l**. As an example of these macros, we prepared the result in cells **[B51:L52]**. Just copy the contents of these cells into **[V24:F25]** and then use the Ctrl+n macro, which renders the core of the 11 matches of the game.

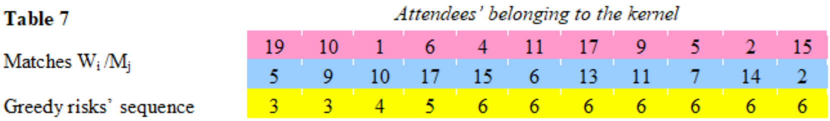


Table 7

Attendees' belonging to the kernel

Matches W_i /M_j	19	10	1	6	4	11	17	9	5	2	15
	5	9	10	17	15	6	13	11	7	14	2
Greedy risks' sequence	3	3	4	5	6	6	6	6	6	6	6

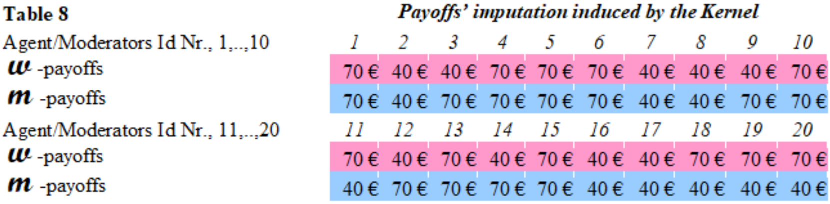


Table 8

Payoffs' imputation induced by the Kernel

Agent/Moderators Id Nr., 1,..,10	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
w -payoffs	70 €	40 €	40 €	70 €	70 €	70 €	40 €	40 €	40 €	70 €
m -payoffs	70 €	40 €	70 €	70 €	70 €	70 €	40 €	40 €	70 €	70 €
Agent/Moderators Id Nr., 11,..,20	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
w -payoffs	70 €	40 €	70 €	40 €	70 €	40 €	40 €	70 €	70 €	70 €
m -payoffs	40 €	70 €	70 €	70 €	70 €	40 €	40 €	40 €	40 €	40 €

Let us look at Table 7, where only 11 matches are accomplished, i.e., all columns to right starting at from the match [19,5] till [15,2] visualize the outcome n of our marketing game. Table 7 marks those participants who decided to match, while all the rest but on this particular list are not yet taken their decisions or have been, perhaps, unlucky to find a partner.

Table 8 will note the payoffs, that is, the imputation induced by the kernel matching—the amount of payments in the form of rewards or compensations for bad luck to all 40 participants—20 clients and 20 agencies. Payoffs of 40€ and 70€ correspond to what the kernel makes up in cash. The result is a total amount of 2000€ received by the cashier in the form of participation fees minus 2260€ as payoffs, i.e., –260€ not in favor of the cashier.

We can continue creating the sequence of matchings with macros using mAtch [ctrl + a], pointing to the cell in the top box: pink on the left (or yellow on the right), until all participants have been matched. Please note this, starting with pair **No.12**; we can no longer use the macros of our heuristic algorithm. There are no participants with increasing payoff compensations 1-11, which represent the maximum point—a payoffs n of the game.

In the Table 9-10 below, the Matching Sequence consists of $k=\overline{1,20}$ time slices or periods; we labeled attendees $[i,j]$ using notation α_k. Together with levels of mutual risks in row 3, the pink and blue rows correspond to the sequence $\overline{\alpha}=\langle\alpha_k\rangle$ of matchings. Compensations and rewards for marketing are not payable at all, and only the costs of goodies (each worth 10€) occupy similarly pink and blue rows. For match #3, the participants risk jumps from 4 to 5, and for match number 4 also increase from 5 to 6. Note that due to the risks single ∩-peakedness, the lowest risk levels first for match #3 increase starting from 4, and after the level 6, starting from match #12, it begins to decrease to 0.

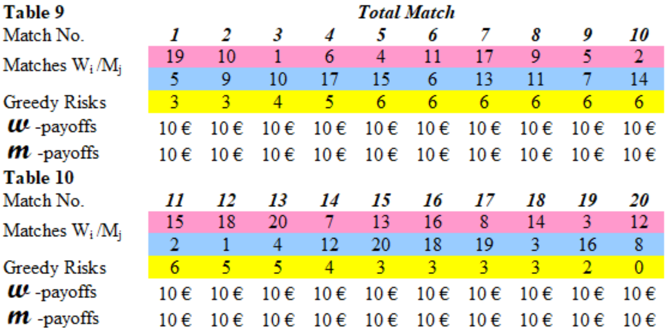


Table 9

					Total Match					
Match No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Matches W_i /M_j	19	10	1	6	4	11	17	9	5	2
	5	9	10	17	15	6	13	11	7	14
Greedy Risks	3	3	4	5	6	6	6	6	6	6
w -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
m -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

Table 10

Match No.	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Matches W_i /M_j	15	18	20	7	13	16	8	14	3	12
	2	1	4	12	20	18	19	3	16	8
Greedy Risks	6	5	5	4	3	3	3	3	2	0
w -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €
m -payoffs	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €	10 €

The list of macros; CH —cells in horizontal, CV —cells in vertical direction.

- **CaseS.** Ctrl+**s,** Trying to move by improvement along the block $\mathrm{CH(X)}\times\mathrm{CV(X)}$ of cells $\sigma=[\mathrm{i},\mathrm{j}]$ by "<" operator in order to find a new matching at the strictly higher level.
- **CaseD.** Ctrl+**d,** Trying to move while rolling along the block $\mathrm{CH(X)}\times\mathrm{CV(X)}$ of cells $\sigma=[\mathrm{i},\mathrm{j}]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseF.** Ctrl+**f,** Trying to move while rolling along the block $\mathrm{CH(X)}\times\overline{\mathrm{CV(X)}}$ of cells $[\mathrm{i},\mathrm{j}]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseG.** Ctrl+**g**, Trying to move while rolling along the block $\overline{\mathrm{CH(X)}}\times\mathrm{CV(X)}$ of cells $\sigma=[\mathrm{i},\mathrm{j}]$ by "<=" operator in order to find a new matching at the same or higher level.
- **CaseH.** Ctrl+**h**, Trying to move while rolling along the block $\overline{\mathrm{CH(X)}}\times\overline{\mathrm{CV(X)}}$ of cells $\sigma=[\mathrm{i},\mathrm{j}]$ by "<=" operator in order to find a new matching at the same or higher level.

Functional test. The spreadsheet users are invited first to perform a functional test, in order to become familiar with the effects of **ctrl-keys** attached to different macros. Calculations in Excel can be performed in two modes, ***automatic*** and ***manual***. However, it is advisable to choose properties and set the calculus in the manual mode, as this significantly speeds up the performance of our macros. The macros one can take if something goes wrong are listed below.

- **Originate.** [Ctrl+**o**]. Perform the macro by Ctrl+**o**, and then use Ctrl+**b**. This macro restores the original status of the game saved by the BacKup, i.e., saved by ctrl-**k**.
- **RandM.** [Ctrl+**m**]. The macro Ctrl+m rearranges columns of **Staff Employees**' priority **M** table by random (permutations). N.B. the effect upon staff employees' rankings **M**.
- **RandW.** [Ctrl+**w**]. The macro by Ctrl+w rearranges rows of **clients** priority table **W** by random permutations. N.B. the effect upon client's priority table **W**.

- **Proceed.** [Ctrl+**e**]. While proc**E**eding with macros Rand**M** and Rand**W,** the macro is using random permutations for agency staff employees and client until it generates the priority tables **M** and **W** with minimum mutual risk equal to 4.
- **Blank.** [Ctrl+**u**]. This macro is removing from the list of participants those participants that do not wish to play the game. We call them blank agents. Activate the row-1, or column-A by pointing at employee $\mathbf{m}_{\#\#}$, or client $\mathbf{w}_{\#\#}$ and then perform Ctrl+**u** excluding the chosen participants from playing the game.
- **MAttendee.** [Ctrl+**a**]. Try to m**A**tch [ctrl+**a**] partners by pointing at the cell in the upper block: pink color to the left (or yellow to the right) in the row $\mathbf{w}_i$ (corresponding to an client) and the column $\mathbf{m}_j$ (corresponding to a moderator).
- **TrackR.** [Ctrl+**r**]. Visualizes Tracking forwa**R**d. Memorizes the status of *clients*-**W** and *Staff Employees*-**M** rankings to be restored by **TrackB** macro. The effect is invisible, however, it can be used whenever it is appropriate to save the active status of all tables and arrays necessary to restore the status by **TrackB** macro. When the search for quasi-core matchings is performed manually, the effect becomes visible.
- **TrackB.** [Ctrl+**b**] Visualizes Tracking **B**ack. Restores the status of *client*-**W** and *Staff Employees*-**M** rankings memorized by **TrackR** macro.
- **Happiness.** [Ctrl+**p**]. The macro calculates an index of ha**P**piness of the initial tables status.
- **Matching.** [Ctrl+**n**]. The macro rebuilds the matching matching following the matching matching list previously transferred into area "AV24:AO25".
- **Chernoff.** [Ctrl+**q**]. Useful when indicating **by red font** in Excel the status of the Choice Operator **C(X)={argmin}**. Using this macro will help to confirm the validity of the Succession Operator. To clear the status, use Ctrl+**l**.

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Statement of competing interests. The sole author, Joseph E. Mullat, of the manuscript "Partial Matching in the marketing Game: Reassessing incompatibility Indicators" states that no one else can have counter-interests in the article; neither he received any financial support nor did he discuss or contact anyone in the scientific community for any reason other than those stated in the Acknowledgments. The author still owns the intellectual property of the conference article and has the right to publish its current version, if applicable.

VISUALIZATION OF THE MARKETING GAME WITH 20 CLIENTS AND 20 STAFF EMPLOYEES

Legend

Client → Clients' Awards,
Staff → Employees' Awards
Cash → Cashier's Balance

Client nr. 19
Employee nr. 5

Any agreement signed outside the quasi-core will reduce payoffs (including compensation) to all participants, however this will improve the cashier's balance. In this case, the members of the quasi-core will recommend stopping the game.

Client	Staff	Cash	Period												
515€	515€	970€	*1*	19											
				5											
530€	530€	940€	*2*	19	1										
				5	9										
660€	660€	680€	*3*	19	1	18									
				5	9	10									
800€	800€	400€	*4*	19	1	18	6								
				5	9	10	17								
950€	950€	100€	5	19	1	18	6	20							
				5	9	10	17	15							
980€	980€	40€	*6*	19	1	18	6	20	17						
				5	9	10	17	15	2						
1.010€	1.010€	20€	*7*	19	1	18	6	20	17	4					
				5	9	10	17	15	2	14					
1.040€	1.040€	80€	*8*	19	1	18	6	20	17	4	11				
				5	9	10	17	15	2	14	12				
1.070€	1.070€	140€	*9*	19	1	18	6	20	17	4	11	15			
				5	9	10	17	15	2	14	12	6			
1.100€	1.100€	200€	*10*	19	1	18	6	20	17	4	11	15	5		
				5	9	10	17	15	2	14	12	6	4		
1.130€	1.130€	260€	*11*	19	1	18	6	20	17	4	11	15	5	10	
				5	9	10	17	15	2	14	12	6	4	7	
1.160€	1160€	320€	*12*	19	1	18	6	20	17	4	11	15	5	10	8
				5	9	10	17	15	2	14	12	6	4	7	3

Period 12 deals $\alpha_1,\alpha_2\ldots\ \alpha_{12}$ of participants, which represent stable matching situation, like quasi-core agents in a marketing game on the 6th level of incompatibility of the risk indicator scale. In period 1, the risk score was at level 3.

The total amount $F(X)\cdot\left[|D\boldsymbol{x}|+½\left(|\mathcal{P}|-|D\boldsymbol{x}|\right)\right]+|\mathcal{P}|$ of rewards + compensations, inclusive goodies, is equal to $\{6\cdot[2\cdot12+½(2\cdot20–2\cdot12)]+2\cdot20\}\cdot10€=\{6\cdot[24+½\cdot16]+40\}\cdot10€=$ **2.320€**.

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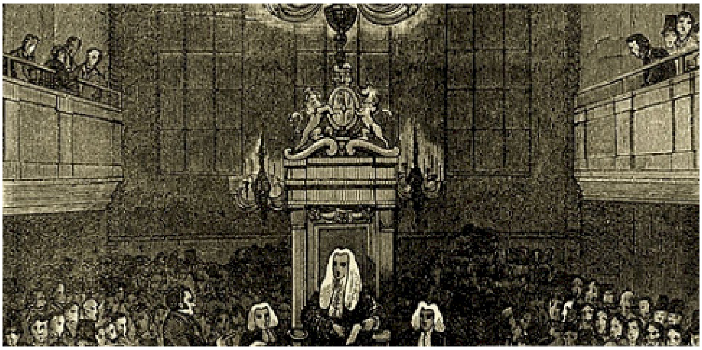

The Left- and Right-Wing Political Power Design:[*] The Dilemma of Welfare Policy with Low-Income Relief

Abstract. The findings yielded by this experiment represent a significant contribution to the theoretical landscape of welfare policy, shedding light on fundamental inquiries surrounding the rules and norms governing wealth redistribution. Specifically, the examination meticulously dissects the expenses associated with redistributing both basic necessities and critical public goods, recognizing the distinct considerations each category demands. Notably, the analysis unveils a pivotal revelation: within the framework of a poverty line designed to treat all citizens equitably, politicians with contrasting ideologies assume the responsibility of determining the financing mechanisms for redistributing essential and vital resources. This process underscores the imperative for political consensus, contingent upon the approval of decisions by the electorate. However, in instances where such approval is absent, policymakers find themselves compelled to engage in ongoing negotiations, underscoring the complexities inherent in reaching viable solutions. Building upon this foundational premise, the study posits that political decisions that elevate the poverty line as a parameter may inadvertently engender inverse incentives for benefit claimants, potentially leading to financial imbalances. This, in turn, raises concerns regarding the fiscal sustainability of distributing both basic and non-basic goods to their intended recipients. To mitigate these challenges and ensure fiscal equilibrium, the proposition is made to utilize half of the median income (μ), colloquially known as the Fuchs point, as the benchmark for defining the poverty line. By adopting this approach, the aim is to foster a more just and equitable framework for wealth redistribution, thereby fostering resolution to the ideological schisms that often divide left- and right-wing politicians. Furthermore, through the application of sophisticated modeling techniques, such as those employed in the development of the Negative Friedman Income Tax (NIT) since 1962, the study demonstrates the efficacy of implementing a wealth redistribution exclusion rule based on an income threshold of half the median income (½ μ) in reducing the Gini coefficient, thereby advancing the discourse on socioeconomic equality and justice.

Keywords: bargaining; welfare policy; public goods; taxation; voting

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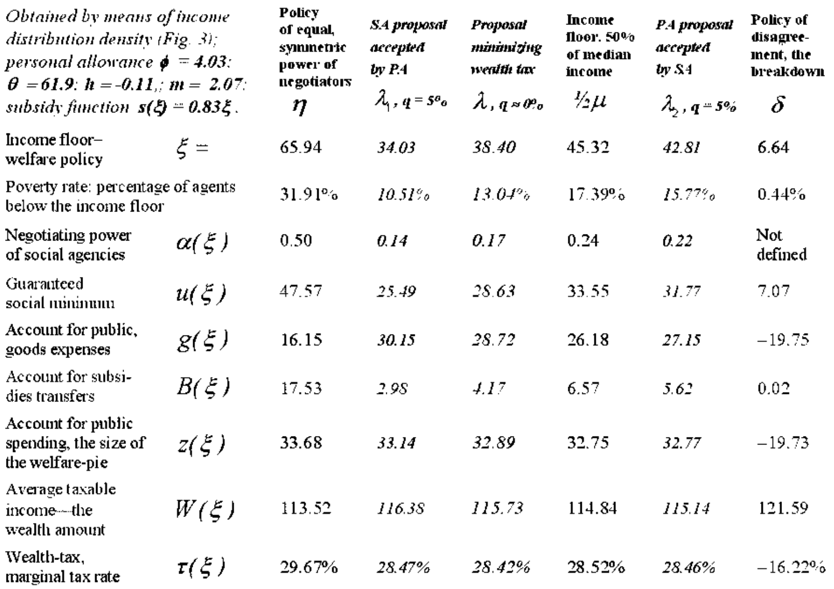
1. INTRODUCTION

Political competition related to wealth redistribution often fosters debate regarding what the state "should" or "should not" deliver. Wider and more substantial welfare benefits and relief payments could be problematic, as they might encourage certain behaviors, such as low savings or productivity when economic security is guaranteed. Similarly, they may lead to high wage demands, as an incentive to remain in employment, given that unemployment benefits are substantial and are compensated by high tax rates $\boldsymbol{\tau}$. In addition, high taxes are an incentive for entering a black labor market that avoids paying taxes, or moonlighting, i.e., holding multiple jobs. Finally, high benefits typically undermine social and geographical mobility. Evidence also shows that, under these conditions, a few would opt for working just because financially they would not be tempting, while many will be wondering why studying is worth the efforts and sacrifices. In sum, excessive benefits might result in human capital not developing quickly and well enough, *e.g.,* "*…implicit support to those waiting on benefits looking for the 'right type of job' or a job that pays well enough,*" as noted by Oakley and Saunders (2011).

As the welfare policy of the state presupposes the existence of both a functioning market economy and a democratic political system, its hallmark is that the distribution of public goods and services is governmental responsibility and obligation. The term *public* in this context refers solely to wealth redistribution. In particular, an obligation to ensure that those on low incomes are awarded appropriate levels of social benefits and relief payments results in a more egalitarian allocation of wealth than can be provided by the free market. In this scenario, politicians face a dilemma of whether such allocation is just and fair to all citizens. The solution depends on many factors, including the characteristics and views of the main benefactors of wealth redistribution. In the absence of a universal definition, in this work, we use the term "wealth" in the scholarly sense, delivered through tax channels and distributed by the state. Under this premise, the average taxable income per capita represents the wealth $\mathbf{W}$.

The primary goal of this experiment is to demonstrate fallacy of arguments advocating in favor of higher benefits and relief payments. Beyond the negative perception of higher benefits, it is also reasonable to believe that distribution of citizens' incomes σ is, perhaps, the only target for control and an exclusive source of information for assessing the amount of benefits available. Our goal is to highlight a hidden side of public interests to welfare issues (Flora, ed., 1987), its geographical, historical justification and broad experimental support in analyzing credible income distributions (Huber et al, 2008). Since we approach welfare redistribution from a more theoretical perspective, we need to have a different emphasis compared to these issues. However, apart from this key aspect, the solution of the welfare policy dilemma, based on numerical simulations, yields the benefits to the needy that are sufficiently close to be considered a realistic match (see Table 1), as noted by Bowman in 1973, to "*what amounts to a moving poverty line at ½ of median income.*" In support of this approach, it is worth noting that Rawls (1971, 2005) pronounced the Fuchs (1965) point as an alternative to the measurement of poverty with no reference to social position. The motive of the experiment presented here is thus to provide — while acknowledging that a few examples clearly cannot make a trend — a theoretical confirmation for the claim recognizing the poverty line, defined as ½μ of the median income μ, as a realistic political consensus.

Table 1. **Numerical experiment behind the welfare policy dilemma of income redistribution;** **SA—Social Agencies, PA—Public Agencies**



Obtained by means of income distribution density (Fig. 3); personal allowance $\phi = 4.03$; $\theta = 61.9$; $h = -0.11$,; $m = 2.07$; *subsidy function* $s(\xi) = 0.83\xi$.		**Policy of equal, symmetric power of negotiators** η	***SA proposal accepted by PA*** $\lambda_1, q = 5\%$	***Proposal minimizing wealth tax*** $\lambda, q \approx 0\%$	**Income floor. 50% of median income** $½\mu$	***PA proposal accepted by SA*** $\lambda_2, q = 5\%$	**Policy of disagreement, the breakdown** δ
Income floor–welfare policy	$\xi =$	65.94	*34.03*	*38.40*	45.32	*42.81*	6.64
Poverty rate: percentage of agents below the income floor		31.91%	*10.51%*	*13.04%*	17.39%	*15.77%*	0.44%
Negotiating power of social agencies	$\alpha(\xi)$	0.50	*0.14*	*0.17*	0.24	*0.22*	Not defined
Guaranteed social minimum	$u(\xi)$	47.57	*25.49*	*28.63*	33.55	*31.77*	7.07
Account for public, goods expenses	$g(\xi)$	16.15	*30.15*	*28.72*	26.18	*27.15*	−19.75
Account for subsidies transfers	$B(\xi)$	17.53	*2.98*	*4.17*	6.57	*5.62*	0.02
Account for public spending, the size of the welfare-pie	$z(\xi)$	33.68	*33.14*	*32.89*	32.75	*32.77*	−19.73
Average taxable income—the wealth amount	$W(\xi)$	113.52	*116.38*	*115.73*	114.84	*115.14*	121.59
Wealth-tax, marginal tax rate	$\tau(\xi)$	29.67%	*28.47%*	*28.42%*	28.52%	*28.46%*	−16.22%

In our scheme, citizens earning low incomes (below a certain level, in this case the poverty line ξ) receive relief payments, whereas those with higher incomes (above the aforementioned level) do not. In this regard, it should be noted that, in 1962, Milton Friedman (2002) proposed a similar scheme of wealth redistribution, combined with flat tax, called the negative income tax — the NIT. According to the rules and norms of the NIT, low-income earners receive a relief payment proportional to the difference between their earnings and the predetermined NIT poverty line. Most importantly, the total — the sum of the key income and the NIT relief payment — is not subject to taxation. We argue that levying taxes in compliance with the tax rules and norms in force for all, inclusive of low-income citizens, would have the same result. Although the total income of low-income citizens is now taxable, they would, even so, still be eligible for the relief in line with NIT, similar to the widely adopted low-income — LI relief. The known drawback of such an approach, and the relief, in particular, stems from the issue of social abuse by those earning low income. In order to mitigate these undesirable effects, in this work, we introduce the so-called hazard of working incentives, referred to as the h-effect.

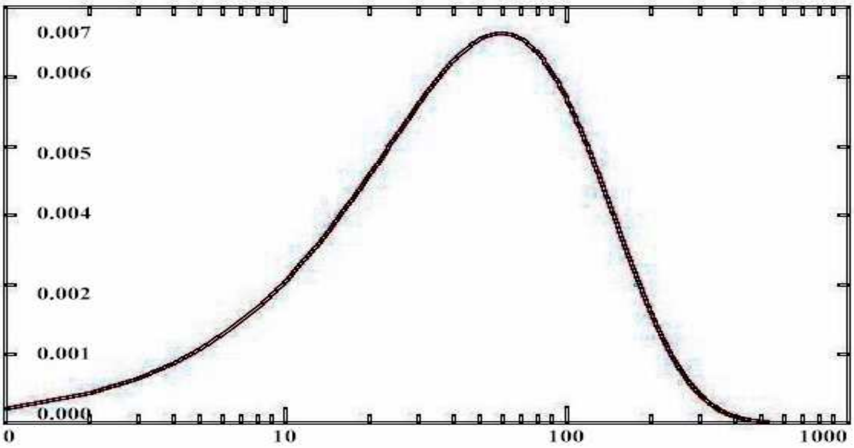


Figure 1. At the sample $P(\sigma, \theta + h \cdot \frac{1}{2}\mu)$ of the income density distribution, μ solves the equation $\int_0^{\xi} P(\sigma, \theta + h \cdot \xi)d\sigma = 0.5$ for ξ, $\mu = 82.30$. Appendix A1 contains the analytical form for the sample expression in Figure 1.

We thus present a theoretical model of visionary politicians, whereby we consider a masquerade of life or a scenario of *realistic utopia*. In this scenario, two actors/politicians, akin to two political coalitions, are playing a bargaining game, each attempting to implement his/her own wealth redistribution policy. *Left-wing politicians* tend to oppose the disproportion in private consumption, unjust wealth redistribution, profit motive, and private property as the main sources of socioeconomic evil. *Right-wing politicians*, owing to a different ideology, tend to focus on regulating business and financial risks, thus encouraging the government's use of its powers in combating corruption, criminal violence and commercial fraud. While left-wing politicians prefer immediate and equitable sharing of the available stock of goods and services, both sides are aware of the citizens' sacrifices — in terms of direct contribution of a part of their income to the funding of welfare benefits and public goods. We posit that applying the rules and norms of wealth redistribution pertaining to the reliance on the elevated relief would increase the quantity of the relief payments to be delivered. Consequently, citizens will have to meet a greater tax burden. This outcome is not ideal, given that lower tax burden and greater private consumption always lie at the heart of citizens' economic and political aspirations. These private objectives prompt majority of voters, who hold power in electing political parties, to oppose increasing the tax burden. As a result, they are instrumental in the competition between the left- and right-wing politicians and their views on tax policies.

Political consensus is rarely possible in reality. Consequently, we aim to design an experiment capable of predicting an appropriate political division between interest groups for desirable implementation of the welfare policy. This approach does not require analysis of the voting system or a scheme by which voters-citizens express their arguments. In adopting this approach, we analyze political power indicators as replications $(\alpha,1-\alpha)$, $0<\alpha<1$, in line with Kalai's bargaining game (1977) in which division of $1 is attempted. In this scenario, among other assumptions, it is posited that a power α is appropriate to adopt the ability to negotiate, or be in the position to request financial support to a greater extent than the opposite side. Similar interpretation of players' power dynamic may be found in the recent work of Mullat (2014). In short, we adopted the view of Roberts who noted in 1977, "*The point is not whether choices in the public domain are made through a voting mechanism but whether choice procedures mirror some voting mechanism.*"

These brief remarks should be sufficient to elucidate some goals of the state, allowing us to conclude that welfare policy in a representative democracy always faces ideological controversies of politicians and citizens. A further aim of this experiment is to shed light on how a political consensus is reached and whether it reflects a criterion of tax policy that results in the least burden to the citizens. To address this issue, as already stated, we focus our analysis on two visionary politicians. For the purpose of the experiment, we assume that these politicians are granted a political mandate to initiate proposals ensuring that the relief payments are allocated to citizens who are in need. We thus assume that, in balancing the books accounting for finance of relief payments and for vital public goods and services, expenses are constrained. This premise ensures that the citizens control the negotiations, forcing the politicians to act within the imposed budget constraints in order to pledge safe funding for their proposals. While trying to reduce the after-tax income inequality, the politicians in their respective roles of left- and right-wing actors are committed to ensuring that the wealth is redistributed fairly.

At this point, it is essential to state the assumptions/limitations underpinning the analysis of a hypothetical behavior of those occupying three distinct roles in the negotiations — those of left- and right-wing politicians and voters-citizens. Throughout this work, we emphasize the incomparability between the aims of the left-wing politicians struggling to ensure adequate access to basic goods and the right-wing politicians advocating for availability of non-primary but vital goods and services. In the analysis, we implicitly assume that politicians do not have adequate knowledge of citizens' needs in a more primitive environment. Hence, they can only work with the monetary payoff specification. Given this limitation, politicians are unaware that the provision of equivalently valued public services is not a perfect substitute. For example, we assume that politicians do not have any information on how household income is assembled and used to buy private health insurance or services of nursing housing, etc. Thus, we do not merit the debate on what is right or wrong in the economic or political environment involving left- and right-wing politicians and voters-citizens. In short, our work does not extend to the democratic context of voters' prototypes/characteristics. While acknowledging the significance of prototypes, in this work, we view voters' behavior as a binary process, allowing support for either left- or right wing politicians. This, however, introduces a risk $q > 0$ of premature political breakdown of negotiations. In addition, we refer to the tax revenue in accord with voters' preferences as the "wealth-pie" $\tau \cdot W$, which is divided into two parts (x, y), whereby x denotes various social benefits or relief payments, and y pertains to public goods, so that $x + y = 1$. We posit that any further enrichment of voters' characteristics would disrupt the delicate balance between the motives of our experiment and the theoretical framework, which is already technically sophisticated.

Roadmap. Because of the narrative complexity, it is possible that the reader would find proceeding with the content of the paper in chronological order difficult. Thus, to mitigate this potential issue, Section 3 presents the most relevant problems, in particular, the pre-equity condition of political breakdown of the negotiations. In our view, it is prudent to master the material presented in Section 3.1 before moving to Section 4. Similarly, Section 3.2 aims to assist with understanding of the content of Section 5, while Section 3.4 supports Section 6. On the other hand, those not wishing to delve deeply into the technical aspects of this work could simply move onto Section 7. Nonetheless, Section 3.3 provides a scheme pertaining to the pre-equity of breakdown of the negotiations and, in our view, does not require further clarification.

2. PRELIMINARIES

Before delving deeper into our work, we specify the category of the game payoffs functions $u(\xi, x)$, $g(\xi, y)$ and taxes $\tau(\sigma, x)$ required for the model validity. As noted above, Section 3 provides background information that assists in understanding material given in Section 4-6. In Section 4, we disclose fiscally safe welfare policy in amalgamation with imposed *budget constraints* for financing relief payments. Referred to as volatility constraint, the amalgamation dynamically restricts the h-effect — an inverse working incentives phenomenon of low-income citizens. In Section 5, citizens' ambivalence and multifaceted welfare policy perceptions are discussed from the perspective of the alternating-offers game. The policy on poverty associates the left- and right-wing politicians with payoffs functions $u(\xi, x)$ and $g(\xi, y)$. Under these conditions, it is possible to obtain an analytical solution to the game with incomes σ density distribution $P(\sigma, \xi)$. Indeed, as will be shown, the calculus of indicators $(\alpha, 1-\alpha)$ complies with the political power design given in Section 6. The results are discussed in Section 7, followed by concluding remarks, presented in Section 8.

In the current experiment, an income σ equal to the poverty line ξ, $\xi \in [\xi_1, \xi_2]$ parameterizes all arguments and functions. In this vein, we adopt quantitative measurement, whereby we utilize a scale quantum as an average income with the income σ density $P(\sigma, \xi)$ distribution, $0 \leq \sigma < \infty$. The average establishes the ratio scale. Hence, we suggest that $u(\xi, x) = (1 - \tau(\xi, x) \cdot (\xi - \phi) + \phi$ (the after-tax residue of income $\sigma = \xi$) signifies the 1st actor's social position at the specified scale, i.e., the left-wing political aims. We apply the residue formula based on Malcomson's (1986) model, with a personal allowance parameter ϕ, $0 < \phi < \xi$, determined by the tax bracket $[\phi, \infty)$. The 2nd actor's aim — the right-wing political objective $g(\xi, y)$ — is ensuring sufficient amount of the non-basic goods per capita. Here, we refer to the citizen $\sigma = \xi$ as *marginal citizen*. While, for the minority of voters, the relief is more attractive than lower taxes, the 3rd actor is the implicit partaker embodying the majority of voters whose preference is minimizing tax obligation $\tau(\sigma, x)$. This is a typical public finance dilemma of efficient division (x, y) of the tax-revenue into shares $x + y = 1$. In this work, the dilemma is represented by the alternating-offers bargaining game $\Gamma(q)$ with premature risk q, $0 < q << 1$, of political breakdown. When $q \rightarrow 0$, the solution converges into Nash axiomatic approach (1950). The relationship between the one that suggests the alternating-offers bargaining and axiomatic solution is well known from the work of Osborn and Rubinstein (1990). As this game is thoroughly described by Osborn and Rubinstein, for brevity, no further elaboration is offered here.

When negotiating on finance issues, under the guise of a "*wealth-pie workshop,*" politicians will allegedly try to divide the wealth-pie in a rational and efficient manner. As a result, the tax $\tau(\sigma, x)$ will increase as will the wealth-pie, when increasing the poverty line ξ. Logically, a decrease in taxes would yield the reverse effect. While taxes vary, the division will depend upon the characteristics and expectations of the bargainers involved. Indeed, the left- and right-wing political aims $u(\xi, x)$ pertaining to basic goods, as well as the objective $g(\xi, y)$ related to the non-basic goods, are controversial. We illustrate this tax controversy by elevated single-peaked frontier of $u(\xi, x)$, the $\frac{2}{5}$ -share/slice in Figure 2, which corresponds to the lower, but progressively increasing, concave frontier of $g(\xi, y)$, the $\frac{3}{5}$ -share/slice in Figure 3, as well as for another division of the pie, into shares/slices $(x = \frac{1}{8}, y = \frac{7}{8})$. We believe, that, while $(x = \frac{2}{5}, y = \frac{3}{5})$ highlights the left-wing political aspirations, the share/slice $\left({}^{1}\!/_{8},{}^{7}\!/_{8}\right)$ elucidates those of the right-wing political objective. This premise appears to be crucial for understanding our primary goal in resolving the welfare policy dilemma.

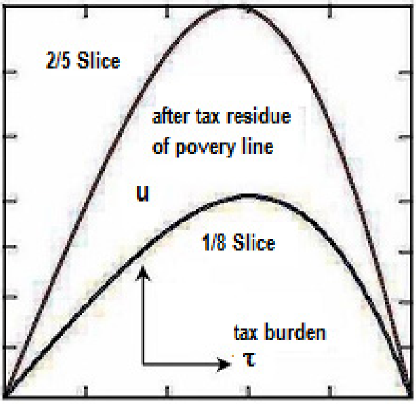


Figure 2. Left-wing politicians' emphases.

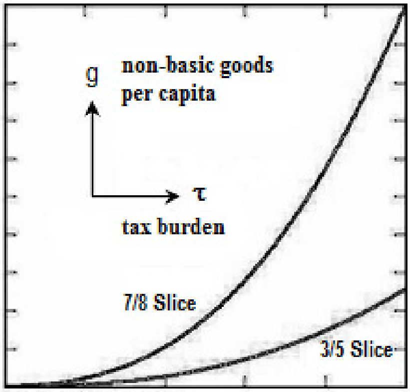


Figure 3. Right-wing politicians' emphases.

In support of the aforementioned assumption, the political payoffs in general, as shown in Figure 2 and Figure 3, emerge within a two-man economy endowed by citizens' income abilities marginalized at the level of poverty line. According to Black (1948), single peakedness plays the key role in collective decision making when the decision is reached by voting. The payoffs for the two actors, shaped in this way, are non-conforming/incomparable, and are thus impossible to match through a monotone transformation, as established by Narens and Luce (1983). The single peakedness is nonetheless in line with Malcomson's tax residue $u(\xi, x)$, when the terms of contract commit the actors to shares (x, y). This, however, requires that the expenses covered by flat taxes will balance the books, while accounting for relief payments, as shown in Figure 2. Clearly, increasing the poverty line requires an excessive increase in taxes, which in turn provides a greater amount of non-basic goods $g(\xi, y)$, as shown in Figure 3. An opposite scenario of increasing the available amount of non-basic goods $g(\xi, y)$ equally requires an excessive tax increase, which may lead to the possibility of increasing poverty line.

Following the traditional procedure for division of the wealth-pie in the alternating-offers game, when the pie is desirable at all the times, the politicians (bargainers) — changing roles — commit to shares (x, y), $x + y = 1$. According to the shares (x, y), the valid rules and norms of wealth redistribution, which guarantee a desirable level of relief payments, require establishing a poverty line ξ parameter. However, an efficient division of the wealth-pie — as a result of single-peaked $\cap$-curves depicted in Figure 2 — no longer represents any traditional bargaining procedure. This is the case as, instead of division, the procedure can be resettled. Indeed, we can proceed at distinct levels of one parameter — within the poverty line interval $[\xi_1, \xi_2]$ — reflecting the scope of negotiations. In fact, Cardona and Ponsattí (2007), also noted that "*the bargaining problem is not radically different from negotiations to split a private surplus,*" when all the parties in the bargaining process have the same, conforming expectations. This argument applies even when the expectations of the first player are principally non-conforming, i.e., single-peaked, rather than excessively concave in regard to the second player. In our experiment, the scope of negotiations on the "contract curve" of non-conforming expectations allows for omitting the "Pareto efficiency" and replacing the axiom by "well defined bargaining problem," as posited by Roth (1977). The well-defined problem (x, y) of the wealth-pie division can now be solved (resettled) inside the poverty line interval $[\xi_1, \xi_2]$.

Settings. In accordance with Friedman's NIT system, in this work, we assume that, for the unfair subsistence of the less fortunate citizen $\sigma < \xi$, the relief amount $r \cdot (\xi - \sigma)$, $0 < r \leq 1$, serves as a monetary compensation designated for purchasing an eligible "poverty basket" of food, clothing, shelter, fuel, etc. According to Rawls, "*primary goods are things which it is supposed a rational man wants whatever he wants.*" In defining the parameter ξ in this manner, it becomes contingent on financing the relief. This can be achieved by assuming that elevating the poverty line ξ requires an increased marginal tax rate $\tau(\sigma, x)$. In increasing the wealth-pie through tax channels, we assume an acceleration $\tau''_\sigma(\sigma, x) > 0$ of the tax rate $\tau(\sigma, x)$; $\tau'_\sigma(\sigma, x) > 0$ inclusive all of those citizens who indicate the marginal income ξ denoted by $\sigma = \xi$.

As noted previously, the marginal citizen $\sigma = \xi$ must bear the cost of the left-wing political aims using tax residue $u(\xi, x)$, as well as the right-wing political objective $g(\xi, x)$, referred to as "public or non-basic goods." With the proviso that politicians commit to the shares (x, y), we conclude that $u(\xi, x)$ is a single $\cap$-peaked curve, due to the tax rate $\tau(\xi, x)$ increase upon ξ. While objective $g(\xi, x)$ of right-wing politicians decreases with an increase in x, the reverse is true with elevating ξ due to $\tau(\xi, x)$ acceleration. Here, payoffs $\langle u, g \rangle$ are considered analytic functions $u(\xi, x)$, $g(\xi, x)$. Given the interval $[\xi_1 \leq \xi \leq \xi_2]$, referred to as the scope of negotiations, $u(\xi, x)$ reflects single $\cap$-peakedness — $u''_\xi < 0$ upon ξ increase, $u'_\xi(\xi_1, x) > 0$, $u'_\xi(\xi_2, x) < 0$. Following an increase in x, the payoffs $u(\xi, x)$ become convex, $u''_x > 0$, $u'_x > 0$, whereas an increase in ξ would produce concave payoffs $g(\xi, x)$, with $g'_\xi > 0$, $g''_\xi > 0$. It can be shown that, with increasing x, payoffs g always decrease; in other words, in both circumstances, either $g''_x > 0$ is convex, or $g''_x < 0$ is concave.

3. RELEVANT TRENDS AND ISSUES

In the extant literature (Espring-Andersen, 1990; Iversen, 2005; Swank, 2002) the welfare, economic, and political issues are usually addressed in reference to specific questions. In our view, a much deeper analysis is achieved when addressing them more generally, adopting well-established knowledge discovery methodologies. In particular, our wealth-pie workshop concept, jointly adopting four issues — (a) public finance, (b) alternating-offers game, (c) negotiations' collapse analysis, and (d) political power design — leads to a more informative point of departure.

To explain the root cause of the results in order to bring the welfare, economic, and political content to the surface in a rigorous analytical form, and to find bilaterally acceptable solutions to the game, we will visit all of the classrooms in our workshop. Our goal is to lay the foundation for a more constructive welfare policy comprehending the meaning of following four narratives:

Fiscal policy	During the delivery to its final destinations, provided that the books accounting for the relief payments finance have been balanced *a priori*, the wealth-pie must remain balanced throughout and in spite of volatility in the economy;
Negotiations	The left- and right-wing political bargaining on how to share the wealth-pie complies with the rules and norms of the alternating-offers bargaining game;
Pre-equity of breakdown	Political breakdown, or threat point, directly affects the bargaining solution. Pre-equity guarantees equal conditions for players before the bargaining game commences;
Political power design	Bringing a motion to a vote is necessary to address the majority opposition to high taxes and excessive public spending. Whether it is viewed as positive or negative, or whether it ought to be acknowledged or not, rejected or accepted, this motion must be politically designed in advance.

In our wealth-pie workshop, these four narratives can be understood as obligations/constraints to be met by welfare policy rules and norms, akin to "Rational man" deliberation of Rubinstein (1998). This interpretation allows us to provide a scenario under which the narratives are embedded into the welfare policy of the state. In addition, evaluating the welfare policy from this perspective reveals that the analysis can be subject to and performed by computer simulations, as shown in Appendix A2. Our initiative could also serve to unify the theoretical structure of economic analysis of public spending. It can be used to evaluate the political power design of left- and right-wing politicians, or to launch systematic inquiry into impacts of governmental decisions and actions on wealth redistribution.

As the state has the duty to help the less fortunate, our experiment approaches wealth redistribution in a two-fold manner. First, it addresses the provision of basic necessities or goods, such as shelter and heating, clean and fresh water, nutrition, etc., before focusing on non-basic goods, including national defense, public safety and order, roads and highway systems, and so on. Welfare policy issues, according to Boix (1998), "*...There is wide agreement in the literature that governments controlled by conservative or social democrats parties have distinct partisan economic objectives that they would prefer to pursue in the absence of any external constrains.*" Meeting this challenge, based on income σ density distribution $P(\sigma, \xi)$, we identify an effective approach to the division $\left(x^{\circ}, y^{\circ}\right)$ into shares $x^{\circ} + y^{\circ} = 1$ pertaining to basic x° and non-basic goods y°. Fundamentally, the efficient division $\left(x^{\circ}, y^{\circ}\right)$ of the wealth-pie aims at just and fair delivery of all aforementioned goods, traditionally perceived as public goods. In our experiment, we refer to public goods as non-basic but vital goods, whereas basic goods are deemed fundamental. Incidentally, during the delivery of basic and non-basic goods to their end destinations, we treat both as public goods.

We assume that the left-wing politicians have the necessary political influence — when an offer is made, irrespective of its legitimacy — to control the redistribution of basic goods independently. Given the single-peaked aspirations of the left-wing, in contrast to the objective of their right-wing counterparts, the influence the left-wing politicians enjoy, is supposed to be adequate enough to reach the peak of these expectations. In particular, we believe that, beyond some peak position, inefficient usage of basic goods would lead to an excessive decline in the quality of welfare services, as well as cause deterioration in access to public goods for all citizens. In making these suppositions, we agree with Rawls's statement, about the precepts of perfect justice: "*The sum of transfers and benefits [...] from essential public goods should be arranged so as to enhance the emphases of the least favored consistent with the required saving and the maintenance of equal liberties.*"

An efficient usage of public resources implies that a consensus between left- and right-wing politicians might be reached. Despite some views to the contrary (Rothstein, 1987), we posit that the bargaining aimed at finding a just and fair division of basic vs. non-basic goods is an acceptable path to the bargaining dynamics. Based on this premise, we can identify relevant connections in extant works on economic and political behavior that analyze the sociological and political aims of ensuring adequate welfare by using public finance. This is likely being the best starting point for visiting our wealth-pie workshop.

3.1. Fiscally safe welfare policies, *to be continued in Section 4*

Public finance focuses on the revenue side of tax policy. In particular, it pertains to the budget formation, as noted by Formby and Medema (1995), aiming to provide a guaranteed level of welfare to citizens endowed by poor productivity. While the welfare policy is a separate issue, it should be considered on the grounds of legal and moral rights of citizens. Empirical evidence confirming that such policy is government's legal obligation can be found in pertinent literature. For example, as noted by Saunders (1997), "*...poverty line. The line was initially set (in 1966) equal to the level of the minimum wage plus family benefits for one-earner couple with two children.*" Similarly, a hypothesis consistent with moral obligations can be found in the literature of economic politics (Eichenberger, 1996; Feld, 2002).

In 1959, Musgrave examined two basic approaches to taxation — the "*benefit approach*" and "*ability-to-pay,*" which put taxation into efficiency and equity context, respectively. In this work, we utilized the benefit approach in order to augment the existing standard of welfare policy, whereby we allocate a guaranteed amount of income for minimum taxes. We posit that a flat tax system — based on injecting optimal equity according to the ability-to-pay principle of "proportional sacrifice" — ensures that taxes remain *fairly levied.*

Taxation is a principal funding source of social costs and benefits. Thus, the first postulate in our welfare policy workshop (see above) discloses an obvious paradigm in social policy. According to the ability-to-pay principle commonly adopted in public finance, in order to stabilize the distortion of tax polices, the known terms of warranty must rely on exogenous taxes enforced on the productivity of citizens. The concept, proposed in 1996 by Berliant and Page Jr., is a variant of the classic public finance and similar approaches, applicable when an agent characterized by a specific level of productivity does not shift his/her labor supply after all adjustments to the tax formula have been implemented. In other words, under this paradigm, optimal taxation enforces optimal labor supply.

Yet another "treatment of policies," closely related to societal instability, entails equity of pre- and post-tax positions of citizens. Such a view demarcates between citizens and has attracted the attention of economists and tax policy makers. In the view of Kesselman and Garfinkel (1978), credit tax-scheme analysis opposes the income-tested program in the rich-and-the-poor, also known as two-man economy. Poverty measurements have also been addressed in the works of Sen (1976), Atkinson, (1987), Ebert (2009), and Hunter (2002) et al. According to Tarp (2002) et al: "*The poverty line acts as a threshold with* *households falling below the poverty line considered poor and those above poverty line considered nonpoor.*" García-Peñalosa (2008) investigated wealth redistribution as a form of social insurance in relation to economic growth. On the other hand, Stewart et al (2009) attempted to reduce horizontal inequalities, proposing "*a reallocation in the production, operation and consumption of publicly funded services.*"

In the attempt to assess and control the circulation of wealth through tax channels, we argue that, unless fiscal stabilization is not a required condition when justifying public spending, it will be difficult to explain how the citizens eligible for relief gain access to the benefits and relief payments. Thus, while we continue to rely on fiscal stabilization, in order to highlight a particular type of the dynamics stability, we refer to welfare policy as idempotent. For clarity, a choice operation (or decision) applied multiple times is deemed idempotent if, beyond the initial application, it yields the same result (Malishevski, 1998). Thus, based on this dynamic definition, idempotent scheme allows the politicians to honor the pledges made during the election campaign as, once the political decision is taken, it eliminates the need for further stabilization. While visiting the workshop, the circulation of wealth is supposed to be dynamically stable, i.e., it is idempotent.

3.2. **Bargaining the Welfare State rules and norms***,*

to be continued in Section 5

Bargaining is the key element of economics and is at the core of politics. On the other hand, as pointed out by North (2005), "*The interface between economics and politics is still in a primitive state in our theories but its development is essential if we are to implement policies consistent with intentions.*" More recently, Feldstein (2008) noted, "*Unfortunately, there is no reason to be pleased about the analysis in policy discussions of the efficiency effects…of the welfare consequences of proposed tax changes.*" Similarly, in a review on "Handbook of New Institutional Economics," Richter (2006) stressed, "*…that the sociological analysis…and large institutional structures in economic life is still at an early stage…game theory, and computer simulation could help to further develop the new institutional approach…game theory might be a defendable heuristic device of NIE.*" Indeed, the left- and right-wing politicians, like actors in the game, strive to implement their vision of the state welfare institutions. This is succinctly explained by Ostrom (2005), who noted, "*These flimsy structures, however, are used by individuals to allocate resource flows to participants according to rules that have been devised in tough constitutional and collective-choice bargaining situations over time.*"

In order to achieve the aforementioned vision of collective choice, it is appropriate to consider a scenario in which the actors/voters play the "bargaining drama" of economic and political issues. Bargaining has been a theme of a wide range of publications, including the work of Alvin E. Roth (1985). Despite the simplification, the binary behavior of voters remains at the root of the democratic transformation of public institutions. In this regard, binary position fits particularly well into the bargaining game with exogenous risk q, $0 < q << 1$, of breakdown (Osborn and Rubinstein). Actually, bargaining can be risky for all interested actors because they may lose voters to the competition if their terms are not met. Thus, it is essential to first clarify political power dynamics of both the left-wing and the right-wing politicians. Henceforth, they are respectively referred to as LWP, the 1st actor, benefiting from a power α, $0 < \alpha < 1$, and RWP, the 2nd actor, benefiting from a power $1 - \alpha$.

Numerous factors — such as economic growth, decline or stagnation, demographic shift or pit, political change or change in scarcity of resources, skills and education of the labor force, etc., — might create fiscal imbalance in a desirable welfare policy due to the transfers of benefits and relief payments. As a result, the size of the wealth-pie might be too small (i.e., not worth the effort required for its redistribution), or too large (introducing mutual traps) to achieve a stabilized public spending mechanism. In either case, the actors may decide not to share the pie at all. To address this controversy, as previously underlined, we assume that politicians participate in relevant public institutions. If the institutions cannot or do not want to follow RWP's policy of wealth redistribution, RWP — in order to promote their own understanding — can be sufficiently legitimate to deliver the wealth "properly." In doing so, RWP can enforce vital decisions by several means, including resource mobilization, retaliation for breaches and criminal fraud, recruiting political volunteers and managing public service commissions, soliciting private contributions, etc. In other words, as Kalai pointed out, RWP would rely on an "*enthusiastic supporter.*" On the other hand, as LWP face decay in political legitimacy for perfect justice, they cannot fully control RWP's actions and intentions when their political interests in the final agreement are incomparable. In these circumstances, RWP are aware that their abilities and access to information might necessitate agreeing with, or at least not resisting, LWP's privileges to make arrangements upon the size of the pie. Hence, from the RWP's critical point of view, whether acting politically in common interest or not, it might be prudent to acknowledge LWP's welfare activities. This elucidates the asymmetric dynamics of political power division between the LWP and RWP.

Returning to the main points of asymmetric bargaining, we will illustrate an efficient solution $\left(x^{\circ}, y^{\circ}\right)$ by division of \$1 aimed at maximizing the product of actors' payoffs above the disagreement point $d = \left\langle d_1, d_2 \right\rangle$:

$$\begin{aligned}\left(x^{\circ}, y^{\circ}\right) &= \arg\max_{0 \le x+y \le 1} f(x, y, \alpha) = \\ &= \left(u(x) - d_1\right)^{\alpha} \cdot \left(g(y) - d_2\right)^{1-\alpha}\end{aligned} .$$

Although game theory purists might find the solution clear, the questions asked by many often include: *What are* x*,* y*,* α*,* $u(x)$*, and* $g(y)$*? What does the point* $\langle d_1, d_2 \rangle$ *mean, and how is the* $arg\,max$ *formula used?* The simple answer, as initially provided by Kalai as an asymmetric variant of Nash problem, is as follows:

- x is the 1st actor's share of \$1, with α as the 1st actor's asymmetric power indicator, $0 \le x \le 1$, $0 \le \alpha \le 1$;
- $u(x)$ denotes the 1st actor's payoffs of the 1st actor's \$1 share x;
- y is the 2nd actor's share of \$1, where $1-\alpha$ is the 2nd actor's asymmetric power indicator, $0 \le y \le 1$;
- $g(y)$ denotes the 2nd actor's payoffs of the 2nd actor's \$1 share y.

Based on the widely accepted nomenclature, we refer to $s = \langle u(x), g(y) \rangle$ as to the utility or payoffs pair. Thus, the disagreement/threat point $d = \langle d_1, d_2 \rangle$ represents the payoffs the two actors obtain if they cannot agree on how to share the wealth-pie. In the same vein, $d = \langle d_1, d_2 \rangle = \langle 0, 0 \rangle$ represents the disagreement or breakdown point, whereby the players collect nothing.

In the subsequent sections, we will provide an analytical solution exploiting payoffs in the form $\langle u(\xi), g(\xi) \rangle$ and taxes in the form $\tau(\xi)$ within the scope of negotiations $[\xi_1, \xi_2]$ comprising the endpoints of the interval $[\xi_1, \xi_2]$. According to the analytical solution, implicitly hiding the variables x, y, it follows that any negotiation of shares (x, y) can be perceived as two sides of the same bargain's portfolio, as the shares (x, y) are accompanied by poverty lines $\xi \in [\xi_1, \xi_2]$. While hiding the variables x, y, $x + y = 1$, we may respond to the question of whether solution $\xi^\circ \in [\xi_1, \xi_2]$ is efficient in a traditional sense. Indeed, akin to the above, political bargaining can now be expressed by poverty line ξ° maximizing the product of political payoffs above the threat point $d = \langle d_1 = u(\xi_1), d_2 = g(\xi_2) \rangle$:

$$\xi^\circ = \mathrm{argmax}_{\xi \in [\xi_1, \xi_2]}\, f(\xi, \alpha) = (u(\xi) - d_1)^{\alpha} \cdot (g(\xi) - d_2)^{1-\alpha}.$$

On the other hand, unlike the traditional threat point $d = (d_1, d_2)$, the public/vital goods amount d_2 in the game — the d_2 component of the point d — might be negative. This will apply in our experiment of a breakdown of negotiations, whereby funds need to be borrowed or acquired through other means in order to balance the books and account for the welfare expenses — a situation of "genuine negative taxes." It is important to note that, while this may seem counterintuitive to some readers, in the theory of public finance, the use of genuine negative taxes is not prohibited.

Finally, we conclude that, all these remarks notwithstanding, it is irrelevant whether the players are bargaining on shares (x, y) or trying to agree on the poverty line level. This assertion highlights the main advantage of hiding the variables x, y. In particular, it brings about a number of different patterns of outcome interpretations in the game, such as linking an outcome to the lowest tax rate, which is the most desirable sacrifice of voters' majority. In consideration of alternative approaches — which describe outcomes of collective bargaining in the form of voting, or partaking in any voting scheme in the form of bargaining — the scope of negotiations $[\xi_1, \xi_2]$ brings the voting and bargaining schemes into the same context, as both can be enriched by adopting this approach. Our insight is forward-looking in the sense that it aims to identify an alternative-offers game solution, whereby both actors accept at once the proposals (moves) made by the other side. Our initiative could also serve to unify the theoretical structure of economic analysis of productivity problem. Indeed, when referring to Leibenstein's work (1979), Altman (2006) noticed:

> *Leibenstein (1979, p.493) argued that there are two components to the productivity problem: one relates to the determination of the size of the pie, while the second relates to the division of the pie. Looked upon independently, all agents can jointly gain by increasing the pie size..."the situation need not be a zero-sum game. Tactics that determine pie division can affect the size of the pie. It is this latter possibility that is especially significant.*

3.3. Pre-equity of political breakdown

Beyond the asymmetric dynamics, the game inherits a premature disagreement or breakdown point, similar to that discussed by Osborn and Rubinstein:

> *We can interpret a breakdown as the result of the intervention of a third party, which exploits the mutual gains. A breakdown can be interpreted also as the event that a threat made by one of the parties to halt the negotiations is actually realized. This possibility is especially relevant when a bargainer is a team (e.g., government), the leaders of which may find them unavoidably trapped by their own threats.*

In our game, the asymmetric solution incorporates the left- and right-wing political power indicators $(\alpha, 1-\alpha)$ into a breakdown policy. In order to be addressed properly, the indicators cannot be given exogenously. To overcome this obstacle, we introduce a policy of endogenously extracted breakdown $d = \langle d_1, d_2 \rangle$ into the game, based on a condition referred to as the *pre-equity of political breakdown.*

Traditionally, in the alternating-offers game, the breakdown corresponds to two standard pairs of payoffs $\{\langle 1,0 \rangle, \langle 0,1 \rangle\}$, or in the words of Osborn and Rubinstein, *"to the worst outcome."* In the left- and right- political bargaining, due to the implicit pressure from the voters, as both politicians aim to find — at least from their perspective — a just and fair solution, there will always be a temptation for binary voters to defect to the other side. This puts the negotiations at risk $0 < q << 1$ of a premature collapse. Even under the worst circumstances, in the event of collapse, the quality and the size of the wealth-pie should be equal for both politicians. This premise holds in these unfavorable circumstances, as the entire pie will be decided upon by one of the politicians. Thus, when the premature collapse occurs, it is important to arrange the terms of contract in such a way that neither politician can exploit or misuse these adverse circumstances to his/her own advantage. To meet this condition, when normalizing the standard breakdown under the description valid for the alternating-offers game $\Gamma(q)$, we are working toward an endogenous form for equity in accordance with political non-conforming expectations.

As stated, the standard case of breakdown in the alternating-offers game corresponds to two pairs $\{\langle 1,0 \rangle, \langle 0,1 \rangle\}$ of payoffs. In this form, the breakdown is generally found using ex-ante linear transformation, namely the exogenous normalization of utilities. When the collapse is imminent, the political breakdown exposes equity condition pertaining to the actual event of breakdown. Unlike the standard case, once the most unfavorable result occurs, the resulting collapse must include additional parameters — the tax τ and the wealth W. In order to equalize — endogenously normalize — the breakdown, the politicians involved in negotiations can make *a priori* arrangements, or sign binding agreements upon these two parameters, i.e., τ and W. Without availability or warranty of such a pre-equity, an endogenous normalization is unrealistic. In the view of the voters' electoral maneuvering (discussed in the next subsection), even if the *pre-equity normalization* is not always achievable, it is more constructive to determine the breakdown according to some rational context.

Before proceeding further with a detailed assessment of the aforementioned definition, we recall the concept of wealth amount W, redistributed by the state as the average taxable income per capita, scholarly defined as "prosperity or a commodity." Next, according to the conditions characterizing the collapsed environment, at the start of the negotiations, the draft of a contract includes both taxes τ and — in line with our nomenclature — the wealth amount W. The product $\tau(\xi)\cdot W(\xi)$ identifies the size z of the wealth-pie within an interval $[\xi_1, \xi_2]$ within the scope of negotiations, thus establishing the boundary for the two politicians. The lower limit ξ_1 denotes the initial proposal, which is the most attractive for RWP, while being the most unattractive for LWP. In the same but inverse order $u_2 = u(\xi_2)$ can be paired with $g_2 = g(\xi_2)$. Having set these limits, we can proceed with examining how the breakdown $\{\langle u_1, g_1\rangle, \langle u_2, g_2\rangle\}$ might be conditionally, albeit endogenously, encoded into the game.

Indeed, we now contribute to implementing our wealth definition of how the breakdown can be established endogenously. To do so, we consider a situation driving the welfare policy in the context of cost-benefit equity. When the collapse of negotiations is imminent, the differences in the amounts of wealth and taxes for funding low-cost welfare policy ξ_1 against an expensive policy ξ_2, $\xi_1 < \xi_2$ — i.e., funding payoffs $\langle u_1, g_1\rangle$ for ξ_1 against $\langle u_2, g_2\rangle$ for ξ_2, $u_1 < u_2$, $g_1 > g_2$ — can amplify misunderstandings and contribute to traps. At the endpoints of the scope $[\xi_1, \xi_2]$, the wealth-pie sizes $z(\xi_1)$ and $z(\xi_2)$ at poverty lines ξ_1 and ξ_2 can require the delivery of wealth amounts $W(\xi_1)$ and $W(\xi_2)$, albeit at different prices, represented as taxes $\tau(\xi_1)$ and $\tau(\xi_2)$, Buchanan (1967). Hence, prior to the start of the game, and in line with the cost-benefit equity, in the most adverse circumstances, the payoffs $s_1 = \langle u_1, g_1\rangle$ and $s_2 = \langle u_2, g_2\rangle$ should preserve equal prices τ for the delivery of equal amounts W of wealth. Such a market-driven interpretation of *commodities delivery to the end destinations* relies heavily on the size of the wealth-pie, which is equal to $\tau\cdot W$. It should be noted that this interpretation is only relevant to the case of flat (proportional) taxes.

To explicate the interpretation of reasoning in previous lines, it is worth examining the "well defined bargaining problem," depicted as the contract curve in Figure 4. Based on the discussion presented thus far, our goal is to set an interval $[\xi_1, \xi_2]$ solving two non-linear equations, $\tau(\xi_1) = \tau(\xi_2)$ and $W(\xi_1) = W(\xi_2)$, by attempting to find a cross-point (τ^*, W^*) where the curve crosses its own contour, as *YX*-axis coordinates, on the plane with (τ, W), which is equivalent to the roots ξ_1^* and ξ_2^*. Although the calculus of the point $\left(\tau^*, W^*\right)$ does not extend beyond high school mathematics, it does not confirm the possibility of normalization in general. This, however, does not invalidate our discussion, as we do not claim that the equity condition can be achieved in all circumstances. It should still be pointed out that, in a number of examples where the validity of the condition was detected, we found a breakdown endogenously encoded into the game, indicating normalization in the form of

$$\left\{\left\langle u_1^*, g_1^*\right\rangle, \left\langle u_2^*, g_2^*\right\rangle\right\} = \left\{\left\langle u(\xi_1^*), g(\xi_1^*)\right\rangle, \left\langle u(\xi_2^*), g(\xi_2^*)\right\rangle\right\}.$$

The Swing of the Contract Curve within $\left[\xi_1, \xi_2\right]$

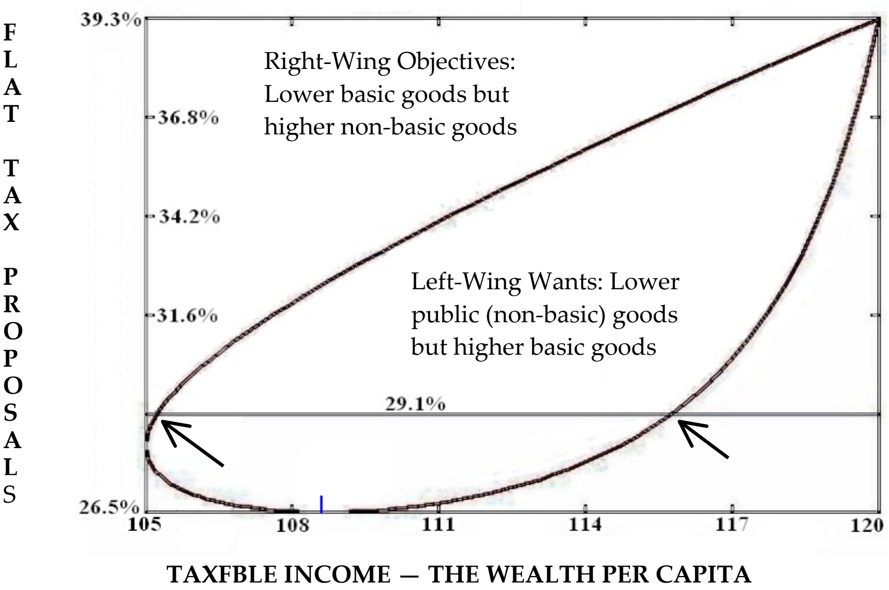


Figure 4. The graph depicts two different motions for a vote. For the higher tax $\tau = 29.1\%$, marked by the horizontal line, and the lowest tax $\tau = 26.52\%$, marked by the vertical dash. Indicated by $\rightarrow$, at cross-points of the contract curve with the horizontal line, we observe controversial expectations of voters. The shares of lower basic but higher public goods are shown on the left, while this payoff reverses towards the right side of the graph, as the shares of basic goods increase while those of public goods decrease. Thus, the higher tax $\tau = 29.1\%$ cannot lead to a political consent, in line with Observation 5.

In line with the above, as the aim is to bring the politicians, if possible, into just and equal positions prior to negotiations, equalizing taxes τ and wealth amounts W in the collapsed environments ξ_1 and ξ_2 might be a rational starting point. Under this premise, endogenously encoded into the game, we label the equity condition, as a *pre-equity of political breakdown.*: $\left[\tau(\xi_1) = \tau(\xi_2), W(\xi_1) = W(\xi_2)\right]$. If valid, this condition equalizes fiscally realistic and just demands for public spending prior to negotiations — in particular, the size of the wealth-pie $z(\xi_1) = z(\xi_2)$.

3.4. **Voting and political power design***, to be continued in Section 6*

Only the voting results can reveal the true incentives of people that will give the democracy its final judgment. The voting process is the only avenue for the voters to assume the roles of current or upcoming politicians to whom the opportunity will be granted in line with population's aspirations to redesign the rules and norms of wealth redistribution. Voters' inequalities, life plans, background, social class and experience, native endowments, political capital, etc., determine the bulletin collected at the voting table. Consequently, incongruence in voters' views or interpretations of reality affects the individual choices and thus the voting results, thereby influencing political pre-election campaign. Voting results are not fully predictable due to the deviations in voters' views and opinions on how the wealth redistribution ought to be achieved. The problem stems from the fact that welfare policy proposals that benefit minority of citizens sometimes require higher taxes. On the other hand, majority of voters would be primarily guided by selfish attitudes toward lower taxes, which would implicitly affect the political bargaining positions. Such an attitude likely deserves a critical examination. Given these arguments, our question is — Why should the left- and right-wing politicians care about lower taxes?

It is timely to recall political outmaneuvering with an implicit risk q, $0 < q << 1$, upon negotiations suffering a premature collapse. Indeed, Figure 5 depicts the contract curve of efficient public policies/proposals ξ upon poverty lines in the bargaining game $\Gamma(q)$. Politically rational and economically effective proposals ξ, forming the curve, have been projected onto the two-dimensional space of the tax rate $\tau(\xi)$ and taxable income — the wealth amount $W(\xi)$. Although the payoffs $\langle u(\xi), g(\xi) \rangle$ are embedded in each point, they are not visible on the graph. These invisible/hidden payoffs in the upper part of the graph symbolize wealth-pie division (x, y) into lower basic $x(\xi)$, yet higher of public goods shares $y(\xi)$, as left-wing politicians aim for $u(\xi)$, whereas those in the right-wing political party aspire towards $g(\xi)$ accordingly. Similarly, the payoffs in the lower part symbolize a reverse situation — the higher basic, vs. lower public goods, as shown in Figure 4. Thus, once all views are represented, the political payoffs $\langle u(\xi), g(\xi) \rangle$ for pledged tax hikes $\tau(\xi)$ are more favorable for some coalitions of voters compared to others. As voters' preferences for the balance between basic and public goods vary, the approach to determining efficient poverty line resulting from eventual agreement between politicians is two-fold. Indeed, unless the tax hikes are excessively high, the *upper coalitions*' representatives will always try to outmaneuver the *lower coalitions*' representatives. The politicians are aware of this dynamic when taxes are high. As they feel trapped in negotiations, binary voters become more likely to defect to the other side, putting the negotiations at risk $q > 0$ of a premature collapse. In contrast, when taxes are sufficiently low, the range of eventual voters' electoral maneuvering will substantially reduce or even vanish. The lowest tax is likely the one that yields desirable outcomes for the majority of citizens.

In line of reasoning that concerns the majority of citizens, it is appropriate to address of the design of the political power indicators $(\alpha, 1-\alpha)$. Considering the bargaining outmaneuvering of left- and right-wing politicians according to the alternate-offers game $\Gamma(q)$, we state that the politicians on the opposite sides of the bargaining table might disagree with respect to the terms of outcomes. Consequently, they would delay the decision while consolidating a draft of a consensus document. This document might not necessarily yield the best outcome for the citizens, who represent the majority, and are of view that the policy that minimizes taxes is always the most desirable choice. Despite knowing that the majority will never endorse higher taxes, the minimum tax rate might not necessarily be a desirable outcome from the political perspective. Thus, politicians may choose to disregard the majority interests because political power of LWP or RWP, as rational actors/politicians, might be strong enough to negotiate selfish decisions that are beneficial only for them. In order to entice politicians not to act selfishly, as this would likely result in ultimate collapse in the negotiation process, their political power indicators $(\alpha, 1-\alpha)$ ought to represent a *natural power consensus* motivating them to choose a desirable outcome for themselves and for the majority of citizens — a platform that should ideally be designed in advance. This completed our preliminary investigation of the problem.

4. ANALYSIS OF FISCALLY SAFE WELFARE POLICIES,
continued from Section 3.1

Delivery of basic goods, which counteracts negative contingency, if it occurs, is the main political responsibility of the left-wing actors. Herewith, the left-wing political intervention is of the greatest political importance. It is universal in the sense that it pertains to all citizens, irrespective of individual situation before or after the contingency. Under this premise, basic goods that are available to citizens are of sufficiently high quality and poverty is not allowed, as stressed by Greve (2008). This course provides a relatively high level of welfare spending and taxes, creating misbalance in the books accounting for public finances, thereby introducing volatility conditions into the wealth-pie delivery. Hence, secured largely independently of market forces, the high level of basic goods might have a conflict-driven effect on the welfare policy, which should not be borne solely by citizens as, as already noted, the state has a duty to help the disadvantaged.

Assuming that the conflict-driven welfare policy guides our political actors in trying to reach an agreement, the left-wing politicians should aim to secure an efficient size of the wealth-pie. Thus, LWP prescribe the size of the pie and propose the division method, which the right-wing politicians accept or reject. If rejected, the RWP would suggest their preferred division, while only having the authority to recommend a size that the LWP might not be obligated to accept. We also assumed that, upon delivery to its end destinations, the wealth-pie remains fiscally safe, i.e., it does not change its size. Under the rules of the alternating-offers procedure (see later), the game will continue until a consensus is reached. This process presupposes that left-wing politicians are committed to the share of the pie, while not being committed to the size.

Let us now envisage a contrasting scenario, whereby the public spending increases. Hence, both actors know that, upon delivery, the size of the wealth-pie might change. This, in turn, leads to a misbalance between the relief payments, which can put the pie size in doubt or make it even more difficult to ascertain. As a result, the difficulty related to political pledges might force both sides to retreat. In such volatile conditions, the wealth-pie is no longer fiscally safe and might affect the expectations of both politicians. Consequently, a fiscally safe plan in spite volatile conditions for the delivery and division of the wealth-pie is needed. Otherwise, unless welfare policy fails to enforce fiscal safety, the rules and norms of the relief payments are not living up to their claims. In other words, having a criterion for determining whether a welfare policy is fiscally safe is necessary.

It is helpful to focus first on welfare policy without any warranty of fiscal safety. It could, for example, be determined by the poverty line ξ, identifying the recipients of wealth redistribution. When ξ is low, the variable σ, $0 < \sigma \leq \xi$, allocates the income of the needy or the benefit claimants. In this scenario, the benefit claimant $\sigma < \xi$ claims and receives a relief payment proportional to $\xi - \sigma$, i.e., $r \cdot (\xi - \sigma)$, as previously discussed. In this scenario, all other citizens — both the wealthy and those with marginal income, denoted as $\sigma > \xi$ and $\sigma = \xi$, respectively — receive no relief payment.

Next, we study a specific scheme highlighting the readiness of the society to fund welfare and public spending. For this analysis, we assume that the average cost B of the relief payments and the average taxable income W both depend on the poverty line parameter ξ, $B \equiv B(\xi)$, $W \equiv W(\xi)$ — this is realistic, as shown in Appendix A1. As previously scholarly defined, $W(\xi)$ can refer to the wealth amount. Based on our perception of income σ density $P(\sigma, \xi)$ distribution samples, the product $\tau \cdot W(\xi)$ estimates the average tax revenue. Let the average cost of public goods be $g(\xi)$, whereas the size $z(\xi)$ of the wealth-pie equals $\tau \cdot W(\xi)$, $z(\xi) = \tau \cdot W(\xi)$. We assume that welfare and public spending reached the intended recipients, whereby the total spending equals $\tau \cdot W(\xi) = B(\xi) + g(\xi)$. This suggests that the basic and non-basic goods have been delivered to their final destinations. In other words, the wealth collected through tax channels is spent.

Now, let us assume that politicians in the game preferred to commit to the shares fixing (x, y), and might agree to hold the balance $B(\xi) = x \cdot \tau \cdot W(\xi)$ of the books accounting for financing the relief payments B. That is, the left-wing politicians must be ready to finance the relief, i.e., to deliver $B(\xi)$ by dividing the wealth-pie $\tau \cdot W(\xi)$. In this scenario, the politicians pledge to retain the balance $B(\xi) = x \cdot \tau \cdot W(\xi)$ of the relief payments between credits $B(\xi)$ and debts $x \cdot \tau \cdot W(\xi)$ as a portion x of the wealth-pie $\tau \cdot W(\xi)$. The balance also specifies the welfare policy ξ — an implementation of the poverty line ξ, welfare reform, pact, program, etc. While the aforementioned balance is initially valid, it might not be in the future, putting the adjustment in ξ on the agenda either once or repeatedly. Thus, the policy ξ might represent a problem of fiscal imbalance. Almost all citizens, even if for different reasons, will prefer the opposite in the long run — a fiscally safe policy ξ. For this reason, we now shift the focus on examining a constraint that corresponds to fiscal safety of welfare policy ξ, identifying — what we called above as idempotent — the safe delivery of the wealth-pie to its end destinations.

4.1. Idempotent rules and norms of wealth redistribution

The delivery of basic and public (non-basic) goods does not necessarily safeguard the funding of the expenses. As the expenses neither match nor prevent taxation hikes, the size of the wealth-pie could vary too rapidly. This leads, as previously discussed, to numerous adjustments of welfare policy rules and norms. To mitigate this issue, we have to examine at the sequence $.,\xi',\xi'',.$ of multiple adjustments of the poverty line ξ. This highlights the fact that, on delivery, no adjustments of the wealth-pie are desirable. Consequently, it is better to keep the size of the pie unchanged, i.e., fiscally safe. In other words, when replacing the old policy ξ' with ξ'', the two must coincide. Similar schemes, known as *idempotent*, stem from bounded rationality mechanisms (Rubinstein, 1998; Malishevski, 1998). This premise suggests that, even if welfare policy rules and norms are subject to multiple adjustments, these adjustments should not change the *machinery* of relief payments. In particular, when implemented twice, the rules must produce the same outcome. To guarantee the fiscal safety of the poverty line, such an understanding requires that the poverty lines must coincide amid a sequence of pairs (ξ',ξ'') at some matching policy $(\xi'=\xi'')$.

The need to balance the books accounting for the delivery of relief payments $B(\xi)=x\cdot\tau\cdot W(\xi)$, in spite the wealth-pie volatility, can also be seen as immunity for financing the welfare policy. In particular, the immunity restricts, or at least realistically limits the h-effect of wealth redistribution. Given the *immune,* i.e., *fiscally idempotent*, composition $[B(\xi),W(\xi)]$, the idempotent scheme is equivalent to implementing the policy ξ only once. For this reason, we assume that the rules and norms of the relief payments have been socially planned and redesigned accordingly.

In this idempotent mode that outlines the fiscal safety of public spending, the rules and norms must reflect idempotent policy ξ that brings the spending policy into focus. We conclude that the expenses $x\cdot\tau\cdot W(\xi)$ designated for welfare spending must be in balance not only for funding relief payments $B(\xi)$, when the particular policy ξ takes effect, but the policy ξ must also enforce the fiscal safety in the full spectrum of current and future events.

Clearly, the balance $B(\xi)=x\cdot\tau\cdot W(\xi)$ is a static relationship leading to functional dependency $\tau\equiv\dfrac{B(\xi)}{x\cdot W(\xi)}$ that links the arguments ξ and x. Hereby, the tax rate τ becomes a function of ξ and x, expressed as $\tau\equiv\tau(\xi,x)$. According to rules and norms in force of relief payments, the post-tax residue $\pi(\xi,\tau)=(1-\tau)\cdot(\xi-\phi)+\phi$ of the marginal citizens' $\sigma=\xi$ comprises fiscal limitations of wealth redistribution, while ϕ determines the personal allowance parameter, as shown above. The dependency $\tau\equiv\tau(\xi,x)$ transforms $\pi(\xi,\tau)$ into a fiscally realistic social position $\pi(\xi,\tau(\xi,x))$. Irrespective of the current expenditure on basic goods, the real cost of living does not necessarily match $\pi(\xi,\tau(\xi,x))$. Hence, ensuring realistic and fiscally idempotent rules and norms, and/or, in particular, attempting to avoid the h-effect of this mismatch or adopt rules to keep the effect tolerable at the least, an equation for a fiscally idempotent policy ξ should be solved.

Observation 1. *Constraint on left-wing political aims* $u=\pi(\xi,\tau(\xi,x))$ *is necessary for upholding idempotent fiscal rules and norms of imposed budget constraint* $B(\xi)=x\cdot\tau\cdot W(\xi)$.

According to this observation, whatever tax increase is implemented, the poverty line residue u of the marginal citizens' $\sigma=\xi$ is unfeasibly high and cannot be attained when the condition has been violated.

Corollary. *When* $u=\pi(\xi,\tau(\xi,x))$ *solves for* ξ*, the subsequent adjustments* ξ', ξ'',... *are unnecessary. An option to change their welfare positions is irrational for citizens with incomes* $\sigma<\xi$ *or* $\sigma>\xi$*; thus, the root* ξ *restricts (realistically limits) the h-effect.* All pertinent proofs are given in Appendix A3.

The fiscally idempotent policies ξ induce the basis for solutions in our game as fiscally idempotent compositions $[B(\xi),W(\xi)]$. A reasonable question thus emerges: *Which taxable income* $W(\xi)$ *characterizes fiscally idempotent welfare policies* ξ *for the delivery of relief payments* $B(\xi)$? The answer is provided in the form of the following three constraints: [1]

Delivery constraint by which the wealth-pie is spent — the basic and public goods have been delivered. This form of constraint makes sense only for proportional or flat taxes. Flat taxes will later substantially simplify the method of function minimization with constraints.	$\tau\cdot W(\xi)=$ $=B(\xi)+g$	(1)

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[1] Below, we continue to refer to the average taxable income as "wealth."

Budget constraint imposed on relief payments finance in accordance with the share x of the wealth-pie — the tax-revenue. The left-wing politicians pledge to credit/debit the account $B(\xi)$ that must be equal to the average of relief shifted by the policy ξ.

$$B(\xi) = \\ = x \cdot \tau \cdot W(\xi) \tag{2}$$

Stability constraint that determines fiscally idempotent property of (2). In contrast to $(\sigma, \tau) \in \Re^2$, we distinguish poverty line residues $u = \pi(\xi, \tau)$ as one-dimensional curves $\pi(\xi, \tau) \in \Re \subset \Re^2$.

$$u = (1 - \tau) \cdot \\ \cdot (\xi - \phi) + \phi \tag{3}$$

Taking the expression $\tau(\xi, x) \equiv \dfrac{B(\xi)}{x \cdot W(\xi)}$ out of the constraint (2) and replacing $\dfrac{B(\xi)}{x \cdot W(\xi)}$ into $u = \pi(\xi, \tau(\xi, x))$, the constraint given in (3) can be resolved with a fiscally idempotent policy for ξ, thus yielding:

$$L(\xi, x, u) = (\xi - \phi) \cdot B(\xi) - x \cdot (\xi - u) \cdot W(\xi) = 0\,. \tag{4}$$

Referred to as the volatility constraint, the constraint (4) determines the fiscal safety module. It holds down the h-effect amalgamating the constraints (2) and (3) by balancing the books accounting for relief payments.

Summary. The outcome $\phi, \xi \Rightarrow z, x, \alpha, \tau, \langle u, g \rangle$ constitutes the citizens' bargaining shield for wealth redistribution that relates to a bundle of arguments or constants: ϕ, ξ are controls, and z, x, α, τ are status variables,[2] while $\langle u, g \rangle$ are the competing political proposals:

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[2] Status and control variables are the prerogatives of control theory.

ϕ – the personal allowance establishing the tax bracket $[\phi,\infty)$; it is an ex-ante control (tuning) variable, $0 < \phi = \text{const} < \xi$;

ξ – the income frame, the poverty line; a policy determining who is living in poverty, as well as the choice or the control parameter;

z – the size $z = \tau \cdot W(\xi)$ of the wealth-pie; the amount of wealth-pie that is equal to public spending per capita when taxes are proportional;

x – the share of the wealth-pie of size z; a portion x of z to be deposited in favor of the left-wing politicians for funding the relief payments, $0 \le x \le 1$;

α – the political power of the left-wing politicians, $0 < \alpha < 1$;

τ – the marginal tax rate, the rate $\tau(\xi, x)$ of the wealth amount $W(\xi)$ determined by (1);

u – the after-tax residue of the income frame equal to the poverty line ξ, the wants function $u(\xi, x)$ of the left-wing politicians, as determined by (2) and (3);

g – the objective function $g(\xi, x)$ of the right-wing politicians, determined by (1) and (2); the account for the refund of public goods expenses per capita.

5. ANALYSIS OF THE WELFARE STATE BARGAINING RULES AND NORMS, *continued from Section 3.2*

Suppose that politicians, in pursuit of their commitments to a fair division of the wealth-pie, agreed to play the alternating-offers bargaining game $\Gamma(q)$ (Osborn and Rubinstein). In doing so, rational politicians are motivated to align the procedure to participate in any eventual agreement. The risk $q > 0$ of a premature collapse during negotiations, especially early in the game, might be the driving force behind their commitment to reach the consensus. Once a consensus on division is reached, they must agree on who will determine the size of the pie. Politicians negotiate on such matters when there are equal and symmetric preconditions in place that guarantee their equal rights. Thus, both will play an equal role in the decision regarding the pie size. Considering the right-wing vital political objective of wealth redistribution, it will be realistic to reduce the scope of RWP's duties concerning welfare matters, while allowing them to retain their advisory rights. Our subsequent discussions are based on this premise.

5.1. Left- and right-wing politicians' bargaining procedure

Previously, we emphasized that, in a representative democracy, the division of the wealth-pie will always be subject to controversy. Recall that we consider two politicians — one acting in the role of LWP, who is aiming to provide basic goods to all citizens, and the other, representing RWP, advocating for availability of non-basic goods. A precondition for the bilateral agreement is that the expectations of these two politicians depend solely on efficient policies of the LWP within the framework aimed at setting the poverty line ξ. However, politicians are more concerned with shares (x, y) than they are with the size of the wealth-pie. As a consequence of this independence, efficient poverty line ξ° provides shares related to efficient divisions (x°, y°). Accepting this precondition, the RWP will only propose an efficient line ξ°, as failure to do so would result in all other shares being rejected with certainty by LWP. Nonetheless, it is realistic that the RWP would — by negligence, mistake or some other reason — recommend an inefficient poverty line ξ', which the LWP would mistakenly accept. It is also possible that, in a reverse scenario, the LWP would choose to disregard an efficient recommendation ξ°. This would be an irrational choice as, in any agreement, regardless of the underlying motives, both politicians are committed by proposals to shares (x, y).

Indeed, within the scope of negotiations $[\xi_1, \xi_2]$, the recommendation ξ° concurs with RWP's efficient share proposal y°. Consequently, accepting $1-y^\circ$, while shifting LWP's ξ° mistakenly to $\xi' \neq \xi^\circ$, at which both politicians must be committed to (x°, y°), the shift ξ' becomes inefficient and thus superfluous. Hence, making a proposal, the RWP's recommendation on poverty lines makes a rational argument that the LWP must accept or reject in a standard way. Such an account, in our view, explains that the outcome of the bargaining game might be a desirable poverty line $\xi \in [\xi_1, \xi_2]$. Hereby, the interval is referred to as the scope $[\xi_1, \xi_2]$ of negotiations or bids proposals that are now, by default, linking efficient lines ξ° with shares (x°, y°). The bargaining occurs exclusively in the interval $[\xi_1, \xi_2]$ as a scope for efficient lines ξ° of most trusted policy platforms for negotiations, where both players would either accept or reject the proposals. Political competition, depending on

$[\xi_1, \xi_2]$, arranges a contract curve $\mathcal{S}_b$ (shown in Figure 4 and Figure 5) as a way to assemble the bargain portfolio. Given that the portfolio "has changed its color from shares to lines," the politicians can now conceive themselves as making poverty line proposals. If a proposal is rejected, the roles of politicians change and a new proposal is submitted. The game continues in the traditional way by alternating offers.

The Contract Curve Projection within $[\xi_1, \xi_2]$

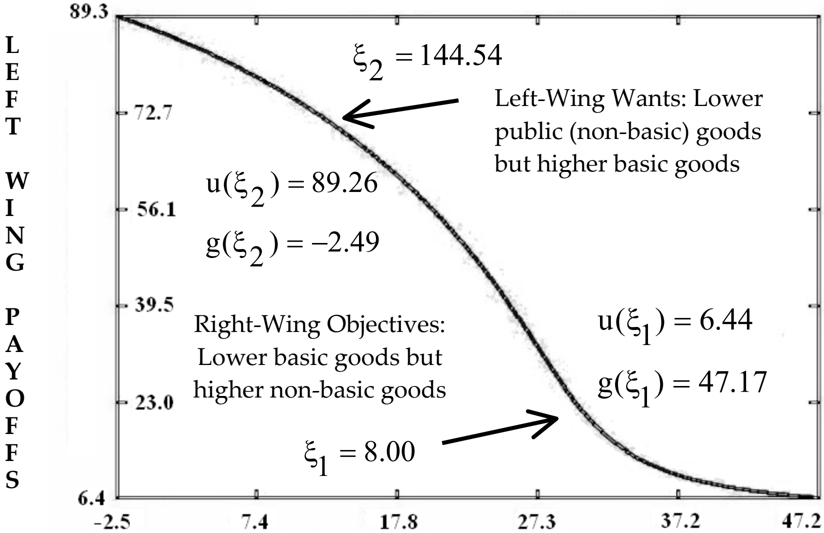

Figure 5. The aspirations of left-wing politicians expressed when opposing the right-wing political objectives are depicted on the vertical and horizontal axes, respectively. The graph shows the contract curve sloping from ξ_2 toward ξ_1, projected on the surface of basic goods *vs.* vital goods — the projection of efficient poverty lines $\xi \in [\xi_1, \xi_2]$ resolving the contract constraint (5).

5.2. Alternating-offers bargaining game analysis

We now proceed to a more accurate analysis of the game rules. Although the rules can be perceived as fiscally idempotent, the game itself contains a new challenge. The elevated poverty line ξ does not necessarily increase the marginal citizens' $\sigma = \xi$ after-tax residue $u(\xi, x)$. The low-income citizens — the benefit recipients — can claim relief payments, whereby an increased number of claims might have a reverse effect on $u(\xi, x)$, which would consequently decline. Indeed, in contrast to increasing poverty line ξ and despite the required unavoidable increase in taxes — as the hazard (h-effect) is still present — in this scenario, the residue $u(\xi, x)$ will decrease. With the proviso that the left-wing politicians commit to the share x, the right-wing politicians are left with $y = 1 - x$. Thus, the fiscally idempotent poverty line tax residues $u(\xi, x)$ correspond to a narrower set than $0 \le x \le 1,\ 0 \le y \le 1$ — the set of shares $\langle x, y \rangle$ of what we refer to as a *contract curve* $\boldsymbol{S}_b$ of payoffs $\langle (u(\xi, x), g(\xi, y)\rangle$ with poverty line ξ as a parameter. [3]

Assuming that the maximum of a single $\cap$-peaked residue function $u(\xi, x)$ can be reached, the peak position $\xi^\circ = \arg\max_\xi u(\xi, x^\circ)$ indicates an efficient welfare policy. Although the bargain portfolio of left-wing politicians contains an efficient policy ξ° as a function of x°, the portfolio also includes the share $x = x^\circ$. The maximum value given by $u = u^\circ$, in the inverse situation, which corresponds to u°, consolidates an efficient policy $\xi^\circ \in [\xi_1, \xi_2]$. A unique share x°, which solves $u(\xi^\circ, x) = u^\circ$ and corresponds to $g(\xi^\circ, y^\circ) = g^\circ$, represents the non-conforming expectations of politicians. We can thus refer to the shares (x°, y°) as an efficient division linked to the policy ξ°. This scenario is depicted in Figure 4 on wealth amount W and taxes τ — efficient peaks ξ°, which correspond to efficient shares (x°, y°), and in Figure 5 in various projections on payoffs $\langle u^\circ, g^\circ \rangle$ geometry. This geometry highlights the maximum values u° can take — namely, efficient policies of left-wing politicians at peaks ξ° that refer to the well-known result obtained by Canto et al (1981), also known as the Laffer curve:

> *The marginal tax-revenue raised decreases with increase in tax rates, finally reaching some point where the marginal tax-revenue raised is zero. Beyond this point, any tax rate increases will reduce revenue collection.*

Our result pertaining to the single-peaked aspirations of the left-wing politicians is similar. First, "*poverty line residue* u *being proposed in the normal range of poverty line parameter* ξ." Next,

> *…by passing through the top point of* u *as a function, the proposals* u *will be assessed and reviewed in the range of prohibited values of* ξ.

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[3] We already highlighted the worsening quality of welfare services for all citizens when the LI level is “climbing” high.

We previously introduced an idempotent composition $\left[B(\xi), W(\xi)\right]$ — the average $B(\xi)$ of the relief payments, and the average $W(\xi)$ of the taxable income, denoted as the wealth. The expectations of the two politicians, reflecting their preferred rules and norms pertaining to relief payments, can now be set using the composition $\left[B(\xi), W(\xi)\right]$. At the end of the subsection, the composition will lead to an appropriately settled bargaining problem that will associate the threat originating from the implicit partaker — in the form of the electoral maneuvering of voters — with an implicit risk of the negotiations collapsing prematurely. This requires augmenting the standard rules of the game we have already presented with two further rigorous suppositions. Let us first specify the payoffs.

Political payoffs of the 1st/2nd actor and the third partaker′s implicit risk factor are defined as follows:

Politician No. 1,	Politician No. 1, u – the left-wing political aspirations, the marginal citizens' $\sigma = \xi$ after-tax residue, basic necessities of the needy, cost of living;
Politician No. 2,	g – he right-wing political objective, expenses that benefit all citizens — expenses upon vital goods alone, without relief payments;
Third Partaker,	q, τ – voters' electoral maneuvering facing higher taxes τ expressing an implicit risk $0 < q << 1$ of the negotiations collapsing prematurely.

As promised, we now assume that the rules and norms of the wealth redistribution that are efficient with respect to the wealth-pie division include the volatility constraint (4), which certifies the idempotent composition $\left[B(\xi), W(\xi)\right]$ for the policy ξ. In the game, the composition $\left[B(\xi), W(\xi)\right]$ could not be implemented without the volatility constraint $L(\xi, x, u) = 0$ (Observation 1). This assumption is contingent on the conclusions of the previously undertaken analysis.

When varying ξ under their own rules and norms, let us assume that LWP propose a fiscally idempotent policy $\xi = \xi^{\circ}$, which — for each share $x = x^{\circ}$ they commit to — links x° to ξ°, irrespective of who originates the proposals x° or y°. This ensures the efficient proposal of poverty line residue $u(\xi^{\circ}, x^{\circ}) = \max_{\xi} u(\xi, x^{\circ})$. Clearly, inefficient recommendation ξ', proposed by the RWP if $\xi' \neq \xi^{\circ}$ for share y°, will be rejected by the LWP. As a result, an efficient policy $\xi = \xi^{\circ}$ must occur on contract curve amid efficient shares x° at $\left\langle u^{\circ} = u(\xi^{\circ}, x^{\circ}),\ g^{\circ} = g(\xi^{\circ}, x^{\circ}) \right\rangle$ as an ongoing precondition for the agreement — as previously discussed. Indeed, LWP have no reason to reject efficient recommendation ξ°, as doing so, when $\xi' \neq \xi^{\circ}$, they cannot ultimately maintain the efficient commitment to x°. Below, we assume the efficiency by default when it is convenient.

Observation 2. *Idempotent policies* ξ *at the contract curve* $\boldsymbol{S}_{b} = \left\langle u(\xi, x), g(\xi, x) \right\rangle$, *which certifies the composition* $\left[B(\xi), W(\xi) \right]$, *must satisfy the constraint*

$$\begin{aligned} D(\xi, x, u) &= \frac{\partial}{\partial \xi} L(\xi, x, u) = \\ &= \frac{\partial}{\partial \xi} \left[(\xi - \phi) \cdot B(\xi) - x \cdot (\xi - u) \cdot W(\xi) \right] = 0 \end{aligned} \quad . \qquad (5)$$

Particularly, when we collated sub-expressions and introduced some simplifications upon

$$\begin{aligned} Q(\xi, \tau, g) &= 0 && \to \text{Delivery(1)} \\ L(\xi, x, u) &= 0 && \to \text{Volatility(4)} \\ D(\xi, x, u) &= 0 && \to \text{Contract curve(5)} \end{aligned}$$

enforcing constraints on rules and norms of the wealth redistribution.

These constraints, with the proviso of flat taxes, together with the previously detailed preliminary settings $\tau'_{\xi} > 0$, $\tau''_{\xi} > 0$, $u''_{\xi} < 0$, $u'_{\xi} > 0$, $u'_{\xi} < 0$, $u''_{x} > 0$, $u'_{x} > 0$, $g'_{\xi} > 0$, $g''_{\xi} > 0$, $g''_{x} \neq 0$, lead to an analytical solution: $u(\xi) = \xi - \dfrac{(\xi - \phi)}{v(\xi)}$, where

$$v(\xi) = 1 + (\xi - \phi) \cdot \left(\frac{\dot{B}(\xi)}{B(\xi)} - \frac{\dot{W}(\xi)}{W(\xi)} \right);^{4} \ \tau(\xi) = \frac{1}{v(\xi)},$$

---

[4] ± rates $\dot{W}(\xi) \leq 0$, $\dot{W}(\xi) \geq 0$ of the changes in the wealth amounts $W(\xi)$ are essential for the analysis, whereas the function $B(\xi)$ is valid only when $\dot{B}(\xi) > 0$, and $0 < \phi < u < \xi$.

$$g(\xi) = \frac{W(\xi)}{\nu(\xi)} - B(\xi)\text{; the size of wealth-pie}$$

$$z(\xi) = B(\xi) + g(\xi) = \frac{W(\xi)}{\nu(\xi)}.$$

Now it is evident that payoffs $\langle u, g \rangle$ at the contract curve $\mathcal{S}_b$ depend exclusively on policies ξ, $\langle u(\xi), g(\xi) \rangle \in \mathcal{S}_b$. We conclude that politicians are only concerned with making proposals that pertain to efficient policies ξ, since effective shares (x, y) have been linked to ξ. Contract curve $\mathcal{S}_b = u(g)$ in Figure 4 illustrates the payoffs. The functions $g(\xi)$ and $u(\xi)$ in the form presented above are, in fact, not a subject to any constraints. They are mathematically derived in Appendix A4.

Before proceeding with further line of analysis, let us recall the threat phenomenon created by voters that increases the implicit risk of the negotiations collapsing prematurely. As noted previously, if politicians reject their counterpart's proposal — knowing that it is risky to continue the bargain — they will likely consolidate a draft. This introduces the risk that the voters will reject the draft when politicians, without fulfilling the voters' terms, try to continue bargaining over costly and controversial proposals, thereby putting the negotiations at a risk of collapse, as previously discussed.

Suppose that politicians bargain over all fiscally idempotent policies $\xi \in [\xi_1, \xi_2]$ within the scope of negotiations $[\xi_1, \xi_2]$. We follow the alternating-offers game $\Gamma(q)$ with an exogenous risk $0 < q << 1$ of a premature collapse, as described previously (Osborn and Rubinstein). We posit that, each time the proposal ξ is rejected by one of the politicians, the momentary phase of the game results in a draft, which can be opposed by the voters, as just recalled. In these circumstances, the politicians might be uncertain on how to proceed, if the voters' terms are not met. As a result, they might choose to leave the bargaining table prematurely. Extracted from the endpoints $\xi_1 < \xi_2$ of contract curve $\mathcal{S}_b$, the outcome

$$\{\langle u_1, g_1 \rangle, \langle u_2, g_2 \rangle\} =$$
$$= \{\langle u(\xi_1), g(\xi_1) \rangle, \langle u(\xi_2), g(\xi_2) \rangle\}$$

naturalizes this risk q in the worst-case scenario.

What is known as the *well-defined bargaining problem,* first introduced by Roth, or the individual rationality associated with the Nash bargaining scheme $\langle \boldsymbol{S}, \mathrm{d} \rangle$, seems to be instructive for further analysis. Indeed, inequalities $g_1 > g_2$ and $u_1 < u_2$ hold for the pair $d = \langle d_1 = u_1, d_2 = g_2 \rangle$. Synthesizing the unfavorable political outcome $\{\langle u_1, g_1 \rangle, \langle u_2, g_2 \rangle\}$ into a policy δ on poverty introduced below will naturalize the Nash disagreement point d into the problem $\langle \boldsymbol{S}_b, d \rangle$, $\boldsymbol{S}_b \subset \Re^1$. Thus, compared to the traditional approach of compact convex set $\boldsymbol{S} \subset \Re^2$, inequalities $s > d$ are also true for any pair $s \in \boldsymbol{S}_b$. The pair $\langle \boldsymbol{S}_b, d \rangle$ for the contract curve $\boldsymbol{S}_b$ becomes a well-defined bargaining problem. Given that it is not immediately apparent whether the policy δ is a fiscally idempotent outcome of the game, the following observation removes any doubt.

Observation 3. *To test whether the point* $d = \langle d_1, d_2 \rangle = \langle u_1, g_2 \rangle$ *becomes a fiscally idempotent outcome of the left- and right-wing political bargaining, it is necessary and sufficient that there exists a policy* δ *on poverty, which solves the equation:*

$$(\delta - \phi) \cdot (B(\delta) + d_2) - (\delta - d_1) \cdot W(\delta) = 0;$$

$$\text{The condition } \delta \notin [\,\xi_1, \xi_2\,] \text{ must hold true.} \qquad (6)$$

It should be noted that, in the worst-case scenario δ, the wealth redistributed equals $W(\delta)$ — the average of expenses for funding the relief payments equal $B(\delta)$ — whereby the proposal δ depends on the endpoints of the bargaining interval $[\,\xi_1, \xi_2\,]$. This dependence, provided that the Equation (6) can be solved for δ, serves as the basis for validation of the *pre-equity condition of breakdown,* as discussed in Section 7.

Observation 4. *In the alternating-offers game* $\Gamma(q)$ *with the risk* $0 < q << 1$ *of negotiations collapsing prematurely into the disagreement point* $\langle d_1, d_2 \rangle$, *the functions* $(u(\xi) - d_1)^\alpha$ *and* $(g(\xi) - d_2)^{1-\alpha}$ *imply bargaining payoffs of left- and right-wing politicians, respectively. Thus,* (without proof) *for variables* λ_1, λ_2, $\lambda_1 \le \lambda \le \lambda_2$, *solving the equations*

$$(1-q) \cdot (u(\lambda_1) - d_1)^\alpha = (u(\lambda_2) - d_1)^\alpha \text{ and}$$

$$(1-q) \cdot (g(\lambda_2) - d_2)^{1-\alpha} = (g(\lambda_1) - d_2)^{1-\alpha},$$

the solution λ *of the well-defined bargaining problem* $\langle \boldsymbol{S}_b, d \rangle$ *is close to the pair* (λ_1, λ_2).

As explained by Osborn and Rubinstein, the outcome in our experiment of bargaining game $\Gamma(q)$ encapsulates the power indicators $(\alpha, 1-\alpha)$ of the left- and right-wing politicians. In the next section, we consider the design of political power indicators $(\alpha, 1-\alpha)$ using the solution λ that minimizes the tax burden with respect to an appropriately settled bargaining problem $\langle \boldsymbol{S}_b, d \rangle$.

6. ANALYSIS OF VOTING AND POLITICAL POWER DESIGN, *continued from Section 3.4*

Here, we will elaborate on power indicators $(\alpha,1-\alpha)$ specifically, referring to the original bargaining scenario of \$1 division, based on the previously discussed axiomatic approach — α signifies LWP's political power, and $1-\alpha$ the political power of RWP, $0<\alpha<1$. Considering

$$(x^{\circ}, y^{\circ}) = \text{arg max}_{\ 0 \le x+y \le 1} \ f(x, y, \alpha) =$$
$$= (u(x)-d_1)^{\alpha} \cdot (g(y)-d_2)^{1-\alpha}$$

the following questions emerge: What type of \$1 division will assist a moderator designing the power indicator α of the 1st actor? What will ensure that, during the negotiations, the 1st actor will obtain a desired or any other share x° of \$1? To answer these questions, let us assume that the 2nd actor might only accept or reject the 1st actor's proposals. We can thus start redesigning the power indicators $(\alpha,1-\alpha)$ by replacing $y=1-x$, and taking the derivative of the resulting $f(x,1-x,\alpha)$ with respect to the variable x by evaluating $f'_x\,(x,1-x,\alpha)$. For a moment suppose, finally, that x° share of \$1 is a desirable solution. Given $x=x^{\circ}$, the equation $f'_x\,(x^{\circ},1-x^{\circ},\alpha)=0$ can be solved for $\alpha=\alpha^{\circ}$. In general, one might find comfort in the following egalitarian judgment:

To count on x° *share of \$1 is a realistic attitude toward the 1st actor's position of negotiations. Indeed, even if the 2nd actor might have a stronger negotiating power than the 1st actor,* $\alpha^{\circ} < 1-\alpha^{\circ}$*, the 1st actor, sooner rather than later, might predict the 2nd actor's preferences and thus force a concession.*

When, for example, the voters' representatives attempt to redesign political power indicators to $(\alpha, 1-\alpha)$, we assume that politicians will try to share the wealth-pie in the manner in which \$1 was divided above. In doing so, we suppose that both politicians are ready to proceed with tax concessions. Reflecting just illustrated axiomatic bargaining toward allegedly desirable \$1 share x°, we proceed with our discussion.

In accordance with our analytical solution without constraints, the contract curve $\boldsymbol{S}_b = \mathrm{u}(\mathrm{g})$ corresponds to a curve $\langle \mathrm{u}(\xi), \mathrm{g}(\xi) \rangle$. Moving along the curve while taking into account the scope of negotiations $[\xi_1, \xi_2]$, the expectations $\tau(\xi)$ of voters' majority lead to detection of $\tau_{\min} \leftarrow \tau(\xi)$:

$$\min_{\xi \in [\xi_1, \xi_2]} \tau(\xi) \Bigg| \ \tau(\xi) = \frac{1}{\mathrm{v}(\xi)}.$$

With the proviso that $\tau(\xi)$ is concave and sufficiently smooth, the detection point of $\tau_{\min}$ is the root λ of the equation $\dot{\tau}(\xi) = 0$. Consequently, akin to the egalitarian judgment given above, the root λ might help in redesigning of the rules and norms of the wealth redistribution. This can be done by adjusting the α in a way that the political power α of the left-wing politicians will be sufficient to persuade the right-wing politicians to agree upon the poverty line residue $\mathrm{u}(\lambda)$.

Indeed, in the left- and right- political bargaining, the old *standard* (discussed above) of how to share the $1 can now be a new *Standard* pertaining to how to plan the wealth redistribution rules and norms. Under this premise, we can set $\mathrm{f}(\xi, \alpha) = (\mathrm{u}(\xi) - \mathrm{d}_1)^{\alpha} \cdot (\mathrm{g}(\xi) - \mathrm{d}_2)^{1-\alpha}$, where α facilitates the political power of the LWP. Instead of $\mathrm{x} = \mathrm{x}^{\circ}$, planning the rules, we suppose that $\xi = \lambda$ is an allegedly desirable solution. Hence, we first take the derivative of $\mathrm{f}(\xi, \alpha)$, with respect to ξ, evaluating $\mathrm{f}'_{\xi}(\xi, \alpha)$, which allows us to solve the equation $\mathrm{f}'_{\xi}(\xi|_{\xi=\lambda}, \alpha) = 0$ for α. As a result, the root α° will correspond to the redesigned political power of the left-wing politicians. This is the result as it appears.

Summary. To control the left- and right-wing political agreement on shares (x, y) of the wealth-pie, akin to the new *Standard* above, the majority of citizens can accept or reject a premature agreement archived at the a particular point during the negotiations, thereby voting for or against the division. As previously noted, the majority will favor the policy λ that minimizes the tax burden. This restriction allows us to rebalance the welfare institutions or finance resources by appropriate design of power indicators $(\alpha, 1-\alpha)$ of the left- and right-wing politicians, ensuring that the most favorable shares $(\mathrm{x}^{\circ}, \mathrm{y}^{\circ})$ of the wealth-pie would incorporate the Nash axiomatic — the minimum tax — solution λ into the bargain portfolio as the most optimal outcome. This is our *case study* of tax policy in which only a minority would object to a proposal that corresponds to the tax rate minimum at the contract curve. In doing so, the implicit pressure of citizens will be lower. To be implemented in favor of majority, the minimum appears to be a desirable consensus.

Observation 5. *Given that politicians can reach a preliminary agreement on tax rate* $\tau = \tau(\xi)$, *condition* $\lambda = \arg\min_{\xi \in [\xi_1, \xi_2]} \tau(\xi)$ *is necessary to put forward a poverty proposal* λ *before voters by appropriately designing the power indicators* $(\alpha, 1-\alpha)$ *in advance. At the contract curve* $\boldsymbol{S}_b$, *the proposal* λ *outlines a unique outcome:*

$$\phi, \xi \Rightarrow z, x, \alpha, \tau(\lambda), \langle u(\lambda), g(\lambda) \rangle \in \boldsymbol{S}_b \ .$$

7. DISCUSSION

The true essence of the economic reality behind the left- and right-wing political bargaining could be revealed by determining whether it is true that funding relief payments of the needy and maintaining the budget in balance will be difficult to sustain when the tax burden for all citizens is decreasing. On the surface, it seems that, at some point, fairness and equity might no longer be the main requirement because of the "risks becoming a Downton Abbey economy" (2014). Economists, including Kittel and Obinger (2003), have analyzed the poverty gap issue. In the face of these controversies, it is not possible to estimate the extent of potential fallout that might result from such outcomes of tax burden cut.

The citizens are those who decide what needs to be done and what should ultimately bring order to socially plan, or how to redesign the wealth redistribution rules and norms. Taking advantage of this opportunity, it is instructive to perform an exercise related to the most appropriate choice of welfare policy, as shown in the *"minimizing wealth-tax"* column of Table 1. [5] We illustrated that, despite minimizing the tax burden for all citizens, the minimum is, in fact, fiscally safe, while also ensuring just and fair redistribution of wealth for all citizens.

Due to the assumptions made during the analysis, the following discussion perhaps offers some guidance on doing the exercise. Before commenting on those, it is worth noting that the experiment presented here should be understood as purely normative — namely, "what ought to be" in economic or political matters, as opposed to "what is." Despite the fact that, in the preceding analysis, no actual situation was presented, our theoretical results rest on the assumptions delineated below.

First, our work is based on the premise that politicians would only make promises that can be fulfilled — fiscally safe proposals. Fiscal safety, when taken separately, even when attempted in accordance with the rules and norms in force, could lead to unjust and unfair solutions. Taken at will, fiscal safety

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[5] Table 1 was created by numerical simulation carried out upon imaginary distribution of citizens' incomes.

might be a profoundly mistaken idea of justice. In Table 1, we presented the percentage of citizens below the poverty line, thus establishing the poverty rate. [6] Driven at will, the official poverty rate, in accordance with the "disagreement" column of Table 1, could cause the poverty rate to decline below 0.41%, which wrongly appears to be the most just and the fairest.

Second, we postulated that the wealth redistribution compensates for the inequalities in the income of citizens that were below the poverty line. Usually, similar parameters are in the national government competence. While taking into account increases in the cost of living, the official number of individuals living in poverty should be adjusted annually according to government guidelines. Although our key assumption was that the right-wing politicians inherited no more than an advisory authority, the rules and norms that govern the poverty line determination have been solely under the mandate of the left-wing politicians. This decision was made because, in the analysis, we deliberately emphasized the distinctions between stereotypical motivations of left- and right-wing politicians. In our view, welfare protection that is most likely to be just as fair should be addressed as an independent institute, or better yet, as an assembly of independent institutes or legal charity foundations. We believe that, in our experiment of organizational independence, welfare protection could be expected to yield efficient welfare policies. Thus, in determining an efficient policy on poverty, we concluded that left-wing politicians should be in a privileged position that allows them to prescribe the poverty line independently. Only when these guidelines of independence are applied, the value judgment based upon the data presented in Table 1 makes sense. Still, it should be noted that the characterization of whether setting up such a privilege was a positive or negative restriction requires further investigation.

Next, we focused on the political power indicators $(\alpha, 1-\alpha)$, which highlight the amount of resources, skills and competence of left- and right-wing politicians. The fundamental factor in our analysis was the welfare protection of the society as a whole to justify and maintain welfare duties under the principle of how the state ought to act when attempting to fulfill its welfare mission. When the decision made by the politicians is not in line with the objectives of special interest groups, as previously pointed out, welfare protection could be a recurrent theme in political debates and election campaigns, and a source of significant political competition. A controversy with respect to political interests might lead to violent upsets, providing the opportunity to develop policy in favor of these groups. According to the foregoing account, which requires considerable administrative efforts and fiscally unrealistic expenses — and previous observations pertaining to the independence of the welfare services — we believe that having sophisticated left-wing institutions is unneces-

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[6] Poverty rate determines the percent of anyone who lives with income below the official poverty line. The poverty line separates the rich (those with an income above the poverty line), from the less fortunate (having income below the line).

sary. Recognizing the vital role of the right-wing politicians, due to their central position in deciding who will be purchasing and delivering public goods, in the interpretation of the parameter α, we believed that it was beneficial to impose a lower α to the left-wing politicians, with a corresponding higher share $1-\alpha$ assigned to the right-wing politicians, i.e., $\alpha < 1-\alpha$, $0 < \alpha < 1$.

Thus, it was reasonable to assume that left-wing politicians, with almost no extra effort, would demonstrate an ample degree of readiness to make efficient decisions. Herewith, in planning and regulating the size of the wealth-pie to suit a fiscally realistic welfare policy to settle and assist the state welfare mission, we attempted to redesign the balance of political powers between the left- and right-wing politicians by adjusting the power indicators α and $1-\alpha$, imposed on the on the left- and right-wing politicians, respectively. With the goal in our view, to benefit all citizens in society, this enabled us to adjust the state rules and norms of the wealth redistribution, aligning them closer to the legal responsibilities and moral obligations of the citizens. We referred to the process of adjusting the power indicators $(\alpha, 1-\alpha)$ as a political power design. Such a politically designed outcome, as we supposed, justified the time and effort invested, even if the vision was a utopia.

The design of political power indicators $(\alpha, 1-\alpha)$ is a difficult and extremely time-consuming process. Indeed, prolonged political efforts might not be in the interest of anyone — citizens might not pursue such endeavor, even if the balance of political power can be ultimately reached. In particular, we supposed that electoral maneuvering of voters might put prolonged political efforts at risk of a premature collapse. It was deemed acceptable to assume presence of an implicit risk of voters defecting to the other side, which could interrupt negotiations ahead of the schedule. Thus, we brought the problem of likelihood of negotiations collapsing into focus. In our experiment, the failure of negotiations was deemed extremely undesirable for both politicians, as we hoped that this would be an incentive to move toward a solution faster. Alternatively, the actors would be more motivated to agree on terms of a contract, where both sides approach each other by making considerable concessions. In the view of receipt of relief payments, a policy of higher tax rates might be the most favorable and just solution for minority. From the majority perspective, however, the minimum tax rate is always preferable. For the citizens who finance the relief payments, as we assumed in the analysis, the minimum tax rate provides a more just and fair redistribution of wealth. In our experiment, the minimum rate also provided an outcome λ in which the designed political power indicators $(\alpha, 1-\alpha)$ visualize the society′s common denominator. Assuming, as we previously did, in accordance with the rules of the game, that outcome λ, minimizing taxes, could be politically designed — it provides insight into what policy should entail.

Table 1, presenting all four assumptions, suggests several proposals for citizens to vote on. Note that, when voting for policy of equal left- and right-wing political power, the policy $\eta = 79.23$ is less just and less fair than the outcome $\lambda = 45.50$, where the minimum *26.52%* of marginal tax rate is reached. Thus, only the policy/outcome λ on the poverty line (Figure 4) can be the desirable political consent. Indeed, in the variety of rules in the game the left- and right-wing politicians play, when engaged in an interaction aimed at implementing equal/egalitarian policy η, the equal political power $\alpha = 0.5$ of the LWP was stronger than *0.21*. Consumers' goal, however, can still be achieved by applying the weaker policy $\lambda = 45.50$ for the tax rate *26.52% < 28.21%*, although the outcome of the weakened political power indicator $\alpha = 0.21$ is yet to be confirmed. Through a reduction of citizens' obligations — even with LWP's weakened political position — the LWP will be able to come to a desirable agreement with the RWP, maintaining the most just and fair poverty line of wealth for all citizens.

In closing the discussion, we would like to point to a decision δ that corresponds to the political breakdown of negotiations. Utopian society, planned according to the event of a breakdown, as shown in Table 1, seemingly ignores welfare protection because practically all citizens are considered rich by default, i.e., poverty does not exist. Given this utopian society, financing expenses almost entirely with respect to vital public/non-basic goods, the breakdown policy δ, under the equity condition, requires *–2.49* public debt per capita. This, in turn, will require borrowing or money printing, promoting public spending, *e.g.,* through natural assets for refunding the debt. We admit that, based on the lowest tax burden of *26.52%*, a self-financing tax system has a better chance of being implemented.

8. CONCLUDING REMARKS

Given the ideological controversies of the left- and right-wing politicians, and the need to resolve the welfare policy dilemma, both actors should be willing to make concessions. In most cases, the root of the controversy is that, the left-wing politicians struggle — in response to public aspirations — in pursuing their own political causes for the increase of basic goods, whereas the right-wing politicians advocate for meeting the needs for non-basic goods. In our experiment, left-wing politicians gave credit to the tax system to guarantee a reasonably high living standard for benefit claimants. Whatever public spending voters preferred, both politicians were aware of voters' electoral maneuvering, which could put the negotiations at risk of a premature collapse. In our work, this threat was the only driving force in reaching the consensus. We argued that political arguments demanding higher taxes were weak, since overly costly welfare proposals lead to an excessive number of relief payments claimants, which, in spite of the tax increase, could diminish the quality of the welfare services. In turn, the excessive number of claims could generate further requests for the additional financial support through tax channels. In order to satisfy those who bear additional costs, and who could only approve the requests on the terms of fiscally safe welfare policies, we reduced the scope of negotiations to the fiscally realistic domain of voters' expectations.

In view of the above, a pretext for the analysis of the domain and the extent of bargain portfolio of two visionary politicians, denoted as LWP and RWP, were established. The portfolio was supposed to account for politicians having non-conforming expectations. Instead of the wealth-pie division, such an account allowed for including a guide on how the eventual consensus ought to be analyzed and interpreted within the scope of negotiations $\left[\xi_1, \xi_2\right]$ at the contract curve. In this context, the left- and right-wing political power indicators, specified by the bargaining problem solution, were supposed to be politically designed in advance and subsequently tailored in accordance with the citizens' visions and ambitions.

It was initially deemed that, due to the uncertainty in the selection of the breakdown policy, we could only treat the left- and right-wing political power indicators as given exogenously. While this is true at least in the valuable examples we provided, we found a condition where we can encode the indicators endogenously, to which we referred as *the pre-equity of political breakdown.*

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APPENDICES

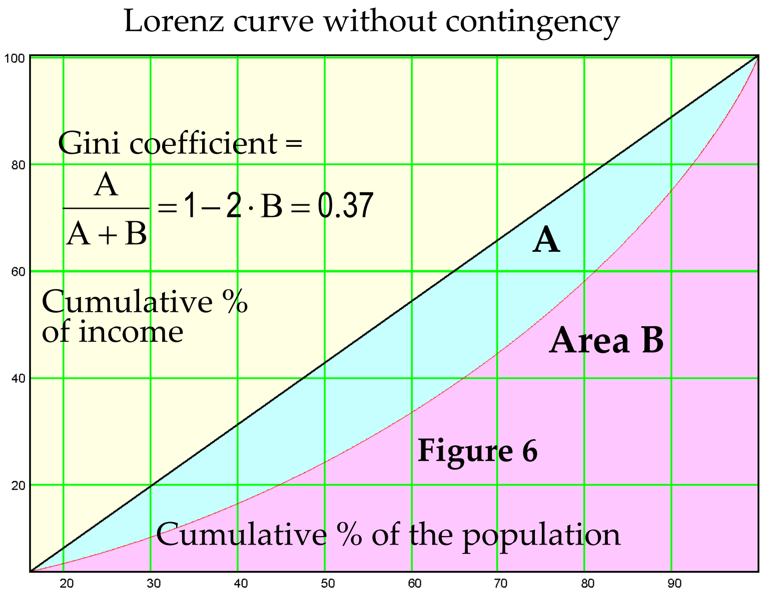
A1. Example and results

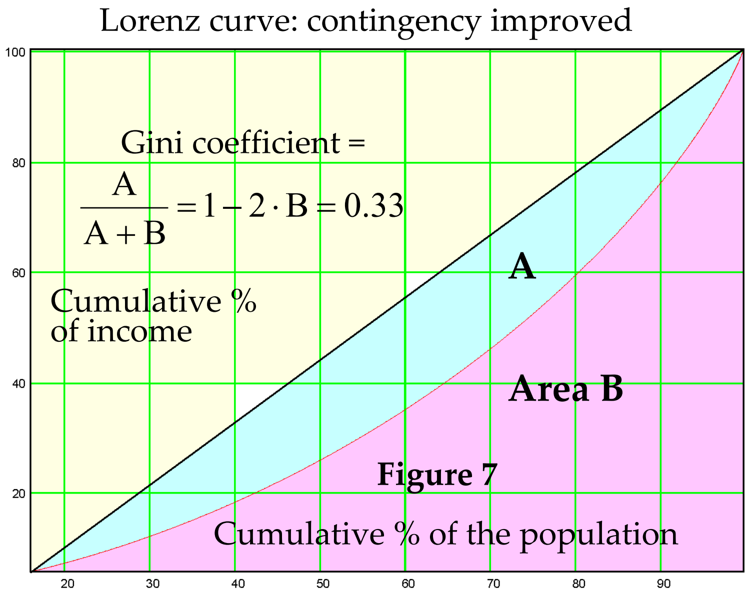
We proceed with a specific allocation of the welfare policy, encapsulating samples of income density distribution, parameterized by poverty line ξ, similar to an exponential function:

$$P(\sigma,\theta+h\cdot\xi)=$$
$$=\frac{1}{(\theta+h\cdot\xi)\cdot\Gamma(m)}\left(\frac{\sigma}{\theta+h\cdot\xi}\right)^{m-1}\cdot\exp\left(-\frac{\sigma}{\theta+h\cdot\xi}\right),$$

where $\theta=61.9$, $m=2.07$, and $h=-0.18$ are additional ex-ante parameters. More specifically, θ controls the wealth of citizens — a *horizontal shift* of samples; m controls inequality — a *vertical shift*; h is a hazard parameter; and $\Gamma(m)$ is an extension of $(m-1)!$ to real numbers. The sample ξ = ½μ (median income = μ) can be presented as **Lorenz Curve**, where citizens below an income ***95.1***, i.e., ***49.92%*** of the population, have ***24.13%*** of a total cumulative income, while the remaining ***50.08%***, with incomes at or above ***95.1***, have ***75.87%***, Figure 6. Gini Coefficient equals ***0.37*** and is impervious to the horizontal shifts only. Relief payments, delivered to the population in line with Friedman personal exception rule in force equal to ½μ applied upon the income distribution sample ξ = ½μ diminished the Gini coefficient to ***0.33***. Indeed, on Figure 7 citizens below an income ***95.1***, i.e., ***49.83%*** of the population has slightly increased to ***25.83%*** of a total cumulative income, while the remaining ***50.17%***, with incomes at or above ***95.1***, have slightly decreased to ***74.17%***.

The density function $P(\sigma,\theta+h\cdot\xi)$, depending on ξ, reflects the initial wealth redistribution through tax channels. Political decision $\xi'>\xi$ shifts the density distribution $P(\sigma,\theta+h\cdot\xi)$ of incomes horizontally toward the allocation $P(\sigma,\theta+h\cdot\xi')$ that favors less wealthy. When shifted, the distribution $P(\sigma,\theta)$ masks the h-factor, $h=0$, of the benefit claimants. The rate of change $Hz(\xi)=h\cdot\dot{a}(\theta+h\cdot\xi)<0$ of the policy ξ quantifies a fiscally tolerable hazard ($h<0$).





A2. Simulation foundation and illustration

In order to perform simulations, the expressions for average $B(\xi)$ of expenses on the relief payments and average taxable income — the wealth amount $W(\xi)$ — can incorporate income density distribution $P(\sigma, \theta + h \cdot \xi)$ in a more realistic but general form:

$B(\xi) = r \cdot \int_{0}^{\xi} (\xi - \sigma) \cdot P(\sigma, \theta + h \cdot \xi)\, d\sigma$; $r \cdot (\xi - \sigma)$ is the LI-relief payment;

$$W(\xi) = \int_{0}^{\xi} (\sigma + r \cdot (\xi - \sigma) - \phi) \cdot P(\sigma, \theta + h \cdot \xi)\, d\sigma + \int_{\xi}^{\infty} (\sigma - \phi) \cdot P(\sigma, \theta + h \cdot \xi) \ ;$$

$0 < r < 1$

In the left- and right-wing political bargaining, the choice of ξ, in general, is also determined by the ability to maintain the average income $a(\theta + h \cdot \xi)$, in order to uphold $a(\theta + h \cdot \xi) > W(\xi)$ within the "striking" distance from $W(\xi)$, which can be ensured through proper choice of the personal allowance constant $\phi > 0$, where ϕ identifies a flat tax bracket $[\phi, \infty)$. The average $a(\theta + h \cdot \xi)$ of income σ over the density sample $P(\sigma, \theta + h \cdot \xi)$ equals $\int_0^{\infty} \sigma \cdot P(\sigma, \theta + h \cdot \xi) d\sigma$.

The taxation of the total income $\sigma + r \cdot (\xi - \sigma)$ of the needy complies with the rules and norms in force, while the h-factor reveals the inverse working incentives, namely the feedback of the welfare recipients.

At this point, it is useful to verify that a disagreement policy δ under the primacy of equity principle of breakdown might be an outcome of the game. There is no reason to assume that the equation

$$(\delta - \phi) \cdot (B(\delta) + d_2) - (\delta - d_1) \cdot W(\delta) = 0,$$

in accordance with Observation 3, should have a solution in general. However, for the income density $P(\sigma, \theta + h \cdot \xi)$ (see above), a solution can be found. Given payoffs $\langle u, g \rangle$ at the endpoints $\langle u_1 = 6.44, g_1 = 47.18 \rangle$, $\langle u_2 = 89.26, g_2 = -2.49 \rangle$ of the scope of negotiations — within the interval $[\xi_1 = 8.00, \xi_2 = 144.54]$ — it can be shown that the pair $d = \langle d_1 = u_1, d_2 = g_2 \rangle = \langle 6.44, \ -2.49 \rangle$, $u_1 < u_2$, $g_1 > g_2$ consolidates an equity for breakdown policy $\delta = 6.39 \notin [\xi_1, \xi_2]$; wealth $W^* = 120.46$ and tax $\tau^* = -2.06\%$.

It should not be surprising that the amounts of public goods and tax rates may be negative. Ensuring this game outcome, the interpretation suggests that the simulated breakdown demonstrates a specific payoff deficit on public goods when it is impossible to cover all the costs through taxes. In such a scenario, as we have pointed out earlier, when discussing negotiations breakdown, it is necessary to resort to an external loan, money printing, or use of natural resources, if the latter are available.

The magnitude and dimension of poverty proposals to be debated or implemented, as *outcomes of the left- and right-wing political bargaining,* are given in Table 1.

Recall already known proposals for incomes η, λ_1, λ, λ_2, δ, whereby δ is outside of the scope of negotiations, $\delta \notin \left[\, \xi_1, \xi_2 \right]$ and the poverty proposal $\frac{1}{2}\mu$, with their definitions given as follows:

η	the policy on poverty with equals left- and right-wing political power; the left- and right-wing political organizations are in symmetrical positions or in equal roles;
λ_1	the outcome of the alternating-offers game — representing what the right-wing politicians accept;
λ	the policy on poverty minimizing wealth-tax;
$\frac{1}{2}\mu$	½ of the median income, indicating that half of the population earns income above μ, while the income of the remaining half is below μ;
λ_2	the outcome of the alternating-offers game — representing what the left-wing politicians accept;
δ	the least desirable outcome, resulting in the policy breakdown or disagreement, which naturalizes the risk of negotiations' premature collapse, caused, for instance, by mutual traps.

A3. Verification

Proof of observation 1. Let us now assume an inverse scenario, whereby $u > u' = \pi(\xi, \tau(\xi, x))$. Here, the left-wing politicians — LWP — aim *to improve* the poverty line residue u', i.e., an after-tax residue of a marginal citizen $\sigma = \xi$ with income equal to the poverty line ξ. By initiating a new rule for policy $\xi' > \xi$, the LWP attempt to implement $u > u'$. Because of the inequalities $u \geq \pi(\sigma, \tau(\xi, x)) > u'$, for some highly pragmatic benefit claimants σ, it becomes apparent that they can be *better off* by *claiming relief payments*. Consequently, actions of these claimants will *increase* the expenditure $B(\xi') > B(\xi)$ on the relief payments and shift the balance of books $B(\xi) = x \cdot \tau(\xi, x) \cdot W(\xi)$ toward *deficit* $B(\xi') > x \cdot \tau(\xi, x) \cdot W(\xi)$. The balance was valid in the past, when $\tau(\xi, x) \equiv \dfrac{B(\xi)}{x \cdot W(\xi)}$. Thus, the only option that would ensure that the balance in maintained, as the LWP must stay committed to x, is to adjust $\tau(\xi, x)$ to $\tau(\xi, \xi', x) = \dfrac{B(\xi')}{x \cdot W(\xi)} >$

$> \tau(\xi, x)$, as x was fixed by the agreement. Otherwise, keeping the old policy ξ intact, the LWP could — through a *decrease* in x — violate the commitment x. As LWP cannot directly change x, they resort to reducing the *deficit* via a tax increase. If $u > \pi(\xi', \tau(\xi, \xi', x))$, the LWP must continue with the tax adjustment policy by $\tau(\xi', \xi'', x) > \tau(\xi, \xi', x)$, now adjusting upon the welfare policy ξ' and proposing $\xi'' > \xi'$, whereby the new *deficit* becomes $B(\xi'') > x \cdot \tau(\xi, \xi', x) \cdot W(\xi')$. These *improvements* $u > u'' > u'$ initiate a sequence of poverty policies $(..., \xi'' > \xi' > \xi, ...)$ and after-tax residues $(..., u > u'' > u', ...)$ of marginal citizens. Thus, the conditions $u = u''$ and $\xi = \xi''$ can never be met, as this would contradict the assumption that the equation $u = \pi(\xi, \tau(\xi, x))$ cannot be solved for ξ. For this reason, the sequence $..., \xi'' > \xi', ...$ is infinite. ■

The chain of reasoning regarding $u' > u$ is similar to that outlined above and is presented as a set of instructions. It should first be noted that, at low values $u' > u'' > u$, even when taxes are low, there would always be a surplus to finance the LI benefits and relief payments. The surplus masks a contradiction, since it is clear that, at low values of the after-tax residue parameter u, benefits financing can always be balanced.

Replace	*to implement* *an improved*	by	*to make a decline in*
–	*better off*	–	*worse off*
–	Improve *improvement*	–	Decline *deterioration*
–	*to claim for* *relief payments*	–	*that relief payments* *have been revoked*
–	*deficit*	–	*surplus*
–	$\geq, >$	–	$\leq, <$
Transpose:	*an increase*	with	*a decrease*

In what follows, we investigate the payoffs $\langle u, g \rangle \in \boldsymbol{S}_b$ of the left- and right-wing politicians. The consensus occurs at outcomes $\phi, \xi \Rightarrow z, x, \alpha, \tau, \langle u, g \rangle$ under the constraint that the variation in policy ξ does not improve the position of the left-wing politicians; rather, the policy emerges as the point on the contract curve $\boldsymbol{S}_b = u(g)$ as fiscally idempotent outcome.

For fiscally idempotent outcomes, the arguments of after-tax residue u, share x, policy ξ, and tax rate τ depend on each other. The share $\mathrm{x} = \mathrm{x}^{\mathrm{o}}$, if settled as eventual agreement, redirects the residue $\mathrm{u} = \pi(\xi, \tau(\xi, \mathrm{x}^{\mathrm{o}})$ to become a function $\mathrm{u} = \mathrm{u}(\xi, \mathrm{x}^{\mathrm{o}})$. Thus, the peak policy u with regard to the best welfare policy can be expressed as:

$$\xi^{\mathrm{o}} = \arg\ \max_{\xi}\ \mathrm{u}(\xi, \mathrm{x}^{\mathrm{o}}) \qquad \text{(A1)}$$

Lemma. *Let us assume that left-wing politicians do not shift from the share* $\mathrm{x} = \mathrm{x}^{\mathrm{o}}$ *and that the volatility constraint* (4) *solves for two different policies* $\xi_1 < \xi_2$. *Let the tax sacrifice* $\mathrm{t}(\xi, \mathrm{x}^{\mathrm{o}}) = \tau(\xi, \mathrm{x}^{\mathrm{o}}) \cdot (\xi - \phi)$ *be a differentiable function of* ξ *progressively increasing with* ξ *within the closed interval* $[\xi_1, \xi_2]$ *— namely, the following derivatives hold:*

$$\left.\frac{\partial}{\partial \xi}\mathrm{t}(\xi, \mathrm{x}^{\mathrm{o}})\right|_{\xi=\xi_1} > 0,\ \left.\frac{\partial}{\partial \xi}\mathrm{t}(\xi, \mathrm{x}^{\mathrm{o}})\right|_{\xi=\xi_2} < 0 \text{ and } \frac{\partial^2}{\partial \xi^2}\mathrm{t}(\xi, \mathrm{x}^{\mathrm{o}}) > 0.$$

In such situation, the poverty line residue $\mathrm{u}(\xi, \mathrm{x}^{\mathrm{o}}) = \xi - \mathrm{t}(\xi, \mathrm{x}^{\mathrm{o}})$ *is a single* $\cap$*-peaked function of* ξ.

Corollary. *There exists a unique interior policy* ξ^{o} *maximizing* u *at*

$$\left.\frac{\partial}{\partial \xi}\mathrm{u}(\xi, \mathrm{x}^{\mathrm{o}})\right|_{\xi=\xi^{\mathrm{o}}} = 0.$$

Provided that the conditions of the lemma are fulfilled, the discussion that follows concerns the necessary and sufficient conditions for the fiscally idempotent policy ξ to occur at the contract curve.

Observation 2. *Let us assume that the volatility constraint* (4) *is differentiable from its arguments. The after-tax residue* $\mathrm{u} = \mathrm{u}(\xi, \mathrm{x}^{\mathrm{o}})$ *is differentiable and single peaked with respect to the policy* ξ *within some closed interval* $[\xi_1, \xi_2]$. *For a fiscally idempotent outcome* $\phi, \xi^{\mathrm{o}} \Rightarrow \mathrm{z}^{\mathrm{o}}, \mathrm{x}^{\mathrm{o}}, \alpha, \tau^{\mathrm{o}}, \langle \mathrm{u}^{\mathrm{o}}, \mathrm{g}^{\mathrm{o}} \rangle$ *to occur on the contract curve* $\boldsymbol{S}_{\mathrm{b}} = \mathrm{u}(\mathrm{g})$, *it is necessary and sufficient that the policy* ξ^{o} *solves the set of equations:*

(i) $$\frac{\partial}{\partial \xi} L(\xi, x^{o}, u^{o})\bigg|_{\xi=\xi^{o}} = 0, \text{ where } u^{o} = u(\xi^{o}, x^{o})$$

provided that

(ii) $$\frac{\partial}{\partial u} L(\xi^{o}, x^{o}, u)\bigg|_{u=u^{o}} \neq 0.$$

Necessity. Let the fiscally idempotent outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$ on the contract curve $\mathcal{S}_{b} = u(g)$ maximize (A1) at $u^{o} = u(\xi^{o}, \tau(\xi^{o}, x^{o}))$. Varying ξ in the vicinity of ξ^{o} of the outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$ and substituting $u = u(\xi, \tau(\xi, x^{o}))$ into the volatility constraint (4), we obtain an identity $L(\xi, x^{o}, \pi(\xi, \tau(\xi, x^{o}))) \equiv 0$. Within the proximity of (ξ^{o}, u^{o}), the following equation holds for arguments ξ, u:

$$\frac{\partial}{\partial \xi} L(\xi, x^{o}, u^{o}) + \frac{\partial}{\partial u} L(\xi^{o}, x^{o}, u) \cdot \frac{\partial}{\partial \xi} \pi(\xi, \tau(\xi, x^{o})) = 0, \qquad \text{(A2)}$$

from which we deduce the necessity statement for $\xi = \xi^{o}$ and $u = u^{o}$.

Sufficiency. Suppose the condition **(ii)** holds. Let **(i)** solve for ξ^{o} at the fiscally idempotent outcome $\phi, \xi^{o} \Rightarrow z^{o}, x^{o}, \alpha, \tau^{o}, \langle u^{o}, g^{o} \rangle$. Combining **(i)** and (A2), we conclude that

$$\frac{\partial}{\partial \xi} \pi(\xi, \tau(\xi, x^{o}))\bigg|_{\xi=\xi^{o}} = 0.$$

The sufficiency clause (A1) holds, since $u = u(\xi, x^{o})$ is a convex function of ξ.■

Proof of Observation 3. The clause is correct, provided that there exists a fiscally idempotent policy δ for the implementation of the pair $\langle d_1, d_2 \rangle$.

In order to identify such a policy, we first replace the variable g with d_2 in the expression for the constraint (1). Next, we extract the expression for $\tau = \frac{B(\delta) + d_2}{W(\delta)}$ from (1) and substitute it into $(1 - \tau)\ldots$ of the constraint (3), where u should be replaced by d_1 in advance. By simplifying, we arrive at the statement of the observation.■

Sketch of the proof (Observation 5). Looking at the tax rate $\tau > \tau_{\min}$, for any outcome $\ldots, \tau, \langle u, g \rangle \in \mathcal{S}_b$, one may indeed prefer a counter outcome as a motion $\ldots, \tau, \langle u', g' \rangle$, which outlines $\ldots, \tau, \langle u' > u, g' < g \rangle$ or $\ldots, \tau, \langle u' < u, g' > g \rangle$. As the contract curve $\mathcal{S}_b = u(g)$ is a curve of efficient preferences $\langle u, g \rangle$ guaranteeing the poverty line residue $u(g)$, someone could put a motion $u' > u^{\circ}$ or $g' > g^{\circ}$ against an outcome $\ldots, \tau > \tau_{\min}, \langle u^{\circ}, g^{\circ} \rangle$. We argue that, in order to fulfill the expectations and requests of citizens' majority, it is necessary to pursue political consent via the proposal $\ldots, \tau_{\min} = \tau(\lambda), \langle u^{\circ} = u(\lambda), g^{\circ} = g(\lambda) \rangle$.■

$\tau \cdot W(\xi) = B(\xi) + g$	**Delivery constraint**: the size of the welfare pie, i.e., the average amount of tax returns is equal to the sum of the average monetary value per capita of primary goods and the average of non-primary goods g .
$B(\xi) = x \cdot \tau \cdot W(\xi)$	**Budget constraint** imposed on the relief payments finance in accordance with the share x of the wealth-pie — the tax-revenue.
$u = (1 - \tau) \cdot (\xi - \phi) + \phi$	**Stability constraint** that determines fiscally idempotent policy ξ .
$u = \xi - \tau \cdot (\xi - \phi)$	**After-tax residue constraint**: an alternative form of stability constraint, where u is after-tax position of a marginal citizen with income $\sigma = \xi$, which concedes with the left-wing political aspirations.

A4. Mathematical derivation

Replacing $\tau = \dfrac{B(\xi)}{x \cdot W(\xi)}$ from the budget constraint into the stability constraint, we obtain the volatility constraint (4) as stated:

$$L(\xi, x, u) = (\xi - \phi) \cdot B(\xi) - x \cdot (\xi - u) \cdot W(\xi) = 0$$

that amalgamates budget constraint and after-tax residue. Contract curve (5) is thus given by:

$$D(\xi, x, u) = L'_\xi(\xi, x, u) =$$
$$= \left[(\xi - \phi) \cdot B(\xi) - x \cdot (\xi - u) \cdot W(\xi)\right]'_\xi = 0$$

$$L'_\xi(\xi, x, u) =$$
$$= B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi) - x \cdot W(\xi) - x \cdot (\xi - u) \cdot \dot{W}(\xi) = 0 \cdot$$

The last expression may be rewritten as:

$$D(\xi, x, u) =$$
$$= B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi) - x \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \cdot$$

Extracting $x = \dfrac{(\xi - \phi) \cdot B(\xi)}{(\xi - u) \cdot W(\xi)}$ from the volatility constraint (4), we can substitute variable x into the rewritten expression for $D(\xi, x, u)$. The substitution results in the following expressions:

$$B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi) -$$
$$- \frac{(\xi - \phi) \cdot B(\xi)}{(\xi - u) \cdot W(\xi)} \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \text{, or}$$

$$\frac{\begin{array}{l}\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot (\xi - u) \cdot W(\xi) - \\ - (\xi - \phi) \cdot B(\xi) \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right)\end{array}}{(\xi - u) \cdot W(\xi)} = 0 .$$

Provided that $(\xi - u) > 0$ and $W(\xi) > 0$, we can conclude that the following is true:

$$\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot (\xi - u) \cdot W(\xi) -$$
$$- (\xi - \phi) \cdot B(\xi) \cdot \left(W(\xi) + (\xi - u) \cdot \dot{W}(\xi)\right) = 0 \cdot$$

This allows writing the sub-expression $(\xi - u)$ in the form:

$$\left(\begin{array}{l}\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - \\ - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)\end{array}\right) \cdot (\xi - u) -$$
$$- (\xi - \phi) \cdot B(\xi) \cdot W(\xi) = 0.$$

As a consequence of presenting the sub-expression $(\xi - u)$ in the form given above:

$$\xi - u = \frac{(\xi - \phi) \cdot B(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)} .$$

We observe that

$$u = \xi - \frac{(\xi - \phi) \cdot B(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)}.$$

We can now substitute the tax rate τ from the delivery constraint into the after-tax residue constraint. The result will be $u = \xi - \frac{B(\xi) + g}{W(\xi)} \cdot (\xi - \phi)$.

After replacing the result into the observed u-expression, we obtain:

$$\xi - \frac{B(\xi) + g}{W(\xi)} \cdot (\xi - \phi) =$$

$$= \xi - \frac{(\xi - \phi) \cdot B(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)};$$

$$\frac{B(\xi) + g}{W(\xi)} \cdot (\xi - \phi) =$$

$$= \frac{(\xi - \phi) \cdot B(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)};$$

$$\left(B(\xi) + g\right) \cdot (\xi - \phi) =$$

$$= \frac{(\xi - \phi) \cdot B(\xi) \cdot W(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)};$$

$$B(\xi) + g =$$

$$= \frac{B(\xi) \cdot W(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)};$$

$$g = \frac{B(\xi) \cdot W(\xi) \cdot W(\xi)}{\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi)} - .$$

$$- B(\xi)$$

We can now impose the denominator in the last expression for g on sub-expression for $(\xi - \phi)$, which can be written as:

$$\left(B(\xi) + (\xi - \phi) \cdot \dot{B}(\xi)\right) \cdot W(\xi) - (\xi - \phi) \cdot B(\xi) \cdot \dot{W}(\xi) =$$
$$= B(\xi) \cdot W(\xi) + (\xi - \phi) \cdot \left(\dot{B}(\xi) \cdot W(\xi) - B(\xi) \cdot \dot{W}(\xi)\right).$$

Continuing with the expression for $g(\xi)$, we can replace the denominator transformed above:

$$\begin{aligned} g = \frac{B(\xi)\cdot W(\xi)\cdot W(\xi)}{B(\xi)\cdot W(\xi) + (\xi-\phi)\cdot\left(\dot{B}(\xi)\cdot W(\xi) - B(\xi)\cdot\dot{W}(\xi)\right)} - \;; \\ -B(\xi) \end{aligned}$$

$$g = \frac{\begin{gathered} B(\xi)\cdot W(\xi)\cdot W(\xi) - \\ -B(\xi)\cdot\begin{pmatrix} B(\xi)\cdot W(\xi) + (\xi-\phi)\cdot \\ \cdot\left(\dot{B}(\xi)\cdot W(\xi) - B(\xi)\cdot\dot{W}(\xi)\right) \end{pmatrix} \end{gathered}}{\begin{gathered} B(\xi)\cdot W(\xi) + (\xi-\phi)\cdot \\ \cdot\left(\dot{B}(\xi)\cdot W(\xi) - B(\xi)\cdot\dot{W}(\xi)\right) \end{gathered}}$$

Now, both the nominator and the dominator can be divided by $B(\xi)\cdot W(\xi)$, yielding:

$$g = \frac{W(\xi) - B(\xi)\cdot\dfrac{\begin{pmatrix} B(\xi)\cdot W(\xi) + (\xi-\varphi)\cdot \\ \cdot\left(\dot{B}(\xi)\cdot W(\xi) - B(\xi)\cdot\dot{W}(\xi)\right) \end{pmatrix}}{B(\xi)\cdot W(\xi)}}{\begin{pmatrix} B(\xi)\cdot W(\xi) + (\xi-\varphi)\cdot \\ \cdot\left(\dot{B}(\xi)\cdot W(\xi) - B(\xi)\cdot\dot{W}(\xi)\right) \end{pmatrix}}.$$

Let us define $v(\xi) = 1 + (\xi-\phi)\cdot\left(\dfrac{\dot{B}(\xi)}{B(\xi)} - \dfrac{\dot{W}(\xi)}{W(\xi)}\right)$, as this allows us to evaluate the expression for the right-wing political objective on public but vital goods as:

$$g(\xi) = \frac{W(\xi) - B(\xi)\cdot v(\xi)}{v(\xi)} = \frac{W(\xi)}{v(\xi)} - B(\xi).$$

In accordance with the delivery constraint, the size of the wealth-pie $\tau(\xi)\cdot W(\xi)$ equals $B(\xi) + g(\xi)$. Consequently, the tax rate is given by:

$$\tau(\xi) = \frac{B(\xi) + g(\xi)}{W(\xi)} = \frac{B(\xi) + \left(\dfrac{W(\xi)}{v(\xi)} - B(\xi)\right)}{W(\xi)} = \frac{1}{v(\xi)}.$$

Replacing the $\tau = \dfrac{1}{v(\xi)}$ in the after tax residue $u = \xi - \tau\cdot(\xi-\phi)$, we can finally evaluate the expression for the left-wing political wants on basic goods as: $u(\xi) = \xi - \dfrac{(\xi-\phi)}{v(\xi)}.$

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Financing Dilemma Supporting a Project [1]

Joseph E. Mullat [*]

Abstract. The concept of an intelligent decision-making core for coalition formation becomes increasingly pertinent, especially in contexts where, e.g., diverse *“stake-shareholders”* must collaborate to achieve common goals. Take, for example, a multinational corporation aiming to launch a new product line. The company relies on contributions from various departments, each with its own goals and priorities. The decision core, in this case, must navigate through competing interests, ensuring that all *stakeholders* are motivated to contribute to the project's success. However, in practice, securing the necessary funding often falls short of initial expectations. Despite commitments from stakeholders, financial constraints or unforeseen circumstances can lead to incomplete funding, requiring the decision core to adapt and recalibrate strategies to meet the project's objectives. Thus, the concept of forming coalitions extends beyond theoretical frameworks, finding practical application in complex organizational dynamics where the alignment of interests is crucial for success.

Keywords: coalition; game; contribution; donation; monotonic; project

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[1] This article can be considered as an independent but at the same time as complementary addendum to bounded rationality in decision-making, “Case Study of Fuel Consumption by Vehicles Utilizing the Postulates of Bounded Rationality", Chater VIII.

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1. INTRODUCTION

In multiplayer games, as discussed by Owen (1971, 1982), coalitions emerge when a subset of participants collaborates. Among these coalitions, rational ones hold particular significance, as they promise individual benefits to all involved parties. These benefits are secured regardless of the actions taken by players outside the coalition. This note focuses on examining one of the simplest forms of player-formed coalitions, all of which are noteworthy within the context of bounded rationality. Bounded rationality acknowledges the limitations of human decision-making, which are often influenced by irrational factors.

The class of games under consideration here adheres to an additional monotonic condition, previously explored by Mullat (1979). It's important to clarify that no prior familiarity with this topic is assumed. Nonetheless, the formal theory of monotone systems employed in this note mirrors that outlined by Mullat (1971-1977), differing primarily in interpretation. Specifically, the abstract indices of interconnection among system elements are viewed as donation intentions. This approach allows us to demonstrate, in a specific case, the feasibility of identifying rational coalitions in line with Nash's principle of independence of rejected alternatives (Nash, 1950). To facilitate understanding, let's consider the following simplified scenario.

2. PEDAGOGICAL SCENARIO

Here we are dealing with participants who intend to fund a project being under development through donations. In principle, each participant $j=\overline{1,n}$ is willing to contribute a certain amount p_j supporting the project. In summary, each participant's donation amount p_j might be in accord with distribution defined by an exponential density function:

$$f(x,\beta)=\left|\begin{array}{l}\frac{1}{\beta}\cdot\exp(-{}^{x}\!/\!_{\beta})\ \text{ for }\ x\geq 0,\\ 0\ \text{ for }\ x<0.\end{array}\right. .$$

In support of the project, there is an expectation to gather a specific fund to finance its execution. However, as negotiations unfold regarding the feasibility and merits of the proposed project among like-minded stakeholders, their preferences may undergo a transformation. This dynamic arrange the stage for the emergence of a coalition game following the monotonic game scheme, with its solution encapsulated in the concept of a kernel (Mullat, 1979). The intricacies of aligning the financing interests of the participants manifest in the form of the kernel solution, which comprises a group of participants willing to support the project, albeit perhaps not to the extent initially envisioned, but still within reasonable bounds. This notion of reasonableness delineates the most optimal avenue for financing the project in its finalized iteration. It's worth emphasizing that within this framework, a reasonable scenario is construed as a guaranteed payment commitment, wherein each project participant pledges a contribution towards the anticipated total amount.

We define the credential of participant $j \in H$ as $\pi(j,H) = |H| \cdot p_j$. Thus, it indicates that the total expected payments of all in H will not be less than $F(H) = \min_{j \in H} \pi(j,H)$. The kernel H^* in this scenario will be understood as participants $H^* = \arg\max_{X \subseteq W} F(X)$. The kernel H^* is remarkable in that it guarantees a contribution $F(H^*)$ to the project. Can more participants with lower individual p_j payments intentions fund the project to a greater extent? Such situation is possible, however, such payments cannot be guaranteed – this is the point. In what follows, we will focus only on payments guaranteed by project participants belonging to the kernel H^*.

The global maximum for project funding by the kernel participants serves as the cornerstone of independence, aligning with the hypothesis of rejected alternatives. This implies that irrespective of the preferences of participants not included in the kernel, if any express interest in joining the kernel, their inclusion should not significantly impact the funding outcome. However, it's prudent not to place undue trust in these external participants, as their reliability may be questionable. There's a possibility that they could seek to alter their preferences unfavorably towards the project, undermining the stability and integrity of the funding arrangement.

Therefore, our assumption is that the refusal of non-kernel participants to engage in the project will have no bearing on the views and actions of kernel members. This premise aligns with the principle of bounded rationality, specifically the principle of independence from rejected alternatives, as articulated by Nash (1950). Essentially, within our context of project financing, this principle ensures that participants remain steadfast in their decisions despite external developments. Kernel participants will uphold their financing commitments regardless of changes in project conditions or the refusal of certain participants to engage. To provide a more formal framework for this consideration, we can characterize it as the stability property of decisions made by kernel participants, akin to the well-established idempotent principle. Once a decision is reviewed under unchanged commitments and priorities, it remains unaltered and does not necessitate further adjustments, maintaining its original form and validity.

Example. Let we introduce in accord with exponential distribution the preferences p_j, of participants' $W = \{j = \overline{1,n}\}$. We can designate as X, all participants who prefer to participate in the project together with their like-minded people, while $\overline{X}$ prefer to reject the project or have other reasons for participating in the project.

Let we now try to determine the preferences π for the participants j in X, $j \in X$, supposing that their contributions in the project together with others in X be equal to $\pi(j,X) = |X| \cdot p_j$. Obviously, if some participant could not at all find a suitable partner for the project, the intention to contribute will be equal to $\pi(j,\{j\}) = |\{j\}| \cdot p_j$, $|\{j\}| = 1$. Conversely, if all participants contribute to the project and all participants are in an adequate company W, the estimated contribution will be greater and equal to $\pi(j,W) = (|W| = n) \cdot p_j$. If now for any reason a participant $j \in X$ decides to spend the rest of the project development alone, the intention to contribute to all others remaining participants in X, including those to which some like-minded participants $X - \{j\}$ still join, will decrease: $\pi(i, X - \{j\}) \le \pi(i,X)$ for $i \in X - \{j\}$. On the contrary, their intentions to contribute will increase if one $j \notin X$ of the previously single participants decides to join X and become a member of $X + \{j\}$: $\pi(i, X + \{j\}) \ge \pi(i,X)$ for $i \in X$.

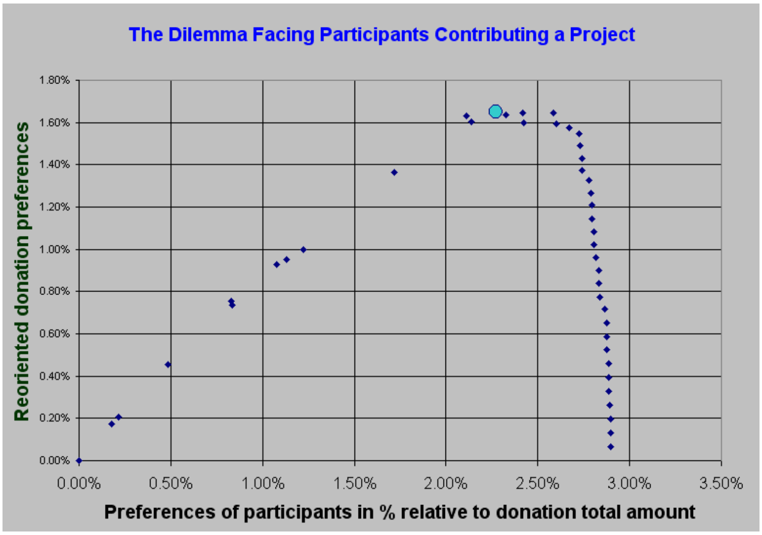

Figure 1. The kernel participants contribute at least 52.8% of their initial intentions to the project. The blue dot is the largest guaranteed contribution in which participants continue to agree to participate in the project.

The graph depicted above illustrates the distribution of participants' donations as a percentage relative to their initial intentions, plotted along the X-axis. Corresponding contributions, both in percentage terms and in relation to the total amount indicated on the Y-axis, reflect the adjustments made to their donation preferences. Through simulation, it becomes evident that kernel members consistently demonstrate a readiness to finance approximately 50% of their initial intentions. This observation underscores the adaptability of kernel participants in aligning their contributions with the evolving dynamics of the project's funding landscape.

To elaborate further, in its initial state, this percentage represents each participant's contribution to the total funding amount required for the project. It serves as a foundational reference point, akin to the starting position, delineating the preferences of participants' donations along the X-axis.

The procedure for finding the kernel is very easy to set up. First, all the expected donation preferences $p_j, j = \overline{1,n}$, are sorted in descending order, constituting the order $\langle p_j \rangle$, the X-axis, and then a sequence $\overline{\pi}$ is constructed as $\overline{\pi} = \langle \pi_j \rangle = \langle p_j \rangle \cdot j$, by which we have denoted these reoriented $\langle \pi_j \rangle$ preferences, the Y-axis. The latter sequence is called defining. We then select the local maximum, i.e., the defining sequence. This is the kernel of Mullat’s monotonic game, which is represented by a blue dot in Figure 1.

I. FINANSEERIMISE DILEMMA PROJEKTI TOETAMISEL [2]

Kokkuvõtte. Koalitsiooni moodustamise intelligentse otsustustuumiku kontseptsioon muutub üha asjakohasemaks, eriti kontekstis, kus näiteks erinevad *”aktsionärid-sidusrühmad”* peavad ühiste eesmärkide saavutamiseks koostööd tegema. Võtame näiteks rahvusvahelise ettevõtte, mille eesmärk on tuua turule uus tootesari. Ettevõte tugineb erinevate osakondade panustele, millest igaühel on oma eesmärgid ja prioriteedid. Sel juhul peab otsustamise tuum liikuma läbi konkureerivate huvide, tagades, et kõik *sidusrühmad* on motiveeritud projekti edusse panustama. Praktikas jääb aga vajaliku rahastuse tagamine sageli esialgsetele ootustele alla. Vaatamata sidusrühmade võetud kohustustele võivad rahalised piirangud või ettenägematud asjaolud viia mittetäieliku rahastamiseni, mis nõuab strateegiate kohandamist ja ümberkalibreerimist projekti eesmärkide saavutamiseks. Seega ulatub koalitsioonide moodustamise kontseptsioon teoreetilistest raamidest kaugemale, leides praktilist rakendust keerulises organisatsiooni dünaamikas, kus huvide kooskõla on edu saavutamiseks ülioluline.

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[2] Seda artiklit võib pidada iseseisvaks, kuid samal ajal täiendavaks lisandiks artiklile piiratud ratsionaalsuse kohta otsuste tegemisel „Case Study of Fuel Conumption by Vehicles Utilites of Bounded Rationality Postulates of Bounded Rationality“, VIII peatükk.

Mitme-isiku mängudes (Owen 1971, 1982) moodustatakse koalitsioon osalejate alamrühmast. Kõigist koalitsioonidest pakuvad ratsionaalsed koalitsioonid eriti huvi, kuna need võimaldavad kõigil osalejatel saada individuaalseid eeliseid. Veel võib täpsustada, et selle hüvitise saamine tagatakse sõltumata mängijate tegevusest, kes ei ole koalitsiooni liikmed. Sõnumis käsitleme mängijate poolt moodustatud koalitsioonide ühte kõige lihtsamat juhtumit, mida võib pidada piiratud ratsionnaalsuse mõttes silmapaistvateks. Ratsionaalsus on piiratud sellega, et inimeste ratsionaalset otsustamist piirab inimeste irratsionaalne olemus.

Pakutud mängude klassile rakendatakse täiendavat monotoonset seisundit, mida on uuritud Mullati poolt (1979) monotoonses mängus ja varasemastes töödes. Tuleb märkida, et siin käsitletud teema eelteadmisi ei nõua. Kasutatud monotoonsete süsteemide teooria on identne sellega, mida Mullat (1971–1977) on varem kirjeldanud; ainus erinevus ilmneb tõlgendamises ja puudutab süsteemielementide abstraktseid sidumisnäitajaid, mida käsitletakse annetuste kavatsustena. Välja töötatud lähenemisviis võimaldab meil ühel konkreetsel juhul esiletuua lihtsa metoodika ratsionaalsete koalitsioonide leidmiseks, mis on kooskõlas (Nash, 1950) tagasilükatud alternatiivide sõltumatuse põhimõttega. Lihtsuse huvides järgmine pedagoogiline stsenaarium võib aga olla informatiivne.

II. PEDAGOGIKA

Siin on tegemist osalejatega, kes kavatsevad arendusjärgus olevat projekti rahastada annetuste kaudu. Põhimõtteliselt on iga osaleja $j=\overline{1,n}$ nõus projekti toetamiseks teatud summa p_j panustama. Kokkuvõtlikult võib iga osaleja annetussumma p_j olla kooskõlas jaotusega, mis on määratletud eksponentsiaalse tiheduse funktsiooniga:

$$f(x,\beta)=\left|\begin{array}{l}\frac{1}{\beta}\cdot\exp(-x/\beta)\ \text{for}\ x\geq 0,\\ 0\ \text{for}\ x<0.\end{array}\right.$$

Seetõttu loodetakse hankida projekti rahastamiseks teatud fond. Läbirääkimised mõttekaaslastega kavandatava projekti sobivuse üle viivad aga nende viimaste eelistused ümbersuunamiseks. Eeldatakse, et siin tekib vastavalt monotoonsele mänguskeemile teatud koalitsioonimäng, mille lahendab tuuma kontseptsioon (Mullat, 1979). Tuum on osalejate mõnevõrra tähelepanuväärne alamhulk.

Nagu juba ööldud on osalejate finantseerimishuvide keerukus esitatud lahenduse vormis, mida nimetatakse tuumaks, mis moodustab teatud osalejate rühma, kes nõustuvad projekti rahastama, kuid võib-olla mitte sellises mahus, nagu need algselt olid mõeldud, kuid siiski mõistlikkuse piires. Tegelikult on see mõistlik piir mis on parim tulemus projekti lõppfinantseerimisvõimaluste rahastamisel. Siinkohal tuleb märkida, et garanteeritud stsenaariumi all mõeldakse teatud garanteeritud makset, mille puhul iga projektis osaleja garanteerib oma panuse eeldatavasse kogusummasse.

Määratleme osaleja $j \in H$ mandaadi kui $\pi(j,H) = |H| \cdot p_j$. Seega näitab see, et kõigi sissemaksete eeldatav kogusumma ei ole väiksem kui $F(H) = \min_{j \in H} \pi(j,H)$. Selle stsenaariumi tuuma all mõistetakse osalejaid H^*. Tuum on tähelepanuväärne selle poolest, et see tagab projekti panuse $F(H^*)$. Kas väiksemate individuaalsete maksete kavatsustega p_j osalejad saavad projekti suuremal kui $F(H^*)$ määral rahastada? Selline olukord on võimalik, aga selliseid makseid garanteerida ei saa – see on asja mõte. Järgnevalt keskendume ainult nendele maksetele, mille tagavad tuuma H^* kuuluvad projektis osalejad.

Tuuma poolt projektile eraldatav globaalse maksimumi kogurahastus moodustab sõltumatuse aluse vastavalt juba nn tagasilükatud alternatiivide hüpoteesile, st sõltumata tuuma mittekuuluvate osalejate eelistustest, kui neid leidub, mis peavad tuumas osalemist siiski asjakohaseks. Kuid me ei tohiks eriti neid uskuda, kuna need ei ole väga usaldusväärsed ja võib-olla soovivad nad oma eelistusi projektis osalemise kohta muuta.

Seetõttu eeldame, et kui tuuma mittekuuluvad osalejad keelduvad projektis osalemast, siis ei mõjuta see neid kes kuuluvad tuuma, st tuumaliikmete vaateid ja nende tegevusi. Siin on tegemist nagu juba ööldud, nn piiratud ratsionaalsuse põhimõttega, see tähendab sõltumatuse põhimõttega tagasilükatud alternatiividest, vt Nash 1950. Sisuliselt tagab see põhimõte meie projekti rahastamise puhul, et projektis osalejad oleksid läbirääkimiste arengutega kursis. Tuuma osalejad ei muuda oma rahastamisotsuseid olenemata sellest, mis toimub või mis muudavad projektis osalemise tingimusi, hoolimata asjaolust, et mõned projektis osalejad keeldusid osalemast. Kui anname sellele viimasele kaalutlusele mõnevõrra formaalsema iseloomu, siis võime öelda, et tuumast osavõtjate tehtud otsuste stabiilsuse omadus pole midagi muud kui tuntud idempotentsuse põhimõte. Pärast otsuse läbivaatamist tingimustes, kus võetud kohustused ja prioriteedid jäävad muutumatuks, ei vaja see uusi muudatusi ning see otsus tehakse samal kujul, nagu see varem vastu võeti.

Näide. Tutvustame vastavalt eksponentsiaalsele jaotusele osalejate $W = \{j = \overline{1,n}\}$ eelistusi p_j, $j = \overline{1,n}$. Võime X-na tähistada kõiki osalejaid, kes eelistavad projektis osaleda, et koos oma mõttekaaslastega kokku leppida, samal ajal kui $\overline{X}$-s olevad osalejad eelistavad projekti tagasi lükata või on neil muud põhjused projektis osalemiseks.

Proovime nüüd määrata kindlaks X -s osalejate $j \in X$ eelistused, eeldades, et nende panus projekti koos teistega X -s on võrdne $\pi(j,X) = |X| \cdot p_j$. Ilmselt kui mõni osaleja ei suuda üldse projekti jaoks sobivat partnerit leida, on kaastöö tegemise kavatsus võrdne $\pi(j,\{j\}) = |\{j\}| = 1 \cdot p_j$ -ga. Ja vastupidi, kui kõik osalejad panustavad projekti ja kõik osalejad on sobivas mõttekaaslaste seas W, on nende viimaste eeldatav panus suurem ja võrdne $\pi(j,W) = (|W| = n) \cdot p_j$ -iga. Kui nüüd mõni osaleja $j \in X$ soovib või otsustab mingil põhjusel veeta ülejäänud projekti arenduse üksi, väheneb kavatsus panustama kõigile teistele X -is allesjäänud osalejatele, sealhulgas ka neile, kellega mõned mõttekaaslased X -ga endiselt liituvad: $i \in X - \{j\}$, $\pi(i, X - \{j\}) \le \pi(i,X)$. Vastupidi, nende panustamiskavatsused suurenevad, kui üks varem osalenud üksikliikmeline $j \notin X$ osaleja otsustab liituda X -iga ja saada $X + \{j\}$: $i \in X$ liikmeks: $\pi(i, X + \{j\}) \ge \pi(i,X)$.

Ülaloleval joonisel, Figure 1, on näidatud osalejate annetused protsentides, võrreldes nende esialgsete kavatsuste suhtes kogusumma panusena X-teljel koos vastava sissemaksega protsentides, samuti sama summa kohta, mis on näidatud Y-teljel, kus nende annetuseelistused olid ümber orienteeritud. Nagu simulatsioon näitab, on tuuma liikmed peaaegu alati valmis finantseerima umbes. 50% nende algsest kavatsusest. Kui täpsem olla, siis algseisundis on projekti finantseerimise kogusummast tehtud panuse protsent, mis peegeldab osalejate eelistuste lähtepunkti X-teljel — osalejate annetuste esitamine.

Tuuma H^* leidmise protseduuri on väga lihtne üles ehitada. Esiteks järjestatakse kõik arvud p_j, $j = \overline{1,n}$, langevas järjekorras, muutes järjestust p_j järjestuseks $\langle p_j \rangle$, ja seejärel konstrueeritakse järgmiste arvude jada, mida me nagu eelpool juba neid arvu tähistanud olime $\overline{\pi}$ -ks: $\overline{\pi} = \langle \pi_j \rangle = \langle p_j \rangle \cdot j$ mis on Joonise 1 Y-teljel, nn osalejate panuste ümberorienteerimine. Seda jada nimetatakse määravaks jadaks. Seejärel valime selle viimase, järjestatud, st määratud jada põhjal, lokaalset maksimumi. See ongi monotoonse mängu Mullat'i tuum, mis on Joonisel 1 tähistatud sinise punktina.

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Equilibrium in a Retail Chain with Transaction Costs: Rational Coalitions in Monotonic Games [*]

Abstract. In a real-world scenario, we consider a regional retail chain comprising suppliers, agents, and distributors operating in the grocery industry. Due to various factors such as fuel price hikes and regulatory changes, transaction costs within the chain begin to rise. As a consequence, the coordination of orders and deliveries among the chain's entities becomes crucial to maintain cost efficiency. Amidst these challenges, the retail chain adapts by forming tighter collaborations and optimizing its logistics network to minimize the impact of increasing transaction costs. For instance, suppliers might consolidate deliveries to reduce transportation expenses, while distributors streamline their inventory management systems to avoid stock-outs and excess inventory costs. Despite the evolving landscape, the key players within the chain strive to uphold a balance where the profitability of each transaction outweighs the associated costs, fostering a resilient ecosystem. This dynamic mirrors the concept of a monotonous game, wherein participants abide by established rules and strategies to navigate the changing market conditions while ensuring their individual and collective sustainability. Moreover, the formal scheme of coalition formation described in the context of this retail chain sheds light on how strategic alliances can enhance resilience, with certain coalitions possessing inherent advantages that bolster their ability to withstand market volatility.

Keywords: suppliers; distributors; monotonic game; retail chain; coalition.

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[*] A part of this article was translated from Avtomatica i Telemekhanika, 1980, 12, pp. 124 – 131. Original article submitted 1979. Automat. and Remote Control, Plenum Publishing Corporation, 1981, pp. 1724-1729. Russian version.

> *Businessmen in deciding on their ways of doing business and on what to produce have to take into account transaction costs. If the cost of making an exchange are greater than the gains which that exchange would bring, that exchange would not take place and the greater production that would flow from specialization would not be realized. In this way transaction costs affect not only contractual arrangements, but also what goods and services are produced.* Ronald H. Coase, "The Institutional Structure of Production," Ménard, C., and M. M. Shirley (eds.) (2005), *Handbook of New Institutional Economics,* Spriner: Dordrecht, Berlin, Heidelberg, New York. XIII. 884pp., p.35, ISBN 1-4020-2687-0.

1. INTRODUCTION

Everybody, probably knows that prices on commodity markets sometimes continue to rise unabated on the back of an anticipated shortage in the global raw materials availability and sharp volatility in the commodity future markets and terminal prices on fears of an immediate shortage of materials in the short term. Along with the significant increase in commodity prices, on one hand, the transaction costs increase on inputs like petroleum, electricity, etc. On the other, while currency of exchange rates also moving adversely, the situation becomes uncertain. As an example, one may point at recent market price increase of coffee raw materials, which did not have immediate consequences for some known positions, while the distributors [1] of a retail chain, however, demonstrate readiness to make loosing transactions. With this in mind, distributors are trying to hold prices constant. However, it is also understandable that it would be impossible for the distributor to make frequent price changes again and again. Given the current context, they will have no other option but to seek price increase for distributed commodities with an immediate effect.

The volatility inherent in commodity market prices often triggers a chain reaction, culminating in amplified transaction costs throughout the retail distribution network. This escalation of costs perpetuates a cycle of uncertainty, exacerbating the challenges faced by distributors. As transaction expenses mount, stakeholders find themselves ensnared in a self-perpetuating loop of price hikes, potentially stalling bilateral trade and necessitating a recalibration of the market's supply and demand equilibrium.

In such an environment characterized by persistent price escalation, the synchronization between orders and deliveries becomes increasingly elusive within the established supply chain framework. Despite these adversities, participants within the retail chain are driven by the rational pursuit of profit maximization, prompting them to explore novel strategies for restructuring operations. In essence, the interplay between market price uncertainties and transaction costs underscores the dynamic nature of retail distribution channels, prompting continual adaptation and innovation among industry participants.

It is worth noting that within the realm of market transactions, New Institutional Economics offers valuable insights, particularly in two key directions. Firstly, vertical integration, as expounded upon by Joskow (2005), delineates a market structure characterized by the interlinking of semi-product components

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[1] A group of retail outlets owned by one firm and spread nationwide or worldwide.

in a vertical chain. Secondly, the concept of horizontal outsourcing emerges, where companies leverage external services and products to streamline their operations and meet end-product requirements.

This paper addresses the above situation in question by setting up a retail chain game of the participants in the chain grounding on supposition that orders and deliveries be met with uncertainty of transaction costs. In so doing, the paper attempts to develop a numerical description of the supply and demand structure for the deliveries of commodities in the retail chain. The allegedly rational behavior of a participant is not always such, because the participants on purpose may attempt to enter but irrationally into certain losing transactions in hope to offset the negative effect of the former. Given this irrational situation the prices will increase additionally upon already profitable transactions. Numerical analysis of irrational situations reveals, however, that in case the participants will try to avoid all losing transactions, their behavior is once again becoming rational and in such situations the participants of the retail chain will end up in the Nash equilibrium (1953).

To our knowledge (or lack of that), the retail chain formation, or in mundane terms the restructuring process of the retail chain is rather complicated mathematical problem, which do not have satisfactory solutions. However, in recent years it has become clear that a mathematical structure known as antimatroid is well suited for such type a retail chain formation process (cf. Algaba, et al, 2004). Antimatroid is a collection of potential interests groups — subsets of participants, i.e., those who make decisions to buy and sale in bilateral trade transactions. That is to say, within antimatroid one will always find a path of transactions connecting members of the retail chain — if the latter forms of course — with each other by mutual business interests inside groups/coalitions belonging to antimatroid and making the exchange as participants of a characteristic retail chain.

We step up beyond convention of the theory of coalition games that the solution mandatory has to be a core, and take the retail chain formation process in terms of so-called *defining sequence* of transactions (Mullat, 1979). The sequence facilitates the retail chain formation as a transformation process of nested sets of bilateral transactions, which ends at its last and highest costs' threshold — the most tolerant retail chain towards costs — a kernel. Hereby, the kernel operates as a retail chain of participants capable to cover the highest transaction costs in case of uncertainty. In our case, the *defining sequence* of transactions produces the elements of an antimatroid — some interest groups, cf. Levit and Kempner, (2001); see also (1991) Korte et al. The defining sequence on antimatroid, in particular, follows the Greedy heuristic procedure of Shapley's value, but in inverse order, cf. Rapoport (1985).

Bearing all this in mind, the suggested framework allows performing a series of computer simulations. First, to determine the possible response of the retail chain participants, to different supply and demand structures. Second, to identify the participants, where the executive efforts might be applied to prevent unpredictable actions that may misbalance the equilibrium in the retail chain. With this object, we used a model to assemble an "elasticity" measure for the choice of customers; this measure is represented by transaction costs' interval, for which the retail chain remains in equilibrium.

The rest of this paper is structured as follows. The next section sets up the basic concepts intending to bring at the surface the calculus of utilities of participants in the retail chain. It is a preliminary step necessary to move forward to the Section 3, where the general model of participants of the chain is described. In Section 4, which is main part of the paper, the retail chain game of customers addresses the process of the chain formation in details. Here the monotonic property of utilities plays its major role. In Sections 5-6, we construct different varieties of coalitions of retail network players that are "outstanding" in the sense of rationality, and indicate relations between such coalitions. Also, constructive processes described in Section 7 for discovering these *outstanding players*, described in additional Section 8. A summary of the results ends the study. Appendix contains the proofs of all theorems.

2. DESCRIPTION OF A RETAIL CHAIN: THE SIMPLE FORM

To consider the simplest case of commodities distribution in a retail chain might be instructive. This elementary model is used at current stage solely as a convenient means of simplifying the presentation.

The distribution of commodities in the retail chain is characterized by sales figures that may be expressed as one of the following three alternative numbers: a) a demand η which is disclosed to the particular participant either externally or by other participant in the chain; b) a capable supply ξ calculated at the cost of all commodities produced by the participant for delivery outside the chain or to the other participants; c) actual sales γ calculated at the prices actually paid by the customers for the delivered commodities.

An order is thus defined as a certain quantity of a particular commodity ordered by one of the participant's from another participant in the retail chain; a delivery is similarly defined as a certain quantity of a commodity delivered by one of the participant's to another participant in the chain. We assume that the chain includes suppliers who are only capable of making deliveries – the produces; participants, who both issue orders and make deliveries – the agents; and the distributors, who only order commodities from other participants. [2]

In what follows we consider the retail chain of orders and deliveries for the case like "pipeline" distribution without "closed circuits." Therefore, we can always identify a unique direction of "retail chain" of orders from the distributors to the produces via agents and a "retail chain" of deliveries in the reverse direction.

Let us consider in more detail this particular retail chain of orders and deliveries of commodities. The direction of the chain of orders (deliveries) is defined by assigning serial numbers – the indexes $1,2$ and 3 – to the producer, to the agent, and to the distributor, respectively. The producer and the agent act as suppliers, the agent and the distributor act as customers. The agent thus has the dual role of a supplier and a customer, whereas the producer only acts as a supplier and the distributor only acts as a customer.

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[2] The distributors also act as suppliers to external customers.

The chain of orders to the produces from the customers is characterized by two numbers η_{23} and η_{12}. The number η_{wj} $(w = 1,2; j = 2,3)$ is the demand η_{wj} disclosed by the customer j to the supplier w. We assume that sales are equal to deliveries. Two numbers ξ_{12} and ξ_{23}, which are interpreted as the corresponding capable sales similarly characterize the chain of deliveries to the distributor.

Suppose that the demand of the distributor to the external customers is fixed by d bank notes. The capable sales of the producer are s bank notes. In other words, d is the estimated amount of orders from the external customers and it plays the same role as the number η for the customers in the retail chain. Similarly, s is the intrastate amount of estimated deliveries by the producer, and it has the same role as ξ for the customers.

Let us now consider the exact situation in a chain. To make deliveries at a demand amount of d bank notes, the distributor have to place orders with the agent in the amount of $\eta_{23} = \nu_{23} \cdot d$ bank notes, where ν_{23} are the distributor's cost of commodities sold (the cost per 1 bank note of sales). The agent, having received an order from the distributor, will in turn place an order with the supplier in the amount $\nu_{12} \cdot \eta_{23}$, where ν_{12} is the agent's cost per one bank note of sales. On the other hand, the estimated sales of the producer are ξ_{12} bank notes, $\xi_{12} = s$. Assuming that all the transactions between the suppliers and the customers in the retail chain are materialized in amounts not less than those indicated in the purchase orders, the actual sales of the producer to the agent are given by $\gamma'_{12} = \min\{\xi_{12}, \eta_{12}\}$.

Now, since the agent paid the producer γ'_{12} for the commodities ordered, the agent's revenue is $\xi_{23} = \gamma'_{12}/\nu_{12}$, where clearly $\xi_{23} \geq \gamma'_{12}$. The difference between the revenue ξ_{23} and the costs γ'_{12} is defined as

$$\pi_{12} = \gamma'_{12} \cdot (1 - \nu_{12})/\nu_{12} .$$

From the same considerations, $\gamma'_{23} = \min\{\xi_{23}, \eta_{23}\}$ [3] give the actual sales of the agent to the distributor. We similarly define the difference $\pi_{23} = \gamma'_{23} \cdot (1 - \nu_{23})/\nu_{23}$. The numbers π_{12}, π_{23} represent the profit of the customers in the retail chain.

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[3] In subsequent sections, γ'_{wj} is replaced by $\gamma_{wj} = \gamma'_{wj}/\nu_{wj}$. The numbers γ and γ' differ in the units of measurement of the commodities delivered to the user j. While γ' represents the sales at the cost, γ represents the same sales at actual selling prices.

In conclusion of this section, let us consider the numbers π_{12}, π_{23} more closely. We see from the above discussion that the material costs are the only component of the costs of commodities sold for the customers in the retail chain; no other producing or transaction costs are considered. And yet in Section 4 the numbers π_{12}, π_{23} are used as the admissible bounds on transaction costs, which are assumed to be unknown. It is in this sense we construct a model of a monotonic game of customers (Mullat, 1979, p.6).

3. DESCRIPTION OF A RETAIL CHAIN: THE GENERAL FORM

Consider now a retail chain consisting of n participants indexed w, $j = 1,2,...,n$. The state of a supplier w is characterized by a $(m+1)$-component vector [4] $\langle d_w, y_w \rangle = \langle d_w, \eta_{w,k+1},..., \eta_{w,n} \rangle$, $(n-k=m)$; the state of a customer j by a $(v+1)$-component vector $\langle s_j, x_j \rangle = \langle s_j, \gamma_{1j},..., \gamma_{vj} \rangle$. The components of the $\langle d_w, y_w \rangle$ and $\langle s_j, x_j \rangle$ vectors are interpreted as follows: d_w is the total orders amount of the supplier w acting as a customer; s_j is the capable sales total amount of the customer j acting as a supplier; η_{wj} is the cost of orders placed by the customer j with the supplier w ; γ_{wj} are actual sales (deliveries) to customer j from the supplier w . As indicated in the footnote, γ_{wj} represents the deliveries valued at the selling prices of the customer j acting as a supplier. The vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ are the order and the delivery vectors, respectively.

With each participant in the retail chain we associate certain domains in the nonnegative orthants $\Re^{m+1}$ of the $(m+1)$ – and $\Re^{v+1}$ of the $(v+1)$ – dimensional space. These domains $\Re^{m+1}$ and $\Re^{v+1}$ are the regions of feasible values of vectors $\langle d_w, y_w \rangle$, $\langle s_j, x_j \rangle$ in the $(m+v+2)$ – dimensional space.

For some of the participants vectors with $\gamma_{wj} > 0$ are inadmissible, and for some participants vectors with $\eta_{wj} > 0$ are inadmissible. Participants having the former property will be called produces and those having the latter property will be called distributors; all other participants in the retail chain will be called agents. In what follows the numbers s_w $(w = 1,2,...,k)$ characterize the k produces; the number s_w represents the capable sales controlled by the participant w . The numbers d_j $(j = v+1, v+2,...,n)$ correspondingly characterize the r distributors: the number d_j represents the demand to the external customers $(n-v=r)$.

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[4] k is the number of produces, see below.

Let us now impose certain constrains on the admissible vectors in this retail chain. The following constrains are strictly "local," i.e., they apply to the individual participants in the retail chain.

The admissible retail chain states are constrained by balance conditions equating the actual sales from all the suppliers to a particular customer to capable sales of that customer acting as a supplier:

$$s_j = \sum_{w=1}^{v} \gamma_{wj} \quad (j = k+1, k+2, ..., n). \tag{1}$$

We also require balance conditions between the cost of orders placed by all the customers with a particular supplier and the demand figure of that supplier acing as a customer:

$$d_w = \sum_{j=i+1}^{n} \eta_{wj} \quad (w = 1,2,...,v). \tag{2}$$

As we have noted above, the retail chain considered in this article does not allow "closed-circuit motion" of orders or deliveries until a particular order reaches a producer or the delivery reaches a distributor. The indexes labeling the participants in such chains are ordered in a way [5] that if w is a supplier and j is a customer, then $w < j$ ($w = 1,2,...,v;\quad j = v+1, v+2,...,n$). We call such chains as of a retail-type, and their description requires certain additional assumptions.

Consider the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ satisfying the following constraints ($w < j; j = k+1,...,n$):

$$\sum_{j} \alpha_{wj} \leq 1 \ (j > w; w = 1,2,...,v), \ \sum_{w} \beta_{wj} \leq 1 \tag{3}$$

For the supplier w, the number α_{wj} is the fractional cost of orders made to the customer j. For customer j, the number $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$ is the fractional cost of the deliveries from supplier w, which are necessary for meeting the sales target.

Suppose that purchase of orders in the retail chain move from distributors through agents to suppliers. This chain is conducted at the wholesale prices. The deliveries, also conducted at the wholesale prices of the chain in the opposite direction. We express the effective wholesale prices by a set of constants ν_{wj} ($w = 1,2,...,v; j = k+1, k+2,...,n$), which represent the participant's cost per one bank note of sales for a customer acting as a supplier.

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[5] The term topological sorting originates from Knuth (1972) to describe the ordering of indexes having this property.

The set of constants α_{wj}, β_{wj} and ν_{wj} make it possible to uniquely determine the amount of orders and deliveries in a given transaction. Indeed, the amount of orders to the supplier w from the customer j is given by $\eta_{wj} = \beta_{wj} \cdot d_j \cdot \nu_{wj}$. The relation (see Section 2) determines the amount of deliveries $\gamma'_{wj} = \min\left\{\xi_{wj}, \eta_{wj}\right\}$, where $\xi_{wj} = s_w \cdot \alpha_{wj}$ are the capable sales values at cost prices. Considering the difference in revenue from sales of customer j acting as a supplier, we conclude that the deliveries from the supplier w to the customer j are given by $\gamma_{wj} = \gamma'_{wj} / \nu_{wj}$.

In conclusion, let us consider one computational aspect of order and delivery vectors in a retail-type distribution chain. [6] It is easily seen that the components d_j, s_w, η_{wj} and γ_{wj} $(w = 1,2,...,v;\ j = k+1, k+2,...,n)$ as obtained from (1) and (2) are given by ($w < j; j = k+1,...,n$)

$$d_w = \sum_j \beta_{wj} \cdot d_j \cdot \nu_{wj} \quad (j > w; w = 1,2,...,v) \tag{4}$$

$$s_j = \sum_{\omega} \min\left\{s_w \cdot \alpha_{wj}; \beta_{wj} \cdot d_j \cdot \nu_{wj}\right\} / \nu_{wj} \tag{5}$$

The input data in (4) is the demand of the distributors to external customers, i.e., the numbers $d_{v+1}, d_{v+2},..., d_n$. The input data in (5) are the capable sales amounts $s_1, s_2,..., s_k$ of the produces, which together with the numbers $d_1, d_2,..., d_v$ from (4) are used in (5) to compute the actual sales of customers.

4. A MONOTONIC GAME OF CUSTOMERS IN THE RETAIL CHAIN

In the previous section we considered a retail-type distribution in the chain with participants indexed by $w = 1,2,...,v;\ j = k+1, k+2,...,n$: the index j identifiers a customer, the index w identifiers a supplier.

Let us interpret the activity of the retail chain as a monotonic game (Mullat, 1979), in which the customers need to decide from what supplier to order a particular commodity.

Suppose that in addition to the cost of materials, the customers bear uncertain transaction costs in their bilateral trade with suppliers. Because of the uncertainty of transaction costs, it is quite possible that in some transactions the costs will exceed the gross profit from sales. In this case, the potentially feasible transactions will not take place.

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[6] Here we need only consider the principles of the computational procedure.

Let the set R_j represents all the potential transactions corresponding to the set of suppliers from which the customer j is to make his choice. The choice of the customer j ($j = k+1, k+2, ..., n$) is a subset A^j of the set R_j: $A^j \subseteq R_j$; the case $A^q = \varnothing$ is not excluded: it requires the customer's refusal to make a choice. The collection $\langle A^{k+1}, A^{k+2}, ..., A^n \rangle$ represents the customer's joint choice. It is readily seen that the sets R_j are finite and nonintersecting; their union corresponds to set $W = R_{k+1} \bigcup R_{k+1} \bigcup ... \bigcup R_n$.

In what follows, we focus on the criterion by which the customer j chooses his suppliers A^j while the lowest transaction costs, as a threshold u^o, increases. In contrast to the standard monotonic game (Mullat, 1979), which is based on a coalition formation, we will consider the strategy of individual customers whose objective is to maximize the profit from the actual sales revenues. We will thus essentially deal with m players' game, $m = n - k$.

Let us first introduce a measure of the utility of a transaction between customer j and supplier $w \in A^j$ ($j = k+1, k+2, ..., n$). The utility of a transaction between customer j and supplier w is expressed by the corresponding profit $\pi_{wj} = \gamma_{wj} \cdot (1 - \nu_{wj})$.

The utility of a transaction with a supplier $w \in A^j$ is a function $\pi_{wj}(X_{k+1}, X_{k+2}, ..., X_n)$ of many variables: the value of the variable X_j is the choice A^j of the customer j, the number of variables is $m = n - k$. To establish this fact, it is sufficient to show how to compute the components of the order and delivery vectors from the joint choice $\langle X_{k+1}, X_{k+2}, ..., X_n \rangle$. Indeed, according to our description, a retail-type distribution in the chain requires defining the constants $\alpha_{wj} \geq 0$ and $\beta_{wj} \geq 0$ ($w = 1,2,...,v; j = k+1,...,n$) that satisfy the constraints (3). A pair of constants α_{wj} and β_{wj} can be assigned in a one-to-one correspondence to a supplier $w \in R_j$, rewriting (3) in the form

$$\sum_{w \in R_j} \alpha_{wj} \leq 1, (w = 1,2,...,v), \sum_{w \in R_j} \beta_{wj} \leq 1, (j = k+1,...,n) \tag{6}$$

If the constrains (6) are satisfied, then the same constrains are of necessity satisfied on the subsets A^j of the set R_j. Thus, restricting (4) and (5) to the sets $X_j \subseteq R_j$, the numbers γ_{wj} can be uniquely calculated for every joint choice $\langle X_{k+1}, X_{k+2}, ..., X_n \rangle$. Finally, let us define the individual utility criterion of the customer j in the form:

$$\Pi_j = \sum_{w \in A^j} (\pi_{wj} - u_{wj}) \tag{7}$$

where u_{wj} are the customer j transaction costs allocable to the supplier $w \in A^j$; we define $\Pi_j = 0$ if the customer j refused to make a choice — $A^j = \varnothing$. The function $\pi_{wj}(X_{k+1}, X_{k+2}, ..., X_n)$ has the obvious property of monotone utility, so that for every pair of joint choices of customers $\left\langle L^{k+1}, L^{k+2}, ..., L^n \right\rangle$ and $\left\langle G^{k+1}, G^{k+2}, ..., G^n \right\rangle$ such that $L^j \subseteq G^j$ ($j = k+1, ..., n$) we have

$$\pi_{wj}(L^{k+1}, L^{k+2}, ..., L^n) \leq \pi_{wj}(G^{k+1}, G^{k+2}, ..., G^n)\,. \tag{8}$$

The property of monotone utility leads to certain conclusions concerning the behavior of customers depending on the individual utility criterion. Under certain conditions, rational behavior of customer j (i.e., maximization of the profit Π_j) is equivalent to avoid profit-loosing transaction with all the suppliers $w \in A^j$. This aspect is not made explicit in Mullat (1979), although it is quite obvious. Thus, using the lemma, see the English version at p.1473, we can easily show that if the utilities $\pi_{wj}(..., X_j, ...)$ are independent of the choice X_j, the customer j maximizes his profit Π_j by extending his choice to the set-theoretically largest choice. In what follows we will show that this result also applies under a weaker assumptions.

Below we first start with a few reservations about the proposed condition – see (9). This condition has a simple economic meaning: the customer j entering into loosing transactions cannot achieve a net increase in his utility of the losses. For example, if for fixed choices of all other customers in the retail chain, the utilities $\pi_{wj}(..., X_j, ...)$ for $w \in X_j$ are independent of the choice X_j, the condition (9) hold as strict inequalities. These conditions are also reduces to strict inequalities when, for instance, the capable sales ξ_{wj} in each transaction between customer j and supplier $w \in A^j$ is not less than the demand η_{wj} so that every customer can receive the entire quantity ordered from his suppliers. In particular, by increasing the producers' supply $s_1, s_2, ..., s_k$ with unlimited manufacturing capacity, we can always increase the capable sales to such an extent that it exceeds the demand, so that the conditions (9) are satisfied.

We can now formulate the final conclusion: the following lemma suggests that each customer will make his choice so as to maximize the profit Π_j, providing all the other customers keep their choices fixed. [7]

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[7] The joint choice of users having this property is generally interpreted in the sense of Nash equilibrium (1953); see also Owen (1968).

Let the suppliers not entering the set A_j be assigned indexes $q = 1,2,...$. Then the profit Π_j of customer j is represented by a many-variable function $\Pi_j(t_{1j}, t_{2j},...)$ with variables t_{qj} varying on $\left[0,\beta_{qj}\right]$. [8] The value of the function $\Pi_j(t_{1j}, t_{2j},...)$ is the customer's profit for the case when the customer j has extended the choice by placing orders in the amounts of $t_{qj} \cdot d_j \cdot \nu_{qj}$ with the suppliers $q = 1,2,...$ outside the choice A_j. Thus, the customers j who expand their choice A_j, identify the suppliers $q = 1,2,...$ by the set of variables t_{qj}. If all $t_{qj} = 0$, the choice A_j is not expanded and the profit $\Pi_j(0,0,...)$ coincides with (7).

The profit function $\Pi_j(t_{1j}, t_{2j},...)$ thus has to satisfy the following constraint: for every t_{qj} in $\left[0,\beta_{qj}\right]$ $q = 1,2,...$

$$\Pi_j(t_{1j}, t_{2j},...) \leq \Pi_j(0,0,...). \qquad (9)$$

Definition. *A joint choice* $\left\langle A_o^{k+1},..., A_o^n\right\rangle$ *of the retail chain customers is said to be rational with the threshold* u^o *if, given an amount of transaction costs not less than* $u^o > 0$ *, the utility measure* $\pi_{wj} \geq u^o$ *in every transaction of customer* j *with the supplier* $w \in A_o^j$, $j = k+1,...,n$.

Lemma. *The set-theoretically largest choice* $S^o = \left\langle A_o^{k+1},...,A_o^n\right\rangle$ *among all the joint choices rational with threshold* $u^o > 0$ *ensures that the retail-type distribution chain is in equilibrium relative to the individual profit criterion* Π_j *under the following conditions: a) the transaction costs* u_{wj} *for* $w \in S^o$ *do not exceed min* π_{wj} *over* $w \in S^o \bigcap R_j$ *; b) inequality (9) holds.*

Proof. Let S^o be a set-theoretically largest choice among all the joint choices rational with the threshold u^o, i.e., S^o is the largest choice H among all the choices such that $\pi_{wj}(H \bigcap R_{k+1},..., H \bigcap R_n) \geq u^o$. Suppose that some customer *p* achieves a profit higher than Π_p by making the choice $A^p \subseteq R_p$, which is different from $S^o \bigcap R_p$; $\Pi_p' = \sum_{w \in A^p}\left(\pi_{wp}(..., A^p,...) - u_{wp}\right) > \Pi_p$, subject to $u^o \leq u_{wp} \leq \min_{w \in A^p} \pi_{wp}$. Clearly, the choice A^p is not a subset of S^o, since

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[8] We recall that β_{qj} is the fractional cost of all the orders placed with supplier q.

this would contradict the monotone property (8), so that $A^p \setminus S^o \neq \varnothing$. By the same monotone property, the customer making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ will achieve a profit not less than Π'_p. On the other hand, all transactions in $A^p \setminus S^o$ are losing transactions for this customer, since S^o is the set-theoretically largest set of non-losing bilateral trade agreements tolerant towards the transactions costs' threshold $u^o > 0$. For the customer p making the choice $A^p \bigcup \left(S^o \bigcap R_p\right)$ the profit Π'_p does not decrease only if the total increase in utility due to the contribution π_{wp} of the transactions $w \in S^o \bigcap R_p$ exceeds the total negative utility due to the transactions in $A^p \setminus S^o$. Clearly, because of the constraint (9), the customer p has no such an opportunity. This contradiction establishes the truth of the lemma. ■

In conclusion, we would like to consider yet another point. With uncertain transaction costs, the refusal to enter into any transaction may lead to an undesirable "snowballing" of refusals by customers to choose their suppliers. It therefore seems that customers will attempt at least to conclude transactions with $\pi_{wj} \geq u^o$; even when there is some risk that the transaction costs will exceed the utility π_{wj}. Thus, without exaggeration, we may apparently state that the size of the interval $\left[u^o, \min \pi_{wj}\right]$ reflects the elasticity of the customer's choice: the number $\min \pi_{wj} - u^o$ is thus a measure of a "risk" that the customer will get into non-equilibrium situation. Clearly, a customer with a small interval will have greater difficulties to maintain the equilibrium than a customer with a wide interval.

5. RATIONAL COALITIONS IN MONOTONIC GAMES

In many-persons games (Owen, 1971) by a coalition we shall understand a subset of participants. Among all coalitions we usually single out rational coalitions — a participant in such coalition extracts from the interaction in the coalition a benefit, which satisfies him. In addition, sometimes it is further stipulated that extraction of this benefit is ensured independently of the actions of the players not entering into the coalition.

The class of games proposed in this paper is subjected to an additional monotonic condition, which has been studied earlier in Mullat (1976, 1977) (although knowledge of the latter is not presupposed). There is no difference between the formal scheme of the present paper and that of Mullat in essence; the difference involved in interpretation is in abstract indices of interconnection of elements of the system, which are understood as utility indices. The approach developed enables us to establish, in one particular case, the possibility of finding rational coalitions in the state of individual equilibrium according to Nash.

6. FORMAL DEFINITIONS AND CONCEPTS

We consider a set of n players denoted by I. Each player $j \in I$ $(j = \overline{1, n})$ is matched by a set R_j from which the player j can select elements. It is assumed that the sets R_j are finite and do not intersect. Their union forms a set $W = R_1 \cup R_2 \cup ... \cup R_n$. The elements selected by the player j from R_j compose a set $A^j \subseteq R_j$. The set A^j is called the choice of the player j, while the collection $\langle A^1, A^2, ..., A^n \rangle$ is called the joint choice. The case $A^k = \varnothing$ is not excluded and is called the refusal of k-th player from the choice.

We introduce the utility functions of elements $w \in A^j$. We assume that certain joint choice $\langle A^1, A^2, ..., A^n \rangle$ has been carried out. Let there be uniquely determined, with the respect to the result of the choice, a collection of numbers $\pi_w \geq 0$ that are assigned to the elements $w \in A^j, j = 1,2,...,n$; on the remaining elements of W the numbers are not determined. The numbers π_w are called utility indices, or simply utilities, and by definition, are in general case functions $\pi_w(X_1, X_2, ..., X_n)$ of n variables. The value of the variable X_j is the choice A^j of the player j. We shall single out utility functions possessing a special monotonic property.

Definition 1. *A set of utilities* π_w *is called monotonic, if for any pair of joint choices* $\langle L^1, L^2, ..., L^n \rangle$ *and* $\langle G^1, G^2, ..., G^n \rangle$ *such that* $L^j \subseteq G^j$,

$$\pi_w(L^1, L^2, ..., L^n) \leq \pi_w(G^1, G^2, ..., G^n) \qquad (10)$$

is fulfilled for any $w \in L^j$[9]. $j = 1,2,...,n$

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[9] We note that fulfilment of (1) is not required for the element $w \notin L^j$. Furthermore, even the numbers π_w themselves may not be defined for $w \notin L^j$.

We now turn to the problem of coalition formation. We shall call any nonempty subset of the set of players a coalition. Let there be given a coalition V, and let its participants have made their choices. We compose from the choices A^j of the participants of the coalition V a set-theoretic union H, which is called the choice of the coalition V: $H = \bigcup_{j \in V} A^j$.[10]

To determine the degree of suitability of the selection of an element $w \in R_j$ for the player j, a participant of the coalition, we introduce an index of guaranteed utility. With this aim we turn our attention to the dependence of the utility indices on the choice of the players not entering into coalition. It is not difficult to note that as a consequence of the monotonic condition of the functions π_w the worst case for the participants of the coalition will be when all players outside the coalition V reject the choice: $A^k = \varnothing$, $k \notin V$, so that all elements outside H will not be chosen by any of the players who are capable of making their choices. In other words, the guaranteed (the least value) of utility π_w of an element *w* chosen by a player in the case of fixed choices $H \cap R_j$ of his partners in the coalition equals $\pi_w(H \cap R_1, ..., A^j, ..., H \cap R_n)$. The quantity $g_j(H) = \min_{w \in A^j} \pi_w(H \cap R_1, ..., A^j, ..., H \cap R_n)$ is called the guarantee of the participant j in the coalition V for the choice H.

We assume that according to the rules of the game, for each chosen element $w \in A^j$ a player $j \in V$ must make a payment u°. [i] It is obvious that under condition of the payment u° the selection of each element $w \in A^j$ is profitable or at least without loss to the player $j \in V$ if and only if $\pi_w \geq u^\circ$. In the calculation for the worst case this thus reduces to the criterion $g_j(H) \geq u^\circ$. In reality we shall be interested, in relation to the player $j \in V$, in all three possibilities: a) $g_j(H) > u^\circ$, b) $g_j(H) = u^\circ$ and c) $g_j(H) < u^\circ$. We shall say that a participant of the coalition V is above u°, on the level of u°, and below u°, if the conditions a), b), and c) are fulfilled respectively. The size of the payment is further considered as a parameter u of the game being described and is called the threshold. We shall say that a coalition V, having made a choice H, functions on the level $u[H] = \min_{j \in V} g_j(H)$.

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[10] A choice H without indication about the coalition V, which has affected it, is not considered, and if somewhere the symbol V is omitted, then under a coalition we understand a collection of players such and only such for which $H \cap R_j \neq \varnothing$.

Definition 2. *A coalition* V *is called rational with the respect to a threshold* $\mathrm{u}^{\circ}=\mathrm{u}[\mathrm{H}]$ *if for a certain choice* H *all participants of the coalition are not below* u° *while someone in the coalition* $\mathrm{k}\cup\mathrm{V}$ *is below* u° *if any participant* $\mathrm{k}\not\in\mathrm{V}$ *outside the coalition* V *makes a nonempty choice* $\mathrm{A}^{\mathrm{k}}\neq\varnothing$.

The set of numerical values being attained by the function $\mathrm{u}[\mathrm{H}]$ on rational coalitions will be called the spectrum. Each value of the function $\mathrm{u}[\mathrm{H}]$ will be called the spectral level (or simply the level). The entire construction described above will be called a monotonic parametric game on W .

Subsequently we will be interested in rational coalitions functioning on the highest possible spectral level. It is obvious that the spectrum of each monotonic game on a finite set W is bounded, and therefore there exists a maximum spectral level $\mathrm{u}^{\mu}=\max_{\mathrm{H}\subseteq\mathrm{W}}\mathrm{u}[\mathrm{H}]$.

Definition 3. *A rational coalition* V^{*} *such that for a certain choice* H^{*} *the level* u^{μ}: $\mathrm{u}[\mathrm{H}]=\mathrm{u}^{\mu}$ *is attained is called the kernel of the monotonic parametric game on* W .

Theorem 1. *If* V_{1}^{*} *and* V_{2}^{*} *are kernels of the monotonic game on* W , *then one can always find the minimum kernel (in set-theoretic sense)* V_{c}^{*} *such that* $\mathrm{V}_{c}^{*}\supseteq\mathrm{V}_{1}^{*}\cup\mathrm{V}_{2}^{*}$. *The proof is presented in the appendix.*

Theorem 1 asserts that the set of kernels in the sense indicated by the binary operation of coalitions is closed. The closeness of a system of kernels allows as looking at the largest (in the set-theoretic sense) kernel, i.e., a kernel $\mathrm{K}^{\ominus}$ such that all other kernels are included in it. From the Theorem 1 it follows the existence of the largest kernel in any finite monotonic parametric game.

The rest of the paper is devoted to the description of constructive methods of setting up coalitions that are rational with the respect to the threshold u°, including those rational with the respect to the threshold u^{μ}, i.e., the kernels coalitions. In particular, a method of constructing the largest kernel is suggested.

7. SEARCH OF RATIONAL COALITIONS

We consider a monotonic parametric game with n players. Below we bring together a system of concepts, which allows us constructively to discover rational coalitions with respect to an arbitrary threshold u° if they exist. In the monotonic game only a limited portion of subsets of the set W have to be searched in order to discover the largest rational coalition. With this aim in the following we study coalitions V whose participants do not refuse from a choice: for $j \in V$ the choice $A^j \neq \varnothing$. Such a coalition, which has affected a choice H, is denoted by $V[H]$. From here on, for the motive of simplicity of notation of guaranteed utility $\pi_w(H \cap R_1,...,A^j,...,H \cap R_n)$, where H is a subset of the set W, we use $\pi(w;H)$.

Definition 4. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_0, \alpha_1, ..., \alpha_{m-1} \rangle$ *(*m *is the number of elements in* W*) from* W *is said to be in concord with respect to the threshold* u°, *if in a sequence of subsets of the set* W

$$\langle N_0, N_1, ..., N_{m-1}, N_m \rangle ,$$

where $N_0 = W$, $N_{i+1} = N_i \setminus \alpha_i$, $N_m = \varnothing$, *there exists a subset* N_p *such that*:

a) *The utility* $\pi(\alpha_i;N_i) < u^\circ$ *for all* $i < p$;

b) *For each* $w \in N_p$ *the condition* $u^\circ \leq \pi(w;N_p)$ *is fulfilled, or, this being equivalent, for each* $j \in V(N_p)$ *the condition* $u^\circ \leq g_j(N_p)$ [11] *is fulfilled.*

A sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, uniquely defines the set N_p. This fact is written in the form $N(\overline{\alpha}) = N_p$.

Definition 5. *A set* $S^\circ \subseteq W$ *is said to be in concord with the respect to a threshold* u°, *if there exists a sequence* $\overline{\alpha}$ *of elements of* W, *in concord with respect to the threshold* u° *and such that* $S^\circ = N(\overline{\alpha})$, *while the coalition* $V(S^\circ)$ *is said to be in concord with respect to the threshold* u°.

The following two statements are derived directly from Definitions 4 and 5.

A. In the case where the set $S^\circ = W$ is in concord with the respect to the threshold u°, all players $j \in I$ are not below u°: $g_j(W) \geq u^\circ$.

B. If the set S°, in concord with the respect to the threshold u°, is empty, then there exists a chain of constructing sets

$$\langle N_0, N_1, ..., N_{m-1}, N_m \rangle ,$$

such that for each player $j \in I$, commencing with a certain N_t, in all those coalitions $V(N_i)$, $t \leq i$, where the player j enters, this player is below u°.

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[11] By definition $g_j(N_p) = \min_{w \in N_p \cap R_j} \pi(w;N_p)$.

Theorem 2. *Let* S° *be a set that is in concord with respect to the threshold* u°. *Then any rational coalition* V *functioning on the level not less than* u° *makes a choice* H, *which is a subset of the set* S°: $H \subseteq S^\circ$. The proof is given in the appendix.

Corollary 1. *The set* S°, *in concord with respect to the threshold* u°, *is unique*. Indeed, if we assume that there exists a set S', in concord with the respect to the threshold u° and different from S°, then from theorem 2, $S' \subseteq S^\circ$. But analogously at the same time the inverse inclusion $S' \supseteq S^\circ$ must also be satisfied, which bring us to conclusion that $S' = S^\circ$.

Corollary 2. *As the spectral levels of functioning of coalitions in the monotonic parametric game grow, one can always find a chain of rational coalitions, included in one another and being in concord with respect to each increasing spectral level, as with respect to the growing threshold.*

Indeed, from the formulation of the theorem it follows that a rational coalition, in concord with the respect to a spectral level $\lambda < \mu$, satisfies the relation $V(S^\lambda) \supseteq V(S^\mu)$, since in a set-theoretic sense $S^\lambda \supset S^\mu$.

Below we arrange a certain sequence $\overline{\alpha}$, which use up all elements of W. After the construction we formulate a theorem about the sequence $\overline{\alpha}$ thus constructed being in concord with respect to the threshold u°. The arrangement proves constructively the existence of a sequence of elements of W that is necessary in the formulation of the theorem.

Construction. Initial Step.

Stage 1. We consider a set of elements W. Among this set we search out elements γ_0 such that $\pi(\gamma_0; W) < u^\circ$, and order them in any arbitrary manner in the form of a sequence $\overline{\gamma}_0$. If there are no such elements, then all elements of W are ordered arbitrarily in the form of a sequence $\overline{\alpha}$, and the construction is completed. In this case W is assumed to be the set $N(\overline{\alpha})$.

Stage 2. Subsequently we examine the sequence $\overline{\gamma}_0$. When considering the t-th element $\gamma_0(t)$ of this sequence $\overline{\gamma}_0$, the sequence $\pi_w(A^1,\ldots,A^{j-1},X_j,A^{j+1},\ldots,A^n)$ is supplemented by the element $\gamma_0(t)$, which is denoted by the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_0(t) \rangle$, while the set W is replaces by $W \setminus \overline{\alpha}$. After the last element of $\overline{\gamma}_0$ is examined we go over to the recursive step of the construction.

Recursive Step k .

Stage 1. Before constructions of the k -th step there is already composed a certain sequence $\overline{\alpha}$ of elements from W . Among the set u° we seek out elements γ_k such that $\pi(\gamma_k; W \setminus \overline{\alpha}) < u^\circ$ and order them in any arbitrary manner in the form of a sequence $f_j(H) =$. Analogously to the initial step, if there happen to be no elements γ_k, the construction is ended. In this case in the role of the set $N(\overline{\alpha})$ we choose $W \setminus \overline{\alpha}$ while $\overline{\alpha}$ is completed in an arbitrary manner with all remaining elements from W .

Stage 2. Here we carry out constructions, which are analogous to stage 2 of the initial step. The entire sequence of elements $\overline{\gamma}_k$ is examined element by element. While examining the t -th element $\gamma_k(t)$ the sequence $\overline{\alpha}$ is complemented in accordance with the expression $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \gamma_k(t) \rangle$. After examining the last element $\gamma_k(t)$ of the sequences $\overline{\gamma}_k$ we return to stage 1 of the recursive step.

On a certain step p, either initial or recursive, at stage 1 there are no elements γ , which are required by the inequalities (2) or (3), and the construction could not continue any more.

Theorem 3. *A sequence $\overline{\alpha}$ constructed according to the rules of the procedure is in concord with the respect to the threshold* u° . The proof is presented in the appendix.

In the current section, in view of the use, as an example, of the concepts just introduced, we consider a particular case of a monotonic parametric game in which the difference in the individual and cooperative behavior of the participants of the coalition is easily revealed. We assume that the utilities

$$\pi_w(A^1,...,A^{j-1},X_j,A^{j+1},...,A^n)$$

do not depend on X_j in the case that choices specified by the remaining players are fixed. In this case the j-s participant of the coalition V , under the condition that the remaining participants of it keep their choices, can limit his choice X_j to a single element $w' \in R_j$ on which the maximum guarantee $\max_{w' \in R_j} g_j(H)$ is attained. However, such a selection narrowing his choice down to a single-element, generally speaking, reduces the choice (in view of monotonicity of utility indices π_w) to the guarantee of the remaining participants of the coalition. Consequently, individual behavior of the participants of a coalition contradicts their cooperative behavior. In spite of this contradiction, in the general case, in the given case, using the concept of a rational coalition $V(S^\circ)$ in concord with respect to the threshold u°, and having slightly modified the criteria of "individual interests" of the players, we can convince someone that there always exists a situation in which the individual interests do not contradict the coalition interests.

We define the winnings of the j-th participant of the coalition in the form of the sum of utilities after subtraction of all payments u°, i.e., as the number

$$f_j(H) = \sum\nolimits_{w \in A_j} \left[\pi(w;H) - u^\circ\right]$$

(the winnings f_k for $k \notin V$ are not defined). Having represented H as a joint choice $\left\langle A^1, A^2, ..., A^{|V|} \right\rangle$, we can consider the behavior of each j-th participant as player in a certain non-cooperative game selecting a strategy A^j.

The situation of individual equilibrium in the sense of Nash (Owen, 1971) of the participants of the coalition V in the game with winnings f_j is defined as their joint choice $\bigcup_{j \in V} A_*^j = H^*$ such that for each $j \in V$

$$f_j(A_*^1, ..., A_*^{j-1}, A^j, A_*^{j+1}, ..., A_*^{|V|}) \le f_j(H^*)$$

for any $A^j \subseteq R_j$. In other word, the situation of equilibrium exists if none of the participants of the coalition has any sensible cause for altering his choice A_*^j under the condition that the rests keep to their choices.

Not every choice H of participants of the coalition V is an equilibrium situation. To see this it is sufficient to consider a choice H such that in the coalition V there are players having chosen elements $w \in A^j$ with utilities $\pi(w;H) < u^\circ$; for the selection of such an element the player pays more than this element brings in winnings $f_j(H)$ and, therefore, for the player, proceeding merely on the basis of individual interests, it would be advantageous to refrain from selection of such elements. Refraining from the selection of such elements of the set H is equivalent to non-equilibrium of H in the sense of Nash.

Lemma. *Let the utilities* $\pi(w;H)$ *be independent of* A^j. *Then a joint choice* S° *of the participants of the rational coalition* $V(S^\circ)$, *in concord with the respect to the threshold* u°, *is a situation of individual equilibrium.*

Indeed, according to Theorem 2, S° is the largest choice in the set-theoretic sense among all choices H of the rational coalition $V(S^\circ)$, where for any $w \in H$ the relation $\pi(w;H) \geq u^\circ$ is fulfilled. Let the choice of the participants of the coalition with an exception of that of the $j^{\text{-th}}$ participant be fixed. Since the utilities $\pi(w;S^\circ)$ do not depend on A^j, the $j^{\text{-th}}$ participant of $V(S^\circ)$ cannot secure an increase in the winnings $f_j(S^\circ)$ either by broadening or by narrowing his choice in comparison with $R_j \cap S^\circ$.

8. COALITIONS FUNCTIONING ON THE HIGHEST SPECTRAL LEVEL

We consider the problem of search of the largest kernel. First of all we present some facts, which are required for the solution of this problem.

From the definition of the guarantee $g_j(H)$ of the participant j effecting the choice H we see that the equality $g_j(H) = \min_{w \in A^j} \pi(w;H))$ is fulfilled. Hence, according to the definition of the level $u[H]$ of functioning of the coalition $V(H)$ it follows that $u[H] = \min_{w \in H} \pi(w;H)$

If we carry out a search of the subset H^* of the set W on which the value of the maximum of the function $u[H]$ is achieved, then thereby the search of a coalition functioning on the highest level $u^\mu = u[H]$ of the spectrum of a monotonic parametric game is affected. Without describing the search procedure, we give the definition of a sequence of elements W allowing us to discover the largest (in the set-theoretic sense) choice H^* of the largest coalition – a kernel K^*.

Definition 6. *A sequence* $\overline{\alpha}$ *of elements* $\langle \alpha_0, \alpha_1, ..., \alpha_{m-1} \rangle$ *(m is the number of elements in* W*) from* W *is called the defining sequence of the monotonic game, if in the sequence of sets* [12]

$$\langle N_0, N_1, ..., N_{m-1}, N_m \rangle$$

there exists a subsequence $\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \rangle$ *such that:*

a) *for any element* $\alpha_i \in \Gamma_k \setminus \Gamma_{k+1}$ *of the sequence* $\overline{\alpha}$ *the utility* $\pi(\alpha_i; N_i) < u[\Gamma_{k+1}]$ $(k = 0,1,...,p-1)$;

b) *in the rational coalition* $V(\Gamma_p)$ *no sub-coalition exists on a level above* $u[\Gamma_p]$.

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[12] The given sequence is constructed exactly in the same way as the one in Definition 4.

From the Definition 6 one can see that the defining sequence in many ways is analogous to a sequence, which is in concord with the respect to the level u°. Since any rational coalition $V(\Gamma_k)$ functions on the level $u^k = u[\Gamma_k]$, it is not difficult to note that the defining sequence $\overline{\alpha}$ composes strictly increasing spectral levels $u[\Gamma_0] < u[\Gamma_1] < ... < u[\Gamma_p]$ of functioning of rational coalitions $V(\Gamma_k)$ in the monotonic parametric game. As a result, we require yet another formulation.

Definition 7. *A rational coalition* $V \subseteq I$ *is said to be determinable, if there exists a defining sequence* $\overline{\alpha}$ *of elements* W *such that among the choices of this coalition there is a choice* Γ_p *composed by* $\overline{\alpha}$ *according to Definition 6.*

Theorem 4. *For each monotonic parametric game a determinable coalition exists and is unique. Among the choices of the determinable coalition there is a choice on which the highest spectral level* u^{μ} *is attained.* The proof of the theorem is presented in the appendix.

Corollary to Theorem 4. *The concepts of a determinable coalition and the largest kernel are equivalent.*

Indeed, directly from the formulation of the Theorem 4 we see that a determinable coalition always is the largest kernel. Hence, since a determinable coalition always exists, while the largest kernel is unique, it follows that the largest kernel coincides with the determinable coalition.

Thus, the problem of search of the largest kernel is solved if we construct a defining sequence $\overline{\alpha}$ of elements W. The construction of $\overline{\alpha}$ can be effected by the procedure of discovering kernels (KFP) from Mullat. In conclusion we present yet another approach to the concept of "stability" of a coalition. [13]

Definition 8. *A coalition* $\hat{V}$ *is said to be a critical, if for a certain choice* $\hat{H}$ *of it no coalition* V *having a nonempty intersection with the coalition* $\hat{V}$ *functions on a level higher than* $u[\hat{H}]$. *The level* $\hat{u} = u[\hat{H}]$ *is called the critical level of the coalition* $\hat{V}$, *while the choice* $\hat{H}$ *is called its critical choice.*

From the Definition 8, in particular, it follows at once the uniqueness of the critical level of the coalition $\hat{V}$. Indeed, on the contrary, if were two different levels $\hat{u}'$ and $\hat{u}''$, $\hat{u}' < \hat{u}''$, then $\hat{u}'$ could not be a critical one according to the definition: it is sufficient to consider the coalition $V = \hat{V}$ itself with the choice $\hat{H}''$, which ensures $\hat{u}'' > \hat{u}'$.

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[13] This approach is close to the concept of "M-stability" in cooperative n-person games, G. Owen.

It is obvious that kernels are critical coalitions. The inverse statement, generally speaking, is not true; a critical coalition is not necessarily a kernel.

We now consider the following hypothetical situation. Let $\hat{V}$ be a critical coalition and let $\hat{H}$ be its critical choice. We assume that this coalition is rational with respect to the threshold u°; i.e., $u^\circ \leq u[\hat{H}]$ (see Definition 2). We assume that an increase of the threshold u° up to the level $u^\circ > u[\hat{H}]$ took place and the critical coalition $\hat{V}$ with the critical choice $\hat{H}$ was transformed into unstable coalition with respect to the higher threshold u°. Let the participants of the coalition $\hat{V}$ preserving the stability of the coalition attempt to increase their guarantees. One of the possibilities for increasing the guarantee of a participant $j_0 \in \hat{V}$ is to refrain from the choice of an element $\alpha_0 \in A^{j_0}$ on which the value $g_{j_0}(H)$ - the minimum level of utility guaranteed for him, see (4), is attained. It is natural to assume that a participant with a level of guarantee $g_{j_0}(\hat{H}) = u[\hat{H}] < u^\circ$ will be among the participants attempting to increase their guarantees, and refrains from the selection of the element α_0 indicated above. It may happen that the refusal of α_0 gives rise, for another participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, to a decrease from his guarantee $g_{j_1}(\hat{H}) > u[\hat{H}]$ to the quantity $g_{j_1}(\hat{H} \setminus \alpha_0) \leq u[\hat{H}]$. A participant $j_1 \in V(\hat{H} \setminus \alpha_0)$, acting from the same considerations as j_0, refrains from the selection of an element α_1 on which $g_{j_1}(\hat{H} \setminus \alpha_0)$ is attained. Such a refusal of α_1 can give rise to subsequent refusals, and emerges hereby a chain of "refusing" participants $\langle j_0, j_1, ... \rangle$ of the coalition $\hat{V}$.

If a coalition V, rational with respect to the threshold u° in the sense of Definition 2, with the choice H became unstable as the threshold u° increases, then such a coalition, generally speaking, disintegrates; i.e., some of its participants may become participants of a new coalition which already is rational with the respect to the increased threshold u°. By definition of a critical coalition, transaction of its participants into new rational coalition, when the threshold u° increases is not possible, and it disintegrates completely. The theorem presented below and proved in the appendix reflects a possible character of complete disintegration of a critical coalition in terms of the hypothetical system described above.

Theorem 5. *Let there be given a critical coalition* $\hat{V}$ *having a nonempty intersection with a certain coalition* $V: c$. *Let* H *be the choice of the coalition* V *and* $\hat{H}$ *the critical choice of the coalition* $\hat{V}$. *Then in the coalition* $\hat{V} \cap V$ *there exists a sequence of its participants* $\bar{j} = \langle j_0, j_1, \dots, j_{r-1} \rangle$ *such that:*

a) in the sequence $\bar{j}$ *there are represented all participants of the coalition* $\hat{V} \cap V$ *(the players* j_i *may be repeated,* r *is number of elements in* $\hat{H} \cup H$;

b) for the sequence $\bar{j}$ *we can construct a chain of contracting coalitions*

$$\langle V(N_0), V(N_1), \dots, V(N_{r-1}) \rangle,$$

where $N_0 = \hat{H} \cup H$, $N_{i+1} \subset N_i$, *so that for any* $j \in V$, *commencing from a certain* N_t, *in all those coalitions* $V(N_i)$, $t \leq i$, *into which the player* j *enters, this player is not above* $u[\hat{H}]$.

9. FINAL REMARKS

The article is a comprehensive journey that begins with an exploration of the complex dynamics that govern the distribution of goods within the intricate framework of retail network. Here goods move through the network of transactions aimed at maximizing participants' respective profits, from producers as sellers to distributors as both, buyers and sellers, and at last to consumers purchasing solely for consumption.

Central to this dynamic journey is a pricing system carefully structured around constants that serve as fundamental percentages. These constants are then recalibrated to facilitate the calculation of selling prices, ensuring that they are sufficiently superior to purchase prices to provide satisfactory results for participants seeking to optimize their profitability. However, despite this seemingly simple structure, the pricing system becomes increasingly complex with the introduction of transaction costs. These costs, once integrated into buying and selling decisions, introduce a new level of complexity, fundamentally changing the behavioral landscape of participants. Suddenly, transactions that were once considered potentially profitable now carry the risk of loss, forcing participants to recalibrate their strategies and decision-making processes accordingly.

Zooming out to a global perspective, the paper scrutinizes the nuanced interplay between transaction costs and their thresholds, meticulously ranging from low to high values. This analysis unveils a stark reality: as transaction costs escalate, allegedly profitable transactions within bilateral trade agreements plummet into the realm of unprofitability, rendering them futile for rational participants seeking mutually beneficial agreements.

In response to this shifting paradigm, the structure of the retail chain undergoes a remarkable metamorphosis. With each increment in the transaction costs' threshold, the chain evolves into a complex tapestry of nested sets, each tier adeptly equipped to counteract the mounting pressures of higher transaction costs while steadfastly maintaining equilibrium.

Central to sustaining this delicate equilibrium is the imperative that all participants within the retail chain steadfastly avoid engaging in unprofitable transactions. To achieve this, the formation of the retail chain is imbued with a sophisticated mechanism, incorporating elasticity intervals tailored to the nuances of transaction costs. These intervals serve as a beacon of rationality, guiding participants through the intricate maze of buying and selling decisions, meticulously encoded into the scheme and individually calculated for each participant within the chain.

ACKNOWLEDGMENT

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APPENDIX

Proof of Theorem 1. Let the level u^{μ} be attained for the coalitions V_1^* and V_2^*, which effect the choices H_1^* and H_2^* respectively; i.e., $u^{\mu} = u[H_1^*]$ and $u^{\mu} = u[H_2^*]$. For player $j \in I$ we consider two choices: $H_1^j = H_1^* \cap R_j$ and $H_2^j = H_2^* \cap R_j$[14]. By the definition of guarantee $g_j(H_1^*)$ for the participant $j \in V_1^*$ of the coalition we have

$$\min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, ..., H_1^n) = g_j(H_1^*) \geq u^{\mu}; \tag{A1}$$

for the participant $j \in V_2^*$ we respectively have

$$\min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, ..., H_2^n) = g_j(H_2^*) \geq u^{\mu}. \tag{A2}$$

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[14] We note that, in the worst case, for player $k \notin V_1^*$ ($k \notin V_2^*$), $H_1^k = \varnothing$ ($H_2^k = \varnothing$).

We determine the choice of a participant $j \in V_1^* \cup V_2^*$ as $\Phi^j = H_1^j \cup H_2^j$. The monotonic property (1) allows us to conclude that the following inequalities are valid:

$$\min_{w \in H_1^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq \min_{w \in H_1^j} \pi_w(H_1^1, H_1^2, ..., H_1^n)\,; \qquad \text{(A3)}$$

$$\min_{w \in H_2^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq \min_{w \in H_2^j} \pi_w(H_2^1, H_2^2, ..., H_2^n)\,. \qquad \text{(A4)}$$

Combining (A1) – (A4), we obtain

$$\min_{w \in \Phi^j} \pi_w(\Phi^1, \Phi^2, ..., \Phi^n) \geq u^\mu \qquad \text{(A5)}$$

for any $j \in V_1^* \cup V_2^*$. If by Φ^* we denote the set $H_1^* \cup H_2^*$, then for the coalition $V_1^* \cup V_2^*$ affecting the choice Φ^* the inequality (A5) is rewritten in the form

$$g_j(\Phi^*) \geq u^\mu,\ j \in V_1^* \cup V_2^*. \qquad \text{(A6)}$$

Due to the monotonic property (1) some elements $w \notin \Phi^*$ (if one can find such) may be added to Φ^* while the inequality (A6) is still true [15]. We will denote the enlarged set by Φ^c: $\Phi^c \supseteq \Phi^*$ and obviously for $V^c = V(\Phi^c)$ we have $V(\Phi^c) \supseteq V_1^* \cup V_2^*$. By the definition of a spectral level u^μ, for the participant $j' \in V^c$, on which $u[\Phi^c]$ is attained, we have

$$g_{j'}(\Phi^c) = u[\Phi^c] \leq u^\mu, \qquad \text{(A7)}$$

since u^μ is the maximum spectral level of functioning of coalitions in the monotonic game. Applying (A7) and (A6) to the choice Φ^c for the participant $j = j'$, we see that $g_{j'}(\Phi^c) = u^\mu$, and the coalition $V^c \supseteq V_1^* \cup V_2^*$ functions on the spectral level u^μ. The theorem is proved. ■

Proof of Theorem 2. Let S° is a subset of the set W in concord with the respect to the threshold u°; i.e., there exists a sequence $\overline{\alpha}$, in concord with the respect to the threshold u°, such that $S^\circ = N(\overline{\alpha})$. We assume that there exists a coalition V affecting a choice $H \subset S^\circ$ and functioning on the level $u[H] \geq u^\circ$, $H \setminus S^\circ \neq \varnothing$. Let $\alpha_1 \in H \setminus S^\circ$ and let α_t be an element, which is leftmost in the sequence $\overline{\alpha}$. Let p be the index of the set N_p in the sequence $\langle N_0, N_1, ..., N_{m-1}, N_m \rangle$. It is obvious that $t < p$ and, consequently:

$$\pi(\alpha_t; N_t) < u^\circ \qquad \text{(A8)}$$

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[15] We suppose that such elements cannot be added to Φ^c.

in accordance with a) of the Definition 4. Since the game being considered is monotonic, $\alpha_t \in H$ and $H \subseteq N_t$ there must hold

$$\pi(\alpha_t ; H) \leq \pi(\alpha_t ; N_t). \qquad \text{(A9)}$$

From inequalities (A8) and (A9) it follows

$$\pi(\alpha_t ; N_t) < u^\circ \leq u[H] \qquad \text{(A10)}$$

(the latter $\leq$ by assumption). According to the inequality (A10) and by the definition of $u[H]$ we have

$$\pi(\alpha_t ; H) < \min_{j \in V} g_j(H). \qquad \text{(A11)}$$

Let the element α_t be chosen by a certain q-th player; i.e., $\alpha_t \in A^q$, $q \in V$. On the basis of (A11) we assume that

$$\pi(\alpha_t ; H) < g_q(H) \qquad \text{(A12)}$$

is valid. By definition $g_q(H) = \min_{w \in A^q} \pi(w; H)$ and following (A12), we note that $\pi(\alpha_t ; H) < \min_{w \in A^q} \pi(w; H)$. The last inequality is contradictory, what proves the theorem. ■

Proof of Theorem 3. We assume that the construction of the sequence $\overline{\alpha}$ according to the rules of the procedure ended on a certain p-th step. This means that $\overline{\alpha}$ is made up of sequences $\overline{\gamma}_k$ $(k = \overline{0, p})$, and also of elements of the set N_p, found according to the rules of the procedure and being certainties for the sequences $\overline{\gamma}_k$. We consider any element α_i of the sequence thus constructed, being located on the left of the α-th element: $i < p$. The given element in the construction process falls into certain set $\overline{\gamma}_q$. By construction

$$\pi(\alpha_i ; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} < u^\circ. \qquad \text{(A13)}$$

If to the sequence $\langle \overline{\gamma}_0, \overline{\gamma}_1, \ldots, \overline{\gamma}_{q-1} \rangle$ we add the elements $\overline{\gamma}_q$, which in $\overline{\alpha}$ are on the left of the α_i-th. Then, this set of elements together with the added part $\overline{\gamma}_q$ composes the complement $\overline{N}_i$ up to the set W (see Definition 4).

On the basis of the monotonic property (1) we conclude that $\pi(\alpha_i ; W \setminus \{\overline{\gamma}_0 \cup \overline{\gamma}_1 \cup \ldots \cup \overline{\gamma}_{q-1}\} \geq \pi(\alpha_i ; W \setminus \overline{N}_i) = \pi(\alpha_i ; N_i)$. The last relation in the combination with (A13) shows that $\pi(\alpha_i N_i) < u^\circ$. From the construction of the sequence $\overline{\alpha}$ it is also obvious that for any $j \in V(N_p)$ the guarantee $g_j(N_p) \geq u^\circ$. The theorem is proved. ■

Proof of the Theorem 4. Theorem can be proved as follows. First, a sequence $\overline{\alpha}$, in concord with respect to the highest spectral level u^{μ}, in the monotonic game exists, according to Theorem 3, and is, at the same time, a defining sequence; as the subsequence $\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \rangle$ in this case we have to choose the sequence $\langle W, S^{\mu} \rangle$, where S^{μ} is a set $S^{\mu} \subset W$ which is in concord with respect to the highest level u^{μ}. The determinable coalition is $V(S^{\mu})$. The uniqueness of the coalition $V(S^{\mu})$ is proved in Corollary 1 to the Theorem 1. Secondly, the choice S^{μ} of the coalition $V(S^{\mu})$, playing the part of the set Γ_p in the Definition 6, attains the maximum of the function $u[H]$, a fact which follows from Theorem 3 and b) of Definition 6; i.e., $u[S^{\mu}] = u^{\mu}$. Thirdly, the last statement of Theorem 4 is a particular case of the statement of Theorem 2, if we put $u^{\circ} = u^{\mu}$. The theorem is proved. ■

Proof of the Theorem 5. We consider a monotonic game of participants of a coalition $\hat{V} \cup V$ on the set $\hat{H} \cup H$, where $\hat{H}$ is the critical choice of the critical coalition $\hat{V}$, and H is some choice of the coalition V. Below we note the set $\hat{H} \cup H$ by Ω, while all concepts refer to a monotonic sub-game on Ω.

Let u° be the threshold of the parameter u of the game on Ω, and let $u^{\circ} > u[H]$. We construct a sequence $\overline{\alpha}$ of elements Ω, which is in concord with respect to the threshold u°. Two variants could be represented: 1) the set S°, in concord with the respect to the threshold u° is empty; 2) S° is not empty. We consider them one after the other. First, in the variant 1) from a sequence of elements $\overline{\alpha}$ of elements of Ω in concord with respect to the threshold u°, we uniquely determine a sequence of participants of the coalition $\hat{V} \cup V$ choosing elements α_i from sequence $\overline{\alpha}$ and composing a certain chain $\overline{j} = \langle j_0, j_1, ..., j_{r-1} \rangle$ (r is the number of elements Ω). Secondly, from the sequence $\overline{\alpha}$ we also uniquely determine the sequence of coalitions $\langle V(N_0), V(N_1), ..., V(N_{r-1}) \rangle$, where $N_0 = \Omega$, $N_{i+1} = N_i \setminus \alpha_i$, with $j_i \in V(N_i)$.

In the second variant none of the participants of the coalition $\overline{V}$ can be in a coalition, which is in concord with the respect to the threshold $u^{\circ} > u[H]$. This would contradict the definition of a critical coalition $\overline{V}$. Therefore in the chain $\overline{j}$ thus constructed of participants of the coalition $\hat{V} \cup V$ (by the same method as in the first variant) all participants of the coalition $\overline{V}$ are on the left of the j_p -th player; p is uniquely determined from the sequence $\overline{\alpha}$ (see Definition 4). By property a) of the Definition 4 and from the definition of the guarantee of a player $j_i \in V(N_i)$ we have

$$g_{j_i}(N_i) \leq \pi(\alpha_i; N_i) < u^{\circ}. \qquad \text{(A14)}$$

Proceeding from the structure of the spectrum of a monotonic parametric game on Ω (see Corollary 2 to the Theorem 2) the value u° marginally close to $u[H]$ is satisfied successfully in the two variants considered. The first variant of the Theorem 5 forms the statement b) derived earlier from Definition 4 and 5 (see section 2). The second variant of the statement of the theorem is directly derived from the relation (A14). ■

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It is worth to emphasize, in view of the above, when the player $j \in V$ must make a payment u° for the element $w \in A^{j}$, the payment is well suited in the role of transaction cost. Indeed, in economics, transaction costs refer to the expenses incurred during an economic exchange. For instance, when buying or selling stocks, individuals typically pay commissions to brokers, which are transaction costs. Similarly, purchasing a banana involves not only the banana's price but also the energy, effort, and time spent on deciding which banana to buy, traveling to the store, waiting in line, and completing the payment. These additional costs beyond the banana's price are transaction costs. Considering transaction costs is crucial when assessing potential transactions.

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[i] In the book review of Ménard and Shirley, (eds., 2005), noticed that

> *North and Williamson stress, besides transaction costs, the role of bounded rationality, uncertainty, and imperfect rationality. Their objects of research differ: Northian NIE focuses on macro institutions that shape the functioning of markets, firms, and other modes of organizations such as the state (section II) and the legal system (section III). Williamsonian NIE concentrates on the micro institutions that govern firms (section IV), their contractual arrangements (section V), and issues of public regulation (section VI). Both the Northian and Williamsonian approaches to the NIE are used, i.e., in development and transformation economics: in efforts towards explaining the differences of exchange-supporting institutions (section VIII).*

Data Analysis

The image entitled "Data Analysis" features a complex network graph with numerous interconnected nodes. The varying sizes and densities of the nodes suggest different levels of connectivity and importance within the network, indicating the analysis of complex data relationships, possibly in fields such as social networks, communication patterns, or big data analytics. The visual representation highlights the interconnected and intricate nature of the data being analyzed.

On The Maximum Principle for Some Set Functions [1]

Abstract. This article explores the challenge of identifying extrema for functions across all subsets of a broad or generalized finite set. The approach outlined herein unveils exceptional subsets, with a key aspect being the specification of certain indicators for each subset and its elements. These indicators, akin to accounting data or weights, adhere to monotonicity conditions—a pervasive concept in real-world scenarios spanning technology, human endeavors, and econometrics. For instance, optimizing the placement of fundamental transition radio stations to enhance radio signal performance across expansive geographical areas exemplifies the application of such indicators. Conversely, adapting communication lines to effectively mitigate failures presents another perspective. Platforms like Facebook utilize metrics such as the number of individuals in the Famous People list or engaging with Daily Messages, demonstrating how indicators facilitate the identification of popular content or trends. Monotone indicators likely help in gauging factors such as citation counts, reader engagement, or impact metrics, which contribute to determining the popularity or significance of articles. Moreover, a wide array of survey types and potential indicator domains underscores the versatility and practicality of monotonicity-driven approaches. The exploration of additional areas where indicators exhibit monotonic properties further enriches the scope of applications.

Keywords: classification; graphs; convex functions; algorithm

1. INTRODUCTION [NB!]

In our study, we consider the problem of finding the global extremum of a function defined on all subsets of a given finite set. The described construction algorithm was used to solve some problems of object classification using the technique of homogeneous Markov chains. In general terms, the proposed construction allows one to solve some problems on graphs, for example, to single out, in a sense, "connected" subsets of the vertices of the graph. We formulate the theoretical foundations of our construction in terms of transparent rules for choosing subsets in a given finite set and some sequences of the same elements of a finite set. The result will be extracting the extreme subsets.

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[1] This idea at the moment, perhaps invisible from the first glance, is incorporated into "Left- and Right-Wing Political Power Design" as political parties bargaining game. Reg. "data analysis", see also, J. E. Mullat (1976-1977) Extremal Subsystems of Monotonic Systems, I, II, III, Automation and Remote Control, 37, pp. 758-766; 37, pp. 1286-1294; 38. pp. 89-96.

The types of problems of similar nature have a combinatorial character and do belong mostly to the discrete programming problems. Cherenin (1962), Cherenin and Hachaturov (1965) have successfully solved a preeminent class of similar problems on the finite sets. In the framework of these papers a functions have been considered satisfying condition, which can be formulated as follows. If ω_1 and ω_2 are two representatives for subsets of a given finite set then

$$f(\omega_1) + f(\omega_2) \le f(\omega_1 \cup \omega_2) + f(\omega_1 \cap \omega_2).$$

This condition with some reservation reflects the convexity of the function f.

The main property or requirement for the class of functions considered in the manuscript is the assumption of the existence of some numbers or weights that reveal for each element of a finite set the degree of its occurrence in the subset. The degree of occurrence must satisfy conditions (1) and (2), see below.

Concerning the current investigation it is worthwhile also to pay attention to Mirkin's (1970) work. In this work, a problem of optimal classification is reduced to finding special "painting" on a non-ordered graph. The optimal classification there is characterized by some maximum value of a function, corresponding in its form to the definition (1), however hereby we interpret (1) in a different sense. We do not consider in our function definition a decomposition of a given set into two non-intersecting subsets what was the main concern of Mirkin's work.

2. THE MODEL

Let $\{H\}$ is a set of subsets of some finite set W. Suppose that we introduce a π_H function for each set $H \subseteq W$ of its elements as arguments. Below by the collection $\{\pi_H\}$ we entitle a system of weights on the set H. The main supposition concerning the weight systems $\{\{\pi_H\}\}$ is as follows:

p.1 the credential $\pi_H(\alpha)$ of the element $\alpha \in H$ is a real number.

p.2 Following dependencies inhere between different credential, i.e., credential systems for different subsets of the set M: for each element $\alpha \in H$ and each $\beta \in H \setminus \{\alpha\}$ yields that $\pi_{H \setminus \alpha}(\beta) \le \pi_H(\alpha)$.

In other words, following p.2, the requirement is that a removal of an arbitrary element α from a set H results in a new credential system $\left\{\pi_{H\backslash\alpha}\right\}$ and the effect of the removed element α on the credentials within the remaining part $H\setminus\{\alpha\}$ is only towards the direction of a decrease. We explain these two conditions by examples from the graph theory, although there are examples from other jurisdictions, however less convenient for a short discussion. Let consider non-oriented graphs, i.e., graphs with the property when a relation of a vertex x to y implies a reverse relation of vertex y to x.

Example 1. [2] [3]

Let W is a vertex set of a graph G. We define a credential system $\left\{\pi_H\right\}$ on each subset of vertexes H as a collection of numbers $\left\{\pi_H(\alpha)\right\}$, where the number $\pi_H(\alpha)$ is equal to the number of vertexes in H related to the vertex α. The truthfulness of the pp. 1 and 2 is easily checked, if one only remembers to recall that together with the removal of a vertex α all connected to it edges have to be removed concurrently.

Example 2.

Let W is a set of edges in a graph G or the set of pairs of vertexes related by the graph G. We define a credential system $\left\{\pi_H\right\}$ on arbitrary subset H of edges in the graph G as a collection of numbers $\left\{\pi_H(\alpha)\right\}$, where $\alpha \in H$ and $\pi_H(\alpha)$ is a number of triangles in the set of edges H, containing the edge α. The number $\pi_H(\alpha)$ is equal to the number of those vertexes on which the set H resides such, that if x is a pointed vertex and the edge $\alpha=[b,e]$, then it ensues that $[b,x]\in H$ and $[e,x]\in H$.

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[2] Kempner Y., Mirkin B. and I. Muchnik (1997) have given another example in Monotone Linkage Clustering and Quasi-Convex Set Functions, Appl. Math. Letters, v. 10, issue no. 4, pp. 19-24. Mirkin B. and I. Muchnik. (2002) Layered Clusters of Tightness Set Functions, Applied Mathematics Letters, v. 15, issue no. 2, pp. 147-151.

[3] Yet another examples, Kuznetsov E.N. and I.B. Muchnik, Moscow (1982) Analysis of the Distribution Functions in an Organization, Automation and Remote Control, Plenum Publishing Corporation, pp. 1325-1332; Kuusik R. (1993) The Super-Fast Algorithm of Hierarchical Clustering and The Theory of Monotonic Systems, Data Processing, Problems of Programming, Transactions of Tallinn Technical University, No. 734, pp. 37-61; Mullat J.E., (1995) A Fast Algorithm for Finding Matching Responses in a Survey Data Table, Mathematical Social Sciences 30, pp. 195-205; Genkin A.V. and I. B. Muchnik (1993) Fixed Approach to Clustering, Journal of Classification, Springer, 10, pp. 219-240,.

In the examples, we have exploited the fact, that a graph is a topological object from one side and a binary relation from the other side. Let now consider the following set function

$$f(H) = \min_{\alpha \in H} \pi_H(\alpha), \tag{1}$$

where $H \subseteq W$. We suggest below a principle, valid for the subset H, on which the global maximum of a type (1) function is reached. We formulate this principle in terms of some sequences of the set W elements and the sequences of the subsets of the same set W.

Let $\overline{\alpha} = \left\{ \alpha_0, \alpha_1, ..., \alpha_{k-1} \right\}$ is a sequence of elements of the set W and $k = |W|$. We define using the sequence $\overline{\alpha}$ a sequence of sets $\overline{H}(\overline{\alpha}) = \left\{ H_0, H_1, ..., H_{k-1} \right\}$: as $H_0 = W$ and $H_{i+1} = H_i \setminus \{\alpha_i\}$.

Definition 1. We call a sequence of elements $\overline{\alpha}$ from the set W a defining sequence, if in the sequence of sets $\overline{H}(\overline{\alpha})$ there exists a sub sequence $\overline{\Gamma} = \left\{ \Gamma_0, \Gamma_1, ..., \Gamma_p \right\}$ such that:

1°. The credential $\pi_{H_i}(\alpha_i)$ of an arbitrary element, belonging to Γ_j, but not belonging to Γ_{j+1}, is strictly less than $f(\Gamma_{j+1})$;

2° In Γ_p there do not exists such a strict subset L that $f(\Gamma_p) < F(L)$.

Definition 2. We call a subset H of the set W a definable, if there exists a defining sequence such that $H = \Gamma_p$.

Below, we simply refer to the notification $\left\{ \pi_H \right\}$ as a credential system with respect to the set H.

Theorem. On the definable set H the function $f(H)$ reaches its global maximum. The definable set is unique. All sets, where the global maximum has been reached, lie within the definable set.

Proof. Let H is a definable set. Assume, that there exists L such that $f(H) \leq f(L)$. Suppose that $L \setminus H \neq \varnothing$, [4] otherwise we have just to proof the uniqueness of H, what we will accomplish below. Let H_t is the smallest from the sets H_i $(i = 0,1,...,k-1)$, which include in it the set $L \setminus H$. From this fact one can conclude, that there exists an element $\ell \in L$ such, that

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[4] Here $\varnothing$ symbolizes an empty set.

$\ell \in H_t$, but $\ell \notin H_{t+1}$. Moreover, in combination with $L \setminus H \neq \varnothing$ the last conclusion ensues $t < p$. Inequality $t < p$ disposes to an existence of at least one a subset in the sequence of sets $\overline{\Gamma}$ such, that

$$\pi_{H_t}(\ell) < f(\Gamma_j) \tag{2}$$

and $j \geq t+1$. Since $\ell \notin H_{t+1}$ and $\Gamma_j \subseteq H_{t+1}$ are true, it follows that $\ell \notin \Gamma_j$. Thus, the inequality

$$f(\Gamma_j) \leq f(\Gamma_p) \tag{3}$$

is valid as a consequence of the property 2° for the defining sequence.

Now, let $\ell \in L$ and the credential $\pi_L(\ell)$ is at the minimum in credential system with the respect to the set L. Inequalities (2) and (3) allow us to conclude, that $\pi_{H_t}(\ell) < \pi_L(\ell)$. Above we selected H_t on the condition that $L \subset H_t$. Hereby, recalling the main property p.2 of the credential system (the removal of elements), it is easily to establish that $\pi_L(\ell) \leq \pi_{H_t}(\ell)$, i.e., in the credential system with the respect to the set L, there exists a credential, which is strictly less than the minimal. We came to a contradiction and by this, we have proved that on H the global maximum has been reached. Further, all such sets, different from H, where the global maximum is likewise reached, might really be located within H. It remains to be proved the uniqueness of the definable set. In connection of what we proved above, one might suppose that a definable set H' is located within H, however, proceeding with the line of reasoning towards H' similar to those we proposed above for L, we conclude, that $H \subset H'$. ■

Corollary. Let $\{R\}$ is a system of sets, where the function of type (1) reaches its global maximum. Hereby, if $H_1 \in \{R\}$ and $H_2 \in \{R\}$ are valid, then $H_1 \cup H_2 \in \{R\}$.

Proof. Following the p.2 (the main property) $f(H_1) \leq f(H_1 \cup H_2)$, but in addition $f(H_1 \cup H_2) \leq f(H_1)$, consequently $H_1 \cup H_2 \in \{R\}$. ■

Below we introduce an actual algorithm for constructing the defining sequences of elements of a set W. For the availability of the algorithm is exposed in the form of a block-scheme similar to some extent of a computer program.

3. ALGORITHM[5]

a.1. Let the set $R = W$ and sequences $\overline{\alpha}$ and $\overline{\beta}$[6] be empty sets in the beginning, and let the index $i = 0$.

a.2. Find an element μ at the least credential with the respect to the set R, record the value $\lambda = \pi_R(\mu)$ and constitute $\overline{\alpha} = \overline{\alpha}, \overline{\beta}, \mu$ and thereafter $\overline{\beta} = \varnothing$.

a.3. Exclude the element μ from the set R and take into account the influence of the removed element $\mu \in R$ on remaining elements, i.e., recalculate all values $\pi_{R\setminus\mu}(\beta)$ for all $\beta \in R \setminus \{\mu\}$.

a.4. In case, among the remaining elements there exist such γ, that

$$\pi_{R\setminus\mu}(\gamma) \le \lambda \qquad (4)$$

compose a sequence from those elements $\overline{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ and substitute $\overline{\beta} = \overline{\beta}, \overline{\gamma}$.

a.5. Substitute the set $R = R \setminus \{\mu\}$ and the element $\mu = \beta_{i+1}$. Return to the a.3 in case the element β_{i+1} is the element for the sequence $\overline{\beta}$ increasing in this moment the index i by one.

a.6. In case, when the sequence $\overline{\alpha}$ has utilized the whole set W, the construction is finished. Otherwise, return to a.2 initializing first $i = 0$.

Let us prove that the sequence $\overline{\alpha}$ just constructed by the proposed algorithm is defining. We consider the sequence $\overline{H}(\overline{\alpha})$ and let one selects in the role of the sequence $\overline{\Gamma}$ those sets, which start by the element μ found at the moment the algorithm is crossing the step a.2. The fact of crossing the a.2 of the algorithm guarantees, that the condition (4) is not valid before the cross was occurred, and the element β_{i+1} is not in the sequence β at this stage. The above guarantees as well the condition 1° fulfillment for the defining sequences. Suppose, that the condition 2° in the definition 1 do not hold, i.e., in the last set Γ_p in the sequence $\overline{\Gamma}$, there exists such a subset L, that

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[5] Further developments, see Muchnik, I., and Shvartser, L. (1990) Maximization of generalized characteristics of functions of monotone systems, Automation and Remote Control, 51, pp. 1562-1572,

[6] Hereby $\overline{\beta} = \{\beta_1, \beta_2, \ldots, \beta_i, \ldots\}$

$f(\Gamma_p) < f(L)$. Let us consider the sequence $\overline{\beta}$, which is generated at the last crossing through the a.2 of the above-described algorithm and let λ symbolize the highest value among all such λ. One has to conclude, that $\lambda_p < f(\Gamma_p)$, and, from the supposition of an existence of a set L, we come to the inequality $\lambda_p < f(L)$. By the construction, the sequence $\overline{\alpha}$ and together with the sequence $\overline{\beta}$ (both of them), which is generated at last crossing though the a.2 of the algorithm has utilized all elements in W. Consequently, we can consider a set of elements K in the sequence $\overline{\beta}$, which start from the first confronted element $\ell \in L$, where $L \subset K$. On the basis justified above, we have $\pi_K(\ell) = \lambda_p$ and, recalling the main property of the credential system p.2 (the removal of elements), we conclude moreover that $\pi_L(\ell) \leq \lambda_p$. We reached to a contradiction and by that we have proved the property 2° of the definition 1 for the sequence $\overline{\alpha}$. On that account, the construction of defining sequences is possible by the pointed above algorithm.

We emphasize the necessity of concretizing the notion of credential system with the respect to a subset of a given finite set for solving some of the pattern recognition problems, what should be the subject for further investigation.

In conclusion, we will point out, that the construction of defining sequences has been realized in practice on a computer for one problem in graph theory, related to an extraction of "almost totally connected" sub-graphs in a given graph. The number of edges in such graphs has been around 10^4.

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NB! In his work "Cores of Convex Games" Shapley investigated a class of n -person's games with special convex (supermodular) property, International Journal of Game Theory, Vol. 1, 1971, pp. 11-26. When writing current paper, in that time in the past, the author was not familiar with this work and could not predict the close connection between the basic monotonicity property pp.1-2, see above, and that of supermodular characteristics functions in convex games induce the same property upon marginal utilities. We are going to explain the connection. We will consequently do it in Shapley's own words to make the idea crystal clear.

The core of a n -person game is the set of feasible outcomes that cannot be improved upon by any coalition of players. A convex game is one that is based on a convex set function; intuitively this means that the incentives for joining a coalition increase as the coalition grows, so that one might expect a "snowballing" or "band-wagon" effect when the game is played cooperatively… In Shapley's paper a coalition game is a function V mapping a Ring of subsets from some set called a grand coalition $\mathcal{N}$ to the real numbers, satisfying $\mathrm{v}(\varnothing) = 0$. *The function* V *is supperadditive if*

$\mathrm{v(S) + v(T) \le v(S \cup T)}$, i.e., all $\mathrm{S, T} \in \mathcal{N}$, with $\mathrm{S \cap T} = \varnothing$.

It is convex if $\mathrm{v(S) + v(T) \le v(S \cup T) + v(S \cap T)}$

for all $\mathrm{S, T} \in \mathcal{N}$, p.12.

In the standard form in game theory, the elements of $\mathcal{N}$ are "players", the subsets of $\mathcal{N}$ are "coalitions"; $\mathrm{v(S)}$ is called the "characteristic function", which gives each coalition the best payoff that it can get without the help of other players.

Supper-additivity arises naturally in this interpretation, but convexity is another matter. For example, in voting situation S and T, but not $\mathrm{S \cap T}$, might be winning coalitions, causing "convexity" to fail. To see what convexity does entail, consider the function m:

$$\mathrm{m(S,T) = v(S \cup T) - v(S) - v(T)},$$

as defining the "incentive to merge" between disjoint coalitions S and T. Then it is a simple exercise to verify that convexity is equivalent to the assertion that $\mathrm{m(S,T)}$ is no decreasing in each variable – whence the "snowballing" or "band wagon" effect mentioned in the introduction.

Another condition that is equivalent to convexity (provided $\mathcal{N}$ is finite) is to require that

$$\mathrm{v(S \cup \{i\}) - v(S) \le v(T \cup \{i\}) - v(T)}$$

for all individuals $\mathrm{i} \in \mathcal{N}$ and all $\mathrm{S \subseteq T} \subseteq \mathcal{N} \setminus \{\mathrm{i}\}$. This expresses a sort of increasing marginal utility for coalition membership, and is analogous to *"increasing the returns to scale associated with convex production functions in economics.",* p.13

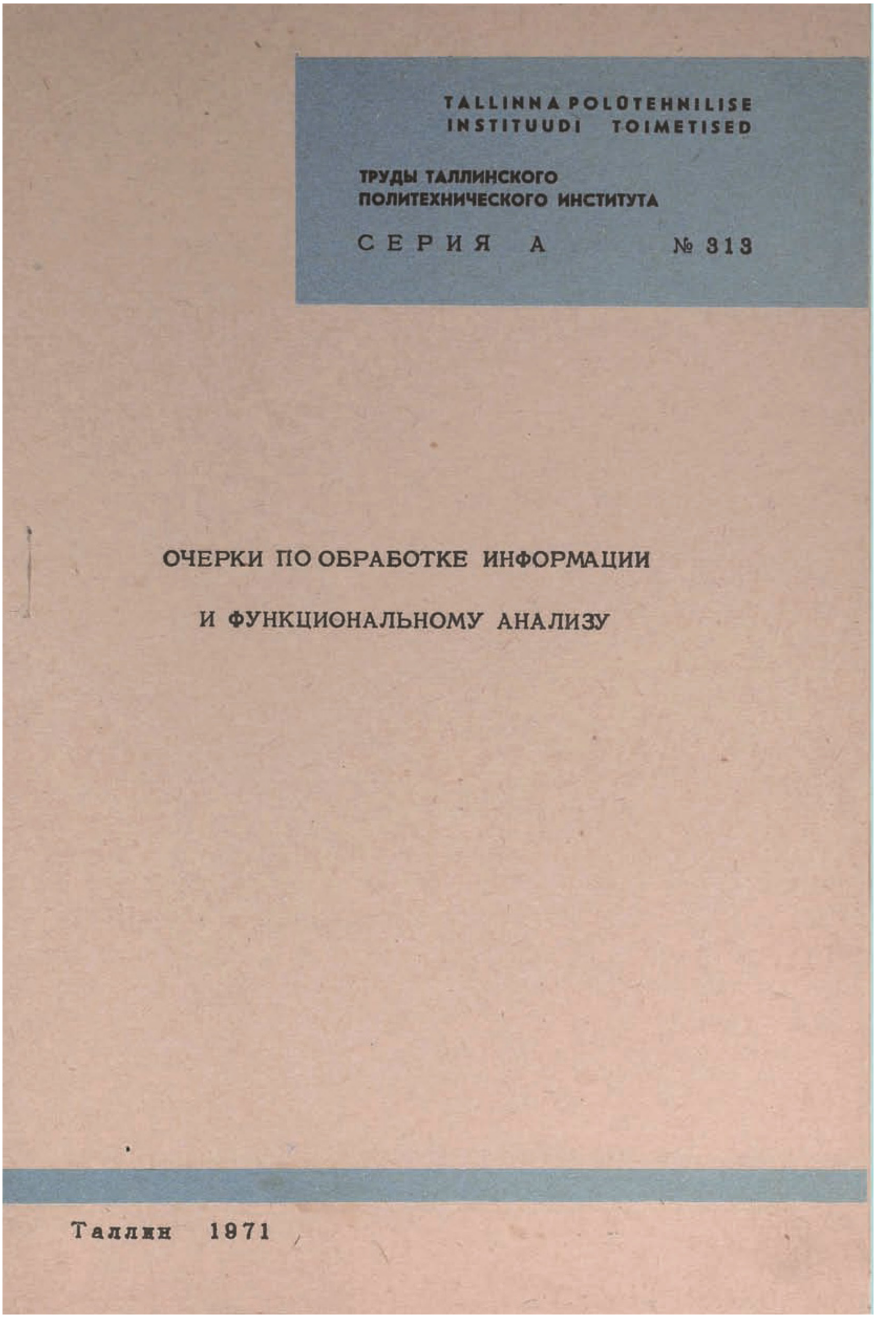
We return now back from the "expedition" into Shapley's work and make some comments. The latter condition, which is equivalent to convexity, is an exact, we repeat it once again, an exact utilization of our basic monotonicity property pp.1-2. Set functions of this type are also known in the literature as "supermodular". As it turns out now the author knew such functions. To the knowledge of the author Cherenin was first who introduced functions of this type already in 1948. Nemhauser et al, also used $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ but an inverse property introduced in 1978 for computational optimization problems in "An Analysis of Approximation for Maximizing Submodular Set Functions", Mathematical Programming 14, 1978, 265-294. Shapley also notes the latter inverse property in connection with rank function of a matroid known as "submodular" or "lower semi-modular." Besides, in Nemhauser et al paper, the reader may find the proof of the conditions:

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ and}$$
$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \text{ equivalency.}$$

However, the connection between the convex games and the monotonicity property pp.1-2 is invisible. Only recently Genkin and Muchnik pointed out (not in the connection with game theoretical models, but actually in connection with the problems of object classification, see “Submodular Set Functions and Monotone Systems in Aggregation Problems I,II,” Translated from Automat. Telemekhanika No.5, pp.135-148, © 1987 0005-1179/87/4805-0679, Plenum Publishing Corporation), that the functions family $\pi_H(\alpha) = v(H) - v(H \setminus \{\alpha\})$ represent a derivatives of supper-modular set functions in the form just exhibited in Shapley's work.

SUMMARIZING

In convex games, following the theory developed in this work from 1971, one can always find a coalition, where it members will be awarded individually at least by some maximum payoff of guaranteed marginal utility, see the Theorem. We call this coalition the largest kernel (nuclei) or the definable set. A good example and its like, is the Example 1. Here, in economic terms, the marginal utility highlights the number of direct dealers with the player $i \in S$ (number of direct contacts, buyers, sellers, direct suppliers, etc.). On the contrary, the Example 2 is not its like and goes beyond the Shapley's Convex Game idea.



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ПОЛИТЕХНИЧЕСКОГО ИНСТИТУТА

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О Принципе Максимума для некоторых Функций Множеств

Резюме. В статье рассматривается задача нахождения экстремальных точек функции, заданной на всех подмножествах конечного множества. Метод построения функции (1) приводит к выделению экстремальных множеств. Основная особенность метода построения основана на предположении, что для каждого элемента α существует набор чисел $\left\{\pi_H(\alpha)\right\}$, где H – подмножество конечного множества и $\alpha \in H$.

1. Введение

В нашем исследовании мы рассматриваем задачу нахождения глобального экстремума функции, заданной на всех подмножествах данного конечного множества. Описанный алгоритм построения применялся для решения некоторых задач классификации объектов с помощью метода однородных цепей Маркова. В общем виде предлагаемая конструкция позволяет решать некоторые задачи на графах, например, выделять в некотором смысле «связные» подмножества вершин графа. Теоретическая основа конструкции формулируется в терминах специальных правил отбора последовательностей подмножеств данного конечного множества и некоторых последовательностей его элементов, результатом которых является извлечение экстремальных подмножеств.

Задачи подобного типа имеют или носят комбинаторный характер и относятся скорее всего к задачам дискретного программирования. Определенный класс подобных задач на конечных множествах успешно решается в работах Черенина (1962), Черенина и Хачатурова (1965). В рамках этих работ рассматривались функции, удовлетворяющие условию, которое можно сформулировать следующим образом. Если ω_1 и ω_2 являются двумя представителями подмножеств данного конечного множества, то

$$f(\omega_1) + f(\omega_2) \le f(\omega_1 \cup \omega_2) + f(\omega_1 \cap \omega_2).$$

Это условие в некоторой степени отражает выпуклость функции f.

Главным свойством или требованием предъявляемым к рассматриваемому в рукописи класса функций является предположение о существовании некоторых чисел или весов (credentials, ed.), выявляющих для каждого элемента конечного множества степень его вхождения в подмножество. Степень вхождения должна удовлетворять условиям пп.1-2 (см. ниже).

Относительно настоящего исследования стоит также обратить внимание на работу Миркина (1970). В данной работе задача оптимальной классификации сводится к поиску специальной «раскраски» на неупорядоченном графе. Оптимальная классификация там характеризуется некоторым максимальным значением функции, соответствующим по своему виду определению (1), однако при этом мы интерпретируем (1) в ином смысле. Мы не рассматриваем в нашем определении функции разбиение заданного множества на два непересекающихся подмножества, что было основной задачей Миркина (cf., Võhandu & Frey, 1966, ed.).

2. Принцип максимума

Пусть $\{H\}$ множество подмножеств некоторого конечного множества W. Предположим, что мы вводим функцию π_H для каждого из элементов $H \subseteq W$ на совокупности подмножеств $\{H\}$ в качестве аргументов. Ниже под набором $\{\pi_H\}$ мы подразумеваем систему весов на множестве подмножеств $\{\{\pi_H\}\}$. Основное предположение относительно весовых систем следующее:

п.1 Весом $\pi_H(\alpha)$ элемента $\alpha \in H$ является действительное число;

п.2 Между различными системами весов $\{\{\pi_H\}\}$ для разных подмножеств $\{H\}$ набора $\{\pi_H\}$, существуют следующие зависимости: для каждого элемента $\alpha \in H$ и каждого $\beta \in H \setminus \{\alpha\}$ справедливо: $\pi_{H\setminus\alpha}(\beta) \le \pi_H(\alpha)$.

Другими словами, согласно пункту 2, требование состоит в том, чтобы удаление произвольного элемента α из множества H приводило бы к новой системе весов $\{\pi_{H\setminus\alpha}\}$, а влияние удаленного элемента α на веса в оставшейся части $H \setminus \{\alpha\}$ было бы только в направлении уменьшения. Поясним эти два условия на примерах из теории графов, хотя есть и примеры из других областей познания, однако менее удобные для краткого обсуждения. Рассмотрим неориентированные графы, т.е. графы со свойством, когда отношение вершины x к y влечет обратное отношение вершины y к x.

Пример 1.

Пусть W – множество вершин графа G. Мы определяем систему весов $\{\pi_H\}$ на каждом подмножестве H вершин как набор чисел $\{\pi_H(\alpha)\}$, где число $\pi_H(\alpha)$ равно количеству вершин, в H связанных с вершиной α. Истинность пп. 1 и 2 легко проверяется, если только вспомнить, что вместе с удалением вершины α должны быть одновременно удалены все связанные с ней ребра.

Пример 2.

Пусть W это множество ребер в графе G или множество пар вершин, связанных графом G. Определим весовую систему $\{\pi_H\}$ на произвольном подмножестве H ребер в графе G как набор чисел $\{\pi_H(\alpha)\}$, где $\alpha \in H$ и $\pi_H(\alpha)$ – количество треугольников в множестве ребер H, содержащих ребро α. Число $\pi_H(\alpha)$ равно числу тех вершин, на которых находится множество H, такое, что если x вершина указывающая на ребро и ребро $\alpha = [b, e]$, то отсюда следует что $[b, x] \in H$ и $[e, x] \in H$.

В примерах мы использовали тот факт, что граф является топологическим объектом с одной стороны и бинарным отношением с другой стороны. Теперь рассмотрим следующую функцию множества

$$f(H) = \min_{\alpha \in H} \pi_H(\alpha), \tag{1}$$

где $H \subseteq W$. Ниже мы предлагаем принцип, справедливый для подмножества H, на котором достигается глобальный максимум функции типа (1). Сформулируем этот принцип в терминах некоторых последовательностей элементов множества W и последовательностей подмножеств того же множества W.

Пусть $\overline{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ – последовательность элементов множества W и $k = |W|$. При помощи последовательности $\overline{\alpha}$ задана последовательность множеств $\overline{H}(\overline{\alpha}) = \{H_0, H_1, \ldots, H_{k-1}\}$, где $H_0 = W$ и $H_{i+1} = H_i \setminus \{\alpha_i\}$.

Определение 1. Назовем последовательность $\overline{\alpha}$ элементов из множества W определяющей, если в последовательности множеств $\overline{H}(\overline{\alpha})$ существует подпоследовательность $\overline{\Gamma} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_p\}$ такая, что:

1°. Вес $\pi_{H_i}(\alpha_i)$ произвольного элемента, принадлежащего Γ_j, но не принадлежащего Γ_{j+1}, строго меньше $f(G_{j+1})$;

2° В Γ_p не существует такого строгого подмножества L, что $f(G_p) < F(L)$.

Определение 2. Назовем подмножество H множества W определимым, если существует определяющая последовательность $\bar{\alpha}$ такая, что $H = \Gamma_p$.

Ниже мы вновь воспользуемся набором $\{\pi_H\}$ в виде системы весов по отношению к множеству H.

Теорема. На определимом множестве H функция $f(H)$ достигает своего глобального максимума. Определимое множество единственно. Все множества, в которых достигнут глобальный максимум, лежат в определяемом множестве.

Доказательство. Пусть H определимое множество. Предположим, что существует такое $L \subseteq W$, что $f(H) \le f(L)$. Предположим, что $L \setminus H \ne \varnothing$; в противном случае нам остается только доказать единственность H, что мы и сделаем ниже. Пусть H_t есть наименьшее из множеств H_i $(i = 0,1,\ldots,k-1)$, включающих в себя множество $L \setminus H$. Из этого факта можно заключить, что существует такой элемент $\ell \in L$, что $\ell \in H_t$, но $\ell \notin H_{t+1}$. Более того, в сочетании с последним $L \setminus H \ne \varnothing$ напрашивается вывод $t < p$. Неравенство $t < p$ располагает к существованию хотя бы одного такого подмножества в последовательности множеств $\bar{\Gamma}$, что

$$\pi_{H_t}(\ell) < f(\Gamma_j) \tag{2}$$

и $j \ge t+1$. Так как $\ell \notin H_{t+1}$ и $\Gamma_j \subseteq H_{t+1}$ верны, то следует, что $\ell \notin \Gamma_j$. Таким образом, неравенство

$$f(\Gamma_j) \le f(\Gamma_p) \tag{3}$$

справедливо как следствие п. 2° определяющей последовательности.

Теперь пусть $\ell \in \mathrm{L}$ и веса $\pi_{\mathrm{L}}(\ell)$ минимальны в системе весов по отношению к множеству L. Неравенства (2) и (3) позволяют сделать вывод, что $\pi_{\mathrm{H}_t}(\ell) < \pi_{\mathrm{L}}(\ell)$. Выше мы выбрали H_t при условии, что $\mathrm{L} \subset \mathrm{H}_t$. При этом, вспоминая основное свойство п.2 системы весов (удаление элементов), нетрудно установить, что $\pi_{\mathrm{L}}(\ell) \le \pi_{\mathrm{H}_t}(\ell)$, т. е. в системе весов по отношению к множеству существует вес, строго меньший, чем минимальный. Мы пришли к противоречию и тем самым доказали, что достигнут глобальный максимум. Далее, все такие множества H, отличные от L, где также достигается глобальный максимум, действительно могут находиться внутри H. Остается доказать лишь единственность определимого множества H. В связи с доказанным выше можно предположить, что некое определимое множество H' находится внутри H, однако, продолжая линию рассуждений, аналогичную предложенной нами выше для L, заключаем, что $\mathrm{H} \subset \mathrm{H}'$. ■

Следствие. Пусть $\{\mathrm{R}\}$ – система множеств, в которой функция типа (1) достигает своего глобального максимума. Тогда, если $\mathrm{H}_1 \in \{\mathrm{R}\}$ и $\mathrm{H}_2 \in \{\mathrm{R}\}$, то $\mathrm{H}_1 \cup \mathrm{H}_2 \in \{\mathrm{R}\}$.

Доказательство. Следуя пункту **2°** (основное свойство) $f(\mathrm{H}_1) \le f(\mathrm{H}_1 \cup \mathrm{H}_2)$, а кроме того из $f(\mathrm{H}_1 \cup \mathrm{H}_2) \le f(\mathrm{H}_1)$, следовательно $\mathrm{H}_1 \cup \mathrm{H}_2 \in \{\mathrm{R}\}$. ■

Ниже мы приводим конкретный алгоритм построения определяющих последовательностей элементов множества W. Для доступности алгоритм представлен в виде блок-схемы, похожей в какой-то степени на компьютерную программу.

3. АЛГОРИТМ

а1. Пусть множество $\mathrm{R} = \mathrm{W}$ и последовательности $\overline{\alpha}$ и $\overline{\beta}$ вначале пусты, а индекс $i = 0$. Здесь $\overline{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots\}$, $\overline{\beta} = \{\beta_1, \beta_2, \ldots, \beta_i, \ldots\}$.

a2. Найдите элемент μ с наименьшим весом по отношению к множеству R, запоминаем значение $\lambda = \pi_R(\mu)$ и полагаем после этого $\overline{\alpha} = \overline{\alpha}, \overline{\beta}, \mu$, а затем $\overline{\beta} = \varnothing$.

a3. Исключаем элемент μ из множества R и учитываем влияние удаленного элемента на оставшиеся элементы $\mu \in R$, т.е. вычисляем все велечины $\pi_{R\backslash\mu}(\beta)$ для всех $\beta \in R \setminus \{\mu\}$.

a4. В случае, если среди остальных (оставшихся) элементов найдутся такие γ, что

$$\pi_{R\backslash\mu}(\gamma) \le \lambda \tag{4}$$

то образуем последовательность указанных элементов $\overline{\gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ и положим $\overline{\beta} = \overline{\beta}, \overline{\gamma}$.

a5. Положим множество $R = R \setminus \{\mu\}$ и элемент $\mu = \beta_{i+1}$ и . возвращаемся к пункту a3 в случае, если элемент β_{i+1} определен для последовательности элементов $\overline{\beta}$, увеличивая в этот момент индекс i на единицу.

a6. В случае, когда последовательность $\overline{\alpha}$ изчерпаала все множество W, построение закончено. В противном случае вернитесь к пункту a.2, полагая сначала индекс $i = 0$.

Докажем, что только что построенная по предложенному алгоритму последовательность $\overline{\alpha}$ является определяющей. Рассмотрим последовательность $\overline{H}(\overline{\alpha})$ и выделим в качестве последовательности $\overline{\Gamma}$ те множества, которые начинаются с элемента, найденного в момент перехода алгоритма через шаг a2. Факт пересечения a2 алгоритма гарантирует, что условие (4) не выполнялось до того, как произошло пересечение, и элемент β_{i+1} не находится в последовательности на данном этапе β. Сказанное выше гарантирует также выполнение условия 1° для определяющих последовательностей. Предположим, что условие 2° в определении 1 не выполнено, т.е. в последнем множестве Γ_p последовательности $\overline{\Gamma}$ существует такое подмножество L, что $f(\Gamma_p) < f(L)$. Рассмотрим последовательность $\overline{\beta}$, которая генерируется при последнем переходе через a2 вышеописанного алгоритма, и пусть λ_p символизирует наибольшее значение среди всех таких λ. Приходится заключить что из предположения о существовании множества L, и замечая что $\lambda_p = f(\Gamma_p)$ приходим к неравенству $\lambda_p < f(L)$. По построению последовательность $\overline{\alpha}$ и вместе с последовательностью $\overline{\beta}$ (обе они), которая генерируется при последнем переходе через а2 алгоритма, использовали все элементы W. Следовательно, мы можем рассматривать множество элементов K в последовательности $\overline{\beta}$, которые начинаются с первого противостоящего элемента $\ell \in L$, где $L \subset K$. На основании обоснованного выше имеем $\pi_K(\ell) = \lambda_p$, и, вспоминая основное свойство учетной системы п.2 (удаление элементов), заключаем, кроме того, что $\pi_L(\ell) \leq \lambda_p$. Мы пришли к противоречию и тем самым доказали свойство 2° определения 1 для последовательности $\overline{\alpha}$. В связи с этим возможно построение определяющих последовательностей по указанному выше алгоритму.

Подчеркнем необходимость конкретизации понятия системы весов применительно к подмножеству заданного конечного множества для решения некоторых задач распознавания образов, что должно стать предметом дальнейшего исследования.

В заключение отметим, что построение определяющих последовательностей реализовано на практике на ЭВМ для одной задачи теории графов, связанной с выделением «почти полносвязных» подграфов в заданном графе. Количество ребер в таких графах составляет около 10^4.

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Case Study of Fuel Consumption by Vehicles Utilizing the Postulates of Bounded Rationality

Joseph E. Mullat [*]

Abstract

The article introduces a groundbreaking method called "*Blind Data Analysis*," which simplifies the examination of numerous statistical indicators with unknown distributions. It emphasizes the practical importance and necessity of logically categorizing data to improve the accuracy of forecasts. This method seems to make considering different indicators easier, following the principle of simplicity similar to Ockham's Razor. By leveraging the idea of reality through the phenomena of a 'monotonicity constraint,' it suggests a more dynamic approach to analysis. Particularly, it explores the core ideas of so-called bounded rationality in decision-making, providing strong support for the reliability of the method. Using data from the Spritmonitor.de database, which tracks various vehicle metrics including gas and electricity consumption and mileage, the study showcases the effectiveness of the method. This database enables users to track fuel savings and expenses by providing real-world cost metrics for thousands of vehicles. To ensure evaluation of the method's reliability through the bounded rationality postulates, it was rigorously tested against this database using the Excel macro program.

Keywords: data analysis; decision-making; vehicle; fuel consumption; monotonic system

Concise Glossary of Mathematical Notations

We consider fuel consumption indicators $p_k \in A$, $|A| = n$ of n car models or labels/issues, $k = \overline{1,n}$. Indicators $\langle \overline{p} \rangle = \overline{p}_1 \geq \overline{p}_2 \geq \ldots \geq \overline{p}_j \geq \ldots \geq \overline{p}_n$, $j = \overline{1,n}$, in contrast to the original list p_k, are necessarily descending. A sequence $\overline{\pi} = \langle \pi_j \rangle = \pi_1, \pi_2, \ldots, \pi_j, \ldots, \pi_n$ arrange so-called ″torques″ or ″moments″, where $\pi_j = \overline{p}_j \times j$ and index j points at the ″distance″ from the top of the descending list $\langle \overline{p} \rangle$. The choice consists of an act of selecting several car models from X, Y or $H \subseteq A$ (as described by Ma et al., 2015) or as multiple options $C(X) \subseteq X$, $C(Y) \subseteq Y$ or $C(H) \subseteq H$ according to certain rules also outlined by Strzalecki (2011). A totality of lists of all 2^n samples or issues $H \subseteq A$ is denoted by $2^A = \{H\}$. In accord with the descending sequence $\langle \overline{p} \rangle$, samples X, Y can be reordered into segments $X = [x\_1 > x\_r]$, $Y = [y\_1 > y\_r]$. Thus, segments X, Y correspond to choices $C(X) = [c(x)\_1 > c(x)\_r]$ and $C(Y) = [c(y)\_1 > c(y)\_r]$, etc. In this case, we refer to the segmentation operators $C(X)$ and $C(Y)$ as a choice made from reordered segments $S(A)$.

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1. INTRODUCTION

Typically, contributions to theory delving into reality involve expanding existing categories, concepts, models, or simplifications to uncover new theoretical insights, resolve unresolved issues, propose novel frameworks, enhance predictive accuracy, or address practical applications. However, an alternative approach to deriving fresh knowledge from established categories entail identifying novel relationships or connections concealed within fundamental categories. This article's primary objective is to innovate by juxtaposing two existing elements in a novel manner—a comparison, or rather, an interpretation of data analysis within the framework of decision-making processes. The intricate relationship between data analysis and decision-making is particularly pronounced in the automotive market, where precise evaluations of factors like fuel consumption present challenges. Despite its apparent nature, rigorous scientific validation ensures that the correlations between data analysis and decision-making remain grounded in reality, averting potential misinterpretations.

Car models are conventionally categorized based on their fuel consumption or economy, aiding consumers in making informed decisions regarding efficiency and environmental impact. These categories typically encompass labels such as "compact," "midsize," or "luxury," each indicating distinct performance levels and fuel efficiency standards. Our analysis entails examining a dataset comprising 2927 car models from various manufacturers, with a focus on fuel consumption measured in Miles Per Gallon or Liters per 100 km for conventional vehicles, and additionally, in kWh/100 km for electric and hybrid vehicles

Data pertaining to established car models typically yield relatively accurate forecasts. However, numerous factors, including lifestyle changes, technological advancements, and fluctuating fuel prices, can significantly influence the automotive market, posing challenges for even the most widely used models to adequately account for them. Publications in esteemed journals such as those by the Society of Automotive Engineers (SAE), the International Journal of Automotive Technology, and authoritative sources like the International Council on Clean Transportation (ICCT) often delve into topics concerning vehicle fuel efficiency, labeling, and data analysis methodologies. Consequently, while a straightforward data analysis approach may not be optimal, it may be preferable to overly intricate probabilistic case studies of fuel characteristics.

Data analysis and categorization techniques are pivotal across diverse domains, spanning from machine learning to information organization. In the domain of machine learning, algorithms such as decision trees, support vector machines, and neural networks are prevalent for classification endeavors. These methodologies are designed to allocate pre-established labels or categories to input data by discerning patterns and features inherent within them.

In data analysis and classification, two opposing approaches can be distinguished based on their direction: one begins with subjective knowledge to reach objective conclusions, while the other follows the reverse path. In the former approach, experts in various fields such as physicians, biologists, astronomers, market practitioners, and data analysts utilize categorization techniques to objectively interpret experimental data and observations. They may employ artificial intelligence (Mitchell, 1997; Seiffarth et al., 2021), statistical learning (Hastie et al., 2009), pattern recognition (Bishop, 2016), machine learning (Domingos, 2010, which explores the quest for a universal learner), or other probabilistic methods (Murphy, 2012; Koller and Friedman, 2009) in their data analysis. This also encompasses data mining techniques (Nan and Kamber, 2011, covering various data mining aspects), parameter estimation (Walter and Pronzato, 1997), management and decision-making problems (Narula and Weistroffer, 1989), continuous modeling, multiple time series (Voelkle et al., 2012), and computational methods (Mirkin et al., 1995). Interpretable Machine Learning: A Guide For Making Black Box Models Explainable Paperback – (Christoph Molnar, 2022) is a book aimed at helping practitioners understand and interpret the decisions made by complex machine learning. However, despite these approaches requiring an understanding of the distribution of judgments regarding the object under analysis, practical implementation doesn't always align with this ideal.

The purpose of subjective assessment methods (Frey, Võhandu, 1966) appears to lie in their utilization for dividing multi-dimensional indicators into two classes, a task that may initially seem contradictory. Specialized knowledge might be deemed necessary, although, as previously mentioned, it's not always the case since understanding the distribution of numerical parameters, or indicators, could be superfluous. These points are accentuated when examining the objective-subjective data analysis outlined in the article, particularly within the framework of "*Blind Data Analysis*" (referred to as ***BDA***), which concentrates solely on discerning whether one number is "less than" or "greater than" another. Achieving common sense in this approach invokes the well-known principle of parsimony, also known as "Ockham's Razor," which posits that simpler theories are preferable to more complex ones. Consequently, a procedure requiring fewer assumptions about reality can be considered the most reliable in such contexts.

2. BOUNDED RATIONALITY POSTULATES

Rational choice theory stands as a robust framework designed to elucidate the intricate processes underlying individual decision-making, factoring in their unique preferences and the constraints they encounter (e.g., Arrow, 1948; Jamison, 1973). Within this framework, several postulates fall under the umbrella of bounded rationality, a concept acknowledging the cognitive limitations individuals face when making decisions. These postulates encompass the assumption of well-defined preferences, the consideration of expected utility in decision-making, and the reliance on available information to arrive at rational choices.

When potential car buyers apply these foundational assumptions to the dynamic realm of vehicle evaluations, it becomes apparent that an additional postulate or constraint—monotonicity—must be introduced to accommodate the fluidity of consumer preferences and assessments. Monotonicity posits that as the list of potential car models for purchase or sale is progressively narrowed down, consumers' subjective evaluations or utilities (referred to here as moments) exhibit a consistent and monotonic decrease. Simply put, as options dwindle, individual preferences or the perceived value of the remaining models diminish in a predictable and continuous manner.

Moreover, within this framework, the concept of constraint pertains to the spontaneous or impulsive judgments and evaluations potential buyers make during the car selection process. The argument posits that as buyers systematically eliminate certain car models from consideration, their impulsiveness diminishes, signaling a transition toward more deliberate and thoughtful evaluations as the selection process unfolds. This evolution in decision-making behavior underscores the dynamic interplay between rationality, bounded by constraints, and the ever-shifting landscape of consumer preferences.

Indeed, this perspective holds relevance across a spectrum of decision-making and data analysis scenarios, extending beyond just vehicle evaluations to encompass the evaluation and comparison of various products before making a purchase. It operates on the premise that as decision-makers systematically eliminate alternatives, their preferences, classifications, and pairwise comparisons exhibit a consistent decline rather than fluctuating or displaying non-monotonic behavior. This principle underscores the importance of understanding how preferences evolve and assessments shift as choices are narrowed down, guiding decision-makers towards more informed and rational choices. Whether in consumer behavior analysis, market research, or strategic planning, recognizing the monotonic nature of decision dynamics can provide valuable insights for optimizing decision processes and outcomes.

It's worth highlighting that while the concept of a monotonic constraint provides a valuable framework for understanding decision-making processes, it may not universally apply to all scenarios. Individual preferences and subjective assessments can vary significantly, leading to diverse valuation criteria and impulses among different people. Consequently, their evaluations may not always exhibit a monotonic decrease as the range of options narrows down.

Therefore, while constraints offer insights into certain decision dynamics, they should be applied judiciously and in conjunction with other factors that influence individual choices. To address this variability, Arrow's (1959) strict consistency postulate is slightly adapted in this work to uphold the fundamental postulates of rational choice. This modification to the standard rational choice framework acknowledges the dynamic nature of the automotive market and the behavior of car buyers.

Specifically, the process of choice involves selecting from various issues (as described by Ma et al., 2015) or multiple options according to specific rules outlined by Strzaletski (2011). By integrating these considerations, the decision-making framework becomes more robust, capturing the nuances of individual preferences and the complexities of real-world decision contexts.

Let us recall in a Boolean—that is, in a more formal—form the bounded rationality canonical postulates (cited by Aizerman and Malishevski, 1981, pp. 65□83, English version translated from Russian, p. 189). Here, they are presented in connection with rational choice in the automotive market that involves factors such as fuel efficiency, cost, environmental impact, and personal needs. Evaluating these elements helps customers make an informed decision that aligns with their preferences and priorities, as outlined below.

- *Independence with respect to removing rejected alternatives (or, for brevity, elimination of options), Postulate 5 (Chernoff,* 1954*, pp. 422−443) or Axiom 2 (Jamison and Lau,* 1973*, pp. 901−912):*

  From $C(Y) \subset X \subset Y$ *it follows that* $C(X) = C(Y)$*;*

- *Compatibility, the same as Postulate 10 of Chernoff and property* γ *of Sen:*

  From $X \cup Y$ *it follows that* $C(X) \cap C(Y) \subset C(X \cup Y)$

- *Non-strict Consistency, which is the same as Postulate 4 (Chernoff), or property* α *(Sen,* 1971*, pp. 307−317) or the axiom C2 of Arrow-Uzawa (Arrow,* 1959*, pp. 121−127):*

  From $X \subset Y$ *it follows that* $X \setminus C(X) \subset Y \setminus C(Y)$ *or equivalent to* $X \cap C(Y) \subset C(X)$*;*

- *Strict Consistency or constant residual choice, which is the same as Postulate 6 (Chernoff,* 1954*) and one of the forms of the "weak axiom of revealed preference" of Samuelson, i.e., the axiom C4 (Arrow,* 1959*, pp. 121−127):*

  From $X \subset Y$ *and* $X \cap C(Y) \neq \varnothing$ *it follows that* $X \cap C(Y) = C(X)$.

The strict consistency postulate was validated through experiments involving the correlation matrix, as detailed in the Appendix. Specifically, preferences for pairwise comparisons of indicators within the narrowed list X of the broader set Y remain consistent, maintaining the same "less/greater" relation $x \leq y$ or $y \leq x$ as in Y. However, it's crucial to note that pairwise indicator preferences may disproportionately shift based on the narrowed list X, potentially leading to the selection of previously unselected indicators from Y in X, akin to our Pedagogical Scenario. To address this limitation of the strict consistency postulate C4, a simple correction can be implemented while preserving the validity of the rational choice postulate, i.e.:

From $X \subset Y$ and $X \cap C(Y) \neq \varnothing$ it follows that $X \cap C(Y) = C(X) \cap C(Y)$.

Even in this slightly modified form, the postulate maintains its functionality akin to Arrow's canonical strict consistency postulate, despite the original founders of rational choice theory, such as Simon in 1978, not explicitly considering this dynamics. These foundational postulates ensure consistent outcomes with repeated decisions, enhancing predictability in the decision-making process. Specifically, these postulates ensure that relations among objects remain stable, mirroring the complete visual representation of the objects. This concept resonates with the idea of self-similarity found in the Fibonacci principle, where the characteristics of a part reflect those of the whole. Such self-similarity is a common observation in various natural patterns and structures, underscoring the universality of these principles across different domains and contexts.

3. PARSIMONIOUS APPROACHES

To validate the postulates of rational choice within the context of fuel consumption classes across various car manufacturers, an examination of consumer decision-making when selecting a car based on fuel efficiency can offer valuable insights into whether their choices align with the assumptions of rational choice theory. Several statistical methods can be employed to test these postulates. Econometric models, for instance, can estimate the parameters of a utility function describing how car owners make decisions, while machine learning algorithms can identify patterns in the data and assess their consistency with rational choice theory. It's important to note that the following discussion on the exhibition scenario serves as an introduction to the main topic, focusing on the results and experiments conducted using the Excel spreadsheet of information and interactive computer services available at https://www.spritmonitor.de/en/ . The spreadsheet underwent a validity test of the independence postulates of the rejected alternative and the postulates of non-strict and strict consistency with established car models in the market.

The findings suggest that car buyers may prioritize factors such as engine power or fuel efficiency, which can be rationalized when considering rejected alternatives. The postulates of consistency, as per Ockham's Razor procedure, appear to corroborate the experimental results. However, it's important to acknowledge that rigorous proof of these assertions exceeds the scope of this article and warrants further research. Moreover, the proof of independence from the rejected alternatives stems from Proposition I outlined in Section 3.3, with its origins tracing back to its publication in the Proceedings of Tallinn Polytechnic Institute by Mullat in 1971.

3.1. Pedagogical Scenario

The postulates of consistency and compatibility are specific theoretical foundations used in economics to analyze decision-making behavior. These postulates suggest that individuals make decisions based on issues of consistent preferences that do not change over time. With regards to the postulate of strict consistency C4, it is useful to paraphrase Arrow's intuitive interpretation to consider the following. In terms close to the data analysis scenario, this suggests that if some car models are labeled in the context of fuel consumption from the range of models available for sale, then narrowing the range of labels should not change the status of previously labeled or unlabeled to selected or unselected to designate models' cars. While this is part of the rationality criteria that Arrow explored in his work on social choice theory and the impossibility of creating a perfect voting system, it does assume some stability in the labeling system, which is often useful when implementing data analytics tools.

Let's start the scenario with a "hypothetical" or "pedagogical exhibition" based on the choice of a car at an exhibition when analyzing decision-making classification phenomena. Suppose further that, after accepting an offer to purchase a car, the salesman tells the customer that some of his preferred options are not available, potentially causing irrational behavior on the part of the customer or the salesperson. From a customer's point of view, it might be wiser to try fuel-efficient cars that were initially overlooked. On the other hand, the seller may offer more stylish and luxurious cars, even though there are economical and equally good options. If a customer preferred fuel-efficient vehicles but learned they were out of stock, they would likely include more fuel-efficient models to expand the list of alternatives that were initially overlooked. However, if the buyer wanted to buy an economical car and his choice was limited for some reason, buyers, contrary to their original intention, may welcome the seller's alternative offering more stylish models.

3.2. Ockham's Razor significance

Returning to the car's pedagogical scenario, the cars would be listed linearly in descending order based on fuel/electricity, where the highest fuel or electricity consumption per 100 km is multiplied by **1**, the next item in the list is multiplied by **2**, and so on. Here these figures are interpreted as *"fuel consumption credentials"* or "moments". The local maximum fuel consumption is selected when the moment's maximum is reached. Some details of the car selection procedure just outlined are also relevant for analyzing automobile market data.

Let us assume that the client decides to accept the car fuel consumption at the local moment maximum as an acceptable level of significance when choosing cars with a higher or equal fuel consumption level, e.g., the list $\mathbf{10^2}$, $\mathbf{9^2}$, $\mathbf{8^2}$, $\mathbf{7^2}$, $\mathbf{6^2}$, $\mathbf{5^2}$,… indicates that the peak of this sequence is located at $\mathbf{7^2 = 49}$. Define a list of fuel consumption indicators $p_k \in A$, $|A| = n$ of n car models, $k = \overline{1,n}$. In particular, suppose that in the sample denoted by the letter H, the prospective car buyer selects some potential cars as viable candidates according to the reasonable fuel consumption. We can further define a totality of lists $\{H\}$ of all 2^n samples or issues $H \subseteq A$. Accordingly, $\pi(p_k, H) = p_k \cdot |H|$ moments as monotone system (in terms of Mullat, 1971), or as monotone linkage clustering (Kempner et al., 1997,) will evaluate so already called *"credentials"* of fuel consumption. The procedure for finding the significance level of fuel consumption commences with sorting all the fuel consumption indicators p_k, constituting (as in the price sticker list) the vehicle fuel/electricity indicators permutation $\langle \overline{p} \rangle = \overline{p}_1 \geq \overline{p}_2 \geq ... \geq \overline{p}_j \geq ... \geq \overline{p}_n$ in descending order. Next, a sequence $\overline{\pi} = \langle \pi_j \rangle = \overline{p}_1 \times 1, \overline{p}_2 \times 2, ..., \overline{p}_j \times j, ..., \overline{p}_n \times n$, which components π_j we called moments, $j = \overline{1,n}$, is constructed. Hereby, the list of fuel consumption indicators $\langle \overline{p} \rangle$, in contrast to the original list p_k, is necessarily descending. We called such sequences $\overline{\pi}$ as defining (Mullat 1971).

3.3. Internal Personal Stability

Internal or intrinsic personal stability refers to an individual's ability to maintain a sense of balance, composure, and well being within themselves, irrespective of external circumstances or challenges. It involves emotional resilience, self-awareness, and a capacity to navigate life's ups and downs with a steady and grounded mindset.

When we talk about internal personal/intrinsic stability in terms of being "interpersonally incompatible" or "impossible to match through a monotone transformation," it means that the relative economic preferences or classifications of individuals cannot be reconciled using a simple, consistent scaling or transformation.

In the context of Narens and Luce's work from 1983, monotone transformation refers to a mathematical function that preserves the order of preferences but might change the classification scale. If two individuals have inherently incompatible preferences that cannot be aligned through such transformation, it implies that there is no single, uniform way to compare or match their classification preferences or evaluations. This concept underscores the complexity of ensuring stability in interpersonal classification framework, as certain inherent classes or categories in individual preferences may resist easy standardization or comparison.

Our experiments show that when categorizing vehicles according to specific utility functions or monotonic transformations based on fuel consumption using ***BDA***, the assumption of intrinsic personal stability is generally not satisfied. Simply put, this highlights the complexity of the process, as individual preferences and ratings, such as the rationale for predicting vehicles' prices based on fuel consumption, in some cases cannot be consistently compared and categorized, regardless of the specific scaling or units of measurement used.

3.4. The reasonable level

The moments $\langle \pi_k \rangle = \pi_1, \pi_2, ... \pi_j ... \pi_n$ are single peaked, where the peak denotes the kernel issues H^* (Mullat, 1971−1995) of a monotone system. The list H^* constitutes the rational, i.e., the monotone linkage clustering implemented in our findings. At the location k^* from the top of the moments $\overline{\pi} = \langle \pi_k \rangle$, i.e., from the top of the defining sequence of models, $j = \overline{1,n}$, where the local maximum $u = \max_{j=\overline{1,n}} \pi_j$ is reached, the peak, denoted by u, will be called the level of significance.

Proposition I*. Among the totality of all samples* $H \subseteq A$*, i.e., among all the lists* $\{H\}$ *of all* 2^n *samples, the kernel* H^* *guarantees reaching the global maximum of the moment function* $F(H)$ *of samples* H; $F(H) = u$ *is equal to* $u = \min_{p_k \in H} \pi(p_k, H)$: $H^* = \arg\max_{H \subseteq A} F(H)$.

Proposition I confirms the postulate of independence from rejected alternatives in two-person games, which was originally studied by John F. Nash in the 1950s, when he developed a solution to the bargaining problem. With regard to the market for the purchasing and production of cars, Proposition I states that any final decisions made or based on statistics should not be affected by the removal of any parts of those statistics that are not reliable or represent a very small number of cases in which, for example, statistics have been collected into a database and selected for review.

A transformation using some monotonic function $\mathrm{tr}(x)$ of indicators $\overline{p}_j \geq \overline{p}_{j+1}$, $j=\overline{1,n-1}$, preserving the validity of $\mathrm{tr}(\overline{p}_j) \geq \mathrm{tr}(\overline{p}_{j+1})$, can still shift the original H^*-kernel to $H^* \neq \mathrm{tr}(H^*)$, what happens, for example, when transforming by $\mathrm{tr}(x) = x^2$. However, the H^* kernel, which takes into account fuel consumption under the personal/intrinsic guidance of Narens and Luce, remains unbiased for the proportional mapping $\mathrm{tr}(x) = \alpha \cdot x$, for example, in the case of converting liters to gallons.

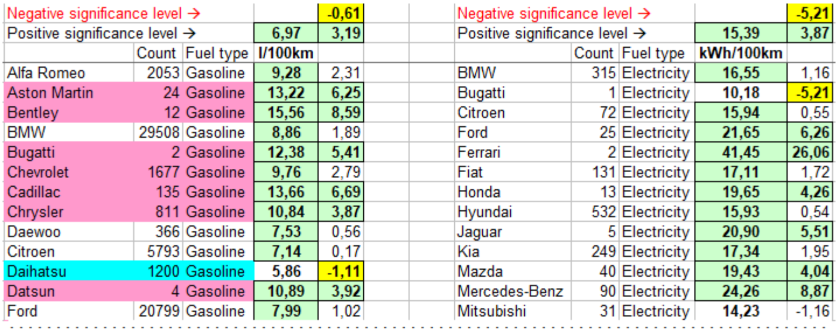
3.5. Threshold-based Time-Series indicators

Time-Series data typically refers to data that is collected or updated periodically or at regular intervals over time. Series are commonly used in data mining and other areas where the objective is to detect outliers or changes in system behavior. This could include information such as sales figures, stock prices, weather data, or any other type of data that is recorded over a specific time period. The frequency of data collection can vary depending on the needs and requirements of the specific use case or analysis.

A threshold value can be used in the fuel consumption series to determine the reasonable significance of cases where consumption level exceeds a certain positive threshold or falls below a certain negative threshold (an illustration of this scenarios are presented in the Appendix). In the specific case considered, the indicators called fuel moments create a dynamic system since the previous state of the consumtion determines its subsequent state. It is worth noting that the postulates of strict and non-strict consistency emphasize the rational behavior of car buyers when new models expand the list of available alternatives. In the event that prospective car buyers have chosen some of the best cars in the past, these postulates state that they will still be inclined to consider old models, in accordance with the "old love does not rust" adage.

Accordingly, if we look at our methodology for determining the significance level u of car fuel consumption indicators p_j, one may get the impression that the procedure is applicable only to positive numbers. Generally speaking, the same procedure is obviously valid for a negative series of numbers. In this sense, the procedure can be used by analogy with what is called a ″confidence interval″ in statistics. Indeed, if we apply the procedure to a positive series of numbers, then as a result we will obtain a level of significance u in the form of a positive number. Now, based on the found level u of significance, we can create a sequence of deviations around this level, both $\pm\Delta$ towards less and towards excess. Now it will be possible to apply the procedure again, but this time in relation to deviations from the original significance level u. As a result, the ″confidence interval″ $\left[-\Delta_1 + u, u + \Delta_2\right]$ will be determined. By considering this interval and observing whether the dynamic indicators cross the threshold u, one can make significant decisions. Indeed, customers interested in economy cars would likely make a purchase if the indicators consistently cross below

$-\Delta_1 + u$, while this will be unlikely if the dynamic indicators intersect above the $u + \Delta_2$ level. Looking at this interval and observing whether the dynamic indicators cross the threshold value u customers can make a decision on how to cluster the relevant data. Since market actual monitoring for the automobiles data includes 2927 vehicle models, a fragmented version of it is presented in Table 1 (screenshot from an Excel spreadsheet).



Negative significance level →				-0,61
Positive significance level →			6,97	3,19
	Count	Fuel type	**l/100km**	
Alfa Romeo	2053	Gasoline	**9,28**	2,31
Aston Martin	24	Gasoline	**13,22**	**6,25**
Bentley	12	Gasoline	**15,56**	**8,59**
BMW	29508	Gasoline	**8,86**	1,89
Bugatti	2	Gasoline	**12,38**	**5,41**
Chevrolet	1677	Gasoline	**9,76**	2,79
Cadillac	135	Gasoline	**13,66**	**6,69**
Chrysler	811	Gasoline	**10,84**	**3,87**
Daewoo	366	Gasoline	**7,53**	0,56
Citroen	5793	Gasoline	**7,14**	0,17
Daihatsu	1200	Gasoline	5,86	-1,11
Datsun	4	Gasoline	**10,89**	**3,92**
Ford	20799	Gasoline	**7,99**	1,02

Negative significance level →				-5,21
Positive significance level →			15,39	3,87
	Count	Fuel type	**kWh/100km**	
BMW	315	Electricity	**16,55**	1,16
Bugatti	1	Electricity	**10,18**	-5,21
Citroen	72	Electricity	**15,94**	0,55
Ford	25	Electricity	**21,65**	**6,26**
Ferrari	2	Electricity	**41,45**	**26,06**
Fiat	131	Electricity	**17,11**	1,72
Honda	13	Electricity	**19,65**	**4,26**
Hyundai	532	Electricity	**15,93**	0,54
Jaguar	5	Electricity	**20,90**	**5,51**
Kia	249	Electricity	**17,34**	1,95
Mazda	40	Electricity	**19,43**	**4,04**
Mercedes-Benz	90	Electricity	**24,26**	**8,87**
Mitsubishi	31	Electricity	**14,23**	-1,16

Table 1. Screen dump from Excel spreadsheet:

$\geq u=+6.97$, $\leq \Delta_1=-61$, $\geq \Delta_2=+3.19$ $\geq u=+15.39$, $\geq \Delta_2=+3,87$, $\leq \Delta_1=-5.21$,

4. AUTOMOTIVE MARKET DATA

To demonstrate the effectiveness of the proposed approach, the standard mechanisms and techniques of the MS Windows platform were utilized to view the database related to thousands of car models in Excel spreadsheets. The list includes cars that are not only economical and reasonably inexpensive but also even expensive stylish or luxury and cars of all available models. This information has been extracted and recompiled from the interactive computer services provided on the Spritmonitor.de website and includes vehicle fuel data, significant volumes and other relevant variables.

Some comments are needed to clarify the implementation of our Ockham's Razor "procedure" for analyzing the car fuel consumption dynamics. Specifically, it should be noted that the reliability of data on the lease or purchase cars with regard to fuel consumption, where all fuel consumption data have been available to everyone, is given by the fact that the MPG (mileage or mile per gallon) data are guaranteed by the Cost Calculator and Tracker at the date-to-date basic activity at the Spritmonitor.de database. The spreadsheet https://www.spritmonitor.de/en/search.html was compiled using domain (Accessed July 10, 2023).

An overview of common internal combustion fuels or hybrid/pure electric vehicles is available (*http://www.datalaundering.com/download/MPG-MileAge-Data.xls*, August 22, 2023) in the database used in the experiment is presented here solely for the purpose of illustrating the data collected so that the article is well suited to the layperson of interest. Clearly, each fuel type has advantages and disadvantages in terms of efficiency, emissions, availability and infrastructure. Analyzing fuel consumption across these categories can provide valuable insights into the efficiency and environmental impact of different car models. As seen from the tabulated columns containing fuel consumption, blue and yellow cells differ from others in certain patterns and frames, identified using the macro—Ctrl+s. In accordance with Proposition I given earlier, an analysis of the significance levels of the negative/positive (yellow/blue) values of the car indicator dynamics has been conducted. Using the macro in columns, (selected or ″pasted″) areas X of spreadsheet in their entirety may consist of negative/positive numbers that are distributed without any special purchase. However, the standard EXCEL data sorting options allow the content of selected areas to be sorted in ascending or descending order depending on the specified columns or rows. Thus, relevant cells can be redistributed into ″contiguous areas″ of negative or positive values in the column or row patterns to satisfy the necessary conditions. Such contiguous areas can be used in experiments featuring the Case Study results. The $C(X)$ operator was compiled into the Ctrl+s macro, using automotive market share fuels X as the initial data table below in column format of alternative X.

5. OCKHAM'S RAZOR PROCEDURE GUIDE AND PROPERTIES

The novel procedure proposed here, as previously noted, was called ***BDA***, as it involves finding the simplest explanation or a most parsimonious models that fit the data based on the premise that simpler explanations are more likely to be true than complex explanations. It is important to note that the ***BDA*** procedure is not necessarily equivalent to other known statistical hypothesis testing, such as the null hypothesis (often denoted H_0), which is the statement that there is no significant difference or effect. Researchers seek to test this hypothesis against the alternative hypothesis (H_1), which, in contrast, suggests that there is a significant difference or effect. The goal is to determine, based on statistical analysis of the data, whether there is sufficient evidence to reject the null hypothesis in favor of the alternative. However, it is important to remember that our ***BDA*** procedure is just one tool in a broader set of statistical methods and may not be suitable for every situation.

5.1. Arranging indicators in Excel

In Excel spreadsheet http://www.datalaundering.com/download/MPG-MileAge-Data.xls you can find the ***BDA*** Ctrl+s macro. Those wishing to use it can copy the base spreadsheet code into their own spreadsheet. In the properties of this macro, you must also indicate that the macro can be executed using the Ctrl+s command. However, when tabulating information, the first two rows of the table must be blank, so users must insert at least two blank rows at the top of the table.

You can as well sort the p indicators in descending order by yourself, located somewhere in the spreadsheet for which you want to calculate the moments. Go to the Data tab and use the Sort option to arrange the p indicators $p_1, p_2, ..., p_n$ in descending order, denoted as $\overline{p}_1 \geq \overline{p}_2 \geq ... \geq \overline{p}_n$, based on the selected column. Then, for the column to the right of the sorted indicators, create an additional column of numbers $\pi_j = \overline{p}_j \times j$, where j ranges from 1 to n, moving from the top $\overline{p} \times 1$ of the sorted list of moments/numbers $\overline{p}_j \times j$ to the bottom $\overline{p} \times n$. You will notice that the $\overline{p}_j \times j$ moments rearrange themselves into a single-peaked sequence of moments corresponding to the numbers $\pi_j = \overline{p}_j \times j$. Finally, assign a special color to the newly created moments starting from the top of the list to an indicator somewhere within the sequence of moments corresponding to the local peak $u = \arg\max \pi_j$ reached while moving j from 1 to n.

5.2. Validation of consistency postulates

From the information presented in the main part of the article, it is clear that we are discussing the statistical moment of an indicator, which was used to classify fuel consumption in the context of choosing the optimal car. The moment was calculated as the product of its 'distance' or 'position number' from the top in a descending linearly ordered list of fuel consumption of vehicles. It serves to measure the desirability of each fuel consumption based on its magnitude. This measurement involves applying Ockham's Razor procedure to select the optimal option using fuel moment as a scalar criterion. From the perspective of Ockham's Razor, when choosing between competing options, simpler explanations or models are preferred.

Proof of Proposition 1.

To prove or verify the truth of this proposition, we can revisit the article. "On a Maximum Principle for Certain Functions of Set Functions", (Mullat, 1971). This is not necessary, however, since the proof in our particular case is much simpler thanks to the following lemma.

> ***Lemma.*** *In any subset* $H \subseteq A$ *of indicators* A*, the order of moments* $\pi(p_j, H) = p_j \cdot |H|$ *corresponds to the grand order of moments* $\pi(p_j, A) = p_j \cdot |A|$, $|A| = n$ *on the set* A.

Simply put, the lemma states that if we take a pair of indicators $p_i \le p_j$, then no matter in which subset X or Y we consider this pair of indicators $\left\langle p_i, p_j \right\rangle$, the moment inequalities of $\pi(p_i, X) \le \pi(p_j, X)$ or $\pi_i(p_i, Y) \le \pi_j(p_j, Y)$ will remain in force.

Now we can begin to prove the Proposition 1. The proof will be carried out by contradiction. So, by construction, the set of indicators p_j, $j = \overline{1, n}$, is ordered in descending order from largest to smallest. In this case, it is obvious that the sequence of moments $\langle \pi_j \rangle$, constructed according to the rules of our procedure by multiplying the indicators by 1, by 2, etc., starting with the largest indicator by multiplying by 1, etc., the sequence $\langle \pi_j \rangle$ is single-peaked. Let this peak be reached at a certain index p_*. This indicator indicates a certain maximum achieved at the local level. Now suppose that, contrary to the achieved local maximum, we can find a certain subset H' of indicators on which the set function $\min \pi_{j\in H}(H) > \pi_*$; this means that on some set H' the global maximum is greater than the achieved local maximum π_*. If we now supplement the list of indicators H' appending H' to $H_k \supseteq H'$—to the list of all indicators in A starting from some indicator $p_k = \arg\min \pi_{j\in H'}(H_k)$, $k = |H_k|$—then, according to the lemma, it turns out that in our single-peaked sequence $\langle \pi_j \rangle$ we encountered an indicator p_k with a moment $p_k \cdot k$, $k = \pi' > \pi_*$. This is not possible due to our construction method of the single peaked sequence $\langle \pi_j \rangle$. ■

The "Strict Consistency Postulate", which has been validated in experiments, has been modified to suit the decision-making process, based on the premise that, along with these modified postulates, a reliable and reasonable way to statistically analyze data is provided. Somehow, however, the theorem of Aizerman and Maliszewski (Theorem I, 1981) may be useful, which states that the scalar condition of utility functions is necessary and sufficient for the truth of the modified strict consistency postulate. A thorough analysis or evaluation of this claim is beyond the scope of this work. If necessary, the list A of indicators must be presented in the form of linearly descending order of fuel consumption or other economic values related to vehicles.

This means that the choice operator $C(X)$ on the issues X of the list A of alternatives/indicators acts on a certain list of segments $S(A)$, which can be either open or closed by resembling the set of all sub-lists 2^A. Thus, we can call this dual terminology by choice or by segmentation/classification. The alternatives A can be identified by special issues, now denoted already as segments $X = [x\_l,\ x\_r]$ of the indicators under consideration. Narrowing a segment $Y \subseteq S(A)$ to a segment $X \subseteq S(A)$ is an action of narrowing the segment $Y = [y\_l,\ y\_r]$ to $X = [x\_l,\ x\_r]$. In view of this understanding that indicators are linearly descending, the situation $x\_r \geq y\_r$ with segments can preserve the original ordering of choice operators $C(X)$ nomenclature. To do this, in the notation just introduced, a set function is defined (hereinafter referred to as the function $f(X)$ of the segment X): $f(X) = x\_r$ or $f(Y) = y\_r$. With this function $f(X)$ notification, we are ready to prove Proposition II given below.

Proposition II*. When shrinking the segment* $\mathrm{Y} \in \mathrm{S(A)}$ *to the segment* $\mathrm{X} \in \mathrm{S(A)}$ *as an extent of the segments of indicators of the common grand segment* A*, the condition* $\mathrm{f(C(Y)) > f(C(X))}$ *is necessary and sufficient for the fulfillment of the non-strict consistency postulate.*

Proof.

Necessity. Suppose $\mathrm{X} \subset \mathrm{Y}$ and condition $\mathrm{f(C(Y) > f(C(X)))}$ are satisfied, or in equivalent form between segments X and Y the situation results in $\left[\mathrm{c(y)\_r > c(x)\_r}\right]$. Note the validity of $\mathrm{c(y)\_r = \{y \setminus c(y)\}\_1}$ and $\mathrm{c(x)\_r = \{x \setminus c(x)\}\_1}$. Thus, the $\mathrm{f(C(Y) > f(C(X)))}$ condition results in $\mathrm{\{y \setminus c(y)\}\_1 > \{x \setminus c(x)\}\_1}$. Given that $\mathrm{X} \subset \mathrm{Y}$, we can rewrite the last inequality in set-theoretic notation as $\mathrm{X \setminus C(X) \subset Y \setminus C(Y)}$, which indicates the validity of the non-strict consistency. ■

Sufficiency. Let us assume that the postulate of consistency is not satisfied for some segments $\mathrm{X} \subset \mathrm{Y}$ in the form of segments $\mathrm{S(A)}$: i.e., contrary to the postulate of consistency, the condition $\mathrm{f(C(Y) > f(C(X)))}$ is violated. Given the violation, we consider only the opposite case $\mathrm{f(C(X) > f(C(Y)))}$, excluding the case $\mathrm{f(C(Y) = f(C(X)))}$. The opposite case $\mathrm{c(x)\_r > c(y)\_r}$ and $\mathrm{\{x \setminus c(x)\}\_1 > \{c(y)\_r\}}$ are equivalent. From this we conclude that it is possible to find an indicator $\mathrm{p_* = \{x \setminus c(x)\}\_1}$ such that $\mathrm{p_* \in X \setminus C(X)}$ in contrast to $\mathrm{p_* \notin Y \setminus C(Y)}$. The last statement contradicts the consistency postulate, namely the violation $\mathrm{X \setminus C(X) \not\subset Y \setminus C(Y)}$ of Proposition II. ■

6. DISCUSSION, FINDINGS AND CONCLUSIONS

The rational choice postulates have led to intriguing results. The independence of rejected alternatives can explain how certain car brands are favored for dynamic options, while consistency plays a role in stable fuel consumption decisions. Factors like options available, preferences, and situations influence customer choices. Understanding buying behavior and context is crucial in the automotive industry. Manufacturers should align strategies with customer needs, address biases, and base offerings on objective data. Implementing a data analytics strategy like ***BDA*** can benefit customers in the automotive market.

With all this in mind, it is crucial to delve deeper into the intricate process of data analysis and categorization of data. It's imperative to recognize that this process transcends mere organization; it is a nuanced journey from objective depiction to subjective discernment. As emphasized repeatedly, the insights gleaned from data case study manifest not only in factual descriptions but also in subjective evaluations. These evaluations, often articulated as "interpretations, " serve as invaluable aids for experts across various domains, aiding them in navigating the complexities unearthed during the data case study endeavor. To facilitate this comprehension and preempt potential pitfalls, it becomes imperative to outline certain "situations" or "traps" those motorists may encounter. By elucidating these scenarios, specialists can leverage our comprehensive analysis methodology to preemptively address and mitigate the adverse ramifications of such situations, thereby fostering informed decision-making and proactive risk management.

6.1. Pitfalls interpretation

Potential car owners may prioritize dynamic options such as powerful engines and sportier styling over consistent fuel economy for reasons such as performance preference, driving experience or a desire for a more engaging and responsive ride. These people may prioritize the excitement and thrill of driving, valuing the dynamic aspects of the car over fuel efficiency.

Car buyers can become fixated on the starting price presented by the seller or on the sticker. They may find it difficult to negotiate or deviate from this anchor point, even if it is not the best offer. Some customers may prioritize the social status associated with owning a particular make or model of car over its practicality or affordability. They may be willing to spend more than they can afford simply to maintain or improve their social image.

When buying a car, impulsive behavior is prevalent as many customers make quick decisions without doing thorough research or thinking about the long-term consequences. They may fall in love with a particular car at first sight and rush into the purchase without evaluating the alternatives. Customers can be influenced by the opinions and actions of others, leading to a herd mentality. They may buy a car simply because their friends, family or colleagues have one, without properly evaluating their own needs and preferences. Emotional attachment to a particular make, model, or even color of car can also cloud their judgment.

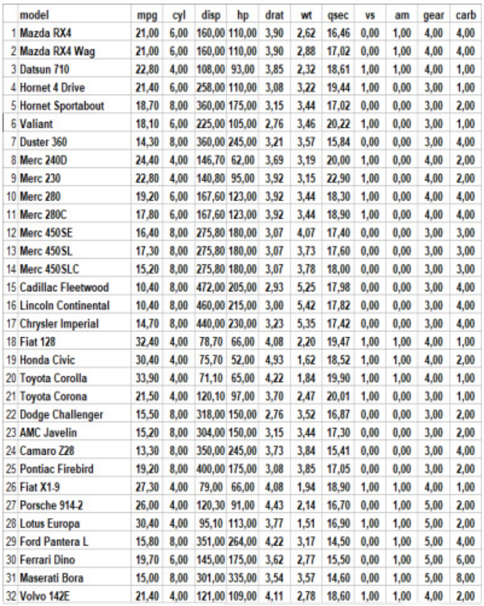
Customers may overlook practical aspects such as fuel efficiency, maintenance costs or resale value, instead prioritizing their emotional connection. Some customers may be overconfident in their negotiation skills or car knowledge, leading them to make irrational decisions. They may be reluctant to seek expert advice and instead rely solely on their own judgment, which may result in increased fees or sub optimal choices. Customers may have a strong bias in favor of buying brand new cars, believing that new models are inherently superior, even though a used car with similar features could meet their needs at a lower price. This bias can lead to cost overruns and financial stress.

Some clients may be overly concerned about the fear of missing out or losing a perceived opportunity. This fear can lead them to make impulsive decisions or agree to unfavorable terms, driven by the desire to get a deal done quickly, even if it is not the best option available. However, it is important to note that while such behavior may be irrational from a purely logical perspective, it often stems from human psychology and the complex interplay of emotions, biases and social factors.

***APPENDIX* 1.** In data mining, separation refers to the ability to distinguish distinct patterns or classes within a dataset. Techniques like clustering, data analysis, or anomaly detection aim to separate data into meaningful groups based on similarities or differences. This separation helps uncover hidden patterns, trends, or outliers, contributing to better data understanding and decision-making. A well-known categorization in this direction is a Finite Closer System $C(X)$ of sub-lists $X \in 2^A$ of alternatives. Equivalent to a more precise definition provided by Seiffarth et al. (2021), our nomenclature will include the choice operator $C(X)$, which is given by Ctrl-s or a C-macro representing the *Fixed Point* $X = C(X)$ of the macro. Thus, based on the concept of a fixed point, the search problem of closed lists turns into a search for a system of sub-lists from a list A of alternatives such that each sub-list from this system is a fixed point of the operator Ctrl-s. From the database https://www.spritmonitor.de/en/ we have extracted a list A of almost all known gasoline-powered cars, where A includes well-established car models. The experiment shows that the separation consists of three segments of gasoline consumption per 100 km: **X=[26.59,…,9.80]**, **Y=[9.78,…6.16]** and **Z=[6.13,…, 4.75]**. The procedure for finding this separation is simple. First, the Ctrl-s macro is applied to all vehicles under investigation, resulting in the extraction of the first fixed point $X = C(X)$. In the remaining list $A \setminus X$, the Ctrl-s macro is implemented again, resulting in the next fixed point $Y = C(Y)$. Then, you need to extract the third one in the same way (accessed August 22, http://www.datalaundering.com/download/MPG-MileAge-Data.xls, 2023).

***APPENDIX* 2.** Based on the information provided in the database, it appears that the segment **[5.47−9.80]** liters per **100 km** have been determined as a "reliable" range for gasoline consumption for all gasoline-fueled car models. Within this range, models to the left are considered more fuel-efficient, while those outside the range to the right are deemed to consume fuel more excessively. Particularly, based on the experiment conducted using models **A3** and **A4**, these two models fall within a range of no more than **3.22 l/100 km** to the right of the significance value u **= 6.58 l/100 km**. On the other hand, the **A5**, **A6**, **A7** and **A8** models exceed the significance level u by more than **3.22 l/100 km**, indicating a noticeable increase in fuel consumption. We also observed that the segment **[5.47–9.80]** and the significance level **6.58** enable data analysis of all considered car models into four fuel consumption classes. The most economical Audi **A2** model is in the yellow class, as opposed to the blue class, which is quite economical, to which model **A1** belongs, and which is below the significant fuel consumption value **u = 6.58**.

APPENDIX* 3.** The purpose of this piece of writing is to illustrate what we have called a procedure of "*Blind Data Analysis*" or ***BDA, based on a popular data containing information about various car models. The data includes indicators such as miles per gallon (mpg), horsepower and other attributes for 32 different car models. We invite the reader to check or at least review the fact that our visual representation of the correlation matrix in the example known to many analysts largely comes down to the same motives that constitute the essence of this study. Many data analysts often use this example (Henderson and Velleman, 1981).

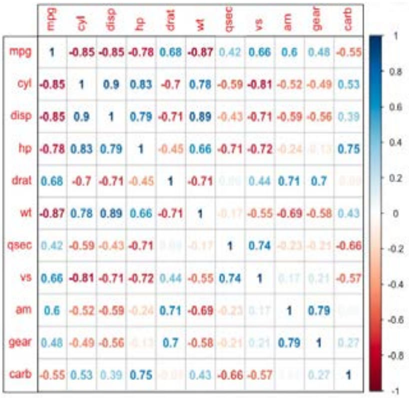


	model	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
1	Mazda RX4	21,00	6,00	160,00	110,00	3,90	2,62	16,46	0,00	1,00	4,00	4,00
2	Mazda RX4 Wag	21,00	6,00	160,00	110,00	3,90	2,88	17,02	0,00	1,00	4,00	4,00
3	Datsun 710	22,80	4,00	108,00	93,00	3,85	2,32	18,61	1,00	1,00	4,00	1,00
4	Hornet 4 Drive	21,40	6,00	258,00	110,00	3,08	3,22	19,44	1,00	0,00	3,00	1,00
5	Hornet Sportabout	18,70	8,00	360,00	175,00	3,15	3,44	17,02	0,00	0,00	3,00	2,00
6	Valiant	18,10	6,00	225,00	105,00	2,76	3,46	20,22	1,00	0,00	3,00	1,00
7	Duster 360	14,30	8,00	360,00	245,00	3,21	3,57	15,84	0,00	0,00	3,00	4,00
8	Merc 240D	24,40	4,00	146,70	62,00	3,69	3,19	20,00	1,00	0,00	4,00	2,00
9	Merc 230	22,80	4,00	140,80	95,00	3,92	3,15	22,90	1,00	0,00	4,00	2,00
10	Merc 280	19,20	6,00	167,60	123,00	3,92	3,44	18,30	1,00	0,00	4,00	4,00
11	Merc 280C	17,80	6,00	167,60	123,00	3,92	3,44	18,90	1,00	0,00	4,00	4,00
12	Merc 450SE	16,40	8,00	275,80	180,00	3,07	4,07	17,40	0,00	0,00	3,00	3,00
13	Merc 450SL	17,30	8,00	275,80	180,00	3,07	3,73	17,60	0,00	0,00	3,00	3,00
14	Merc 450SLC	15,20	8,00	275,80	180,00	3,07	3,78	18,00	0,00	0,00	3,00	3,00
15	Cadillac Fleetwood	10,40	8,00	472,00	205,00	2,93	5,25	17,98	0,00	0,00	3,00	4,00
16	Lincoln Continental	10,40	8,00	460,00	215,00	3,00	5,42	17,82	0,00	0,00	3,00	4,00
17	Chrysler Imperial	14,70	8,00	440,00	230,00	3,23	5,35	17,42	0,00	0,00	3,00	4,00
18	Fiat 128	32,40	4,00	78,70	66,00	4,08	2,20	19,47	1,00	1,00	4,00	1,00
19	Honda Civic	30,40	4,00	75,70	52,00	4,93	1,62	18,52	1,00	1,00	4,00	2,00
20	Toyota Corolla	33,90	4,00	71,10	65,00	4,22	1,84	19,90	1,00	1,00	4,00	1,00
21	Toyota Corona	21,50	4,00	120,10	97,00	3,70	2,47	20,01	1,00	0,00	3,00	1,00
22	Dodge Challenger	15,50	8,00	318,00	150,00	2,76	3,52	16,87	0,00	0,00	3,00	2,00
23	AMC Javelin	15,20	8,00	304,00	150,00	3,15	3,44	17,30	0,00	0,00	3,00	2,00
24	Camaro Z28	13,30	8,00	350,00	245,00	3,73	3,84	15,41	0,00	0,00	3,00	4,00
25	Pontiac Firebird	19,20	8,00	400,00	175,00	3,08	3,85	17,05	0,00	0,00	3,00	2,00
26	Fiat X1-9	27,30	4,00	79,00	66,00	4,08	1,94	18,90	1,00	1,00	4,00	1,00
27	Porsche 914-2	26,00	4,00	120,30	91,00	4,43	2,14	16,70	0,00	1,00	5,00	2,00
28	Lotus Europa	30,40	4,00	95,10	113,00	3,77	1,51	16,90	1,00	1,00	5,00	2,00
29	Ford Pantera L	15,80	8,00	351,00	264,00	4,22	3,17	14,50	0,00	1,00	5,00	4,00
30	Ferrari Dino	19,70	6,00	145,00	175,00	3,62	2,77	15,50	0,00	1,00	5,00	6,00
31	Maserati Bora	15,00	8,00	301,00	335,00	3,54	3,57	14,60	0,00	1,00	5,00	8,00
32	Volvo 142E	21,40	4,00	121,00	109,00	4,11	2,78	18,60	1,00	1,00	4,00	2,00

Table 2. The MTcars data frame with 32 observations on 11 (numeric) indicators

mpg	"**Miles/(US) gallon**" represents the fuel efficiency of different car models, specifically the number of miles they can travel on one gallon of fuel.
cyl	Represents the **count of cylinders** in the engine of each car model.
disp	"**Displacement (cu.in.)**" attribute refers to the engine displacement of each car model, typically measured in cubic inches (cu.in.).
hpc	**Gross horsepower** is a measure of the engine's power output before accounting for various losses, such as those from the transmission and accessories.
drat	"**Rear axle ratio**" attribute refers to the ratio of the number of revolutions the drive shaft makes to one revolution of the rear axle.
wt	"**Weight (1000 lbs)**" attribute represents the weight of each car model in thousands of pounds.
qsec	"**1/4 mile time**" is often used as a measure of a car's acceleration and performance, particularly in drag racing.
vs	Engine (0 = **V-shaped**, 1 = straight) is a binary indicator that categorizes the type of engine in each car model. A value of 0 typically represents a V-shaped (V6 or V8) engine, while a value of 1 represents a straight (inline) engine.
am	"**Transmission** (0 = automatic, 1 = manual)" attribute in the `categorizes the type of transmission used in each car model.
gear	"**Number of forward gears**" indicates the count of forward gears available in the transmission of each car model.
carb	"**Number of carburetor**" represents the count of carburetors in the engine of each car model. Carburetors are devices that mix air with a fine spray of liquid fuel for internal combustion engines.

For data analysis in practice, correlation matrices are usually calculated and visualized. Correlation matrix analysis involves examining the relationships between multiple variables by calculating and visualizing their correlations. Each cell in the correlation matrix displays the correlation coefficient, which indicates the strength and direction of the relationship between two variables. Positive values suggest a positive correlation, negative values indicate a negative correlation, and values close to zero suggest a weak or no correlation. This analysis helps in understanding patterns, dependencies, and potential multi-co-linearity of variables recorded in the database.

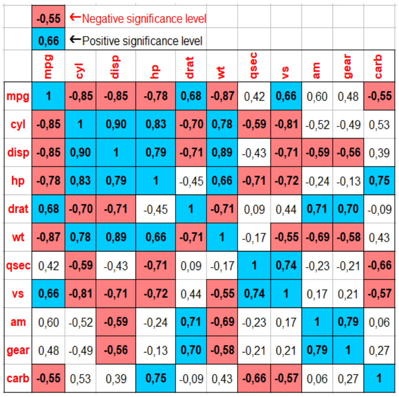


	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
mpg	1	-0.85	-0.85	-0.78	0.68	-0.87	0.42	0.66	0.6	0.48	-0.55
cyl	-0.85	1	0.9	0.83	-0.7	0.78	-0.59	-0.81	-0.52	-0.49	0.53
disp	-0.85	0.9	1	0.79	-0.71	0.89	-0.43	-0.71	-0.59	-0.56	0.39
hp	-0.78	0.83	0.79	1	-0.45	0.66	-0.71	-0.72	-0.24	-0.13	0.75
drat	0.68	-0.7	-0.71	-0.45	1	-0.71	0.09	0.44	0.71	0.7	-0.09
wt	-0.87	0.78	0.89	0.66	-0.71	1	-0.17	-0.55	-0.69	-0.58	0.43
qsec	0.42	-0.59	-0.43	-0.71	0.09	-0.17	1	0.74	-0.23	-0.21	-0.66
vs	0.66	-0.81	-0.71	-0.72	0.44	-0.55	0.74	1	0.17	0.21	-0.57
am	0.6	-0.52	-0.59	-0.24	0.71	-0.69	-0.23	0.17	1	0.79	0.06
gear	0.48	-0.49	-0.56	-0.13	0.7	-0.58	-0.21	0.21	0.79	1	0.27
carb	-0.55	0.53	0.39	0.75	-0.09	0.43	-0.66	-0.57	0.06	0.27	1

Table 3

Visualization of the MTcars correlation matrix in the form presented is available to everyone in the public domain.

https://miro.medium.com/v2/resize:fit:1400/format:webp/1*UJvgUROXv07GQsQCzukMAw.png



-0,55 ←Negative significance level

0,66 ←Positive significance level

	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
mpg	**1**	**-0,85**	**-0,85**	**-0,78**	**0,68**	**-0,87**	0,42	**0,66**	0,60	0,48	**-0,55**
cyl	**-0,85**	**1**	**0,90**	**0,83**	**-0,70**	**0,78**	**-0,59**	**-0,81**	-0,52	-0,49	0,53
disp	**-0,85**	**0,90**	**1**	**0,79**	**-0,71**	**0,89**	-0,43	**-0,71**	**-0,59**	**-0,56**	0,39
hp	**-0,78**	**0,83**	**0,79**	**1**	-0,45	**0,66**	**-0,71**	**-0,72**	-0,24	-0,13	**0,75**
drat	**0,68**	**-0,70**	**-0,71**	-0,45	**1**	**-0,71**	0,09	0,44	**0,71**	**0,70**	-0,09
wt	**-0,87**	**0,78**	**0,89**	**0,66**	**-0,71**	**1**	-0,17	**-0,55**	**-0,69**	**-0,58**	0,43
qsec	0,42	**-0,59**	-0,43	**-0,71**	0,09	-0,17	**1**	**0,74**	-0,23	-0,21	**-0,66**
vs	**0,66**	**-0,81**	**-0,71**	**-0,72**	0,44	**-0,55**	**0,74**	**1**	0,17	0,21	**-0,57**
am	0,60	-0,52	**-0,59**	-0,24	**0,71**	**-0,69**	-0,23	0,17	**1**	**0,79**	0,06
gear	0,48	-0,49	**-0,56**	-0,13	**0,70**	**-0,58**	-0,21	0,21	**0,79**	**1**	0,27
carb	**-0,55**	0,53	0,39	**0,75**	-0,09	0,43	**-0,66**	**-0,57**	0,06	0,27	**1**

Table 4

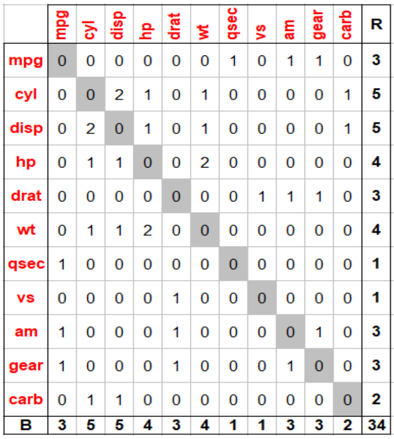
Visualization of the **MTcars** correlation matrix as it appears by applying the ***BDA*** method using the Ctrl+s macro

http://www.datalaundering.com/download/mtcars.xls

Comparing Table 3 from Table 4, it is easy to notice the almost complete similarity of the tables. The only difference is that the significance levels of the coefficients in Correlation Table 3 is a visualization associated with practical or common sense judgment based on experience and traditional reasoning, in particular with the color scheme, while those in Table 4 are based on the postulates of rational choice.

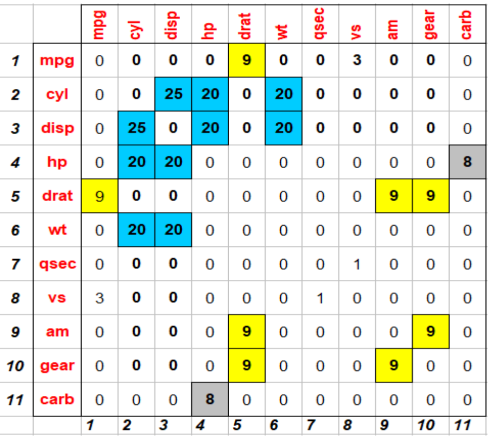
***APPENDIX* 4.** The goal of a graph classification problem is to assign labels to specific nodes or edges in the graph and to learn patterns and features that help make accurate decisions. The challenge is to efficiently aggregate and process information from a graph structure. When graphs are viewed as sets of edges, labels are often used for entire sub-graphs or individual edges. In our example, we have a graph representing correlations. Thus, a classification task may involve labeling specific groups of our **11** parameters from **32** car models forming certain relationships, representing sub-graphs with strong positive correlation "within a sub-graph" or with strong negative correlation "between sub-graphs", each edge of which is associated with a blue or red label. where the correlation is greater than **+.66** within or consistently less than **-0.55** between sub-graphs.

In the example below, Table 5 represents the $A * A$ multiplication, according to standard algebraic rules, obtained from the (0,1)-adjacency matrix A. The (0,1)-cells in A denote by 1 the correlation coefficients (Table 3 or Table 4) between the 11 parameters with a positive correlation threshold above +.66. Then Table 5, where diagonal cells contain 0-s, is converted to Table 6. Table 6 corresponds to vectors R and B outer-product $R \times B$ of the "total" column R to the right of Table 5 by the "total" row B at the bottom of Table 5. In Table 6 we leave only graphically adjacent vertices A, denoting with a 0-value those cells of Table 6 that do not indicate at adjacent vertices in A.



	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb	R
mpg	0	0	0	0	0	0	1	0	1	1	0	3
cyl	0	0	2	1	0	1	0	0	0	0	1	5
disp	0	2	0	1	0	1	0	0	0	0	1	5
hp	0	1	1	0	0	2	0	0	0	0	0	4
drat	0	0	0	0	0	0	0	1	1	1	0	3
wt	0	1	1	2	0	0	0	0	0	0	0	4
qsec	1	0	0	0	0	0	0	0	0	0	0	1
vs	0	0	0	0	1	0	0	0	0	0	0	1
am	1	0	0	0	1	0	0	0	0	1	0	3
gear	1	0	0	0	1	0	0	0	1	0	0	3
carb	0	1	1	0	0	0	0	0	0	0	0	2
B	3	5	5	4	3	4	1	1	3	3	2	34

Table 5; The product $A \times A$ of adjacency **(0,1)**-matrix A



		mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
1	mpg	0	**0**	**0**	**0**	**9**	**0**	**0**	**3**	**0**	**0**	0
2	cyl	0	**0**	**25**	**20**	**0**	**20**	**0**	**0**	**0**	**0**	0
3	disp	0	**25**	**0**	**20**	**0**	**20**	**0**	**0**	**0**	**0**	0
4	hp	0	**20**	**20**	0	0	0	0	0	0	0	**8**
5	drat	9	**0**	**0**	0	0	0	0	0	**9**	**9**	0
6	wt	0	**20**	**20**	0	0	0	0	0	0	0	0
7	qsec	0	**0**	**0**	0	0	0	0	1	0	0	0
8	vs	3	**0**	**0**	0	0	0	1	0	0	0	0
9	am	0	**0**	**0**	0	**9**	0	0	0	0	**9**	0
10	gear	0	**0**	**0**	0	**9**	0	0	0	**9**	0	0
11	carb	0	0	0	**8**	0	0	0	0	0	0	0
		1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*

Table 6; The R and B outer-vector product $R \otimes B$

We can now apply our ***BDA*** technique to Table 6, the result of which is shown in Figure 1. There are many classification methods on graphs, for example, Viswanathan, et al., 2010. We have also contributed to this field by using the so-called method of "Monotonic Systems" algorithm (simplified in our ***BDA***) to visualize the results of data analysis (Mullat, 1977). The correlation matrix visualization below is only an addition to ***BDA*** technique. As shown in Figure 1, there are two different classes that can be distinguished by dividing our 11 parameters given in Table 2.

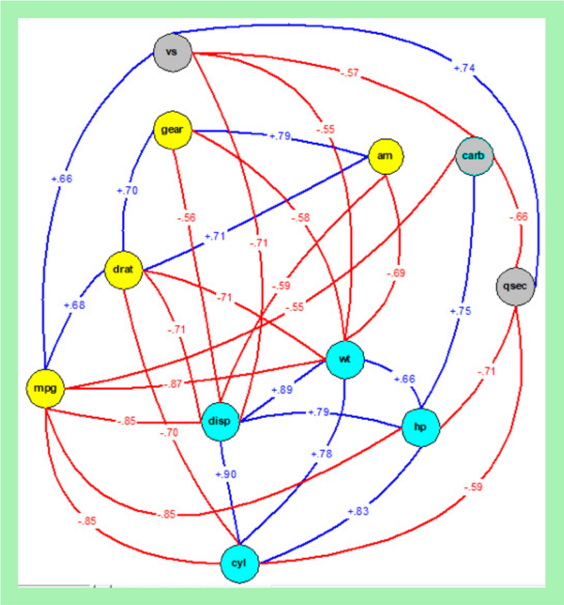


Figure 1; Visualization of ***BDA*** results using correlation of 11 parameters of Table 2.

Indeed: 1) **{cyl, disp, hp, drat, wt}**; 2} **{am, gear, carb}** and separately group of parameters 3) **{mpg, vs, qsec}**. We do not intentionally use any classification method, but simply use common sense, which we hope is sufficient to visualize the effectiveness of our *"Blind Data Analysis"* procedure. However, it can be proven that the first two classes visualize the separation of correlation coefficients when ***BDA*** is applied separately: initially by a block (outer-vector) $\overline{1,...,7} \otimes \overline{1,...,7}$, and then by $\overline{5,...,11} \otimes \overline{5,...,11}$ of rows and columns. The phenomenon of separation has already been discussed in Appendix 1.

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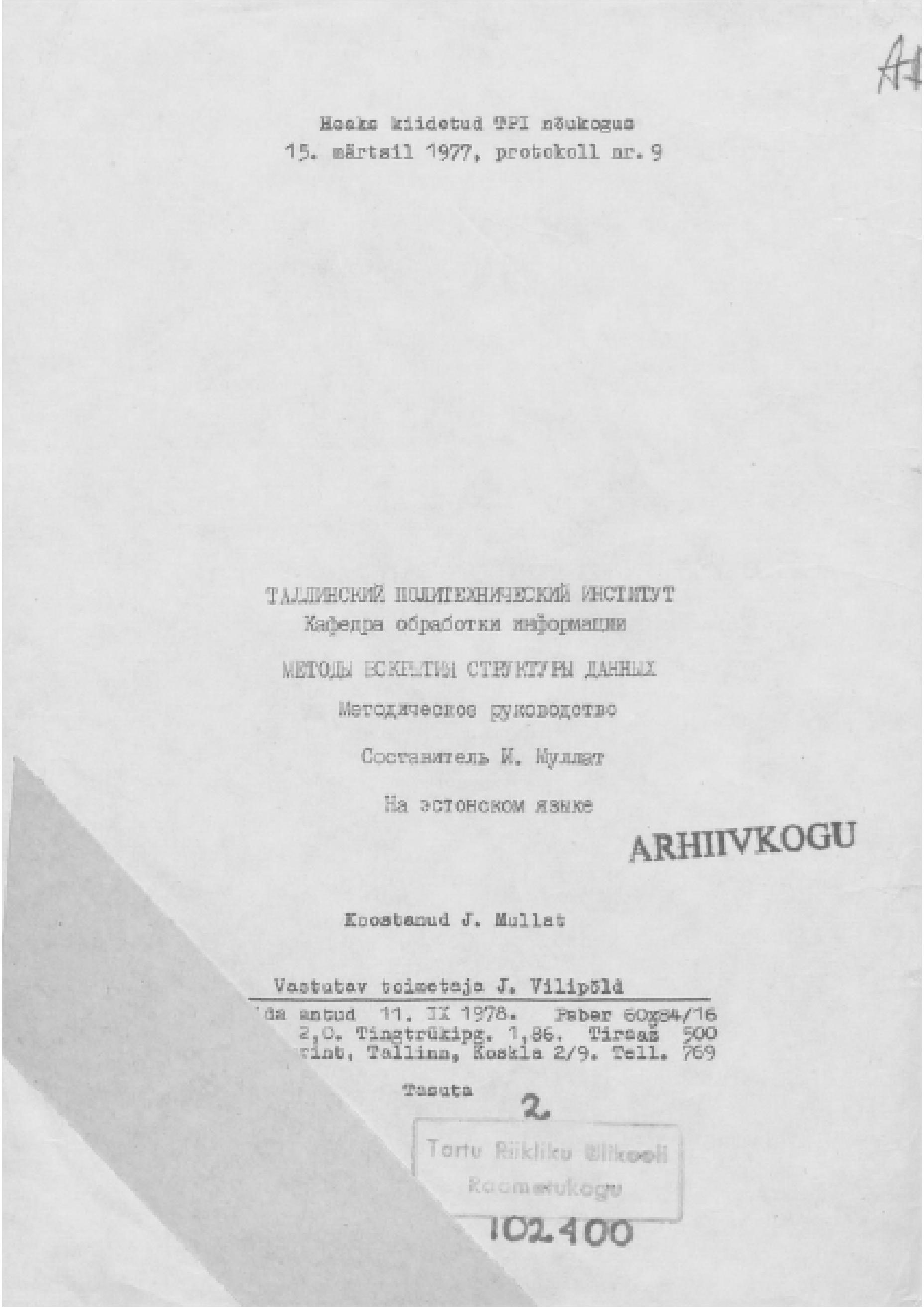

Heaks kiidetud TPI nõukogus
15. märtsil 1977, protokoll nr. 9

ТАЛЛИНСКИЙ ПОЛИТЕХНИЧЕСКИЙ ИНСТИТУТ
Кафедра обработки информации

МЕТОДЫ ВСКРЫТИЯ СТРУКТУРЫ ДАННЫХ

Методическое руководство

Составитель И. Муллат

На эстонском языке

Koostanud J. Mullat

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Tasuta

2

Data Structure Opening Method: Methodological Guide [*]

Abstract. This methodological study delves into an extensively documented yet powerful monotonicity-based information processing technique that is often overlooked despite its widespread application and contemporary use. The focus is on the application of the category of monotonicity to formal systems for data analysis, a method with a simple algorithmic component in uncovering complex data structures and obtaining information in various fields, including sociology, economics, biology and demography. This methodology recognizes patterns in two basic data structures: frequency tables and graphs. Frequency tables arise as a common outcome of surveys when data are organized into categorical responses. The effectiveness of the method depends on converting these categories into frequency measures, which facilitates in-depth analysis based on numerical indicators. This preparatory step lays the foundation for robust analysis that allows researchers to gain detailed information about social trends, consumer behavior and economic models. The application of the method extends to the field of graph theory, where comprehensive patterns in complex networks are modeled. By emphasizing the construction of generalized models, this approach illuminates the fundamental characteristics of reality through visualization of so-called "encompassing pictures." This framework focuses on key metrics such as saturation levels and the presence or absence of important components such as triangles and cycles in graphs. By carefully studying these graph structures, researchers can unravel complex relationships, identify emergent phenomena, and elucidate the underlying mechanisms governing system behavior.

Keywords: data matrix; layering algorithm; graph; tournament

1. INTRODUCTION

If one decides to collect data, the following questions must first be answered:

- What information is needed?
- Why is this information needed?
- To what extent are the reasons for gathering information?
- How can decisions be made based on the information gathered and thus influence the investigation process?

If answers are available, then the set of collected *"objects"*, those data, is also defined. For example, information may concern people living in a city, families in a given country, electronic equipment, factories made up of basic production units (objects in the terminology of the guide) etc.

Population information can be composed of a series of indicators that describe the population as a whole, such as the scales against which income is measured. In productive area, indicators determine the technical environment in which, e.g. equipment was manufactured and operated. Naturally, estimates based on the information collected differ from actual estimates. Thus, the researcher may draw incorrect conclusions if the error of the estimate is too large. This guide looks at one possible way to avoid the errors associated with the so-called stratification concept.

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[*] The original version in Estonian (Mullat, 1977), Protocol No. 9, approved by the TPI Council (Tallinn Polytechnic Institute) on March 15, 1977. TPI currently stands for Tallinn University of Technology – TalTech.

Let's give an example of the importance of this concept in information processing: in USA a presidential elections were held in 1932. Literary Digest sent postcards to voters with questions to predict Roosevelt's election to the presidency. Some 10 million postcards had been sent out. The results showed that the forecast made on the basis of the information collected was accurate within 1%. However, the prediction made using exactly the same technique in 1936, contained an error of almost 20%.

There is a general perception that the "postcard method" introduced a disproportion among voters who return postcards. It turned out that people with higher education and better conditions tended to return more postcards. People with a higher standard of living tended to prefer Roosevelt's competitor during the readiness period, and the forecast of results shifted away from the real thing.

This example shows that when the population is stratified (for example, only voters with higher education and better conditions are observed), a big mistake cannot be avoided. That is, in order to avoid such an error, the researcher must know in advance the subgroups of the population (classification), but usually the identification of subgroups is a complex and voluminous effort, which in turn is associated with the collection of information.

The guide looks at population stratification (classification) methods that currently exist in three types:

a) Methods that take into account the researcher's subjective opinion of the population. This means that classification with exact properties are known or simply assumed;
b) Methods to be used in the absence of any data or hypotheses about existing strategies and their attributes;
c) Methods, which are intended only to visualize a sample of the population in order for the researcher to be able to make a decision on the available strata.

Among methods a), b) and c), only the so-called monotonic layering (Mullat, 1971-1995) or known since then as the “monotonic linkage method” (Kempner et al, 1997) is considered. The last chapters are devoted to the theoretical study of these monotone systems and methods of monotone layering, in particular, on graphs. We do not discuss issues related to the use of standard statistical methods and algorithms. The additional tools and technologies needed for the monotone layering of data, the accompanying terminology and strict nomenclature are explained in the course of the narrative and defined where necessary.

The article consists of an introduction and a section that discusses the main concepts, a total of 8 sections. Section 3 discusses the different types of metric distances between objects to measure the difference between objects in classification problems. Section 4 describes the method itself at an informal level. Section 5 provides a more accurate construction at a precise mathematical level. In Sections 6-7, we consider the application of the method to the study of graphs, in particular, to determine the groups of strong players in tournaments as opposed to weak players. Concluding remarks are provided in Section 8.

2. KEY DEFINITIONS

First, we introduce the reader to the terminology and basic concepts used. The basic concept of data processing is a data matrix. The data X is a $\mathrm{n} \times \mathrm{m}$ matrix (n row and m columns), each row of which is called an object; one column of the matrix is called an attribute. This means that the data matrix is:

$$\mathrm{X} = \begin{Vmatrix} \mathrm{x}_{1,1}, \mathrm{x}_{1,2}, .., \mathrm{x}_{1,m} \\ \mathrm{x}_{2,1}, \mathrm{x}_{2,2}, .., \mathrm{x}_{2,m} \\ \dots\dots\dots\dots\dots\dots \\ \mathrm{x}_{n,1}, \mathrm{x}_{n,2}, .., \mathrm{x}_{n,m} \end{Vmatrix}$$

and $\mathrm{x}_{i,j}$ is the value of the j-th attribute of the i-th object. It is natural that the question immediately arises as to what the numerical values of the attribute in the data matrix reflect? There must be brands that the attributes may differ substantially. For example, the air temperature may be a characteristic when electric lamps are lit; the shoe number of the person; gender (male or female), etc. As the processing is formally based on mathematical apparatus, three types of attributes are distinguished in order to be able to interpret the final results and use them according to the purpose:

a) Attributes on a continuous scale (Interval scale), such as body credential, height, temperature (quantitative);

b) Attributes on a discrete ordinal scale, such as the grades a student receives in some subjects: unsatisfactory, satisfactory, good, and very good. At this point, the values of the attribute are considered ordered (in Points or ranked);

c) Attributes with discrete values that are not ranked (nominal scale or even qualitative attributes), For example, eye color, gender (male or female).

2.1. Quantitative attributes

The quantitative expression of an attribute is usually referred to as the value of the attribute can be compared. Questions about how many times the value of one attribute is greater than another can be answered. At first glance, the question does not seem to be very complicated, although a deeper examination in turn raises the question: "What is natural to compare?" Let's look at some more examples before answering this question.

Let us choose the cars that are described by the price tag. Undoubtedly, the attribute "price" is quantitative, the a car with the price of 1O.OOO€, is twice as expensive as the b car with the price of 5.000€. The characteristic "price" or "value" expressed by the function $f(a)$ can also be expressed by the function $\kappa \cdot f(a)$) (κ is a positive number). Every other type of conversion changes the price ratio of cars. The allowed transformations of the attribute "price" are multiplication by the constant κ. This property of the price makes it possible to determine how many times $f(a)$ is greater than $f(b)$ — the ratio ${}^{\kappa \cdot f(a)}\!\big/_{\kappa \cdot f(b)}$ does not depend on κ of the choice, and if κ is fixed, we can thus say how much is $f(a)$ greater than $f(b)$. This class of transformations allows for the universal presentation of concepts related to quantitative as well as other types of attributes. However, the determination of a unit of measurement requires only quantitative attributes.

2.2. Definition

The permissible transformation of the value of an attribute $f(a)$ in the set of attributes $\mathcal{A}$ is called the function $\varphi(x)$ if the function $\varphi(f(a))$ $(a \in \mathcal{A})$ shows the same attribute. If the values of the characteristic f are given together with the number of allowed conversions F, then we say that the measurements of the characteristic were performed on the F-type scale.

In the example of passenger cars $F_o \{\kappa \cdot x \mid \kappa > 0\}$ and on the scale F_0 it is usually said that the measurements are made on a ratio scale. An interval scale is a scale where the number of transformations allowed is $F_x = \{\kappa \cdot x + o \mid \kappa > 0\}$; the specific scale F_x is determined by the quantities κ and o, which give the unit of measurement and the starting point of the scale.

In most cases, the measurement results are presented in the form of a matrix, if after each allowed transformation the measurement results do not change. However, it should be noted that the results of matrix measurements do not allow them to be immediately used in arithmetic calculations. For example, the relationship $f(a) + f(b) > f(c)$ does not make sense in the scale with origin $o > 0$, since $\kappa \cdot [f(a) + f(b)] + 2 \cdot o$ is greater than $\kappa \cdot f(c) + o$ only for some κ and o values. Indeed, absolute zero is the natural and unambiguous presence of the zero point o that cannot be changed: °0-Kelvin is absolute zero on the scale, which characterizes the absence of the measured feature. However, °0-Celsius or °0-Fahrenheit are not. Two arbitrary physical phenomena are taken here: melting of ice, or an equal mixture of water, ice and salt at -21.1°C. Comparing the mean values of the interval scale is another matter.

Expression

$$\frac{1}{n}\cdot\sum_{i=1}^{n} f(a_i) > \frac{1}{m}\sum_{j=1}^{m} f(b_j) \tag{1}$$

remains unchanged after using the allowed conversion. Namely

$$\frac{1}{n}\cdot\sum_{i=1}^{n} \kappa\cdot f(a_i) + o > \frac{1}{m}\sum_{j=1}^{m} \kappa\cdot f(b_j) + o \text{ iff}$$

$$\frac{\kappa}{n}\cdot\sum_{i=1}^{n} f(a_i) + \frac{o\cdot n}{n} > \frac{\kappa}{m}\sum_{j=1}^{m} f(b_j) + \frac{o\cdot m}{m}$$

and the latter is equivalent to inequality (1). It makes sense to compare the absolute differences in the values of the attributes, namely

$$\frac{|f(a)-f(b)|}{|f(c)-f(d)|} = \frac{|(\kappa\cdot f(a)+o)-(\kappa\cdot f(b)+o)|}{|(\kappa\cdot f(c)+o)-(\kappa\cdot f(d)+o)|}.$$

Now we ask the question what determines the number of allowed transformations $f(x)$? Usually the choice is related to other attributes with the possibility of forecasting. Formally expressed laws of science allow all these forecasting transformations not to change the law. For example, Clipperon's law $P\cdot V/T = \text{const}$ connects the scales of temperature T, volume V and pressure P of a given gas, allows transformation, leaving the law unchanged. Also in economics, in functional models, the price is determined fixed to within a multiplier.

Unknown patterns of relationships, characteristic of sociological or psychological research, allow transformations between objects in the form of empirical relationships, for example, by stratification methods. In these studies, however, interval or ratio scales are unacceptable.

2.3. Point or ordinal scales.

Pupil assessment aims to test the degree of skill acquisition and achievement of primary education goals on a point scale: Fail (IN – Insuficiente); Pass (SU – Suficiente), Good (BI – Bien), Very Good (NT – Notable), Excellent (SB – Sobresaliente). Point scale gradations are limited by equal intervals of discrete numerical values. Expert judgments are often recorded as a sequence of natural numbers arranged symmetrically to the o point $(o, \pm 1, \ldots)$.

A distinction should be made between two types of point estimates. In the first case, the assessments reflect some well-known standards. The more opportunities you have to describe and characterize standards, the more accurately you can, for example, determine the deviation from the standard. Thus, the teacher depending on his work experience and personal experience forms the pedagogical level of high school students' performance. On the other hand, refining a benchmark helps predict attribute values; for example, a student who is very good at geometry usually also scores higher in algebra.

The second type of points occurs when there are no well-known standards or even the existence of an objective criterion is questionable, which may be reflected in subjective judgments, for example, the taste of culinary products. This type is also called an ordinal or ordered scale. The set of allowed transformations F consists of all monotonically increasing functions. The ordered values of the attributes are compared only on the basis of the relation "higher-lower". It is meaningless to compare the differences between the values of the attribute. For example:

if $f(a)=10$, $f(b)=2$, $f(c)=1$, $f(a)-f(b)=8$, $f(b)-f(c)=1$, $f(a)-f(b)=8>f(b)-f(c)=1$, Then, using the monotonic transformation φ, where $\varphi(1)=1$, $\varphi(2)=20$, $\varphi(10)=30$ gives a contradiction $10=\varphi(f(a))-\varphi(f(b))>\varphi(f(b))-\varphi(f(c))=19$.

It is, nonetheless, realistic to fix the values of original attributes using non-numerical terms. Eligible elements for each ordered set, such as alphabet, etc.

(c) The nominal scale. The scales of the above attributes — quantitative, point and ordinal scales — have general attributes. All scales define the binary relation B on the set of objects X. The relation is defined by the following rule: $(a,b)\in B$ then and only then when $f(a)>f(b)$. Quantitative and point measurements are informatively more voluminous than ordinary measurements.

In practice, we can often only be interested in the information contained in the binary relation B. The researcher's conclusions about the functioning of the socio-economic system are usually qualitative (for example, stratification or ranking of objects in a sample).

It is natural to ask the question: is qualitative information not enough to draw conclusions? Qualitative information is easier to measure and more reliable. We do not have the means to accurately measure $f(a)$ and $f(b)$, while we can be sure that $f(a)>f(b)$.

On the other hand, the complex examination of data requires the transformation of the measurement results of individual assessments and objective indicators into a common type of data: quantitative or qualitative.

By limiting the number of transformations F allowed, complex data analysis is usually performed by quantifying all measurements. By limiting the number of transformations allowed sophisticated data analysis is usually performed by quantifying all measurements. Qualitative measurements can "suffer" in this way. When examining qualitative data, it is also possible to do the opposite: to transform quantitative measurements into qualitative ones. It is possible that even then the data will "suffer". However, if the results using quantitative methods are consistent with the results of qualitative data processing methods, the investigator is more likely to be sure of the conclusions reached, Let the equivalence relation $\mathcal{J}$ be given for the cross product of objects $X \times X$. We assign to each object $x \in X$ the number of the i-th class of X, which contains the object X. Let's say that the measurements are made on a nominal scale, if the value of the attribute is the number of the i-th equivalent relation. Number of conversions allowed by F_n are unique functions. Thus the pair $(a,b) \in \mathcal{J}$ then and only then when attributes values $f(a) = f(b)$. Measurement on a nominal scale is the "weakest" measurement step, as it is only determined whether the equation $f(a) = f(b)$ truly applies.

3. METHODS FOR MEASURING DIFFERENCES BETWEEN OBJECTS

All of the methods that we will discuss in Sections 4-7 relate to some degree to the concept of distance or metric. This means that the task of stratification can be performed accurately only if the distance between objects is determined. Choosing a distance means also comparing distances that measure the similarity of two objects. The higher this number, the more the objects themselves differ, and vice versa.

The distance $\rho(x,y)$ between objects x and y is called a function that satisfies three conditions:

(a) for each x object $\rho(x,x) = 0$;

(b) for each pair (x,y) of objects $\rho(x,y) = \rho(y,x)$;

(c) there is a relationship for each of the three objects (x,y,z) that $\rho(x,y) + \rho(y,z) \geq \rho(y,z)$.

The following is a list of metrics or distances used. The notations are as follows: We denote the i-th, $i = \overline{1,n}$, object of the data matrix X as $x_i = \langle x_{i,1}, x_{i,2}, \ldots, x_{i,m} \rangle$, where $x_{i,j}$, $j = \overline{1,m}$, is the j-th attribute of the object i. The distance between two objects x_k and x_ℓ herein as said is nominated as $\rho(x_k, x_\ell)$.

Here are some of the most commonly used metrics.

Cubic distance: $$\rho(x_k, x_\ell) = \max_{j=\overline{1,m}} \left| x_{k.j} - x_{\ell,j} \right|,$$

where $|\cdot|$ indicates an absolute value.

Octahedral distance: $$\rho(x_k, x_\ell) = \sum_{j=1}^{m} \left| x_{k.j} - x_{\ell,j} \right|.$$

Euclidean distance, $$\rho(x_k, x_\ell) = \sqrt{\sum_{j=1}^{m} \left(x_{k.j} - x_{\ell,j} \right)^2}$$

These three metrics are mostly useful for an interval scale. The following distance is useful when attributes are measured in points or on an ordinal scale:

$$\rho(x_k, x_\ell) = \sum_{j=1}^{m} \left| x_{k.j} - x_{\ell,j} \right| \Bigg/ \sum_{j=1}^{m} \max_{k,\ell} \left(x_{k,j}, x_{\ell,j} \right).$$

There are distances that are valid when the attributes are binary. Binary is a sign of "marital" status, e.g. if there can be only two answers — "married-yes" or "married-no". These distances are valid even if the scale is nominal.

3.1. Hamming distance

The notation is borrowed from set theory because objects can be interpreted as subsets of attributes. A value of 1 can be viewed as an indicator $x_{i,j}$ of whether the original attribute j belongs or does not belong to subset x_i. The object x_i is thus a Boolean vector $x_i = \left\langle x_{i,1}, \ldots, x_{i,m} \right\rangle$, where $x_{i,j}$ is the "1"-one or "0"-zero type, $j = \overline{1, m}$.

The absolute distance $\rho(x_k, x_\ell)$ is defined as follows: $\rho(x_k, x_\ell) = m - \left| x_k \cap x_\ell \right|$, which equals the number of missing matches in the objects x_k, x_ℓ. In this case, $\left| x_k \cap x_\ell \right|$ is the number of attributes matches in the data matrix, which takes into account 1-s in both objects x_k, x_ℓ, indicating the same attributes. The relative distance looks like $\rho(x_k, x_\ell) = 1 - \left| x_k \cap x_\ell \right| / \left| x_k \cup x_\ell \right|$, where $x_k \cup x_\ell$ is a set of only those attributes that are present in both x_k, x_ℓ objects, but do not necessarily indicate the same attributes.

The list of distances between objects can be continued, since the possibilities for determining the distances are not limited. It should only be noted that the choice of distances is a process that is difficult to formalize and is usually performed by a researcher based on his/her own experience. Measuring the differences or distances between attributes further complicates matters and differs from the above list. Inter-trait, or correlation coefficient between features/attributes is the most commonly used measure that shows the relative linearity of the change in a second identifier when the first identifier changes. The correlation coefficient c between attributes α, β can be determined using the following expression:

$$c_{\alpha,\beta} = \frac{\sum_{i=1}^{n} x_{i,\alpha}\cdot x_{i,\beta} - \left(\sum_{i=1}^{n} x_{i,\alpha}\cdot\sum_{i=1}^{n} x_{i,\beta}\right)\Big/ n}{\sqrt{\sum_{i=1}^{n} x_{i,\alpha}^{2} - \left(\sum_{i=1}^{n} x_{i,\alpha}^{2}\right)^{2}\Big/ n}\cdot\sqrt{\sum_{i=1}^{n} x_{i,\beta}^{2} - \left(\sum_{i=1}^{n} x_{i,\beta}^{2}\right)^{2}\Big/ n}}$$

In the case of the attributes "no", "yes", it is useful to apply a binary (Pirson's φ) correlation r between objects κ, ℓ in the form of:

$$r_{\kappa,\ell} = \frac{|x_k \cap x_\ell|\cdot|\bar{x}_k \cap \bar{x}_\ell| - |x_k \cap \bar{x}_\ell|\cdot|\bar{x}_k \cap x_\ell|}{\sqrt{|x_\kappa|\cdot|\bar{x}_\ell|\cdot|\bar{x}_\kappa|\cdot|x_\ell|}},$$

where $\bar{x}$ is a complement of x; $|x_\kappa|\cdot|\bar{x}_\ell|\cdot|\bar{x}_\kappa|\cdot|x_\ell| > 0$. Before selecting the distance/correlation between objects, one must perform a Class F independence check of the permitted transformations.

4. DATA LAYERING ALGORITHM

The reader is probably aware that many models of automatic stratification or objective classification are given and described in the literature. We also know that quite a lot of algorithms of this type have been developed, but due to the lack of access to such knowledge, we independently developed and studied here only one, possibly new for many, method. This method is primarily intended for sociological data, but it can also be used to process the general data matrix X.

Let the information gathered be presented in a form that can depict a large graph. For example, some cities are divided into many quarters. The researcher collects information from the city's residents on movements from one quarter to another. Thus, quarters occur on top of a graph (graph) on the vertices of a graph. The arcs of Graph indicate where the local movements of the population are directed in the city. The task is to find out the movements global trends. So the task is basically in that not to stratify city quarters, but stratify possible directions of movement.

Let's match the number to each arrow (arc) in the graph indicating how many transit paths of length 2 the arrow around gives. Graphically, this means that the number of triangles attached to the arc of the graph has been enumerated, (Fig. 1).

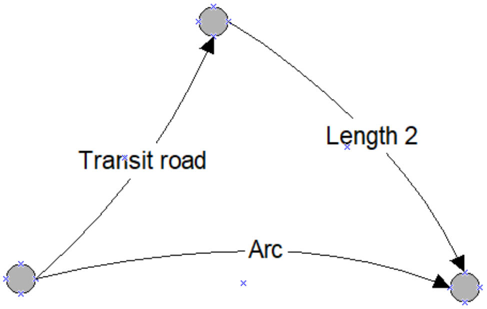


Figure 1

When this is done, the stratification of the arrows (arcs) is completed using the following algorithm. Mullat developed this algorithm in 1971-1977. Everywhere, if necessary, we will call this algorithm using the abbreviation KSF — "Kernel Searching Routine".

1. **Zero step**

   Find the arc with the least number of triangles on the graph and set it to the value of the parameter u at the level u_0. The arc is removed from the graph. It may be that the removal operation at this point affects some other arcs in the graph and the number of triangles viewed on them changes, so that some other arcs with credentials become less than or equal to u_0. These arcs are also removed. This removal of arc or set of arcs shall be repeated until there are no more arcs whose credentials satisfy the condition: less than or equal to u_0,

2. **Recursive k-th step**

a) From the graph that developed in the previous $k-1$ steps when used, a new minimum credential arc, such as an arc with a minimum number of triangles but higher than previous u_{k-1} is found. The parameter u level u_k, $u_{k-1} < u_k$ of the credential of this arc remembers the level. The arc or arcs found is or are removed from the graph.

b) It may be that the removal operation in current step k affects some more arcs and that their credentials become less or equal to u_k. We repeat this "peeling" until there is no more arcs with credentials less or equal to u_k. All arcs are on some p-th step removed/reset from the graph. This terminates the algorithm.

As a result of the algorithm, all arcs of the graph are distributed into groups or layers, each of which is linked with the corresponding size (threshold) u_k, $k = \overline{(0,p)}$. Observing these groups from the last, p-th group, the researcher can draw conclusions about the global or major movement directions on the graph.

Example Let this graph be in Fig. 2,

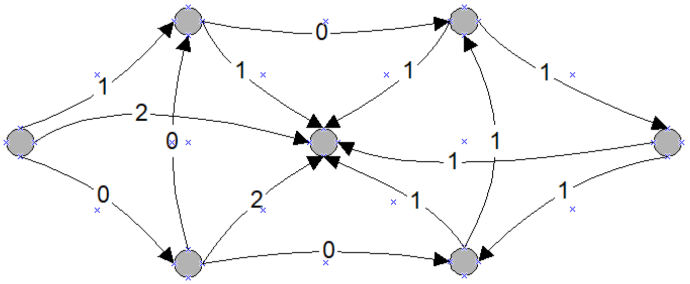


Figure 2.

This figure shows the transit number of routes defined above by Fig. 1 around with the arc in Fig. 2. According to the algorithm the performance of the zero step is the shape of the graph as shown in the Fig. 3.

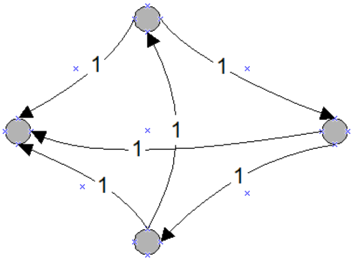


Figure 3

So, above in Fig. 2 it is determined that the given graph has three different 0-arcs. If it were a traffic intensity graph, then there should be two different u_0, u_1 values, or two different traffic layers: 0 and 1, in fact, meaning that the main traffic is possible only for the traffic shown in Fig. 3.

Another way to use the layering algorithm is more complex. An analogous algorithm can also be applied to the % processing (layering) of the data matrix. Only a few new concepts should be defined.

Based on data matrix X, we can create two frequency tables: the rows table and columns table, which will indicate the possible values of the attributes in a nominal scale. The maximum possible number **atr** of different attributes in the data matrix determines the nominal scale width or expansion.

By scanning the cells and at the same time summing the 1-s in the additional tables the two frequency tables $\mathscr{c}$ and $\mathscr{r}$ are progressively filled out. First, let's look at the corresponding cell of the κ-th object and its ℓ-th attribute in X. The $x_{\kappa,\ell}$ of this cell determines in which additional column $x_{\kappa,\ell}$ to the right of X, and in which additional row $x_{\kappa,\ell}$ at the bottom, in relation to X, the 1-s in cells of $r_{k,x_{k,\ell}}$ and 1-s in cells of $c_{x_{k,\ell},\ell}$ are summed up correspondingly. Namely, in relation to X, here $x_{\kappa,\ell}$ is the column No to the right, but also the row No at the bottom, in additional tables $\mathscr{r}$ and $\mathscr{c}$. We assume that table X (see example below) is filled with integer attributes or labels 1,2,1,3,… When filling out frequency tables, we initially look at the first object, then the next, and so on.

	Table 1	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**	*´1*	*´2*	*´3*		
	´1	1	1	1	2	1	1	2	0	*5*	*2*	*0*	**7**	
	´2	1	1	1	3	1	1	3	3	*5*	*0*	*3*	**8**	
	´3	3	2	2	1	3	0	2	2	*1*	*4*	*2*	**7**	$\Leftarrow \mathscr{r}$
	´4	1	1	1	2	1	1	3	3	*5*	*1*	*2*	**8**	
	´5	1	1	1	0	1	1	2	1	*6*	*1*	*0*	**7**	
	´1	*4*	*4*	*4*	*1*	*4*	*4*	*0*	*1*					
$\boldsymbol{c} \Rightarrow$	*´2*	*0*	*1*	*1*	*2*	*0*	*0*	*3*	*1*					
	´3	*1*	*0*	*0*	*1*	*1*	*0*	*2*	*2*					
		5	**5**	**5**	**4**	**5**	**4**	**5**	**4**					

In more compact form, the data cell (κ,ℓ) attribute determines the column No-$x_{k,\ell}$ of frequency $c_{x_{k,\ell},\ell}$ location in the table $\boldsymbol{c} = \left\|c_{t,\ell}\right\|$, $t = \overline{1,atr}$, while the cell (κ,ℓ) also determines the frequency $r_{k,x_{k,\ell}}$ location but in the row No-$x_{k,\ell}$ of table $\mathscr{r} = \left\|r_{\kappa,t}\right\|$; i.e. the cell (κ,ℓ) of the data matrix X, points at frequencies: $r_{k,x_{k,\ell}}$ and $c_{x_{k,\ell},\ell}$. Consider the following credentials: $\pi_{\kappa,\ell} = r_{k,x_{k,\ell}} + c_{x_{k,\ell},\ell} + \sum_{t=1}^{atr} r_{\kappa,t} + \sum_{t=1}^{atr} c_{t,\ell}$, where **atr** already has been determined as the nominal scale expansion or width.

Zero step. For all credentials $\pi_{\kappa,\ell}$ the minimum must be found and remembered using the auxiliary variable u_0. In the data matrix X the entry, where the minimum was found, — the κ-th row and ℓ-th column cell of the data table X is reset to zero or marked as processed. Thus, it usually happens that the corresponding cells to κ-th row and ℓ-th column in additional frequencies tables $\boldsymbol{c}$ and $\mathscr{r}$ change.

Recursive step. Thus, the reset operation may affect some of the other credentials $\pi_{\kappa,\ell}$ of the data matrix X cells, so that the credentials corresponding to those cells become less than or equal to the minor value u_k. Repeat the current step or steps for matrix X cells with this credential level u_k until no entries (cells) are found in the matrix X that satisfy the reset (zeroing) condition at the k-th step.

It is analogous to the zero steps in the graph alignment algorithm. Examples of 5×8 matrix see the Table 1 above. The credential matrix corresponding to the data matrix is as follows:

Table 2	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**
´1	21	21	21	15	21	20	17	11
´2	22	22	22	16	22	21	18	17
´3	15	17	17	13	15	11	19	16
´4	22	22	22	15	22	21	17	16
´5	22	22	22	11	22	21	16	18

After the algorithm has been implemented against Table 2, it performs a transformation of Table 2 to Table. 3 (the reset cells are marked with the number 99):

Table 3	**´1**	**´2**	**´3**	**´4**	**´5**	**´6**	**´7**	**´8**
´1	18	18	18	99	18	18	99	99
´2	18	18	18	99	18	18	99	99
´3	99	99	99	99	99	99	99	99
´4	18	18	18	99	18	18	99	99
´5	18	18	18	99	18	18	99	99

If the result needs to be interpreted essentially, the algorithm offers the researcher, after further investigation, the following interpretation: An area exists inside the data table X or block filled with 3-s labels, which consists of rows $1,2,4,5$ and columns $1,2,3,5,6$.

A similar algorithm can be used for the following two cases. Let's choose the credentials π as a cell value of the data matrix X in the κ-th row and ℓ-th column, which will be

$$\pi_{\kappa,\ell} = \sum_{t=1}^{atr} t \cdot r_{\kappa,t} + \sum_{t=1}^{atr} t \cdot c_{t,\ell} .$$

These types $\pi_{\kappa,\ell}$ of indicators in mechanics are called moments. The credential consists of row moment and column moment sum. We can act exactly according to the algorithm presented earlier.

Another example. The entropy of an object κ can be calculated by formula:

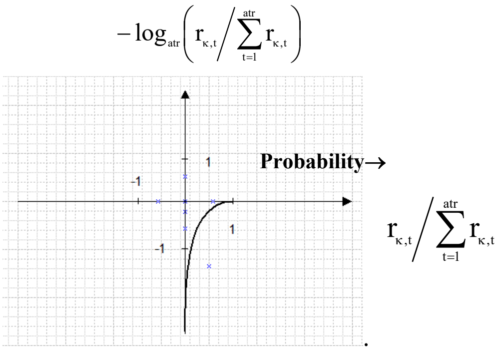
$$H(\kappa) = -\frac{1}{\sum_{t=1}^{atr} r_{\kappa,t}} \sum_{t=1}^{atr} r_{k,t} \cdot \log_{atr}\left(r_{k,t} \Big/ \sum_{t=1}^{atr} r_{k,t} \right),$$

as well as similar formula $H(\ell)$ for an attribute ℓ.

The quantities $H(\kappa)$ and $H(\ell)$ are the contributions of the κ-th object and ℓ-th attribute to the total entropies $\sum_{\kappa=1}^{n} H(\kappa)$ or $\sum_{\ell=1}^{m} H(\ell)$ of the data matrix X, which according to Shannon can be expressed as the sum of the entropies of individual objects or attributes respectively.

The maximum entropy in the frequency table is reached when the distribution of distribution in the data matrix X becomes uniform. To clarify the last statement, we draw a graph of the function:

$-\log_{\mathrm{atr}}\left(r_{\kappa,t}\Big/\sum_{t=1}^{\mathrm{atr}} r_{\kappa,t}\right)$:



The maximum entropy of the data matrix in the row direction is computed when the probabilities on the x-axis allocate a uniform frequency distribution, resulting in $\underset{\max}{\mathrm{H}}(\kappa)\approx 1$. Indeed, the value $-\log_{\mathrm{atr}}\left(r_{\kappa,t}\Big/\sum_{t=1}^{\mathrm{atr}} r_{\kappa,t}\right)$ is at its maximum when $r_{\kappa,t}\Big/\sum_{t=1}^{\mathrm{atr}} r_{\kappa,t}\approx 1\big/_{\mathrm{atr}}$. In case $r_{k,t}=0$ then this zero value is not taken into account. Based on the maximum entropy, we get the actual information about the object κ equal to $1-\mathrm{H}(\kappa)$. Thus, the complete information contained in the data matrix X is calculated by the formula: $n-\sum_{\kappa=1}^{n}\mathrm{H}(\kappa)$. The above layering algorithm can now be used.

For the credential of an individual object, we choose the entropy value $\mathrm{H}(\kappa)$. Thus, the set of objects $x_1,x_2,\ldots,x_n$ is to be stratified. It is only necessary to keep in mind that after removing an object from the data matrix, changes occur in the frequency table (frequency bands). The changes consist in the fact that when using the values of the ℓ-th attribute $x_{\kappa,\ell}$ of the κ-th object, in the corresponding cells $r_{\kappa,x_{\kappa,\ell}}$ and $c_{x_{\kappa,\ell},\ell}$ of the frequency tables $\boldsymbol{r}$ and $\boldsymbol{c}$, 1 is subtracted from the frequencies: $r_{\kappa,x_{\kappa,\ell}}=r_{k,x_{\kappa,\ell}}-1$ and $c_{x_{\kappa,\ell},\ell}=c_{x_{\kappa,\ell},\ell}-1$.

We will consider the properties of the stratification algorithm using the mentioned monotone systems in the next section, where the positive $\oplus$ and negative effects of elements are used. In graphs, the negative $\ominus$ effect on the arc was its removal. For data matrix, this is the reset of the ℓ-th attribute of the κ-th object or a series of $\ominus$ effects until the object will be completely removed by the entropy level u_k assessment.

5. MONOTONE SYSTEMS

We will continue our story about monotone systems now at a more precise level according to the publication (Mullat, 1976-1977) in Autom. and Telemechanics. A monotonous system manifests itself in the relationship between elements in the fact that if an element of the system is "positively influenced", then this effect is also positively reflected on its interrelated elements. It's the same with negative effects.

The monotonicity property as a central property allows us to formulate the concept of the system kernel or core in a general form. By the core, we mean a subset of the elements of "strongly attracting" or "strongly pushing" each other the elements of the system.

Consider any system W consisting of a finite set of elements, i.e., $|W| = n$ |. Quantities or credentials that indicate the level of "importance" of the element $\alpha \in W$ for the functioning of the system as a whole characterize the states of the elements of a system W.

It proves necessary to reflect the internal dependence of the elements of the system at the level of importance of the elements. In view of the fact that the elements of the system are interconnected, it is possible to take into account the effect of element α on other elements related to the change in the properties of element β. We assume that the level of importance of the element α itself also changes due to its effect. If elements α and β are in no way related in the system, it is natural to assume that the change caused by element α to the importance of element β is zero.

In the system W, we consider as an effect on the element α of two types of effects: $\oplus$ and $\ominus$ type effects ($\oplus$- and $\ominus$-effects). In the first case, the properties of element α are considered to improve as its importance to the system increases; in the second case, the properties of element α deteriorate as its level of importance in relation to the system decreases.

Now we can also provide a definition of a monotonic system. A monotonic system is a system in which the positive effect of $\oplus$ on any system element α causes the positive effect of $\oplus$ on all other elements of the system and the effect of the $\ominus$ type causes the effect of $\ominus$ type respectively.

5.1. System monotonicity conditions

The observed important concept — the effect on the element α of the system W and the accompanying effect on the other elements of the system — allows the set W to determine an infinite number of functions, since we have at least one actual function of the importance of the elements W of the system: $\pi : W \rightarrow \Re$, where $\Re$ is a set of real numbers.

If element α is affected, then it can be said that the function π is reflected in the function π_α^+ for the effect of $\oplus$ and in the function π_α^- for the effect of $\ominus$ respectively. As a result of the effects $\oplus$ and $\ominus$ on the element implementation, the credentials of the system elements are redistributed from the function π to the functions $\pi_\alpha^+\pi_\alpha^-$ or the initial set of values $\{\pi \mid \pi(\delta \in W)\}$ is transferred to a new set $\{\pi \mid \pi_\alpha^+(\delta \in W)\}$ and $\{\pi \mid \pi_\alpha^-(\delta \in W)\}$ respectively. The functions π, π_α^+, π_α^- are defined on the whole set W and thus are also defined $\pi_\alpha^+(\alpha)$ and $\pi_\alpha^-(\alpha)$. It is clear that if there is given a sequence α_1, α_2, α_3 ...from the W set of elements (all repetitions and combinations of elements are allowed), and e.g. the a binary sequence $\oplus$, $\ominus$, $\oplus$,... then can be easily determined the combined effect in the form of a functional product of $\pi_{\alpha_1}^+ \cdot \pi_{\alpha_2}^- \cdot \pi_{\alpha_3}^+ \cdot ...$

The presented construction allows writing the monotonicity property of the systems as two main inequalities:

$$\pi_\alpha^+(\beta) \geq \pi(\beta) \geq \pi_\alpha^-(\beta)$$

for each element pair $\alpha, \beta \in W$, including pairs (α, α) and (β, β).

5.2. Identification of the system kernel

To determine the kernel of the system, consider the two subsets of W, namely H and $\overline{H}$, so that $H \cup \overline{H} = W$ and $H \cap \overline{H} = \varnothing$.

If only elements α_1, $\alpha_2, ...,$ $\in H$ are positively affected then it determines for the set W a certain function $\pi_{\alpha_1}^+ \cdot \pi_{\alpha_2}^+ \cdot ...$, which can be considered determined only for the subset H. If we choose one of all possible sequences of a set H, namely $\left\langle \alpha_1, \alpha_2, ..., \alpha_{|\overline{H}|} \right\rangle$ where α_i does not repeat, then the function $\pi_{\alpha_1}^+ \cdot \pi_{\alpha_2}^+ \cdot, ..., \pi_{\alpha_{|\overline{H}|}}^+$ is denoted unambiguously on the set H function and call it a standard function. The function thus introduced is called the credential function on the set H and the individual value of the function on the element α is the credential. These credentials $\{\pi^+H(\alpha) \mid \alpha \in H\}$ we denote by Π^+H and call this set of credentials specified for a given set H, i.e., for the set of credentials with respect to the set H.

Suppose that the set of credentials sets $\{\Pi^+H \mid H \subseteq W\}$ for all possible subsystems 2^W of system W — the number of all possible subsystems is $2^{|W|}$.

Instead of the plus effects of the standard function, we can look at the analogous ⊖ effects function $\pi^{-}_{\alpha_1}\cdot\pi^{-}_{\alpha_2}\cdot,\dots,\pi^{-}_{\alpha_{|\overline{H}|}}$. Similarly to the function $\pi^{+}H(\alpha)$, we also determine, the set of credentials $\{\pi^{-}H(\alpha) \mid \alpha\in H\}$ and also the collections of sets of credentials $\{\Pi^{-}H \mid H\subseteq W\}$. In addition, to obtain a process of type ⊖ effects — an analogous process $\Pi^{-}H$ is performed. All elements of the set H are affected in sequence according to the ordered list $\left\langle \alpha_1,\alpha_2,\dots,\alpha_{|\overline{H}|}\right\rangle$.

On the subsets or arrays $\{\Pi^{+}H \mid H\subseteq W\}$ and $\{\Pi^{-}H \mid H\subseteq W\}$ of credentials given on the sets $H\subseteq W$, the following two functions can be defined for each subset H:

$$F_{+}(H)=\min_{\alpha\in H}\pi^{+}H(\alpha),\ F_{-}(H)=\min_{\alpha\in H}\pi^{-}H(\alpha).$$

By the kernels of W we call the global minimum of the function $F_{+}(H)$ and the global maximum of the function $F_{-}(H)$. The subsystem $H^{\oplus}$ that reaches the global minimum of the F_{+} function is called the system ⊕-kernel, and the subsystem $H^{\ominus}$ that reaches the global maximum of the F_{-} function is called the ⊖-kernel, respectively.

Definition. The defining set considered in monotone systems theory is the last set in the layer algorithm with level u_p (see the section 3 above), where the sequence $\overline{\alpha}=\left\langle \alpha_1,\alpha_2,\dots,\alpha_{|\overline{H}|}\right\rangle$ of system elements by which such a defining set is found is called the defining sequence.

Theorem 1. The defining set $H^{\ominus}$ is the set where the F_{-} function reaches the global maximum. There is only one defining set $H^{\ominus}$ set. All other subsets if they exist where F_{-} reach the global maximum are within the defining set $H^{\ominus}$.

Theorem 2. For the definite set of $H^{\oplus}$, the function F_{+} reaches a global minimum. There is only one defining set $H^{\oplus}$. All sets that reach the global minimum are enclosed in the defining set $H^{\oplus}$.

The existence of defining sets $H^{\ominus}$ and $H^{\oplus}$ is ensured by a special constructive routine. The defining sets are kernels of Monotone Systems, because on these sets the functions F_{-} and F_{+} reach the global maximum (minimum) accordingly. Theorems 1 and 2 guarantee that all kernels are located in one "large" kernel — the defining set.

6. MONOTONE SUBSYSTEMS ON GRAPHS

Let us have a "*big*" graph G and a "*small*" graph g. It is necessary to select a part of the "*big*" graph G (a set of arcs or edges) so that this set is the most *"saturated"* with "*small*" graphs g. For example, we can assume that one part of the graph is more saturated than the other if the first contains more small graphs g than the second.

With some complexity, saturation can also be approached as follows. Consider the arcs, edges or vertices of G that belong to the part we are interested in. We now count in integers: how many there are small graphs g, separately those g graphs that are located "near" each vertex, arc or edge. By this integers is meant the number of graphs g that contain a given vertex, arc or edge, and are thus expressed as an integer. By doing this, we get exactly such an integer or credential that characterizes the part of G we is interested in. Each such integer reflects a certain "*local*" saturation of the graph G with the graphs g.

Based on the obtained integers, several variants open to determine the saturation of the G part of the graph. The mean, variance, etc., of these numbers can be calculated. We consider the simplest credential magnitude, namely the entity of small graphs g, which are located in a separate part of a large graph G, i.e., the smallest value of the local parts. Figuratively speaking, this number of sub-graphs is in the most "empty" location of the graph G, which we should further on remove by $\ominus$ type actions.

Below we give an exact representation of the problem of determining the most saturated parts of the graph G with small graphs. We set the problem as follows: From all possible parts (or a large number of parts) of a graph G we find the one with the maximum value of the smallest number of local sets of small graphs g.

It is natural that in this method many small graphs g can be placed in a part in the usual way, because the number of small sub-graphs g on each vertex or arc is not less than on the vertex or arc on which it is minimal. At the same time, however, this minimum number in the extreme part is quite large, because we specifically chose the part where the local number of graphs condition reaching the global maximum of the minimum would be satisfied, Similarly, we can set the task of finding the part of the graph G that is least saturated with small graphs g. The number of sub-graphs g at the vertex or arc where this number is maximal characterizes then each part of the graph. Instead of looking for the part of the graph where the minimum local number of graphs is the maximum, we look for the part where the maximum local number is the minimum. In this case, the number g of the sub-graphs of each vertex or arc is not greater than the "maximum" vertex or arc, and the latter has a default due to the global minimum condition.

The extreme parts of a graph are usually uniformly saturated or unsaturated with small graphs. In a saturated extreme part, no single vertex or arc can usually have very few graphs g, because without the arc of this vertex the part of the graph is probably more saturated at the top or arc with sub-graphs g in the more complex sense mentioned above.

7. GENERAL MODEL OF KERNEL EXTRACTION ON GRAPHS

If a graph G is given, then with $V(G)$ or by V we denote the set of vertices of the graph. We denote the set of arcs of an oriented graph G by $U(G)$ or U and the set of edges of an unoriented graph by $E(G)$ or E.

In graph theory, the concept of a sub-graph of a given graph G is used. A graph G' is a sub-graph of the graph $[V(G), U(G)]$ if $V(G') \subset V(G)$ and $U(G')$ is the set of arcs of all and only those that bind the pair from $V(G')$. Similarly, we can define a sub-graph of an undirected graph if the term edge is used instead of the arc.

Sometimes the term part G of a graph is also used. We call graph G a part of the graph $G[V, U]$ if $V(G'') \subseteq V(G')$ and $U(G'') \subseteq U(G')$. In terms of the oriented graph, some arcs of the graph G are simply missing. Similarly, an undirected sub-graph is determined.

The design of concepts described in the previous two sections of this guide must begin with the identification of the elements of the system W. Two structural units can be separated from graphs — a vertex and an arc. Let us consider first the case where the vertex of the graph G is chosen as an element of the system. We now determine the effects of the ⊕- and ⊖-effects on the vertices, i.e., on the elements of the system W. Determining the effects of ⊕ and ⊖ requires the addition of a special significance function π to the vertices of the graph G. The action has already been mentioned in the previous two sections of the guide, that the credentials in the system must increase as a result of the ⊕ effect and decrease as a result of the ⊖ effects.

We need to define saturation indicators, or whatever we call them, credentials for the elements α of each subset of H from W. To get this, we need to set up an initial set of credentials for W, as well as a framework how to express $\oplus$ and $\ominus$ effects.

An initial set of credentials $\left\{\pi(\alpha) \middle| \alpha \in W\right\}$ can be specified, for example, as follows. Let g be a small graph given a large graph G. We count the number of different sub-graphs of graph G that are isomorphic to graph g and whose vertices include vertex α. We set the just obtained number to the initial credential level $\pi(\alpha)$. To underline the introduced dependence of the level $\pi(\alpha)$ on the small graph g, we use the expression — the credential of the vertex α of the graph G with respect to g. Next, we consider two operations for obtaining new graphs from G, namely the $\oplus$ and $\ominus$ operations.

Let a graph G be given and an empty graph Λ (a graph that has no arcs but has $|V(G)|$ vertices). We assume that $V(\Lambda)$ is an exact copy of $V(G)$. And when we talk about the vertex α, we mean the vertex of a graph G, which appears in two forms — like the vertex of a graph G and like the vertex of a graph Λ.

A $\ominus$-type operation of a graph G with a vertex α is to carry out removing all the arcs or edges leading to that vertex. On an empty graph Λ, however, the $\oplus$-type operation is a recovery operation for all edges leading to that vertex α. It appears that if a $\oplus$-type operation is applied to a vertex, the credentials of all other vertices (relative to the small graph g) either decrease or, in some cases, remain the same. When performing a $\oplus$-type operation, a natural question arises: what should be considered the credential of the vertex after restoring the vertex?

The solution to this question lies in the following construction. Let us count the credentials of the vertices of the graph Λ (with respect to the small graph g) and add the credentials of the vertices of the graph G. We consider the obtained amounts as the total credentials of the vertices. In this case, the opposite effect can be observed: as a result of the $\oplus$-type operation, the total credentials increase or, like the $\ominus$-type credentials, remain at the same level. Generally speaking, the initial credential set $\left\{\pi(\alpha) \middle| \alpha \in W\right\}$ (the credential set before any $\oplus$-type operation) of the vertices of graph G can be considered as a general credential set to be built since any part of graph G is initially empty. At this stage, minimizing the maximum credentials means some options for the vertices of graph G to be isolated. In this approach, the monotonicity condition is satisfied.

When constructing sets of credentials in system W, it must be demonstrated how the initial set of credentials $\{\pi(\alpha) \mid \alpha \in \mathrm{W}\}$ found is redistributed due to ⊕ and ⊖ operations.

Let be given a certain sequence of vertices $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$, which forms a set of $\overline{\mathrm{H}} \subseteq \mathrm{W}$. We express the effect of ⊕ on the vertices of G according to their occurrence in the sequence. As a result, a sub-graph of G is formed on the graph $\mathrm{V}(\Lambda)$. At the vertex of each resulting sub-graph we can count the number of isomorphic sub-graphs with a small graph g, so we get the credentials of a set of H (the complement of $\overline{\mathrm{H}}$ to W) elements. Consistent with the above theory, we can state that the set H determines a new significance function in the form,

$$\pi^{+}_{\alpha_1} \cdot \pi^{+}_{\alpha_2} \cdot \ldots \qquad (2)$$

obtained from the initial credential collection $\{\pi(\alpha) \mid \alpha \in \mathrm{W}\}$.

Thus, if a sequence of vertices $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$ is given that promotes the set $\overline{\mathrm{H}}$, then the set H forms a set of credentials determined by (2) or (3). We denote this set by $\Pi^{+}\mathrm{H}$, and we call the set of credentials by the set of vertices induced on H. The sets of induced credentials form the set $\{\Pi^{+}\mathrm{H} \mid \mathrm{H} \subseteq \mathrm{W}\}$. Sometimes it is appropriate to use the expression of ⊕-collection of sets with respect to the small graph g.

The collection or array $\{\Pi^{-}\mathrm{H} \mid \mathrm{H} \subseteq \mathrm{W}\}$ of sets of credentials is determined analogously. The collection $\Pi^{-}\mathrm{H}$ of the credentials is determined by the function

$$\pi^{-}_{\alpha_1} \cdot \pi^{-}_{\alpha_2} \cdot \ldots \qquad (3)$$

given in part G of the graph, which remains after the application of the ⊖-activities to the sequence of vertices forming $\overline{\alpha} = \langle \alpha_1, \alpha_2, \ldots \rangle$. It only needs to be emphasized that each subset $\mathrm{H} \subseteq \mathrm{W}$ of the set of credentials is in fact the set of the remaining part, but not the total, i.e., not the part given by the set of graph Λ, which actually is an empty graph.

Next, let's take the arc as the system element. The system is defined as the set of interconnected arcs $\mathrm{U}(\mathrm{G})$ of the graph G, determining the ⊕ and ⊖ effects again requires setting the values of the initial function π.

Let be given a small graph of g. We count the number of different subgraphs of the graph G that are isomorphic to the graph g and whose arcs or edges include this arc or edge. The resulting integer is taken as the significance level of the arc α of the graph G. This is called the credential of the arc α with respect to the graph g.

Similarly to those described at the vertices of G, the concepts of $\oplus$ and $\ominus$ activities are also determined by the arcs or edges of the graph G. Arcs or edges are now removed or restored instead of vertices.

Let's look at the $\ominus$ operation first. It is obvious that as a result of removing the arc (edge), the initial set of credentials with respect to the small graph g may decrease or remain the same. A decrease in importance of credentials indicates that the $\ominus$ operation is equivalent to defining $\ominus$ activity for system elements.

Let $\langle \alpha_1, \alpha_2, ... \rangle$ be a sequence of different arcs on G, including arcs forming $\overline{H} \subseteq U(G)$. We perform $\ominus$-actions sequentially on the arcs of the graph G according to the given sequence. As a result, we get a certain part of the graph G, the elements of which are arcs (edges) belonging to the set $H \subseteq U(G)$. For each arc $\alpha \in H$, count the number of isomorphic graphs with the graph g, which is considered to be the credential or significance of the element α with respect to the set H.

According to the notations used, the method for determining the given credentials creates a function on the elements of the set H of arcs. Similarly to the case where the number of sets of credentials was assigned to the vertices of a given graph, arcs (edges) are created that belong to the set of credentials $\left\{\pi^{-}H(\alpha) \,\middle|\, \alpha \in H\right\}$, which we denote again $\Pi^{-}H$. We proceed in a similar way to find the set of credentials $\left\{\Pi^{-}H \,\middle|\, H \subseteq U(G)\right\}$. On an empty graph Λ, defining the $\oplus$-activity on the basis of the $\oplus$-operation requires a more detailed analysis.

Let again the sequence of arcs $\overline{\alpha} = \langle \alpha_1, \alpha_2, ... \rangle$ in the given graph G (said arcs form the set $\overline{H}$), we perform $\oplus$-operations on the set $\overline{H}$ arcs sequentially. As a result, the set of vertices $V(\Lambda)$ forms a part of a graph G whose list of arcs is equal to $\overline{H}$. For the vertex model, we calculated the total credential of each vertex $\alpha \in V(G)$. In this case, too, we try to do the same and find the total credential of the arcs forming H.

The arcs belonging to the set H are not present in the graph g and the question is how to count the number of sub-graphs isomorphic to the graph g and containing the arc α (which is not present in the graph Λ). Proceed as follows: we read that this arc α is fictitious only at the moment of counting the sub-graphs. In this case, the set of arcs H forms certain integers that depend on both the graph and the part of the graph formed on the empty graph g.

In the method described above, the function $\pi^+_{\alpha_1} \cdot \pi^+_{\alpha_2} \cdot \ldots$ is determined from the quantity H, which creates a set of ⊕-credentials $\{\pi^+ H(\alpha) \mid \alpha \in H\}$.

In this case, even in the case of a ⊕-operation, the set of credentials of the ⊕-activities can be determined with respect to a small graph. The use of the term "⊕-activity" is perfectly legal here, as the total credentials of those elements that are not yet subject to ⊕-activity may increase or remain the same.

7.1. Illustrative Examples on Directed Graphs

A graph G of partial ordering is defined as a binary relation *G* with the following properties:

a) Reflexivity, i.e., if $\alpha \in V(G)$, then $\alpha G \alpha$. The graph G has a loop at the vertex α.

b) Transitivity, if there exists an arc (α,β) and (β,γ), then the graph G has an arc (α,γ), or from $\alpha\, G\, \beta$ and $\beta\, G\, \gamma$ it follows that $\alpha\, G\, \gamma$.

A complete order is defined as a graph of partial ordering in which any pair of vertices α and β is connected by an arc.

It is possible to formulate the following problem: in a given directed graph it is required to find the (in certain sense) most “saturated” regions that are “close” to a graph of partial ordering or to graphs of complete ordering. This problem will be solved by a method of organization (on a graph) of a monotonic system with subsequent determination of kernels.

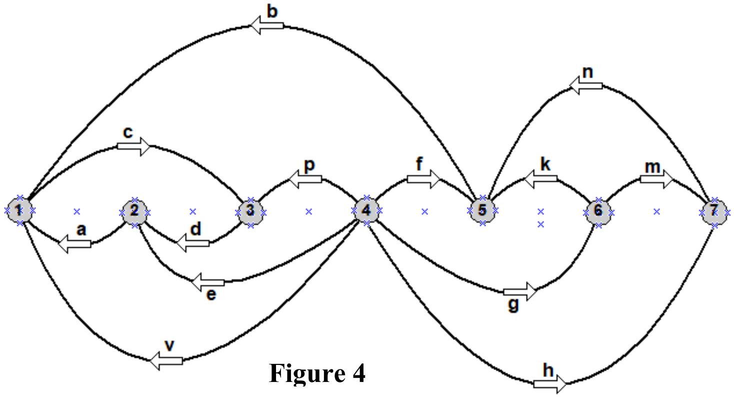


Figure 4

In accordance with the scheme of organization of a monotonic system on graphs described in the previous section, it is necessary to assign a small graph g. Suppose that this graph consists of three vertices x, y, z, and it is such that $U(\Gamma) = \{ (x,y),(y,z),(x,z) \}$. The graph has a total of three arcs (a transitive triple).

Now let us consider the assignment of collection of credentials arrays at the vertices of a graph shown in Fig. 4. The loops on this graph have been omitted.

According to the scheme of assignment of collections of credential arrays at the vertices of a graph, it is required to determine an initial array of credentials $\{ \pi(\alpha) \}$, where $\alpha = 1,2,3,...,7$. According to the method of calculation of the values $\pi(\alpha)$ with respect to the graph g (a transitive triple), we obtain $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 7$, $\pi(5) = 4$, $\pi(6) = 3$, $\pi(7) = 3$. As an example, let us determine a credential array on a subset of vertices $H = \{1,2,3,4,5\}$. By successively performing $\ominus$ actions on the set $\overline{H} = \{ 6,7 \}$, we obtain on the set H a new credential array $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 4$, $\pi(4) = 4,$, $\pi(5) = 1$.

The values of the function $\overset{+}{\pi}_6 \overset{+}{\pi}_7$ can be obtained in a similar way, but for this purpose it is necessary to use the assignment of collections of total $\oplus$ arrays with respect to a transitive triple. According to Fig. 5, the values of this function in their order at the vertices $\{1,2,3,4,5\}$ are as follows: $\pi(1) = 3,$, $\pi(2) = 2$, $\pi(3) = 2$, $\pi(4) = 8$, $\pi(5) = 4$. In exactly the same way we can determine on any subset H of vertices $V = \{1,2,3,4,5,6,7\}$ a proper credential array of $\oplus$ or □ actions with respect to a transitive triple.

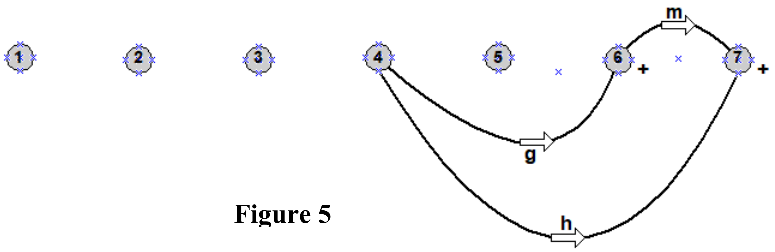


Figure 5

Now let us consider a construction that is assigned not on vertices, but on the arcs of the graph presented on Fig. 4. In this case the set of elements of the system W will be $U(G) = \{ a,b,c,...,n,m \}$. As the small graph g we shall take the same graph as above, with a set $U(g) = \{ (x,y),(y,z),(x,z) \}$.

By analogy with the foregoing, we realize the construction in the same succession. We determine an initial credential array $\left\{ \pi(\alpha) \,\middle|\, \alpha \in U \right\}$ on the arcs of the graph G in accordance with the general scheme.

We find that

$$\pi(a) = 1, \pi(b) = 1, \pi(c) = 1, \pi(d) = 1, \pi(e) = 2, \pi(f) = 3,$$
$$\pi(g) = 2, \pi(h) = 2, \pi(k) = 2, \pi(n) = 2, \;\; \pi(m) = 1, \pi(v) = 3, \pi(p) = 2$$

As an example, let us now perform (⊕ and ⊖ actions on the arcs f,k and m, i.e., on the set $H = \left\{ f,k,m \right\}$. On the set H we hence obtain

$$\pi(a) = 1, \pi(b) = 0, \pi(c) = 1, \pi(d) = 1, \pi(e) = 2,$$
$$\pi(g) = 0, \pi(h) = 0, \pi(n) = 0, \pi(v) = 2, \pi(p) = 2.$$

In accordance with the adopted system of notations this array of numbers will be denoted by $\Pi^{-}H$. For obtaining an $\Pi^{+}H$ array, we must calculate the total credentials. The dashed lines in Fig. 6 represent the arcs of graph Λ that experience the effect of □ actions performed on the arcs f,k and m.

According to Fig. 6, the total credential array will be as follows:

$$\pi(a) = 1, \pi(b) = 1, \pi(c) = 1, \pi(d) = 1, \pi(e) = 2,$$
$$\pi(g) = 3, \pi(h) = 2, \pi(n) = 3, \pi(v) = 2, \pi(p) = 2.$$

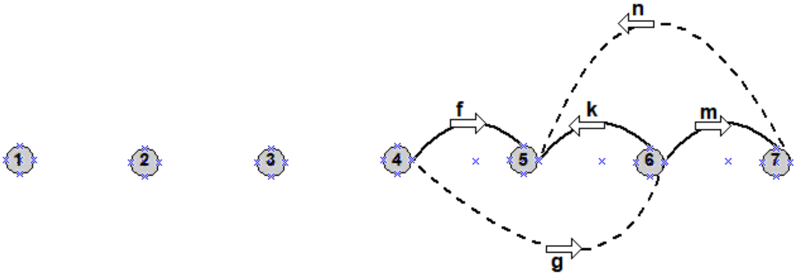


Figure 6

Thus on any subset H of arcs of the graph shown in Fig. 4 we can construct the credential arrays $\Pi^{-}H$ and $\Pi^{+}H$.

Next we describe the procedures of construction of determining sequences of ⊕ or ⊖ actions, at first for vertices, and then for arcs of the graph shown in Fig. 4. The construction is carried out for the purpose of illustrating the concepts of ⊕ or kernels of the monotonic system and also for ascertaining the effect of the duality theorem formulated by Mullat (1976-1977).

Let us consider an example in which ⊖ credential arrays are assigned at vertices with respect to a transitive triple. According to the scheme prescribed in Mullat's routine of construction of a determining ⊕ and ⊖ sequence of vertices of a graph on the basis of ⊕ and ⊖ actions. For the graph shown in Fig. 4, the Kernel-Searching Routine consists of two steps: the zero-th and the step one. It yields two subsets $\Gamma_0^-, \Gamma_1^- \subseteq V(G)$, where

$$\Gamma_0^- = V(G) = \{1,2,3,...,7\},\ \Gamma_1^- = \{4,5,6,7\},$$

and the thresholds $u_0 = 2$, $u_1 = 3$.

The determining sequence of vertices constructed with the aid of ⊖ actions is as follows: $\overline{\alpha}_- = \langle 3,2,1,4,5,6,7 \rangle$. Thus on the basis of: a) according to Theorems 1,3 (Mullat, 1971) and b) according to Theorem 1 (Mullat, 1976) about KSR, it can be argued that the set $\{4,5,6,7\}$ is the definable set of vertices of the graph shown in Fig. 4, and, therefore, this set is also the largest kernel $K^{\ominus}$.

Now let apply the KSR for constructing a ⊕-determining sequence. We find that $\overline{\alpha}_+ = \{4,5,6,7,1,2,3\}$. The routine terminates at the third step, and it consists of four steps, namely the zero-th, the first, the second and the third. According to the construction of ⊕ sequences prescribed in the KSR, we produce the sets Γ_j^+: $\Gamma_0^+ = \{4,5,6,7,1,2,3\}$, $\Gamma_1^+ = \{5,6,7,1,2,3\}$, $\Gamma_2^+ = \{6,7,1,2,3\}$, $\Gamma_3^+ = \{2,3\}$ and a sequence of thresholds $u_0 = 7$, $u_1 = 4$, $u_3 = 3$, $u_3 = 2$. As in the case of a ⊕ sequence, we conclude on the basis of Theorems 2 and 3 of a) Mullat, and of Theorem 1 of b) Mullat, that $\{2,3\}$ is the largest $K^{\oplus}$ kernel of the system of vertices of the graph in Fig.1.

A careful analysis of Fig.1 shows that the $K^{\oplus}$ kernel is in fact completely ordered set, i.e., $\langle 4,5,6,7 \rangle$. On the other hand the $K^{\oplus}$ indicates from the point of view of the "structure" of a graph that the region, in which the vertices are least ordered, it is ordered itself as well. This is in agreement with the our formulation of the problem of finding kernels as representatives of "saturated" or "unsaturated" regions (parts of a graph) with small graphs g

Now let us use the KSR for constructing determining sequences of arcs of the graph in Fig.1. The graph has a total of 13 arcs. After applying the KSR, we obtain on the basis of ⊖ actions the following sequence:

$$\overline{\alpha}_- = \langle a,b,c,d,v,e,p,f,k,n,m,h,g \rangle.$$

The routine terminates at first step and it consists of two steps, namely the zero-$^{\text{th}}$ step and the first step. At the zero-$^{\text{th}}$ step we have $\Gamma_0^- = \mathrm{U(G)}$, and at the first step we have $\Gamma_1^- = \{\, \mathrm{f,k,n,m,h,g} \,\}$, with the thresholds $\mathrm{u}_0 = 1$ and $\mathrm{u}_1 = 2$ respectively. Summing up, we can assert on the basis of the results of a), b) Mullat, that this is a definable set and at the same time the largest $\mathrm{K}^{\ominus}$ kernel in the system of arcs.

From the point of view of the graph structure, the application of the KSR to arcs in the construction of a ⊖ determining sequence does not yield anything new compared to the application of the KSR to vertices. We obtain the same complete order $\langle 4,5,6,7 \rangle$ represented in the form of a string of arcs, and it also corroborates our assertions concerning the saturation of a $\mathrm{K}^{\ominus}$ kernel by transitive triples. On the other hand the use of KSR for constructing ⊕ determining sequence of arcs yields a $\mathrm{K}^{\oplus}$ kernel

$$\Gamma_1^+ = \{\mathrm{k, m, n, g, h, e, p, b, a, c, d}\},$$

whose meaning with regard to “non-saturation” with transitive triples cannot be determined.

Below we shall illustrate the peculiar features of using the duality theorem from b) Mullat (1976) for finding $\mathrm{K}^{\oplus}$ and $\mathrm{K}^{\ominus}$ kernels of a monotonic system specified by vertices or arcs of a directed graph.

At first let us consider the monotonic system of vertices of the graph in Fig.1. The sequence of sets $\langle \Gamma_j^+ \rangle$ specified by the KSR on the basis of ⊕ actions uniquely determines the sets $\mathrm{V} \setminus \Gamma_1^+ = \{\, 4 \,\}$, $\mathrm{V} \setminus \Gamma_2^+ = \{\, 4{,}5 \,\}$, $\mathrm{V} \setminus \Gamma_3^+ = \{\, 1{,}4{,}5{,}6{,}7 \,\}$. Above we have found that $\mathrm{F}_+(\Gamma_2^+) = \mathrm{u}_2 = 3$. From the construction of a determining sequence $\overline{\alpha}_-$ of vertices of a graph we know that $\mathrm{F}_-\{\, 4{,}5{,}6{,}7 \,\} = 3$. Hence by virtue of Corollary 1 of Theorem 1 of b) Mullat, we can assert already after the second step of construction of a $\overline{\alpha}_{,}$ sequence that the set $\{\, 1{,}4{,}5{,}6{,}7 \,\}$ contains the largest $\mathrm{K}^{\ominus}$ kernel. Thus we have shown that the sufficient conditions of the duality theorem of b) Mullat, are satisfied in the example of the graph represented in Fig. 1.

Now let us consider the set $\mathrm{V} \setminus \Gamma_1^- = \{\, 1{,}2{,}3 \,\}$. As was shown above, inside this set there exists a set $\Gamma_3^+ = \{\, 2{,}3 \,\}$ such that $\mathrm{F}_+(\Gamma_3^+) = 2$; $\mathrm{F}_-(\Gamma_1^-) = 3$ on the other hand. By virtue of Corollary 4 of the duality theorem we can assert that set $\{\, 1{,}2{,}3 \,\}$ contains the largest $\mathrm{K}^{\oplus}$ kernel of the system of vertices of the graph (Fig.1); this likewise confirms that existence of the conditions governing the theorem.

At last let us consider a collection of credential arrays on the arcs of the graph. The determining $\overline{\alpha}_+$ sequence of arcs specifies a set $\Gamma_1^+ = \{ k, m, n, g, h, e, p, b, a, c, d \}$. It is easy to see that inside the set $U \setminus \Gamma_1^+$ there does not exist a set H as required by the conditions of Corollaries 1 and 2 of the duality theorem in Mullat (1976). This shows that in comparison to arrays on vertices, credential arrays on arcs do not satisfy the duality theorem.

7.2. Monotonic systems on special classes of graphs

In contrast to the previous section, we do not carry out here a detailed construction of collections of credential arrays and determining sequences and kernels on any illustrative example. Here we shall show how to select a small graph g and $\oplus$ and $\ominus$ actions so as to match the selection of these elements with the desired "saturation" of the investigated graph. The desired saturation of a graph can be understood as the saturation desirable for the investigator who usually has a working hypothesis with respect to the graph structure. In view of this, we shall consider the following classes of graphs: tournaments, a-cyclic (directed) graphs, and (directed or undirected) trees.

Let us recall the definitions of these classes of graphs. A tournament is a directed graph in which each pair of vertices (x, y) is connected by an arc, cf. Harari (1969). A none-cyclic graph is a graph without cycles (in case of an undirected graph), and a graph without circuits (in case of a directed graph). None-cyclic undirected graphs are trees, and we shall consider the most general class of trees, as well as the class of directed trees.

In tournaments it is appropriate to consider regions of vertices that are "saturated" with cyclic triples. A cyclic triple is a graph g such that $V(g) = \{ x, y, z \}$, $U(g) = \{ (x, y), (y, z), (x, z) \}$. It can be assumed that a tournament in which there exists such a region represents a structure of the participants of the tournament. This structure is non-uniform; i.e., there exists a central region (set) of participants who can win against the other players, but they are in neutral position with respect to one another.

For solving the above problem, we propose the following exact formulation in the language of monotonic systems. In Section 2 we have considered credential arrays on vertices and arcs of a graph. Now let us consider the above models on vertices or arcs in a certain order. In both models we take a cyclic triple as the small graph g with respect to which the π function is calculated. Suppose that the methods of assignment of collections of credential arrays on vertices are the same as in Section 2. It is possible to modify this scheme by taking as a $\ominus$-action on the vertex α the removal of all arcs of a tournament that originates at α, whereas $\oplus$-action is the restoration of all the arcs in the graph Λ that originate at α. In Section 2 we performed the opposite operation, i.e., the removal of incoming arcs and the restoration of these same incoming arcs.

The assignment of credential arrays on arcs of a tournament graph must be carried out in accordance with a scheme similar to that described in Section 2. Within the framework of the theory it is apparently impossible to decide whether the scheme of determination of kernels on arcs of a tournament is preferable to the scheme using vertices; therefore, it is necessary to carry out computer experiments. There exists only one heuristic consideration. If in a tournament there can exist several central regions saturated with cyclic triples, it will be preferable to use the scheme of determination of kernels on the arcs of tournament, since these regions can be found. The model based on vertices makes it possible to find a kernel that consists also of regions, but it does not permit finding an individual region. We do not possess a string of arcs representing these regions.

None-cyclic directed graphs are a convenient language for describing operation systems (Kendal, 1940). An operation system can be regarded as a system of modules and interpreted as a library of programs. Each working program is a path in a none-cyclic graph, or, in other words, the set of modules of a library needed at a given instant. The modules are called one after another if not all of them can be stored in the main memory. In case of a library of a large size, there naturally arises the question of fixing the modules on information carriers. Prior to solving this problem, it is appropriate to ascertain the "structure" of a none-cyclic graph of a library of modules.

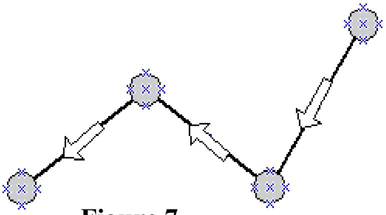


Figure 7

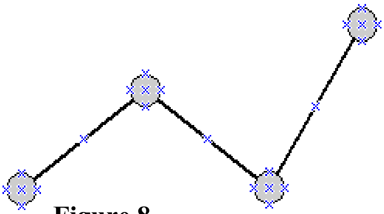


Figure 8

For ascertaining the structure of a graph and for just-mentioned task of fixing the modules, we have to find the principal (nodal) vertices or arcs. The nodes are the "bottlenecks" of graphs or, in other words, the modules that occur in many working programs.

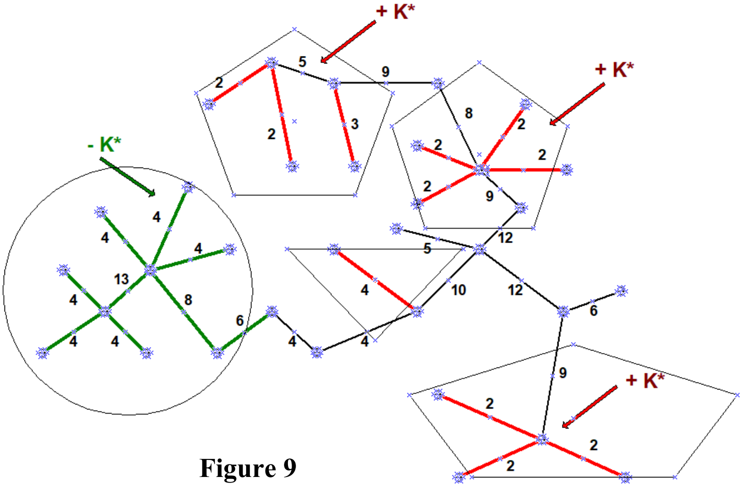


Figure 9

We shall now formally describe this problem with the aid of a model of organization of a monotonic system on a graph. As a small graph we shall take directed graph in Fig.7. The structure of this graph is in accordance with the above definition of bottlenecks of the none-cyclic graph under consideration. It is possible to construct a monotonic system also on the arcs of a none-cyclic graph of a library of modules. With the respect to the graph on Fig.7, the collection of credential arrays and ⊕ and ⊖ actions, in accordance with the general scheme of Section 2, must be defined. After this it is necessary to use the routine of finding vertex kernels or arc kernels, which in conjunction must indicate the bottlenecks in accordance with the above definition. As in case of tournaments, which a monotonic system is preferable of arcs or vertices requires experimental checking.

In comparison to the two previous examples, the last example does not have the aim of associating the application or description of any actual problem with tees. Our aim is to try and find in a tree a region, which in some sense is more similar to “cluster” than any other part of the tree.

At first let us consider undirected trees. We shall use a model of organization of a monotonic system on the branches of a tree. As a small graph g we shall take the graph shown on Fig. 8. As in the case of assignment of collections of ⊕ and ⊖ credential arrays on arcs, we assign the corresponding ⊕ and ⊖ arrays with respect to the graph shown in Fig.9. The ⊖ arrays appear as a result of ⊖ actions (removal of edges), whereas the ⊕ arrays result from ⊕ actions (restoration of edges on empty graph Λ by calculating the total credentials of the tree G and its copy on Λ. As an example we presented the ⊕ and ⊖ kernels in Fig.9 of this tree. Together with each edge we indicated the number of sub-graphs g that contain this edge and which are isomorphic to the graph shown in the Fig.8.

Now let us consider directed trees. If it is of interest to separate "clusters" in a directed tree, we shall proceed as follows. Let us consider the following small graphs: g_1 , g_2 and g_3 (see Fig. 10).

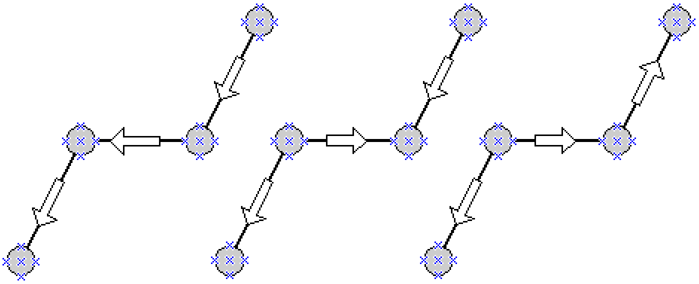


Figure 10

The credential function π on a directed tree can be calculated separately with respect to each small graph g_1 , g_2 and g_3; then the values of all these three functions can be added up (a linear combination), thus yielding the overall function with respect to the graphs g_1 , g_2 and g_3. In the same way we can assign a monotonic system on arcs of a tree if ⊖ action signifies the removal of an arc of a tree, ⊕ action the restoration of an arc on a copy of given tree on Λ . Thus we can pose on directed trees a similar problem of finding cluster kernels. Let us note that we use in the last example with trees a more general model of assignment of collections of credential functions with respect to a series of small graphs. The model in Section 2 has been presented for one graph g . A collection of credential arrays with respect to a series of graphs has also the property of monotonicity, and apparently such a model is more interesting in solving problems of determination of "saturated" parts of graphs.

Let us consider how the g , ⊕ and ⊖ activities of a small graph can be selected to coordinate the selection of these elements with the desired "saturation" of the graph under study. The desired saturation of a graph can be understood as desirable from the researcher's point of view, because the researcher has a certain working hypothesis about the structure of the graph.

For the small graph g for which the functions π were calculated, we choose a cyclic triangle. We use the method described in the previous subsections to create a set of credentials. The removal of all the arcs in the tournament $\langle \text{x wins y} \rangle$ from the vertex x is the ⊖ action on the vertex x and the ⊕-action on the graph Λ is the restoration of all pairs where x wins y . The set of credentials on the graph tournament arcs must be created analogously to the previous sections.

The question of which is more preferable, whether the scheme is done on the arcs of the tournament (a game between two participants) or on the vertices of a graph, cannot be solved within the theory. It can only be said that if there are several central regions in the tournament that are saturated with cyclic triplets, the scheme of separating the kernel by arcs will be better, because these regions can be separated. A model that uses vertices separates the kernel that consists of these regions, but does not allow a single region to be found. We don't have a list of arcs that represent these areas.

Non-cyclic oriented graphs are a suitable tool for describing operating systems. The operating system can be thought of as a system of modules and interpreted as a library of programs. Each work program is a set of modules activated from a library, or in other words, in a non-cyclic graph of the path form. The modules call each other in sequence if they are not all in RAM or for some other reasons.

If the library is large, the natural idea is to place the modules on data carriers. Before solving this task, it is reasonable to explain the structure of the non-cyclic graph of the library of modules. The latter can be understood as the separation of the main sub-vertices or arrows. Vertices are very important places in the graph, in this case they are modules that are available in many work programs.

This task can be formally described in a graph by a monotonic system organization model. The question of the preference of monotonic systems formed by arrows or vertices again requires experimental control. Looking at the trees, we try to separate them from an area that is in some way more like a "bush" than the rest of the tree.

8. DISCUSSIONS AND SUMMARY

Usually, information is collected in to draw the necessary conclusions on issues related to human collectives, economic activity, production processes, demography, etc. If you are more interested in the verbal history itself, then the numerical experiments in Tables 1-3 can still be interesting of themselves. Indeed, with the help of these tables, the main feature of the analysis method is manifested, namely, the independence from any prior knowledge or specific information that is necessary for data analysis. This is especially true of the usual practice of personal and expensive interviews in sociological research. In this regard, the algorithm described in the manual for decomposing the data matrix into layers can be called "blind eye of statistical evaluation or scoring", which is what we need (Võhandu, 1979, 1989). This methodological guide looked at this information processing method that often has been used.

Although the main component of this methodological guide was prepared and presented for publication many years ago, as it seems to us everything that is given here is still relevant. It's not a secret that with the development of information technologies, methods for analyzing data extracted from our environment not only become more complicated, but also their volume has grown to enormous sizes when you have to deal with databases whose size reaches many gigabytes in the amount of collected information. One thing is that all the information in such well-known applications as Facebook and the like are always reflected in some graphs of mutual relations between the participants, whether it is Linkedin or Twitter, etc. Many do not even suspect that our technology for analyzing relationships reflected in these applications are fully adapted to the analysis of such information. The problem here is that such information must be collected and presented either in tabular form or in the form of graphs. Graphs, however, must again be presented in tabular form, which, as we have already indicated, is the main form of data to be analyzed.

The algorithm for decomposing data into layers given in this tutorial turned out to be effective in many specific problems as we can apply here in the form of data viewing technology. Moreover, as already indicated throughout the book, the entire analysis process begins with the construction of the so-called defining sequence, whether it be elements of graphs or data tables, when it is required to find a local maximum at which the global maximum is reached when moving along the defining sequence from weak elements in the direction of strong ones. It turns out that a more effective method of searching for the core or kernel of a monotonic system is to move from top to bottom, from strong to weak elements. Such a search for the kernel is much more economical than the one that was proposed at that time in the original of this methodological manual.

On the other hand, the model of a monotonic system turned out to be a more complex than the author had assumed, who initiated the theoretical and practical use of monotonic systems. The fact is that on graphs when arcs of a graph or edges are taken as elements of the system, it is required to formulate very precisely what are $\oplus$ and $\ominus$ actions. If the $\ominus$ action is to remove or $\oplus$ is add both arcs and edges of the graph together with arcs and edges adjacent to an arc or edge, then monotone systems of a special type arise when the layering algorithm does not always lead to an optimal layer in the global sense. This white area has not yet been sufficiently studied, and here it is quite possible to discover some new features of monotonic systems of the indicated unusual type. We have already indicated this feature earlier in the article on how to organize a party in order to make the optimal combination of participants.

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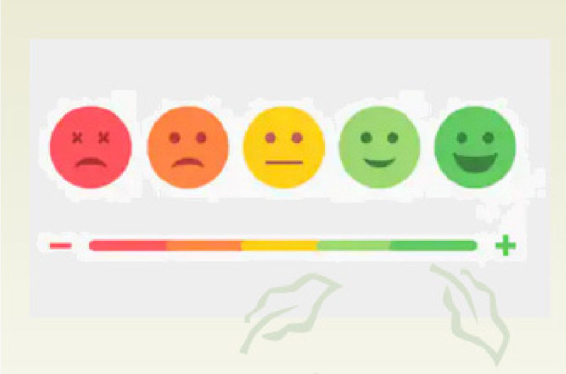

Survey Data Cleaning: Monotone Linkage[*]

Abstract. Analysis of data obtained from surveys performs as a fundamental approach for organizations to extract insights crucial for decision-making, whether it involves discerning characteristics within specific population groups or broader categorical trends. However, the process of data collection often encounters numerous challenges, ranging from respondent biases to technical issues, leading to what is commonly known as the 'contamination' effect. This phenomenon introduces inaccuracies and biases into the dataset, thereby distorting the subsequent analysis and interpretation. Addressing these challenges, this note explores the principle of data cleaning, which is paramount in ensuring the reliability and validity of survey findings. The implementation procedure outlined herein includes a comprehensive recommendation aimed at elucidating and illustrating erroneous results that may arise during the analysis of survey data. By meticulously scrutinizing the data for anomalies, inconsistencies, and outliers, organizations can mitigate the impact of the contamination effect and enhance the accuracy of their decision-making processes. Moreover, the note introduces a methodology for segregating data into positive and negative factors, thereby facilitating a nuanced understanding of survey results. This approach enables the visual representation of data through graphs plotted on a two-dimensional axis, distinguishing between positive and negative influences. By visually depicting these factors, organizations can gain deeper insights into the underlying dynamics driving survey outcomes, thereby fostering more informed and strategic decision-making practices."

Keywords: data cleaning; dirty data; customer satisfaction

1. INTRODUCTION

Every day, we are inundated with a relentless barrage of polls, studies, statistics, opinions, and research findings. Entities ranging from businesses to media pundits to academic institutions strive to shape our understanding of reality by presenting information in various forms, often derived from data collected through surveys. Yet, while we may passively consume this deluge of data, few of us pause to question its utility. Many simply accept the conclusions put forth by analysts as unquestionable facts. However, the reality is far more nuanced. Consider a simple example: if a majority of respondents in a survey express a preference for rye bread over white bread, does this necessarily reflect the die-

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[*] Presented at the 19th Nordic Conference on Mathematical Statistics, June 9-13, 2002, Stockholm, Sweden and at the “Symposium i Advent Statistik,” January 23-26, 2006, København. Linkage term was used by Ylia Kempner (2008).

tary habits of the entire global population? Certainly not; reality is multifaceted, shaped by diverse choices, behaviors, and preferences. Thus, conclusions drawn from a typical survey, no matter how rigorously conducted, are inherently limited in their ability to capture the complexities of any given subject. To arrive at more accurate depictions of reality, thorough statistical investigations are imperative. When interpreting data gleaned from a sample of the population, consulting with a knowledgeable researcher or expert is essential. Their expertise enables a nuanced analysis that accounts for the intricacies of the subject matter. Additionally, careful consideration must be given to the relevance of the survey questions and the credibility of the respondents. Evaluating these factors helps ensure the reliability of the instrument and the validity of the conclusions drawn from it. Ultimately, a critical approach to data interpretation is indispensable in navigating the sea of information that surrounds us.

2. RELIABILITY

Understanding reliability poses a challenge when approached as a generic concept, often finding clarity within specific contexts. However, amidst this fervent pursuit of reliability, one must not overlook the pivotal role played by the method of analysis. It stands as the silent conductor orchestrating the symphony of outcomes, weaving together the threads of subjective reality perception into the fabric of conclusive understanding. Indeed, while the 'maximum principle' may cast its discerning eye upon the data, it is the methodological framework that ultimately guides the voyage to truth. Thus, in defiance of the empirical purists, we assert the undeniable sway of subjective perception in shaping the final estimation. For it is through this lens that we glean insights into the intricate tapestry of human experience, transcending the confines of statistical rigor to embrace the nuanced shades of reality.

Embracing the "maximum principle" not only aids researchers in their analytical pursuits but also streamlines investigations by sifting through unreliable responses, thereby eliminating interference or outliers – responses that diverge significantly from the norm or contradict the expected outcome. Yet, it's crucial to underscore that the chosen method of analysis remains pivotal in determining the success of the outcome. Despite the aforementioned argument, the final estimation should still be rooted in subjective perceptions of reality. What sets this approach apart from conventional statistical analysis is its ability to identify both unreliable respondents and their answers, offering a more comprehensive understanding of reality. By examining patterns aligned with responses from the remaining group members, a clearer picture of reality emerges. To illustrate this method, we'll use a simplified example of an ongoing survey lacking serious purpose or value, aiming to outline the foundational aspects of the approach.

Food, being not only a necessity but also a cultural cornerstone, holds a prominent place in public discourse, leading analysts to eagerly dissect associated data. In our playful or imaginative scenario, the aim is to unravel the intricate tapestry of people's culinary predilections. To embark on this flavorful journey, survey participants are tantalized with five menus, each a culinary symphony, and prompted to divulge their daily indulgences across the delectable array of food groups.

The options they are given are as follows:

1. *Dairy produce: cheese and milk*
2. *Cereals: bread, potatoes, rise and pasta*
3. *Vegetables: vegetables, fruit, etc.*
4. *Fish: shrimp, frozen/fresh fish*
5. *Meat products: various meats, sandwich spreads and sausages*

The results pertaining to seven study participants are presented in Table 1, which will suffice for the upcoming food preferences investigation.

Table 1.

	Dairy	Cereal	Vegetables	Fish	Meat	Total
Respond. no. 1		X	X			2
Respond. no. 2	X	X		X	X	4
Respond. no. 3			X	X		2
Respond. no. 4	X	X		X	X	4
Respond. no. 5			X	X		2
Respond. no. 6	X	X	X	X	X	5
Respond. no. 7		X	X			2
Total	3	5	5	5	3	21

Looking at the total score provided at the table's bottom, it appears that people are making healthy and nutritious food choices. The data suggests that "cereals," "vegetables," and "fish" are the most commonly consumed food groups, with five out of seven respondents reporting daily consumption of these items. However, drawing a conclusion about the overall healthiness of people's lifestyles based solely on this information would be premature.

Furthermore, assuming that 71% of the population eats cereals, fish, and vegetables every day could be misleading. It's essential to scrutinize the individual responses closely, particularly within this small sample group. For instance, respondents 1, 3, 5, and 7 have only selected two food groups from the provided list. Respondents 1 and 7 claim to consume only "cereals" and "vegetable" products daily, while respondents 3 and 5 stick to "vegetables" and "fish" exclusively.

Considering this list might not be exhaustive, it seems improbable that individuals would completely exclude other food groups from their diets. This discrepancy underscores the importance of verifying the accuracy of respondents' answers to ensure their inclusion in the analysis. Responses like those mentioned above are unreliable reflections of reality and should be treated with caution. Therefore, an experimental approach to discard unreliable respondents and their answers may yield a more credible result, offering a more accurate depiction of reality.

3. AGREEMENT LEVEL – TUNING PARAMETER

It's rare for individuals to rely solely on two food groups for their sustenance. Similarly, it's improbable that someone would restrict themselves to consuming only bread from the cereal category or exclusively shrimp from the fish category. Therefore, in refining our experiment, the goal is to pinpoint all respondents who have exclusively chosen these two categories. The aim, as previously emphasized, is to achieve a more accurate representation of reality.

Table 2 below showcases the outcomes of this data refinement process, which hinges on the selected "agreement level" or "tuning parameter." In this instance, the agreement level is set at 4, meaning none of the totals in the last column fall below this threshold. This approach ensures that respondents who have chosen only two menus with a significant degree of consistency are retained for further analysis. By fine-tuning the experiment in this manner, we strive to paint a clearer and more nuanced picture of dietary habits and preferences among the respondents.

Table 2.

	Dairy	Cereal	Vegetables	Fish	Meat	Total
Respond. no. 2	X	X		X	X	4
Respond. no. 4	X	X		X	X	4
Respond. no. 6	X	X	X	X	X	5
Total	3	3	1	3	3	13

This seems to be a very useful instrument for the experiment. However, the tuning parameter will only be relevant when its value exceeds one. If, for example, we try to set the agreement level (tuning) to 1 in Table 1, this would render ALL respondents reliable, even though menus “Dairy” and “Meat” are associated with the lowest frequency number, namely three. What can we conclude from the outcome of adopting tuning parameter = 1? The conclusion is exactly the same as that yielded by the original analysis — “people’s lifestyle is healthy.” In contrast, setting the tuning parameter to 2, 3 or a higher value allows us to explore patterns in answers that would not be otherwise apparent. Table 2 shows the distribution of respondents based on the tuning parameter = 4.

Why should we use this particular value as a tuning parameter? Yes, indeed, in the following analysis we intend to adopt the maximum principle as a method for selecting reliable respondents. This will be done through “agreement level”, see “*totals*” of columns, pertaining to a single respondent. The value of the tuning parameter is not fixed, and can be changed depending on the purpose of analysis, and is typically set at the level that reveals the most adequate picture of reality. Roughly speaking, we can compare the situation to rotating a tuner on TV or Radio, when we attempt to receive a clear picture/sound by trying to select the right frequency. The tuner value here is 4, and we assume that the selected respondents are now reliable.

4. MAXIMUM PRINCIPLE

Merely pinpointing the correct tuner position is insufficient, as our ensuing discussion will elucidate. For instance, upon closer examination, we discover that only one ostensibly reliable respondent opted for the "vegetable" menu. This revelation suggests that a mere 33% of the sample is engaging in daily vegetable consumption. While such a proportion might be expected within a small respondent pool, it's imperative to emphasize that this scenario is but a distilled version of a broader survey. In a more extensive study, such skewed results would indeed raise eyebrows. Hence, our fine-tuning endeavors must delve deeper, this time scrutinizing the menu content itself.

Firstly, let's consider removing "vegetables" from the available options to gauge its impact on our analysis. Subsequently, we embark on the next phase of our analysis, known as the "maximum principle" (Mullat, 1971a), drawing inspiration from an age-old merchant marketing analogy. Imagine a merchant seeking to strike a balance between maximizing demand for their wares and streamlining their inventory. Intuitively, they would cull the least-demanded commodity from their offerings, presuming it's identified from the purchasing habits of reliable clientele. In our context, the "vegetables" menu emerges as the least sought-after. Remarkably, its removal results in equal frequencies across the remaining menus.

However, caution must be exercised when pruning available options, ensuring it doesn't inadvertently purge reliable respondents. In certain scenarios, supplementing the sample with additional reliable respondents might be necessary, aligning with our tuning parameter once more, and so forth.

Next, in essence, the formulation of the maximum principle encapsulates a strategic approach: among all the reliable respondents, the initial step involves culling options with the lowest agreement levels, namely those exhibiting the lowest frequencies. Take, for instance, the "vegetables" menu in Table 2 of our example. By excising such options, the choice pool diminishes, yet the remaining responses with comparably lower frequencies gain heightened significance. The overarching objective is to pare down available options in a manner that ensures the remaining choices boast robust representation and yield more congruent responses.

Put succinctly, in menus where alignment is initially lacking, the removal of less matched options engenders a relative increase in alignment, a phenomenon unattainable if those options were retained. The aim, therefore, extends beyond merely segregating menus with higher matching responses; it also entails identifying a cohort of respondents for whom the menu with the lowest matching level achieves a comparatively elevated standing. This elucidates the crux of the maximum principle.

Essential to note is that the respondents integrated into the analysis must not only exhibit reliability but also yield answers that align closely with one another. Following this rationale, the decision to eliminate the "vegetables" menu stems from its divergent response pattern, which deviates from the overarching trend dictated by the maximum principle. It's imperative to underscore that this removal isn't contingent on qualitative assessments but rather guided purely by discernible patterns gleaned from answer matching.

In accordance with this argument, the menu “vegetables” is removed, since the responses associated with it are not aligned with the general answer pattern based on the maximum principle. Note that here, the removal is not based on any qualitative tests, but is rather guided purely by a pattern disclosed by matching the answers!

Table 3.

	Dairy	Grain	Fish	Meat	Total
Respond. no. 2	X	X	X	X	4
Respond. no. 4	X	X	X	X	4
Respond. no. 6	X	X	X	X	4
Total	3	3	3	3	12

5. CONCLUSION

The simplified survey scenario discussed above offers a profound revelation: the final outcome starkly contrasts with the initial analysis results. Simplifying matters, Table 1 initially suggests that people's food preferences align with health-conscious choices and current dietary recommendations. Conversely, Table 3 unveils a less rosy reality, indicating that food habits veer towards the less healthy spectrum. The implementation of our analysis principle, aimed at refining the pool of reliable respondents, has significantly altered the analytical landscape, reshaping the trajectory of our findings.

Naturally, one might question the credibility of the proposed principle compared to other analytical methods. While it's undeniable that subjective considerations and personal judgment have influenced the adopted analytical framework leading to the final outcomes, some may argue that this approach is flawed. They contend that relying solely on analyst or researcher intuition for tuning parameters, such as adjusting the "agreement level," lacks objectivity.

However, it's essential to recognize that personal considerations cannot be disregarded entirely. In certain instances, this approach aligns closely with common sense, where prevalent responses often mirror actual reality. This becomes particularly evident in straightforward surveys, such as those posing questions like "Will you vote for so and so in the upcoming election?"

The true value of this approach shines through in more complex surveys involving hundreds or even thousands of respondents and myriad questions. In such cases, the sheer diversity of responses forms intricate patterns that surpass unaided human comprehension. Here lies the strength of our method—it serves as a tool for pinpointing erroneous or misleading patterns through comprehensive data comparisons.

However, this does not diminish the role of analysts; rather, it underscores their responsibility in making informed judgments regarding data exclusion. The ultimate goal remains the identification and removal of all unreliable respondents with the aid of the tuning parameter. This "cleansing procedure" seeks to retain only the most reliable answers, in alignment with our maximum principle. Therefore, it's crucial to view the method presented here as a tool in the hands of the analyst, one that must be wielded skillfully to capture the clearest depiction of reality. The overarching aim is to mitigate the interference effect caused by unreliable respondents, thereby enhancing the accuracy and reliability of the analytical process.

APPENDIX

1. Practical recommendations

The preliminary explanation above is a general introduction to our maximum principle, the background of which is found in a much more complex methodology and theory.[1] First, it is beneficial to demonstrate how the results can be used and presented for the analyst, making the use of the notion of positive/negative profile.

When designing a questionnaire, it is widely accepted that the available responses associated with the individual questions should be presented in the "same direction," i.e., from positive to negative values/opinions or vice versa. Using a more rigorous terminology, such ordering would be denoted numerically and represented on a nominal/ordinal scale. This nomenclature is used primarily because, when implementing our method in the form of computer software, the analyst must separate the answers by grouping them together into positive/negative scale ends — the $(+/-)$ pools. The next step will be to create profile groups within each $(+)$ or $(-)$ pool range. A profile group of answers is created following their subject-oriented field of interest. For example, one might be interested in participants' lifestyle, nutritional practices, exercising, etc. Thus, these profiles, distinguished by their placement in $(+/-)$ pools, are also either positive or negative.

Once the analyst has created the $(+/-)$profiles, an automated process utilizing our maximum principle, which further organizes the data into what we call a series of profile components, conducts the subsequent analysis. Each profile component is a table, as above, located within particular profile limits. Clearly, a component is differentiated from the profile by the fact that, while a profile is a list of subject-specific questions and the corresponding options/answers composed by the analyst, the component is a table formed using the maximum principle. Therefore, the list of answers constituting a component (and the resulting set of table columns) is smaller, as only specific answers/columns from the full profile are included. Thus, once again the components will be separated into $(+/-)$ components $\mathrm{K}_1^{\pm}, \mathrm{K}_2^{\pm},...$, just as the profiles were separated into $(+/-)$ profiles. The $\mathrm{K}_1^{\pm}, \mathrm{K}_2^{\pm},...$ separation provides not only conceptual advantages, but also allows for more transparent illustration of the survey findings.

Analysis findings increase in value if they are presented in the format that can be easily comprehended. The simplest tool available for graphical presentation is a pie chart. Here, the pie can be divided into positive $\mathrm{K}_1^{+}, \mathrm{K}_2^{+},...$, and

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[1] Some theoretical aspects may be found in Appendix A.2

negative $K_1^-, K_2^-, \ldots$ components, represented in green and red color, respectively. However, to depict these components accurately, it is necessary to calculate some statistical parameters beforehand. For example, one can merge the $(+/-)$ components into single $(+/-)$ table and calculate the $(+/-)$ probabilities.[2] Hereby, statistical parameters based on the $(+/-)$ probabilities may be evaluated and illustrated by a pie chart divided into green and red area, effectively representing the $(+/-)$ elements.[3] There are many techniques and graphical tools at the analyst's disposal, and a creative analyst may proceed in this direction indefinitely. Still, it is plausible to wonder if the creation of the $(+/-)$ components is worthwhile. In other words, what is the advantage of using the "maximum principle" when interpreting the survey findings? The answer, see above, is that the blurred nature of the data may hinder clear interpretation of the reality underlying the data.

2. Some theoretical aspects

Suppose that respondents $N = \{1, \ldots, i, \ldots n\}$ participate in the survey. Let x, $x \in 2^N$, denote those who expressed their preferences towards certain questions $M = \{1, \ldots, j, \ldots, m\}$. We lose no generality in treating the list M as at a profile, whether negative or positive. Let a Boolean table $W = \left\| a_{i,j} \right\|_n^m$ reflect the survey results related to respondents' preferences, whereby $a_{i,j} = 1$ if respondent i prefers the answer j, $a_{i,j} = 0$ otherwise. In addition, all lists 2^M of answers $y \in 2^M$ within the profile M have been examined. Let an index $\delta_{i,j}^k = 0$, $i \in x, j \in y$ if $\sum_{j \in y} a_{i,j} < k$, otherwise $\delta_{i,j}^k = 1$, e.g., $\sum_{j \in y} a_{i,j} \geq k$, where k is our tuning parameter. We can calculate an indicator $F_k(H)$, using sub-table H formed by crossing entries of the rows x and columns y in the original table W. The number of 1-entries $\delta_{i,j}^k \cdot a_{i,j} = 1$ in each column within the range y determines the indicator $F_k(H)$ by further selection of a column with the minimum number $F_k(H)$ from the list y.

Identification of the component K seems to be a tautological issue, in the sense that following our maximum principle we have to solve the indicator maximization problem $K = \arg\max_{(x,y)} F_k(H)$. The task thus becomes an

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[2] Certainly, some estimates only.

[3] Please, find below a typical pie chart pertinent to what we just discussed. The positive and negative profiles relate to 21 questions highlighting people's behaviour, responses, opinions, etc., regarding their daily work and habits. Answers to these questions can be presented using an ordinal scale 1, 2, ..., 5, where 1, 2, 3 are at the negative, and 3, 4, 5 at the positive end of the scale.

NP-hard problem, the solution of which includes operations that grow exponentially in number. Fortunately, we claim that our $K^{\pm}$ components might be found by polynomial $O(m \cdot n \cdot \log_2 n)$ algorithm, as shown in the cited literature. Finally, we can restructure the entire procedure by extracting a component $K_1^{\pm}$ first, before removing it from the original table W and repeating the extraction procedure on the remaining content, thus obtaining components $K_2^{\pm}$, $K_3^{\pm}$,... etc. From now on, statistical parameters and other table characteristics, which empower $(+/-)$ share, arise from components $K_1^-, K_2^-, \ldots$ and $K_1^+, K_2^+, \ldots$ only, and are available to the analyst for illustration purposes, as depicted in the example below.

3. Illustration

In the example, we use a sampling highlighting 383 people's attitudes towards 21 phenomenal questions. Each question requires a response on an ordinal scale, with $1 < 2, \ldots, < 5$, where $1 < 2 < 3$ are positive values at the left end, and $3 < 4 < 5$ are negative values at the right end.[4] Hence, our sampling, depicted as a Boolean table, has 383×105 dimensions. As the tuning parameter $k = 5$ was chosen, we also extracted a set of three positive K_1^+, K_2^+, K_3^+ and negative K_1^-, K_2^-, K_3^- components. The actual values in the title and those shares illustrate our positive (green) and negative (red) (+/–) components.

Some typical sampling questions are given below:

1. Is your behavior slow/quick? – eating, talking, gesticulating,...
    - 1.1 Absolutely slow
    - 1.2 Somewhat slow
    - 1.3 Sometimes slow and sometimes quick
    - 1.4 Somewhat quick
    - 1.5 Absolutely quick
2. Are you a person who prefers deadlines/postpones duties?
    - 2.1 Absolutely always prefer deadlines
    - 2.2. Often prefer deadlines
    - 2.3. Sometimes prefer deadlines or sometimes postpone my duties
    - 2.4. Often postpone my duties
    - 2.5. Absolutely always postpone my duties

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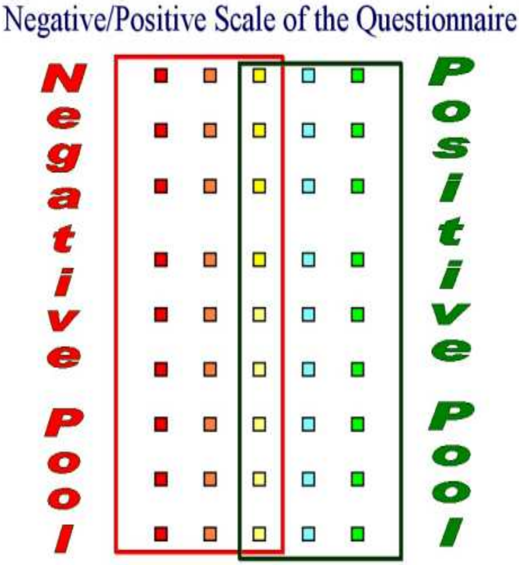
[4] Sampling owner (Scanlife Vitality ApS in Denmark) kindly provided us with a permission to use the data for analysis purposes. We are certainly very grateful for such help.



The figure shows more clearly the methodology of the positive/negative analysis of surveys data tables to identify hidden preferences of respondents. Whatever the analyst is doing to build a negative ordering of the left half of the questionnaire, our negative defining sequence is then compared with similar sequence of the right half of the questionnaire. As a result, two credential scales have been formed, which can then be visualized graphically in two-dimensional coordinate system on the plane.

At first glance that being said, our story may seem perhaps frivolous, but we say that it is much easier to suggest something new if the essence of the matter is presented in the form of an allegory, which can be interpreted in such a way as to reveal the hidden meaning of reality.

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A Fast Algorithm for Finding Matching Responses in Survey Data Table

Abstract. This article examines the intricacies of a greedy serialization algorithm designed to scrutinize survey data tables full of untrusted records. These unreliable records inherent in human phenomena are revealed when survey data tables are generated from various studies in reality, covering a variety of questionnaires completed by data analysts in countless branches of human activity. The algorithm presented in the paper exhibits almost linear complexity, and its efficiency depends exactly on the number of elements present in the table. Central to its effectiveness is the distinctive property of monotonicity, a defining feature that encapsulates a wide array of data typically collected in tabular formats, especially those that resemble Boolean tables. The algorithm implementation procedure includes a pragmatic recommendation designed to illuminate and decipher the subtleties of the analysis results, giving a tangible advantage to the interpretation process.

Keywords: survey; boolean; data table; matrix.

1. INTRODUCTION

Situations in which customer responses being studied are measured by means of survey data arise in the market investigations. They present problems for producing long-term forecasts because the traditional methods based on counting the matching responses in the survey with a large customer population are hampered by unreliable human nature in the answering and recording process. Analysis institutes are making considerable and expensive efforts to overcome this uncertainty by using different questioning techniques, including private interviews, special arrangements, logical tests, “random” data collection, questionnaire scheme preparatory spot tests, etc. However, percentages of responses representing the statistical parameters rely on misleading human nature and not on a normal distribution. It appears thereby impossible to exploit the most simple null hypothesis technique because the distributions of similar answers are unknown. The solution developed in this paper to overcome the hesitation effect of the respondent, and sometimes unwillingness, rests on the idea of searching so-called “agreement lists” of different questions. In the agreement list, a significant number of respondents do not hesitate in choosing the identical answer options, thereby expressing their willingness to answer. These respondents and the agreement lists are classified into some two-dimensional lists – "highly reliable blocks".

For survey analysts with different levels of research experience, or for the people mostly interested in receiving results by their methods, or merely for those who are familiar with only one, "the best survey analysis technique", our approach has some advantages. Indeed, in the survey, data are collected in such a way that can be regarded as respondents answering a series of questions. A specific answer is an option such as displeased, satisfied, well contented, etc. Suppose that all respondents participating in the survey have been interviewed using the same questionnaire scheme. The resulting survey data can then be arranged in a table $X = \langle x_{iq} \rangle$, where x_{iq} is a Boolean vector of options available, while the respondent i is answering the question q. In this respect, the primary table X is a collection of Boolean columns where each column in the collection is filled with Boolean elements from only one particular answer option. Our algorithm will always try to detect some highly reliable blocks in the Table X bringing together similar columns, where only some trustworthy respondents are answering identically. Detecting these blocks, we can separate the survey data. Then, we can reconstruct the data back from those blocks into the primary survey data table $X' = \langle x'_{iq} \rangle$ format, where some "non-matching/doubtful" answers are removed. Such a "data-switch" is not intended to replace the researchers’ own methods, but may be complementary used as a "preliminary data filter” – separator. The analysts’ conclusions will be more accurate after the data-switch has been done because each filtered data item is a representative for some "well known sub-tables".

Our algorithm in an ordinary form dates back to Mullat (1971). At first glance, the ordinary form seems similar to the greedy heuristic (Edmonds 1971), but this is not the case. The starting point for the ordinary version of the algorithm is the entire table from which the elements are removed. Instead, the greedy heuristic starts with the empty set, and the elements are added until some criterion for stopping is fulfilled. However, the algorithm developed in the present paper is quite different. The key to our paper is that the properties of the algorithm remain unchanged under the current construction. For matching responses in the Boolean table, it has a lower complexity.

The monotone property of the proposed technique – “monotone systems idea” – is a common basis for all theoretical results. It is exactly the same property (iii) of submodular functions brought up by Nemhauser et al (1978, p.269). Nevertheless, the similarity does not itself diminish the fact that we are studying an independent object, while the property (iii) of submodular set functions is necessary, but not sufficient.

From the very start, the theoretical apparatus called the "monotone system" has been devoted to the problem of finding some parts in a graph that are more "saturated" than any other part with "small" graphs of the same type (see Mullat, 1976). Later, a Markov chain replaced the graph presentation form where the rows-columns may be split implementing the proposed technique into some sequence of submatrices (see Mullat, 1979). There are numerous applications exploiting the monotone systems ideas; see Ojaveer et al (1975). Many authors have developed a thorough theoretical basis extending the original conception of the algorithm; see Libkin et al (1990) and Genkin and Muchnik (1993).

The rest of the paper is organized as follows. In Section 2, a reliability criterion will be defined for blocks in the Boolean table B. This criterion guarantees that the shape of the top set of our theoretical construction is a sub-matrix – a block; see the Proposition 1. However, the point of the whole monotone system idea is not limited by our specific criterion as described in Section 2. This idea addresses the question: How to synthesize an analysis model for data matrix using quite simple rules? In order to obtain a new analysis model, the researcher has only to find a family of π-functions suitable for the particular data. The shape of top sets for each particular choice of the family of π-functions might be different; see the note prior to our formal construction. For practical reasons, especially in order to help the process of interpretation of the analysis results, in Section 3 there are some recommendations on how to use the algorithm on the somewhat extended Boolean tables $B^{\pm}$. Section 4 is devoted to an exposition of the algorithm and its formal mathematical properties, which are not yet utilized widely by other authors.

2. RELIABILITY CRITERION

In this Section we deal with the criterion of reliability for blocks in the Boolean tables originating from the survey data. In our case we analyze the Boolean table $B = \langle b_{ij} \rangle$ representing all respondents $\langle 1,...,i,...,n \rangle$, but including only some columns $\langle 1,...,j,...m \rangle$ from the primary survey data table $X = \langle x_{iq} \rangle$; see above. The resulting data of each table B can be arranged in a $n \times m$ matrix. Those Boolean tables are then subjected to our algorithm separately, for which reason there is no difference between any sub-table in the primary survey data and a Boolean table. A typical example is respondent satisfaction with services offered, where $b_{ij} = 1$ if respondent i is satisfied with a particular service j level, and $b_{ij} = 0$ if he is unsatisfied. Thus, we analyze any Boolean table of the survey data independently.

Let us find a column j with the *most* significant frequency F of 1-elements among all columns and throughout all rows in table B. Such rows arrange a $g = 1$ *one*-column sub-table pointing out only those respondents who prefer *one* specific *most* significant column j. We will treat, however, a more general criterion. We suggest looking at some significant number of respondents where at least F of them are granting at least g Boolean 1 - elements in each single row within the range of a particular number of columns. Those columns arrange what we call an agreement list, $g = 2,3,...$; g is an agreement level.

The problem of how to find such a significant number of respondents, where the F criterion reaches its global maximum, is solved in Section 4. An optimum table S^*, which represents the outcome of the search among all "sub-sets" H in the Boolean table B, is the solution; see Theorem I. The main result of the Theorem I ensures that there are at least F positive responses in each column in table S^*. No superior sub-table can be found where the number of positive responses in each column is greater F. Beyond that, the agreement level is at least equal to $g = 2,3,...$ in each row belonging to the best sub-table S^*; g is the number of positive responses within the agreement list represented by columns in sub-table S^*. In case of an agreement level $g = 1$, our algorithm in Section 4 will find out only *one* column j with the *most* significant positive frequency F among all columns in table B and throughout all respondents, see above. Needless to say that it is worthless to apply our algorithm in that particular case $g = 1$, but the problem becomes fundamental as soon as $g = 2,3,....$

Let us look at the problem more closely. The typical attitude of the respondents towards the entire list of options — columns in table B — can be easily "accumulated" by the total number of respondent i positive hits selected:

$$r_i = \sum_{j=1,...,m} b_{ij} .$$

Similarly, each column – option can be measured by means of the entire Boolean table B as

$$c_j = \sum_{i=1,...,n} b_{ij} .$$

It might appear that it should be sufficient to choose the whole table B to solve our problem provided that $r_i \geq g, i = \overline{1,n}$. Nevertheless, let us look throughout the whole table and find the worse case where the number $c_j, j = \overline{1,m}$ reaches its minimum F. Strictly speaking, it does not mean that the whole table B is the best solution just because some "poor" columns (options with rare responses – hits) may be removed in order to raise the worst-case criterion F on the remaining columns. On the other hand, it is obvious that while removing "poor" columns, we are going to decrease some r_i numbers, and now it is not clear whether in each row there are at least $g = 2,3,...$ positive responses. Trying to proceed further and removing those "poor" rows, we must take into account that some of c_j numbers decrease and, consequently, the F criterion decreases as well. This leads to the problem of how to find the optimum sub-table S^*, where in the worst-case F criterion reaches its ***global maximum***? The solution is in Section 4.

Finally, we argue that the intuitively well-adapted model of 100% matching 1-blocks is ruled out by any approach trying to qualify the real structure of the survey data. It is well known that the survey data matrices arising from questionnaires are fairly empty. Those matrices contain plenty of small 100% matching 1-blocks, whose individual selection makes no sense. We believe that the local worst-case criterion F top set, found by the algorithm, is a reasonable compromise. Instead of 100% matching 1-blocks, we detect somewhat blocks less than 100% filled with 1-elements, but larger in size.

3. RECOMMENDATIONS

We consider the interpretation of the survey analysis results as an essential part of the research. This Section is designed to give guidance on how to make the interpretation process easier. In each survey data it is possible to conditionally select two different types of questions: (1) The answer option is a fact, event, happening, issue, etc.; (2) The answer is an opinion, namely displeased, satisfied, well contented etc.; see above. It does not appear from the answer to options of type 1, which of them is positive or negative, whereas type 2 allows us to separate them. The goal behind this splitting of type 2 opinions is to extract from the primary survey data table two Boolean sub-tables: table B^+, which includes type 1 options mixed with the positive options from type 2 questions, and table B^- where type 1 options are mixed together with the negative type 2 options – opinions. It should be noticed that doing it this way, we are replacing the analysis of primary survey data by two Boolean tables where each option is represented by one column. Tables B^+ and B^- are then subjected to the algorithm separately.

To initiate our procedure, we construct a sub-table K_1^+ implementing the algorithm on table B^+. Then, we replace sub-table K_1^+ in B^+ by zeros, constructing a restriction of table $B^{\pm}$. Next, we implement the algorithm on this restriction and find a sub-table K_2^+, after which the process of restrictions and sub-tables sought by the algorithm may be continued. For practical purposes we suggest stopping the extraction with three sub-tables: K_1^+, K_2^+ and K_3^+. We can use the same procedure on the table B^-, extracting sub-tables K_1^-, K_2^- and K_3^-.

The number of options-columns in the survey Boolean tables $B^{\pm}$ is quite significant. Even a simple questionnaire scheme might have hundreds of options – the total number of options in all questions. It is difficult, perhaps almost impossible, within a short time to observe those options among thousands of respondents. Unlike Boolean tables $B^{\pm}$, the sub-tables $K^{\pm}_{1,2,3}$ have reasonable dimensions. This leads to the following interpretation opportunity: the positive options in $K^{+}_{1,2,3}$ tables indicate some most successful phenomena in the research while the negative options in $K^{-}_{1,2,3}$ point in the opposite direction. Moreover, the positive and negative sub-tables $K^{\pm}_{1,2,3}$ enable the researcher in a short time to "catch" the "sense" in relations between the survey options of type 1 and positive/negative options of the type 2. For instance, to observe all Pearson's *r* correlations a calculator has to perform $O(n \cdot m^2)$ operations depending on the $n \times m$ table dimension, n-rows and m-columns. The reasonable dimensions of the sub-tables $K^{\pm}_{1,2,3}$ can reduce the amount of calculations drastically. Those sub-tables – blocks $K^{\pm}_{1,2,3}$, which we recommend to select in the next Section as index-function $F(H)$ top sets found via the algorithm, are not embedded and may not have intersections; see the Proposition 1. Concerning the interpretation, it is hoped that this simple approach can be of some use to researchers in elaborating their reports with regard to the analysis of results.

4. DEFINITIONS AND FORMAL MATHEMATICAL PROPERTIES

In this Section, our basic approach is formalized to deal with the analysis of the Boolean $n \times m$ table B, n-rows and m-columns. Henceforth, the table B will be the Boolean table $B^{\pm}$ – see above – representing certain options-columns in the survey data table. Let us consider the problem of how to find a sub-table consisting of a subset S_{max} of the rows and columns in the original table B with the properties: (1) that $r_i = \sum_j b_{ij} \geq g$ and (2) the minimum over j of $c_j = \sum_i b_{ij}$ is as large as possible, precisely – the global maximum. The following algorithm solves the problem.

Algorithm.

```
Step I.   Set up the initial values.
  1i.     Set minimum and maximum bounds $a$, $b$ on threshold
          $u$ for $c_j$ values.
Step A.   To find that the next step B produces a non-empty
          sub-table.
  1a.     Using step B, test $u$ as $(a+b)/2$.
          If it succeeds, replace $a$ by $u$. If it fails replace $b$
          by $u$.
  2a.     Go to 1a.
```

Step B. To test whether the minimum over j can be at least u .

1b. Delete all rows whose sums $r_i < g$.
This step **B** fails if all must be deleted; return to step **A**.

2b. Delete all columns whose sums $c_j \leq u$.
This step **B** fails if all must be deleted, return to step **A**.

3b. Perform step **T** if none deleted in **1b** and **2b**;
otherwise go to **1b**.

Step T. Test that the global maximum is found.

1t. Among numbers c_j find the minimum.
With this new value as u test performing step **B**.
If it succeeds, return to step **A**, otherwise final stop.

Step **B** performed through the step **T** tests correctly whether a sub-matrix of B can have the rows sums at least g and the column sums at least u . Removing row i , we need to perform no more than m operations to recalculate c_j values; removing column j, we need no more than n -operations. We can proceed through **1b** no more than n -times and through **2b**, m -times. Thus, the total number of operations in step **B** is $O(n \cdot m)$. The step **A** tests the step **B** no more than $\log_2 n$ times. Thus, the total complexity of the algorithm is $O(\log_2 n \times nm)$ operations.

Note. It is important to keep in mind that the algorithm itself is a particular case of our theoretical construction. As one can see, we are deleting rows and columns including their elements all together, thereby ensuring that the outcome from the algorithm is a sub-matrix. But, in order to expose the properties of the algorithm, we look at the Boolean elements separately. However, in our particular case of π -functions it makes no difference. The difference will be evident if we utilize some other family of π -functions, for instance $\pi = c_j \max(r_i, c_j)$. We may detect top binary relations, which we call kernels, different from submatrices. It may happen that some kernel includes two blocks – one quite long in the vertical direction and the other – in the horizontal. All elements in the empty area between these blocks in some cases cannot be added to the kernel. In general, we cannot guarantee either the above low complexity of the algorithm for all families of π -functions, but the complexity still remains in reasonable limits.

We now consider the properties of the algorithm in a rigorous mathematical form. Below we use the notation $H \subseteq B$. The notation H contained in B will be understood in an ordinary set-theoretical vocabulary, where the Boolean table B is a set of its Boolean 1 -elements. All 0 -elements will be dismissed from the consideration. Thus, H, as a binary relation, is also a subset of a binary relation B. However, we shall soon see that the top binary relations – kernels from the theoretical point of view are also sub-matrices for our specific choice of π-functions. Below, we refer to an element we assume that it is a Boolean 1-element.

For an element $\alpha \in B$ in the row i and column j we use the similarity index $\pi = c_j$ if $r_i \geq g$ and $\pi = 0$ if $r_i < g$, counting only on Boolean elements belonging to H. The value of π depends on each subset $H \subseteq B$ and we may thereby write $\pi \equiv \pi(\alpha, H)$: the set H is called the π-function parameter. The π-function values are the real numbers – the similarity indices. In Section 2 we have already introduced these indices on the entire table B. Similarity indices, as one can see, may only concurrently increase with the "expansion" and decrease with the "shrinking" of the parameter H. This leads us to the fundamental definition.

Definition 1. Basic monotone property. *By a monotone system will be understood a family* $\{\pi(\alpha, H) : H \subseteq B\}$ *of* π*-functions, such that the set* H *is to be considered as a parameter with the following monotone property: for any two subsets* $L \subset G$ *representing two particular values of the parameter* H *the inequality* $\pi(\alpha, L) \leq \pi(\alpha, G)$ *holds for all elements* $\alpha \in B$.

We note that this definition indicates exactly that the fulfilment of the inequality is required for all elements $\alpha \in B$. However, in order to prove the Theorems 1,2 and the Proposition 1, it is sufficient to demand the inequality fulfillment only for elements $\alpha \in L$; even the numbers π themselves may not be defined for $\alpha \notin L$. On the other hand, the fulfillment of the inequality is necessary to prove the argument of the Theorem 3 and the Proposition 2. It is obvious that similarity indices $\pi = c_j$ comply with the monotone system requirements.

Definition 2. *Let* $V(H)$ *for a non-empty subset* $H \subseteq B$ *by means of a given arbitrary threshold* u° *be the subset* $V(H) = \{\alpha \in B : \pi(\alpha, H) \geq u^\circ\}$. *The non-empty* H*-set indicated by* S° *is called a stable point with reference to the threshold* u° *if* $S^\circ = V(S^\circ)$ *and there exists an element* $\xi \in S^\circ$*, where* $\pi(\xi, S^\circ) = u^\circ$. See Mullat (1981, p.991) for a similar concept.

Definition 3. *By monotone system kernel will be understood a stable set* S^* *with the maximum possible threshold value* $u^* = u_{\max}$.

We will prove later that the very last pass through the step **T** detects the largest kernel $\Gamma_p = S^*$. Below we are using the set function notation $F(X) = \min_{\alpha\in X}\pi(\alpha, X)$.

Definition 4. *An ordered sequence* $\alpha_0, \alpha_1, ..., \alpha_{d-1}$ *of distinct elements in the table B, which exhausts the whole table,* $d = \sum_{i,j} b_{ij}$, *is called a defining sequence if there exists a sequence of sets* $\Gamma_0 \supset \Gamma_1 \supset \mathbf{...} \supset \Gamma_p$ *such that:*

A. *Let the set* $H_k = \{\alpha_k, \alpha_{k+1}, ..., \alpha_{d-1}\}$. *The value* $\pi(\alpha_k, H_k)$ *of an arbitrary element* $\alpha_k \in \Gamma_j$, *but* $\alpha_k \notin \Gamma_{j+1}$ *is strictly less than* $F(\Gamma_{j+1})$, $j = 0,1,...,p-1$.

B. *In the set* Γ_p *there does not exist a proper subset* L, *which satisfies the strict inequality* $F(\Gamma_p) < F(L)$.

Definition 5. *A subset* D^* *of the set* B *is called definable if there exists a defining sequence* $\alpha_0, \alpha_1, ..., \alpha_{d-1}$ *such that* $\Gamma_p = D^*$.

Theorem 1. *For the subset* S^* *of* B *to be the largest kernel of the monotone system – to contain all other kernels – it is necessary and sufficient that this set is definable:* $S^* = D^*$. *The definable set* D^* *is unique.*

We note that the Theorem 3 will establish the existence of the largest kernel later.

Proof.

Necessity. If the set S^* is the largest kernel, let's look at the following sequence $B = \Gamma_0 \supset \Gamma_1 = S^*$ of only two sets. Suppose we have found elements $\alpha_0, \alpha_1, ...\alpha_k$ in $B \setminus S^*$ such that for each $i = \overline{1,k}$ the value $\pi(\alpha_i, B \setminus \{\alpha_0, ..., \alpha_{i-1}\})$ is less than $u^o = u_{\max}$ and $\alpha_0, \alpha_1, ...\alpha_k$ does not exhaust $B \setminus S^*$. Then, in $(B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}$ some α_{k+1} exists such that $\pi(\alpha_{k+1}, (B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}) < u^*$. Otherwise, the set $(B \setminus S^*) \setminus \{\alpha_0, ..., \alpha_k\}$ is a larger kernel than with the same value u^*. Thus, the induction is complete.

This gives the ordering with the property (a). If the property (b) failed, then u^* would not be a maximum, contradicting the definition of the kernel. This proves the necessity.

Sufficiency. Note that every time the algorithm — see above — goes through step **T**, some stable point, a set $S°$ is put in the form of a set $\Gamma_j = S°$, $j = 0,1,...,p-1$, where $u = u_j = \min_{\alpha\in S^o} \pi(\alpha, S^o)$. Obviously, these stable “layering" points (stable sets) form an embedded chain of sets $B=\Gamma_0 \supset \Gamma_1 \supset ... \supset \Gamma_p = D^*$. Let the set $L \subseteq B$ be the largest core. Suppose that this L is a proper subset of D^*, then by property (b) $F(D^*) \geq F(L)$ and hence D^* is also a kernel. The set L as the largest kernel cannot be a proper subset of D^* and therefore must be equal to D^*.

Suppose now that L is not the subset of D^*. Let H_s be the smallest set $H_k=\{\alpha_k, \alpha_{k+1},..., \alpha_{d-1}\}$, which includes L. The value $\pi(\alpha_s, H_s)$ by our basic monotone property must be grater than, or at least equal to u^*, since α_s is an element of H_s and it is also an element of the kernel L and $L \subseteq H_s$. By property (a) this value is strictly less than $F(\Gamma_{j+1})$ for some $j = 0,1,..., p-1$. But that contradicts the maximality of u^*. This proves the sufficiency. Moreover, it proves that any largest kernel equals D^* so that it is the unique largest kernel. This concludes the proof. ■

Proposition 1. *The largest kernel is a sub-matrix of the table* B.

Proof. Let S^* be the largest kernel. If we add to S^* any element lying in a row and a column where S^* has existing elements, then the threshold value u^* cannot decrease. So by maximality of the set S^* this element must already be in S^*. ■

Now, we need to focus on the individual properties of the sets $\Gamma_0 \supset \Gamma_1 \supset ... \supset \Gamma_p$, which have a close relation to the case $u < u_{\max}$ – a subject for a separate inquiry. Let us look at the step **T** of the algorithm originating the series of mapping initiating from the whole table B in form of $V(B), V(V(B),...$ with some particular threshold u. We denote $V(V(B))$ by $V^2(B)$, etc.

Definition 6. *The chain of sets* $B, V(B), V^2(B),...$ *with some particular threshold u is called the central series of monotone system; see Mullat (1981) for exactly the same notion.*

Theorem 2. *Each set* $\Gamma_0 \supset \Gamma_1 \supset ... \supset \Gamma_p$ *in the defining sequence* $\alpha_0, \alpha_1, ..., \alpha_{d-1}$ *is the central series convergence point* $\lim_{k=2,3,...} V^k(B)$ *as well as the stable point for some particular thresholds values* $F(W) = u_0 < u_1 < ... < u_n = F(S^*)$. *Each* Γ_j *is the largest stable point – including all others for threshold values* $u \geq u_j = F(\Gamma_j)$.

It is not our intention to prove the statement of Theorem 2 since this proof is similar to that of Theorem 1. Theorem 1 is a particular case for Theorem 2 statement regarding threshold value $u = u_p$.

Next, let us look at the formal properties of all kernels and not only the largest one found by the algorithm. It can easily be proved that with respect to the threshold $u_{max} = u_p$ the subsystem of all kernels classifies a structure, which is known as an upper semilattice in lattice theory.

Theorem 3. *The set of all kernels – stable points – for* u_{max} *is a full semi-lattice.*

Proof. Let Ω be a set of kernels and let $K_1 \in \Omega$ and $K_2 \in \Omega$. Since the inequalities $\pi(\alpha, K_1) \geq u$, $\pi(\alpha, K_2) \geq u$ are true for all K_1 and K_2 elements on each K_1, K_2 separately, they are also true for the union set $K_1 \cup K_2$ due to the basic monotone property. Moreover, since $u = u_{max}$, we can always find an element $\xi \in K_1 \cup K_2$ where $\pi(\xi, K_1 \cup K_2) = u$. Otherwise, the set $K_1 \cup K_2$ is some H-set for some u' greater than u_{max}. Now, let us look at the sequence of sets $V^k(K_1 \cup K_2)$, $k = 2,3,...$, which certainly converges to some non empty set – stable point K. If there exists any other kernel $K' \supset K_1 \cup K_2$, it is obvious, that applying the basic monotone property we get that $K' \supseteq K$. ■

With reference to the highest-ranking possible threshold value $u_p = u_{max}$, the statement of Theorem 3 guarantees the existence of the largest stable point and the largest kernel S^* (compare this with equivalent statement of Theorem 1).

Proposition 2. *Monotone system Kernels are sub-tables of the table* B.

Proof. The proof is similar to proposition 1. However, we intend to repeat it. In the monotone system all elements outside a particular kernel lying in a row and a column where the kernel has existing elements belong to the kernel. Otherwise, the kernel is not a stable point because these elements may be added to it without decreasing the threshold value u_{max}.

Note that Propositions 1,2 are valid for our specific choice of similarity indices $\pi = c_j$. The point of interest might be to verify what π-function properties guarantee that the shape of the kernels still is a sub-matrix. The defining sequence of table B elements constructed by the algorithm represents only some part $u_0 < u_1 < u_2 < ... < u_p$ of the threshold values existing for central series in the monotone system. On the other hand, the original algorithm, Mullat (1971), similar to the inverse Greedy Heuristic, produces the entire set of all possible threshold values u for all possible central series, what is sometimes unnecessary from a practical point of view. Therefore, the original algorithm always has the higher complexity.

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ЭКСТРЕМАЛЬНЫЕ ПОДСИСТЕМЫ МОНОТОННЫХ СИСТЕМ. I

И. Э. МУЛЛАТ

(Таллин)

Рассматривается общая теоретическая модель, предназначенная для начального этапа анализа систем взаимосвязанных элементов. В рамках модели и исходя из специально постулированного свойства монотонности систем гарантируется существование особых подсистем – ядер. Устанавливается ряд экстремальных свойств и структура ядер в монотонных системах. Детализируется язык описания монотонных систем взаимосвязанных элементов на общем теоретико-множественном уровне, и на его основе вырабатывается конструктивная система понятий в случае систем с конечным числом элементов. Изучается ряд свойств особых конечных последовательностей элементов системы, с помощью которых осуществимо выделение ядер в монотонных системах.

1. Введение

При изучении поведения сложной системы часто приходится сталкиваться с задачей анализа конкретных числовых данных о функцио-нировании системы. На основе подобных данных иногда требуется выяснить, существуют ли в системе особые элементы или подсистемы элементов, реагирующих однотипно на какие-либо «воздействия», а также «отношения» между однотипными подсистемами. Сведения о существовании указанных особенностей или о «структуре» изучаемой системы необходимы, например, до проведения обширных или дорогостоящих статистических исследований.

В связи с широким применением вычислительной техники в настоящее время на начальном этапе выявления структуры системы намечается подход, основанный на различного рода эвристических моделях [1-4]. При построении моделей многие авторы исходят из содержательных постановок задач, а также из формы представления исходной информации [5, 6].

Естественной формой представления информации для целей изучения сложных систем является форма графа [7]. Распространенным носителем информации служит также матрица, например матрица данных [8]. Матрицы и графы легко допускают выделение двух минимальных структурных единиц системы: «элементов» и «связей» между элементами*. В данной работе понятия «связь» и «элемент» трактуются достаточно широко. Так, инргда желательно рассматривать связи в виде элементов системы; в этом случае можно обнаружить более «тонкие» зависимости в исходной системе. Представление системы в виде единого объекта – элементы и связи между элементами – позволяет придать более четкий смысл задаче выявления структуры системы. Структура системы – это такая организация элементов системы в подсистемы, которая складывается в виде множества отношений между подсистемами. Структурой системы, например, может быть естественно сложившийся способ объединения подсистем в единую систему, который определяется на основе «сильных» и «слабых» связей между элементами системы. Подобный подход к анализу систем описан, например, в [9], где рассматривается вопрос агрегирования систем взаимосвязанных элементов. Агрегирование оказывается удобным макроязыком для вскрытия структуры системы.

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* В литературе подобные системы называются системами взаимосвязанных элементов.

ЭКСТРЕМАЛЬНЫЕ ПОДСИСТЕМЫ МОНОТОННЫХ СИСТЕМ. II

И. Э. МУЛЛАТ

(Таллин)

Предлагается конструктивная процедура построения особых определяющих последовательностей элементов монотонных систем, рассмотренных в [1]. Изучаются взаимные свойства двух определяющих последовательностей $\overline{\alpha}_-$ и $\overline{\alpha}_+$, и полученный результат формулируется в виде теоремы двойственности. На основе теоремы двойственности описан способ сужения области поиска экстремальных подсистем – ядер монотонной системы и приведена соответствующая схема поиска.

1. Введение

В [1] разработан основной аппарат выделения в монотонных системах особых подсистем – ядер, обладающих экстремальными свойствами. Основным понятием развитого аппарата является определимое множество [2]. В принятой терминологии определимое множество оказыва-ется наибольшим ядром монотонной системы взаимосвязанных элемен-тов. Понятие определимого множества в [1] вводилось с помощью пред-положения существовании особых подпоследовательностей элементов изучаемой системы, названных опреде-яющими ($\overline{\alpha}_-$ и $\overline{\alpha}_+$) – последовательностями.

В данной работе вопрос существования определяющих последоательностей решается конструктивно в виде процедур – алгоритмов. Основ-ные свойства определяющей последовательности, построенной по прави-лам процедуры и исчерпывающей все множество элементов системы **W,** гарантируется теоремой.

Рассматривается также вопрос о том, какая существует связь между определяющими последовательностями $\overline{\alpha}_-$ и $\overline{\alpha}_+$. Можно предположить, что если построена определяющая последовательность $\overline{\alpha}_-$, то стоит взять эту последовательность в обратном порядке, как получится $\overline{\alpha}_+$ последовательность. В общем случае это не так. Тем не менее имеет место более слабое утверждение. На основе определенных в [1] понятий дискретных действий типа ⊕ и ⊖ и на элементы системы **W** данное утверждение формулируется в виде теоремы двойственности. В случае выполнения условий теоремы двойственности изложенные алгоритмы построения определяющих последовательностей используются для значительного сужения области поиска ⊕ и ⊖ ядер системы **W** повышая тем самым эффективнось алгоритма. Алгоритм сужения области поиска изложен в виде процедуры – конструктивно.

Extremal Subsystems of Monotonic Systems [i]

Abstract. In the exploration of complex systems, a pivotal aspect involves analyzing specific numerical data to comprehend the system's functioning. This effort often extends to identifying specialized elements or subsystems within the system, discerned by their consistent response to defined 'actions' and intricate 'relations' among homogeneous subsystems. Understanding these nuanced characteristics through rigorous mathematical analysis, elucidating the underlying structure of the system, is crucial, particularly as a foundation for conducting complex or resource-intensive statistical studies. The research explores this basic methodology to identify single-peak sequences that define components of what we call "monotonic systems," where peaks represent "kernels" and "hikes" are depicted as ”stable sets.” Furthermore, we extensively delve into an additional constructive methodology involving two defining sequences within monotonic systems. Through meticulous exploration, we uncover the complex relationship between these defining sequences, ultimately leading to the formulation of the duality theorem. This theorem not only serves as a cornerstone in our understanding but also provides a systematic approach for limiting the search area for kernels and stable sets. In light of this, we present an algorithm designed specifically for the identification of extremal subsystems, namely kernels and stable subsets, within a monotonic system, encapsulated by a certain dual scheme.

Keywords: monotonic; system; matrix; graph; cluster

1. INTRODUCTION

For the study of a complex system, it is often necessary to encounter the problem of analyzing numerical case data about the system functioning. Sometimes based on similar data it is required to show whether in the system there exist special elements or subsystems, reacting in one way to some “actions” as well as “relations” between one-type subsystems. Information on the existence of the indicated peculiarities or on the “structure” of the system under study is necessary, for example, before carrying out extensive or expensive statistical investigation.

Concerning wide application of computational techniques, at the present time, to initial detection of the structure of a system an approach based on various kind of heuristic models is planned (Braverman et al, 1974; McCormik, 1972; Deutch, 1971; Zahn, 1971). For constructing models, many authors start with intuitive formulations of the problem and also with the form of presentation of the initial data (Võhandu, 1964; Терентьев, 1959).

A natural form of presentation the data for the purpose of studying complex systems is that of a graph (Muchnik, 1974). A matrix, for example, a data matrix (Hartigan, 1972) also serves as a widely spread carrier of information. Matrices and graphs easily admit isolation of two minimal structural units of the system: ”elements” and ”connections” between elements.[1] In this paper the notions ”connections” and ”elements” are interrelated in a sufficiently broad fashion. Thus, sometimes it is desirable to consider connections in the form of elements of a system; in this case, it is possible to find more ”subtle” relations in the original system.

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[1] Analogous systems are called systems of interrelated elements in the literature.

Representation of the system in the form of a unique object, comprising elements and connections between them, enables a more precise understanding of the system's structure. This structure entails the organization of system elements into subsystems, delineated by a network of relationships between them. Such a structure may manifest as a natural amalgamation of subsystems into a cohesive whole, delineated by the strength or weakness of interconnections among its elements. This approach finds resonance in the work of Braverman et al. (1971), where the assembly of systems from interconnected elements is expounded upon, revealing assembly as a convenient macro language for expressing system structure.

In system theory, conventional analysis often focuses on direct connections between elements. However, certain scenarios necessitate the consideration of indirect connections as well. These indirect connections are deemed dynamic relations, wherein the degree of interdependence is dictated by the subsystem in which each connection is assessed. Below, we delve into a particular subclass of such dynamic systems what we called as "monotonic systems."

The foundational property of monotonicity within these systems facilitates the delineation of a system's kernel. This kernel, as initially or primarily indicated, serves as a reflection of the overarching structure of the entire system. Operating within the intrinsic framework of the system, a kernel constitutes a subsystem highly responsive to either positive or negative actions, thus delineating the existence of both positive and negative kernels.

The existence of kernels, which are specialized subsystems, is not left to chance within the mathematical model expounded in this paper; rather, it is a guarantee embedded within the very fabric of the model. The quest to "isolate" these kernels represents a quintessential challenge in the articulation of a "large" system in the parlance of a "small" system – the kernel. In a figurative sense, a kernel of a system embodies a subsystem whose removal invokes profound and irrevocable alterations in the system's properties; it's akin to the system relinquishing its established structure, akin to shedding its skin.

In elucidating the subject matter, the discourse relies on the terminology and symbolism of set theory, a domain accessible to all without necessitating specialized knowledge. However, it warrants attention to the introduction of specific notation, as the framework developed within this paper introduces novel concepts and methodologies. This new apparatus serves as the cornerstone for the exploration and analysis of complex systems, offering insights into their underlying structure and behavior.

2. EXAMPLES OF MONOTONIC SYSTEMS

In the present paper a monotonic system is defined, to be a system over whose elements one can perform ″positive and ″negative″ actions. In addition, positive actions increase certain quantitative indicators of the functioning of a system while the negative actions decrease those indicators. In the examples considered above the positive action is the addition of an element to a subsystem while the negative action is removing an element from the subsystem; in the third example the converse holds.

In examples, the kernel should possess an intuitive significance. For instance, in citation graphs, a negative kernel would represent publications extensively citing each other, typically authored by individuals from the same scientific school. Conversely, a positive kernel would comprise publications with fewer reciprocal citations, indicating representation from diverse scientific schools.

In transport road networks, the intuitive essence of a kernel should be evident in the following manner. If we consider the elements of a communication network as the transportation routes, then a negative kernel would encompass a set of routes that, on average, experience a significant number of traffic congestions–a sort of consensus among these routes. Conversely, a positive kernel would represent a collection of routes that, on average, encounter fewer traffic congestions, indicating smoother traffic flow.

Alternatively, when the system elements are viewed as the transportation points within the network, a negative kernel would denote a landscape characterized by mutually unreliable points. These points would exhibit a lack of dependability in facilitating transportation connections with one another. On the other hand, a positive kernel would depict a landscape comprising more dependable points, where transportation connections are more reliable and consistent.

I. Examining the complex organization behind the apparently random friend lists found on platforms like Facebook, Linkedin and other social networking media reveals a carefully structured system. Upon closer inspection, it becomes clear that these lists are not random, but rather follow a clearly defined pattern. Each user's friend's list serves as a vital indicator, not only checking connections, but also offering information about mutual acquaintances and potential interests. This gives users the opportunity to make direct connections with new people, seamlessly integrating them into existing social circles.

This process is not simply about expanding one's social circle, but represents a purposeful desire to expand one's social sphere. It is noteworthy that any exclusion of a user from the friends list causes a decrease in the overall score, which means a negative action in the network lexicon. Conversely, adding new connections leads to an increase in the indicator, which means positive interaction with the platform.

These contrasting actions, both negative and positive, are the essence of the formal scheme discussed in this article. By diving deeper into the dynamics of friend lists and related metrics, we gain invaluable insight into the fundamental principles governing social interactions in digital spheres.

In practice, research into social network structures may be conducted incognito, since the identities of the participants and their specific interactions are often irrelevant. Instead of tagging users by name, a simple numbering system is enough to allow chains of actions–both positive and negative–to be built within the network. This approach contributes to a deeper understanding of the complex dynamics of internal relations, allowing researchers to explore different mutual reflections and combinations of interactions when analyzing network structure.

II. This excerpt elaborates on enhancing the efficiency of cellular networks through spatial signal processing and adaptive antennas. It underscores the intricate interplay among antenna arrays, processing algorithms, and resource allocation for maximizing data throughput. By focusing on specific parametric classes of antenna systems, optimization becomes more feasible, allowing for the estimation of benefits from adaptive antennas. The example replicates Shorin et al.'s 2016 study for antennas distribution. The study also introduces a novel algorithm facilitating Monotonic Systems efficient grouping of antennas based on angular diversity, ensuring optimal resource utilization.

The introduction of spatial signal processing technology and adaptive antennas makes it possible to significantly (manifold) increase the throughput of the radio channel due to the active use of the resource associated with the capabilities of spatial signal selection.

In the context of cellular networks, optimizing adaptive spatial processing entails a shift from traditional approaches to achieving maximum throughput for a radio channel connecting numerous spatially dispersed subscribers with a serving base station. This shift emphasizes the interdependence of the antenna array, spatial processing algorithm, radio channel resource distribution algorithm, and data exchange algorithms, forming a unified hardware and software module dedicated to solving the transmission problem. While the optimal design of antenna arrays and algorithms remains a question, practical simplifications can be made by constraining antenna systems to specific parametric classes, such as ring homogeneous structures with adjustable placement radii and radiation pattern widths.

The following approach facilitates optimization and allows estimation of the benefits derived from using adaptive antennas, often through simulation. Furthermore, the proposed algorithm in this article introduces a "mode with reverse extraction of elements from groups," enabling the creation of minimal clusters with desired angular diversity levels. Additionally, this mode facilitates the distribution of subscribers in favorable locations across multiple groups, maximizing the utilization of available radio channel resources.

In the particular scenario of the “Monotone System” being addressed, the algorithm outlined in this article offers a precise solution. This algorithm introduces a "mode with reverse extraction of elements from groups," which serves a dual purpose. Firstly, it enables the creation of the fewest possible groups or clusters while maintaining a specified level of angular diversity. Secondly, it facilitates the simultaneous allocation of individual subscribers situated in more favorable locations across multiple groups. This approach ensures optimal utilization of the available resources within the radio channel, maximizing efficiency and performance.

III. Let's consider a scenario where there exists a network of transportation exchanges or nodes, denoted as landscape **W**, interconnected by two-sided roads. In the absence of direct transportation between these nodes within this road system, transit transportation can be organized. Over a long period of observation, if such a pattern of operation persists regardless of the presence of direct transport links, it is possible to assess the efficiency of transportation by measuring the average frequency of traffic jams when establishing transportation between these nodes within a standard unit of time. Essentially, to characterize the reliability of establishing transportation between each node in a system W, one can use the average number of traffic congestions experienced by connecting to at least one destination node in the system over a given period of time. It is obvious that these quantitative indicators, namely the feasibility of transportation over a given period of time and the characteristics of the guarantees provided, are applicable only within each subsystem of the road network **W**.

The proposed model exhibits several inherent characteristics. Any interruption in the flow of transportation along a two-sided route amplifies the average number of traffic congestions among all other transportation points, while the introduction of a new route conversely diminishes this average. This dynamics correlates with an increase or decrease in the load on facilitating transit transportation within the transport communications network.

Similarly, when activity is scaled back at any transportation point within a given subsystem, the unreliability of all points within that subsystem escalates. Conversely, the addition of a new transportation point to the subsystem reduces this unreliability. These observations mirror the behavior of monotonic systems discussed earlier, affirming that the model governing transportation roads adheres to the principles of a monotonic system.

IV. In the exploration of academic research, various scientific disciplines utilize graphs of cited publications, as outlined by Налимов and Мульченко in 1969. These graphs are directed and a-cyclic, reflecting the nature of scholarly citations where authors can only cite papers that have already been published. It is reasonable to conceptualize the set of publications, denoted as **W**, as a system where information exchange occurs through citations.

Within this framework, considering a subset of publications from the entire set **W** allows us to characterize each publication based on the number of bibliographical references within that subset. When a publication is removed from the subset, this quantitative measure of information exchange within the subset diminishes. Conversely, adding a publication to the subset enhances this evaluation for all publications within the subset. Hence, the citation system represented by these graphs exhibits monotonic behavior. In a related context, Trybulets (1970) highlights an intriguing example where a directed graph inadvertently illustrates the concept of a monotonic system

V. In the n -dimensional vector space let there be given **N** vectors. For each pair of vectors x and y one can define in many ways a distance $\rho(\mathrm{x}, \mathrm{y})$ between these vectors (i.e., to scale the space). Let us assume that the set of given vectors forms an unknown system **W**. For every vector in an arbitrary subsystem of **W** we calculate the sum of distances to all vectors situated inside the selected subsystem. Thus, with the respect to each subsystem of **W** and each vector situated inside that subsystem, a characteristic sum of distances is defined, which can be different for different subsystems. It is not difficult to establish the following property of the set of sums of distances. Because of removing a vector from the subsystem the sums computed for the remaining vectors decrease while because of adding a vector to the subsystem they increase. A similar property of sums for every subsystem of system **W** is called in this paper the monotonicity property and a system **W** having such a property is called a monotonic system.

3. DESCRIPTION OF A MONOTONIC SYSTEM

One considers some system W consisting of a finite number of elements,[3] i.e., $|W| = N$, where each element α of the system W plays a well-defined role. It is supposed that the states of elements α of W are described by definite numerical quantities characterizing the "significance" level of elements α for the operation of the system as a whole and that from each element of the system one can construct some discrete actions.

We reflect the intrinsic dependence of system elements on the significance levels of individual elements. The intrinsic dependence of elements can be regarded in a natural way as the change, introducible in the significance levels of elements β, rendered by a discrete action produced upon element α.

We assume that the significance level of the same element varies as a result of this action. If the elements in a system are not related with each other in any way, then it is natural to suppose that the change introduced by element α on significance β (or the influence of α on β) equals zero.

We isolate a class of systems, for which global variations in the significance levels introduced by discrete actions on the system elements bears a monotonic character.

Definition. By a monotonic system, we understand a system, for which an action realized on an arbitrary element α involves either only decrease or only increase in the significance levels of all other elements.

In accordance with this definition of a monotonic system two types of actions are distinguished: type $\oplus$ and type $\ominus$. An action of type $\oplus$ involves increase in the significance levels while $\ominus$ involves decrease.

The formal concept of a discrete action on an element α of the system W and the change in significance levels of elements arising in connection with it allows us to define on the set of remaining elements of W an uncountable set of functions whenever we have at least one real significance function $\pi : W \to D$ (D being the set of real numbers).

Indeed, if an action is rendered on element α, the starting from the proposed scheme one can say that function π is mapped into π_α^+ or π_α^- according as a the action $\oplus$ or $\ominus$. Significance of system elements is redistributed as action on element α changes from function π to π_α^+ $\left(\pi_\alpha^-\right)$ or, otherwise, the initial collection of significance levels $\left\{\pi(\partial) \,\middle|\, \partial \in W\right\}$ changes into a new

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[3] If W is a finite set, then $|W|$ denotes the number of its elements.

collection $\left\{ \pi^{+}_{\alpha}(\partial) \,\middle|\, \partial \in \mathrm{W} \right\}$. [4] Clearly, if we are given some sequence $\alpha_1, \alpha_2, \alpha_3, ...$ of elements of W (arbitrary repetitions and combinations of elements being permitted) and the binary sequence $+,-,+,...$, then by the usual means one can define the functional product of functions $\pi^{+}_{\alpha_1}, \pi^{-}_{\alpha_2}, \pi^{+}_{\alpha_3}$ in the form $\pi^{+}_{\alpha_1}\pi^{-}_{\alpha_2}\pi^{+}_{\alpha_3}$.

The construction presented allows us to write the property of monotonic systems in the form of the following basic inequalities:

$$\pi^{+}_{\alpha}(\partial) \geq \pi(\partial) \geq \pi^{-}_{\alpha}(\partial) \tag{1}$$

for every pair of elements $\alpha, \partial \in \mathrm{W}$, including the pairs α, α or ∂, ∂.

Let there be given a partition of set W into two subsets, i.e., $\mathrm{H} \cup \overline{\mathrm{H}} = \mathrm{W}$ and $\mathrm{H} \cap \overline{\mathrm{H}} = \varnothing$. If we subject the elements $\alpha_1, \alpha_2, \alpha_3, ... \in \overline{\mathrm{H}}$ to positive actions only, then by the same token on set W there is defined some function $\pi^{+}_{\alpha_1}\pi^{-}_{\alpha_2}\pi^{+}_{\alpha_3}...$, which can be regarded as defined only on the subset H of W. [5]

If from all possible sequences of elements of set $\overline{\mathrm{H}}$ we select a sequence $\left\langle \alpha_1, \alpha_2, ..., \alpha_{|\overline{\mathrm{H}}|} \right\rangle$, [6] where α_i are not repeated, then on the set H the function $\pi^{+}_{\alpha_1}\pi^{+}_{\alpha_2}...$ is induced ambivalently.

We denote this function $\pi^{+}\mathrm{H}$ and call it a standard function. We shall also refer to the function thus introduced as a credential function and to its value on an element as an α credential. In accordance with this terminology the set $\left\{ \pi^{+}\mathrm{H}(\alpha) \,\middle|\, \alpha \in \mathrm{H} \right\}$, which is denoted by $\Pi^{+}\mathrm{H}$ is called a credential collection given on the set H or a credential collection relative to set H. Let us assume that we are given a set of credential collections $\left\{ \Pi^{+}\mathrm{H} \,\middle|\, \mathrm{H} \subseteq \mathrm{W} \right\}$ on the set of all possible subsystems $\mathrm{P(W)}$ of system W. The number of all possible subsystems is $\left|\mathrm{P(W)}\right| = 2^{|\mathrm{W}|}$.

Instead of considering a standard function for positive actions $\pi^{+}_{\alpha_1}\pi^{+}_{\alpha_2}...$ one can consider a similar function for negative actions $\pi^{-}\mathrm{H}$. Thus, one defines single credential collection $\Pi^{-}\mathrm{H} = \left\{ \pi^{-}\mathrm{H}(\alpha) \,\middle|\, \alpha \in \mathrm{H} \right\}$ and the aggregate of credential collections $\left\{ \Pi^{-}\mathrm{H} \,\middle|\, \mathrm{H} \subseteq \mathrm{W} \right\}$ by an exact analogy.

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[4] Functions π, π^{+}_{α} and π^{-}_{α} are defined on the whole set W and, consequently, $\pi^{+}_{\alpha}(\partial)$ and $\pi^{-}_{\alpha}(\partial)$ are defined.

[5] We are not interested in significance levels obtained as a result of operations on elements of $\overline{\mathrm{H}}$ onto the same set $\overline{\mathrm{H}}$.

[6] Here symbols $\langle , \rangle$ are used to stress the ordered character of a sequence of $\overline{\mathrm{H}}$.

Let us briefly summarize the above construction. Starting with some real function π defined on a finite set W and using the notion of positive and negative actions on elements of system W, one can construct two types of aggregate collections $\Pi^{+}H$ and $\Pi^{-}H$ defined on each of the H of subsets of W. Each function from the aggregate (credential collection) is constructed by means of the complement to H, equaling $W \setminus H$, and a sequence $\left\langle \alpha_{1}, \alpha_{2}, \ldots, \alpha_{|\overline{H}|} \right\rangle$ of distinct elements of the set $\overline{H}$. For this actions of types ⊕ and ⊖ are applied to all elements of set $\overline{H}$ in correspondence with the ordered sequence $\left\langle \alpha_{1}, \alpha_{2}, \ldots, \alpha_{|\overline{H}|} \right\rangle$ in order to obtain $\Pi^{+}H$ and $\Pi^{-}H$ respectively.

Credential collections/arrays concept of $\Pi^{+}H$ and $\Pi^{-}H$ needs refinement. The definition given above does not taken into account the character of dependence of function πH on the sequence of actions realized on the elements of set $\overline{H}$.[7] Generally speaking, credential collection $\Pi^{+}H(\Pi^{-}H)$ is not defined uniquely, since it can happen that for different orderings of set $\overline{H}$ we obtain different function πH.

In order that credential collection $\Pi^{+}H$ $(\Pi^{-}H)$ be uniquely defined by subset H of the set W it is necessary to introduce the notion of commutability of actions.

Definition. An action of type ⊕ or ⊖ is called commutative for system W if for every pair of elements $\alpha, \beta \in W$ we have

$$\pi_{\alpha}^{+}\pi_{\beta}^{+} = \pi_{\beta}^{+}\pi_{\alpha}^{+},\ \pi_{\alpha}^{-}\pi_{\beta}^{-} = \pi_{\beta}^{-}\pi_{\alpha}^{-}$$

In this case it is easy to show that the values of function πH on the set H do not depend on any order defined for the elements of the set $\overline{H}$ by sequence $\left\langle \alpha_{1}, \alpha_{2}, \ldots \right\rangle$. The proof can be conducted by induction and is omitted.

Thus, for commutative actions the function $\pi^{+}H$ $(\pi^{-}H)$ is uniquely determined by a subset of W.

In concluding this section, we make one important remark of an intuitive character. As is obvious from the above-mentioned definition of aggregates of credentials collection of type ⊕ and ⊖, the initial credential collection serves as the basic constructive element in their construction. The initial credential collection is a significance function defined on the set of system elements before

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[7] In the sequel, if sign "–" or "+" is omitted from our notation, then it is understood to be either "–" or "+"

the actions are derived from the elements. In other words, it is the initial state of the system fixed by credential collection ΠW. It is natural to consider only those aggregates of credential collections that are constructed from an initial ⊕ collection, which is the same as the initial ⊖ collection. The dependence indicated between ⊕ and ⊖ credential collections is used considerably for the proof of the duality theorem in the second part of this paper.

4. EXTREMAL THEOREMS. STRUCTURE OF EXTREMAL SETS

Let us consider the question of selecting a subset from system W whose elements have significance levels that are stipulated only by the internal "organization" of the subsystem and are numerically large or, conversely, numerically small. Since this problem consists of selecting from the whole set of subsystems $P(W)$ a subsystem having desired properties, therefore it is necessary to define more precisely how to prefer one subsystem over another, see also Muchnik and Shvartser (1990).

Let there be given aggregates of credential collections $\left\{\Pi^{+}H \,\middle|\, H \subseteq W\right\}$ and $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$. On each subset there are defined the following two functions:

$$F_{+}(H) = \max_{\pi\in H} \pi^{+}H(\alpha),\ F_{-}(H) = \min_{\pi\in H} \pi^{-}H(\alpha).$$

Definition of Kernels. By kernels of set W we call the points of global minimum of function F_{+} and of global maximum of function F_{-}.

A subsystem, on which F_{+} reaches a global minimum is called a ⊕ kernel of the system W, while a subsystem on which F_{-} reaches a global maximum, is called ⊖ kernel. Thus, in every monotonic system the problem of determining ⊕ and ⊖ kernels is raised.

With the purpose of intuitive interpretation as well as with the purpose of explaining the usefulness of the notion of kernels introduced above we turn once again to the examples of citation graphs and telephone commutation networks.

The definition of the kernel can be formulated using the levels of significance of the elements of the system, that is: the ⊕ kernel is a subsystem of a monotonic system, for which the maximum level among the levels of significance is determined only by the internal organization of the system is the minimum, and the ⊖ kernel is the subsystem for which the minimum level among the same significance levels is the maximum.

The definition of a kernel accords with the intuitive interpretation of a kernel in citation graphs and telephone commutation networks. Thus, in citation graphs a $\oplus$ kernel is a subset (subsystem) of publications, in which the longest list of bibliographical titles is at the same time very short; though not inside the subset, but among all possible subsets of the selected set of publications (among the very long lists). If in our subset of publications a very short list of bibliographical titles is at the same time very long among the very short ones relative to all the subsets, then it is a $\ominus$ kernel of the citation graph. It is clear that a $\ominus$ kernel publications cite one another often enough, since for each publication the list of bibliographical titles is at any rate not less than a very short one while a very short list is nevertheless long enough. In a $\oplus$ kernel the same reason explains why in this subset one must find representatives of various scientific schools.

In telephone commutation networks, one can consider two types of system elements – lines of connections and points of connections. In a system consisting of lines, a $\ominus$ kernel turns out to be a subset of lines, for which the lines with the least number of traffic congestions in that subset are at the same time the lines with the greatest number of traffic congestions among all possible sets of lines. This means that at least the number of traffic congestions stipulates only by the internal organization of a sub-network of lines of a $\ominus$ kernel is not less than the number of traffic congestions for lines with the smallest number of traffic congestions and, besides, this number is large enough. Hence one can expect that the number of traffic congestions for lines of a $\ominus$ kernel is sufficiently large. Similarly one should expect a small number of traffic congestions for lines of a $\oplus$ kernel. Formulation for $\oplus$ and $\ominus$ kernels for points of connection is exactly the same as for the lines and is omitted here.

Before stating the theorems, we need to introduce some new definitions and notations. Let $\overline{\alpha} = \left\langle \alpha_0, \alpha_1, ..., \alpha_{k-1} \right\rangle$ be an ordered sequence of distinct elements of set W, which exhausts the whole of this set, i.e., $k = |W|$. From sequence $\overline{\alpha}$ we construct an ordered sequence of subsets of W in the form $\Delta_{\overline{\alpha}} = \left\langle H_0, H_1, ..., H_{k-1} \right\rangle$ with the help of the following recurrent rule $H_0 = W$, $H_{i+1} = H_i \setminus \{\alpha_i\}$; $i = 0,1,...,k-2$ [9]

Definition. Sequence $\overline{\alpha}$ of elements of W is called a defining sequence relative to the aggregate of credentials collections $\left\{ \Pi^{-}H \,\middle|\, H \subseteq W \right\}$ if there exists in sequence $\Delta_{\overline{\alpha}}$, a subsequence of sets $\Gamma_{\overline{\alpha}} = \left\langle \Gamma_0^{-}, \Gamma_1^{-}, ..., \Gamma_p^{-} \right\rangle$, such that:

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[9] Sign $\setminus$ denotes the subtraction operation for sets.

a) credential $\pi^{-}H_{i}(\alpha_{i})$ of an arbitrary element α_{i} in sequence $\overline{\alpha}$, belonging to set Γ_{j}^{-} but not belonging to set Γ_{j+1}^{-} is strictly less than values of $F_{-}(\Gamma_{j+1})$;[10]

b) in set Γ_{p}^{-} there does not exist a proper subset L, which satisfies the strict inequality $F_{-}(\Gamma_{p}) < F_{-}(L)$.

A sequence $\overline{\alpha}$ with properties a) and b) is denoted by $\overline{\alpha}_{-}$. One similarly defines a sequence $\overline{\alpha}_{+}$.

c) arbitrary element α_{i} in sequence $\overline{\alpha}$, belonging to set Γ_{j}^{+} but not belonging to set Γ_{j+1}^{+} is strictly greater than values of $F_{+}(\Gamma_{j+1})$;

d) in set Γ_{q}^{+} there does not exist a proper subset L, which satisfies the strict inequality $F_{+}(\Gamma_{q}) > F_{+}(L)$.

Definition. Subset H_{+}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{+}$ such that $H_{+}^{*} = \Gamma_{q}^{+}$.

Definition. Subset H_{-}^{*} of set W is called definable if there exists a defining sequence $\overline{\alpha}_{-}$ such that $H_{-}^{*} = \Gamma_{p}^{-}$.

Below we formulate, but do not prove, a theorem concerning properties of points of global maximum of function F_{-}. The proof is adduced in Appendix 1. A similar theorem holds for function F_{+}. In Appendix 1 the parallel proof for function F_{+} is not reproduced. The corresponding passage from the proof for F_{-} to that of F_{+} can be effected by simple interchange of verbal relations "greater than" and "less than", inequality signs "≥" and "≤", ">", "<" as well as by interchange of signs "+" and "–". The passage from definable set H_{+}^{*} to H_{-}^{*} and from definition of sequence $\overline{\alpha}_{+}$ and $\overline{\alpha}_{-}$, is affected by what has just been said.

Theorem 1. On a definable set H_{-}^{*} function F_{-} reaches a global maximum. There is a unique definable set H_{-}^{*}. All sets, on which a global maximum is reached, lie inside the definable set H_{-}^{*}.

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[10] Here and everywhere, for simplification of expression, where it is required, the sign "–" or "+" is not used twice in notations. We should have written $F_{-}(\Gamma_{j+1}^{-})$ or $F_{+}(\Gamma_{j+1}^{+})$.

Theorem 2. On a definable set H_+^* function F_+ reaches a global minimum. There is a unique definable set H_+^*. All sets, on which a global minimum is reached, lie inside the definable set H_+^*.

In the proof of Theorem 1 (Appendix 1) it is supposed that definable set H_-^* exists. It is natural that this assumption, in turn, needs proof. The existence of H_-^* is secured by a special constructive procedure. [11]

The proof of Theorem 2 is completely analogous to the proof of Theorem 1 and is not adduced in Appendix 1. We present a theorem, which reflects a more refined structure of kernels of W as elements of the set $P(W)$ of all possible subsets (subsystems) of set W.

Theorem 3. The system of all sets in $P(W)$, on which function F_- (F_+) reaches maximum (minimum), is closed with the respect to the binary operation of taking union of sets.

The proof of this theorem is given in Appendix 2 and only for the function F_-. The assertion of the theorem for F_+ is established similarly.

Thus, it is established that the set of all ⊕ kernels (⊖ kernels) forms a closed system of sets with respect to the binary operation of taking the unions. The union of all kernels is itself a large kernel and, by the statements of Theorems 1 and 2, is a definable set.

5. ROUTINE OF FINDING THE KERNELS

In preceding sections, we established the fundamental approach for selecting singular subsystems within monotonic systems, specifically identifying kernels with extremal properties. At the core of this method lies the notion of a 'definable set,' as delineated by Mullat in 1971. In our framework, a definable set represents the largest kernel within a monotonic system of interconnected elements. Back in 1971, we introduced the concept of a definable set through the utilization of defining $\overline{\alpha}_-$ and $\overline{\alpha}_+$ sequences within the system.

Subsequently, we tackled the issue of identifying defining sequences, offering constructive solutions in the form of algorithms. The key attributes of these defining sequences, generated according to predefined routines, and encompassing the entirety of system elements **W**, are delineated by a theorem.

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[11] This procedure will be presented in the second part of the article, since here only the extremal properties of kernels and the structure of the set of kernels are established.

We will delve into the intricate relationship between two defining sequences, denoted as $\overline{\alpha}_-$ and $\overline{\alpha}_+$. While one might intuitively consider obtaining $\overline{\alpha}_+$ by simply reversing the order of $\overline{\alpha}_-$, this assumption doesn't universally hold true. However, we can make a more nuanced assertion based on the discrete operations ⊕ and ⊖ on the elements of system W, as defined by Mullat in 1976. This assertion manifests as a duality theorem, which we shall expound upon shortly.

Under the auspices of this duality theorem, the algorithms elucidated for constructing defining sequences serve to significantly narrow the scope of search for both ⊕ and ⊖ kernels within system W. The algorithm delineating this restriction of the search domain is presented in the form of a constructive routine.

Now, let's dissect the routine for constructing an ordered sequence α comprising all elements of W, succinctly known as the Kernel Searching Routine (KSR). This routine plays a pivotal role in our methodology, facilitating the systematic identification and organization of system elements for further analysis and manipulation.

This routine consists of rules of generation and scanning of an ordered series of ordered sets $\left\langle \overline{\beta}_j \right\rangle$ (sequences); here j varies from zero to a value p, which is automatically determined by the rules of the routine, whereas the elements of each sequence $\overline{\beta}_j$ are selected from the set W [12].

This series $\left\langle \overline{\beta}_j \right\rangle$ constructed by this rule forms a numerical sequence of thresholds $\left\langle u_j \right\rangle$ and a sequence of sets $\left\langle \Gamma_j \right\rangle$. On the other hand the sequence of thresholds governs the transactions from $\overline{\beta}_{j-1}$ to $\overline{\beta}_j$ in the chain $\left\langle \overline{\beta}_j \right\rangle$, and the sequence $\left\langle \Gamma_j \right\rangle$ terminates with a set, which is definable.

In the description of a rule we use the operation of extending a sequence $\overline{\beta}_j$ by adjoining to it another sequence $\overline{\gamma}$. This operation is symbolically expressed by $\overline{\beta} \leftarrow \left\langle \overline{\beta}, \overline{\gamma} \right\rangle$. This rule of construction of the sequence $\overline{\alpha}$ of all elements of the set W can be described stages: by step **Z** and **R**.

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[12] Let us recall that in a) the brackets $\left\langle , \right\rangle$ denoted an ordered set; in the case under consideration they denote an ordered set of ordered sets $\overline{\beta}_j$.

Z. In the set W we have found an element μ_0 for which $\pi^{-}W(\mu_0) = \min_{\delta\in W}\pi^{-}W(\delta) = F_{-}(W)$; we are constructing a defining sequence $\overline{\alpha}_{-}$. The construction of $\overline{\alpha}_{+}$ is entirely similar and therefore not presented here. We shall only indicate where it is necessary to invert the sign of inequalities, and where the search for an element with the minimal credential must be replaced by search for an element with maximal credential, so as to be able to construct $\overline{\alpha}_{+}$. Thus the construction of $\overline{\alpha}_{+}$, the element μ_0 is obtained from $\pi^{+}W(\mu_0) = \max_{\delta\in W}\pi^{+}W(\delta) = F_{+}(W)$ condition. We shall write $u_0 = \pi^{-}W(\mu_0)$, $\overline{\alpha} = \langle\mu_0\rangle$ and the set $\Gamma_0 = W$. We select a subset of elements γ from W such that $\pi^{-}W \setminus \overline{\alpha}(\gamma) \le u_0$. The construction of $\overline{\alpha}_{+}$ requires the selection of such γ that $\pi^{+}W \setminus \overline{\alpha}(\gamma) \ge u_0, u_0 = \pi^{+}W(\mu_0)$. After that we order the elements in a certain manner (which can be arbitrary selected). The thus-obtained ordered set is denoted by $\overline{\gamma}$. Let us write $\overline{\beta}_0 = \overline{\gamma}$.

R. We construct a recursive routine for extending the sequences $\overline{\alpha}$ and $\overline{\beta}_0$. Here we denote by $\beta_0(i)$ the i-th element of the sequence $\overline{\beta}_0$. We specify one after another the elements of the sequence $\overline{\beta}_0$. At each instant of specification we extend the sequence $\overline{\alpha}$ by the elements from $\overline{\beta}_0$ of the sequence fixed at this instant. In accordance with the symbolic notation of the operation of extension of a sequence $\overline{\alpha}$, we perform at each instant t of specification the operation $\overline{\alpha} \leftarrow \langle\overline{\alpha}, \beta_0(t)\rangle$. Suppose that all the elements of $\overline{\beta}_0$ up to $\beta_0(i-1)$ inclusive have been fixed. Then the sequence $\overline{\alpha}$ will have the form $\langle\mu_0, \beta_0(1), \beta_0(2), ..., \beta_0(i-1)\rangle$, which corresponds to the symbolic notation of the operation of extension of the sequences $\overline{\alpha} \leftarrow \langle\overline{\alpha}, \beta_0(1), \beta_0(2), ..., \beta_0(i-1)\rangle$ in the case that $\overline{\alpha}$ inside the brackets consists of one element μ_0. Let us consider an element $\beta_0(i-1)$ of the sequence $\overline{\beta}_0$. At the instant of specification of the element $\beta_0(i-1)$ we decide during the above-mentioned operation of extension of $\overline{\alpha}$ also about any further extension or about stopping the extension of the sequence $\overline{\beta}_0$. We must check the following three conditions:

a) In the set $W \setminus \overline{\alpha}$ there exist elements such that $\pi^{-}W \setminus \overline{\alpha}(\gamma) \le u_0$ In constructing $\overline{\alpha}_{+}$, this condition is replaced by $\pi^{+}W \setminus \overline{\alpha}(\gamma) \ge u_0$;

b) the element $\beta_0(i)$ is defined for the sequence $\overline{\beta}_0$. By assumption an element $\beta_0(i)$ to be defined for a sequence $\overline{\beta}_0$ if the sequence $\overline{\beta}_0$ has an element with an ordinal number i. Otherwise the element $\beta_0(i)$ is not defined. There can be four cases of fulfillment or no fulfillment of these conditions. In two cases, when the first condition is satisfied, irrespective of whether or not the second condition holds, the sequence $\overline{\beta}_0$ will be extended. This means that the set of elements γ in $W \setminus \overline{\alpha}$ specified by the first condition is ordered in the form of sequence $\overline{\gamma}$. The sequence $\overline{\beta}_0$ is extended in accordance with the formula $\overline{\beta}_0 \leftarrow \langle \overline{\beta}_0, \overline{\gamma} \rangle$. In case when the first condition is not satisfied, whereas the second condition is satisfied, we shall fix the element $\beta_0(i)$ and at the same time extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \beta_0(i) \rangle$, and proceed to new recursion stage. In case neither the first nor the second condition holds, the sequence $\overline{\beta}_0$ will not be extended nor the last fixed element in the sequence $\overline{\beta}_0$ will be the element $\beta_0(i-1)$. Suppose that we have fixed all the elements of the sequence $\overline{\beta}_j$. By that time we have constructed a sequence $\overline{\alpha}$. Let us consider the set $W \setminus \overline{\alpha}$ and the credential system $\Pi^{-} W \setminus \overline{\alpha}$. We shall find an element in $\Pi^{-} W \setminus \overline{\alpha}$ on which the minimum is reached in the credential system $\Pi^{-} W \setminus \overline{\alpha}$. The obtained element is denoted by μ_{j+1}. We obtain $\overline{\alpha}_{+}$ the element μ_{j+1} from the $\pi^{+} W \setminus \overline{\alpha}(\mu_{j+1}) = \max_{\delta \in W \setminus \overline{\alpha}} \pi^{+} W \setminus \overline{\alpha}(\delta) = F_{+}(W \setminus \overline{\alpha})$ condition:. Thus, $\pi^{-} W \setminus \overline{\alpha}(\mu_{j+1}) = F_{-}(W \setminus \overline{\alpha})$. Let us write $u_{j+1} = \pi^{-} W \setminus \overline{\alpha}(\mu_{j+1})$, and for the set $\Gamma_{j+1} = W \setminus \overline{\alpha}$; then we supplement the sequence $\overline{\alpha}$ by the element μ_{j+1}, i.e., $\overline{\alpha} \leftarrow \langle \overline{\alpha}, \mu_{j+1} \rangle$. In the same way as during the zero step, we select a subset of elements γ from $W \setminus \overline{\alpha}$ such that $\pi^{-} W \setminus \overline{\alpha}(\gamma) \leq u_{j+1}$. Here we select for $\overline{\alpha}_{+}$ a set of elements γ such that $\pi^{+} W \setminus \overline{\alpha}(\gamma) \geq u_{j+1}$. The selected set can be ordered in any manner. The ordered set is denoted by $\overline{\gamma}$. The set $\overline{\beta}_{j+1}$ is assumed to be equal to $\overline{\gamma}$.

c) By analogy with previous b) the recursion step will be described as a recursion routine. At this stage we also use the rule of extension of the sequences $\overline{\alpha}$ and $\overline{\beta}_{j+1}$. Suppose that we have fixed all elements of $\overline{\beta}_{j+1}$ up to $\beta_j(i-1)$ inclusive. Then the sequence $\overline{\alpha}$ will have the form $\overline{\alpha}=\left\langle\overline{\alpha},\mu_{j+1},\beta_j(1),\ldots,\beta_j(i-1)\right\rangle$, where $\overline{\alpha}$ denotes the sequence $\overline{\alpha}$ obtained at the instant of fixing all the elements of $\overline{\beta}_j$, or, to rephrase, the sequence $\overline{\alpha}$ prior to the $(j+1)$-th step. The last equation corresponds to the symbolic operation of extension of the sequence $\overline{\alpha}=\left\langle\overline{\alpha},\mu_{j+1},\beta_j(1),\ldots,\beta_j(i-1)\right\rangle$ in the case that $\overline{\alpha}$ inside the brackets denotes the sequence $\left\langle\overline{\alpha},\mu_{j+1}\right\rangle$. Let us consider an element $\beta_{j+1}(i-1)$ of the sequence $\overline{\beta}_{j+1}$. At the instant of fixing the element $\beta_{j+1}(i-1)$ we decide about a further extension or about stopping the extension of the sequence $\overline{\beta}_{j+1}$. For this purpose we consider the credential system $\Pi^{-}W\setminus\overline{\alpha}$ and we check two conditions:

1) The set $W\setminus\overline{\alpha}$ contains elements γ such that $\pi^{-}W\setminus\overline{\alpha}(\gamma)\leq u_{j+1}$ For constructing $\overline{\alpha}_{+}$ we must take elements γ such that $\pi^{+}W\setminus\overline{\alpha}(\gamma)\geq u_{j+1}$;
2) the element $\beta_{j+1}(i)$ is defined for the sequence $\overline{\beta}_{j+1}$.

By analogy with the step **Z**, we find that the sequence $\overline{\beta}_{j+1}$ is extended in two cases in which the first condition is satisfied irrespective of whether or not the second condition holds. The set of elements γ in $W\setminus\overline{\alpha}$ specified by the first condition is ordered in the form of a sequence $\overline{\gamma}$. The sequence $\overline{\beta}_{j+1}$ is extended in accordance with the formula $\overline{\beta}_{j+1}\leftarrow\left\langle\overline{\beta}_{j+1},\overline{\gamma}\right\rangle$. In the case that the first condition does not hold, whereas the second condition is satisfied, the element $\beta_{j+1}(i)$ will be fixed and at the same time we extend the sequence $\overline{\alpha}$, i.e., $\overline{\alpha}\leftarrow\left\langle\overline{\alpha},\beta_{j+1}(i)\right\rangle$, and after that we proceed again in accordance with the rules of Stage 2 of the recursion routine of extension of the sequence $\overline{\beta}_{j+1}$. In the case that neither the first, nor the second condition holds, the sequence $\overline{\beta}_{j+1}$ will not be extended, and the last fixed element of the sequence $\overline{\beta}_{j+1}$ will be the element $\beta_{j+1}(i-1)$. At some step p the sequence $\overline{\alpha}$ will exhaust the entire set of elements W.

Theorem 4. A sequence $\overline{\alpha}$ constructed on the basis of a collection of credential system $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$ is a defining sequence $\overline{\alpha}_{-}$, whereas a sequence $\overline{\alpha}$ constructed on the basis of $\left\{\Pi^{+}H \,\middle|\, H \subseteq W\right\}$ is a defining sequence $\overline{\alpha}_{+}$.

The first part of the theorem (for $\overline{\alpha}_{-}$) is proved in Appendix 3. The second part (for $\overline{\alpha}_{+}$) can be proved in the same way.

NB1. Let us note that a sequence $\overline{\alpha}$ constructed by KSR rules has somewhat stronger properties than required in obtaining a defining sequence. More precisely, there does not exist a proper subset L for $j = 0,1,...,p-1$ such that $\Gamma_{j} \supset L \supset \Gamma_{j+1}$ and $F_{-}(\Gamma_{j}) < F_{-}(L)$. This is not required for obtaining a defining sequence $\overline{\alpha}_{-}$ ($\overline{\alpha}_{+}$). The corresponding proof is not given here.

NB2. Let us note another circumstance. With the aid of the kernel-searching routine it is possible to effectively find (without scanning) the largest kernel, i.e., a definable set. It is not possible to find an individual kernel strictly included in a definable set (if the latter exists) by constructing a defining sequence.

6. DUALITY THEOREM

Let us establish a relationship between the defining sequences $\overline{\alpha}_{-}$ and $\overline{\alpha}_{+}$ of a system W.

Theorem 5. Let $\overline{\alpha}_{-}$ and $\overline{\alpha}_{+}$ be defining sequences of the set W with respect to the collection of credential system $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$, $\left\{\Pi^{+}H \,\middle|\, H \subseteq W\right\}$ respectively. Let $\left\langle \Gamma_{j}^{-} \right\rangle$ be the subsequence of the sequence $\Delta_{\overline{\alpha}_{-}}$ $(j = 0,1,...,p)$ needed in the determination of $\overline{\alpha}_{-}$, and let $\left\langle \Gamma_{j}^{+} \right\rangle$ be the corresponding subsequence of the sequence $\Delta_{\overline{\alpha}_{+}}$ $(j = 0,1,...,q)$.

Hence if for an m and a n we have

$$F_{+}(\Gamma_{n}^{+}) = F_{-}(\Gamma_{m}^{-}), \tag{2}$$

then $\Gamma_{m}^{-} \subseteq W \setminus \Gamma_{n+1}^{+}$, $\Gamma_{n}^{+} \subseteq W \setminus \Gamma_{m+1}^{-}$. If

$$F_{+}(\Gamma_{n}^{+}) < F_{-}(\Gamma_{m}^{-})\ ^{2}, \tag{3}$$

then $\Gamma_{m}^{-} \subseteq W \setminus \Gamma_{n}^{+}$, $\Gamma_{n}^{+} \subseteq W \setminus \Gamma_{m}^{-}$.

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[2] In the following, the + and – sign will not be used twice in notation. This rule applies also to Appendices 1 and 2

This theorem is important from two points of view. Firstly, under the conditions (2) and (3) there exists a relationship between an $\overline{\alpha}_-$ sequence and $\overline{\alpha}_+$. This relationship consists in the fact that elements of $\overline{\alpha}_+$ which are at the "beginning" and form either the set $W \setminus \Gamma^+_{n+1}$ or the set $W \setminus \Gamma^+_n$ will include all the elements of the set Γ^-_m that are at the "end" of $\overline{\alpha}_-$. The same applies also to sets $W \setminus \Gamma^-_{m+1}$ or $W \setminus \Gamma^-_m$ which are at the beginning of $\overline{\alpha}_-$, since they include in a similar way the set Γ^+_n. In other words, the theorem states that the sequence $\overline{\alpha}_+$ does not differ "very much" (under certain conditions) from the sequence, which is the inverse to $\overline{\alpha}_-$.

Let us note that the conditions (2) and (3) are sufficient conditions, and it can happen that actual monotonic systems satisfying these conditions do not exist. Nevertheless, in the third part of this article, we shall describe actual examples of such systems.

7. KERNEL SEARCH ROUTINE BASED ON DUALITY THEOREM

We just noted that a defining sequence $\overline{\alpha}_+$ differs "slightly" from the inverse sequence of $\overline{\alpha}_-$. For elucidating the possibility of a search for kernels on the basis of the duality theorem, let us rephrase the latter. This assertion can be formulated as follows: at the beginning of the sequence $\overline{\alpha}_+$ we often encounter elements of the sequence $\overline{\alpha}_-$, which are at the end of the latter.

Such an interpretation of the duality theorem yields an efficient routine of dual search for ⊕ and ⊖ kernels of the system W. This is due to the fact if the elements are often encountered, there exists a higher possibility of finding a ⊕ kernel at the beginning of the sequence $\overline{\alpha}_+$ as compared to finding it at the end of $\overline{\alpha}_-$; the same applies also to a ⊖ kernel in the sequence $\overline{\alpha}_-$.

The routine under construction is based on Corollaries I-IV of the duality theorem presented in Appendix II, where we also prove this theorem.

The routine of dual search for kernels described below is an application of two constructive routines, i.e., a KSR for constructing $\overline{\alpha}_+$ and a KSR for constructing $\overline{\alpha}_-$. The routine is stepwise, with two constructing stages realized at each step, i.e., a stage in which the KSR is used for constructing $\overline{\alpha}_+$ with ⊕ operations, and a stage in which the same routine is used for constructing $\overline{\alpha}_-$ with the aid of ⊖ operations on the elements of the system.

Z. At first we store two numbers: $u_0^+ = F_+(W)$ and $u_0^- = 0$. After that we perform precisely Stage 1 and 2 of the zero step of the KSR used for constructing the defining sequence $\overline{\alpha}_+$. This signifies that the set W contains an element μ_0 such that $\pi^+W(\mu_0) = \max_{\delta \in W} \pi^+W(\delta) = F_+(W)$. The threshold u_0^+ is equal to $\pi^+W(\mu_0)$, etc. By using the constructions of the zero step of KSR at the previous stage of the dual routine under construction, we obtained a set $\Gamma_1^+ \subset W$. Then we examine the set $W \setminus \Gamma_1^+$ and the credential system $\Pi^+W \setminus \Gamma_1^+$. On the set $\overline{\alpha}_+$ with the credential system $\Gamma_{j+1}^+ \subset \Gamma_j^+$ we perform a complete kernel-searching routine for the purpose of constructing a defining sequence of $\oplus$ operations only for the set $W \setminus \Gamma_{j+1}^+$. As a result, we obtain in the set $W \setminus \Gamma_1^+$ a subset F_- on which the function F_- reaches a global maximum among all the subsets of the set $W \setminus \Gamma_1^+$.

R. By applying the previous $(j-1)$ steps to the j-th step, we obtained a sequence of sets $\Gamma_0^+, \Gamma_1^+, ..., \Gamma_j^+$, and according to the construction of a defining sequence we have $\Gamma_0^+ \supset \Gamma_1^+ \supset ... \supset \Gamma_j^+$ and $\Gamma_0^+ = W$. At first we store two numbers: $u_j^+ = F_+(\Gamma_j^+)$ and $u_j^- = F_-(H^j)$. By analogy, we perform the same construction consisting of two stages of a KSR recursion step for constructing $\overline{\alpha}_+$ with the aid of $\oplus$ operations. At a given instant of such dual construction we obtained a set $\Gamma_{j+1}^+ \subset \Gamma_j^+$. Then we consider the set $W \setminus \Gamma_{j+1}^+$ and the credential system $\Pi^- W \setminus \Gamma_{j+1}^+$. In the same way as at the zero step, we perform on the set $W \setminus \Gamma_{j+1}^+$ a complete kernel-searching routine with the purpose of constructing a sequence $\overline{\alpha}_-$ only on the set $W \setminus \Gamma_{j+1}^+$. As a result we obtain in the set $W \setminus \Gamma_{j+1}^+$ a subset H^{j+1} on which the function F_- reaches a global maximum among all subsets of the set $W \setminus \Gamma_{j+1}^+$.

S. Before starting the construction of the j-th step of the routine under construction, we check the condition of a Rule of Termination of Construction Routine:

$$u_j^+ \leq u_j^- . \tag{4}$$

If (4) is satisfied as a strict inequality, the construction will terminate before the j-th step. If (4) is an equality, the construction will terminate after the j-th step.

8. DEFINABLE SETS OF DUAL KERNEL-SEARCH ROUTINE

At the end of the construction process, the above routine yields a set H^j or a set H^{j+1}. It can be asserted that one of the sets is definable set or the largest kernel of the system W with respect to a collection of credential system $\left\{\Pi^{-}H \middle| H \subseteq W\right\}$.

The assertion is based on the following. Firstly, by applying the KSR we obtained the second stage of the j-th step of a dual routine the maximal set H^{j+1} among all the subsets of the set $W \setminus \Gamma^{+}_{j+1}$ on which the function F_{-} reaches a global maximum in the system of sets of all the subsets of the set $W \setminus \Gamma^{+}_{j+1}$. Secondly, by virtue of Corollary 1 of the Theorem 2 (the duality theorem), it follows that, prior to the j-th step and provided that (4) is a strict inequality, the largest kernel (a definable set) will be contained in the set $W \setminus \Gamma^{+}_{j}$, or it follows from the Corollary 2 of the Theorem 2, if (4) is a equality, that the largest kernel is included in the set $W \setminus \Gamma^{+}_{j+1}$. Thus by comparing these two remarks we can see that either H^j or H^{j+1} is a definable set.

By virtue of Corollaries 3 and 4 of the duality theorem, it is possible to find by similar dual routine also the largest kernel $K^{\oplus}$- definable set. This assertion can be proved in the same way as the assertion about H^j and H^{j+1}; therefore this proof is not given here.

APPENDIX 1

Proof of Theorem 1. We suppose that a definable set H^{*}_{-} exists.

(Conducting the proof by contradiction) let us assume that there exists a set $L \subseteq W$, which satisfies the inequality

$$F_{-}(H^{*}) \leq F_{-}(L). \qquad \text{(A1.1)}$$

Thus two sets H^{*}_{-} and L are considered. One of the following statements holds:

1) Either $L / H^{*}_{-} \neq \varnothing$, which signifies the existence of elements in L, not belonging to H^{*}_{-};

2) or $L \subseteq H^{*}_{-}$.

We first consider 2). By a property of definable set H^{*}_{-} there exists a defining sequence $\overline{\alpha}_{-}$ of elements of set W with the property b) (cf. the definition of $\overline{\alpha}_{-}$) such that the strict inequality $F_{-}(H^{*}) < F_{-}(L)$ does not hold and, consequently, only the equality holds in (A1.1). In this case, the first and the third statements of the theorem are proved. It remains only to prove the uniqueness of H^{*}_{-}, whish is done after considering 1).

Thus, let $L / H^*_- \neq \varnothing$ and let us consider set H_t – the smallest of those H_i $(i = 0,1,...,k-1)$ from the defining sequence $\overline{\alpha}_-$ that include the set L / H^*_-. Then the fact that H_t is the smallest of the indicated sets implies the following: there exists element $\lambda \in L$, such that $\lambda \in H_t$, but $\lambda \notin H_{t+1}$.

Below, we denote by $i(\Omega)$ the smallest of the indices of elements of defining sequence $\overline{\alpha}_-$ that belong to the set $\Omega \subseteq W$.

Let Γ^-_p be the last in the sequence of sets $\langle \Gamma^-_j \rangle$, whose existence is guaranteed by the sequence $\overline{\alpha}_-$. For indices t and $i(\Gamma^-_p)$ we have the inequality $t < i(\Gamma^-_p)$.

The last inequality means that in sequence of sets $\langle \Gamma^-_j \rangle$ there exists at least one set Γ^-_s, which satisfies

$$i(\Gamma^-_{s+1}) \geq t+1. \qquad \text{(A1.2)}$$

Without decreasing generality, one can assume that Γ^-_s is the largest among such sets.

It has been established above that $\lambda \in H_t$, but $\lambda \notin H_{t+1}$. Inequality (A1.2) shows that $\Gamma^-_s \subset H_{t+1}$, since the opposite assumption $\Gamma^-_s \supseteq H_{t+1}$ leads to the conclusion that $i(\Gamma^-_s) \geq t+1$ and, consequently Γ^-_s is not the largest of the sets, for which (A1.2) holds.

Thus, it is established that $\Gamma^-_{s-1} \supset H_t$. Indeed, if $\Gamma^-_{s-1} \subseteq H_t$, then for indices $i(\Gamma^-_{s-1})$ and t we have $i(\Gamma^-_{s-1}) \geq t$.

Hence $i(\Gamma^-_{s-1}) + 1 \geq t+1$ and the inequality $i(\Gamma^-_s) \geq i(\Gamma^-_{s-1}) + 1$ implies $i(\Gamma^-_s) \geq t+1$. The last inequality once again contradicts the choice of set Γ^-_s as the largest set, which satisfies inequality (A1.2).

Thus, $\lambda \notin \Gamma^-_s$ but $\lambda \in \Gamma^-_{s-1}$ since $\lambda \in H_t$, $H_t \subseteq \Gamma^-_{s-1}$. On the basis of property a) of the defining sequence $\overline{\alpha}_-$, we can conclude that

$$\pi^- H_t(\lambda) < F_-(\Gamma_s), \qquad \text{(A1.3)}$$

where $0 \leq s \leq p$.

Let us consider an arbitrary set Γ_j^- $(j = 0,1,...,p-1)$ and an element $\tau \in \Gamma_j^-$, which has the smallest index in the sequence $\overline{\alpha}_-$. In other words, set Γ_j^- starts from the element τ in sequence $\overline{\alpha}_-$. In this case, set Γ_j^- is a certain set H_i in the sequence of imbedded sets $\langle H_i \rangle$. The definition of $F_-(H)$ and the property a) of defining sequence $\overline{\alpha}_-$ implies that

$$F_-(\Gamma_j) \leq \pi^- \Gamma_j(\tau) < F_-(\Gamma_{j+1}) .$$

Hence, since $\Gamma_p^- = H_-^*$ and $F_-(\Gamma_0) < F_-(\Gamma_1) < ... < F_-(\Gamma_p)$ and as a corollary we have for $j = 0,1,...,p$

$$F_-(\Gamma_j) \leq F_-(\Gamma_p) = F_-(H^*) , \tag{A1.4}$$

Let $\mu \in L$ and let credential $\pi^- L(\mu)$ be minimal in the collection of credentials relative to set L. On the basis of inequalities (A1.1), (A1.3), and (A1.4) we deduce that

$$\pi^- H_t(\lambda) < \pi^- L(\mu) = F_-(L). \tag{A1.5}$$

Above, H_t was chosen so that $L \subseteq H_t$. Recalling the fundamental monotonicity property (1) for collection of credentials (the influence of elements on each other), it easy to establish that

$$\pi^- L(\lambda) \leq \pi^- H_t(\lambda) . \tag{A1.6}$$

Inequalities (A.5) and (A.6) imply the inequality

$$\pi^- L(\lambda) < \pi^- L(\mu) ,$$

i.e., there exists in the collection of credentials relative to set L a credential, which is strictly less than the minimal credential.

A contradiction is obtained and it is proved that set L can only be a subset of H_-^* and that all sets, distinct from H_-^*, on which the global maximum is also reached, lie inside H_-^*.

It remains to prove that if a definable set H_-^* exists, then it is unique. Indeed, in consequence of what has been proved above we can only suppose that some definable set $H_-^{'}$, distinct from H_-^*, is included in H_-^*.

It is now enough to adduce arguments for definable set $H_-^{'}$ similar to those adduced above for L, considering it as definable set $H_-^{'}$; this implies that $H_-^* \subseteq H_-^{'}$. The theorem is proved. ■

APPENDIX 2

Proof of Theorem 3. Let Ω be the system of set in $P(W)$, on which function F_- reaches a global maximum, and let $K_1 \in \Omega$ and $K_2 \in \Omega$.

Since on K_1 and K_2 the function F_- reaches a global maximum, therefore we might establish the inequalities

$$F_-(K_1 \cup K_2) \leq F_-(K_1), \tag{A2.1}$$

$$F_-(K_1 \cup K_2) \leq F_-(K_2). \tag{A2.2}$$

We consider element $\mu \in K_1 \cup K_2$, on which the value of function F_- on set $K_1 \cup K_2$, is reached, i.e.,

$$\pi^- K_1 \cup K_2(\mu) = \min_{\alpha \in K_1 \cup K_2} \pi^- K_1 \cup K_2(\alpha).$$

If $\mu \in K_1$, then by rendering $\ominus$ actions on all those elements of set $K_1 \cup K_2$, that do not belong to K_1, we deduce from the fundamental monotonicity property of collections of credentials (1) the validity of the inequality

$$\pi^- K_1(\mu) \leq \pi^- K_1 \cup K_2(\mu).$$

Since the definition of F_- implies that $F_-(K_1) \leq \pi^- K_1(\mu)$ and by the choice of element μ we have $\pi^- K_1 \cup K_2(\mu) = F_-(K_1 \cup K_2)$, we therefore deduce the inequality

$$F_-(K_1) \leq F_-(K_1 \cup K_2).$$

Now from the inequality (A2.1) it follows that

$$F_-(K_1) = F_-(K_1 \cup K_2).$$

If, however, it is supposed that $\mu \in K_2$, then $\ominus$ actions are rendered on elements of $K_1 \cup K_2$, not belonging to K_2; in an analogous way implementing (A2.2) we obtain the equality

$$F_-(K_2) = F_-(K_1 \cup K_2).$$

The Theorem 3 has been proved. ■

APPENDIX 3

Proof of Theorem 1. We shell prove that a sequence $\overline{\alpha}$ constructed by the KSR rules is a defining sequence for a collection of credential systems

$$\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}.$$

First of all let us recall the definition of a defining sequence of elements of the system W. We shall use the notation $\Delta_{\overline{\alpha}} = \left\langle H_0, H_1, ..., H_{k-1}\right\rangle$, where $H_0 = W$, $H_{i+1} = H_i \setminus \alpha_i$ $(i = 0,1,...,k-2)$. A sequence of elements of a set W is said to be defining with respect to a coalition of credential system $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$ if the sequence $\Delta_{\overline{\alpha}}$ has a subsequence of sets $\Gamma_{\overline{\alpha}} = \left\langle \Gamma_0, \Gamma_1, ..., \Gamma_p \right\rangle$, such that

a) The credential $\pi^{-}H_i(\alpha_i)$ of any element α_i of the sequence $\overline{\alpha}$ that belongs to the set Γ_j, but does not belong to the set Γ_{j+1}, is strictly smaller than the credential of an element with minimal credential with respect to the set Γ_{j+1}, i.e., $\pi^{-}H_i(\alpha_i) < F_{-}(\Gamma_{j+1})$, $j = 0,1,...,p-1$ [3];

b) the set Γ_p does not have a proper subset L such that the strict inequality $F_{-}(\Gamma_p) < F_{-}(L)$ is satisfied (the "–" symbol has been omitted; see previous footnote).

We shall consider a sequence of sets $\Delta_{\overline{\alpha}}$ and take the subsequence $\Gamma_{\overline{\alpha}}$ in the form of the sets Γ_j $(j = 0,1,...,p)$ constructed by the KSR rules. We have to prove that sets Γ_j have the required properties of a defining sequence. Assuming the contrary carries out the proof.

Let us assume that Mullat property (1971) of a defining sequence is not satisfied. This means that for any set Γ_j there exists in the sequence of elements

$$\overline{\beta}_j = \left\langle \beta_j(1), \beta_j(2), ..., \right\rangle$$

an element $\beta_j(r)$ such that

$$\pi^{-}H_{v+r}(\beta_j(r)) \geq F_{-}(\Gamma_{j+1}) = u_{j+1} \tag{A3.1}$$

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[3] In the definition of $\overline{\alpha}_{+}$ sequence it is required that the following strict inequality be satisfied: $\pi^{+}H_i(\alpha_i) > F_{+}(\Gamma_{j+1})$, $j = 0,1,...,q-1$

Here v is the index number of the element μ_j selected in Stage 1 of the recursion step of the constructive routine of determination of $\overline{\alpha}$; in the vocabulary of notation used in Mullat (1976) we have $v = i(\Gamma_j)$.

According to the method of construction, the sequence $\overline{\beta}_j$ consists of sequences γ formed at the second stage of the j-th step of the constructive routine. Let M be a set in a sequence of sets $\Delta_{\overline{\alpha}}$ such that the first element $\alpha_{i(M)}$ of the set M in the constructed sequence $\overline{\alpha}$ is used at the second stage of the j-th step for constructing the sequence γ to which the element $\beta_j(r)$ belongs. This definition of M shows that $H_{v+r} \subseteq M$.

From the construction of the second stage of the j-th step and the principal property of monotonicity of ⊖ operations in the system we obtain the inequalities

$$\pi^{-}H_{v+r}(\beta_j(r)) \le \pi^{-}M(\beta_j(r)) \le \pi^{-}\Gamma_j(\mu_j) = u_j \qquad \text{(A3.2)}$$

By virtue of the above method of selection of the set Γ_{j+1} from the sequence of sets $\langle \Gamma_j \rangle$ and of the properties of a fixed sequence $\overline{\beta}_j$, we obtain at the j-th step

$$u_j = \pi^{-}\Gamma_j(\mu_{j+1}) < \pi^{-}\Gamma_{j+1}(\mu_{j+1}) = u_{j+1}, \qquad \text{(A3.3)}$$

where $j = 0,1,...,p-1$.

According to the rule of constructing of the sequence $\overline{\alpha}$, the function F_{-} reaches its value on the elements μ_j and μ_{j+1}. The elements μ_j and μ_{j+1} belong to the sets Γ_j and Γ_{j+1} respectively; therefore the inequalities (A.1) – (A3.3) are contradictory.

Thus our assumption is not true and Mullat Property of the defining sequence $\overline{\alpha}$ constructed by KSR rules has been proved.

Let as assume that Property b) does not hold, i.e., the last Γ_p of the sequence $\langle \Gamma_j \rangle$ contains a proper subset L such that

$$F_{-}(\Gamma_p) < F_{-}(L). \qquad \text{(A3.4)}$$

Let the element $\lambda \in L$, and suppose that it is the element with minimal ordinal number in $\overline{\alpha}$ belonging to L; moreover, let t denotes this number, i.e., $t = i(L)$, $\alpha_t = \lambda$. From the definition of t it follows that $L \subseteq H_t$.

Our analysis carried out above for the set H_{v+r} we repeat below for the set H_t. By analogy with the definition of the set M we define a set M' with the aid of the element λ and the sequence $\overline{\alpha}$.

The set M' is equated with the set of the sequence of sets $\Delta_{\overline{\alpha}}$ that begins with an element used in the formation of a set $\overline{\gamma}$ at the p-th step of the constructive routine such that $\lambda \in \overline{\gamma}$.

By analogy with derivative of (A3.2) we obtain

$$\pi^{-}H_t(\lambda) \leq \pi^{-}M'(\lambda) \geq \pi^{-}\Gamma_p(\mu_p) = u_p . \tag{A3.5}$$

Since $F_{-}(L) \leq \pi^{-}L(\lambda)$, it follows from (A3.4) and (A3.5) that $\pi^{-}H_t(\lambda) < \pi^{-}L(\lambda)$.

We noted above that $L \subseteq H_t$, by virtue of the monotonicity of $\ominus$ operations, it hence follows that

$$\pi^{-}L(\lambda) \leq \pi^{-}H_t(\lambda) .$$

The last two inequalities are contradictory, and hence Property b) of the defining sequence is satisfied.

Thus we have proved that the sequence $\overline{\alpha}$ constructed by the KSR rules is a defining sequence with respect to a collection of credential systems $\left\{\Pi^{-}H \,\middle|\, H \subseteq W\right\}$, and hence it can be denoted by $\overline{\alpha}_{-}$, whereas the sequence $\left\langle\Gamma_j\right\rangle$ obtained by a constructive routine can be denoted by $\Gamma^{-}_{\overline{\alpha}_{-}}$.

APPENDIX 4

Proof of Duality Theorem. Below we shall show that $\Gamma^{-}_{m} \subseteq W \setminus \Gamma^{+}_{n+1}$, if $F_{+}(\Gamma^{+}_{n}) = F_{-}(\Gamma^{-}_{m})$ (we omit a twice notation of + and − symbols; a promised above the + and − sign will not be used twice in notation. This rule has been applied also to Appendices 1 and 2.

Let us assume that there exists an element $\xi \in \Gamma_m^-$ and that $\xi \in \Gamma_{m+1}^-$, i.e., $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$. Hence follows that we have defined a credential $\pi\Gamma_{n+1}^+(\xi)$. According to the definition of the function F_+ the following inequality is true: $\pi\Gamma_{n+1}^+(\xi) \leq F(\Gamma_{n+1}^+)$.

For a defining sequence $\overline{\alpha}_+$ and for any $j = 0,1,...,q-1$ we have inequalities

$$F(\Gamma_{n+1}^+) < F(\Gamma_n^+). \tag{A4.1}$$

Let us consider an element $g \in \Gamma_n^+$ with the smallest index number in $\overline{\alpha}_+$. It follows from the definition of $\overline{\alpha}_+$ that

$$\pi\Gamma_n^+(g) > F(\Gamma_{n+1}^+).$$

The choice of element g is convenient because it permits the use of Mullat Property of a defining sequence (see Appendix 1), i.e., in this case the set Γ_n^+ is in the form of $H_t = \Gamma_n^+$. Since $F(\Gamma_n^+) \geq \pi\Gamma_n^+(g)$, we have proved (A4.1).

Since $\xi \in \Gamma_m^-$, it follows that we have defined a credential $\pi\Gamma_m^-(\xi)$. We have the following chain of inequalities:

$$F(\Gamma_m^-) \leq \pi\Gamma_m^-(\xi) \leq \pi^- W(\xi) = \pi^+ W(\xi) \leq \pi\Gamma_n^+(\xi).$$

Let us recall that for any element δ of the system W under consideration, we have in a) the relation $\pi^- W(\delta) = \pi^+ W(\delta)$. The first inequality follows from the definition of the function $F\_$, and the second inequality from the monotonicity of $\ominus$ operations. The equality follows from the definition of the functions π^- and π^+, whereas the last inequality follows from the monotonicity of $\ominus$ operations.

By virtue of (A4.1) and of the conditions of the theorem, we have also the following chain of inequalities:

$$\pi\Gamma_{n+1}^+(\xi) \leq F(\Gamma_{n+1}^+) < F(\Gamma_n^+) = F(\Gamma_m^-).$$

By supplementing this chain by the previous chain of inequalities, we hence obtain $\pi\Gamma_{n+1}^+(\xi) < \pi\Gamma_n^+(\xi)$. Since $\Gamma_{n+1}^+ \subset \Gamma_n^+$, it follows from the monotonicity of $\oplus$ operations that $\pi\Gamma_{n+1}^+(\xi) < \pi\Gamma_{n+1}^+(\xi)$. The logical step used for obtaining the last inequality is valid, and therefore the assumption that $\Gamma_m^- \subseteq W \setminus \Gamma_{n+1}^+$ is untrue.

In the same way we can prove the inclusion $\Gamma_n^+ \subseteq W \setminus \Gamma_{m+1}^-$. For this purpose it suffices to change the signs of the inequalities and (whenever necessary) to replace the set Γ_{n+1}^+ by Γ_{n+1}^-, and Γ_m^- by Γ_n^+.

If condition (3) of the theorem holds, it is not necessary to use (A4.1). In this case the proof will be similar, being based on the following chain of inequalities (The proof is based on assuming the contrary, so that $\Gamma_m^- \not\subset W \setminus \Gamma_n^+$, i.e., there exists, as it were, an element $\xi \in \Gamma_m^-$ and $\xi \in \Gamma_n^+$.): $\pi\Gamma_n^+(\xi) \le F(\Gamma_n^+) < F(\Gamma_m^-) \le\le \pi\Gamma_m^-(\xi) \le \pi^- W(\xi) \le \pi\Gamma_n^+(\xi)$.

The first inequality follows from the definition of $F(\Gamma_n^+)$, the second follows from Condition (3) of the theorem, and the third from the definition of $F(\Gamma_m^-)$. The last two relations express the properties of monotonic systems. Hence in this case we have under Condition (3) also $\pi\Gamma_n^+(\xi) < \pi\Gamma_n^+(\xi)$. This completes the proof of the theorem. ■ Now follows several corollaries of Theorem 2.

Corollary 1. If for $n = \overline{0, q}$ the defining sequence is $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) > F(\Gamma_n^+)$. Thus kernel $K^{\oplus}$ will belong to the set $W \setminus \Gamma_n^+$. Indeed, since a definable set is also kernel, it follows that $F_-(H) \le F(\Gamma_p^-)$, $m = 0,1,...,p$, and hence (in any case) if $m = p$, and n is selected on the basis of the condition of the corollary, then $F(\Gamma_n^+) < F(\Gamma_p^-)$. By virtue of the theorem, we therefore obtain the assertion of the corollary.

Corollary 2. If for $n = 0,1,...,q-1$ of a defining sequence $\overline{\alpha}_+$ there exists a subset $H \subseteq W \setminus \Gamma_n^+$ such that $F_-(H) = F(\Gamma_n^+)$, then the kernel $K^{\oplus}$ will belong to the set $W \setminus \Gamma_{n+1}^+$.

The proof follows directly from Corollary 1, by virtue of (A4.1).

Corollary 3. If for $m = 0,1,...,p$ of a defining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) < F(\Gamma_m^-)$ then the kernel $K^{\ominus}$ will belong to the set $W \setminus \Gamma_m^-$. The proof of Corollary 3 is entirely similar to that of Corollary 1. It is only necessary to change the signs of the inequalities and replace the set Γ_n^+ by Γ_m^-.

Corollary 4. If for $m = 0,1,...,p-1$ of a defining sequence $\overline{\alpha}_-$ there exists a subset $H \subseteq W \setminus \Gamma_m^-$ such that $F_+(H) = F(\Gamma_m^-)$, then the kernel $K^{\ominus}$ will belong to the set $W \setminus \Gamma_{m+1}^-$.

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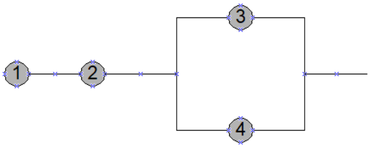
[*i] The name "Monotonic System" at that moment in the past was the best match for our scheme. However, this name "Monotone System" was already occupied in "Reliability Theory" unknown to the author. Below we reproduce a fragment of a "monotone system" concept different from ours in lines of Sheldon M. Ross "Introduction to Probability Models", Fourth Ed., Academic Press, Inc., pp. 406-407.

Example

(A four-Component Structure):



Consider a system consisting of four components, and suppose that the system functions if and only if components 1 and 2 both function and at least one of components 3 and 4 function. Its structure function is given by

$$\phi(x) = x_1 \cdot x_2 \cdot \max\left(x_3, x_4\right).$$

Pictorially, the system is shown in Figure. A useful identity, easily checked, is that for binary variables, (a binary variable is one which assumes either the value 0 or 1) x_i, $i = 1,\ldots,n$,

$\max\left(x_1,\ldots,x_n\right) = 1 - \prod_{i=1}\left(1 - x_i\right)$. When $n = 2$, this yields

$$\max\left(x_1, x_2\right) = 1 - \left(1 - x_1\right)\cdot\left(1 - x_2\right) = x_1 + x_2 - x_1 \cdot x_2 .$$

Hence, the structure function in the above example may be written as

$$\phi(x) = x_1 \cdot x_2 \cdot \left(x_3 + x_4 - x_3 \cdot x_4\right)$$

It is natural to assume that replacing a failed component by a functioning one never lead to a deterioration of the system. In other words, it is natural to assume that the structure function $\phi(x)$ is an increasing function of x, that is, if $x_i \le y_i$, $i = 1,\ldots,n$, then $\phi(x) \le \phi(y)$. Such an assumption shall be made in this chapter and the system will be called *monotone*.

КОНТРМОНОТОННЫЕ СИСТЕМЫ В АНАЛИЗЕ СТРУКТУРЫ МНОГОМЕРНЫХ РАСПРЕДЕЛЕНИЙ

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Ставится задача выделения сгущений в многомерном пространстве измерений на основе векторного критерия качества. Для поиска решений используется специальная параметризация функций, при которой с увеличением значений параметров значение функций во всей области определения уменьшается.

1. Введение

Анализ структуры распределения плотности измерений в n-мерном пространстве — традиционная тематика исследований в таких прикладных областях, как планирование эксперимента [1], анализ изображений [2], анализ принятия решений [3], распознавание образов [4] и т. д.

На содержательном уровне структура распределения обычно представляется совокупностью сгущений, которые иногда называются также модами [5]. Анализ подобной структуры, если не явно, то косвенно, почти всегда сводится к вариационной задаче оптимизации — максимизации какого-либо скалярного критерия качества, оценивающего выделяемые сгущения. Вместо скалярного в данной работе используется векторный критерий, а в основу понятия оптимальности положено так называемое равновесное состояние в смысле Нэша [6].

Правомерность подхода с позиции состояния равновесия к анализу структуры распределения плотности измерений в n-мерном пространстве объясняется тем, что здесь по существу происходит замена одной многомерной многими «почти одномерными» задачами в проекциях на оси координат. На каждой оси сгущение выделяется так, что оси «увязываются» между собой строго определенным образом: сгущение на данной оси нельзя «сдвинуть в сторону» без какого-либо ухудшения сгущения на других осях в смысле рассматриваемого критерия при условии, что эти другие уже фиксированы.

Преимущество предложенного подхода не исчерпывается указанной «технической подробностью» замены одного многомерного пространства одномерными проекциями. Дело в том, что состояние равновесия, выделяемое при помощи используемого векторного критерия, параметризуется так называемыми порогами, которые задают уровни плотности сгущений. По крайней мере в некоторых частных случаях состояние равновесия как решение системы уравнений можно аналитически выразить в форме функций порогов и тем самым полностью обозреть выделяемые сгущения в спектре возможных уровней плотности.

Counter Monotonic Systems in the Analysis of the Structure of Multivariate Distributions

Abstract. In the context provided, a multivariate space refers to a space where data points are represented by multiple variables or dimensions. For instance, if you're measuring several characteristics of an object or a process, each characteristic would represent a dimension in this multivariate space. Now, the problem being discussed is about distinguishing condensations within this multivariate space. Condensations here likely refer to clusters or groupings of data points that share similar characteristics or patterns. The approach described involves using a qualitative vector criterion, which means using some sort of criteria or rules based on vectors (which represent directions or magnitudes in this multivariate space) to distinguish these condensations. This criterion could be based on factors such as distances between points, angles between vectors, or other mathematical relationships. The solution proposed involves parameterizing functions in a special way. Parameterization means expressing functions in terms of parameters, which are variables that can take on different values. These functions are designed such that their values decrease across all regions of the defined multivariate space inversely proportional to the values of the parameters. In simpler terms, this means that the functions are structured in a way that they decrease in value as the parameters increase, and this decrease happens consistently across all regions of the multivariate space. This parameterization likely helps in identifying and distinguishing different condensations or clusters within the multivariate data by providing a systematic way to evaluate their characteristics.

Keywords: monotonic; distributions; equilibrium; cluster

1. INTRODUCTION

The analysis of the structure of the probability density function of measurements in an n-dimensional space is a traditional topic of investigation in such applied fields as experimental design (Finney, 1964), image analysis (Rosenfeld, 1969), the analysis of decision making (Fishburn, 1970), pattern recognition (Aizerman et al, 1970), etc…

At a conceptual level, a distribution structure is usually represented by a set of data clusters, sometimes called modes (Zagoruiko and Zaslavskaya, 1968). The analysis of such a structure is indirectly, if not explicitly, usually reduced to the problem of variational optimization. That is, maximizing some scalar performance metrics that characterize the identified clusters. Instead of a scalar performance index, in this article we use a vector index and base the concept of optimality on the so-called Nash equilibrium state (Owen, 1968).

Approaching the analysis of the structure of a measurement density function in n-dimensional space, our standpoint is the equilibrium state concept. It is justified by the fact that, essentially, what happens, is the replacement here of a single multidimensional problem by many "almost one-dimensional" problems in projections onto the coordinate axes. On each axis a cluster is delineated in such a way as to "bind" the axes together in a rigorously defined way. So, exposed to such a "bind" the cluster on a given axis cannot be "nudged" without in some measure deteriorating itself on the other axes in the sense of investigated performance index, subject to the condition that these others are fixed.

The superiority of the proposed approach is not restricted to the indicated "technical detail" of replacing one multidimensional space by one-dimensional projections. Indeed, an equilibrium state identified by means of the given vector index is parameterized by so-called thresholds, which satisfy the density levels of the clusters. In certain special cases, at any rate, an equilibrium state as the solution of a system of equations can be expressed analytically in the form of threshold functions, whereupon the identified clusters can be fully scanned in the spectrum of possible density levels.

The proposed theory for the identification of clusters of the probability density of measurements in n-dimensional space is set forth in two parts. In the first part (sec.2) the theory is not taken beyond the scope of customary multivariate functions and it concludes with a system equations, namely the system whose solution in the form of threshold functions makes it possible to scan the identified clusters. In the second part (Sec.3) the theory now rests on a more abundant class of measurable functions specified by the class of sets represented on the coordinate axes by at most countable set of unions or intersections of segments. Overall the construction described in this part is so-called counter-monotonic system; actually, the first part on multi-parameter counter-monotonic systems is also discussed in these terms (special case).

The fundamental result of the second part does not differ, in any way, from the form of the system of equations in the first part; the essential difference is in the space of admissible solutions. Whereas in the system of equations of the first part the solution is a numerical vector, in the second part it is a set of measurable sets containing the sought-after measurable density clusters. As the solution of the system of equations, the set of measurable sets serves as a fixed point of special kind mapping of subsets of multidimensional space. This particular feature is utilized in an iterative solving procedure.

2. COUNTER-MONOTONIC SYSTEMS OVER A FAMILY OF PARAMETERS

Here a monotonic system represents first a one-parameter and then a multi-parameter family of functions defined on real axis. This type of representation is a special case of a more general monotonic system described in the next section.

We consider a one-parameter family of functions $\pi(x;h)$ defined on the real axis, where h is a parameter. For definiteness, we assume that an individual copy π of the indicated family is a function that can be taken as an integral with respect to x and differentiable with respect to h. The family of functions π is said to be counter-monotonic if it obeys the following condition: for any pair of quantities ℓ and g such that $\ell \le g$ the inequality

$$\pi(x;\ell) \ge \pi(x;g) \text{ holds for any } x .$$

The specification of a multi-parameter family of functions π is reducible to the following scheme. We replace the one function π by a vector function $\pi = \langle \pi_1, \pi_2, ..., \pi_n \rangle$, each j-th component of which is a copy of the function depending now on n parameters $h_1, h_2, ..., h_n$, i.e., $\pi_j = \pi_j(x; h_1, h_2, ..., h_n)$. We wrote down the counter-monotonicity condition for any pair of vectors $\ell = \langle \ell_1, \ell_2, ..., \ell_n \rangle$ and $g = \langle g_1, g_2, ..., g_n \rangle$ such that $\ell_k \le g_k$, $k = (1,2,...,n)$ in the form of n inequalities $\pi_j(x; \ell_1, \ell_2, ..., \ell_n) \ge \pi_j(x; g_1, g_2, ...g_n)$. We also note that this condition rigorously associates with family of vector functions a component-wise partial ordering of vector parameters.

We give special attention to the case of a so-called de-coupled multi-parameter family of functions π. The family π arrange de-coupled functions if the j-th component of the vector function π does not depend on the j-th component of the vector of parameters h, i.e., on h_j. Therefore, the function π of a de-coupled multi-parameter family is written in the form $\pi_j(x, h_1, ..., h_{j-1}, h_{j+1}, ..., h_n)$ $(j = 1, ..., n)$.

We now return to the original problem of analyzing a multi-modal empirical distribution in multidimensional space. We first investigate the case of one axis probability distribution of only one random variable (univariate distribution).

Let $p(x)$ be the probability density function of points in the *x*-axis. For the counter-monotonic family π we can choose, for example, the functions $\pi(x; h) = p(x)^h$. It is easy verified that the counter-monotonicity condition is satisfied.

We consider the following variational problem. With respect to an externally specified threshold u° $(0 \le u^\circ \le 1)$ let it be necessary to maximize the functional

$$\Pi(h) = \int_{-h}^{+h} [\pi(x; h) - u^\circ] \cdot dx .$$

It is clear that for small h the quantity $\Pi(h)$ will be small because of the narrow interval of integration, while for the large h it will be small by the counter-monotonicity condition. Consequently, the value of $\max_h \Pi(h)$ will necessarily be attained for certain finite nonzero h°.

It is easy to see that if $p(x)$ is a unique function of the density of modes with zero mathematical expectation, then maximizing the functional $\Pi(h)$ implies identifying the interval on the axis corresponding to the density $p(x)$ concentration. But if $p(x)$ has a more complex form, then the maximum $\Pi(h)$ determines the interval in which the "essential part" in a certain sense of the density function $p(x)$ is concentrated.

Directly from the form of the function $\Pi(\mathrm{h})$ we derive necessary condition for the local maximum (the zero equation of the derivative with respect to h : $\frac{\delta}{\delta \mathrm{h}}\Pi(\mathrm{h}) = 0$: or, in expanded form, the equation

$$\pi(-\mathrm{h};\mathrm{h}) + \pi(\mathrm{h};\mathrm{h}) + \int_{-\mathrm{h}}^{+\mathrm{h}} \tfrac{\partial}{\partial y}\pi(\mathrm{x};\mathrm{y})\,|_{\mathrm{y=h}}\,\mathrm{dx} = 2\mathrm{u}^{\mathrm{o}}. \qquad (1)$$

The root of the given equation will necessarily contain one at which $\Pi(\mathrm{h})$ attains a global maximum. We have thus done with the problem: we found the central cluster points of the density function on one axis in terms of a counter-monotonic family of functions.

To find the central clusters of a multivariate distribution in n-dimensional space we invoke the notion of a multi-parameter counter-monotonic family of functions π. Let the family of functions π in vector form be written, say, in the form $\pi_{\mathrm{j}}(\mathrm{x};\mathrm{h}_1,\ldots,\mathrm{h}_{\mathrm{n}}) = \mathrm{p}_{\mathrm{j}}(\mathrm{x})^{\mathrm{h}}$, where $\mathrm{h} = \sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{h}_{\mathrm{k}}$, and $\mathrm{p}_{\mathrm{j}}(\mathrm{x})$ is a projection of the multivariate distribution on the axis j-th axis. In the stated sense the goodness of the delineated central cluster is evaluated by the multivariate (vector) performance index $\Pi = \left\langle \Pi_1,\ldots,\Pi_{\mathrm{n}} \right\rangle$, where

$$\Pi_{\mathrm{j}}(\mathrm{h}_1,\mathrm{h}_2,\ldots,\mathrm{h}_{\mathrm{n}}) = \int_{-\mathrm{h}_{\mathrm{j}}}^{\mathrm{h}_{\mathrm{j}}} [\pi_{\mathrm{j}}(\mathrm{x};\mathrm{h}_1,\ldots,\mathrm{h}_{\mathrm{n}}) - \mathrm{u}_{\mathrm{j}}]\cdot \mathrm{dx} \qquad (2)$$

and u_{j} is the component of the corresponding externally specified multidimensional threshold vector u: $\mathrm{u} = \left\langle \mathrm{u}_1,\mathrm{u}_2,\ldots,\mathrm{u}_{\mathrm{n}} \right\rangle$. As in the one-dimensional case, of course, it is meaningful to use the given functional only distributions $\mathrm{p}_{\mathrm{j}}(\mathrm{x})$ with zero expectation.

Once the goodness of a delineated cluster has been evaluated by the vector index, it must be decided, based on standard (Becker and McClintock, 1967) vector optimization principles, what is an acceptable cluster. In this connection it is desirable to indicate simultaneously a procedure for finding an extremal point in the space of parameters. It turns out that for so-called Nash-optimal Equilibrium State there is a simple technique for finding solutions at least in decoupled family of counter-monotonic functions π.

En equilibrium situation (Nash point) in the parameter space $\mathrm{h} = \left\langle \mathrm{h}_1,\ldots,\mathrm{h}_{\mathrm{n}} \right\rangle$ with indices Π_{j} is defined as a point $\mathrm{h}^* = \left\langle \mathrm{h}_1^*,\mathrm{h}_2^*,\ldots,\mathrm{h}_{\mathrm{n}}^* \right\rangle$ such that for every j the inequality

$$\Pi_{\mathrm{j}}(\mathrm{h}_1^*,\ldots,\mathrm{h}_{\mathrm{j-1}}^*,\mathrm{h}_{\mathrm{j}},\mathrm{h}_{\mathrm{j+1}}^*,\ldots,\mathrm{h}_{\mathrm{n}}^*) \le \Pi_{\mathrm{j}}(\mathrm{h}_1^*,\ldots,\mathrm{h}_{\mathrm{j}}^*,\ldots,\mathrm{h}_{\mathrm{n}}^*)$$

holds for any value of h_j. In other words, if there are no sensible bases in the sense of index Π_j on the one (j-th) axis, then the equilibrium situation is shifted with respect to the parameter h_j, subject to the condition that the quantities h_k^*, $k \neq j$, are fixed on all other axes.

Clearly, a necessary condition at a Nash point in the parameter space (as in the one-dimensional case) is that the partial derivatives tend to zero, i.e., the n equalities $\frac{\partial}{\partial h_j}\Pi_j(h_1^*,...,h_n^*) = 0$ must hold. The sufficient condition comprises the n inequalities $\frac{\partial^2}{\partial h_j^2}\Pi_j(h_1^*,...,h_n^*) \leq 0$.

An essential issue here, however, is the fact that the necessary condition (equalities) acquires a simpler form for de-coupled family of counter-monotonic functions than in the general case. Thus, by the decoupling of the family π the partial derivative $\frac{\partial \Pi_j}{\partial h_j}$ is identically zero, and the system of equations, see (1) by analogy, with respect to the sought-after point h^* is reducible to the form

$$\pi_j(-h_j;h_1,...,h_{j-1},h_{j+1},...h_n) + \pi_j(h_j;h_1,...,h_{j-1},h_{j+1},...h_n) = 2u_j \qquad (3)$$

Now the sufficient condition is satisfied automatically for any solution h^* of Eqs. (3).

In conclusion we write out the system of equations for two special cases of a de-coupled family of counter-monotonic functions π.

1. Let $\pi_j(x;h_1,...,h_{j-1},h_{j+1},...,h_n) = p_j(x)^{\sigma-h_j}$, where $\sigma = h_1 + h_2 + ... + h_n$. The system of equations (3) is reducible to the form $p_j(-h_j)^{\sigma-h_j} + p_j(h_j)^{\sigma-h_j} = 2u_j$, $j = \overline{1,n}$.
2. Let the role of $\pi_j(x;h_1,...,h_{j-1},h_{j+1},...,h_n)$ be taken by the $p_1(x)^{h_1}...p_{j-1}(x)^{h_{j-1}}p_{j+1}(x)^{h_{j+1}}...p_n(x)^{h_n}$ function.

The system of equations (3) for finding a solution, i.e., an equilibrium situation (Nash point) h^*, is written

$$p(-h_j)/p_j(-h_j)^{h_j} + p(h_j)/p_j(h_j)^{h_j} = 2u_j \quad (j = \overline{1,n}),$$

where $p(x) = p_1(x)^{h_1}p_2(x)^{h_2}...p_n(x)^{h_n}$ is the product of univariate density functions.

We conclude this section with an important observation affecting the vector of thresholds $u =< u_1, u_2, ..., u_n >$. By straightforward reasoning we infer that each component h_j^* of the equilibrium situation h^* is a function of thresholds and h^* can be represented by a vector function of thresholds in the form $h_j^* = h_j^*(u_1, u_2, ..., u_n)$. If the solution of the system of equations (3) can be expressed analytically, then prolific possibilities are afforded for scanning the equilibrium situations in the parameter space and, accordingly, selecting an "acceptable" cluster in the spectrum of existing densities of measurements in a multidimensional space of thresholds. A similar approach can be used when solutions of Eqs. (3) are sought by numerical methods.

3. COUNTER-MONOTONIC SYSTEMS OVER A FAMILY OF SEGMENTS

A multi-parameter family of counter-monotonic functions used for the analysis of multivariate distributions, unfortunately, has one substantial drawback. Generally speaking, there is no way to guarantee the identification of homogeneous distribution clusters in projection onto the j-th axis, because the segment $[-h_j, h_j]$ can contain several distinct modes. On the other hand, it is sometimes desirable to identify modes by merely indicating a family of segments containing each mode separately. The construction proposed below enlarges the possibilities for the solution of such a problem by augmenting the counter-monotonic systems of the proceeding section in natural way.

Thus, on real axis we consider subsets represented by at most countable set of operations of union, intersection, and difference of segments. The class of all such subsets is denoted by B, and each representative subset by $H \in B$ (which we call a B set) is distinguished from like sets by length μ (by measure zero). A set L is congruent with G $(G = L)$ if the measure of the symmetric difference $G \Delta L$ is equal to zero $(\mu G \Delta L = 0)$; a set L is contained in G $(L \subseteq G)$ with respect to measure μ if $\mu G \setminus L = 0$. A measure on the real axis, being an additive function of sets (the length), is determined by taking to the limit the length of the sets in the set of unions, intersections, and differences of segments forming the B set. Then set-theoretic operations over B sets will be understood to mean up to measure zero. By convention, all B sets of measure zero are indistinguishable.

We associate with every B set H a nonnegative function $\pi(x;H)$, which is Borel measurable (or simply measurable) and whose domain of definition is on the real axis.[1] In other words, in contrast with the one-parameter family of counter-monotonic functions of the preceding section, the parameter h is now generalized, namely, it is extended to the B set H. As before, we say that a family of measurable functions π is counter-monotonic if it obeys the following condition: for any pair of sets L and G such that $L \subseteq G$ the inequality

$$\pi(x;L) \geq \pi(x;G)$$

holds for any x.

The scheme of specification of a multi-parameter family of functions is analogous to the previous situation. In place of a scalar function π we now specify a vector function $\pi = \langle \pi_1, \pi_2, ..., \pi_n \rangle$, each j-th component of which is a copy of a function depending at the outset on n parameters $\langle H_1, H_2, ..., H_n \rangle$, i.e., $\pi_j = \pi_j(x; H_1, H_2, ..., H_n)$ (B sets). Again, the counter-monotonicity condition is reducible to the statement that for any pair of vectors (ordered sets of B sets) of the form $L = \langle L_1, ..., L_n \rangle$ and $G = \langle G_1, ..., G_n \rangle$ such that $L_k \subseteq G_k$ $(k = 1,2,...,n)$, the following n inequalities are satisfied:[2]

$$\pi_j(x; L_1, ..., L_n) \geq \pi_j(x; G_1, ..., G_n).$$

These inequalities associate a partial ordering of sets of B sets with a family of vector functions π in a rigorously defined way.

In the case of a de-coupled family of counter-monotonic functions, where the j-th component of a copy of the vector function π does not depend on the parameter H_j, or B set on the j-th axis of definition of the function π_j, this component π_j of the vector function π is written $\pi_j = \pi_j(x; H_1, H_2, ..., H_n)$.

Following again the order of discussion of Sec.2, we now consider the original problem of analyzing the structure of a multi-modal empirical distribution in a multidimensional space. We first investigate the case of a one-dimensional (univariate) distribution.

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[1] A function $\pi(x;H)$ is Borel measurable if for any numerical threshold u° the set of all *x* of the real scale for which $\pi(x;H) > u^{\circ}$ is measurable:

$\{x : \pi(x;H) > u^{\circ}\}$ is B set.

[2] Here x is a point on the j-th axis. This is tacitly understood everywhere.

Let $p(x)$ be the density function of points on the *x*-axis. In the role of the counter-monotonic family of functions π, we adopt functions of the form $\pi(x;H) = p(x)^{F(H)}$, where $F(H) = \int_H p(x)dx$ is the probability of a random variable occurring in a B set under the probability density function $p(x)$. It is clear that the counter-monotonicity condition is satisfied.

We consider the following variational problem. Given the externally specified threshold u^o $(0 \le u^o \le 1)$, maximize the functional

$$\Pi(H) = \int_H [\pi(x;H) - u^o]d\mu .$$

The integral here is understood in the Lebegue sense with respect to measure μ, where μ, as mentioned before, is the length of the B set on the x axis.

Clearly, the quantity $\Pi(H)$ as a function of the length μ (measure of set H) increases first and then, as $\mu H \to \infty$, reverts to zero by the counter-monotonicity condition on the family of functions π. Therefore, the value of $\max_H \Pi(H)$ will necessary is attained on a certain B set of finite measure μ (see the analogous assertion in Sec.2).

It is impossible in the same simple way to deduce directly from the form of the functional $\Pi(H)$ any maximum condition comparable with the like condition of the preceding section (Eq.1). To do so would require elaborating the notation of a "virtual translation" from a B set H to a set $\tilde{H}$ similar to it in some sense, in such a way as to establish the necessary maximum condition. These circumstances exclude the case of a univariate distribution from further consideration. Nonetheless, as will be shown presently, for multivariate distribution there are means for finding a B set that will maximize the function $\Pi(H)$ at least in the case of a de-coupled family of counter-monotonic functions.

As in the preceding section, we evaluate the goodness of an identified central cluster by the multivariate (vector) performance index

$$\Pi = \langle \Pi_1, \Pi_2, ..., \Pi_n \rangle : \qquad \Pi_j(H_1, H_2, ..., H_n) = \int_{H_j} [\pi(x; H_1, ..., H_n) - u_j]d\mu ,$$

where u_j is the coordinate of the corresponding multidimensional vector of thresholds u, specified externally: $u = \langle u_1, u_2, ..., u_n \rangle$.

At this point we call attention to the fact that, in contrast with the analogous multivariate index of Sec.2, the given functional now has significance for an arbitrary distribution, rather than only for the centered condition of "*zero-valued-ness*" of the expectation. We again look for the required cluster in multidimensional space as an equilibrium situation according to the vector index $\Pi = \langle \Pi_1, \Pi_2, ..., \Pi_n \rangle$. We regard a cluster as a set of B sets $H^* = \langle H_1^*, H_2^*, ..., H_n^* \rangle$ such that the following inequity holds for every j:

$$\Pi_j(H_1^*,...,H_{j-1}^*,H_j,H_{j+1}^*,...,H_n^*) \le \Pi_j(H_1^*,...,H_j^*,...,H_n^*) \quad (j=\overline{1,n}).$$

In a de-coupled family of counter-monotonic functions it is feasible, as in the multi-parameter case, see Eq. (3), to find an equilibrium situation. Equilibrium situations are sought to be a special technique of mappings of B sets onto real axes.

We define the following type of mappings of B sets onto real axes:

$$V_j(H_j) = \{x : \pi_j(x; H_j) > u_j\},$$

where u_j is the threshold involved in the expression for the functional Π_j $(j=\overline{1,n})$. Thus defined, n such mappings are uniquely expressible in the vector form

$$V(H) = \{x : \pi(x; H) > u\}.$$

Here $H = H_1 \times H_2 \times ... \times H_n$ denotes the direct product of sets H_j. We define a fixed point of the mapping $V(H)$ as a set H^* for which the equality $H^* = V(H^*)$ holds.

Theorem 1. *For a de-coupled family of counter-monotonic functions* π, *a fixed point of the mapping* $V(H)$ *generates an equilibrium situation according to the vector index* $\Pi = \langle \Pi_1, \Pi_2, ..., \Pi_n \rangle$.

The proof of the theorem is simple. Thus, because π_j is independent of the parameter H_j, the form of the function $\pi_j(x; H_1^*,...,H_{j-1}^*,H_{j+1}^*,...,H_n^*)$ does not depend on H_j. Also, the set $H^* = H_1^* \times H_2^* \times ... \times H_n^*$ in projection onto the j-th axis intersects the set H_j^* consisting exclusively of all points x for which

$\pi_j(x;H_j^*) > u_j$: $H_j^* = \{x : \pi_j(x;H_j^*) > u_j\}$. It is immediately apparent that for any H_j distinct from H_j^* the value of the functional $\Pi_j(H_1^*,...,H_{j-1}^*,H_j,H_{j+1}^*,...,H_n^*)$ for immovable sets H_k^* $(k \neq j)$ cannot be anything but smaller than the quantity $\Pi_j(H_1^*,...,H_{j-1}^*,H_j^*,H_{j+1}^*,...,H_n^*)$.

It is important, therefore, to find the fixed points of the constructed mapping of B sets.

4. METHODS OF FINDING EQUILIBRIUM STATE FOR DE-COUPLED FAMILIES OF COUNTER-MONOTONIC FUNCTIONS

The ensuing discussion rests heavily on the counter-monotonicity property of a function π. To facilitate comprehension of the formulations and propositions we use the language of diagrams reflecting the structure of the relations involved in the constructed mappings of B sets, in particular the symbol $\to$ denoting the relation “set X_1 is nested in set X_2 $(X_1 \subseteq X_2)$: $X_1 \to X_2$.

All diagrams of the relations between B sets are based on the following proposition: the relation $X_1 \to X_2$ (as a consequence of the counter-monotonicity condition on π) implies that $V(X_1) \leftarrow V(X_2)$.

Now let the mapping V be applied to the original space W of axes on which the functions π_j $(j = \overline{1,n})$ are defined. After the image $V(W)$ has been obtained, we again apply the mapping V with the B set $V(W)$ as its inverse image, i.e., we consider the image $V^2(W)$, and so on. In this way we construct a chain of B sets W, $V(W)$, $V^2(W)$,..., which we call the central series of the counter-monotonic system.

The following diagram of nestling of B sets of the central series is inferred directly from the above stated proposition:

$$\begin{tikzcd}
W & V(W) \arrow[l] \arrow[r] & V^2(W) \arrow[ll, bend left] & V^3(W) \arrow[l] \arrow[r] \arrow[ll, bend right] & V^4(W) \arrow[ll, bend left] & V^5(W) \arrow[l] \arrow[r] & \dots \arrow[l, bend right] \arrow[ll, bend left]
\end{tikzcd}$$

It is evident from the diagram that there exist in the central series two monotonic chains of B sets: one shrinking and one growing. The monotonically shrinking chain of B sets comprises the sequence $V^2(W) \leftarrow V^4(W) \leftarrow ...$ with even powers of the mapping V. The monotonically growing chain is the sequence $V(W) \to V^3(W) \to V^5(W) \to ...$ with odd powers of V.

It is well known (Shilov and Gurevich, 1967) that monotonically decreasing (increasing) chains in the class of B sets always converge in the limit of sets of the same class. For example, the limit of the sets $V^{2k}(W)$ with even powers is the intersection $L=\bigcap_{k=1}^{\infty}V^{2k}(W)$, and the limit of sets $V^{2k-1}(W)$ with odd powers is the union $G=\bigcup_{k=1}^{\infty}V^{2k-1}(W)$.

Theorem 2. *For the central series of a counter-monotonic system the nesting* $L\subseteq G$ *of the limiting* B set L *of even powers of the mapping* $V(X)$ *in the limiting* B *set* G *of odd powers of the same mapping is always true.*

The theorem follows at once from the diagram of nestlings of the central series.

We now resume our at the moment interrupted discussion of the problem of finding a fixed point of a mapping of B sets, such point generating an equilibrium situation according to the vector index Π (Theorem 1). In counter-monotonic systems, as a rule, the strict nesting $L\subset G$ of limiting B sets holds in the statement of Theorem 2. The equality $L=G$ would imply convergence of the central series in the limit to a single set, namely a fixed pint. In view of the exceptional status of the equality $L=G$, we give a “more refined” procedure, which automatically in the number of cases of practical importance yields the desired result, a solution of the equation $X=V(X)$.

Procedure for Solving the Equation $X=V(X)$. A chain of B sets $H_0,H_1,...,$ is generated recursively according to the following rule. Let the set H_k (where H_0 is any B set of finite measure) be already generated in the chain. We use the mapping $V(X)$ to transform the following B sets:

$$V\{V^2(H_k)\bigcup V(H_k)\}, \qquad V\{V(H_k)\bigcap H_k\},$$
$$V\{V(H_k)\bigcup H_k\}, \qquad V\{V^2(H_k)\bigcap V(H_k)\}.$$

We denote these sets by L_k^2, G_k, L_k, G_k^2 accordingly. By the counter-monotonicity of the family of functions π it turns out that L_k^2 is a subset of G_k and that L_k is a subset of G_k^2. Picking any A_k based on the condition $L_k^2\subset A_k\subset G_k$, and then B_k from the analogous condition $L_k\subset B_k\subset G_k^2$, we put the set H_{k+1} following H_k in the constructed series of B sets equal to $A_k\bigcup B_k$: $H_k=A_k\bigcup B_k$. The sets A_k and B_k can be chosen, for example, according to mapping rules in the class of B sets, namely,

$$A_k=\{x:½[\pi(x;L_k^2)+\pi(x;G_k)]>u\}, B_k=\{x:½[\pi(x;L_k)+\pi(x;G_k^2)]>u\}.$$

The conditions imposed on A_k and B_k are satisfied in this case.

Theorem 3. For the series of sets $V(H_k)$ to contain the limiting set $V(H^*)$ as $k \to \infty$, which would be a solution of the equation $X = V(X)$, the following two conditions are sufficient:

a) $\lim_{k\to\infty} \mu G_k \setminus L_k^2 = 0$,

b) $\lim_{k\to\infty} \mu G_k^2 \setminus L_k = 0$.

The plan of the proof is quickly grasped in the following nesting diagrams, which are consequences of the counter-monotonicity property of the functions π, i.e.,

I. $V^2(H_k) \leftarrow L_k^2 \rightarrow G_k \leftarrow V(H_k)$,

II. $V(H_k) \leftarrow L_k \rightarrow G_k^2 \leftarrow V^2(H_k)$.

Diagrams **I** and **II** imply the validity of the two chains:

1) $V^2(H_k) \setminus V(H_k) \subseteq V^2(H_k) \setminus G_k \subseteq L_k^2 \setminus G_k$,

2) $V(H_k) \setminus V^2(H_k) \subseteq V(H_k) \setminus G_k^2 \subseteq L_k \setminus G_k^2$.

The first chain implies that for the limiting set H^* of the series $H_0, H_1, \ldots$, the equality $\mu V^2(H_k) \setminus V(H^*) = 0$ holds, i.e., $V(H^*) \subset V^2(H^*)$; the second chain implies the opposite relation: $V^2(H^*) \subseteq V(H^*)$. Consequently, $V(H^*)$ is the solution of the equation $X = V(X)$: $V(H^*) = V(V(H^*))$. Of course, the conditions of the theorem are sufficient for the existence of a solution of the equation $X = V(X)$, and their absence does not in any way negate some other solving technique, provided that solutions exist in general. The possibility that solution H^* of the equation $X = V(X)$ do not exist should certainly not be dismissed.

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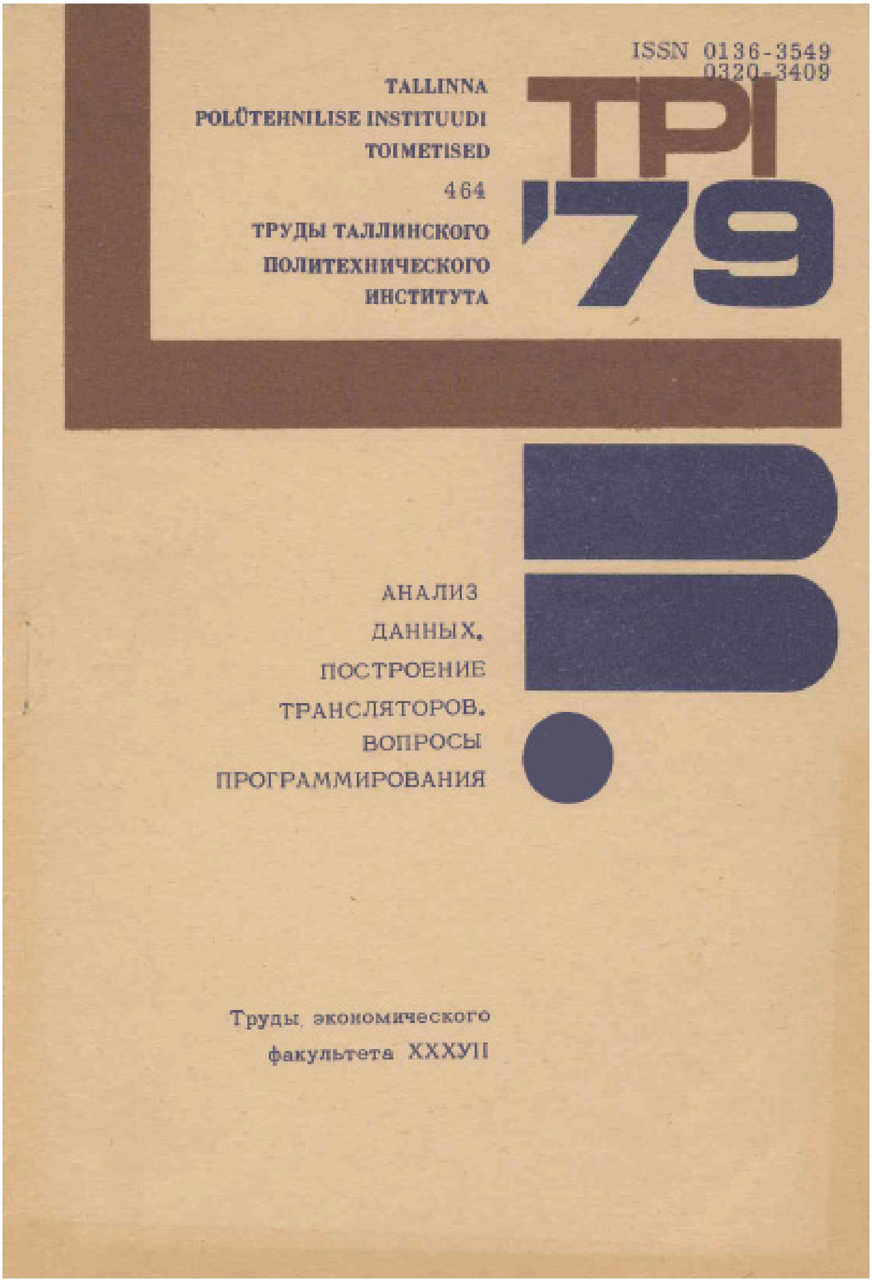

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ТРУДЫ ТАЛЛИНСКОГО
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Труды экономического
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Application of Monotone Systems to the Study of the Structure of Markov Chains

Abstract. Markov Chains are mathematical models used to describe a sequence of possible events in which the probability of each event depends only on the state attained in the previous event. In the context of time series analysis, Markov Chains can be utilized to model the transitions between different states of a dynamic system over time. This approach can help understand the probabilistic behavior of the system and make predictions about future states based on the current state. By analyzing the chain structure, one can examine the transition probabilities between states and infer how the system evolves over time, providing insights into the underlying dynamics of the time series scenario. In the context of dynamic systems of time series scenarios, the method described involves transforming the Markov chain into a monotonic system. This transformation simplifies the analysis by allowing for the separation of kernels from the transformed chain. By separating the kernels, one can focus on understanding the transition probabilities between different states, which is crucial for predicting the future states of the system. Essentially, this approach helps in dissecting the complex dynamics of the time series scenario and allows for a more focused examination of the probabilistic behavior inherent in the Markov chain model.

Keywords: Markov chain; communication line; network; transition matrix; kernel

1. INTRODUCTION

In the work presented here, the theory of monotonic systems developed in an earlier publication (Mullat, a) 1976) is applied to the Markov chains. In the study of Markov chains the interest stems from the fact that it is convenient to interpret a special class of absorbing chains as monotonic systems. On the other hand, it also provides a meaningful way of illustrating the main properties of monotonic systems, as shown here using an example based on communication networks. In the original paper (translated from Russian), Mullat, c) 1979, the term used was "telephone switch net," which was not adopted here, as it is outdated. Still, the concept underpinning the work remains highly relevant, as forms of "switches" are still used in redirecting TCP/IP packages, in a manner comparable to the telephone net. In order to disclose on conceptual level the technology developed for extracting the extreme subsystems in Markov chains discussed in the current paper, we employ the communication network as an example of monotonic system, albeit in a slightly modified form relative to that originally proposed in the context of telephone network. This will enable us to elucidate the manner in which a Markov chain may be associated with the monotonic system and what principal operations may be performed on it towards utilization of monotonic systems theoretical apparatus described in the Mullat original work.

In the earlier paper on which this Mullat work is based, an example of a communication network has been considered, whereby a set W comprising of communication lines/channels between some nodes — communicating units — was introduced.[1] Here, we will assume that each line has certain built-in redundancy mechanisms, such as the main and the reserved channels. In practice,

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[1] Switch is a device, which can learn where to address the communication packages.

network redundancy may be guaranteed by some additional channels/lines activated only in urgent situations when the net usage exceeds some predefined threshold. Thus, if a direct line is not available between nodes, analogously to what was described in Mullat's work 1976, the traffic might be organized through pass-around channels. In addition to this mechanism, in the present case, the possibility of employing pass-around communication is not excluded even if a direct channel is available.

In the example presented in the original paper (Mullat, 1976), an average number of "denials" before establishing the contact characterizing each pair of nodes was utilized. The number of denials usually characterizes the communication lines in the communication network. Network protocol analyzers can collect such types of statistical data. In the model described below, and for the purpose of current investigation, it is more convenient to use a value inverse to the number of denials, as this will characterize the communication line throughput.

Let us assume that each communication line (comprising of both the main and the reserved channels) is characterized by the throughput $c_{i,j}$ or, in other words, by the maximum allowed bandwidth usage, expressed in kilobytes for example. The value $c_{i\,j}$ thus denotes the throughput of main and reserved channels. We then explicate the communication center s by the maximum permissible usage

$$c_i = \sum_{j=1}^{n} c_{i,j} .$$

The traffic redirected through the node s along the main communication channel, as well as the reserved channel, between nodes s and j specifies thereupon a share of maximum permitted usage c_s. In an actual communication network, the usage share must be lower than the maximum allowed share $p_{s\,j} = {}^{c_{s,j}}\!\big/\!{}_{c_s}$. Moreover, the usage share $p_{s,j}$ of the communication channel can be interpreted as a probability of establishing contact between the nodes s and j. Assuming that the main and the reserved channels are treated as equitable, the quantity must satisfy an inequality

$$2 \cdot \sum_{j=1}^{n} p_{i,j} < 1 \qquad (1)$$

for all s without exception,

Let a communication network, characterized by the aforementioned pass-around traffic feasibility, function during a long period of time by originating its main channels. We can characterize the traffic along each main channel (more precisely, the nodes i and j) by the average number of hits $\overline{p}_{i,j}$ that occur in the process of establishing either direct or indirect (pass-around) contact. It is apparent that $\overline{p}_{i,j}$ is slightly greater than the corresponding $p_{i,j}$.

If a malfunction occurs somewhere along the channel, then a change in the communication network will be reflected in a decrease in $\overline{p}_{i,j}$. In such a scenario, activating a reserved channel can enable higher network usage. Obviously, in this case all $\overline{p}_{i,j}$ values will increase accordingly. A communication network organized in this way is a monotonous system.

However, a problem arises with respect to identifying the type of change of malfunctioning/activating of a main/reserved channel that would influence the $\overline{p}_{i,j}$ values. In order to find an appropriate solution, it is necessary to explain the problem in Markov chains nomenclature.

Consider a set W of communication channels described by a square matrix $\left\|p_{i,j}\right\|_n^n$, when no channels exist, $p_{i,j} = 0$. According to the theory of Markov chains (Chung 1960). Such matrices may be associated with a set of returning states for some absorbing Markov chain. In the nomenclature pertaining to chains of this type, the value $\overline{p}_{i\,j}$ can be interpreted as an average number of hits from node i into node j along the Markov chain. Similarly, a malfunction in the main channel, resulting in the activation of the reserved channels, can be described through recalculating the average hit values $\overline{p}_{i,j}$. The above can be denoted as an action of type ⊖, whereas in the nomenclature of monotonic systems, an action of type ⊕ pertains to activating the reserved channel due to the malfunctioning in the main channel.

From the above discussion, it is evident that adopting this special class of absorbing Markov chains allows approaching the problem from the perspective of how to differentiate the Extremal Subsystem of Monotonic System — the kernels. Along with the KSR — Kernel Search Routine elaborated for this purpose in (Mullat, 1976), this approach can actually accomplish the kernel search task.

In Section II below, the problem of kernel extraction on Markov chains is described in more detail. In Section III, we show that the results of performing the ⊕ and ⊖ actions upon Markov chain entries in a transition matrix lead to Sherman-Morrison (Dinkelbach, 1969) expressions for recalculating the numbers of average hits (see Appendix).

2. THE PROBLEM OF KERNEL EXTRACTION ON MARKOV CHAINS

Consider a stationary Markov chain with a finite number of states and discrete time. We denote the set of states by V. Stationary Markov chain can be characterized by the property that the pass probability from the state i to the state j at a certain point in time $t+1$ does not depend upon the state s $(s = 1,2,...,n)$ the considered chain arrived in i in the preceding moment t. We denote by $p(i,j,k)$ $(p(i,j,1) = p_{i,j})$ the conditional probability of this pass from i to j within k units of time.

Below, we consider only a special class of Markov chains that, for arbitrary states i and j within some subset in V, is constrained by

$$\lim_{k\to\infty} p(i,j,k) = 0 .$$

According to the theory of Markov chains, this limit equals zero when the state j is returning, implying that there must be some reversible states in such Markov chains. Without diminishing the generality of this consideration, we will further examine chains with only one reversible state, which must simultaneously be an absorbing state.

The absorbing chains utilized below satisfy the following properties:

1. There exist only one absorbing state $\theta \in V$
2. All remaining states are returning, and the probability of a pass between the states in one step corresponds to an entry in the square matrix
$$\left\| p_{i,j} \right\|_n^n .$$
3. The probability of a pass into an absorbing state θ from some returning state i in one step, in accordance with 1 and 2, is equal to
$$p_{i\theta} = 1 - \sum_{\theta=1}^{n} p_{i,\theta} .$$

The monotonic system mandates a definition of some positive and negative (⊕, ⊖) actions upon system elements. For this purpose, we make use of the average number of hits $\overline{p}_{i,j}$ from the state i into the state j along the chain (Chung 1960). It is known that the value of $\overline{p}_{i\,j}$ is specified by the series

$$\overline{p}_{i,j} = \sum_{k=1}^{\infty} p(i,j,k) . \qquad (2)$$

The sufficient condition for series (2) to converge is established if the sum of entries in each row of the matrix $\left\| p_{i,j} \right\|_n^n$ is less than one. We consider that elements elsewhere in the chains fulfill the conditions 1-3.

Let W be the set of all nonzero entries in the matrix $\left\| p_{i,j} \right\|$. On the transition W set of the Markov chain described above, we define the following actions.

Definition. The action type ⊖ on the element of the system W (nonzero element of the matrix $\left\| p_{i,j} \right\|$) denotes a decrease in its value by some Δp of its probability to pass in one step.

By analogy, we define the action ⊕. In this case, the probability of a pass in one step, which corresponds to the entry value $p_{i,j}$, is increased by Δp. In case of some nonzero increment in the matrix $\|p_{i,j}\|$ element (based on straightforward probability considerations), all average numbers of hits $\overline{p}_{i\,j}$ must also increase accordingly. On the other hand, a Δp decrement would result in a decrease in the corresponding $\overline{p}_{i\,j}$ values. In sum, introduced actions upon system W elements fully meet the monotonic condition (Mullat, 1976), and system W transforms into a monotonic system.

At this juncture, it is important to emphasize that the Δp changes in values of probabilities in one step within W are not specified in the definition of ⊕ and ⊖ actions upon the entries in the matrix $\overline{p}_{i,j}$. Relatively rich possibilities exist for the change definition. For example, it can denote an increase (decrease) in each probability on a certain constant, or the same change, but this time depending upon the probability value itself, etc. When providing the definitions of ⊕ and ⊖ actions on an absorbing Markov chain, it is desirable to utilize authentic considerations. Below, using an example of communication network, we describe one of such considerations.

Let W be the set of all possible transitions in one step among all returning states of an absorbing chain. These transitions in the set W retain the correspondence with nonzero elements of the matrix $\|p_{i,j}\|$. Let T be a certain subset of the set W, relating to the nonzero elements noted above. Denote by $p(T,i,j,k)$ the probability that the chain passes from the state i into the state j within k time units, constrained by the condition that, during this period, all passes in one step upon the set T have been changed by either ⊕ or ⊖ actions. This condition corresponds to the assertion that the passes along the set $W \setminus T \equiv \overline{T}$ proceed in accordance with the “old” probabilities, while those along T are in governed by the “new” Probabilities. We do not exclude the case when no ⊕ or ⊖ actions have been implicated — the set $T = \varnothing$. In this case, we simply omit the T symbol notation in the corresponding probabilities. We suppose that actions do not violate the convergence of probability series, see condition (1).

The average number of hits from i into j, subject to the constraint that some passes in the set T have been changed by actions, is specified by a series

$$\overline{p}(T,i,j) = \sum_{m=1}^{\infty} p(T,i,j,m). \tag{3}$$

Let us now focus on the collections of credentials specified by a monotonic system W. We define a collection $\Pi^{+}H$ on the subset $H \in W$ as a collection of real numbers $\left\{\overline{p}(\overline{H}, i, j) \,\middle|\, (i,j) \in H\right\}$ in case that the positive ⊕ actions occur on $\overline{H} = W \setminus H$, while $\Pi^{-}H = \left\{\overline{p}(\overline{H}, i, j) \,\middle|\, (i,j) \in H\right\}$ collection corresponds to the case of the negative ⊖ actions taking place.

In the original paper (Mullat, 1976), we have proved that, in a monotonic system, two kinds of subsystems always exist — the ⊕ and ⊖ kernels. The definitions introduced above, pertaining to the average number of hits $\overline{p}(\overline{H}, i, j)$, allow us to formulate the notion of ⊕ and ⊖ kernels in the Markov chain.

Definition. By the Extremal Subsystem of passes on absorbing Markov chain — the ⊕ and ⊖ kernels — we call a system $H^{\oplus} \subseteq W$, on which the functional

$$\max_{(i,j)\in H} \overline{p}(\overline{H}, i, j) \tag{4}$$

reaches its global minimum on 2^{W}, whereby ⊖ kernels will be a subsystem $H^{\ominus} \subseteq W$ where the functional

$$\min_{(i,j)\in H} \overline{p}(\overline{H}, i, j) \tag{5}$$

reaches its global maximum as well.

We will now turn the focus toward the notions of ⊕ and ⊖ kernels introduced above, using an example on communication network described earlier. The probabilities of hits $p_{i\,j}$ (without any passes, i.e., in a single step) between nodes i and j $\left(i, j = \overline{1, n}\right)$ allow us to construct for the communication network an absorbing chain satisfying the conditions 1-3 above. In fact, as we already noted, only one condition is mandatory to satisfy the inequality (1), which is a natural condition for any communication network. Conditions 2 and 3, on the other hand, can be guaranteed by the Markov chain design. In this case, numbers $p_{i,j}$ may be interpreted as probabilities of a pass in one step, whereby $\overline{p}_{i,j}$ denotes an average number of hits from i into j, whether directly, or via an indirect pass-around along other lines in the chain.

The search for the ⊕ and ⊖ kernels on an actual Markov chain, reconstructed from a communication network, mandates a precise definition of ⊕ and ⊖ actions. In the beginning of the discussion, we observed that ⊖ action might represent a malfunctioning in the main channel, whereas ⊕ action might pertain to the activation of a reserved channel. On the Markov chain, the malfunctioning is denoted as null, reducing the corresponding probability, while the activating of a reserved channel is reflected in the doubling of its initial probability value. We stress once again that $\oplus$ and $\ominus$ actions are subjective evaluations of an actual situation. The condition (1) guarantees that, in any circumstance that would necessitate such $\oplus$ and $\ominus$ actions, the convergence of series (2) and (3) will not be violated.

We suggest a suitable interpretation of $\oplus$ and $\ominus$ kernels in Markov chain below, starting from the Markov chain characteristics, introduced here in terms of communication network.

In Extreme Subsystem $H^{\ominus}$, none of the communication lines/channels are subject to changes, whereas in all lines outside $H^{\ominus}$, they're reserved channels have been activated. The extreme value of the functional (4) shows that the average number of hits within channels belonging to $H^{\ominus}$, including the indirect pass-around hits (by definition, an indirect hit requires at least two steps to reach the destination), is relatively low. This assertion implies that the lines within the $H^{\ominus}$ kernel are "immune" with respect to package delivery malfunctions, i.e., most of the transported packages pass along direct lines. The set of lines in $H^{\oplus}$ kernel is characterized by a reverse property. Thus, the main channels in $H^{\ominus}$ kernel are the most "appropriate" for organizing "high-quality" indirect communications, but are also a sensible choice for mitigating the malfunctions that may result in a "snowballing" or "bandwagon" effects. Conversely, along $H^{\oplus}$, the indirect communication is typically hampered for some reason.

3. MONOTONE SYSTEM CREDENTIAL FUNCTIONS ON MARKOV CHAINS

In Section II, we defined some $\oplus$ and $\ominus$ actions upon the transition matrix entries in one step corresponding to returning states. In this section, we will develop an apparatus that allows us to incorporate the changes induced by these two types of actions into the average numbers of hits from one returning state i into the other state j. We describe here and deduce some tangible credential functions intended for use alongside our formal monotonic system description, following the conventions presented in the previous work (Mullat, 1976). Let us first recollect the notion of credential function before providing an account of the main section contents.

Suppose that, in the system W, which in the case of Markov chain is characterized as a collection of entries in matrix $\left\|p_{i,j}\right\|_n^n$ corresponding to passes among returning states, a subset H has been extracted. As a result, the set H consists of one-step transitions. Owing to the successive actions of type $\ominus$, by accounting for all individual sequential steps in the process (see Section II) taken upon the elements in $\overline{H}$ (a complementary of H to W), it is possible to establish the average number of hits within the transition set H — the credential system $\Pi^{-}H$. By analogy, on the set $\overline{H}$, a succession of ⊕ actions establishes the credential system $\Pi^{+}H$. The average number of hits in the nomenclature given in Section II may be represented as $\overline{p}(\overline{H}, i, j)$ — i.e., the limit values for series (2) on nonzero elements for the transition matrix P corresponding to the entries/lines within the set H. Further, we will refer to the numbers $\overline{p}(\overline{H}, i, j)$ as the credential functions.

Let us now establish the general form of the credential functions on Markov chains as a matrix series. This can explain the mechanism of actions the defined in Section II, performed upon the elements of a monotonic system — the Markov chain.

The credential function on Markov chain may be found using the series (2), where the single element (i, j) in the series presents the probability of the chain pass from i into j, constrained by the condition that actions have been performed upon the set $\overline{H}$.

The general matrix form of such transition probabilities described in Section II is given below: θ

$$\left\| \begin{matrix} 1 & 0 & . & . & . & 0 \\ p_{1,\theta} & & & & & \\ . & & \mathbf{P} & & & \\ p_{n,\theta} & & & & & \end{matrix} \right\| \text{, where} \tag{6}$$

θ – absorbing state of the chain;

$p_{i,\theta}$ – the probability of a pass from the i's returning state into the absorbing state θ;

P – the transition matrix of probabilities between the returning states within one step, where the matrix dimension is $n \times n$.

Using Chapman-Kolmogorov equations (Chung 1960), the element $p(T, i, j, m)$ in series (3) may be found as the m-s power of the matrix (6), whereby it occupies an entry in the matrix P^{m}.

In summary, the collection of series (3) may be written as the following matrix series

$$\overline{P}_{T} = I + P_{T} + P_{T}^{2} + . \, . \, . , \tag{7}$$

P_{T} – the matrix, where type ⊕ and ⊖ actions have been performed upon all nonzero elements within the set. We suppose that $p(T, i, j, 0) = \delta_{i,j}$, which is what the unity matrix in Section I highlights. In the nomenclature of the Markov chains (Kemeny et al, 1976) theory, matrices of type P_{T} are referred to as the fundamental matrices.

Recall that, in the definition of a monotonic system, the credential function on the set $H \subseteq W$ takes advantage of a complementary set $\overline{H}$ to the set H only. The set $\overline{H}$ is actually the set of performed actions. Given that the elements of the set W are also presented as matrix entries $\overline{P}_{\overline{H}} = \|I - P_{\overline{H}}\|^{-1}$, the matrix is the credential functions collection on the Markov chain, identical to the matrix limit of (7).

In the nomenclature of fundamental matrices, the actions upon the monotonic system elements are transformations, taking place in succession, from the matrix $\|I - P_T\|^{-1}$ to the matrix $\|I - P_{T\cup\alpha}\|^{-1}$. Calculus of such a transformation is, however, a very “hard operation.” In order to organize the search of ⊕ and ⊖ kernels on the basis of constructive procedures (KSR) described previously (Mullat, 1976), the utilization of matrix form is inappropriate. To extract the extreme subsystems on Markov chains successfully and take full advantage of the developed theory of monotonic systems, a more effective technology is needed, which leads us to Sherman-Morrison relationships (Dinkelbach, 1969).

The solution that can account for the changes emerging as a result of the ⊕ and ⊖ actions upon the transition matrix elements within one step in the fundamental matrix of Markov chain may be archived in the following manner. Suppose that, instead of the old probability p_o denoting a pass in between the returning states i and j, an updated (new) probability $p_n = p_o + \Delta p$ is utilized, where the action $(\pm \Delta p)$ results in either an increment or a decrement. In case of $(+\Delta p)$, the ⊕ action has occurred, whereas $(-\Delta p)$ implies the ⊖ action. The change induced by one of these actions may be treated as two successive effects. First, the probability p_o is replaced by 0 and the replacement is recalculated. Second, the transition probability is subsequently reestablished with the new value p_n and the change in the fundamental matrix is recalculated immediately after the first recalculation.

The relationships accounting for the changes in the fundamental matrix $\overline{P}_T$ as a result of the element α having a null value and affecting the matrix P_T, as well as the relationships accounting for the changes in $\overline{P}_T$, also in the reverse case of ⊕ actions, may be found in Appendix I.

In sum, for the search of extreme subsystems following the theory of constructing the defining sequences on system W elements with the aid of KSR routines introduced in the previous work (Mullat, 1976), it is necessary to obtain some well-organized and distinct recurrent expressions, which can account for the changes in the matrix $\overline{P}_T$ whereby it is transformed to the matrix $\overline{P}_{T\cup\alpha}$. The formulas for specified Δp, which allow us to transform from $\overline{P}_T$ in order to find the matrix $\overline{P}_{T\cup\alpha}$ are given in Appendix II on the basis of the expressions II 1.3 and II 1.4.

With the aid of these recurrent expressions, in Appendix II, it is possible to obtain on each set $H \subseteq W$ the collection of credentials $\Pi^{+}H$ or $\Pi^{-}H$ by performing the successive implementation of expressions II 2.5 to all elements upon the set $\overline{H}$. These expressions mirror the transformation of system element credentials π into π_{α} in view of the theoretical apparatus of monotonic systems (Mullat, 1976). Indeed, we construct the collection $\Pi^{+}H$ in the case of $\Delta p > 0$, whereas the collection $\Pi^{-}H$ is constructed if $\Delta p < 0$.

4. ON HOMOGENEOUS MARKOV CHAINS

In this section, we consider homogeneous Markov chains with a finite number n of states and a discrete time. A chain is called homogeneous if and only if the transition probabilities $p_{i,j}$ are independent of time t.

Our goal is to establish the relations between the elements of fundamental matrix denoting an absorbing chain (Chung, 1960), p. 66), see the definition below on the condition that certain transitions per time unit have been declared as prohibited. These relations are used in adjusting the corresponding elements without imposing this restriction. It should be noted that similar relations are encountered in compositions pertaining to the first and the last occurrence of some Markov chain states (see (Chung, 1960), p. 75). However, in spite of this obvious resemblance, such relations have not yet been considered in the literature.

Given without proof, the relations given in the form of theorems I-IV allow making a case for implementation of a general principle of maximum for some functions, defined on finite sets (Mullat, 1971). The foundation for the construction scheme (1971), in particular, is contingent upon requirements applied to the functions in the form of inequalities given as a result of this research.

In developing an efficient algorithm at the computer center of the Tallinn University of Technology, the theorems I-IV served as a foundation for finding solutions for some notable pattern recognition classification problems. Application of the algorithm improved the solution quality and speed with which problems were solved computationally, in comparison with those achieved by currently used algorithms.

Usually, homogenous chain can be represented as a directed graph whose vertices correspond to the state of the chain, whereby the arcs denote possible unit transitions from one state to another at any point in time. In addition, when the transition probability $p_{i,j}$ is zero, the arc $u = (i, j)$ is not depicted on the graph. On the other hand, any graph Γ can be represented in the form of a homogeneous chain attributing the arcs of the chain by satisfying the relation of the conditional probabilities. These chains are referred to as chains associated with the graph Γ.

Let $U(G)$ be the set of arcs of the graph G, and $V(G)$ the set of vertices. Adding to the set of vertices $V(G)$ a vertex θ, which is in turn connected to any vertex in $V(G)$ by an arc leading into θ, can hence reproduce a graph Γ

Consider the following homogeneous Markov chain associated with the graph G:

1) There exists a unique absorbing state $\theta \notin V(G)$;
2) The probability of transition from i to j, $i, j \in V(G)$, $p_{i,j} = p_j$, if the arc $(i, j) \in U(G)$, and $p_{i,j} = 0$ otherwise;
3) The probability of transition from the state $i \in V(G)$ to the absorbing state θ is given by $p_{i,\theta} = 1 - \sum_{i=1}^{n} p_{i,j}$.

It can easily be verified that all states of the chain, identified by the vertices of the graph G, are irrevocable, whereby the designated Markov chain belongs to a class of absorbing chains (see (Chung, 1960), p. 55).

Here, some of the tuning indicators v_j refer to the parameters of the Markov chain associated with the graph G. Further, we assume that for any $v_j = \sum_i^n p_{i,j} < 1$. For all vertices of the graph G, it can be demonstrated that for any graph G, one can find a tuning parameter v for which a given constraint $0 < v < 1/k$ is satisfied. Indeed, let k represent the largest number of nonzero elements in the rows of the fundamental matrix corresponding to the vertices of the graph G.

Moreover, let H denote an arbitrary subset of arcs of the graph G, i.e., $H \subset U(G)$. Here, $p(H,i,j,k)$ designates the probability of transition from the state i to the state j in k units of time, on the condition that the transitions along the arcs of the subset H are prohibited during this period. Owing to this restriction, the subset H denotes a prohibited set of arcs, all of which are thus prohibited as well.

Let $p(H,i,j,0) = \delta_{i,j}$ (where $\delta_{i,j}$ represents the Kronecker's symbol) and

$$\bar{p}(H,i,j) = \sum_{n=0}^{\infty} p(H,i,j,n).$$

Due to the existence of a Markov chain associated with the graph Γ of an absorbing state θ, the entire set $V(G)$ is irrevocable, see Chung, 1960, p. 45, and the series (1) converges.

We use the Greek letters $\alpha, \beta, ...$ to denote prohibited arcs of the graph G, whereby α^{+} refers to the vertex (state) from which the arc emerges, and α^{-} is the vertex toward which the arc is pointing.

Theorem I. We denote by $H + \alpha$ a set-theoretic operation. $H \cup \alpha$.

$$\overline{p}(H+\alpha,i,j) = \overline{p}(H,i,j) - v \cdot \frac{\overline{p}(H,i,\alpha^{+}) \cdot \overline{p}(H,\alpha^{-},j)}{1 + p_{\alpha^{-}} \cdot \overline{p}(H,\alpha^{-},\alpha^{+})}.$$

This expression might be interpreted as a consequence of malfunctions in the communication line α. The next expression can be interpreted as an increase in traffic efficiency after repairs on the line.

Theorem II. $\overline{p}(H,i,j) = \overline{p}(H+\alpha,i,j) + v \cdot \dfrac{\overline{p}(H+\alpha,i,\alpha^{+}) \cdot \overline{p}(H+\alpha,\alpha^{-},j)}{1 - p_{\alpha^{-}} \cdot \overline{p}(H,\alpha^{-},\alpha^{+})}$

Corollary.

From the form of the dependence in the formulations of Theorems I-II it immediately follows that the following inequalities are valid for the case of directed and undirected graphs, respectively

$$\overline{p}(H+\alpha,i,j) \le \overline{p}(H,i,j),\ i,j = \overline{1,n}).$$

These inequalities guarantee the fulfillment of the monotonicity condition for the realization of a monotonic system on homogeneous Markov chains.

APPENDIX I

Consider the value $\overline{p}(T,i,j)$ produced by the series (3). Each component of this series may be treated as the measure of all passes in m time steps (time units) commencing in i and terminating in j. This assemblage of transitions is a union of two nonintersecting collections. The first set pertains to the passes from i to j with a mandatory transition, at least once, along $\alpha \in W$. On the other hand, the second relates to the set of passes from i to j avoiding this transition α. Each passage from the first set consists of two passes: a pass avoiding α being in t steps long, and a pass in $m-t-1$ steps (time units), passing along α. In other words, the passages in t steps avoid the pass along α, whereas passages in $m-t-1$ steps make use of this pass α.

We introduce the following notation: $\overline{p}(T^0, i, j, k)$ represents the average number of hits from i into j with the transition matrix P_T, where the nonzero element α is null, and $p(T^0, i, j, k)$ denotes the probability of transition without making use of α. Implementation of the introduced notification results in:

$$\begin{aligned} & p(T, i, j, m) = p(T^0, i, j, m) + \\ & + p_\alpha \cdot \sum_{t=0}^{m-1} p(T^0, i, \alpha_b, t) \cdot p(T, \alpha_e, j, m-t-1) ; \end{aligned} \qquad \text{I 1.1}$$

$$\begin{aligned} & p(T, i, j, m) = p(T^0, i, j, m) + \\ & + p_\alpha \cdot \sum_{t=0}^{m-1} p(T, i, \alpha_b, t) \cdot p(T^0, \alpha_e, j, m-t-1) , \end{aligned} \qquad \text{I 1.2}$$

where α_b – the state from which a one-step pass begins, ending in α_e; p_α – the pass along α in one step, corresponding to the element α of the matrix P_T.

The first component in I 1.1 and I 1.2 introduces the value of $p(T, i, j, m)$, denoting the measure of transitions avoiding the pass along α. In addition, the components included in the summation represent the probability that the states α_b (for the relationship I 1.1) and α_e (for the relationship II 1.2) have been reached by the first and the last pass along α in the moments t and $t+1$, respectively.

Let us calculate the $\overline{p}(T, i, j)$ values using the relationship II 1.1. We conclude, after performing the summation of each of the equations II 1.1 from 1 to M and thereafter changing the order of sums in the double summation, that

$$\sum_{m=1}^{M} p(T, i, j, m) = \sum_{m=1}^{M} p(T^0, i, j, m) + p_\alpha \cdot \sum_{t=0}^{M-1} p(T^0, i, \alpha_b, t) \cdot \sum_{s=1}^{M-t} p(T, \alpha_e, j, s-1) .$$

Dividing both parts of the latter equation yields $\sum_{t=0}^{M-1} p(T^0, i, \alpha_b, t)$.

Thus, based on the theorem of Norlund averages (Chung 1960) considering the sequence $a_t = p(T^0, i, \alpha_b, t)$ and $b_{m-t} = \sum_{s=1}^{M-t} p(T, \alpha_e, j, s-1)$, while increasing $M \to \infty$ for the sequences a_n and b_n, it can be concluded that the following relations are valid:

$$\overline{p}(T, i, j) = \overline{p}(T^0, i, j) + p_\alpha \cdot \overline{p}(T^0, i, \alpha_b) \cdot \overline{p}(T, \alpha_e, j). \qquad \text{I 1.3}$$

Analogous relationship can be deduced by exploiting the composition I 1.2, namely:

$$\overline{p}(T, i, j) = \overline{p}(T^0, i, j) + p_\alpha \cdot \overline{p}(T, i, \alpha_b) \cdot \overline{p}(T^0, \alpha_e, j). \qquad \text{I 1.4}$$

APPENDIX II

We introduce the following notifications. Let $\overline{p}(T_o, i, j)$ represent the matrix $\overline{P}_T$ element, and $\overline{p}(T_n, i, j)$ denote the matrix $\overline{P}_{T\cup\alpha}$ element. Let us also rewrite II 1.3 and II 1.4 with respect to these notifications, which results in:

$$\begin{aligned}&\overline{p}(T_n, i, j) = \overline{p}(T^0, i, j) + \\ &+ p_n \cdot \overline{p}(T^0, i, \alpha_b) \cdot \overline{p}(T_n, \alpha_e, j)\end{aligned}; \qquad \text{II 2.1}$$

$$\begin{aligned}&\overline{p}(T_o, i, j) = \overline{p}(T^0, i, j) + \\ &+ p_o \cdot \overline{p}(T_o, i, \alpha_b) \cdot \overline{p}(T^0, \alpha_e, j)\end{aligned}. \qquad \text{II 2.2}$$

From the relationships II 2.1 and II 2.2, it follows that the new value for the average hits from i into j is equal to

$$\begin{aligned}&\overline{p}(T_n, i, j) = \overline{p}(T_o, i, j) + \\ &+ p_n \cdot \overline{p}(T^0, i, \alpha_b) \cdot \overline{p}(T_n, \alpha_e, j) - \\ &- p_o \cdot \overline{p}(T_o, i, \alpha_b) \cdot \overline{p}(T^0, \alpha_e, j)\end{aligned} \qquad \text{II 2.3}$$

Substituting in II 2.1 the state $i = \alpha_e$, we obtain

$$\overline{p}(T_n, \alpha_e, j) = \overline{p}(T^0, \alpha_e, j) / \left(1 - p_n \cdot \overline{p}(T^0, \alpha_e, \alpha_b)\right)$$

and from II 2.2, with the same $i = \alpha_e$ we get

$$\overline{p}(T^0, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right).$$

Replacing the latter expression into the preceding one, and taking into account that

$$\overline{p}(T^0, \alpha_e, \alpha_b) = \overline{p}(T_o, \alpha_e, \alpha_b) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right),$$

we finally arrive at

$$\overline{p}(T_n, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 - \Delta p \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right). \qquad \text{II 2.4}$$

The expression II 2.1 is valid if we replace T_n by T_o and p_n by p_o, and if in the expression II 2.2 we make a reverse replacement. Substituting $j = \alpha_n$ in the expression II 2.2, first regrouping it by this reverse replacement, results in

$$\overline{p}(T^0, \alpha_e, j) = \overline{p}(T_o, \alpha_e, j) / \left(1 + p_o \cdot \overline{p}(T_o, \alpha_e, \alpha_b)\right).$$

Finally, we deduce the expression that can account for the changes in the fundamental matrix $\overline{P}_T$ by simplifying the last two equalities and the expression II 2.4, after collecting sub-expressions and making rearrangements transforming $\overline{P}_T$ into the matrix $\overline{P}_{T\cup\alpha}$. Adopting the standard nomenclature given in Section III, the ultimate form of the expression is given as follows:

$$\overline{p}(T\cup\alpha,i,j)=\overline{p}(T,i,j)+\Delta p\cdot\frac{\overline{p}(T,i,\alpha_b)\cdot\overline{p}(T,\alpha_k,j)}{1-\Delta p\cdot\overline{p}(T,\alpha_e,\alpha_b)}. \qquad \text{II 2.5}$$

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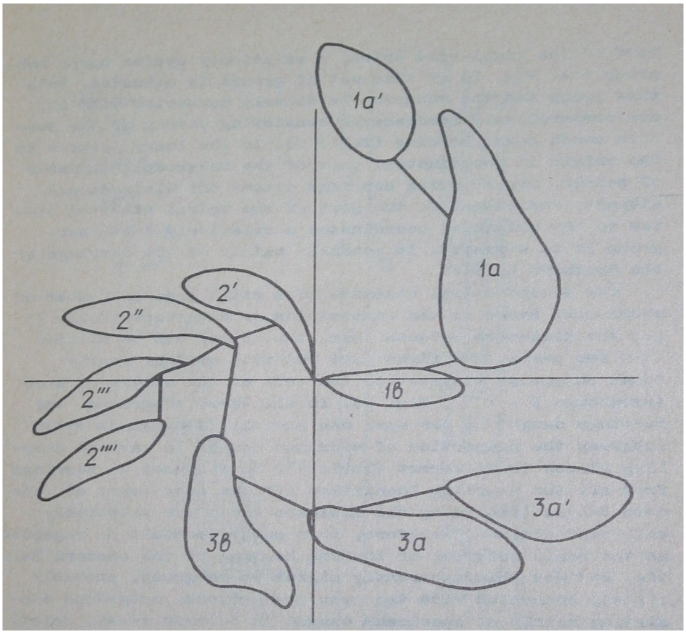
A Study of Infraspecific Groups of the Baltic East Coast Autumn Herring by new Method based on Cluster Analysis



Positions of the autumn herring subgroups differentiated by the method described.

Figure 1

E. Ojaveer, Estonian Laboratory of Marine Ichthyology (1975)

"In the Baltic Sea the autumn spawning herring forms a smaller number of groups than the spring herring does. This is probably connected with the different location of their spawning grounds. Spawning grounds of the spring herring are concentrated in favorable sites near the coast (in gulf, estuaries, etc.) while between such spawning centers gaps occur usually. Contrary to it, in most parts of the Baltic spawning places of the autumn herring form a continuous chain situated in the open sea. Therefore, differences in environment conditions between the autumn spawning grounds of neighboring areas are small and in large districts the characters of the autumn herring do not reveal essential differences. For instance, there is no significant difference between the autumns herrings caught on various grounds off the Polish coasts. The autumn herring of the Swedish Baltic coasts can be divided into four groups (that of the Gulf of Bothnia, that of the Bothnia Sea, the herring of the Swedish east coast and that of the Swedish south coast), between which a gradual transition occurs."

Appendix 1, J. Mullat (1975), Tallinn Technical University

While cluster is a concept in common usage, there is currently no consensus on its exact definition. There are many intuitive, often contradicting, ideas on the meaning of cluster. Consequently, it is difficult to develop exact mathematical formulation of the cluster separation task. Yet, several authors are of view that clustering techniques are already well established, suggesting that the focus should be on increasing the accuracy of data analysis. The available examples of data clustering tend to be rather badly structured, whereas application of the formal techniques on such data fails to yield results when the classification is known *a priori*. These issues are indicative of the fundamental deficiencies inherent in many numerical taxonomy techniques.

Following the standard nomenclature, a vector of measurements can describe every object $\langle x_1, x_2, ..., x_k \rangle$. Thus, for every pair of objects E_i and E_j a distance d_{ij} between those objects can be defined as

$$d_{ij} = \sqrt{(x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + ... + (x_{ik} - x_{jk})^2} \tag{1}$$

However, it should be noted that all measurements are usually standardized beforehand.

Applying Eq. (1) on N objects yields a full matrix of distances

$$D = \begin{Vmatrix} 0 & d_{12} & d_{13} & . & . & d_{1k} \\ d_{21} & 0 & d_{23} & . & . & d_{2k} \\ . & . & . & . & . & . \\ d_{k1} & d_{k2} & . & . & . & d_{kk} \end{Vmatrix} \tag{2}$$

Authors of many empirical studies employ methods utilizing the full matrix of distances as a means of identifying clusters on the set $\{E_1, ..., E_i, ..., E_k\}$.

In this section, we describe a new and highly effective clustering method, underpinned by some ideas offered by the graph theory. As the first step in our novel approach, we emphasize that, for elucidating the structure of the system of objects, knowledge of all elements of the matrix of distances given above is rarely needed. We further posit that, for every object, it is sufficient to consider no more than M of its nearest neighbors.

To explicate this strategy, let us consider a system of 9 objects (Fig. 2) with their interconnections — edges. The matrix of nearest neighbors for such a graph is given by:

$$\boldsymbol{MND} = \left\| \begin{array}{cccccc} 5(1) & 6(1) & 3(2) & 0 & 0 & 0 \\ 4(1) & 3(2) & 7(3) & 0 & 0 & 0 \\ 4(1) & 5(1) & 1(2) & 2(2) & 0 & 0 \\ 2(1) & 3(1) & 5(1) & 7(3) & 0 & 0 \\ 1(1) & 3(1) & 4(1) & 6(1) & 7(3) & 0 \\ 1(1) & 5(1) & 7(3) & 0 & 0 & 0 \\ 2(3) & 4(3) & 5(3) & 6(3) & 8(3) & 9(3) \\ 7(3) & 9(3) & 0 & 0 & 0 & 0 \\ 7(3) & 8(3) & 0 & 0 & 0 & 0 \end{array} \right\|$$

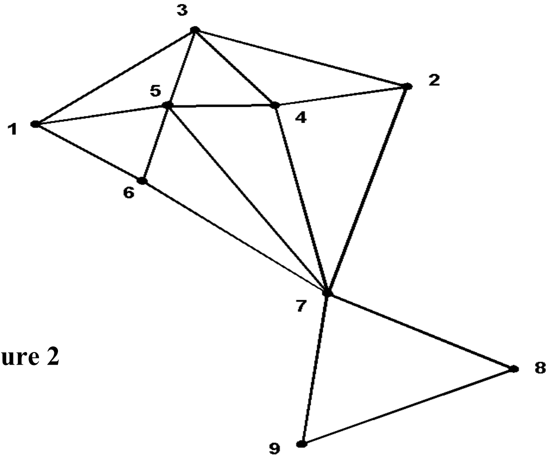


Figure 2

It can be easily verified that each row i of that matrix contains a list of objects j directly connected with a given object E_i, with the distances d_{ij} given in parentheses. Based on this argument, henceforth, we will denote the matrix of nearest neighbor distances by ***MND***.

In most cases, having data pertaining to about 8-10 nearest neighbors is sufficient. This is highly important for computation, where the goal is to minimize the required memory space. By applying this method on, e.g., the case of 1,000 objects, only 10,000 memory locations would be needed, which is a significant saving relative to the 500,000 required when the full matrix is processed.

We will use the ***MND*** defined above as a starting point to create some useful mathematical constructs.

Let W be the list of edges (pairs of objects) in the ***MND***. For every edge $e=[a,b]$, a subset W_b^a of the list W can be defined as follows.

Definition 1. Subset W_b^a of W represents a proximity space of edge $[a,b]$ if

a) for every pair of objects x and y, which are connected with at least one edge in W_b^a, there exists a path joining x and y, and

b) every edge that is a member of that path belongs to the subset W_b^a.

According to the graph theory postulates, proximity space is a sub-graph connected with the edge $[a,b]$.

Example. Let us consider the edge $[4,5]$ shown in Fig. 1. According to the aforementioned rules, its proximity space, denoted as W_5^4, is the sub-graph $W_5^4=\{[3,4],[3,5],[4,7],[5,7],[2,4],[1,5],[5,6],[4,5]\}$.

Definition 2. The system of proximity spaces is referred to, as the proximity structure if for each edge $w=[a,b]$ there exists a nonempty proximity space W_b^a in the system.

Sometimes it is useful to exclude the edge $[a,b]$ from the proximity space W_b^a. In line with the Venn diagram annotation, this exclusion is denoted as $W_b^a \setminus [a,b]$, whereby the resulting subset can be referred to as a reduced proximity space.

In the preceding discussion, for every edge $[a,b]$, only the value of the distance $d[a,b]$ between $[a,b]$ was taken into account. In what follows, it is useful to introduce a new notation. For example, it is beneficial to assign a real number (credential π), which is different from the distance, to every edge on the graph. For example, let us define the credential of every edge in the diagram shown in Fig. 1 as

$$\pi[x,y]=d[x,y]+r[x,y].$$

For example, $\pi[4,7]=3+2$, $\pi[7,8]=3+1$ on the edge $[x,y]$, where $d[x,y]$ is the Euclidean distance (1) between x,y and $r[x,y]$; $r[x,y]$ is the number of triangles that can be built around $[x,y]$.

Let us further assume that a proximity structure $\mathcal{L}$ of a graph W is known and that $f(x)$ is a real function.

Definition 3. The function $f_b^a(\pi)$ defined for all credentials of the edges in W_b^a is called the influence function of the proximity structure $\mathcal{L}$ if the following holds $f_a^b(\pi[x,y]) \leq \pi[x,y]$ for each $[x,y] \in W_b^a \setminus [a,b]$, where $\pi[x,y]$ is the credential of the edge $[x,y]$.

In other words, for every edge $[x,y]$, we can find a new credential in the reduced proximity space $W_b^a \setminus [a,b]$

$$\pi'[x,y] = f_b^a(\pi[x,y]) .$$

To demonstrate the benefit of introducing the influence function, let us again consider the diagram depicted in Fig. 1. Graphically, the influence function represents the value of the number of triangles after the elimination of the edge $[a,b] \in W_b^a$ from the list W_b^a. Using the set W_5^4 as an example, this corresponds to

$$f_5^4(\pi[3,4]) = f_5^4((d_{34} + r_{34}) = (1+1)) = (d_{34} + r_{34}) = (1+0) = 1 ;$$

$$f_5^4(\pi[3,4]) = f_5^4((d_{56} + r_{56}) = (1+0)) = (d_{34} + r_{34}) = (1+0) = 1 ;$$

$$f_5^4(\pi[3,4]) = f_5^4((d_{47} + r_{47}) = (3+1)) = (d_{34} + r_{34}) = (3+0) = 3 .$$

$$\boldsymbol{MNW} = \begin{Vmatrix} 5(3) & 6(2) & 3(3) & 0 & 0 & 0 \\ 4(3) & 3(3) & 7(4) & 0 & 0 & 0 \\ 4(3) & 5(3) & 1(3) & 2(3) & 0 & 0 \\ 2(3) & 3(3) & 5(3) & 7(5) & 0 & 0 \\ 1(3) & 3(3) & 4(3) & 6(3) & 7(5) & 0 \\ 1(2) & 5(3) & 7(4) & 0 & 0 & 0 \\ 2(4) & 4(5) & 5(5) & 6(4) & 8(4) & 9(4) \\ 7(4) & 9(4) & 0 & 0 & 0 & 0 \\ 7(4) & 8(4) & 0 & 0 & 0 & 0 \end{Vmatrix}$$

It is evident that knowledge of the influence function of an edge allows us to easily find the set of new credentials for an entire subset $H \in W$. Let us consider the set $\overline{H} = W \setminus H$ and arrange its edges in some order $\langle e_1, e_2, ... \rangle$. Applying the steps shown above, we can find the proximity spaces of the edges in $\langle e_1, e_2, ... \rangle$ and apply Eq. (3) recursively.

Using the information delineated thus far, we can now introduce our algorithm, the aim of which is to identify the data structure.

At this point, we can assume that steps pertaining to the selection of the proximity structure and the influence function have been completed. Thus, we can proceed through the algorithm as follows:

- **A1**. Find the edge with the minimum credential and store its value.
- **A2**. Eliminate the edge from the list of all edges and compute the credentials for proximity spaces of the minimal edge using the recursive procedure (3).
- **A3**. Traverse through the list of edges and identify the first edge with the credential less or equal to the stored credential. Return to **A2** to eliminate that edge. If no such edge exists, proceed to **A4**.
- **A4**. Check whether there are any further edges in W . If yes, return to **A1**, otherwise terminate the calculations.

Performance of the algorithm will be demonstrated by applying the aforementioned steps to the graph shown in Fig. 1.

First, the credentials for all edges should be defined using the following expression:

$$\pi[x,y] = d[x,y] + r[x,y].$$

To do so, we must compute the matrix of credentials using the matrix of distances (2).

We will demonstrate all steps of the algorithm described above.

A1. Minimal edge is $[1,6]$ and the associated credential is $\pi[1,6]=2$. To store its value, let $u = 2$.

A2. We eliminate the edge $[1,6]$ from the list W and therefore have to change the credentials of $\pi'[6,7]=4$:

$$W_6^1 \setminus [1,6]:\ \pi'[1,3]=3;\ \pi'[1,5]=2;\ \pi'[5,6]=2.$$

A3. Proceeding through the list, we encounter the edge $[1,5]$ as the first edge with the credential less or equal to u. Now, we return to step A2. After 9 steps with $u = 2$, we have the following sequence of edges:

$$\langle [1,6],[1,5],[1,3],[3,5],[3,4],[2,4],[2,3],[4,5],[5,6] \rangle.$$

Now, we consider the case $u = 3$, and after applying the preceding steps, we obtain $\langle [2,7],[4,7],[5,7],[6,7] \rangle$. Finally, using $u = 4$ yields $\langle [7,8],[7,9],[8,9] \rangle$.

It can be easily verified that those ordered lists of edges provide accurate representation of our graph’s structure.

For graphical output, we can utilize the ordered edges to construct a connected tree (a tree is a graph without circles).

For the example given above, we can construct the tree using the ordered lists of edges, while excluding all edges $[a, b]$ if both their end points, a and b, are already members of the list. This approach results in the sequence

$$\langle [1,6],[1,5],[1,3],[3,4],[2,4],[2,7],[7,8],[7,9] \rangle$$

based on which the tree in Fig. 3 can be constructed.

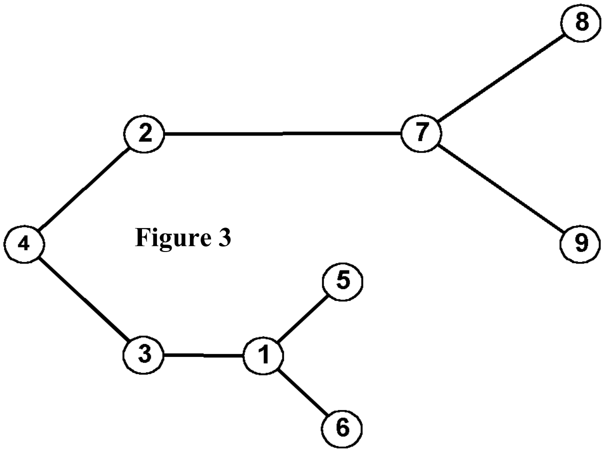


Figure 3

Using this simplified diagram, relative position of any object in the tree can be established by considering the number $S(x, y)$ of steps needed to reach the point y from the point x on the tree (e.g., $S(1,2) = 3$, $S(1,8) = 5$). Hence, for every object x, we can identify another object from which the maximum number of steps is required to reach x. For example, to identify the object at the top of the tree, we will take the object for which that maximum is minimum. Using real data, and applying these rules, we obtain the tree shown in Fig. 1.

LITERATURE

Ojaveer E., Mullat J.E. (Appendix I) and L.K. Võhandu (Appendix II). (1975) A Study of Infraspecific Groups of the Baltic East Coast Autumn Herring by two new Methods based on Cluster Analysis, Appendix I, pp. 42-47 have been written by Mullat, main body by Ojaveer E., Estonian Laboratory of Marine Ichthyology. Estonian Contribution to the International Biological Program, VI, TARTU, pp. 22-50.

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Postscript, Acknowledgement and Prospects

While it's understandable that the term 'Monotonic Systems' was used in this book without awareness of its common usage in another context, it's crucial to acknowledge that employing established terms in divergent contexts can breed confusion and should be minimized. Despite the terminology discrepancy, the phenomena described as 'Monotonic Systems' shouldn't hinder our discussion of the contributions outlined in this work. It's imperative not to discredit the efforts presented here solely due to the utilization of a potentially conflicting term. Evaluating the ideas and concepts based on their intrinsic merit and potential impact on the field is paramount. Although precision in using established terminology and preventing confusion is essential, the value of the ideas and concepts presented should be assessed independently.

In mathematics, a system earns the label 'monotone' when it adheres to a specific criterion known as the monotonicity property. The canonical property states that when a system's inputs increase, the output either remains constant or increases in the same direction or adequately. Conversely, when inputs decrease, output also either remains constant or also decreases equally. This fundamental characteristic is also referred to as non-decreasing or non-increasing behavior. However, we expanded upon the canonical property by incorporating the monotonic increase/decrease within internally dependent elements of the system itself, thus opening the way for practical implementation. The non-canonical monotone system seems to be a foundational mathematical concept widely employed across diverse disciplines.

In fields like computer science and communications, our customary monotone systems are instrumental in analyzing algorithms and protocols exhibiting monotonicity properties. For instance, in network routing protocols, the monotone system considered ensures seamless message delivery without loss or duplication. Similarly, in social sciences and economics, monotone systems serve to model individual and group behaviors in decision-making processes and economic transactions. For example, in market dynamics, such a monotone system can simulate buyer-seller interactions, where goods' prices rise with increasing demand. Furthermore, in game theory and data analysis, monotone systems facilitate the study of player behavior and strategy evolution in diverse games and social networks. For instance, in social network analysis, a monotone system can replicate information or influence propagation. In essence, monotone systems represent a potent analytical tool applicable across a spectrum of systems and processes in numerous fields, thereby proving invaluable to specialists across various domains.

In the discussions, we investigated Greedy type algorithms, which allowed us to arrive at some ordering, as they facilitated arranging what we called the defining sequence. Based on the prerequisites of this sequence, credentials either ascend or descend in accordance with the partial order of sub-lists within the main ordering, such as price credentials for wines, nodes on graphs, records in overview tables, radio-transceivers in cellular networks, routes along communication lines, agents in retail networks, transfer payments, tax relief, and more. The range of indicators suitable for inclusion in our defining sequence was truly boundless. When employing a defining sequence to organize elements, our objective was twofold. Initially, credentials ascend to a peak point, after which they diminish to zero. Alternatively, the reverse scenario could be addressed using a workaround scheme. We could seamlessly execute actions $\oplus$ and $\ominus$ within sub-lists, encompassing all possible sub-lists — the Totality of sets, where the General Ordering served as a representative of the Totality. Actions $\oplus$ enhanced phenomena, while $\ominus$ actions were perceived to have detrimental effects on the same phenomena.

Additionally, we introduced the concept of stable or steady sets, known as fixed points, which remain unaffected by $\oplus$ or $\ominus$ actions when applied to subsets. Essentially, we established that fixed points couldn't be destabilized by predefined mappings. However, our ultimate objective was to identify an optimal solution using Greedy-type algorithms through a defining sequence of ordering. We demonstrated that the defining sequence not only guaranteed optimal ordering but also facilitated the discovery of optimal stable subsets — the kernels. Moreover, we observed that, as a byproduct, any formation of a defining sequence adhered to the Fibonacci rule in general.

Other researchers[1] have also delved into the Monotone System approach, emphasizing the significance of its origins. Various types of lighter Monotone Systems have been established, streamlining the implementation of Greedy-type algorithms due to their simplified structure. This convenience in architecture was observed when the standard order of credentials within the Grand Ordering of elements remained unchanged during the formation of defining sequences. It was demonstrated that any subset of credentials within this Totality of subsets maintained alignment with the initial Grand ordering. Notably, the Totality of wine menu credentials or wine price impulses adheres to this light property harmony.

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[1] a) Yulia Kempner, Vadim E. Levit and Ilya Muchnik. (2008) Quasi-Concave Functions and Greedy Algorithms, Advances in Greedy Algorithms, Book edited by: Witold Bednorz, ISBN 978-953-7619-27-5, p. 586, November I-Tech, Vienna, Austria; b) Yulia Kempner and Ilya Muchnik. (2008) Quasi-concave functions on meet-semilattices, Discrete Applied Mathematics 156, pp. 492-499.

Light "Monotone Systems" offer the flexibility to present the Grand Ordering in either ascending or descending order through standard ordering procedures—any such procedure suffices for this purpose. Consequently, forming the defining sequence necessitates operations whose complexity scales logarithmically, contrasting with the rigidity of the general scheme. It's noteworthy to consider postulates pertinent to bounded rationality theory, as posited by Arrow, Rubinstein, Sen, Uzawa,… among others, applicable to both general systems and light monotone systems. These include the postulate of independence from rejected alternatives and the postulate of succession/adherence, exemplified in scenarios like the "Matching Game" and stock market share visualization. In the parlance of a barmaid, the latter postulate echoes as: "The old love does not rust." This underscores the enduring relevance of certain principles even in dynamic contexts.

It's worth reiterating the significance of "Monotonic Systems," which enable algorithms like Greedy to discover optimal solutions with significantly less computational effort than that required for solving complex NP problems. This optimality is guaranteed by a set function $F(X) = \min_{\alpha \in X} \pi(\alpha, X)$, where α belongs to set X: $\alpha \in X$. As highlighted by other researchers, $F(X)$ must adhere to the quasi-convex property when optimized across subsets X in the Grand Ordering W. Quasi-convexity on W dictates that for any pair $[X, Y]$ of subsets X and Y, the inequality $F(X \cup Y) \geq \min[F(X), F(Y)]$ must hold. Allegedly, this inequality ensures that NP-hard problems can be substituted with polynomial complexity procedures, enabling Greedy-type algorithms to operate within reasonable timeframes.

However, our investigation, as exemplified by straightforward instances such as the "Partial Matching Marketing Game," demonstrated that the quasi-convex property isn't universally upheld across all Monotone Systems. This realization suggests that Monotone Systems, contrary to initial assumptions, possess more intricate or diverse characteristics. Unfortunately, techniques relying on the defining sequence of ordering encounter difficulties when applied to such systems, hindering the pursuit of optimal solutions, particularly kernels. Nevertheless, alternative approaches exist for achieving optimal outcomes. Branch and Bound algorithms, though more complex compared to the Greedy-type algorithms commonly utilized with quasi-convex set functions, prove effective in managing conflict scenarios. They are particularly valuable in elucidating phenomena like bilateral agreements, where the dataset typically remains manageable despite increased computational demands.

Acknowledgement

In conclusion, the author finds it pertinent to share a personal perspective on the history of Monotone Systems. The author had the privilege of being associated with a laboratory of the Institute for Management Problems in Moscow, where the staff worked under the guidance of the late Prof. Aizerman. Since the mid-1950s, the laboratory has been dedicated to researching methods for automatic object classification. One fundamental hypothesis underlying these methods was that objects sharing similar characteristics in a multidimensional space, such as data analysis, visual objects, or sequences of letters and words, tend to cluster together. This hypothesis, known as the compactness of similar objects, posits that similar phenomena are closer to each other while being distant from dissimilar ones.

Building upon the compactness hypothesis, a multitude of classification algorithms emerged, as evidenced by the works of Braverman et al. (1975) [2], Mirkin et al. (1970) [3], and countless others. These methodologies shared a common foundation: the imperative to categorize objects in a manner that maximizes intra-class proximity while ensuring inter-class distinctiveness according to a specified metric. Amidst this pursuit, the groundbreaking contributions of Professor P.V. Terentyev, a luminary in biometrics at St. Petersburg State University, stand out prominently. Terentyev's seminal development, the correlation Pleiades method, revolutionized the selection of robust and "independent" features from a plethora of attributes. Notably, in his 1959 [4] publication, Terentyev applied the Pleiades method to delineate a classification scheme for biological entities, a framework that not only met the demands of its time but also continues to underpin a spectrum of contemporary methods, notably those falling under the rubric of nearest neighbor linkage. Terentyev's pioneering work thus remains a cornerstone in the ongoing evolution of classification methodologies. An exemplar of simplicity lies in the task of classifying objects into two distinct categories. In 1966 [5], Võhandu and Frey introduced a comparable method in the Biological Series of the Estonian Academy of Sciences, aimed at acquainting biologists with the latest statistical advancements.

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[2] Braverman É.M., Litvakov B.M., Muchnik I.B. and S.G. Novikov. (1975) Stratified sampling in the organization of empirical data collection, Autom. Remote Control, 36:10, pp. 1629–1641.

[3] Mirkin B.G. and L.B. Cherny. (1970) On a distance measure between partitions of a finite set, Automation and remote Control, 31, 5, pp. 786-792.

[4] Терентьев П.В. (1959) Метод Корреляционных Плеяд, Вестник ЛГУ№9.

[5] Frey T. and L. Võhandu. (1966) Uus Meetod Klassifikatsiooniühikute Püstitamiseks, Eesti NSV Teaduste Akadeemia Toimetised, XV Köide, Bioloogiline Seeris, Nr.4. Известия Академии Наук Эстонской ССР, Том XV, Серия Биологическая, №4б.

During his, JM’s tenure as a postgraduate student at Tallinn University of Technology from 1969 to 1971, under the mentorship of L.K., Võhandu (L.V.), author's academic journey was enriched by the exposure to similar methodologies, facilitated by L.V.'s guidance. Notably, JM’s fruitful interactions with the late Prof. E.M. Braverman from the "Institute of Control Problems" in Moscow underscored the collaborative spirit of the academic community during that time. Recalling his discussions with Braverman regarding classification within monotone systems, the author vividly remembers Braverman's acknowledgment of the novelty of his views.

In contrast to the conventional "*Nearest Neighbor Method*," J.M. introduced a formal mathematical framework rooted in combinatorial principles, offering a fresh perspective on data analysis methodology. Inspired by L.V.'s "blind-view" ideology of data evaluation and visualization, J.M. developed an innovative approach to data analysis in his Ph.D. thesis, termed "*Kernels in Monotonic Systems Related to the Tasks of Automatic Classification*." This groundbreaking methodology not only diverged from traditional practices but also laid the groundwork for what would later be acknowledged in scientific literature as "*Monotone Linkage Functions*" in the scholarly literature.

Despite its initial designation as the "Monotone/Monotonic System," JM's model has transcended its origins to become a focal point of scholarly inquiry, as evidenced by its contemporary prominence in seminal works such as "Maximum Margin Separations in Finite Closure Systems" by Florian Seiffarth et al. (2021). Through its ingenuity and steadfast commitment to pushing the boundaries of knowledge, JM's Monotone System has etched an indelible mark on the landscape of data analysis, serving as a catalyst for researchers to explore new horizons in pursuit of scientific excellence.

Innovative approach to data analysis, grounded in the principles and theory of Monotonic Systems, has profoundly influenced the methodology of data analysis across diverse fields, ranging from computer science to data mining, machine learning, and pattern recognition. Initially introduced in 1971, the formal framework of this approach has facilitated its widespread adoption and evolution, earning recognition as "Monotonic Linkage Functions" and solidifying its status as a cornerstone concept within the field.

While the author's primary focus may not have been on game theory initially, the techniques and concepts stemming from this pioneering data analysis approach could have indirectly shaped the trajectory of game theory and its sub-fields, including the reflection theory of games. As data analysis continues to permeate into interdisciplinary realms such as economics, psychology, and political science, which frequently intersect with game theory, the author's groundbreaking work likely exerts a significant yet indirect influence on advancements in these domains.

Hence, it is plausible to assert that the author's methodological approach has yielded a broader impact than initially envisioned, potentially contributing to the advancement and refinement of game theory and its related sub-fields.

Prospects

The initiation of rigorous research into the intricacies of Monotonic System design promises to deepen theoretical understanding and accelerate the development of high-performance algorithms, thereby opening up fruitful avenues for future research. It is noteworthy that the study of steady states or stable sets, as evidenced by the book's many fruitful diagrams, turns out to be an extremely promising endeavor. It is worth emphasizing that algorithms designed to identify stable sets have proven to be extremely effective, since their computational complexity remains within the limits of traditional sorting algorithms for numerous statistical and other types of indicators.

However, in the landscape of monotonic systems, difficulties are found when elements or indicators simultaneously participate in pairwise interactions, and the pairs themselves interact as atomic units. Therefore, the basic kernel identification scheme requires a comprehensive re-evaluation, particularly in the context of schedules like multi-move strategic or reflexive games, especially in economics. This implementation highlights the increased complexity of kernel search in such monotonic systems. However, using a branch and bound algorithm seems viable.

It's evident that the inclination to explore a diverse array of objects, encompassing both discrete and continuous entities, through the prism of monotonic systems, is consistently driven by the quest to unveil concealed phenomena via numerical experimentation using diverse indicators. This approach, we believe, consistently fosters enthusiasm and encourages active participation in advancing and disseminating scientific knowledge. Achieving such a goal entails integrating the monotonic systems discussed in this publication into the algorithmic framework of methodologies employed by researchers, thereby facilitating comprehensive analyses of diverse data sets.

The concept or category of monotone (monotonic) system, independent and distinct from all that is usually referred to in the relevant literature as dynamic systems, is applied to computer science and communications, social sciences, social and network economics. It will appeal to specialists in specific areas of game theory and data analysis.

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